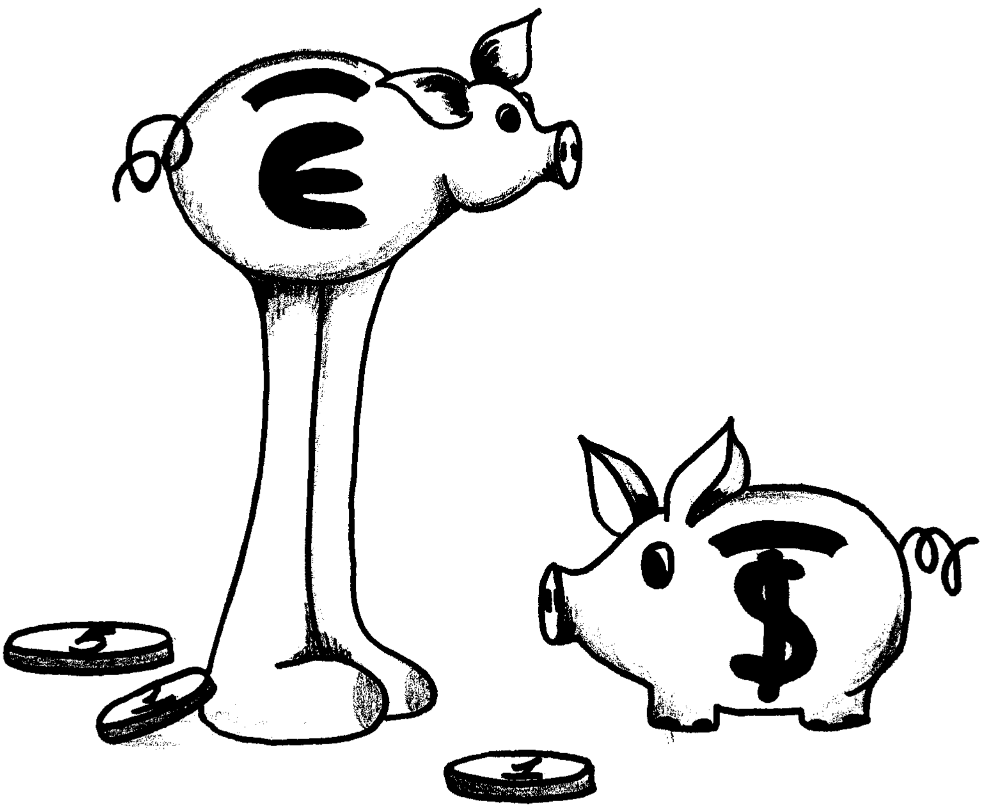
Essays in Option Pricing and Interest Rate Models



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Essays in Option Pricing and Interest Rate Models

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To my family and friends

Acknowledgments

When I came to Sweden I barely spoke English. I did not speak Swedish either. I hadn't been abroad for six years and then only for a very short period. I was not sure I would survive in Sweden. But I did, and not only that, a whole new world opened up before me.

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Stockholm, 2006

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Introduction and Summary

This thesis is devoted to derivatives pricing. **Part I** (Chapter I) addresses option pricing in incomplete markets. The most realistic models of financial markets are by nature and construction highly incomplete. Typical examples of incomplete market models are models for energy and weather derivatives, and stochastic volatility models. If we would like to compute an arbitrage-free price process for a financial derivative in an incomplete market, we are faced with the following well-known fact: there will exist infinitely many arbitrage free price processes for a given derivative. Moreover, the pricing bounds provided by merely requiring an absence of arbitrage are extremely wide, and thus quite useless from a practical point of view. Hence, there is a clear need for "reasonable" pricing bounds for derivative assets. We follow the idea first introduced by Cochrane and Saá Requejo (2000), who suggested ruling out not only the prices that violate the no arbitrage restriction, but also those that in some sense would represent "deals which are too good" and would not normally exist on the market. Cochrane and Saa-Requejo formalized the idea of a good deal essentially as an asset price process with a high Sharpe ratio and posed the problem of finding the upper and lower bounds for all arbitrage-free price processes, given a bound on the Sharpe ratio of the derivative.

Cochrane and Saá Requejo (2000) consider derivatives markets where incompleteness is caused by the presence of non-traded risks. They assume that the derivative payoff depends on several non-traded assets. The price processes of those assets are driven by a set of Wiener processes which are not included in a set driving the price processes of the basic traded assets.

Among these models, the jump models are of particular interest. Firstly, the existence of jumps in asset prices is well documented in the empirical literature. Secondly, the existence of jumps is one of the major reasons a market model can be incomplete, and thirdly, jumps are essential to all reduced-form credit risk models. Models with jumps can be considered fundamentally incomplete since they remain incomplete even without non-traded assets present in the model. Unfortunately, the approach used by Cochrane and Saa-Requejo Cochrane and

Saá Requejo (2000) does not allow us to treat models with asset prices driven by both jump and diffusion processes.

The main contribution of Chapter I is that it allows for jumps in the underlying and derivative processes; this is extremely important from both a theoretical and practical point of view. Using the martingale measure approach instead of the stochastic discount factor framework, we show that the good-deal pricing problem appears directly as a precise, well-formulated standard stochastic control problem. As a by-product, we obtain a generalization of the Hansen-Jagannathan bounds for the case of jump diffusions.

We also offer a relatively detailed study of some concrete point process-driven models, including the classical Merton jump-diffusion model. We present numerical results for these particular models, where we can compare the original Merton pricing formulas (for which the jump risk was assumed to be non-priced), with our derived good-deal bounds.

Part II (including Chapters II and III) is devoted to derivatives pricing on international markets and its particular focus is the term structures of interest rates. The factors driving the yield curves are documented to be international to a large extent. Thus, there is a clear need to take this into account when pricing or hedging financial derivatives. Also, the contagion literature seems to report high correlations between different markets. The development of international markets has highlighted the need to create a two-country model that would favor pricing and estimation procedures on an internationally integrated market.

Chapters II and III present two different approaches. In Chapter III we specify the finite set of factors and provide conditions on the factor dynamics so that the resulting bond prices in the two-country economy would take some predetermined form. This approach leads to the specification of an affine factor dynamics. The approach in Chapter II differs from the approach we employ in Chapter III, and is based on the geometric approach to the interest rate theory first introduced by Björk and Christenssen (1999) and Björk and Svensson (2001). Using the Heath-Jarrow-Morton approach, we choose to model domestic and foreign forward rates and the exchange rate dynamics under the domestic country martingale measure allowing the forward rate volatilities and the volatility of the exchange rate to be arbitrary smooth functionals of both forward rate curves. We do not make any specific assumptions concerning the number of factors driving the forward rates in the two-country economy, nor do we specify any particular dynamics for the factors. The geometric approach allows us to find sufficient (and in simpler cases necessary) conditions in terms of the exchange rate and forward rate volatilities to ensure that we obtain a finite dimensional realization (FDR) for both domestic and foreign forward rates. That is, the approach provides us with the structure of the forward rate volatilities such that it is possible to determine the number of factors and express the forward rate process as a function of the finite set of those factors.

The problem of specifying forward rate models which admit FDR is important from the point of view of model calibration. It has become very common to use the parameterized families of smooth forward rate curves when fitting a forward rate curve to the initial data. At each time step the model needs to be recalibrated, which is natural since the model does not describe the real world perfectly. The problem of the existence of an FDR is equivalent to the problem of determining if there exists a finitely parameterized family of forward rate curves, such that all the forward rate curves produced by the interest rate model are contained within the family. If the interest rate model does not admit an FDR, then recalibration is needed not only because the model is a poor description of reality but simply because the family of forward rates does not go well with the model.

The other problem of the existing interest rate models is that it is difficult to explain term structure movements in terms of outside factors connected with monetary policy if we consider just a simple nominal one-country model. Globalization suggests that the actions of central banks in different countries are very likely to influence term structures of other countries as well.

Chapter III suggests that there exists a finite set of factors that influences the prices of all the objects in the economy and extends a multifactor two-country economy setting so that it models joint affine domestic and foreign term structures, nominal exchange rates, domestic and foreign price levels, domestic and foreign real term structures as well as the real exchange rate between the two countries. This kind of model allows us to explain movements in a term structure in terms of exchange rate changes or macroeconomic shocks or policy-related factors.

However, if we want to take into account effects of macroeconomic shocks and policies in the international markets we might want to consider jump-diffusion factor dynamics. Thus, another contribution of the paper is that we allow the bond prices to be influenced by the multivariate factor whose price process is driven by a standard multidimensional Wiener process and a marked point process (with intensity dependent on the same set of factors). In particular, we provide necessary and sufficient conditions on the factor dynamics, domestic and foreign short rates and/or exchange rate volatilities (alternatively, the form of the exchange rate) and factor jump size in order to obtain affine domestic and foreign, real and nominal, term structures.

In particular, we answer the question of whether the affine framework is consistent with the empirical evidence supporting the Purchasing Power Parity (PPP) hypothesis; that is, under what conditions is the real exchange rate mean stationary in the jointly affine framework? If we ignore this issue we might create a model that right from the modeling stage violates the PPP hypothesis, and hence initially specifies the factor dynamics that would never allow the real exchange rate in the economy to be asymptotically mean stationary; thus, in the economy where PPP holds, we would never be able to obtain a good fit to the data.

We estimate the dynamics of the latent factors in a two-country model using the Kalman filter technique. As compared to all the previous studies, we include in the set of observables real domestic and foreign yields, as well as exchange rate and price levels.

As a practical application of the general international model presented, we construct a European type "real exchange" call option, and demonstrate how to value it. This derivative can be introduced and used by investors to hedge not only exchange rate risks or inflation risks, but joint exchange rate and inflation risks.

Part III (Chapter IV) is devoted to credit risk modeling. Recent empirical studies show that there is a significant systematic risk component in defaultable credit spreads. The purpose of Chapter IV is to present a reduced-form multiple default model that would be able to model the dependence of the credit spread term structure (at the firm level) on macroeconomic risks.

The current theoretical literature considers models where only the default intensity, or equivalently, the *probability of default* (PD) is dependent on a state variable assuming that the *loss given default* (LGD) is either fixed or at least independent of default intensities. We take that analysis one step further and consider the situation in which the same state variable influences both PD and LGD, making these two quantities dependent on one another. In assessing capital at risk, it is extremely important not to ignore the interdependence between PD and LGD, since this would lead to underestimation of the true risk borne by portfolio holders.

We assume that the default intensity and the recovery (given default) depend on the market situation (we consider a market index as a proxy for those). With the PD dependence we try to account for the fact that during bad economic times it is reasonable to expect more defaults, while with the LGD dependence we try to account for the fact that if the entire market is down, the market value of any firm's assets should be lower, and debt holders should recover less if a default occurs.

It is well known that market uncertainty and its level are negatively correlated. That is, periods of recession (low index level) also tend to be periods of high uncertainty (high index volatility) reflecting some sort of market panic, while periods of economic boom are perceived as safe periods with low uncertainty. In setting up the dynamics of the market index, we incorporate this realistic feature by allowing the local volatility of the index to depend negatively on its level.

We derive abstract results for a multiple default reduced-form model when the default events are modeled by a doubly stochastic marked point process, where both intensity and the mark's density depend on some state variable. Using a concrete model, we are able to simulate realistic behaviors of the term structure of credit spreads.

Part I

Option Pricing in Incomplete markets

Chapter 1

Towards a general theory of good-deal bounds

We consider an incomplete market in the form of a multidimensional Markovian factor model, driven by a general marked point process (representing discrete jump events), as well as by a standard multidimensional Wiener process. Within this framework, we study arbitrage-free good-deal pricing bounds for derivative assets, thereby extending the results from Cochrane and Saá Requejo (2000) to the point process case, while, at the same time, obtaining a radical simplification of the theory. To illustrate, we present numerical results for the classic Merton jump-diffusion model. As a by-product of the general theory, we derive extended Hansen-Jagannathan bounds for the Sharpe Ratio process in the point process setting.[1]

1.1 Introduction

The majority of existing financial markets are by nature highly incomplete, because there are not sufficiently many financial instruments on the market to hedge all risks. In some cases, the incompleteness can be considered negligible, but, in a large number of cases, the incompleteness is so severe that it leads to the need to study, on the theoretical level, market *models* that explicitly allow incompleteness.

Typical examples of incomplete market models are models for energy and weather

[1]The paper is coauthored with Tomas Björk, Stockholm School of Economics, and is forthcoming in *Review of Finance*. Presented at Berlin Mathematical Finance workshop for young researchers (01/2004), the Bachelier Finance Society Third World Congress (Chicago 07/2004), Stochastic Finance 2004 (Lisbon, 09/2004), European Finance Association, 32nd Annual Meeting (Moscow 09/2005).

derivatives, stochastic volatility models, models involving mortality risk, and models containing nontrivial jump components.

Among these models, the jump models are of particular interest. Firstly, the existence of jumps in asset prices is well documented in the empirical literature. Secondly, the existence of jumps is one of the major reasons for which a market model can be incomplete, and thirdly, jumps are essential to all reduced-form credit risk models. The object of the present paper is to study derivative pricing in incomplete models that are partially driven by jump processes.

Returning to general incomplete models, suppose now that we would like to compute an arbitrage-free price process for a financial derivative within one of the model classes mentioned above. Then, we face the following well-known facts.

- Since the underlying market is incomplete, there will not exist a unique martingale measure (or a unique stochastic discount factor). Thus there will exist infinitely many arbitrage free price processes for a given derivative.
- In incomplete settings, the pricing bounds that are provided by merely requiring absence of arbitrage are extremely wide, and thus useless from a practical point of view.

There is evidently a need for "reasonable" pricing bounds for derivative assets. In the seminal paper Cochrane and Saá Requejo (2000), the authors (henceforth CSR) introduce the new idea of ruling out, not only those prices which violate the no-arbitrage restriction, but also those prices which, in some sense, represent "deals which are too good" ("good deals"). CSR essentially formalize the idea of a good deal as an asset price process with a high Sharpe ratio and pose the problem of finding the upper and lower bound for all arbitrage-free price processes, given a bound on the Sharpe ratio of the derivative.

In Cochrane and Saá Requejo (2000), this question is analyzed in great detail for discrete time one-period and multi-period models. For continuous time models, the CSR setting is restricted to that of a purely Wiener-driven model, and it seems to be very hard to extend the CSR technique to allow for the existence of jumps (see below). The basic CSR approach is to work under the objective measure P using stochastic discount factors, and their basic technique is dynamic programming. Within this framework, Cochrane and Saá Requejo (2000) derive a pricing PDE, which is then studied in detail and, in some cases, solved numerically.

A similar, but not equivalent, approach to obtain asset price bounds, based on gains-loss-ratios, is presented in Bernardo and Ledoit (2000). See Rodriguez (2000) for an interesting connection of Bernardo and Ledoit (2000) to linear programming. A utility-based approach can be found in Černý (2003).

The principal object of the present paper is to extend the analysis of Cochrane and Saá Requejo (2000) to allow for the possibility of jumps in the random processes that describe the financial market under consideration. Thus, under the setup of the present paper, all processes are allowed to be driven, not only by a multidimensional standard Wiener process, but also by a general marked point process ("MPP"). In view of the discussion of jump models above, we think that this extension is quite important.

We wish to point out that the technique used in the present paper, as opposed to the one used in Cochrane and Saá Requejo (2000), is very much focused on martingale measures, rather than on stochastic discount factors. More precisely, while the authors in Cochrane and Saá Requejo (2000) choose to perform all calculations under the objective measure P using various SDFs, we instead work directly under the relevant equivalent martingale measures using the Girsanov Theorem. Of course, both approaches are logically equivalent, but, for a particular concrete problem, one approach may prove much more efficient than the other from a computational point of view. The general rule is that the method of working with SDFs under P is mainly employed in (empirical) asset pricing, whereas the direct use of martingale measures is the natural choice in derivatives pricing. For the class of problems studied in the present paper, the technique of working directly under the relevant martingale measures is, in our opinion, clearly to be preferred as a technical tool, since the problem at hand (including an MPP) cannot easily be addressed by the SDF approach.

To clarify the contributions -as well as the particular choice of technique- of the present paper, we start by summarizing some limitations of the CSR approach.

Under the SDF formulation in Cochrane and Saá Requejo (2000), the basic pricing problem is never precisely formulated as a standard stochastic control problem, in the sense that it there is no clear distinction between state variables and control variables. In addition, the dynamic programming argument is rather informal.

Cochrane and Saá Requejo make a number of *ad hoc* assumptions concerning the existence and structural properties of the upper and lower good-deal bound processes. Similar assumptions are made about the SDF process.

In the CSR model, the n-dimensional price process S for the underlying market is assumed to be driven by exactly n independent random factors (Wiener processes), and the volatility matrix for the price process S is assumed to have full rank. These constitute, in themselves, minor limitations.

However, the rank assumption is equivalent to assuming that the model is "conditionally complete", namely, incompleteness enters only through the factor process

Y, and that, without the Y process, the model would be complete. This structural property is used crucially in the arguments and it is hard to see how the arguments could be extended to allow for a model that is not conditionally complete.

The main limitation of Cochrane and Saá Requejo (2000) is, however, the absence of jumps. Again, the existence of jumps is one of the major reasons why a market model can be incomplete, so this is a severe limitation from a modeling perspective. Furthermore, in a model without jumps, the SDF automatically remains positive, whereas in a model including jumps, the positivity constraint on the SDF becomes binding. This is already noted in Cochrane and Saá Requejo (2000), and it appears that the CSR technique cannot easily handle a binding positivity constraint on the SDF.

With respect to the specific contributions of the present paper, we begin by noting that, when jumps are introduced into the model, the above assumptions from Cochrane and Saá Requejo (2000) are violated. The implication is that a large and substantial part of the CSR argument leading up to the central pricing equation breaks down.

In contrast, the martingale measure approach of the present paper does not incur the above problems. In particular, it does not require assumptions about the number of driving Wiener processes, nor does it need rank assumptions for the volatility matrix. Furthermore, the technique used in the present paper leads to a drastic simplification of the entire theory, compared to Cochrane and Saá Requejo (2000).

The present paper's major contribution is, however, that it allows for jumps in the underlying and derivative processes; this is extremely important from a theoretical as well as from a practical point of view. In contrast to CSR, our model is generically incomplete even *without* the presence of the factor process Y. This is of special importance when dealing with, for example, reduced-form credit risk models.

We also emphasize that, in the present paper, the good-deal pricing problem appears directly as a precise, well formulated standard stochastic control problem. Indeed, the relevant Bellman equation can be written down immediately, without need of a separate argument, and without need of any *ad hoc* assumptions.

As a by-product of our investigation, we obtain a generalization of the Hansen-Jagannathan (HJ) bounds for the case of jump diffusions.

In summary, the main contribution of the present paper is that we obtain a streamlined, rigorously developed, and surprisingly simple theory of good-deal pricing bounds for a very general class of jump diffusions.

The structure of our paper is as follows. Section 1.2 outlines the probabilistic framework for the rest of the paper. In Section 1.3.1, we present our basic factor market model. The pricing problem is formalized in Section 1.3.2 and, in Section 1.3.3, we derive the fundamental Dynamic Programming Equation for the upper and lower good-deal bounds. Section 1.3.4 discusses the special structure of this equation in some detail and connects the good-deal pricing bounds to the so-called "minimal martingale measure". Section 1.4 is a relatively detailed study of some concrete point process-driven models. The classical Merton jump-diffusion model in Merton (1976) falls within this category, and we present numerical results for that particular model, where we can compare the original Merton pricing formulas (for which the jump risk was assumed to be nonpriced), as well as the pricing formula obtained by using the minimal martingale measure, with our derived good-deal bounds.

In Appendix 1.5.1, we extend the Hansen-Jagannathan bounds from Hansen and Jagannathan (1991) to the general setup of Section 1.2.

For completeness' sake, we conclude by studying, in Appendix 1.5.2, the special case of a purely Wiener-driven model, and show how the Cochrane and Saá Requejo setup is nested within our framework.

1.2 General Setup

Our formal setup (see below for a more intuitive description) consists of a financial market model on a fixed time interval $[0, T]$, living on a stochastic basis (filtered probability space) $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $\mathbf{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$, and where the measure P is interpreted as the objective (or "physical") probability measure. The basis is assumed to carry a d-dimensional standard Wiener process W as well as a marked point process $\mu(dt, dx)$ on the mark space $(X, \mathcal{X})$. The space X can be fairly general, but, in most practical applications, it will be R^k. The filtration $\mathbf{F}$ is assumed to be the internal one, i.e., the filtration generated by W and μ. The predictable (see below for comment) σ-algebra is denoted by $\mathcal{P}$, and we make the definition $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{X}$. We assume that the predictable compensator $\nu(dt, dx)$ admits an intensity, i.e., that we can write $\nu(dt, dx) = \lambda_t(dx)dt$. The compensated point process $\mu(dt, dx) - \lambda_t(dx)dt$ is denoted by $\tilde{\mu}(dt, dx)$.

Remark 1.2.1. *The intuitive interpretation of the point process μ is that we are modeling events which occur at discrete points in time, and a very concrete example could be the modeling of earthquakes (or stock market crashes). As opposed* *to a more standard counting process setting, these discrete events are not all of the same type. Instead, each event is identified by its "mark" $x \in X$. In the earthquake example, a natural mark would be the strength of the earthquake on the Richter scale; in this case, the mark space X is the positive real line. The informal interpretation of the point process μ is that μ is an integer-valued (random) measure such that μ has a unit point mass at the point $(t, x) \subseteq R_+ \times X$, if, at time t, there is an event of the type x. The interpretation of the intensity measure λ is, loosely speaking, that $\lambda_t(dx)$ is the expected number of events with marks in a "small set" dx, per unit time, conditional on the information in $\mathcal{F}_{t-}$. Thus, the compensated point process $\tilde{\mu}(dt, dx) := \mu(dt, dx) - \lambda_t(dx)dt$ is "detrended" and possesses a natural martingale property. The above concept of "predictability" is very important when dealing with jump processes. The formal definition is somewhat complicated but, in most concrete cases, the reader can informally substitute the term "predictable" by "adapted and left continuous".*

We need some mild integrability conditions, which we summarize as follows.

Assumption 1.2.2. *We assume the following:*

- *Defining the underlying counting process N by $N_t = \nu([0, t] \times X)$, we assume that $N_t < \infty$ P-a.s. for all finite t, i.e., μ is a multivariate point process in the terminology of Jacod and Shiryaev (1987). We note that the intensity process of N is given by $\lambda_t = \lambda_t(X)$.*

- *There exists a deterministic increasing real valued function C_t such that, for all $t \geq 0$, we have*

$$\int_0^t \lambda_s ds < C_t, \quad P - a.s. \tag{1.1}$$

- *We assume that the counting process N defined above has exponential moments of all orders, i.e., that for all $t \geq 0$ and all constants $c > 0$ we have*

$$\mathbb{E}^P \left[e^{cN_t} \right] < \infty. \tag{1.2}$$

1.3 A Factor Market Model

We now present our Markovian factor market model, and we also formalize our pricing problem.

1.3.1 The Model

We consider a financial market model built up by the following objects, where * denotes transpose.

- An n-dimensional price process $S = (S^1, \ldots, S^n)^\star$
- A k-dimensional factor process $Y = (Y^1, \ldots, Y^k)^\star$.

The interpretation of this is that $S^1, \ldots, S^n$ are prices of underlying traded assets without dividends, whereas the components of Y are underlying nontraded factors.

Remark 1.3.1. *Note that the underlying asset in the price vector S above can be "standard" underlying assets, like common stocks, but that we also allow S to contain prices of derivatives of arbitrary types, for example, call options, bonds, or bond options. They are only "underlying " in the sense that they, within the model, are considered a priori given. Evidently, they do* **not** *include the derivatives for which we wish to derive the good-deal bounds.*

The precise probabilistic specification of the market model is given by the following standing assumption. See the remark below for an interpretation of the point process integrals appearing in (1.3)-(1.4).

Assumption 1.3.2.

1. *Under the objective measure P, we assume that (S, Y) satisfies the following stochastic differential equations (SDEs):*

$$dS_t^i = S_t^i \alpha_i (S_t, Y_t)\, dt + S_t^i \sigma_i(S_t, Y_t) dW_t + S_{t-}^i \int_X \delta_i(S_{t-}, Y_{t-}, x) \mu(dt, dx), \quad i = 1, \ldots, n \tag{1.3}$$

$$dY_t^j = a_j (S_t, Y_t)\, dt + b_j(S_t, Y_t) dW_t + \int_X c_j(S_{t-}, Y_{t-}, x) \mu(dt, dx). \quad j = 1, \ldots, k \tag{1.4}$$

2. *We assume that, for each i and j, $\alpha_i(s, y)$ and $a_j(s, y)$ are deterministic scalar functions, $\sigma_i(s, y)$ and $b_j(s, y)$ are deterministic row vector functions, and $\delta_i(s, y, x)$ and $c_j(s, y, x)$ are deterministic scalar functions. To avoid negative asset prices, we also assume that $\delta_i(s, y, x) \geq -1$ for all i and all (s, y, x).*

3. *All functions above are assumed to be sufficiently regular to allow for the existence of a unique strong solution for the system of SDEs.*

4. *The point process μ has a predictable P-intensity measure λ. More precisely, we assume that the P-compensator $\nu(dt, dx)$ has the form*

$$\nu(dt, dx) = \lambda(S_{t-}, Y_{t-}, dx)dt. \tag{1.5}$$

For brevity of notation, we often denote $\lambda(S_{t-}, Y_{t-}, dx)$ by $\lambda_t(dx)$. The compensated point process $\tilde{\mu}$ is defined by

$$\tilde{\mu}(dt, dx) = \mu(dt, dx) - \lambda(S_{t-}, X_{t-}, dx)dt, \tag{1.6}$$

i.e., we obtain $\tilde{\mu}$ by "detrending" μ with its mean.

5. *We assume the existence of a short rate r of the form*

$$r_t = r(S_t, Y_t).$$

6. *We assume that the model is free of arbitrage in the sense that there exists a (not necessarily unique) risk-neutral martingale measure Q.*

Remark 1.3.3. *The interpretation of the point process integrals, which appear in (1.3)-(1.4), is very simple. If, at time t, there is an event of type x, then there will be a jump in S^i with relative jump size δ_i, and correspondingly for the Y processes. Note that if the relative jump size δ_i takes the value -1 at some point, then the stock price S^i immediately jumps to zero and stays there for ever. This property of the model is important when dealing with bankruptcy problems.*

The present setup extends the one in Cochrane and Saá Requejo (2000) in two ways. In the CSR setting, the continuous time model is purely Wiener driven, whereas the main contribution of the present paper is that we extend the framework to also include a driving point process. Furthermore, even in the purely Wiener-driven case, our setup extends that of Cochrane and Saá Requejo (2000) because we do not make rank assumptions for the diffusion matrices σ and b, and because we allow the entire vector Wiener process W to drive price vector S. (In the CSR setting, S is only allowed to be driven by an n-dimensional subset of the W vector process.)

For future use, we introduce some more compact notation. Note that all objects below are *functions* of s, y, and (in the case of δ) x, to spaces of vectors and matrices.

Definition 1.3.4. The column vector functions, α, δ, a, and c are defined by

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}. \tag{1.7}$$

The $n \times d$ matrix functions σ and the $k \times d$ matrix function b are defined by

$$\sigma = \begin{bmatrix} -\sigma_1- \\ \vdots \\ -\sigma_n- \end{bmatrix}, \quad b = \begin{bmatrix} -b_1- \\ \vdots \\ -b_k- \end{bmatrix}. \tag{1.8}$$

1.3.2 The Problem

On the market specified above, we consider an arbitrarily chosen contingent T-claim $\mathcal{Z}$ of the form

$$\mathcal{Z} = \Phi(S_T, Y_T). \tag{1.9}$$

The problem is to compute a reasonable price process $\Pi(t; \mathcal{Z})$ for the claim $\mathcal{Z}$.

From general theory, we know that each arbitrage-free price will be given by an expression of the form

$$\Pi(t; \mathcal{Z}) = \mathbb{E}^Q \left[e^{-\int_t^T r_u du} \Phi(S_T, Y_T) \middle| \mathcal{F}_t \right]. \tag{1.10}$$

where Q is a martingale measure for the underlying S.

Since the market in the general case is incomplete, the martingale measure Q (and the stochastic discount factor, see the appendix for details) will generically not be unique, so there will not be a uniquely determined arbitrage-free price for $\mathcal{Z}$. It is also well known that, in incomplete settings like this, the pricing bounds provided by merely requiring absence of arbitrage, i.e., varying Q in the above formula over the entire set of martingale measures, are extremely wide and thus useless from a practical point of view.

There is thus a clear need for reasonable pricing bounds for derivative assets. To this end, we follow the approach in Cochrane and Saá Requejo (2000) and pose the following informal problem.

We wish to compute upper and lower bounds on the price of the derivative by ruling out, not only the prices which violate the no-arbitrage restriction, but also those prices which represent deals that are too good, in the sense that the Sharpe Ratio process is "too high".

This naive formulation of the pricing problem, in terms of a bound on the Sharpe Ratio process on the derivative under study, turns out to be mathematically intractable (see Cochrane and Saá Requejo (2000), and does not rule out the existence of portfolios based on the derivative and the underlying with very high Sharpe ratios.

What we therefore need is a formalization of the pricing problem that gives us a mathematically tractable problem, and that, at the same time, allows us complete control over the Sharpe ratio processes of *all portfolios*, based on the underlying assets and the derivative. Accordingly, we utilize the Hansen-Jagannathan inequality (see Appendix 1.5.1) to replace the bound on the Sharpe ratio with a bound on the norm of the market price of risk vector.

To carry out this scheme, we start by determining the class of martingale measures for the underlying model, and we define, for each potential martingale measure Q equivalent to P, the corresponding likelihood process L through

$$\frac{dQ}{dP} = L_t, \quad \text{on } \mathcal{F}_t; \quad 0 \le t \le T. \tag{1.11}$$

Since L is always a P-martingale, and since every martingale within the present framework, where the filtration is the internal one, admits a stochastic integral representation (see Jacod and Shiryaev, 1987), we know that L must have dynamics of the form

$$\begin{cases} dL_t &= L_t h_t^\star dW_t + L_{t-} \int_X \varphi_t(x) \left\{ \mu(dt, dx) - \lambda_t(dx)dt \right\}, \\ L_0 &= 1, \end{cases} \tag{1.12}$$

where the **Girsanov kernel** processes h and φ (where we view h as a column vector process, hence the transpose *) are predictable, suitably integrable (see Jacod and Shiryaev, 1987 for details), and where φ must satisfy the condition

$$\varphi_t(x) \ge -1, \quad \forall t, x \quad P - a.s. \tag{1.13}$$

in order to ensure the nonnegativity of the measure Q.

Remark 1.3.5. *We note that if we do have $\varphi_t(x) = -1$, then we may lose the equivalence between P and Q, but retain the property that Q is absolutely continuous wrt P. Any ensuing dynamic arbitrage possibilities can be dealt with in two ways. One possibility is to replace the inequality $\varphi_t(x) \ge -1$ by $\varphi_t(x) \ge -1+\epsilon$ for a very small value of ϵ. Another, more natural alternative is to view the good-deal bounds derived below with the constraint $\varphi_t(x) \ge -1$ as* **open** *intervals of good-deal bounds, since the constraint only becomes binding at the boundary.*

The economic interpretation of h and φ is that $-h$ is "the market price vector of diffusion risk", and that $-\varphi_t(x)$ is the "market price of jump risk of type x" (see Appendix 1.5.1 for details). From the Girsanov Theorem, see Jacod and Shiryaev (1987), we also recall the following facts:

- We can write
$$dW = h_t dt + dW^Q, \tag{1.14}$$
where W^Q is a Q-Wiener process.
- The point process μ will, under Q, have an intensity λ^Q, given by
$$\lambda_t^Q(dx) = \{1 + \varphi_t(x)\}\, \lambda_t^P(dx). \tag{1.15}$$

The immediate problem is to find out how the kernel processes h and φ above must be chosen in order to guarantee that Q actually is a martingale measure for S. To this end we use the Girsanov Theorem to obtain the Q-dynamics of S^i, $i = 1, \ldots, n$ as follows, where, for brevity of notation, we write σ_i as shorthand for $\sigma_i(S_t, Y_t)$, and similarly for other terms:

$$dS_t^i = S_t^i \{\alpha_i + \sigma_i h_t\} dt + S_t^i \sigma_i dW_t^Q + S_{t-}^i \int_X \delta_t(x) \mu(dt, dx).$$

We then compensate the point process μ under Q to obtain a representation of S as a Q martingale part, plus a drift part.

$$\begin{aligned} dS_t^i &= S_t \left\{ \alpha_i + \sigma_i h_t + \int_X \delta_i(x) \lambda_t^Q(dx) \right\} dt + S_t^i \sigma_i dW_t^Q \\ &\quad + S_{t-}^i \int_X \delta_i(x) \left\{ \mu(dt, dx) - \lambda_t^Q(dx) dt \right\}. \end{aligned} \tag{1.16}$$

Using (1.15), this can be written as

$$\begin{aligned} dS_t^i &= S_t \left\{ \alpha_i + \sigma_i h_t + \int_X \delta_i(x) \{1 + \varphi_t(x)\} \lambda_t(dx) \right\} dt + S_t^i \sigma_i dW_t^Q \\ &\quad + S_{t-}^i \int_X \delta_i(x) \left\{ \mu(dt, dx) - \lambda_t^Q(dx) dt \right\}. \end{aligned} \tag{1.17}$$

Recalling that the measure Q is a martingale measure if, and only if, the local rate of return of S under Q equals the short rate r, we thus obtain the **martingale condition**

$$\alpha_i + \sigma_i h_t + \int_X \delta_i(x) \{1 + \varphi_t(x)\} \lambda_t(dx) = r_t, \quad i = 1, \ldots, n. \tag{1.18}$$

A Girsanov kernel process (h, φ), which satisfies the positivity constraint (1.13) and the martingale condition (1.18), is referred to as an **admissible** Girsanov kernel.

We can now state our extension of the Hansen-Jagannathan inequality. See Appendix 1.5.1 for the proof and for a precise definition of the Sharpe ratio process SR.

Theorem 1.3.6 (Extended Hansen-Jagannathan Bounds).
For every arbitrage-free price process, derivative or underlying, and for every admissible Girsanov kernel (market price of risk) process (h, φ), the following inequality holds

$$|SR_t|^2 \leq \|h_t\|_{R^d}^2 + \int_X \varphi_t^2(x)\lambda_t(dx), \tag{1.19}$$

where SR denotes the Sharpe Ratio Process.

Remark 1.3.7. *It is important to note that the HJ inequality not only holds for the given underlying asset prices. Specifically, suppose that we choose a fixed pair of Girsanov kernels (h, φ), and that we use the martingale measure induced by these to price various derivatives. Then, the inequality holds for all underlying assets, for all derivatives, and for all self-financing portfolios based on the underlying and the derivatives. In other words, for a given choice of (h, φ), the HJ inequality gives us a uniform upper bound of Sharpe ratios for the entire economy. Note that the Sharpe ratio is not a number but a full fledged random process.*

Since the right hand side of the HJ inequality also trivially bounds the underlying market, we need an assumption of uniform boundedness of the underlying market in order to have a nontrivial problem.

Assumption 1.3.8. *We assume that there exists a real number B such that*

$$\|h_t\|_{R^d}^2 + \int_X \varphi_t^2(x)\lambda_t(dx) \leq B, \tag{1.20}$$

with probability one for all t, and for all Girsanov kernels that are admissible for the underlying market. Henceforth, B_0 will denote the smallest B with this property.

We now express our pricing problem as the problem of finding upper and lower arbitrage-free pricing bounds subject to an upper bound on the norm of the Girsanov kernel (h, φ), or, equivalently, on the norm of market price of risk vector $(-h, -\varphi)$.

Definition 1.3.9. Given a bound B for the market prices of risk, the **upper good-deal price process** is defined as the optimal value process for the following optimal control problem:

$$\max_{h,\varphi} \quad \mathbb{E}^Q\left[e^{-\int_t^T r_u du}\Phi\left(S_T, Y_T\right)\Big| \mathcal{F}_t\right] \tag{1.21}$$

with Q dynamics

$$\begin{aligned} dS_t^i &= S_t^i\left\{r_t - \int_X \delta_i(x)\left\{1+\varphi_t(x)\right\}\lambda_t(dx)\right\}dt + S_t^i\sigma_i dW_t^Q \\ &\quad + S_{t-}^i\int_X \delta_i(x)\mu(dt, dx), \quad i = 1, \ldots, n \end{aligned} \tag{1.22}$$

$$\begin{aligned} dY_t^j &= \left\{a_j + b_j h_t\right\}dt + b_j dW_t^Q \\ &\quad + \int_X c_j(x)\mu(dt, dx). \quad j = 1, \ldots, k. \end{aligned} \tag{1.23}$$

The predictable processes h and φ are subject to constraints

$$\alpha_i + \sigma_i h_t + \int_X \delta_i(x)\left\{1+\varphi_t(x)\right\}\lambda_t(dx) = r_t, \quad i = 1, \ldots, n \tag{1.24}$$

$$||h_t||_{R^d} + \int_X \varphi_t^2(x)\lambda_t(dx) \leq B^2, \tag{1.25}$$

$$\varphi_t(x) \geq -1, \quad \forall t, x. \tag{1.26}$$

Some comments are perhaps in order.

- The expected value in (1.21) is the standard risk-neutral valuation formula for contingent claims.
- In (1.24), we have the martingale conditions from (1.18) on h and φ, guaranteeing that the induced measure Q is indeed a martingale measure for $S^1, \ldots, S^n$.
- The induced Q dynamics of $S^1, \ldots, S^n$ are given in (1.22).
- The induced Q dynamics of $Y^1, \ldots, Y^k$ are given in (1.23).
- The constraint (1.25) is the constraint that rules out good deals. To obtain a nondegenerate problem, we only consider the case when $B \geq B_0$, where B_0 is given in Assumption 1.3.8.

- The constraint (1.26) is needed to ensure that Q is a nonnegative measure.
- The lower pricing bound is defined by the corresponding minimum problem.

Remark 1.3.10. *We also need to check that the likelihood process induced by a pair of admissible Girsanov kernels above is a true martingale and not merely a local martingale. In our case, this is automatically ensured by the good-deal bound constraints. For the Wiener part, the standard Novikov condition is trivially satisfied. For the point process part, the result follows from the good-deal bound constraint, the integrability properties ensured by Assumption 1.2.2, and from Theorem 11 in Chapter VIII of Brémaud (1981).*

1.3.3 The Pricing Equation

To permit us to treat the optimal control problem above with dynamic programming methods, we have to make an extra assumption that ensures that the Markovian structure is preserved under the martingale measure Q.

Assumption 1.3.11. *We assume henceforth that the Girsanov kernel processes h and φ are of the restricted form*

$$h_t = h(t, S_t, Y_t), \tag{1.27}$$

$$\varphi_t(x) = \varphi(t, S_{t-}, Y_{t-}, x). \tag{1.28}$$

Here, with a slight abuse of notation, the right hand side occurences of h and φ denote deterministic functions of the form $h : R_+ \times R^n \times R^k \to R^n$, and $\varphi : R_+ \times R^n \times R^k \times X \to R$, respectively.

Next, we present the basic pricing equation for the upper and lower good-deal bounds - in the present setting, this is quite straightforward. Under Assumption 1.3.11, the optimal expected value in (1.21) can be written as $V(t, S_t, Y_t)$, where the deterministic mapping $V : R_+ \times R^n \times R^k \to R$ is known as the **optimal value function**. Since we are in a standard setting for dynamic programming (DynP), we know from general DynP theory that the optimal value function will satisfy the following Bellman-Hamilton-Jacobi equation on the time interval $[0, T]$:

$$\frac{\partial V}{\partial t} + \sup_{h,\varphi} \mathbf{A}^{h,\varphi} V - rV = 0, \tag{1.29}$$

$$V(T, s, y) = \Phi(s, y), \tag{1.30}$$

where the sup is subject to constraints of the form (1.24)-(1.26), and where $\mathbf{A}^{h,\varphi}$ denotes the infinitesimal operator for the process (S, Y), under the measure Q defined by h and φ.

We recall that, from an operational point of view, the infinitesimal operator $\mathbf{A}^{h,\varphi}$ is merely the integro-differential operator that appears in the dt term in the stochastic differential $dV(t, S_t, Y_t)$ (when the point process increment has been compensated). A standard application of the Itô formula for semimartingales will, in fact, give us the following result.

Proposition 1.3.12. *The infinitesimal operator $\mathbf{A}^{h,\varphi}$ is given by*

$$\begin{aligned}
&\mathbf{A}^{h,\varphi}V(t,s,y) = \\
= \quad & \sum_{i=1}^{n} \frac{\partial V}{\partial s_i}(t,s,y)s_i \left\{ r - \int_X \delta_i(s,y,x)\left\{1+\varphi(t,s,y,x)\right\}\lambda_t(s,y,dx)\right\} \\
+ \quad & \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t,s,y)\left\{a_j(s,y)+b_j(s,y)h(t,s,y)\right\} \\
+ \quad & \int_X \Delta V(t,s,y,x)\left\{1+\varphi(t,s,y,x)\right\}\lambda_t(s,y,dx) \\
+ \quad & \frac{1}{2}\sum_{i,l=1}^{n} \frac{\partial^2 V}{\partial s_i \partial s_l}(t,s,y)s_i s_l \sigma_i^\star(s,y)\sigma_l(s,y) + \frac{1}{2}\sum_{j,l=1}^{k} \frac{\partial^2 V}{\partial y_j \partial y_l}(t,s,y)b_j^\star(s,y)b_l(s,y) \\
+ \quad & \sum_{i,j=1}^{k} \frac{\partial^2 V}{\partial s_i \partial y_j}(t,s,y)s_i\sigma_i^\star(s,y)b_j(s,y). \qquad (1.31)
\end{aligned}$$

Here, ΔV is defined by

$$\Delta V(t,s,y,x) = V\left(t, s(1+\delta(s,y,x)), y + c(s,y,x)\right) - V(t,s,y), \qquad (1.32)$$

where addition and multiplication in $s(1+\delta(s,y,x))$ and $y+c(s,y,x)$ are interpreted componentwise.

Proof. An easy application of the Itô formula. ■

Collecting the facts above, we can finally present the basic equation for the upper good-deal bound.

Theorem 1.3.13. *The upper good-deal bound function is the solution V to the following boundary value problem:*

$$\begin{aligned}
\frac{\partial V}{\partial t}(t,s,y) + \sup_{h,\varphi}\left\{\mathbf{A}^{h,\varphi}V(t,s,y)\right\} - r(s,y)V(t,s,y) &= 0, \qquad (1.33) \\
V(T,s,y) &= \Phi(s,y). \qquad (1.34)
\end{aligned}$$

Here, $\mathbf{A}^{h,\varphi}V$ is given by (1.31), and the supremum in (1.33) is over all functions $h(t,s,y)$ and $\varphi(t,s,y,x)$, satisfying, for all (t,s,y), the constraints

$$\alpha_i + \sigma_i h + \int_X \delta_i(x)\left\{1+\varphi(x)\right\}\lambda_t(dx) \quad = \quad r, \quad i=1,\ldots,n \tag{1.35}$$

$$\|h\|_{R^d} + \int_X \varphi^2(x)\lambda_t(dx) \quad \leq \quad B^2, \tag{1.36}$$

$$\varphi(x) \quad \geq \quad -1. \tag{1.37}$$

The lower bound price function satisfies the same equation with the supremum operator replaced by $\inf_{h,\varphi}$.

1.3.4 On The Structure of the Pricing Equation

The pricing equation (1.33)-(1.34) is a partial integro-differential equation (PIDE,) which generally cannot be solved analytically. However, the equation possesses particular features which we wish to highlight.

As in all applications of stochastic dynamic programming, we note that the stochastic intertemporal optimal control problem (1.21)-(1.26) is reduced to the following two purely deterministic problems:

1. The static optimization problem of finding, for each fixed (t,s,y), the optimal h and φ in the constrained maximization problem

$$\sup_{h,\varphi}\left\{\mathbf{A}^{h,\varphi}V(t,s,y)\right\}, \tag{1.38}$$

   as appears in (1.33).

2. Having solved the static problem above, and denoted the optimal h,φ by $\hat{h},\hat{\varphi}$, we have to solve the PIDE

$$\frac{\partial V}{\partial t} + \mathbf{A}^{\hat{h},\hat{\varphi}}V - rV \quad = \quad 0, \tag{1.39}$$

$$V(T,s,y) \quad = \quad \Phi(s,y). \tag{1.40}$$

Obviously, if we wish to solve the PIDE in step 2 above, we first have to solve the static optimization problem in step 1; it is of great importance to understand the structure of the static problem. We then note that this is, in fact, an infinite-dimensional problem.

More precisely, problem (1.38) has to be solved for every fixed choice of (t, s, y), and the control variables are h and φ. However, whereas the diffusion kernel $h(t, s, y)$ (for fixed t, s and y) is merely a d-dimensional vector, the point process kernel $\varphi(t, s, y, \cdot)$ has to be determined as a function of x and, hence, $\varphi(t, s, y, \cdot)$ is an infinite-dimensional control variable. We thus see that the static optimization problem is not a standard finite-dimensional mathematical programming problem, but a full fledged variational problem. See Section 1.4.6 for a more detailed discussion of how a random jump size affects the good-deal bounds.

The infinite dimensionality of the static optimization problem is intimately connected to the cardinality of the mark space X (or rather, to the cardinality of the support of the measure $\lambda_t(dx)$). If the mark space has an infinite number of elements, then the static problem is infinite dimensional. If, on the other hand, X has a finite number of elements, then the static problem is a finite-dimensional problem. From a modeling point of view, this basically means that if we wish to model a situation with an infinite number of possible jump sizes, then the static problem becomes a variational problem.

Even if the static problem is infinite dimensional, it has a very particular structure. When we examine the expression (1.31) for the infinitesimal operator $\mathbf{A}^{h,\varphi}$, we see that only three terms involve the control variables h and φ and that, in fact, the control variables enter linearly. With notation as in Definition 1.3.4, we formalize this observation in a lemma.

Lemma 1.3.14. *The static optimization problem in (1.33) and (1.38) can be written as*

$$\max_{h,\varphi} \quad \langle \Delta V, \varphi \rangle_{\lambda_t} - V_s D(s) \langle \delta, \varphi \mathbf{1} \rangle_{\lambda_t} + V_y bh, \tag{1.41}$$

subject to the constraints

$$\begin{aligned} \alpha + \sigma h + \langle \delta, \mathbf{1} \rangle_{\lambda_t} + \langle \delta, \varphi \mathbf{1} \rangle_{\lambda_t} &= r\mathbf{1}, & (1.42) \\ \|h\|^2_{R^d} + \|\varphi\|^2_{\lambda_t} &\leq B^2, & (1.43) \\ \varphi &\geq -1. & (1.44) \end{aligned}$$

Here, V_s and V_y denote the gradients of V wrt, the vector variables s, and y, respectively, $D(s)$ denotes the diagonal matrix with the components of s on the diagonal, $\mathbf{1}$ denotes the column vector with 1 in all components, and the inner products $\langle \delta, \varphi \mathbf{1} \rangle_{\lambda_t}$ and $\langle \delta, \mathbf{1} \rangle_{\lambda_t}$ are interpreted componentwise.

When we write the static problem on this form, we see very clearly that we have a linear objective function, an infinite-dimensional linear equality constraint, a scalar quadratic inequality constraint, and an infinite-dimensional linear inequality constraint. Since the set of admissible points is convex, the linearity of the objective function implies that the optimal point is an extremal point of the admissible set. It is also clear that at least one of the inequality constraints has to be binding.

1.3.5 Positivity and the minimal martingale measure

The static problem (1.41) is, apart from the positivity constraint (1.44), a fairly standard linear quadratic problem in the space $L^2[X, \lambda(dx)]$. The really problematic part is the generically infinite-dimensional positivity constraint (1.44). The problem with this constraint is that, although the standard finite-dimensional Kuhn-Tucker theory can, to some extent, be transferred into an infinite-dimensional setting, it requires some nontrivial topological assumptions to be satisfied. These conditions appear only in the truly infinite-dimensional cases, and in those cases the conditions are vital. In our case, the technical condition needed is that the positive cone has a nonempty interior but, unfortunately for us, the positive cone in L^2 does not contain any interior points, which effectively prohibits us from using standard infinite-dimensional Kuhn-Tucker methodology. Below, we discuss the positivity constraint in more detail.

As a first approach to solving Problem (1.41), one hopes (perhaps) that the positivity constraint is not binding in the optimal solution. It is thus natural to solve a relaxed version of Problem (1.41) where the positivity constraint is not present and, having found the optimal solution, to then check whether the positivity constraint is binding or not. If the positivity constraint turns out not to be binding at the relaxed optimal point, then all is well and we have found our optimal solution. If the constraint is violated at the optimal point of the relaxed problem, then the induced measure is not a positive measure. If we use this measure for pricing, it still gives us pricing bounds, but these are wider than the optimal ones. We now formalize these ideas, and we also relate them to the concept of the minimal martingale measure from the theory of local risk minimization.

Definition 1.3.15.

- Denote the optimal upper and lower bound Girsanov kernels from Theorem 1.3.13 by (h^s, φ^s) and (h^i, φ^i), denote the corresponding **optimal martingale measures** by Q^s, Q^i, and define the pricing functions V^s and V^i correspondingly. Here, "s" stands for "sup" and "i" stands for "inf".
- Denote by $(\bar{h}^s, \bar{\varphi}^s)$ the optimal kernels for the relaxed static problem

$$\sup_{h,\varphi} \quad \mathbf{A}^{h,\varphi}V(t,s,y), \tag{1.45}$$

subject to the constraints

$$\alpha_i + \sigma_i h + \int_X \delta_i(x)\{1+\varphi(x)\}\lambda_t(dx) = r, \quad i = 1,\dots,n \tag{1.46}$$

$$\|h\|_{R^d} + \int_X \varphi^2(x)\lambda_t(dx) \leq B^2, \tag{1.47}$$

and denote by $(\bar{h}^i, \bar{\varphi}^i)$ the optimal solutions to the corresponding minimization problem. Denote the solution to the PIDE

$$\frac{\partial V}{\partial t} + \mathbf{A}^{\bar{h}^s,\bar{\varphi}^s} V - rV = 0, \tag{1.48}$$

$$V(T,s,y) = \Phi(s,y), \tag{1.49}$$

by $\bar{V}^s$, and define $\bar{V}^i$ in the same way. The **relaxed martingale measures** induced by $(\bar{h}^s, \bar{\varphi}^s)$ and $(\bar{h}^i, \bar{\varphi}^i)$ are denoted by $\bar{Q}^s$, and $\bar{Q}^i$, respectively.

- Denote by (h^m, φ^m) the Girsanov kernels obtained by solving the problem

$$\min_{h,\varphi} \quad \|h\|^2_{R^d} + \|\varphi\|^2_{\lambda_t} \tag{1.50}$$

subject to

$$\alpha_i + \sigma_i h + \int_X \delta_i(x)\{1+\varphi(x)\}\lambda_t(dx) = r, \quad i = 1,\dots,n. \tag{1.51}$$

Denote the solution to the PIDE

$$\frac{\partial V}{\partial t} + \mathbf{A}^{h^m,\varphi^m} V - rV = 0, \tag{1.52}$$

$$V(T,s,y) = \Phi(s,y), \tag{1.53}$$

by V^m. The measure Q^m, henceforth referred to as the **Minimal Martingale Measure** ("MMM"), is defined as the (possibly signed) measure induced by (h^m, φ^m).

For the measures defined above, the first four depend on the choice of derivative to be priced, whereas the minimal martingale measure Q^m is independent of the choice of derivative. We see that the MMM kernels (h^m, φ^m) are obtained by minimizing the right hand side of the HJ inequality, subject to the "martingale condition" (1.51), but without the positivity constraint, so the MMM is the P-equivalent measure with pointwise minimal L^2 norm satisfying the martingale constraint. It may happen that the MMM is a signed measure, thus assigning negative "probability" to some events. The problem with a signed MMM only occurs in the presence of jumps, so in a pure diffusion setting the MMM will always be a bona fide probability measure. The minimal martingale measure was first defined in connection with local risk minimization (see Schweizer (1995)), where it plays a fundamental role. The original definition of the MMM in Schweizer (1995) is not the one given above, but it is fairly easy to see that, in the present context, the two definitions coincide. We now derive the following easy result.

Proposition 1.3.16.

- *We always have the relations*

$$\bar{V}^i \leq V^i \leq V^s \leq \bar{V}^s, \tag{1.54}$$

  and

$$\bar{V}^i \leq V^m \leq \bar{V}^s. \tag{1.55}$$

- *If the positivity constraint is satisfied by $\bar{h}^i$, $\bar{h}^s$ and h^m, then $\bar{Q}^i$, $\bar{Q}^s$ and Q^m are probability measures (and not just signed measures), and we have*

$$\bar{V}^i \leq V^i \leq V^m \leq V^s \leq \bar{V}^s. \tag{1.56}$$

Proof. Obvious from the definitions. ■

The moral of this can be summarized as follows:

- The minimal martingale measure provides us with a canonical benchmark for pricing any derivative. Furthermore, since the MMM is the solution to a standard minimum norm problem in L^2, it can easily be computed.

- The relaxed martingale measures $\bar{Q}^i$ and $\bar{Q}^s$ are much easier to compute than the optimal measures Q^i and Q^s. The reason for this is that, for the relaxed measures, we disregard the positivity constraints which are highly complex and impossible to handle with Kuhn-Tucker methodology. The corresponding relaxed static problems can be solved explicitly using vector space methods.

- The pricing bounds provided by the relaxed measures $\bar{Q}^i$ and $\bar{Q}^s$ are generically not optimal but, for reasonable values of the Sharpe ratio constraint B, they turn out to be much tighter than the no-arbitrage bounds.

- The bounds obtained by the harder-to-compute optimal measures Q^i and Q^s are, in their turn, considerably tighter than those obtained from $\bar{Q}^i$ and $\bar{Q}^s$.

- Both the minimal martingale measure Q^m and the relaxed measures $\bar{Q}^i$ and $\bar{Q}^s$ can be computed explicitly in terms of input data. Due to the complex nature of the the general formulas, we have chosen not to include them. See Section 1.4.4 for a concrete, worked-out example.

1.4 Point Process Examples

In this section, we study a number of illustrative concrete examples. We restrict ourselves to models which include jumps, in line with the main focus of our paper. See Appendix 1.5.2 for the purely Wiener-driven case.

As opposed to a purely Wiener-driven model, the introduction of a driving point process (together with a Wiener process) produces a nontrivial incomplete market model even without including the factor model Y. For this reason, but also for tractability reasons, we only investigate pure jump-diffusion stock price models without any external factors. Specifically, all models studied in this section will be assumed to have the following structure.

Assumption 1.4.1. *We consider a financial market and a scalar price process S satisfying the SDE*

$$dS_t = S_t \alpha dt + S_t \sigma dW_t + S_{t-} \int_X \delta(x) \mu(dt, dx). \tag{1.57}$$

For this model, we furthermore assume that:

1. *The Wiener process W is one-dimensional.*
2. *The drift α and diffusion volatility σ are deterministic constants.*
3. *The jump function δ is a time invariant deterministic function of x only, i.e., δ is a mapping $\delta : X \to R$.*
4. *The point process μ has a P-compensator of the form*

   $$\nu^P(dt, dx) = \lambda(dx)dt,$$

   where λ is a time-invariant deterministic finite nonnegative measure on $(X, \mathcal{X})$.
5. *The short rate r is constant.*

Under this assumption, the model parameters α, σ, δ, and λ are deterministic objects that do not depend on the stock price S. In particular, the assumption about λ implies that the point process μ has the following properties under P:

- The jump events (disregarding the mark) will occur according to a standard Poisson process with the constant intensity $\lambda(X)$.
- If X_n denotes the mark of event number n, then the sequence $X_1, X_2, \ldots$ is i.i.d. with the common probability distribution

$$\frac{1}{\lambda(X)}\lambda(dx). \tag{1.58}$$

The sequence above is also independent of the interarrival times of the events.

To obtain a feeling for the techniques used, we start with a very simple example and then proceed to consider more complicated cases.

1.4.1 The Poisson-Wiener model

The simplest special case in the jump-diffusion setting above is when we define the point process μ as a standard Poisson process with constant intensity. In terms of the notation above, this means that the mark space X contains a single point, denoted by x_0. Hence, $X = \{x_0\}$, the measure $\lambda(dx)$ is merely a point mass $\lambda(x_0)$ at x_0, and the jump function δ a real number $\delta(x_0)$. For brevity, we denote $\lambda(x_0)$ by λ and $\delta(x_0)$ by δ. Hence, we obtain the following P dynamics of S:

$$dS_t = S_t\alpha dt + S_t\sigma dW_t + S_{t-}\delta dN_t \tag{1.59}$$

where N is Poisson with constant intensity λ.

In this case, the kernel function $h(t,s)$ is scalar, and the kernel $\varphi(t,s)$ does not depend upon x. The upper good-deal bound function $V(t,s)$ is the solution to the following boundary value problem.

$$\begin{aligned}\frac{\partial V}{\partial t}(t,s) + \sup_{h,\varphi}\left\{\mathbf{A}^{h,\varphi}V(t,s)\right\} - rV(t,s) &= 0, &(1.60)\\ V(T,s) &= \Phi(s), &(1.61)\end{aligned}$$

where we -for the moment- suppress the constraints, and where

$$\begin{aligned}\mathbf{A}^{h,\varphi}V(t,s) &= \frac{\partial V}{\partial s}s\left\{r - \delta\lambda(1+\varphi)\right\} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 V}{\partial s^2}\\ &\quad + \left\{V(t,s(1+\delta)) - V(t,s)\right\}\lambda(1+\varphi). \qquad (1.62)\end{aligned}$$

The static optimization problem in Lemma 1.3.14 then becomes

Proposition 1.4.2.

$$\max_{h,\varphi} \quad \lambda \left\{ V(t, s(1+\delta)) - V(t,s) - V_s(t,s)s\delta \right\} \varphi \tag{1.63}$$

subject to the constraints

$$\begin{aligned} \alpha + \sigma h + \delta\lambda \{1+\varphi\} &= r, & (1.64) \\ h^2 + \varphi^2 \lambda &\leq B^2, & (1.65) \\ \varphi &\geq -1. & (1.66) \end{aligned}$$

To study the static problem in more detail, we need some notation.

Definition 1.4.3. Define $(h_{\max}, \varphi_{\max})$ as the optimal solution to the programming problem

$$\max_{h,\varphi} \varphi, \tag{1.67}$$

subject to the constraints (1.64)-(1.66), and $(h_{\min}, \varphi_{\min})$ as the optimal solution to the problem

$$\min_{h,\varphi} \varphi, \tag{1.68}$$

subject to the same constraints.

We will need $h_{\max}$, $\varphi_{\max}$, $h_{\min}$, and $\varphi_{\min}$ below, so we should describe these constants in terms of the given model parameters. This is a simple exercise in constrained optimization theory, but somewhat messy. The result is as follows.

Lemma 1.4.4. *Denote the excess return $\alpha + \delta\lambda - r$ by R. Then, the following hold.*

- *The constants $h_{\max}$ and $\varphi_{\max}$ are given by*

$$\begin{aligned} h_{\max} &= -\frac{\sigma R}{(\sigma^2 + \delta^2\lambda)\lambda} - \frac{\delta\sqrt{B^2(\sigma^2+\delta^2\lambda) - R^2)}}{(\sigma^2+\delta^2\lambda)\sqrt{\lambda}} & (1.69) \\ \varphi_{\max} &= -\frac{\delta R}{\sigma^2+\delta^2\lambda} + \frac{\sigma\sqrt{B^2(\sigma^2+\delta^2\lambda) - R^2)}}{(\sigma^2+\delta^2\lambda)\sqrt{\lambda}} & (1.70) \end{aligned}$$

- *The constants $h_{\min}$ and $\varphi_{\min}$ are given by the following expressions:*

  1. *If*

$$-\frac{\delta R}{\sigma^2+\delta^2\lambda} - \frac{\sigma\sqrt{B^2(\sigma^2+\delta^2\lambda) - R^2)}}{(\sigma^2+\delta^2\lambda)\sqrt{\lambda}} > -1, \tag{1.71}$$

then

$$h_{\min} = -\frac{\sigma R}{(\sigma^2+\delta^2\lambda)\lambda} + \frac{\delta\sqrt{B^2(\sigma^2+\delta^2\lambda)-R^2)}}{(\sigma^2+\delta^2\lambda)\sqrt{\lambda}} \tag{1.72}$$

$$\varphi_{\min} = -\frac{\delta R}{\sigma^2+\delta^2\lambda} - \frac{\sigma\sqrt{B^2(\sigma^2+\delta^2\lambda)-R^2)}}{(\sigma^2+\delta^2\lambda)\sqrt{\lambda}.} \tag{1.73}$$

2. *If*

$$-\frac{\delta R}{\sigma^2+\delta^2\lambda} - \frac{\sigma\sqrt{B^2(\sigma^2+\delta^2\lambda)-R^2)}}{(\sigma^2+\delta^2\lambda)\sqrt{\lambda}} \leq -1, \tag{1.74}$$

then

$$h_{\min} = \frac{r-\alpha}{\sigma}, \tag{1.75}$$

$$\varphi_{\min} = -1. \tag{1.76}$$

Proof. A direct application of Kuhn-Tucker. ■

Now we can present a preliminary description of the optimal kernels.

Proposition 1.4.5. *The optimal kernels $(\hat{h},\hat{\varphi})$ for the static problem (1.63)-(1.66) have the following structure:*

1. *For all (t,s) such that*

$$V(t,s(1+\delta)) - V(t,s) - V_s(t,s)s\delta \geq 0, \tag{1.77}$$

the optimal kernels $(\hat{h},\hat{\varphi})$ are given by

$$\hat{h}(t,s) = h_{\max}, \quad \hat{\varphi}(t,s) = h_{\max}. \tag{1.78}$$

2. *For all (t,s) such that*

$$V(t,s(1+\delta)) - V(t,s) - V_s(t,s)s\delta < 0, \tag{1.79}$$

the optimal kernels $(\hat{h},\hat{\varphi})$ are given by

$$\hat{h}(t,s) = h_{\min}, \quad \hat{\varphi}(t,s) = h_{\min}. \tag{1.80}$$

Proof. Obvious from the arguments above. ■

We thus see that the optimal kernels have a so-called bang-bang structure, i.e., they switch between the extremal choices $(h_{\max}, \varphi_{\max})$ and $(h_{\min}, \varphi_{\min})$. For an arbitrarily chosen problem, switches do indeed occur, and the number of switches depend of course upon the optimal value function V through the conditions (1.77) and (1.79). An interesting special case arises when there are no switches and when the optimal kernels remain constant. Before proving the main result in this direction, we need some preliminary lemmas.

Lemma 1.4.6.

1. *If the optimal value function $V(t,s)$ is convex in the s-variable for all fixed values of t, then*

$$\hat{h}(t,s) = h_{\max}, \quad \hat{\varphi}(t,s) = \varphi_{\max}, \quad \forall t, s. \tag{1.81}$$

2. *If the optimal value function $V(t,s)$ is concave in the s-variable for all fixed values of t, then*

$$\hat{h}(t,s) = h_{\min}, \quad \hat{\varphi}(t,s) = \varphi_{\min}, \quad \forall t, s. \tag{1.82}$$

Proof. If V is convex in s, then (as a function of s) the tangent of V lies below the graph at each point, which implies condition (1.77). The concave case is similar. ∎

Lemma 1.4.7. *Consider the Poisson-Wiener model in (1.59), a fixed martingale measure Q generated by some choice of kernel functions (h, φ), and a contract function $\Phi(s)$. Assume the following:*

1. *The contract function Φ is convex (concave).*

2. *The kernel functions h and φ are deterministic functions of time, i.e., they are of the form*

$$h(t,s) = h(t), \quad \varphi(t,s) = \varphi(t). \tag{1.83}$$

Then, the arbitrage-free pricing function $F(t,s)$ defined by

$$F(t,s) = e^{-r(T-t)} \mathbb{E}_{ts}^{Q} \left[\Phi(S_T) \right], \tag{1.84}$$

which is convex (concave) in the s-variable, and where E_{ts}^{Q} denotes expectation under Q, given $S_t = s$.

Proof. Using the Itô formula it is easy to see that, given $S_t = s$, the SDE (1.59) has the solution

$$S_T = s(1+\delta)^{N_T - N_t} e^{(\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}, \tag{1.85}$$

which we will write as

$$S_T = s \cdot Z, \tag{1.86}$$

with the random variable Z defined in the obvious way above. We thus have

$$F(t,s) = e^{-r(T-t)} \mathbb{E}^Q_{ts}\left[\Phi(s \cdot Z)\right], \tag{1.87}$$

and, from the assumptions on h and φ, it follows that the Q-distribution of Z does not depend on the value of s. The assumed convexity of Φ now immediately implies the convexity of F. ∎

We can now state and prove the main theoretical result concerning the Poisson-Wiener model.

Proposition 1.4.8. *Assume that the contract function Φ is convex. Then, the following hold:*

1. *The optimal upper bound value function V is convex.*
2. *The optimal kernels $\hat{h}$ and $\hat{\varphi}$ are constant and given by*

$$\hat{h} = h_{\max}, \quad \hat{\varphi} = \varphi_{\max}. \tag{1.88}$$

3. *V satisfies the PIDE*

$$\frac{\partial V}{\partial t}(t,s) + \mathbf{A}^{\hat{h},\hat{\varphi}} V(t,s) - rV(t,s) = 0, \tag{1.89}$$

$$V(T,s) = \Phi(s), \tag{1.90}$$

where $\hat{h}, \hat{\varphi}$ are defined by (1.88), and where

$$\begin{aligned} \mathbf{A}^{\hat{h},\hat{\varphi}} V &= \frac{\partial V}{\partial s} s\left\{r - \delta\lambda(1+\hat{\varphi})\right\} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 V}{\partial s^2} \\ &\quad + \left\{V(t, s(1+\delta)) - V(t,s)\right\} \lambda(1+\hat{\varphi}). \end{aligned} \tag{1.91}$$

Instead, if the contract function Φ is concave, then V is concave, and items 2-3 *above still hold, with the only change that $h_{\max}$ and $\varphi_{\max}$ are replaced by $h_{\min}$ and $\varphi_{\min}$.*

Proof. Define the function F as the solution of

$$\begin{aligned} \frac{\partial F}{\partial t}(t,s) + \mathbf{A}^{\hat{h},\hat{\varphi}}F(t,s) - rF(t,s) &= 0, &(1.92)\\ F(T,s) &= \Phi(s), &(1.93)\end{aligned}$$

with $\mathbf{A}^{\hat{h},\hat{\varphi}}$ defined as above. We now want to show that $F = V$, i.e., that F is in fact equal to the optimal value function of the control problem for the upper bound. To do this, we first apply a Feynman-Kac Representation Theorem to deduce that we can write F as

$$F(t,s) = e^{-r(T-t)}\mathbb{E}_{ts}^{\hat{Q}}\left[\Phi(S_T)\right], \qquad (1.94)$$

where $\hat{Q}$ is generated by $\hat{h}$ and $\hat{\varphi}$. From Lemma 1.4.7 we then deduce that F is convex in the s-variable, and this implies - as in the Proof of Lemma 1.4.6- that F also satisfies the PIDE

$$\begin{aligned} \frac{\partial F}{\partial t}(t,s) + \sup_{h,\varphi}\left\{\mathbf{A}^{h,\varphi}F(t,s)\right\} - rF(t,s) &= 0, &(1.95)\\ F(T,s) &= \Phi(s), &(1.96)\end{aligned}$$

with obvious notation and standard constraints on h and φ. We have thus shown that F, defined by (1.92)-(1.93), satisfies the Hamilton-Jacobi-Bellman equation for the optimal control problem for the upper good-deal bound, and we now apply a standard verification theorem (see (Björk 2004), Ch. 19) to deduce that $F = V$. ■

The moral of this result is that, for convex contract functions like European puts and calls, we derive a well-behaved standard pricing equation for the upper bound, and a corresponding lower bound equation, without an embedded supremum operator. For nonconvex contract functions like that of a digital option, the situation is much more complicated and we must solve the full Hamilton-Jacobi-Bellman equation numerically.

1.4.2 The Compound Poisson-Wiener model

We now turn to the general Compound Poisson-Wiener Model specified by Assumption 1.4.1. For this model, the static problem of Lemma 1.3.14 has the following form:

Proposition 1.4.9.

$$\max_{h,\varphi} \quad \int_X \Delta V(t,s,x)\varphi(t,s,x)\lambda(dx) - sV_s(t,s)\int_X \delta(x)\varphi(t,s,x)\lambda(dx), \qquad (1.97)$$

subject to

$$\alpha + \sigma h + \int_X \delta(x)\lambda(dx) + \int_X \delta(x)\varphi(x)\lambda(dx) = r, \tag{1.98}$$

$$h^2 + \int_X \varphi^2(x)\lambda(dx) \leq B^2, \tag{1.99}$$

$$\varphi(x) \geq -1, \tag{1.100}$$

where, as before,

$$\Delta V(t,s,x) = V(t, s(1+\delta(x))) - V(t,s). \tag{1.101}$$

Using, as above, the notation R for the risk premium

$$R = \alpha + \int_X \delta(x)\lambda(dx) - r, \tag{1.102}$$

we express the problem in functional analytical terms, as follows.

Proposition 1.4.10.

$$\max_{h,\varphi} \quad \langle H, \varphi \rangle_\lambda, \tag{1.103}$$

subject to

$$\sigma h + \langle \delta, \varphi \rangle_\lambda + R = 0, \tag{1.104}$$

$$h^2 + \|\varphi\|_\lambda \leq B^2, \tag{1.105}$$

$$\varphi \geq -1, \tag{1.106}$$

where,

$$H(t,s,x) = \Delta V(t,s,x) - V_s(t,s)s\delta(x). \tag{1.107}$$

We now proceed to study this problem in detail.

1.4.3 The minimal martingale measure

We begin by computing the minimal martingale measure Q^m for this model. The measure Q^m is generated by the optimal kernels (h^m, φ^m) for the following programming problem.

Proposition 1.4.11.

$$\min_{h,\varphi} \quad h^2 + \|\varphi\|^2_\lambda, \tag{1.108}$$

subject to

$$\sigma h + \langle \delta, \varphi \rangle_\lambda + R = 0, \tag{1.109}$$

This is a standard minimum norm problem in L^2, so, from general theory, we know that the optimal (h, φ) has to be of the form

$$(h, \varphi) = c \cdot (\sigma, \delta),$$

for some scalar c. When we substitute this into the constraint (1.109), it yields the following result.

Proposition 1.4.12. *The minimal martingale measure Q^m is generated by the kernels (h^m, φ^m), given by*

$$h^m = -\frac{R \cdot \sigma}{\sigma^2 + \|\delta\|_\lambda^2}, \qquad (1.110)$$

$$\varphi^m = - \frac{R \cdot \delta}{\sigma^2 + \|\delta\|_\lambda^2}. \qquad (1.111)$$

1.4.4 The positivity constraint

As we noted in Section 1.3.5, Problem 1.4.10 above is, apart from the constraint (1.106), a fairly standard linear quadratic problem in the space $L^2[X, \lambda(dx)]$. Following the arguments of Section 1.4.4, we are led to study Problem 1.4.10 without the constraint (1.106) in order to compute the "relaxed measures" $\bar{Q}^s$ and $\bar{Q}^i$. The problem for determining $\bar{Q}^s$ is thus as follows.

Proposition 1.4.13.

$$\max_{h,\varphi} \quad \langle H, \varphi \rangle_\lambda, \qquad (1.112)$$

subject to

$$\sigma h + \langle \delta, \varphi \rangle_\lambda + R = 0, \qquad (1.113)$$

$$h^2 + \|\varphi\|_\lambda^2 \leq B^2. \qquad (1.114)$$

This is now a standard optimization problem in $L^2[X, \lambda(dx)]$, which we can solve by finding the extremal points of the associated Lagrangian. See Luenberger (1996) for details and for all unexplained terminology from functional analysis below. First, however, we simplify the problem by using (1.104) to eliminate h. Since the remaining constraint has to be binding, we face the following problem.

Proposition 1.4.14.

$$\max_{\varphi} \quad \langle H, \varphi \rangle, \qquad (1.115)$$

subject to

$$2R\langle \delta, \varphi \rangle + \langle \delta, \varphi \rangle^2 + \sigma^2 \|\varphi\|^2 + R^2 - \sigma^2 B^2 = 0, \qquad (1.116)$$

where we have suppressed λ in $\|\cdot\|_\lambda$ and $\langle \cdot, \cdot \rangle_\lambda$.

This is a standard programming problem in L^2, and the Lagrangian function $\mathcal{L}$ is given by

$$\mathcal{L}(\varphi,\gamma) = \langle H,\varphi\rangle + \gamma\left\{2R\langle\delta,\varphi\rangle + \langle\delta,\varphi\rangle^2 + \sigma^2\|\varphi\|^2 + R^2 - \sigma^2 B^2\right\}. \tag{1.117}$$

From the Kuhn-Tucker Theorem (see Luenberger (1996)) we know that the optimal solution $\hat{\varphi}$ is an extremal point of $\mathcal{L}$, i.e., a point where the Frechet derivative vanishes. Denoting the Frechet derivative of $\mathcal{L}$ w.r.t. φ by $\mathcal{L}_\varphi$, we easily obtain

$$\mathcal{L}_\varphi(\varphi,\gamma) = H + 2\gamma\left\{R\delta + \langle\delta,\varphi\rangle\delta + \sigma^2\varphi\right\}. \tag{1.118}$$

Thus, the first-order conditions for Problem (1.4.14) are

$$H + 2\gamma\left\{R\delta + \langle\delta,\varphi\rangle\delta + \sigma^2\varphi\right\} = 0. \tag{1.119}$$

When we take the inner product with δ in (1.119), it gives us the relation

$$\langle H,\delta\rangle + 2\gamma\left\{R\|\delta\|^2 + \langle\delta,\varphi\rangle\|\delta\|^2 + \sigma^2\langle\delta,\varphi\rangle\right\} = 0, \tag{1.120}$$

and, solving for $\langle\delta,\varphi\rangle$, we obtain

$$\langle\delta,\varphi\rangle = -\frac{\langle H,\delta\rangle + 2\gamma R\|\delta\|^2}{2\gamma\left\{\|\delta\|^2 + \sigma^2\right\}}. \tag{1.121}$$

Inserting this expression into (1.119) gives us the optimal φ as

$$\varphi = \frac{\langle H,\delta\rangle\delta}{2\gamma\sigma^2\left\{\|\delta\|^2 + \sigma^2\right\}} - \frac{R\delta}{\|\delta\|^2 + \sigma^2} - \frac{H}{2\gamma\sigma^2}. \tag{1.122}$$

To determine the Lagrange multiplier γ, we substitute (1.121)-(1.122) into the constraint (1.116). After some calculation, we obtain the following quadratic equation for γ:

$$\gamma^2 = \frac{\sigma^2\langle H,\delta\rangle^2 + \langle H,\delta\rangle\|\delta\|^2 + \|H\|^2K^2 - 2\langle H,\delta\rangle^2K^2}{4\sigma^4K\left\{B^2K - R^2\right\}}, \tag{1.123}$$

where

$$K = \|\delta\|^2 + \sigma^2. \tag{1.124}$$

The Lagrange multiplier is the positive root of this equation. We thus have the following pricing result.

Theorem 1.4.15. *The upper and lower relaxed good-deal bound pricing functions, $\bar{V}^s(t,s)$ and $\bar{V}^i(t,s)$ are given as solutions of the PIDE*

$$\frac{\partial V}{\partial t} + \mathbf{A}^{\varphi} V - rV = 0, \tag{1.125}$$

$$V(T, s) = \Phi(s, y), \tag{1.126}$$

where $\mathbf{A}^{\varphi}$ is given by

$$\mathbf{A}^{\varphi} V(t,s) = \frac{\partial V}{\partial s}(t,s)s\left\{r - \int_X \delta(s,x)\left\{1 + \varphi(t,s,x)\right\}\lambda(dx)\right\}$$
$$+ \int_X \Delta V(t,s,y,x)\left\{1 + \varphi(t,s,x)\right\}\lambda(dx) + \frac{1}{2}s^2\sigma^2\frac{\partial^2 V}{\partial s^2}(t,s),$$

with $\varphi = \bar{\varphi}^s$ and $\varphi = \bar{\varphi}^i$, respectively. The optimal relaxed kernels $\bar{\varphi}^s$ and $\bar{\varphi}^i$ are given by (1.122), where γ is the positive, respectively negative, root of (1.123).

The question now arises whether -at the optimal point- the positivity constraint is binding or not. We have no general theoretical results concerning this question, but our numerical experience (see Section 1.4.5 below) indicates strongly that the constraint is indeed binding in the generic case. The implication of this negative fact is that, in the generic case, the static problem has to be solved numerically. In the next section, we study a concrete numerical example where we discuss the numerical solution of the static problem in more detail.

1.4.5 A numerical example

In the graphs below, we provide the numerical results for a special case of the Wiener compound Poisson model described above. The model under consideration is the Merton jump-diffusion stock price model of Merton (1976), where relative jump size has a shifted lognormal distribution, i.e., it is of the form $e^Z - 1$, where Z is Gaussian. In terms of the notation above this means that $X = [-1, \infty)$, $\delta(x) = x$ and

$$\lambda(dx) = \lambda_0 f(x) dx, \tag{1.127}$$

where λ_0 is the intensity of the underlying Poisson process, and f is the density of the shifted lognormal distribution. The derivatives under considertion are European call options, and in all graphs we have stock price on the horisontal axis and option price on the vertical axis. In the first graph, we see the upper and lower relaxed bounds. In between these, we find the optimal bounds, and in the middle, we have the price generated by the minimal martingale measure. In the second figure, we show how the MMM price curve relates to the price curve generated by the Merton option pricing formula (where, by assumption, the market price of jump risk equals zero).

The MMM price and the relaxed pricing bounds are obtained by inserting the relevant kernels from the previous sections into the pricing PIDE, and then solving the PIDE numerically. To obtain the optimal pricing bounds (including the positivity constraint), we solve the static problem numerically (for each point in the discretized state space), using an interior point algorithm that was kindly provided to us by Mathias Stolpe (Stolpe, 1997). The ensuing kernels are then fed into the PIDE, which has been solved numerically using finite differences.

We use the following parameter values: Maximum grid size $M = 60$, grid stock price step $\delta S = 1$, grid time step $\delta t = 0,0003125$, time to maturity $TT = 0,25$, number of steps $T = 800$, interest rate $r = 0,05$, strike price $K = 40$, volatility $\sigma = 0,15$, Poisson intensity $\lambda_0 = 0,1$, $\alpha = -0,1$, $B = 1$, and the parameters for the normal distribution generating the lognormal jump distribution were: Mean 0,89, and standard deviation 0,45.

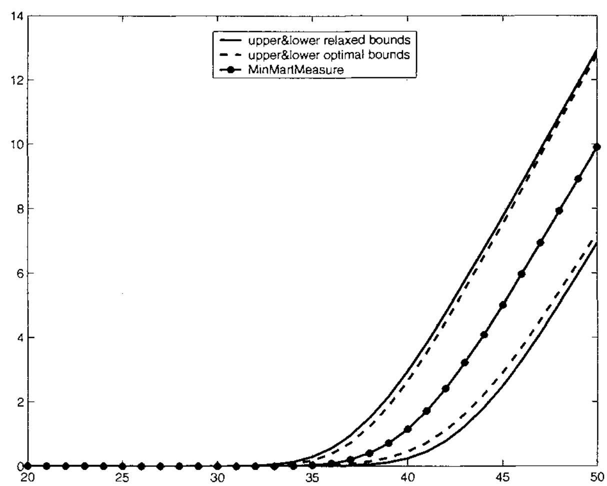

Figure 1.1: Good deal pricing bounds.

1.4.6 The impact of a stochastic jump size

With respect to the examples above, one natural question to ask is how the introduction of a stochastic jump size into a purely Poisson model impacts the

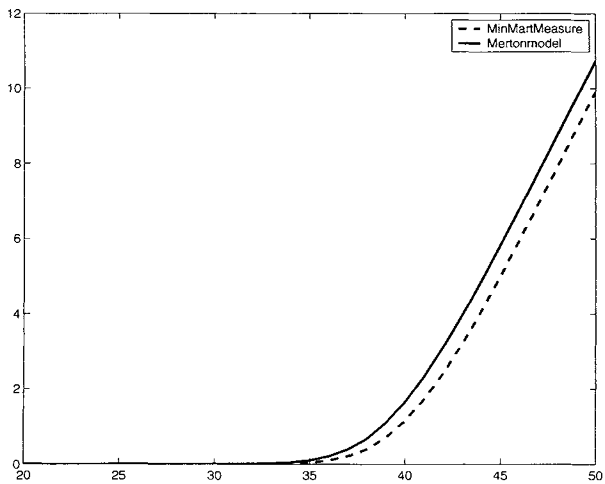

Figure 1.2: The minimal martingale measure and the Merton model.

good-deal bounds. In order to understand this question, let us consider two models: One compound Poisson model with a stochastic jump size, and one pure Poisson model with a deterministic jump size. We assume that the intensity of the underlying event process is the same for both models, and that the jump size of the pure Poisson model equals the expected value of the stochastic jump size of the compound Poisson model. The problem is understanding how the randomness of the jump size affects the length of the good-deal bound interval. The intuition for this problem is as follows.

Since the good-deal bound constraint (i.e., the right hand side of the HJ bound) is quadratic in the control variables h and φ, it is natural to interpret it as a constraint on the control energy (per unit time). Disregarding h for the moment, a given amount of admissible control energy -corresponding to a given good-deal bound B- will, in the pure Poisson case, be used exclusively for changing the intensity process of the underlying Poisson process. In the compound Poisson case, the control energy can be employed to change the intensity as well as the distribution of the jump size. It is thus clear that, from a control perspective, the stochastic jump size case provides more flexibility than does the pure Poisson case. Consequently, we may expect wider good-deal bounds for the compound Poisson model, compared to the pure Poisson model. In the graph below, this intuition is very clearly validated.

We use the following parameter values: Maximum grid size $M = 60$, grid stock price step $\delta S = 1$, grid time step $\delta t = 0,0025$, time to maturity $TT = 0,25$, number of steps $T = 100$, interest rate $r = 0,05$, strike price $K = 45$, volatility $\sigma = 0,15$, Poisson intensity $\lambda_0 = 0,1$, $\alpha = -0,1$, $B = 1$, and the parameters for the normal distribution generating the lognormal jump distribution were: Mean 0,89, and standard deviation 0,45. The jump size for the Poisson process is taken as the mean of the lognormal distribution associated with the normal distribution with the above parameters.

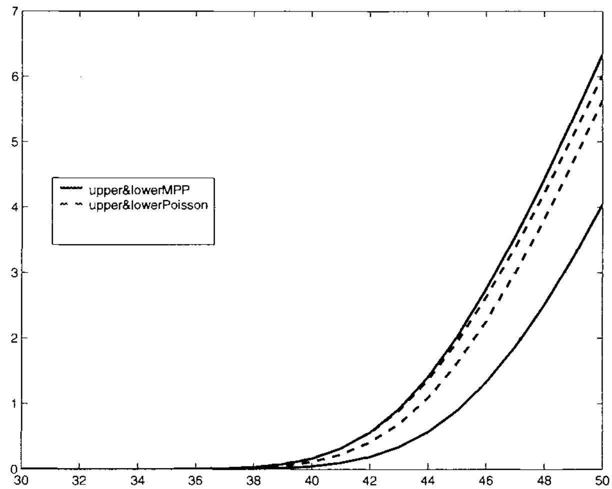

Figure 1.3: The impact of a stochastic jump size.

1.4.7 Conclusion

In this paper, we extend the good-deal bound pricing theory from Cochrane and Saá Requejo (2000) to allow for jumps in the underlying processes. While performing this extension we use a somewhat different technique than the one used in the original CSR paper. The result is a streamlined, rigorously developed, and simple theory of good-deal pricing bounds for a very general class of jump diffusions.

A number of intriguing open problems remain for future research. Firstly, on the theoretical side, we note that the good-deal bound theory is a pure pricing theory. Since the problem of arbitrage pricing is dual to the problem of hedging, one would therefore expect that it should be possible to develop a dual "good-deal hedging theory". In our view, the task of developing such a theory constitutes a highly challenging open problem. Secondly, the problem of numerically determining good-deal bounds is relatively complex in terms of CPU time, so there is a very clear need to develop fast, approximative good-deal pricing algorithms. Thirdly, the theory is now ready for application to real-life problems, and it would be very interesting to analyze, say, a reduced-form credit risk model from a good-deal perspective.

1.5 Appendix

1.5.1 Extended Hansen-Jagannathan Bounds

In this appendix, we derive an extension of the Hansen-Jagannathan bounds (see Hansen and Jagannathan (1991)) to the present point process setting. The HJ bounds provide an inequality for the Sharpe ratio process of any traded asset (underlying or derivative) in terms of the market prices of risk of the driving random sources, and, to make this idea precise, we consider the arbitrage-free price process π of an arbitrary asset (derivative or underlying) with P-dynamics given by

$$d\pi_t = \pi_t \alpha_t dt + \pi_t \sigma_t dW_t + \pi_{t-} \int_X \delta_t(x) \mu(dt, dx). \tag{1.128}$$

Here, α and σ are optional processes, whereas δ is predictable. In order to avoid negative asset prices we must also assume that $\delta_t(x) > -1$.

Compensating the point process μ in (1.128), we obtain the P-semimartingale dynamics of S as

$$d\pi_t = \pi_t \left\{ \alpha_t + \int_X \delta_t(x) \lambda_t(dx) \right\} dt + \pi_t \sigma_t dW_t + \pi_{t-} \int_X \delta_t(x) \tilde{\mu}(dt, dx). \tag{1.129}$$

Since the last two terms in this equation are martingale differentials, we see that the local mean rate of return under P is given by the expression

$$\alpha_t + \int_X \delta_t(x) \lambda_t(dx),$$

so, denoting the possibly stochastic short rate process by r, the **risk premium** process R is given by the formula

$$R_t = \alpha_t + \int_X \delta_t(x) \lambda_t(dx) - r_t. \tag{1.130}$$

We now proceed to define the predictable (total) **volatility** process v, which intuitively should equal the conditional variance of the return of the stock price, i.e., it should roughly be given by the expression

$$v_t^2 dt = \mathrm{Var}^P\left[\left.\frac{d\pi_t}{\pi_{t-}}\right| \mathcal{F}_{t-}\right]. \tag{1.131}$$

We need to make this notion mathematically precise and this is achieved by formally defining v through the relation

$$d\langle \pi, \pi\rangle_t = \pi_{t-}^2 v_t^2 dt, \tag{1.132}$$

where $\langle , \rangle$ denotes the usual predictable bracket process (see Jacod and Shiryaev (1987)). From (1.129), it is not hard to obtain

$$d\langle \pi, \pi\rangle_t = \pi_{t-}^2 \left\{ \|\sigma_t\|_{R^d}^2 + \int_X \delta_t^2(x)\lambda_t(dx)\right\} dt, \tag{1.133}$$

so, by comparing (1.133) with (1.132), we see that the squared volatility process is given by

$$v_t^2 = \|\sigma_t\|_{R^d}^2 + \|\delta_t\|_{\lambda_t}^2, \tag{1.134}$$

where $\|\cdot\|_{\lambda_t}$ denotes the norm in the Hilbert space $L^2\left[X, \lambda_t(dx)\right]$. For future use, we can also express the volatility v as

$$v_t = \|(\sigma_t, \delta_t)\|_{\mathcal{H}}, \tag{1.135}$$

where we view (σ_t, δ_t) as a vector in the Hilbert space $\mathcal{H} = R^d \times L^2\left[X, \lambda_t(dx)\right]$. We finally define the **Sharpe Ratio Process** SR by

$$SR_t = \frac{R_t}{v_t}. \tag{1.136}$$

Our goal is to derive an inequality for the Sharpe Ratio process in terms of the set of market prices of risk that turn up in connection with the class of risk-neutral martingale measures Q. Note again, that the Sharpe ratio process SR is not a number, but an entire random process.

Turning to martingale measures, we now copy the arguments in Section 1.3.1 and see that if the measure Q is generated by the Girsanov kernels (h, φ), then Q is a martingale measure if, and only if, the following conditions are satisfied:

$$\alpha_t + \sigma_t h_t + \int_X \delta_t(x)\left\{1 + \varphi_t(x)\right\}\lambda_t(dx) = r_t. \tag{1.137}$$

A Girsanov kernel process (h, φ) for which the induced measure Q is a martingale measure, i.e., a kernel process satisfying the martingale condition (1.137) and the positivity condition

$$\varphi_t(x) \geq -1, \quad \text{for all } t, x \tag{1.138}$$

is referred to as an **admissible** Girsanov kernel.

We now derive an inequality for the Sharpe ratio process SR, and we start by noting that we can rewrite the martingale condition (1.137) as

$$\alpha_t + \int_X \delta_t(x)\lambda_t(dx) - r_t = -\sigma_t h_t - \int_X \varphi_t(x)\delta_t(x)\lambda_t(dx). \tag{1.139}$$

From (1.130), we recognize the risk premium R in the left hand side of this equation so we can write R as

$$R_t = -\sigma_t h_t - \int_X \varphi_t(x)\delta_t(x)\lambda_t(dx). \tag{1.140}$$

From this expression we see that the Girsanov kernel process (h, φ) has a natural economic interpretation. The component $-h^i$ can be interpreted as the market price of risk for the i:th Wiener process, and $-\varphi(x)$ is the market price of risk for a jump event of type x. Using (1.140), we may also state and prove the main result of this appendix.

Theorem 1.5.1 (Extended Hansen-Jagannathan Bounds).
For any arbitrage-free price processes π and for every admissible Girsanov kernel (market price of risk) process (h, φ), the following inequality holds:

$$|SR_t| \leq \|(h_t, \varphi_t)\|_{\mathcal{H}}. \tag{1.141}$$

Specifically, this inequality can be written as

$$|SR_t|^2 \leq \|h_t\|^2_{R^d} + \int_X \varphi_t^2(x)\lambda_t(dx). \tag{1.142}$$

We note that the inequality (1.141) is an inequality between random processes, not between numbers.

Proof. A closer look at (1.140) reveals that the right hand side can be viewed as an inner product in the Hilbert space $\mathcal{H}$. Denoting this inner product by $\langle , \rangle_{\mathcal{H}}$, we can thus write

$$R_t = \langle (h_t, \varphi_t), (\sigma_t, \delta_{St}) \rangle_{\mathcal{H}}, \tag{1.143}$$

and, from the Schwartz inequality, we obtain

$$|R_t| \leq \|(h_t, \varphi_t)\|_{\mathcal{H}} \cdot \|(\sigma_t, \delta_t)\|_{\mathcal{H}}. \tag{1.144}$$

The inequality (1.141) now follows immediately from (1.135), (1.136), and (1.144). ∎

1.5.2 Purely Wiener-Driven Models

Although the main object of the present paper is to study good-deal bounds in the presence of a marked point process, for completeness' sake, we also study pricing in our model without a driving point process, i.e, we study the special case of a purely Wiener-driven model. This is an extension, although a very modest one, of the model originally considered in Cochrane and Saá Requejo (2000); the main difference is that we do not need a certain rank condition assumed by Cochrane and Saá Requejo (2000). We derive the general HJB equation for the upper (lower) price bounds and, given the appropriate rank condition, we derive the pricing PDE presented by Cochrane and Saá Requejo (2000).

The general case.

We recall our model for the purely Wiener-driven situation.

Assumption 1.5.2.

1. *The price and factor dynamics under objective probability measure P are given by*

$$\begin{aligned} dS_t^i &= S_t^i\alpha_i\left(S_t, Y_t\right)dt + S_t^i\sigma_i(S_t, Y_t)dW_t, \quad i = 1, \ldots, n \\ dY_t^j &= a_j\left(S_t, Y_t\right)dt + b_j(S_t, Y_t)dW_t, \quad j = 1, \ldots, k. \end{aligned}$$

2. *We assume that for each i and j, $\alpha_i(s, y)$ and $a_j(s, y)$ are deterministic scalar functions, and $\sigma_i(s, y)$ and $b_j(s, y)$ are deterministic row vector functions.*

3. *All functions above are assumed to be sufficiently regular to allow for the existence of a unique strong solution for the system of SDEs.*

4. *We assume the existence of a short rate r of the form*

$$r_t = r(S_t, Y_t).$$

5. *We assume that the model is free of arbitrage in the sense that there exists a (not necessarily unique) risk-neutral martingale measure Q.*

Theorem 1.3.13 shows that the upper good-deal bound function $V(t, s, y)$ satisfies the following boundary value problem

$$\frac{\partial V}{\partial t}(t, s, y) + \sup_h \left\{\mathbf{A}^h V(t, s, y)\right\} - rV(t, s, y) = 0, \tag{1.145}$$

$$V(T, s, y) = \Phi(s, y), \tag{1.146}$$

where we momentarily suppress all the constraints, and where the infinitesimal operator $A^{h,\varphi}$ is given by

$$
\begin{aligned}
\mathbf{A}^h V(t,s,y) &= \sum_{i=1}^{n} \frac{\partial V}{\partial s_i}(t,s,y)s_i r \\
&+ \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t,s,y)\left\{a_j(s,y)+b_j(s,y)h(t,s,y)\right\} \\
&+ \frac{1}{2}\sum_{i,l=1}^{n} \frac{\partial^2 V}{\partial s_i \partial s_l}(t,s,y)s_i s_l \sigma_i^\star(s,y)\sigma_l(s,y) \\
&+ \frac{1}{2}\sum_{j,l=1}^{k} \frac{\partial^2 V}{\partial y_j \partial y_l}(t,s,y)b_j^\star(s,y)b_l(s,y) \\
&+ \sum_{i,j=1}^{k} \frac{\partial^2 V}{\partial s_i \partial y_j}(t,s,y)s_i \sigma_i^\star(s,y)b_j(s,y). \qquad (1.147)
\end{aligned}
$$

Thus, we have the following static optimization problem.

Proposition 1.5.3.

$$
\max_h \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t,s,y)\left\{b_j(s,y)h(t,s,y)\right\}, \qquad (1.148)
$$

subject to the constraints

$$
\alpha_i + \sigma_i h = r, \quad i = 1,\dots,n \qquad (1.149)
$$

$$
\|h\|_{R^d}^2 \le A^2. \qquad (1.150)
$$

This is a very simple finite-dimensional optimization problem and, using standard Kuhn-Tucker techniques, we obtain the following result.

Proposition 1.5.4. *Denote the excess return $\alpha - r$ by R.*

- *The upper (lower) good-deal bound function $V(t,s,y)$ satisfies the following boundary value problem:*

$$
\begin{aligned}
\frac{\partial V}{\partial t} + r\sum_{i=1}^{n}\frac{\partial V}{\partial s_i}s_i + \sum_{j=1}^{k}\frac{\partial V}{\partial y_j}\left\{a_j + b_j\hat{h}\right\} + \frac{1}{2}\sum_{j,l=1}^{k}\frac{\partial^2 V}{\partial s_i \partial s_l}s_i s_l \sigma_i^\star \sigma_l& \\
+\frac{1}{2}\sum_{j,l=1}^{k}\frac{\partial^2 V}{\partial y_j \partial y_l}b_j^\star b_l + \sum_{i,j=1}^{k}\frac{\partial^2 V}{\partial s_i \partial y_j}s_i\sigma_i^\star b_j - rV = 0& \quad (1.151) \\
V(T,s,y) = \Phi(s,y).& \quad (1.152)
\end{aligned}
$$

- *For the upper bound, the kernel $\hat{h} = \hat{h}_{\max}$ is given by*

$$\hat{h}_{\max} = b'V_y - \sigma'(\sigma\sigma')^{-1}\sigma b'V_y + \frac{\sigma'(\sigma\sigma')^{-1}R\sqrt{V_y'b\left\{I - \sigma'(\sigma\sigma')^{-1}\sigma\right\}b'V_y}}{\sqrt{A^2 - R'(\sigma\sigma')^{-1}R}}. \tag{1.153}$$

- *For the lower bound, the kernel $\hat{h} = \hat{h}_{\min}$ is given by*

$$\hat{h}_{\min} = b'V_y - \sigma'(\sigma\sigma')^{-1}\sigma b'V_y - \frac{\sigma'(\sigma\sigma')^{-1}R\sqrt{V_y'b\left\{I - \sigma'(\sigma\sigma')^{-1}\sigma\right\}b'V_y}}{\sqrt{A^2 - R'(\sigma\sigma')^{-1}R}}. \tag{1.154}$$

Here, we use the notation $V_y = \left(\frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial y_k}\right)^*$

The Cochrane and Saá Requejo model.

The pricing PDE of Cochrane and Saá Requejo (2000) can now be derived as a particular case of our slightly more general model above. In terms of our notation, the CSR model is specified by the following set of assumptions.

Assumption 1.5.5.

- *The price dynamics are assumed to be of the form*

$$dS_t^i = S_t^i\alpha_i(S_t, Y_t, t)dt + S_t^i\sigma_i(S_t, Y_t, t)dZ_t, \quad i = 1, \ldots, n, \tag{1.155}$$

  where Z is an n-dimensional standard Wiener process.

- *The factor dynamics are assumed to be of the form*

$$dY_t^j = a_j(S_t, Y_t, t)dt + b_j^z(S_t, Y_t, t)dZ + b_j^w(S_t, Y_t, t)dW_t, \quad j = 1, \ldots, k, \tag{1.156}$$

  where W is a k-dimensional standard Wiener process orthogonal to Z.

- *The $n \times n$ volatility matrix σ is assumed to be invertible.*

Given this assumption, we have a d-dimensional driving Wiener process (Z, W), where $d = n + k$, so the kernel process h is now $(n + k)$-dimensional and can be decomposed as $h = (h_z, h_w)$. The martingale condition (1.149) will take the form

$$\alpha_i + \sigma_i h_z = r, \quad i = 1, \ldots, n, \tag{1.157}$$

and the invertibility assumption above allows us to solve for h_z to obtain

$$h_z = \sigma^{-1}(r - \alpha). \tag{1.158}$$

Thus, the static problem is simplified as follows.

Proposition 1.5.6.

$$\max_h \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t, s, y) \{b_j(s, y)h_w(t, s, y)\} \tag{1.159}$$

subject to the constraints

$$h_w' h_w \leq A^2 - R'R.$$

This is a trivial linear-quadratic optimization problem and the optimal h_w is easily found as

$$h_w = \frac{\sqrt{A^2 - R'R}}{\sqrt{b'V_y V_y' b}} \cdot b'V_y.$$

We have thus determined the entire optimal vector $h = (h_z, h_w)$, and substituting this into the pricing equation of Proposition 1.5.4, one obtains the pricing PDE of Cochrane and Saá Requejo (2000).

Part II

Term Structures of Interest Rates on International Markets

Chapter 2

On finite dimensional realizations of two-country interest rate models

In this paper we study a two-country economy and model a derivatives market where domestic T-bonds, foreign T-bonds and exchange rate derivatives are traded. We use the Heath-Jarrow-Morton approach, modeling domestic and foreign forward rates directly under each country martingale measure, assuming that they are both driven by a multidimensional Wiener process. Thus, the interest rate setting for both countries is defined by two infinite dimensional stochastic differential equations (SDEs). In addition we model the exchange rate stochastic process under the domestic martingale measure. The purpose of the paper is to understand when the inherently infinite dimensional domestic and foreign forward rate processes can be realized by means of a Markovian finite dimensional state space model. We assume that both forward rate volatilities and the volatility of the exchange rate can be arbitrary smooth functionals of both forward rate curves. We find necessary and sufficient conditions in terms of the exchange rate and interest rate volatilities to ensure that we obtain a finite dimensional realization (FDR) for both domestic and forward interest rates and we show how the exchange rate depends on such realizations. Finally, for the case when FDRs exist we show how to derive the dynamics of the underlying factors spanning the state space.[1]

[1]I gratefully acknowledge financial support from the Jan Wallander and Tom Hedelius Foundation. I thank Tomas Björk and participants at the Quantitative Methods in Finance Conference, Sydney 2005 for their helpful comments.

2.1 Introduction

The purpose of the paper is to study when the inherently infinite dimensional domestic and foreign forward rate processes in a two-country economy can be realized by means of a Markovian finite dimensional state space model.

We model a derivatives market where domestic T-bonds, foreign T-bonds and exchange rate derivatives are traded. Using the Heath-Jarrow-Morton approach, we choose to model domestic and foreign forward rates and the exchange rate dynamics under the domestic martingale measure allowing forward rate volatilities and the volatility of the exchange rate to be arbitrary smooth functionals of both forward rate curves.

The development of international markets made clear the need for a two-country model that would favor pricing and estimation procedures on internationally integrated markets. The approach of most previous studies has been to specify a finite set of factors and provide conditions for the factor dynamics so that the resulting bond prices in the two-country economy would take some predetermined form. This approach includes affine and quadratic models and leads to the specification of an affine factor dynamics. This paper is based on the geometric approach to the interest rate theory first introduced by Björk and Christenssen (1999) and Björk and Svensson (2001). We do not make any specific assumptions concerning the number of factors driving the forward rates in the two-country economy; neither do we specify any particular dynamics for the factors. The geometric approach allows us to find sufficient, in simpler cases, necessary conditions in terms of the exchange rate and forward rate volatilities to ensure that we obtain a finite dimensional realization (FDR) for both domestic and foreign forward rates. That is, the approach provides us with the structure of the forward rate volatilities that enables us to determine the number of factors and express the forward rate process as a function of the finite set of those factors.

The interest rate setting for both countries is defined by two infinite dimensional stochastic differential equations (SDEs) supplemented by the dynamics of the exchange rate between the two countries. We apply the theoretical results obtained by Björk and Christenssen (1999) and Björk and Svensson (2001) to provide necessary and sufficient conditions on the exchange rate and forward rate volatilities for the two-country interest rate model to admit an FDR.

The problem of identifying models which possess an FDR is important from the point of view of model calibration. It has become very common to use the parameterized families of smooth forward rate curves when it comes to fitting a forward rate curve to the initial data. At each time step, the model needs to be recalibrated, which is natural since it does not describe the real world perfectly. The problem of the existence of an FDR is equivalent to the problem of determining the existence of any finitely parameterized family of forward rate curves, such that all the forward rate curves produced by the interest rate model are contained within the family. If the interest rate model does not admit an FDR, then recalibration is needed not only because the model is a poor description of reality but because the family does not go well with the interest rate model.

Since the theory was first introduced by Björk and Christenssen (1999), the functional analytical framework has been enlarged by Filipović and Teichmann (2003) and Filipović and Teichmann (2002), who extended the class of interest rate models which can be analyzed by means of the geometric approach. Also, Björk and Landén (2001) suggested a way to construct the finite dimensional realization and actually derive the factor dynamics. We use this approach to construct an FDR and derive the factor dynamics in a two-country framework. Later, Björk, Landén, and Svensson (2002) extended the analysis to include stochastic volatility forward rate models. Recently, a number of studies applied the geometric approach to study futures prices Björk, Blix, and Landén (2004) and forward prices Gaspar (2004).

The paper is organized as follows. In Section 2.2 we present the formal setup of the two-country economy, the system of domestic and foreign forward rates and the exchange rate equation, define the objects of our study and describe their dynamics under the domestic martingale measure. We specify the problem we want to address in Section 2.3. Section 2.4 studies the existence of an FDR for constant volatility models and shows how to construct FDRs. In Section 2.5 we address the same question in the more complicated case of constant direction volatility. Our main conclusions are given in Section 2.7 and for the benefit of the reader we include an overview of previous results and a brief mathematical appendix (Section 2.8).

2.2 Two-country economy

Our formal setup consists of a financial market living on a stochastic basis (filtered probability space) $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $\mathbf{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ and where the measure P is interpreted as the objective (or "physical") probability measure. The basis is assumed to carry an m-dimensional standard Wiener process W.

Denote by $p_t(x)$ the time t price of the domestic zero coupon bond maturing at $T = t + x$ and define the domestic instantaneous forward rate as

$$r_t(x) = -\frac{\partial \log p_t(x)}{\partial x} \tag{2.1}$$

Denote by $\tilde{p}_t(x)$ the time t price of the foreign zero coupon bond maturing at $T = t + x$ and define the foreign instantaneous forward rate as

$$\tilde{r}_t(x) = -\frac{\partial \log \tilde{p}_t(x)}{\partial x} \tag{2.2}$$

Using the Musiela parameterization Musiela (1993) and the standard Heath-Jarrow-Morton drift condition (see Heath, Jarrow, and Morton (1992)) we write down the domestic forward rate dynamics under the domestic martingale measure Q^d as follows

$$\begin{aligned} dr_t(x) &= \left\{ \frac{\partial}{\partial x} r_t(x) + \sigma_t(x) \int_0^x \sigma_t^\star(u)du \right\} dt + \sigma_t(x) dW_t, \\ r_0(x) &= r_0^0(x). \end{aligned} \tag{2.3}$$

Analogously under the foreign martingale measure Q^f the foreign forward rate dynamics takes the form

$$\begin{aligned} d\tilde{r}_t(x) &= \left\{ \frac{\partial}{\partial x} \tilde{r}_t(x) + \tilde{\sigma}_t(x) \int_0^x \tilde{\sigma}_t^\star(u)du \right\} dt + \tilde{\sigma}_t(x) d\tilde{W}_t, \\ \tilde{r}_0(x) &= \tilde{r}_0^0(x). \end{aligned} \tag{2.4}$$

The initial forward rate curves, $\{r_0^0(x); x \geq 0\}$ and $\{\tilde{r}_0^0(x); x \geq 0\}$, are taken as given. Volatilities $\sigma_t(x)$ and $\tilde{\sigma}_t(x)$ are any adapted stochastic processes. Assume that the dynamics of the exchange rate process is as follows:

$$dS_t = S_t \alpha_t dt + S_t \delta_t dW_t. \tag{2.5}$$

Then, under the domestic martingale measure Q^d the exchange rate (denoted in units of domestic currency per unit of foreign currency) has the following dynamics:

$$dS_t = S_t(r_t(0) - \tilde{r}_t(0))dt + S_t \delta_t dW_t. \tag{2.6}$$

To specify the foreign forward rate dynamics under the domestic martingale measure we use the standard machinery and perform the Girsanov transformation (see, for example, Björk (2004)) from the foreign to the domestic martingale measure. Denote by $L(T)$ Radon-Nikodym derivative

$$L(T) = \frac{dQ^f}{dQ^d}, \quad \text{on} \quad \mathcal{F}_T,$$

we have

$$L(t) = \frac{S(t)}{S(0)} e^{-\int_0^t (r_s(0) - \tilde{r}_s(0))ds}.$$

Then the likelihood process is a martingale under the domestic martingale measure and has the following dynamics

$$\begin{cases} dL_t &= L_t\delta_t dW_t \\ L_0 &= 1, \end{cases}$$

and Wiener processes under the domestic and the foreign martingale measures are connected via

$$dW_t = d\tilde{W}_t + \delta_t dt.$$

Thus, under the domestic martingale measure Q^d the foreign forward rate dynamics are

$$\begin{aligned} d\tilde{r}_t(x) &= \left\{\frac{\partial}{\partial x}\tilde{r}_t(x) + \tilde{\sigma}_t(x)\int_0^x \tilde{\sigma}_t^\star(u)du - \tilde{\sigma}_t(x)\delta_t^\star\right\} dt + \tilde{\sigma}_t(x)dW_t, \\ \tilde{r}_0(x) &= \tilde{r}_0^0(x), \end{aligned} \tag{2.7}$$

where $\star$ denotes transpose. We rewrite the system of forward rates and the exchange rate equations as follows:

$$\begin{cases} dr_t &= (\mathbf{F}r_t + \sigma_t\mathbf{H}\sigma_t^\star)dt + \sigma_t dW_t \\ d\tilde{r}_t &= (\mathbf{F}\tilde{r}_t + \tilde{\sigma}_t\mathbf{H}\tilde{\sigma}_t^\star - \tilde{\sigma}_t\delta_t^\star)dt + \tilde{\sigma}_t dW_t \\ dS_t &= S_t(\mathbf{B}r_t - \mathbf{B}\tilde{r}_t)dt + S_t\delta_t dW_t \\ r_0 &= r_0^0, \quad \tilde{r}_0 = \tilde{r}_0^0, \quad S_0 = s_0^0, \end{cases} \tag{2.8}$$

where the operators $\mathbf{F}$, $\mathbf{H}$ and $\mathbf{B}$ are given by

$$\mathbf{F} = \frac{\partial}{\partial x} \tag{2.9}$$

$$\mathbf{H}f(x) = \int_0^x f(s)ds \tag{2.10}$$

$$\mathbf{B}f(x) = f(0) \tag{2.11}$$

and we use the following interpretation $\mathbf{H}\sigma_t = [\mathbf{H}\sigma_{1t}, \dots, \mathbf{H}\sigma_{mt}]$.

2.3 Problem formulation

In the previous section we assumed that domestic and forward rate volatilities and exchange rate volatility can be any adapted processes. To study finite dimensional realizations of forward rate models we need to make more specific assumptions concerning the structure of volatilities. It is natural to assume that volatilities of all the objects in the economy can depend on the whole domestic and foreign forward rate curves as well as spot exchange rates. For example, we can assume that if the shape of the forward rate curve predicts a recession, volatility of the forward rate increases demonstrating some sort of market panic. Thus, we make the following assumptions concerning the volatility processes $\sigma_t(x)$, $\tilde{\sigma}_t(x)$ and δ_t.

Assumption 2.3.1. *The adapted processes $\sigma(t,x)$, $\tilde{\sigma}(t,x)$ and $\delta(t)$ have the following functional form:*

$$\sigma(t,x) = \sigma(\hat{r}_t, x), \qquad \tilde{\sigma}(t,x) = \tilde{\sigma}(\hat{r}_t, x), \qquad \delta(t) = \delta(\hat{r}_t),$$

where σ, $\tilde{\sigma}$ and δ denote deterministic mappings

$$\begin{aligned} \sigma &: \mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}^m \\ \tilde{\sigma} &: \mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}^m \\ \delta &: \mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+ \to \mathcal{R}^m, \end{aligned}$$

and where $\hat{r}_t$ is the following infinite dimensional vector process

$$\hat{r}_t = \begin{bmatrix} r_t \\ \tilde{r}_t \\ S_t \end{bmatrix}.$$

The spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are Hilbert spaces of the domestic and foreign forward rate curves[2].

We thus intend to study the following infinite dimensional system of SDE:

$$\begin{aligned} dr_t &= (\mathbf{F}r_t + \sigma(\hat{r}_t)\mathbf{H}\sigma^\star(\hat{r}_t))dt + \sigma(\hat{r}_t)dW_t, \\ d\tilde{r}_t &= (\mathbf{F}\tilde{r}_t + \tilde{\sigma}(\hat{r}_t)\mathbf{H}\tilde{\sigma}^\star(\hat{r}_t) - \tilde{\sigma}(\hat{r}_t)\delta^\star(\hat{r}_t))dt + \tilde{\sigma}(\hat{r}_t)dW_t, \\ dS_t &= S_t(\mathbf{B}r_t - \mathbf{B}\tilde{r}_t)dt + S_t\delta(\hat{r}_t)dW_t, \end{aligned} \tag{2.12}$$

which in the above notations can be rewritten as

$$d\hat{r}_t = \hat{\mu}(\hat{r}_t)dt + \hat{\sigma}(\hat{r}_t)dW_t, \tag{2.13}$$

where $\hat{\mu}$ and $\hat{\sigma}$ are given by

$$\hat{\mu}(\hat{r}) = \begin{bmatrix} \mathbf{F}r + \sigma(\hat{r})\mathbf{H}\sigma^\star(\hat{r}) \\ \mathbf{F}\tilde{r} + \tilde{\sigma}(\hat{r})\mathbf{H}\tilde{\sigma}^\star(\hat{r}) - \tilde{\sigma}(\hat{r})\delta^\star(\hat{r}) \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\|\delta(\hat{r})\|^2 \end{bmatrix}, \quad \hat{\sigma}(\hat{r}) = \begin{bmatrix} \sigma(\hat{r}) \\ \tilde{\sigma}(\hat{r}) \\ \delta(\hat{r}) \end{bmatrix}. \tag{2.14}$$

[2]For the description of the Hilbert spaces see the mathematical appendix.

In the further analysis we switch to Stratonovich dynamics. As opposed to Itô calculus, in Stratonovich calculus the Itô formula takes the form of the standard chain rule for the ordinary calculus that allows for more intuition with the results. The Stratonovich integral is defined as follows:

Definition 2.3.2. For given semimartingales X and Y, the **Stratonovich integral** of X with respect to Y, $\int_0^t X_s \circ dY_s$, is defined as

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_t,$$

where the first term on the RHS is the Itô integral. Where the Wiener process is the only driving noise, we define quadratic variation process $\langle X, Y\rangle_t$ by

$$\langle X, Y\rangle_t = dX_t dY_t$$

with the usual multiplication rules: $dW \cdot dt = dt \cdot dt = 0$, $dW \cdot dW = dt$.

It is easy to see that

$$\hat{\sigma}(\hat{r}_t)dW_t = \hat{\sigma}(\hat{r}_t) \circ dW_t - \frac{1}{2}d\langle\hat{\sigma}(\hat{r}_t), W\rangle_t. \tag{2.15}$$

Indeed, using the infinite dimensional Itô formula we obtain the dynamics of $\hat{\sigma}(\hat{r}_t)$

$$d\hat{\sigma}(\hat{r}_t) = \{\dots\}\, dt + \hat{\sigma}'_{\hat{r}}(\hat{r}_t)[\hat{\sigma}(\hat{r}_t)]dW_t, \tag{2.16}$$

where $\hat{\sigma}'_{\hat{r}}(\hat{r}_t)[\hat{\sigma}(\hat{r}_t)]$ denotes Frechet derivative w.r.t. to the infinite dimensional variable $\hat{r}$ operating on $\hat{\sigma}(\hat{r}_t)$. We use the definition of quadratic variation to compute the Stratonovich correction term

$$d\langle\hat{\sigma}(\hat{r}_t), W\rangle_t = \hat{\sigma}'_{\hat{r}}(\hat{r}_t)\hat{\sigma}(\hat{r}_t)dt. \tag{2.17}$$

Thus the two-country forward rate model in Stratonovich form is given by

$$d\hat{r}_t = \left\{\hat{\mu}(\hat{r}_t) - \frac{1}{2}\hat{\sigma}'_{\hat{r}}(\hat{r}_t)[\hat{\sigma}(\hat{r}_t)]\right\} dt + \hat{\sigma}(\hat{r}_t) \circ dW_t, \tag{2.18}$$

or in expanded form (for our particular case of (2.12))

$$\begin{aligned} d\hat{r} &= \left\{ \begin{bmatrix} \mathbf{F}r + \sigma(\hat{r})\mathbf{H}\sigma^\star(\hat{r}) \\ \mathbf{F}\tilde{r} + \tilde{\sigma}(\hat{r})\mathbf{H}\tilde{\sigma}^\star(\hat{r}) - \tilde{\sigma}(\hat{r})\delta^\star(\hat{r}) \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\|\delta(\hat{r})\|^2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \sigma'_r(\hat{r}) & \sigma'_{\tilde{r}}(\hat{r}) & \sigma'_S(\hat{r}) \\ \tilde{\sigma}'_r(\hat{r}) & \tilde{\sigma}'_{\tilde{r}}(\hat{r}) & \tilde{\sigma}'_S(\hat{r}) \\ \delta'_r(\hat{r}) & \delta'_{\tilde{r}}(\hat{r}) & \delta'_S(\hat{r}) \end{bmatrix} \begin{bmatrix} \sigma(\hat{r}) \\ \tilde{\sigma}(\hat{r}) \\ \delta(\hat{r}) \end{bmatrix} \right\} dt \\ &+ \begin{bmatrix} \sigma(\hat{r}) \\ \tilde{\sigma}(\hat{r}) \\ \delta(\hat{r}) \end{bmatrix} \circ dW_t, \end{aligned} \tag{2.19}$$

where $\sigma'_r(\hat{r})$, $\tilde{\sigma}'_r(\hat{r})$, $\delta'_r(\hat{r})$, $\sigma'_{\tilde{r}}(\hat{r})$, $\tilde{\sigma}'_{\tilde{r}}(\hat{r})$, $\delta'_{\tilde{r}}(\hat{r})$ denote Frechet derivatives with respect to the infinite dimensional variables r and $\tilde{r}$. Since the Wiener process W_t is multidimensional we interpret the above expressions as follows:

$$\sigma'_r(r,\tilde{r},S)\sigma(r,\tilde{r},S) = \sum_{i=1}^{m} \sigma'_{ir}(r,\tilde{r},S)\sigma_i(r,\tilde{r},S).$$

Our task is to understand when the infinite dimensional process $\hat{r}$ can be realized by means of a finite dimensional SDE.

Definition 2.3.3. We say that the system of SDE (2.8) has a d-dimensional **realization** if there exists a point $z_0 \in R^d$, smooth vector fields A, $B_1, \ldots, B_m$ on some open subset $\mathcal{Z}$ of R^d and a smooth map $G : \mathcal{Z} \to \mathcal{H}$, such that $\hat{r}$ has local representation

$$\hat{r}_t = \hat{G}(Z_t, x), \quad \text{or} \quad \begin{bmatrix} r_t \\ \tilde{r}_t \\ S_t \end{bmatrix} = \begin{bmatrix} G_1(Z_t,x) \\ G_2(Z_t,x) \\ G_3(Z_t,x) \end{bmatrix}, \tag{2.20}$$

where Z is the solution of the d-dimensional Stratonovich SDE

$$\begin{aligned} dZ_t &= A(Z_t)dt + B(Z_t) \circ dW_t, \\ Z_0 &= z_0, \end{aligned} \tag{2.21}$$

and where W_t is the same as in (2.12).

Theorem 2.3.4. **(Björk and Svensson)** *Consider the SDE in (2.13) and assume that the dimension, evaluated pointwise, of the Lie algebra* $\{\hat{\mu}, \hat{\sigma}_1, \ldots, \hat{\sigma}_m\}_{LA}$ *is constant near the initial point* $\hat{r}_0 \in \hat{H}$. *Then (2.8) possesses an FDR if and only if*

$$dim\,\{\hat{\mu}, \hat{\sigma}_1, \ldots, \hat{\sigma}_m\}_{LA} < \infty$$

at the neighborhood of $\hat{r}^0$.

The purpose of the paper is to explore the existence of an FDR for the particular case of a two-country economy based on the existing abstract theoretical results in the literature.

In principle, our task is to determine when the infinite dimensional process $\hat{r}$ can be realized by means of a finite dimensional SDE. It is quite difficult, however, to analyze the system of stochastic differential equations for any given domestic forward rate volatility and exchange rate volatility. Instead we will study some particular cases where we are able to provide sufficient (and in some cases necessary) conditions for the system to possess an FDR. We start with the case of constant volatilities.

2.4 Constant forward rate volatilities and constant exchange rate volatility

2.4.1 Existence of an FDR

We consider first the simplest case, assuming that the volatility of the exchange rate process is a constant vector $\delta = (\delta_1, \ldots, \delta_m)$, and that domestic and foreign forward rate volatilities do not depend on $\hat{r}$-variable, that is, $\sigma(\hat{r}, x) = \sigma(x)$ and $\tilde{\sigma}(\hat{r}, x) = \tilde{\sigma}(x)$. We now give the formal assumption.

Assumption 2.4.1. *The volatilities σ and $\tilde{\sigma}$ are C^∞-mappings*

$$\sigma : \mathcal{R}_+ \to \mathcal{R}^m$$
$$\tilde{\sigma} : \mathcal{R}_+ \to \mathcal{R}^m$$

We find that it simplifies the calculations if we consider variable $Y = \ln S$ instead of S. The dynamics of Y are

$$dY_t = \left\{ \mathbf{B}r_t - \mathbf{B}\tilde{r}_t - \frac{1}{2}\|\delta\|^2 \right\} dt + \delta dW_t. \tag{2.22}$$

Since the volatilities are deterministic, the Stratonovich dynamics of r, $\tilde{r}$ and Y coincide with the initial dynamics and can be written as

$$\begin{aligned} dr_t &= (\mathbf{F}r_t + D)dt + \sigma \circ dW_t, \\ d\tilde{r}_t &= (\mathbf{F}\tilde{r}_t + \tilde{D} - \tilde{\sigma}\delta^\star)dt + \tilde{\sigma} \circ dW_t, \\ dY_t &= \left\{ \mathbf{B}r_t - \mathbf{B}\tilde{r}_t - \frac{1}{2}\|\delta\|^2 \right\} dt + \delta \circ dW_t, \end{aligned} \tag{2.23}$$

where the constant fields D and $\tilde{D}$ are given by

$$D(x) = \sigma(x)\mathbf{H}\sigma^\star(x) = \sum_{i=1}^m \sigma_i(x) \int_0^x \sigma_i^\star(s)ds, \tag{2.24}$$

$$\tilde{D}(x) = \tilde{\sigma}(x)\mathbf{H}\tilde{\sigma}^\star(x) = \sum_{i=1}^m \tilde{\sigma}_i(x) \int_0^x \tilde{\sigma}_i^\star(s)ds. \tag{2.25}$$

Then the dynamics of the infinite dimensional column vector process $\hat{r}$ in Stratonovich form can be written as

$$d\hat{r}_t = \hat{\mu}(\hat{r}_t)dt + \hat{\sigma}(\hat{r}_t) \circ dW_t, \tag{2.26}$$

where $\hat{\mu}$ and $\hat{\sigma}$ are given by

$$\hat{\mu}(\hat{r}) = \begin{bmatrix} \mathbf{F}r + D \\ \mathbf{F}\tilde{r} + \tilde{D} - \tilde{\sigma}\delta^{\star} \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\|\delta\|^2 \end{bmatrix}, \quad \hat{\sigma}(\hat{r}) = \begin{bmatrix} \sigma \\ \tilde{\sigma} \\ \delta \end{bmatrix} \tag{2.27}$$

and where $\hat{\sigma}(\hat{r}_t) \circ dW_t = \sum_{i=1}^{m} \hat{\sigma}_i(\hat{r}_t) \circ dW_t^i$ and denotes the Stratonovich integral. In this simplified case we obtain the following result:

Proposition 2.4.2. *Assume that the domestic and foreign forward rate volatilities σ and $\tilde{\sigma}$ are deterministic, i.e. of the form*

$$\sigma(\hat{r}, x) = \sigma(x), \quad \tilde{\sigma}(\hat{r}, x) = \tilde{\sigma}(x),$$

and the volatility of the exchange rate is constant $\delta(\hat{r}) = \delta$. Then there exists a finite dimensional realization if and only if both the foreign and domestic forward rate volatilities $\sigma(x)$ and $\tilde{\sigma}(x)$ are quasi-exponential.

Proof. The two-country model will have a generic FDR at the point $\hat{r}_0$ if and only if

$$dim \{\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}_{LA} < \infty \tag{2.28}$$

(we will denote $dim \{\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}_{LA}$ by $dim \{\hat{\mu}, \hat{\sigma}\}_{LA}$), in a neighborhood of $\hat{r}_0$. Denote $v_0^i = \hat{\sigma}_i$.

We obtain the following matrices for the Frechet derivatives of $\hat{\mu}$ and $\hat{\sigma}_i$, $i = 1, \ldots, m$.

$$\hat{\mu}'(\hat{r}_t) = \begin{bmatrix} \mathbf{F} & 0 & 0 \\ 0 & \mathbf{F} & 0 \\ \mathbf{B} & -\mathbf{B} & 0 \end{bmatrix}, \quad \hat{\sigma}_i'(\hat{r}_t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now we can compute the Lie bracket as follows

$$\begin{aligned} v_1^i &= [\hat{\mu}, \hat{\sigma}_i] = \begin{bmatrix} \mathbf{F} & 0 & 0 \\ 0 & \mathbf{F} & 0 \\ \mathbf{B} & -\mathbf{B} & 0 \end{bmatrix} \begin{bmatrix} \sigma_i \\ \tilde{\sigma}_i \\ \delta_i \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{F}r + D \\ \mathbf{F}\tilde{r} + \tilde{D} - \tilde{\sigma}\delta^{\star} \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\|\delta\|^2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{F}\sigma_i \\ \mathbf{F}\tilde{\sigma}_i \\ \mathbf{B}\sigma_i - \mathbf{B}\tilde{\sigma}_i \end{bmatrix}. \end{aligned} \tag{2.29}$$

In the same way we have

$$v_2^i = [\hat{\mu}, [\hat{\mu}, \hat{\sigma}_i]] = \begin{bmatrix} \mathbf{F}^2\sigma_i \\ \mathbf{F}^2\tilde{\sigma}_i \\ \mathbf{B}\mathbf{F}\sigma_i - \mathbf{B}\mathbf{F}\tilde{\sigma}_i \end{bmatrix}.$$

We continue to calculate the Lie brackets and taking into account that $[\sigma_i, \sigma_j] = 0$ we finally see that the relevant Lie algebra is given by

$$\begin{aligned}
\mathcal{L} &= \{\hat{\mu}, \hat{\sigma}\}_{LA} = span\left\{\hat{\mu}, \hat{\sigma}_1, \ldots, \hat{\sigma}_m, \begin{bmatrix} \mathbf{F}\sigma_i \\ \mathbf{F}\tilde{\sigma}_i \\ \mathbf{B}\sigma_i - \mathbf{B}\tilde{\sigma}_i \end{bmatrix}, \begin{bmatrix} \mathbf{F}^2\sigma_i \\ \mathbf{F}^2\tilde{\sigma}_i \\ \mathbf{BF}\sigma_i - \mathbf{BF}\tilde{\sigma}_i \end{bmatrix}, \ldots \right\} \\
&= span\left\{\hat{\mu}, \hat{\sigma}_1, \ldots, \hat{\sigma}_m, \begin{bmatrix} \mathbf{F}^n\sigma_i \\ \mathbf{F}^n\tilde{\sigma}_i \\ \mathbf{BF}^{n-1}\sigma_i - \mathbf{BF}^{n-1}\tilde{\sigma}_i \end{bmatrix}, n = 1, 2, \ldots; i = 1, \ldots, m \right\} \\
&= span\left\{\hat{\mu}, v_n^i, n = 0, 1, 2, \ldots; i = 1, \ldots, m\right\}. \qquad (2.30)
\end{aligned}$$

We will first prove that in order for the system to possess an FDR, it is *sufficient* that $\sigma_i(x)$ and $\tilde{\sigma}_i(x)$ are quasi-exponential. Assume that $\sigma_i(x)$ and $\tilde{\sigma}_i(x)$ are QE for all i. Assume first that we have only one Wiener driving process and thus only one volatility vector $[\sigma(x), \tilde{\sigma}(x), \delta)]^\star$. Then according to Lemma 2.8.14 (see Appendix), there exists a minimal annihilator polynomial $M(\mathbf{F})$, such that

$$M(\mathbf{F})\sigma = 0, \qquad M(\mathbf{F})\tilde{\sigma} = 0 \tag{2.31}$$

or denoting the degree of the annihilator by n and the coefficients by α_i we expand

$$M(\mathbf{F}) = \mathbf{F}^n + \alpha_{n-1}\mathbf{F}^{n-1} + \ldots + \alpha_1\mathbf{F} + \alpha_0. \tag{2.32}$$

We thus can rewrite (2.31) as follows

$$\mathbf{F}^n\sigma + \alpha_{n-1}\mathbf{F}^{n-1}\sigma + \ldots + \alpha_1\mathbf{F}\sigma + \alpha_0\sigma = 0 \tag{2.33}$$

$$\mathbf{F}^n\tilde{\sigma} + \alpha_{n-1}\mathbf{F}^{n-1}\tilde{\sigma} + \ldots + \alpha_1\mathbf{F}\tilde{\sigma} + \alpha_0\tilde{\sigma} = 0. \tag{2.34}$$

We now notice that

$$\mathbf{F}M(\mathbf{F})\sigma = M(\mathbf{F})[\mathbf{F}\sigma] = 0 \tag{2.35}$$

and the same holds for $\tilde{\sigma}$

$$\mathbf{F}M(\mathbf{F})\tilde{\sigma} = M(\mathbf{F})[\mathbf{F}\tilde{\sigma}] = 0. \tag{2.36}$$

Equations (2.31) also imply that

$$M(\mathbf{F})(\sigma - \tilde{\sigma}) = 0. \tag{2.37}$$

Since the equality holds for every x, then it also holds if evaluated at the point $x = 0$, so

$$\mathbf{B}M(\mathbf{F})(\sigma - \tilde{\sigma}) = 0. \tag{2.38}$$

Written in expanded form equations (2.35), (2.36) and (2.38) imply that

$$\begin{aligned} \mathbf{F}^{n+1}\sigma + \alpha_{n-1}\mathbf{F}^{n}\sigma + \ldots + \alpha_1\mathbf{F}^2\sigma + \alpha_0\mathbf{F}\sigma &= 0 \\ \mathbf{F}^{n+1}\tilde{\sigma} + \alpha_{n-1}\mathbf{F}^{n}\tilde{\sigma} + \ldots + \alpha_1\mathbf{F}^2\tilde{\sigma} + \alpha_0\mathbf{F}\tilde{\sigma} &= 0 \\ \mathbf{BF}^n\sigma - \mathbf{BF}^n\tilde{\sigma} + \alpha_{n-1}(\mathbf{BF}^{n-1}\sigma - \mathbf{BF}^{n-1}\tilde{\sigma}) + \ldots + \alpha_0(\mathbf{B}\sigma - \mathbf{B}\tilde{\sigma}) &= 0. \end{aligned} \tag{2.39}$$

From (2.39) we realize that vector v_{n+1} can be expressed as a linear combination of vectors v_n, $v_{n-1}, \ldots$, and v_1

$$\begin{bmatrix} \mathbf{F}^{n+1}\sigma \\ \mathbf{F}^{n+1}\tilde{\sigma} \\ \mathbf{BF}^n\sigma - \mathbf{BF}^n\tilde{\sigma} \end{bmatrix} = \alpha_{n-1} \begin{bmatrix} \mathbf{F}^{n}\sigma \\ \mathbf{F}^{n}\tilde{\sigma} \\ \mathbf{BF}^{n-1}\sigma - \mathbf{BF}^{n-1}\tilde{\sigma} \end{bmatrix} - \ldots - \alpha_0 \begin{bmatrix} \mathbf{F}\sigma \\ \mathbf{F}\tilde{\sigma} \\ \mathbf{B}\sigma - \mathbf{B}\tilde{\sigma} \end{bmatrix}, \tag{2.40}$$

which means that the resulting Lie algebra is finite dimensional. If the driving Wiener process is m-dimensional we can apply the same reasoning to every volatility vector $\hat{\sigma}_i$, $i = 1, \ldots, m$ to realize that for every $i = 1, \ldots, m$ we can express vector $v^i_{n_i+1}$ as a linear combination of vectors $v^i_{n_i}$, $v^i_{n_i-1}, \ldots$, and v^i_1, where n_i is the degree of the minimal annihilator polynomial $M^i(\mathbf{F})$. Since we have a finite number of volatility vectors $\hat{\sigma}_i$, $i = 1, \ldots, m$, we realize that the Lie algebra is finite and an FDR exists.

We also need to show that QE σ and $\tilde{\sigma}$ are necessary for the system to admit an FDR. Assume that there exists an FDR and thus the Lie algebra $\mathcal{L}$ is finite. Then for each i span $\{v^i_n, n = 0, 1, 2, \ldots\}$ is finite dimensional and there exist n_i and coefficients $\{a^i_k\}^{n_i}_{k=0}$, $i = 1, \ldots, m$, such that

$$v^i_{n_i+1} = \sum_{k=0}^{n_i} v^i_k \alpha^i_k. \tag{2.41}$$

Equations (2.41) in particular imply that

$$\begin{bmatrix} \mathbf{F}^{n_i+1}\sigma_i \\ \mathbf{F}^{n_i+1}\tilde{\sigma}_i \end{bmatrix} = \sum_{k=0}^{n_i} \alpha^i_k \begin{bmatrix} \mathbf{F}^{k}\sigma_i \\ \mathbf{F}^{k}\tilde{\sigma}_i \end{bmatrix}. \tag{2.42}$$

Vector equation (2.42) means that both σ_i and $\tilde{\sigma}_i$ are solutions of linear differential equations with constant coefficients. From Lemma 2.8.14 it then follows that for each i σ_i and $\tilde{\sigma}_i$ are QE. ■

Proposition 2.4.2 shows that any model with QE forward rate volatilities σ and $\tilde{\sigma}$ admits an FDR. From the proof of the above proposition we see that the dimension of the relevant Lie algebra

$$dim\,\{\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}_{LA} \leq 1 + m + \sum_{i=1}^{m} n_i, \tag{2.43}$$

where n_i is the degree of polynomial M^i and m comes from inclusion of additional m vectors $v^i_{n_i+1}$, $i = 1, \ldots, m$[3]. The following result shows that under some conditions on the volatility of the exchange rate, δ, the model admits an FDR with an even smaller dimension of the relevant Lie algebra

$$dim\,\{\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}_{LA} \leq 1 + \sum_{i=1}^{m} n_i. \tag{2.44}$$

In order to understand the proof of Corollary 2.4.3 we now need to introduce the following notation. We notice that each polynomial

$$Q(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \tag{2.45}$$

can be rewritten as follows:

$$\begin{aligned} Q(x) &= x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \\ &= x(x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_1) + a_0 = xQ_1(x) + Q(0), \end{aligned} \tag{2.46}$$

where $Q_1(x)$ is defined as

$$Q_1(x) = x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_2 x + a_1. \tag{2.47}$$

Corollary 2.4.3. *Assume that both foreign and domestic forward rate volatilities $\sigma_i(x)$ and $\tilde{\sigma}_i(x)$ are quasi-exponential with minimal annihilator polynomial M_i of degree n_i for each $i = 1, \ldots, m$. Assume also that none of the volatility vectors v_i can be expressed as a linear combination of other volatility vectors v_j, $j \neq i$. Then there exists an FDR with the dimension of Lie algebra*

$$dim\,\{\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}_{LA} \leq 1 + \sum_{i=1}^{m} n_i. \tag{2.48}$$

if and only if for each i the exchange rate volatility δ and the forward rate volatilities satisfy the relation

$$\mathbf{B}M^i_1(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i) + M^i(0)\delta_i = 0, \qquad M^i(0) \neq 0, \tag{2.49}$$

where M_1 is defined analogous to (2.47).

Proof. Assume first that for every i $M^i(0) \neq 0$ and

$$\mathbf{B}M^i_1(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i) + M^i(0)\delta_i = 0. \tag{2.50}$$

[3]The dimension of the Lie algebra is less then $< 1+m+\sum_{i=1}^{m} n_i$ in case some of the volatility vectors v^i turn out to be a linear combination of other vectors $v^j, j \neq i$

Since for each i $\sigma_i(x)$ and $\tilde{\sigma}_i(x)$ are QE with annihilator polynomial M^i of degree n_i, then

$$M^i(\mathbf{F})\sigma_i = M^i(\mathbf{F})\tilde{\sigma}_i = 0. \tag{2.51}$$

We can rewrite the first two equations of (2.51) as follows

$$\begin{bmatrix} \mathbf{F}^{n_i}\sigma_i \\ \mathbf{F}^{n_i}\tilde{\sigma}_i \end{bmatrix} = \sum_{k=0}^{n_i-1} \alpha_k^i \begin{bmatrix} \mathbf{F}^k\sigma_i \\ \mathbf{F}^k\tilde{\sigma}_i \end{bmatrix}. \tag{2.52}$$

By assumption the following relation between $\sigma_i(x)$, $\tilde{\sigma}_i(x)$ and δ_i is satisfied:

$$\mathbf{BF}^{n_i-1}\sigma_i - \mathbf{BF}^{n_i-1}\tilde{\sigma}_i - \sum_{k=0}^{n_i-2} \alpha_{k+1}^i \mathbf{BF}^k\sigma_i + \sum_{k=0}^{n_i-2} \alpha_{k+1}^i \mathbf{BF}^k\tilde{\sigma}_i - \alpha_0^i\delta_i = 0, \tag{2.53}$$

where n_i is the degree of the annihilator polynomial M^i. This means that for all $i = 1, \ldots, m$ the whole vector $v_{n_i}^i$ (together with the third component) can be expressed as a combination of the other vectors v_k^i, $k = 0, 1, \ldots, n_i - 1$

$$v_{n_i}^i = \sum_{k=0}^{n_i-1} v_k^i \alpha_k^i \tag{2.54}$$

or

$$\begin{aligned} \begin{bmatrix} \mathbf{F}^{n_i}\sigma_i \\ \mathbf{F}^{n_i}\tilde{\sigma}_i \\ \mathbf{BF}^{n_i-1}\sigma_i - \mathbf{BF}^{n_i-1}\tilde{\sigma}_i \end{bmatrix} &= \alpha_{n_i-1}^i \begin{bmatrix} \mathbf{F}^{n_i-1}\sigma_i \\ \mathbf{F}^{n_i-1}\tilde{\sigma}_i \\ \mathbf{BF}^{n_i-2}\sigma_i - \mathbf{BF}^{n_i-2}\tilde{\sigma}_i \end{bmatrix} \\ &- \ldots - \alpha_1^i \begin{bmatrix} \mathbf{F}\sigma_i \\ \mathbf{F}\tilde{\sigma}_i \\ \mathbf{B}\sigma_i - \mathbf{B}\tilde{\sigma}_i \end{bmatrix} - \alpha_0^i \begin{bmatrix} \sigma_i \\ \tilde{\sigma}_i \\ \delta_i \end{bmatrix}. \end{aligned} \tag{2.55}$$

We also note now that for each i all other vectors $v_{N_i}^i$, $N_i > n_i$ can be expressed as a linear combination of vectors v_k^i, $k = 1, \ldots, N_i - 1$. Indeed, construct for each i the polynomial $Q^i(\mathbf{F})$ as follows:

$$Q^i(\mathbf{F}) = M^i(\mathbf{F})P^i(\mathbf{F}), \tag{2.56}$$

where $P^i(\mathbf{F})$ is any polynomial, such that $P^i(0) \neq 0$. Then $Q^i(\mathbf{F})\sigma_i = Q^i(\mathbf{F})\tilde{\sigma}_i = 0$; moreover since

$$\mathbf{B}M_1^i(\mathbf{F})\left(\sigma_i - \tilde{\sigma}_i\right) + M^i(0)\delta_i = 0 \tag{2.57}$$

holds and $P^i(0) \neq 0$ it is also true that

$$\mathbf{B}P^i(0)M_1^i(\mathbf{F})\left(\sigma_i - \tilde{\sigma}_i\right) + P^i(0)M^i(0)\delta_i = 0. \tag{2.58}$$

Since $\mathbf{B}P_1^i(\mathbf{F})M^i(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i)$=0 adding the zero term to (2.58) we obtain that

$$\begin{aligned} &\mathbf{B}P_1^i(\mathbf{F})M^i(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i) + \mathbf{B}P^i(0)M_1^i(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i) + P^i(0)M^i(0)\delta_i \\ &= \mathbf{B}\left(P_1^i(\mathbf{F})M^i(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i) + P^i(0)M_1^i(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i)\right) + P^i(0)M^i(0)\delta_i = 0. \end{aligned} \tag{2.59}$$

We note that $Q_1^i(\mathbf{F}) = P_1^i(\mathbf{F})M^i(\mathbf{F}) + P^i(0)M_1^i(\mathbf{F})$ and thus conclude that

$$\mathbf{B}Q_1^i(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i) + Q^i(0)\delta_i = 0. \tag{2.60}$$

This implies that

$$\begin{aligned} \begin{bmatrix} \mathbf{F}^{N_i}\sigma_i \\ \mathbf{F}^{N_i}\tilde{\sigma}_i \\ \mathbf{B}\mathbf{F}^{N_i-1}\sigma_i - \mathbf{B}\mathbf{F}^{N_i-1}\tilde{\sigma}_i \end{bmatrix} &= b_{N_i-1}^i \begin{bmatrix} \mathbf{F}^{N_i-1}\sigma_i \\ \mathbf{F}^{N_i-1}\tilde{\sigma}_i \\ \mathbf{B}\mathbf{F}^{N_i-2}\sigma_i - \mathbf{B}\mathbf{F}^{N_i-2}\tilde{\sigma}_i \end{bmatrix} \\ &- \ldots - b_1^i \begin{bmatrix} \mathbf{F}\sigma_i \\ \mathbf{F}\tilde{\sigma}_i \\ \mathbf{B}\sigma_i - \mathbf{B}\tilde{\sigma}_i \end{bmatrix} - b_0^i \begin{bmatrix} \sigma_i \\ \tilde{\sigma}_i \\ \delta_i \end{bmatrix}, \end{aligned} \tag{2.61}$$

where b_k^i are the coefficients of the polynomial Q^i of degree p, where $p > m$.

$$Q(x) = x^p + b_{p-1}x^{p-1} + \ldots + b_0.$$

Thus, the Lie algebra $\mathcal{L}$ is finite and there exists an FDR, and the volatility of the exchange rate is determined as in (2.51). The dimension of the relevant Lie algebra

$$dim\,\{\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}_{LA} \leq 1 + \sum_{i=1}^{m} n_i.$$

We prove now that the existence of the Lie algebra with (2.48) implies (2.49). Since for each i $\sigma_i(x)$ and $\tilde{\sigma}_i(x)$ are QE (with annihilator polynomial M^i of degree n_i), according to Proposition 2.4.2 we know that there exists an FDR. Assume that the dimension of the relevant Lie algebra $\mathcal{L}$

$$dim\,\{\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}_{LA} \leq 1 + \sum_{i=1}^{m} n_i.$$

This means that for each i $dim\,\{\hat{\mu}, \hat{\sigma}_i\}_{LA} \leq 1 + n_i$ (since none of the volatility vectors v^i can be expressed as a linear combination of other volatility vectors v^j, $j \neq i$). Since for each i the Lie algebra has the dimension $1 + n_i$, vector $v_{n_i}^i$ can be expressed as a linear combination of vectors $v_0^i, \ldots, v_{n_i-1}^i$, i.e. assume that there exist some coefficients $\{\alpha_k^i\}_{k=0}^{n_i-1}$, $i = 1, \ldots, m$, such that

$$v_{n_i}^i = \sum_{k=0}^{n_i-1} v_k^i \alpha_k^i. \tag{2.62}$$

Equations (2.62) can be written as

$$P^i(\mathbf{F})\sigma_i = P^i(\mathbf{F})\tilde{\sigma}_i = 0 \quad (2.63)$$

$$\mathbf{B}P_1^i(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i) + P^i(0)\delta_i = 0 \quad (2.64)$$

for some polynomial $P^i(\mathbf{F})$. However, since $M^i(\mathbf{F})$ is the minimal polynomial, we must have that $P^i(\mathbf{F}) = C_i M^i(\mathbf{F})$ for some constant C_i. Then (2.63) can be rewritten as

$$M^i(\mathbf{F})\sigma_i = M^i(\mathbf{F})\tilde{\sigma}_i = 0 \quad (2.65)$$

$$\mathbf{B}M_1^i(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i) + M^i(0)\delta_i = 0. \quad (2.66)$$

We note that $M^i(0) \neq 0$ since otherwise $\mathbf{B}M_1^i(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i)$ holds for any choice of $\sigma_i(x)$ and $\tilde{\sigma}_i(x)$, which is impossible since $M_1(\mathbf{F})$ is not the minimal polynomial. Thus we showed that (2.48) leads to (2.49). ■

From the above result we see that to produce a model admitting an FDR with a Lie algebra of dimension less or equal to $1 + \sum_{i=1}^{m} n_i$ we may choose the forward rate volatilities σ and $\tilde{\sigma}$ freely in the class of QE functions. However, given a fixed choice of σ and $\tilde{\sigma}$, the choice of δ is restricted by relation (2.49). We also note that if $M^i(0) \neq 0$ for all i, then (2.49) will in fact determine δ uniquely in terms of σ and $\tilde{\sigma}$.

2.4.2 Construction of an FDR

In this section we study the construction of an FDR in the setup where domestic and foreign forward rate volatilities are deterministic

$$\sigma_i(\hat{r}, x) = \sigma_i(x), \quad \tilde{\sigma}_i(\hat{r}, x) = \tilde{\sigma}_i(x)$$

and exchange rate volatility is constant $\delta_i(\hat{r}) = \delta_i$.

The construction is based on the algorithm proposed by Björk and Landen (2000); they propose that a finite dimensional realization can be constructed in the following way:

- Choose a finite number of vector fields $f_1, \ldots, f_m$, which span the Lie algebra $\{\hat{\mu}, \hat{\sigma}\}_{LA}$.
- Compute the invariant manifold $\hat{G}(z_1, \ldots, z_m)$ using Theorem 2.8.10 (see Appendix).

- We constructed $\hat{r} = \hat{G}(Z)$ and now make the Ansatz for the Stratonovich dynamics of the state space variables Z

$$dZ = a(Z)dt + b(Z) \circ dW_t. \tag{2.67}$$

It must then hold that[4]

$$\hat{G}_\star a = \hat{\mu}, \quad \hat{G}_\star b = \hat{\sigma} \tag{2.68}$$

Solving those equations we find the vector fields a and b and thus the dynamics of the state space variables.

We will consider both cases introduced above. First, we consider the general case that produces a Lie algebra of dimension less or equal to $1+m+\sum_{i=1}^{m} n_i$ and thus requires a larger number of driving factors to be specified. Then, we will consider the specific case when the volatility of the exchange rate satisfies condition (2.49) and thus there exists the realization that requires a smaller number of factors.

The general case

Section 2.4 shows that the two-country model generated by the volatility structure as in Assumption 2.4.1 has an FDR if and only if (according to Proposition 2.4.2) the Lie algebra $\mathcal{L}$ is finite dimensional and spanned by a finite number of vector fields:

$$\{\hat{\mu}, \hat{\sigma}\}_{LA} = span\left\{\hat{\mu}, v_k^i, k = 0, 1, 2, \ldots, n_i; i = 1, \ldots, m\right\}.$$

Since the Lie algebra is finite dimensional then for each i span $\{v_n^i, n = 0, 1, 2, \ldots\}$ is finite dimensional as well and σ_i, $\tilde{\sigma}_i$ and δ_i must satisfy the following vector ODE for some n_i

$$v_{n_i+1}^i = \sum_{k=0}^{n_i} v_k^i \alpha_k^i, \tag{2.69}$$

where for every $i = 1, \ldots, m$

$$v_0^i = \begin{bmatrix} \sigma_i \\ \tilde{\sigma}_i \\ \delta_i \end{bmatrix}, \quad v_1^i = \begin{bmatrix} \mathbf{F}\sigma_i \\ \mathbf{F}\tilde{\sigma}_i \\ \mathbf{B}\sigma_i - \tilde{\mathbf{B}}\sigma_i \end{bmatrix}, \ldots, \quad v_{n_i}^i = \begin{bmatrix} \mathbf{F}^{n_i}\sigma_i \\ \mathbf{F}^{n_i}\tilde{\sigma}_i \\ \mathbf{B}\mathbf{F}^{n_i-1}\sigma_i - \mathbf{B}\mathbf{F}^{n_i-1}\tilde{\sigma}_i \end{bmatrix}.$$

We give the following definitions:

[4] The definition of $\star$ can be found in the Appendix.

Definition 2.4.4. *If μ and σ_i are vector fields on a Hilbert space $\mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+$, we define operator* $\exp\{\hat{\mu}t\}$ *as the solution to the following system of ODE*

$$\begin{cases} \dfrac{d\hat{r}}{dt} = \begin{bmatrix} \frac{\partial r}{\partial t} \\ \frac{\partial \tilde{r}}{\partial t} \\ \frac{\partial Y}{\partial t} \end{bmatrix} = \begin{bmatrix} \mathbf{F}r + D \\ \mathbf{F}\tilde{r} + \tilde{D} - \tilde{\sigma}\delta^\star \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\|\delta\|^2 \end{bmatrix} = \hat{\mu}(\hat{r}) \\ \hat{r}(0) = \hat{r}_0. \end{cases} \tag{2.70}$$

and we define operators $\exp\{v_k^i t\}$ *for* $k = 0, 1 \dots, n_i$ *and* $i = 1, \dots, m$ *as the solutions of the following ODEs*

$$\begin{cases} \dfrac{\partial\hat{r}}{\partial t} = \begin{bmatrix} \frac{\partial r}{\partial t} \\ \frac{\partial \tilde{r}}{\partial t} \\ \frac{\partial Y}{\partial t} \end{bmatrix} = \begin{bmatrix} v_k^{1i} \\ v_k^{2i} \\ v_k^{3i} \end{bmatrix} = v_k^i \\ \hat{r}(0) = \hat{r}_0. \end{cases} \tag{2.71}$$

Using Theorem 2.8.10 we find that the realization is given by

$$\hat{r}_t = \hat{G}(Z_t), \tag{2.72}$$

where the invariant manifold $\hat{G}$

$$\hat{G}(z^0, z_k^i, k = 0, 1, \dots, n_i) = \begin{bmatrix} G_1(z^0, z_k^i, k = 0, 1, \dots, n_i; i = 1, \dots, m) \\ G_2(z^0, z_k^i, k = 0, 1, \dots, n_i; i = 1, \dots, m) \\ G_3(z^0, z_k^i, k = 0, 1, \dots, n_i; i = 1, \dots, m) \end{bmatrix} \tag{2.73}$$

can be constructed as

$$\hat{G}(z^0, z_k^i, k = 0, 1, \dots, n_i) = \prod_{k=0,1,\dots,n_i; i=1,\dots,m} (e^{v_k^i z_k^i}) e^{\hat{\mu}t} \hat{r}_0. \tag{2.74}$$

The solution to the system of ODE (2.70) is given by

$$\begin{aligned} \exp\{\mu_1 t\} r_0 &= e^{\mathbf{F}t} r_0(x) + \int_0^t e^{\mathbf{F}(t-s)} D(x) ds = r_0(x+t) + \int_0^t D(x+t-s) ds \\ &= r_0(x+t) + \frac{1}{2}\left(\|S(x+t)\|^2 - \|S(x)\|^2\right), \end{aligned}$$

$$\begin{aligned} \exp\{\mu_2 t\} r_0 &= e^{\mathbf{F}t} \tilde{r}_0(x) + \int_0^t e^{\mathbf{F}(t-s)} \left\{\tilde{D}(x) - \tilde{\sigma}(x)\delta^\star\right\} ds \\ &= \tilde{r}_0(x+t) + \int_0^t \tilde{D}(x+t-s) ds - \int_0^t \tilde{\sigma}(x+t-s)\delta^\star ds \\ &= \tilde{r}_0(x+t) + \frac{1}{2}\left(\|\tilde{S}(x+t)\|^2 - \|\tilde{S}(x)\|^2\right) - \delta^\star(\tilde{S}(t+x) - \tilde{S}(x)), \end{aligned}$$

$$\exp\{\mu_3 t\} r_0 = y_0 + \int_0^t (r_s(0) - \tilde{r}_s(0)) ds - \frac{1}{2}\|\delta\|^2, \tag{2.75}$$

where

$$S(x) = \int_0^x \sigma(s)ds, \qquad \tilde{S}(x) = \int_0^x \tilde{\sigma}(s)ds.$$

From (2.75) and (2.75) we find that

$$r_s(0) - \tilde{r}_s(0) = r_0(s) - \tilde{r}_0(s) + \frac{1}{2}(\|S(s)\|^2 - \|\tilde{S}(s)\|^2) - \delta^\star \tilde{S}(s) \tag{2.76}$$

and substituting this into equation (2.75) we finally obtain

$$\begin{aligned}\exp\{\mu_3 t\} r_0 &= y_0 + \int_0^t (r_0(s) - \tilde{r}_0(s))ds + \frac{1}{2}\int_0^t (\|S(s)\|^2 - \|\tilde{S}(s)\|^2)ds \\ &- \frac{1}{2}\|\delta\|^2 t - \int_0^t \delta^\star \tilde{S}(s)ds. \end{aligned} \tag{2.77}$$

The solution to the system of ODE (2.71) is given by

$$\begin{aligned}\exp\{v_k^{1i} t\} r_0 &= r_0 + v_k^{1i}(x)t \\ \exp\{v_k^{2i} t\} \tilde{r}_0 &= \tilde{r}_0 + v_k^{2i}(x)t \\ \exp\{v_k^{3i} t\} y_0 &= y_0 + v_k^{3i} t. \end{aligned} \tag{2.78}$$

To construct a finite dimensional realization we compute the invariant manifold $\hat{G}(z^0, z_k^i, k = 0, 1, \ldots, n)$ by means of computing $\exp\{\hat{\mu} t\}\hat{r}_0$ and $\exp\{v_k^i t\}\hat{r}_0$, $k = 0, 1, \ldots n_i$, $i = 1, \ldots, m$.

$$\begin{aligned}G_1(z^0, z_k^i, k = 0, 1, \ldots, n_i) &= r_0(x + z^0) + \frac{1}{2}\left(\|S(x + z^0)\|^2 - \|S(x)\|^2\right) \\ &+ \sum_{i=1}^m \sum_{k=0}^{n_i} \mathbf{F}^k \sigma_i(x) z_k^i \\ G_2(z^0, z_k^i, k = 0, 1, \ldots, n_i) &= \tilde{r}_0(x + z^0) + \frac{1}{2}\left(\|\tilde{S}(x + z^0)\|^2 - \|\tilde{S}(x)\|^2\right) \\ &- \delta^\star(\tilde{S}(z^0 + x) - \tilde{S}(x)) + \sum_{i=1}^m \sum_{k=0}^{n_i} \mathbf{F}^k \tilde{\sigma}_i(x) z_k^i \\ G_3(z^0, z_k^i, k = 0, 1, \ldots, n_i) &= y_0 + \int_0^{z^0} (r_0(s) - \tilde{r}_0(s))ds \\ &+ \frac{1}{2}\int_0^{z^0} (\|S(s)\|^2 - \|\tilde{S}(s)\|^2)ds \\ &- \frac{1}{2}\|\delta\|^2 z^0 - \int_0^{z^0} \delta^\star \tilde{S}(s)ds \\ &+ \delta z_0 + \sum_{i=1}^m \sum_{k=1}^{n_i} (\mathbf{B}\mathbf{F}^{k-1}\sigma_i - \mathbf{B}\mathbf{F}^{k-1}\tilde{\sigma}_i) z_k^i. \end{aligned} \tag{2.79}$$

To find the dynamics of the state vector Z

$$dZ_t = a(Z_t)dt + b(Z_t) \circ dW_t$$

we need to solve the following system of equations for the vector fields a and b

$$\hat{G}_\star a(\hat{r}) = \begin{bmatrix} \mathbf{F}r + D \\ \mathbf{F}\tilde{r} + \tilde{D} - \tilde{\sigma}\delta^\star \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\|\delta\|^2 \end{bmatrix}, \quad \hat{G}_\star b_i(\hat{r}) = \begin{bmatrix} \sigma_i \\ \tilde{\sigma}_i \\ \delta_i \end{bmatrix} \tag{2.80}$$

or

$$\hat{G}'(z)a(z) = \hat{\mu}(\hat{G}(z)), \qquad \hat{G}'(z)b_i(z) = \hat{\sigma}_i(\hat{G}(z)). \tag{2.81}$$

The system of equations $\hat{G}'(z)a(z) = \hat{\mu}(\hat{G}(z))$ can be rewritten as follows:

$$\begin{aligned} & \frac{\partial}{\partial x} r_0(x+z^0)a^0 + D(x+z^0)a^0 + \sum_{i=1}^{m}\sum_{k=0}^{n_i} \mathbf{F}^k \sigma_i(x) a_k^i \\ = \; & \frac{\partial}{\partial x} r_0(x+z^0) + D(x+z^0) + \sum_{i=1}^{m}\sum_{k=0}^{n_i} \mathbf{F}^{k+1} \sigma_i(x) z_k^i, \end{aligned} \tag{2.82}$$

$$\begin{aligned} & \frac{\partial}{\partial x} \tilde{r}_0(x+z^0)a^0 + \tilde{D}(x+z^0)a^0 + (\tilde{\sigma}(x+z^0)\delta^\star - \tilde{\sigma}(x)\delta^\star)a^0 + \sum_{i=1}^{m}\sum_{k=0}^{n_i} \mathbf{F}^k \tilde{\sigma}_i(x) a_k^i \\ = \; & \frac{\partial}{\partial x} \tilde{r}_0(x+z^0) + \tilde{D}(x+z^0) + \delta^\star(\tilde{\sigma}(x+z^0) - \tilde{\sigma}(x)) + \sum_{i=1}^{m}\sum_{k=0}^{n_i} \mathbf{F}^{k+1} \tilde{\sigma}_i(x) z_k^i, \end{aligned} \tag{2.83}$$

$$\begin{aligned} & (r_0(z^0) - \tilde{r}_0(z^0))a^0 + \frac{1}{2}(\|S(z^0)\|^2 - \|\tilde{S}(z^0)\|^2)a^0 - \frac{1}{2}\|\delta\|^2 a^0 - a^0\delta^\star\tilde{S}(z^0) \\ + \; & \delta a_0 + \sum_{i=1}^{m}\sum_{k=1}^{n_i} (\mathbf{B}\mathbf{F}^{k-1}\sigma_i - \mathbf{B}\mathbf{F}^{k-1}\tilde{\sigma}_0)a_k^i \\ = \; & r_0(z^0) - \tilde{r}_0(z^0) - \frac{1}{2}\left(\|S(z^0)\|^2 - \|\tilde{S}(z^0)\|^2\right) - \delta^\star\tilde{S}(z^0) - \frac{1}{2}\|\delta\|^2 \\ + \; & \sum_{i=1}^{m}\sum_{k=0}^{n_i} (\mathbf{B}\mathbf{F}^k\sigma_i - \mathbf{B}\mathbf{F}^k\tilde{\sigma}_i)z_k^i \\ = \; & r_0(z^0) - \tilde{r}_0(z^0) - \frac{1}{2}\left(\|S(z^0)\|^2 - \|\tilde{S}(z^0)\|^2\right) - \delta^\star\tilde{S}(z^0) - \frac{1}{2}\|\delta\|^2 a^0 \\ + \; & \sum_{i=1}^{m}\sum_{k=0}^{n_i-1} (\mathbf{B}\mathbf{F}^k\sigma_i - \mathbf{B}\mathbf{F}^k\tilde{\sigma}_i)z_k^i + \sum_{i=1}^{m}\sum_{k=0}^{n_i-1} \alpha_k^i(\mathbf{B}\mathbf{F}^k\sigma_i - \mathbf{B}\mathbf{F}^k\tilde{\sigma}_i)z_{n_i}^i. \end{aligned} \tag{2.84}$$

The equality should hold for all x with all the coefficients being independent of x; thus, identifying terms we obtain that

$$\begin{aligned} a^0 &= 1 && (2.85)\\ a_0^j &= 0, \quad j=1,\dots,m \\ a_k^j &= \alpha_{k-1} z_{n_j}^j + z_{k-1}^j, \quad j=1,\dots,m; \quad k=1,\dots,n_j. && (2.86) \end{aligned}$$

From equation $\hat{G}'(z)b_i(z) = \hat{\sigma}_i(\hat{G}(z))$ we obtain

$$\begin{aligned} &\frac{\partial}{\partial x} r_0(x+z^0)b^0 + D(x+z^0)b^0 + \sum_{i=1}^{m}\sum_{k=0}^{n_i} \mathbf{F}^k \sigma_i(x) b_k^i = \sigma_i(x), \\ &\frac{\partial}{\partial x} \tilde{r}_0(x+z^0)b^0 + \tilde{D}(x+z^0)b^0 + \delta^\star(\tilde{\sigma}(x+z^0) - \tilde{\sigma}(x)) + b^0\delta^\star \\ + &\sum_{i=1}^{m}\sum_{k=0}^{n_i} \mathbf{F}^k \tilde{\sigma}_i(x) b_k^i = \tilde{\sigma}_i(x), \\ &(r_0(z^0) - \tilde{r}_0(z^0))b^0 + \frac{1}{2}(\|S(z^0)\|^2 - \|\tilde{S}(z^0)\|^2)b^0 - \frac{1}{2}\|\delta\|^2 b^0 - b^0\delta^\star \tilde{S}(z^0) \\ + &\ \delta b_0 + \sum_{i=1}^{m}\sum_{k=1}^{n_i} (\mathbf{B}\mathbf{F}^{k-1}\sigma_i - \mathbf{B}\mathbf{F}^{k-1}\tilde{\sigma}_i) b_k = \delta_i. \qquad (2.87) \end{aligned}$$

Therefore we have

$$\begin{aligned} b^0 &= 0, \\ b_0^j &= 1, \quad \text{if } \ j = i \\ b_k^j &= 0, \quad j \neq i; \quad k=0,\dots,n_i. \qquad (2.88) \end{aligned}$$

Since $\hat{\sigma}$ is constant, the Ito dynamics will look the same as the Stratonovich dynamics, and hence we have just proved the following proposition.

Proposition 2.4.5. *Given the initial forward rate curve $\hat{r}_0$, the two-country forward rate model generated by the volatility vectors as in Proposition 2.4.2 has a finite dimensional realization given by*

$$\hat{r}_t = \hat{G}(Z_t),$$

where $\hat{G}$ was defined in (2.79) and the dynamics of the state space variables are given by

$$\begin{cases} dZ^0 = dt \\ dZ_0^j = dW_t^j, \quad j=1,\dots,m \\ dZ_k^j = (\alpha_k^j Z_{n_j}^j + Z_{k-1}^j)dt \quad j=1,\dots,m; \quad k=1,\dots,n_j. \end{cases} \qquad (2.89)$$

Example 2.4.6. *As a concrete application let us consider a simple case where the domestic forward rate volatility is of the form*

$$\sigma(x) = e^{-\alpha x}, \tag{2.90}$$

and the foreign forward rate volatility of the form

$$\tilde{\sigma}(x) = Ce^{-\alpha x}. \tag{2.91}$$

Assume that the exchange rate volatility, δ, is a constant. Both $\sigma(x)$ and $\tilde{\sigma}(x)$ are quasi-exponential and satisfy the same differential equation

$$Fy = -\alpha y. \tag{2.92}$$

Thus, there exists an FDR with a Lie algebra of dimension equal to 3. Given the initial domestic and foreign forward rate curves r_0 and $\tilde{r}_0$, the forward rate system generated by the volatilities has an FDR given by

$$\hat{r}_t = \hat{G}(Z_t) = \begin{bmatrix} G_1(Z_t) \\ G_2(Z_t) \\ G_3(Z_t) \end{bmatrix}, \tag{2.93}$$

where

$$\begin{aligned} G_1(z^0, z_0, z_1) &= r_0(x + z^0) + \frac{e^{-\alpha x}}{\alpha^2}(1 - e^{-\alpha z^0}) - \frac{e^{-2\alpha x}}{2\alpha^2}(1 - e^{-2\alpha z^0}) + z_0 e^{-\alpha x} \\ &- \alpha z_1 e^{-\alpha x}, \\ G_2(z^0, z_0, z_1) &= \tilde{r}_0(x + z^0) + \frac{e^{-\alpha x}C^2}{\alpha^2}(1 - e^{-\alpha z^0}) - \frac{e^{-2\alpha x}C^2}{2\alpha^2}(1 - e^{-\alpha z^0}) \\ &+ \frac{\delta C e^{-2\alpha x}}{\alpha}\left\{e^{-2\alpha z^0} - 1\right\} + z_0 C e^{-\alpha x} - \alpha C e^{-\alpha x} z_1 \\ G_3(z^0, z_0, z_1) &= y_0 + \int_0^{z^0} (r_0(s) - \tilde{r}_0(s))ds \\ &+ \frac{1}{2\alpha^2}\left\{\frac{z^0}{2} + \frac{e^{-\alpha z^0} - 1}{\alpha} - \frac{e^{-2\alpha z^0} - 1}{4\alpha}\right\}(1 - C^2) \\ &+ \frac{\delta C}{\alpha}(z^0 + \frac{e^{-\alpha z^0} - 1}{\alpha}) + \delta z_0 + (1 - C)z_1 \end{aligned} \tag{2.94}$$

and the dynamics of the state space variables are given by

$$\begin{cases} dZ^0(t) &= dt \\ dZ_0(t) &= dW_t \\ dZ_1(t) &= [-\alpha Z_1(t) + Z_0(t)]dt. \end{cases}$$

Example 2.4.7. *Consider now the case when the domestic forward rate volatility is of the form*

$$\sigma(x) = e^{-\alpha x}, \tag{2.95}$$

and the foreign forward rate volatility is of the form

$$\tilde{\sigma}(x) = Ce^{-2\alpha x}. \tag{2.96}$$

Assume that the exchange rate volatility, δ, is constant. Both $\sigma(x)$ and $\tilde{\sigma}(x)$ obviously satisfy the same differential equation

$$(\mathbf{F} + \alpha)(\mathbf{F} + 2\alpha)y = \mathbf{F}^2 y + 3\alpha \mathbf{F} y + 2\alpha^2 y = 0. \tag{2.97}$$

Thus, the two-country economy has an FDR and the Lie algebra is four-dimensional Given the initial domestic and foreign forward rate curves r_0 and $\tilde{r}_0$, the forward rate system generated by the volatilities has an FDR given by

$$\hat{r}_t = \hat{G}(Z_t) = \begin{bmatrix} G_1(Z_t) \\ G_2(Z_t) \\ G_3(Z_t) \end{bmatrix}, \tag{2.98}$$

where

$$\begin{aligned} G_1(z^0, z_0, z_1, z_2) &= r_0(x + z^0) + \frac{e^{-\alpha x}}{\alpha^2}(1 - e^{-\alpha z^0}) - \frac{e^{-2\alpha x}}{2\alpha^2}(1 - e^{-2\alpha z^0}) + z_0 e^{-\alpha x} \\ &- \alpha z_1 e^{-\alpha x} + \alpha^2 z_2 e^{-\alpha x}, \end{aligned}$$

$$\begin{aligned} G_2(z^0, z_0, z_1, z_2) &= \tilde{r}_0(x + z^0) + \frac{e^{-2\alpha x}C^2}{\alpha^2}(1 - e^{-2\alpha z^0}) - \frac{e^{-4\alpha x}C^2}{\alpha^2}(1 - e^{-4\alpha z^0}) \\ &+ \frac{\delta C e^{-\alpha x}}{\alpha}\left\{e^{-2\alpha z^0} - 1\right\} + z_0 C e^{-2\alpha x} - 2\alpha C e^{-2\alpha x} z_1 + 4\alpha^2 C e^{-2\alpha x} z_2, \end{aligned}$$

$$\begin{aligned} G_3(z^0, z_0, z_1, z_2) &= y_0 + \int_0^{z^0} (r_0(s) - \tilde{r}_0(s))ds \\ &+ \frac{1}{2\alpha^2}\left\{\frac{z^0}{2} + \frac{e^{-\alpha z^0} - 1}{\alpha} - \frac{e^{-2\alpha z^0} - 1}{4\alpha}\right\} \\ &- \frac{C^2}{2\alpha^2}\left\{z^0 + \frac{1}{\alpha}(e^{-2\alpha z^0} - 1) - \frac{1}{4\alpha}(e^{-4\alpha z^0} - 1)\right\} \\ &+ \frac{\delta C}{\alpha}(z^0 + \frac{e^{-2\alpha z^0} - 1}{2\alpha}) + \delta z_0 - (1 - C)\alpha z_1 + (1 - C)\alpha^2 z_2. \end{aligned}$$

and the dynamics of the state space variables are given by

$$\begin{cases} dZ^0(t) &= dt \\ dZ_0(t) &= dW(t) \\ dZ_1(t) &= [-2\alpha^2 Z_2(t) + Z_0(t)]dt \\ dZ_2(t) &= [-3\alpha Z_2(t) + Z_1(t)]dt. \end{cases}$$

A specific case

Assume that the condition for the existence of an FDR with a Lie algebra of smaller dimension is satisfied. According to Proposition 2.4.2 the Lie algebra $\mathcal{L}$ is spanned by a finite number of vector fields

$$\{\hat{\mu}, \hat{\sigma}\}_{LA} = span\left\{\hat{\mu}, v_k^i, k = 0, 1, 2, \ldots, n_i - 1; i = 1, \ldots, m\right\},$$

and thus σ_i, $\tilde{\sigma}_i$ and δ_i satisfy the following vector ODE

$$v_{n_i}^i = \sum_{k=0}^{n_i-1} v_k^i \alpha_k^i, \tag{2.99}$$

where for every $i = 1, \ldots, m$

$$v_0^i = \begin{bmatrix} \sigma_i \\ \tilde{\sigma}_i \\ \delta_i \end{bmatrix}, \quad v_1^i = \begin{bmatrix} \mathbf{F}\sigma_i \\ \mathbf{F}\tilde{\sigma}_i \\ \mathbf{B}\sigma_i - \mathbf{B}\tilde{\sigma}_i \end{bmatrix}, \ldots, \quad v_{n_i-1}^i = \begin{bmatrix} \mathbf{F}^{n_i-1}\sigma_i \\ \mathbf{F}^{n_i-1}\tilde{\sigma}_i \\ \mathbf{B}\mathbf{F}^{n_i-2}\sigma_i - \mathbf{B}\mathbf{F}^{n_i-2}\tilde{\sigma}_i \end{bmatrix}.$$

It can be shown that the realization is given by

$$\hat{r}_t = \hat{G}(Z_t), \tag{2.100}$$

where

$$\hat{G}(z^0, z_k^i, k = 0, 1, \ldots, n_i - 1) = \begin{bmatrix} G_1(z^0, z_k^i, k = 0, 1, \ldots, n_i - 1; i = 1, \ldots, m) \\ G_2(z^0, z_k^i, k = 0, 1, \ldots, n_i - 1; i = 1, \ldots, m) \\ G_3(z^0, z_k^i, k = 0, 1, \ldots, n_i - 1; i = 1, \ldots, m) \end{bmatrix}.$$

The invariant manifold $\hat{G}(z^0, z_k^i, k = 0, 1, \ldots, n_i - 1; i = 1, \ldots, m)$ is given by

$$\hat{G}(z^0, z_k^i, k = 0, 1, \ldots, n_i; i = 1, \ldots, m) = \prod_{k=0,1,\ldots,n_i-1; i=1,\ldots,m} (e^{v_k^i z_k^i}) e^{\hat{\mu} t} \hat{r}_0 \tag{2.101}$$

$$\begin{aligned}
G_1(z^0, z_k^i, k = 0,1,\ldots,n_i - 1) &= r_0(x+z^0) + \frac{1}{2}\left(\|S(x+z^0)\|^2 - \|S(x)\|^2\right) \\
&+ \sum_{i=1}^{m}\sum_{k=0}^{n_i-1} \mathbf{F}^k \sigma_i(x) z_k^i \\
G_2(z^0, z_k^i, k = 0,1,\ldots,n_i - 1) &= \tilde{r}_0(x+z^0) + \frac{1}{2}\left(\|\tilde{S}(x+z^0)\|^2 - \|\tilde{S}(x)\|^2\right) \\
&- \delta^\star(\tilde{S}(z^0+x) - \tilde{S}(x)) + \sum_{i=1}^{m}\sum_{k=0}^{n_i-1} \mathbf{F}^k \tilde{\sigma}_i(x) z_k^i \\
G_3(z^0, z_k^i, k = 0,1,\ldots,n_i - 1) &= y_0 + \int_0^{z^0} (r_0(s) - \tilde{r}_0(s))ds \\
&+ \frac{1}{2}\int_0^{z^0} (\|S(s)\|^2 - \|\tilde{S}(s)\|^2)ds - \frac{1}{2}\|\delta\|^2 z^0 \\
&- \int_0^{z^0} \delta^\star \tilde{S}(s)ds + \delta z_0 \\
&+ \sum_{i=1}^{m}\sum_{k=1}^{n_i-1} (\mathbf{BF}^{k-1}\sigma_i - \mathbf{BF}^{k-1}\tilde{\sigma}_i) z_k^i. \qquad (2.102)
\end{aligned}$$

In order to find the dynamics of the state vector Z

$$dZ_t = a(Z_t)dt + b(Z_t) \circ dW_t$$

we need to solve the following equations for the vector fields a and b:

$$\hat{G}_\star a = \begin{bmatrix} \mathbf{F}r + D \\ \mathbf{F}\tilde{r} + \tilde{D} - \tilde{\sigma}\delta^\star \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\|\delta\|^2 \end{bmatrix}, \quad \hat{G}_\star b = \begin{bmatrix} \sigma_i \\ \tilde{\sigma}_i \\ \delta_i \end{bmatrix}. \qquad (2.103)$$

The system of equations $\hat{G}'(z)a(z) = \hat{\mu}(\hat{G}(z))$ can be rewritten as follows:

$$
\begin{aligned}
& \frac{\partial}{\partial x} r_0(x+z^0)a^0 + D(x+z^0)a^0 + \sum_{i=1}^{m}\sum_{k=0}^{n_i-1} \mathbf{F}^k \sigma_i(x) a_k^i \\
= \; & \frac{\partial}{\partial x} r_0(x+z^0) + D(x+z^0) + \sum_{i=1}^{m}\sum_{k=0}^{n_i-1} \mathbf{F}^{k+1} \sigma_i(x) z_k^i,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial}{\partial x} \tilde{r}_0(x+z^0)a^0 + \tilde{D}(x+z^0)a^0 + \delta^\star(\tilde{\sigma}(x+z^0) - \tilde{\sigma}(x))a^0 + \sum_{i=1}^{m}\sum_{k=0}^{n_i-1} \mathbf{F}^k \tilde{\sigma}_i(x) a_k^i \\
= \; & \frac{\partial}{\partial x} \tilde{r}_0(x+z^0) + \tilde{D}(x+z^0) + \delta^\star(\tilde{\sigma}(x+z^0) - \tilde{\sigma}(x)) + \sum_{i=1}^{m}\sum_{k=0}^{n_i-1} \mathbf{F}^{k+1} \tilde{\sigma}_i(x) z_k^i,
\end{aligned}
$$

$$
\begin{aligned}
& (r_0(z^0) - \tilde{r}_0(z^0))a^0 + \frac{1}{2}(\|S(z^0)\|^2 - \|\tilde{S}(z^0)\|^2)a^0 - \frac{1}{2}\|\delta\|^2 a^0 - a^0 \delta^\star \tilde{S}(z^0) \\
+ \; & \delta a_0 + \sum_{i=1}^{m}\sum_{k=1}^{n_i-1} (\mathbf{B}\mathbf{F}^{k-1}\sigma_i - \mathbf{B}\mathbf{F}^{k-1}\tilde{\sigma}_0) a_k^i \\
= \; & r_0(z^0) - \tilde{r}_0(z^0) - \frac{1}{2}\left(\|S(z^0)\|^2 - \|\tilde{S}(z^0)\|^2\right) - \delta^\star \tilde{S}(z^0) - \frac{1}{2}\|\delta\|^2 \\
+ \; & \sum_{i=1}^{m}\sum_{k=0}^{n_i-1} (\mathbf{B}\mathbf{F}^k \sigma_i - \mathbf{B}\mathbf{F}^k \tilde{\sigma}_i) z_k^i.
\end{aligned}
$$

The equality should hold for all x with all the coefficients being independent of x, thus, remembering that

$$
\begin{bmatrix} \mathbf{F}^{n_i}\sigma_i \\ \mathbf{F}^{n_i}\tilde{\sigma}_i \end{bmatrix} = \sum_{k=0}^{n_i-1} \alpha_k^i \begin{bmatrix} \mathbf{F}^k \sigma_i \\ \mathbf{F}^k \tilde{\sigma}_i \end{bmatrix}
$$

and identifying terms, we obtain

$$
\begin{aligned}
a^0 &= 1 \\
a_0^j &= \alpha_0^j z_{n_j-1}^j, \quad j = 1,\ldots,m \\
a_k^j &= \alpha_k z_{n_j-1}^j - z_{k-1}^j, \quad j = 1,\ldots,m; \quad k = 1,\ldots,n_j-1. \qquad (2.104)
\end{aligned}
$$

From equation $\hat{G}'(z)b_i(z) = \hat{\sigma}_i(\hat{G}(z))$ we obtain

$$\frac{\partial}{\partial x}r_0(x+z^0)b^0 + D(x+z^0)b^0 + \sum_{i=1}^{m}\sum_{k=0}^{n_i-1}\mathbf{F}^k\sigma_i(x)b_k^i = \sigma_i(x),$$

$$\begin{aligned}&\frac{\partial}{\partial x}\tilde{r}_0(x+z^0)b^0 + \tilde{D}(x+z^0)b^0 + \delta^\star(\tilde{\sigma}(x+z^0) - \tilde{\sigma}(x))b^0 \\ + \quad & \sum_{i=1}^{m}\sum_{k=0}^{n_i-1}\mathbf{F}^k\tilde{\sigma}_i(x)b_k^i = \tilde{\sigma}_i(x),\end{aligned}$$

$$\begin{aligned}&(r_0(z^0) - \tilde{r}_0(z^0))b^0 + \frac{1}{2}\left(\|S(z^0)\|^2 - \|\tilde{S}(z^0)\|\right)b^0 - \frac{1}{2}\|\delta\|^2 b^0 - b^0\delta^\star\tilde{S}(z^0) \\ + \quad & \delta b_0 + \sum_{i=1}^{m}\sum_{k=1}^{n_i-1}(\mathbf{B}\mathbf{F}^{k-1}\sigma_i - \mathbf{B}\mathbf{F}^{k-1}\tilde{\sigma}_i)b_k = \delta_i. \qquad (2.105)\end{aligned}$$

Therefore we have

$$\begin{aligned} b^0 &= 0, \\ b_0^j &= 1, \quad \text{if} \quad j = i \\ b_k^j &= 0, \quad j \neq i; \quad k = 0, \dots, n_i. \end{aligned} \qquad (2.106)$$

Since $\hat{\sigma}$ is constant the Ito dynamics will look the same, and we have just proved the following proposition.

Proposition 2.4.8. *Assume that condition (2.49) holds. Then given the initial forward rate curve $\hat{r}_0$, the two-country forward rate model generated by the volatility vectors as in Proposition 2.4.2 has a finite dimensional realization given by*

$$\hat{r}_t = \hat{G}(Z_t),$$

where $\hat{G}$ was defined in (2.102) and the dynamics of the state space variables are given by

$$\begin{cases} dZ^0 = dt \\ dZ_0^j = \alpha_0^j Z_{n_j-1}^j dt + dW_t^j, \quad j = 1, \dots, m \\ dZ_k^j = (\alpha_k^j Z_{n_j-1}^j + Z_{k-1}^j)dt \quad j = 1, \dots, m; \quad k = 1, \dots, n_j - 1. \end{cases} \qquad (2.107)$$

Example 2.4.9. *As a concrete application let us consider a simple case where the domestic forward rate volatility is of the form*

$$\sigma(x) = e^{-\alpha x}, \qquad (2.108)$$

the foreign forward rate volatility is of the form

$$\tilde{\sigma}(x) = (1 + \delta\alpha)e^{-\alpha x}. \tag{2.109}$$

Assume that the exchange rate volatility, C, is a constant. Both $\sigma(x)$ and $\tilde{\sigma}(x)$ obviously satisfy the same differential equation

$$Fy = -\alpha y. \tag{2.110}$$

Thus, the two-country economy has an FDR if and only if the following condition holds

$$\sigma(0) - \tilde{\sigma}(0) = -\alpha C, \tag{2.111}$$

or $1 - (1 + \delta\alpha) = -\alpha C$, which leads to $\delta = C$. Thus, in this simple case we see that if we are given the two forward rate volatilities and know that the model admits an FDR, then we can determine uniquely the volatility of the exchange rate. Then, given the initial domestic and foreign forward rate curves r_0 and $\tilde{r}_0$, the forward rate system generated by the volatilities has an FDR given by

$$\hat{r}_t = \hat{G}(Z_t) = \begin{bmatrix} G_1(Z_t) \\ G_2(Z_t) \\ G_3(Z_t) \end{bmatrix}, \tag{2.112}$$

where

$$\begin{aligned}
G_1(z^0, z_0) &= r_0(x + z^0) + \frac{e^{-\alpha x}}{\alpha^2}(1 - e^{-\alpha z^0}) - \frac{e^{-2\alpha x}}{2\alpha^2}(1 - e^{-2\alpha z^0}) + z_0 e^{-\alpha x}, \quad (2.113) \\
G_2(z^0, z_0) &= \tilde{r}_0(x + z^0) + \frac{e^{-\alpha x}(1 + \delta\alpha)^2}{\alpha^2}(1 - e^{-\alpha z^0}) - \frac{e^{-2\alpha x}(1 + \delta\alpha)^2}{2\alpha^2}(1 - e^{-\alpha z^0}) \\
&+ \frac{\delta(1 + \delta\alpha)e^{-2\alpha x}}{\alpha}\left\{e^{-2\alpha z^0} - 1\right\} + z_0(1 + \delta\alpha)e^{-2\alpha x} \quad (2.114) \\
G_3(z^0, z_0) &= y_0 + \int_0^{z^0} (r_0(s) - \tilde{r}_0(s))ds \\
&+ \frac{1}{2\alpha^2}\left\{\frac{z^0}{2} + \frac{e^{-\alpha z^0} - 1}{\alpha} - \frac{e^{-2\alpha z^0} - 1}{4\alpha}\right\}(1 - (1 + \alpha\delta)^2) \\
&+ \frac{\delta(1 + \alpha\delta)}{\alpha}(z^0 + \frac{e^{-\alpha z^0} - 1}{\alpha}) + \delta z_0 \quad (2.115)
\end{aligned}$$

and the dynamics of the state space variables are given by

$$\begin{cases} dZ^0(t) = dt \\ dZ_0(t) = -\alpha Z_0(t)dt + dW(t). \end{cases}$$

Example 2.4.10. *Consider now the case where the domestic forward rate volatility is of the form*

$$\sigma(x) = e^{-\alpha x}, \tag{2.116}$$

and the foreign forward rate volatility is of the form

$$\tilde{\sigma}(x) = 2(1 + \alpha\delta)e^{-2\alpha x}. \tag{2.117}$$

Assume that the exchange rate volatility, C, is constant. Both $\sigma(x)$ and $\tilde{\sigma}(x)$ obviously satisfy the same differential equation

$$(\mathbf{F} + \alpha)(\mathbf{F} + 2\alpha)y = \mathbf{F}^2 y + 3\alpha\mathbf{F}y + 2\alpha^2 y = 0. \tag{2.118}$$

Thus, the two-country economy has an FDR with a two-dimensional Lie algebra if and only if the following condition holds:

$$\mathbf{F}\sigma(0) - \mathbf{F}\tilde{\sigma}(0) + 3\alpha(\mathbf{F}\sigma(0) - \mathbf{F}\tilde{\sigma}(0)) + 2\alpha^2 C = 0, \tag{2.119}$$

or

$$\delta = C.$$

Thus, in this simple case we see that if we are given the two forward rate volatilities and know that there exists an FDR with a two-dimensional Lie algebra, then we can determine uniquely the volatility of the exchange rate. Then, given the initial domestic and foreign forward rate curves r_0 and $\tilde{r}_0$, the forward rate system generated by those volatilities has an FDR given by

$$\hat{r}_t = \hat{G}(Z_t) = \begin{bmatrix} G_1(Z_t) \\ G_2(Z_t) \\ G_3(Z_t) \end{bmatrix}, \tag{2.120}$$

where

$$
\begin{aligned}
G_1(z^0, z_0, z_1) &= r_0(x+z^0) + \frac{e^{-\alpha x}}{\alpha^2}(1-e^{-\alpha z^0}) - \frac{e^{-2\alpha x}}{2\alpha^2}(1-e^{-2\alpha z^0}) \\
&+ z_0 e^{-\alpha x} - \alpha z_1 e^{-\alpha x},
\end{aligned}
$$

$$
\begin{aligned}
G_2(z^0, z_0, z_1) &= \tilde{r}_0(x+z^0) + \frac{e^{-2\alpha x}2(1+\delta\alpha)^2}{\alpha^2}(1-e^{-2\alpha z^0}) \\
&- \frac{e^{-4\alpha x}2(1+\delta\alpha)^2}{\alpha^2}(1-e^{-4\alpha z^0}) + \frac{\delta(1+\delta\alpha)e^{-\alpha x}}{\alpha}\left\{e^{-2\alpha z^0}-1\right\} \\
&+ z_0(1+\delta\alpha)e^{-\alpha x} - 2\alpha(1+\delta\alpha)e^{-2\alpha x}z_1,
\end{aligned}
$$

$$
\begin{aligned}
G_3(z^0, z_0, z_1) &= y_0 + \int_0^{z^0}(r_0(s) - \tilde{r}_0(s))ds + \frac{1}{2\alpha^2}\left\{\frac{z^0}{2} + \frac{e^{-\alpha z^0}-1}{\alpha} - \frac{e^{-2\alpha z^0}-1}{4\alpha}\right\} \\
&- \frac{(1+\delta\alpha)^2}{2\alpha^2}\left\{z^0 + \frac{1}{\alpha}(e^{-2\alpha z^0}-1) - \frac{1}{4\alpha}(e^{-4\alpha z^0}-1)\right\} \\
&+ \frac{\delta(1+\delta\alpha)}{\alpha}(z^0 + \frac{e^{-2\alpha z^0}-1}{2\alpha}) + \delta z_0 - \delta\alpha z_1.
\end{aligned}
$$

and the dynamics of the state space variables are given by

$$
\begin{cases}
dZ^0(t) &= dt \\
dZ_0(t) &= -2\alpha^2 Z_1(t)dt + dW(t) \\
dZ_1(t) &= [-3\alpha Z_1(t) + Z_0(t)]dt.
\end{cases}
$$

Example 2.4.11. *Consider the multidimensional case with the domestic forward rate volatility vector of the form*

$$\sigma(x) = (\sigma_1(x), \sigma_2(x)) = (C_1 e^{-\alpha x}, 0) \tag{2.121}$$

and the foreign forward rate volatility of the form

$$\tilde{\sigma}(x) = (\tilde{\sigma}_1(x), \tilde{\sigma}_2(x)) = (0, C_2 e^{-2\alpha x}). \tag{2.122}$$

Assume that the exchange rate volatility is constant

$$\delta = (\delta_1, \delta_2). \tag{2.123}$$

Volatility components $\sigma_1(x)$ and $\tilde{\sigma}_2(x)$ satisfy the following differential equations:

$$
\begin{aligned}
(\mathbf{F} - \alpha)\sigma_1 &= 0 \\
(\mathbf{F} - 2\alpha)\tilde{\sigma}_2 &= 0.
\end{aligned}
$$

Thus, the two-country economy has an FDR with a two-dimensional Lie algebra if and only if the following condition holds:

$$\begin{aligned} C_1\sigma(0) &= C_1 = \alpha\delta_1 \\ C_2\tilde{\sigma}(0) &= C_2 = 2\alpha\delta_2. \end{aligned}$$

Thus, in this case we see that an FDR with a two-dimensional Lie algebra exists if and only if the exchange rate volatility is of the form

$$\delta = \left(\frac{C_1}{\alpha}, \frac{C_2}{2\alpha}\right). \tag{2.124}$$

Then, given the initial domestic and foreign forward rate curves r_0 and $\tilde{r}_0$, the forward rate system generated by the volatilities has an FDR given by

$$\hat{r}_t = \hat{G}(Z_t) = \begin{bmatrix} G_1(Z_t) \\ G_2(Z_t) \\ G_3(Z_t) \end{bmatrix}, \tag{2.125}$$

where

$$\begin{aligned} G_1(z^0, z_0) &= r_0(x+z^0) + C_1^2\frac{e^{-\alpha x}}{\alpha^2}(1-e^{-\alpha z^0}) - C_1^2\frac{e^{-2\alpha x}}{2\alpha^2}(1-e^{-2\alpha z^0}) + z_0^1 C_1 e^{-\alpha x} \\ G_2(z^0, z_0) &= \tilde{r}_0(x+z^0) + C_2^2\frac{e^{-\alpha x}(1+\delta\alpha)^2}{\alpha^2}(1-e^{-\alpha z^0}) \\ &- C_2^2\frac{e^{-2\alpha x}(1+\delta\alpha)^2}{2\alpha^2}(1-e^{-\alpha z^0}) + C_2^2\frac{\delta(1+\delta\alpha)e^{-2\alpha x}}{\alpha}\left\{e^{-2\alpha z^0}-1\right\} \\ &+ z_0^2 C_2(1+\delta\alpha)e^{-2\alpha x} \\ G_3(z^0, z_0) &= y_0 + \int_0^{z^0}(r_0(s)-\tilde{r}_0(s))ds + \frac{C_1^2}{2\alpha^2}\left\{\frac{z^0}{2} + \frac{e^{-\alpha z^0}-1}{\alpha} - \frac{e^{-2\alpha z^0}-1}{4\alpha}\right\} \\ &- \frac{C_2^2}{2\alpha^2}\left\{z^0 + \frac{1}{\alpha}(e^{-2\alpha z^0}-1) - \frac{1}{4\alpha}(e^{-4\alpha z^0}-1)\right\} \\ &+ \frac{1}{2}(\frac{C_1^2}{\alpha^2} + \frac{C_2^2}{4\alpha^2})z^0 - \frac{C_2^2}{4\alpha^2}(z^0 + \frac{1}{2\alpha}(e^{-2\alpha z^0}-1)) + \frac{C_1}{\alpha}z_0^1 + \frac{C_2}{2\alpha}z_0^2 \end{aligned}$$

and the dynamics of the state variables are given by

$$\begin{cases} dZ^0(t) &= dt \\ dZ_0^1(t) &= -\alpha Z_0^1(t)dt + dW^1(t) \\ dZ_0^2(t) &= -2\alpha Z_0^2(t)dt + dW^2(t). \end{cases}$$

2.5 Constant direction volatility

In this section we study a more complicated case where the domestic forward rate volatilities are allowed to take a more general form:

$$\sigma_i(r, \tilde{r}, S, x) = \varphi_i(r, \tilde{r}, S)\lambda_i(x), \quad i = 1, \dots, m. \tag{2.126}$$

The foreign forward rate volatilities are also allowed to be of a more general form:

$$\tilde{\sigma}_i(r, \tilde{r}, S, x) = \tilde{\varphi}_i(r, \tilde{r}, S)\tilde{\lambda}_i(x), \quad i = 1, \dots, m, \tag{2.127}$$

where λ_i and $\tilde{\lambda}_i$ are constant vector fields on $\mathcal{H}$ and $\tilde{\mathcal{H}}$ (deterministic functions of x) and φ_i and $\tilde{\varphi}_i$ are scalar vector fields on $\mathcal{H}$ and $\tilde{\mathcal{H}}$ (do not depend on x and depend on the whole current price curve). Thus, σ and $\tilde{\sigma}$ denote deterministic mappings:

$$\begin{aligned}
\sigma &: \mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}^m \\
\tilde{\sigma} &: \mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}^m
\end{aligned}$$

Assume also that the exchange rate volatility is a deterministic mapping:

$$\delta : \mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+ \to \mathcal{R}^m. \tag{2.128}$$

Assumption 2.5.1. *Assume that $\varphi_i(r, \tilde{r}, S) \neq 0$, $\tilde{\varphi}_i(r, \tilde{r}, S) \neq 0$ and $\delta_i(r, \tilde{r}, S) \neq 0$ for all $(r, \tilde{r}, S) \in \mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+$ and for all $i = 1, \dots, m$.*

Again, we change variables $Y = \ln S$; then the dynamics of Y are

$$dY_t = \left(\mathbf{B}r_t - \mathbf{B}\tilde{r}_t - \frac{1}{2}\|\delta(r_t, \tilde{r}_t, Y_t)\|^2\right) dt + \delta(r_t, \tilde{r}_t, Y_t) dW_t. \tag{2.129}$$

We can write the Stratonovich dynamics of $\hat{r}$ as follows:

$$\begin{aligned}
dr &= \left\{\mathbf{F}r + \sum_i^m \varphi_i^2(r, \tilde{r}, Y) D_i\right\} dt - \frac{1}{2} d\langle \sigma(r, \tilde{r}, Y), W\rangle + \varphi(r, \tilde{r}, Y)\lambda \circ dW \\
d\tilde{r} &= \left\{\mathbf{F}\tilde{r} + \sum_i^m \tilde{\varphi}^2(r, \tilde{r}, Y)\tilde{D}_i - \tilde{\varphi}(r, \tilde{r}, Y)\tilde{\lambda}\delta^\star(r, \tilde{r}, Y)\right\} dt - \frac{1}{2} d\langle \tilde{\sigma}(r, \tilde{r}, Y), W\rangle \\
&+ \tilde{\varphi}(r, \tilde{r}, Y)\tilde{\lambda} \circ dW, \\
dY &= \left\{\mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\|\delta(r, \tilde{r}, Y)\|^2\right\} dt - \frac{1}{2} d\langle \delta(r, \tilde{r}, Y), W\rangle + \delta(r, \tilde{r}, Y) \circ dW,
\end{aligned} \tag{2.130}$$

where

$$\begin{aligned} D_i(x) &= \lambda_i(x)\int_0^x \lambda_i(s)ds \\ \tilde{D}_i(x) &= \tilde{\lambda}_i(x)\int_0^x \tilde{\lambda}_i(s)ds, \quad i = 1, \ldots, m. \end{aligned} \tag{2.131}$$

To compute the Stratonovich correction terms we will use the infinite dimensional Itô formula to obtain

$$\begin{aligned} d\sigma(r_t, \tilde{r}_t, Y_t) &= \{\ldots\}\, dt + \sigma_r'(r_t, \tilde{r}_t, Y_t)\sigma(r_t, \tilde{r}_t, Y_t)dW_t + \sigma_{\tilde{r}}'(r_t, \tilde{r}_t, Y_t)\tilde{\sigma}(r_t, \tilde{r}_t, Y_t)dW_t \\ &+ \sigma_Y'(r_t, \tilde{r}_t, Y_t)\delta(r_t, \tilde{r}_t, Y_t)dW_t, \end{aligned} \tag{2.132}$$

$$\begin{aligned} d\tilde{\sigma}(r_t, \tilde{r}_t, Y_t) &= \{\ldots\}\, dt + \tilde{\sigma}_r'(r_t, \tilde{r}_t, Y_t)\sigma(r_t, \tilde{r}_t, Y_t)dW_t + \tilde{\sigma}_{\tilde{r}}'(r_t, \tilde{r}_t, Y_t)\tilde{\sigma}(r_t, \tilde{r}_t, Y_t)dW_t \\ &+ \tilde{\sigma}_Y'(r_t, \tilde{r}_t, Y_t)\delta(r_t, \tilde{r}_t, Y_t)dW_t, \end{aligned} \tag{2.133}$$

$$\begin{aligned} d\delta(r_t, \tilde{r}_t, Y_t) &= \{\ldots\}\, dt + \delta_r'(r_t, \tilde{r}_t, Y_t)\sigma(r_t, \tilde{r}_t, Y_t)dW_t + \delta_{\tilde{r}}'(r_t, \tilde{r}_t, Y_t)\tilde{\sigma}(r_t, \tilde{r}_t, Y_t)dW_t \\ &+ \delta_Y'(r_t, \tilde{r}_t, Y_t)\delta(r_t, \tilde{r}_t, Y_t)dW_t. \end{aligned} \tag{2.134}$$

where σ_r', $\tilde{\sigma}_r'$, δ_r', $\sigma_{\tilde{r}}'$, $\tilde{\sigma}_{\tilde{r}}'$, $\delta_{\tilde{r}}'$ denote Frechet derivatives with respect to the infinite dimensional variables r and $\tilde{r}$. (Further, we will use shorthand notations $\sigma_r'(r_t, \tilde{r}_t, Y_t) = \sigma_r'(r, \tilde{r}, Y)$ omitting the time index.) Thus, we obtain the following expressions for the Stratonovich correction terms:

$$\begin{aligned} d\langle\sigma, W\rangle &= \{\sigma_r'\sigma + \sigma_{\tilde{r}}'\tilde{\sigma} + \sigma_Y'\delta\}\, dt \\ d\langle\tilde{\sigma}, W\rangle &= \{\tilde{\sigma}_r'\sigma + \tilde{\sigma}_{\tilde{r}}'\tilde{\sigma} + \tilde{\sigma}_Y'\delta\}\, dt \\ d\langle\delta, W\rangle &= \{\delta_r'\sigma + \delta_{\tilde{r}}'\tilde{\sigma} + \delta_Y'\delta\}\, dt. \end{aligned}$$

Remark 2.5.2. *Since the Wiener process W_t is multidimensional we interpret the expressions above, for example, as follows:*

$$\sigma_r'(r, \tilde{r}, Y)\sigma(r, \tilde{r}, Y) = \sum_{i=1}^{m} \sigma_{ir}'(r, \tilde{r}, Y)\sigma_i(r, \tilde{r}, Y)$$

Taking into account the functional form of the domestic and foreign forward rates volatilities, we can rewrite the system dynamics as follows:

$$
\begin{aligned}
dr &= \left\{ \mathbf{F}r + \sum_{i}^{m} \varphi_i^2(r,\tilde{r},Y)D_i - \frac{1}{2}\sum_{i=1}^{m} \varphi'_{ri}(r,\tilde{r},Y)[\lambda_i]\varphi_i(r,\tilde{r},Y)\lambda_i \right. \\
&- \left. \frac{1}{2}\sum_{i=1}^{m} \varphi'_{\tilde{r}i}(r,\tilde{r},Y)[\tilde{\lambda}_i]\tilde{\varphi}_i(r,\tilde{r},Y)\lambda_i - \frac{1}{2}\sum_{i=1}^{m} \varphi'_{Yi}(r,\tilde{r},Y)\delta_i(r,\tilde{r},Y)\lambda_i \right\} dt \\
&+ \sum_{i=1}^{m} \varphi_i(r,\tilde{r},Y)\lambda_i \circ dW_i, \\
d\tilde{r} &= \left\{ \mathbf{F}\tilde{r} + \sum_{i=1}^{m} \tilde{\varphi}_i^2(r,\tilde{r},Y)\tilde{D}_i - \frac{1}{2}\sum_{i=1}^{m} \tilde{\varphi}'_{\tilde{r}i}(r,\tilde{r},Y)[\tilde{\lambda}_i]\tilde{\varphi}_i(r,\tilde{r},Y)\tilde{\lambda}_i \right. \\
&- \frac{1}{2}\sum_{i=1}^{m} \tilde{\varphi}'_{ri}(r,\tilde{r},Y)[\lambda_i]\varphi_i(r,\tilde{r},Y)\tilde{\lambda}_i - \frac{1}{2}\sum_{i=1}^{m} \tilde{\varphi}'_{Yi}(r,\tilde{r},Y)\delta_i(r,\tilde{r},Y)\tilde{\lambda}_i \\
&- \left. \sum_{i=1}^{m} \tilde{\varphi}_i(r,\tilde{r},Y)\delta_i(r,\tilde{r},Y)\tilde{\lambda}_i \right\} dt + \sum_{i=1}^{m} \tilde{\varphi}_i(r,\tilde{r},Y)\tilde{\lambda}_i \circ dW_i, \\
dY &= \left\{ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\sum_{i=1}^{m} \delta_i^2(r,\tilde{r},Y) - \frac{1}{2}\sum_{i=1}^{m} \delta'_{ri}(r,\tilde{r},Y)[\lambda_i]\varphi_i(r,\tilde{r},Y) \right. \\
&- \left. \frac{1}{2}\sum_{i=1}^{m} \delta'_{\tilde{r}i}(r,\tilde{r},Y)[\tilde{\lambda}_i]\tilde{\varphi}_i(r,\tilde{r},Y) - \frac{1}{2}\sum_{i=1}^{m} \delta'_{Yi}(r,\tilde{r},Y)\delta_i(r,\tilde{r},Y) \right\} dt \\
&+ \sum_{i=1}^{m} \delta_i(r,\tilde{r},Y) \circ dW_i \qquad (2.135)
\end{aligned}
$$

Our task is to determine when the Lie algebra generated by the following vectors is finite dimensional.

$$\hat{\mu}(\hat{r}) = \begin{bmatrix} \mathbf{F}r + \sum_i^m \varphi_i^2 D_i - \frac{1}{2}\sum_{i=1}^m \varphi'_{ri}[\lambda_i]\varphi_i\lambda_i - \frac{1}{2}\sum_{i=1}^m \varphi'_{\tilde{r}i}[\tilde{\lambda}_i]\tilde{\varphi}_i\lambda_i - \frac{1}{2}\sum_{i=1}^m \varphi'_{Yi}\delta_i\lambda_i \\ \mathbf{F}\tilde{r} + \sum_{i=1}^m \tilde{\varphi}_i^2 \tilde{D}_i - \frac{1}{2}\sum_{i=1}^m \tilde{\varphi}'_{\tilde{r}i}[\tilde{\lambda}_i]\tilde{\varphi}_i\tilde{\lambda}_i - \frac{1}{2}\sum_{i=1}^m \tilde{\varphi}'_{ri}[\lambda_i]\varphi_i\tilde{\lambda}_i - \sum_{i=1}^m \tilde{\varphi}_i\delta_i\tilde{\lambda}_i \\ -\frac{1}{2}\sum_{i=1}^m \tilde{\varphi}'_{Yi}\delta\tilde{\lambda}_i \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\sum_{i=1}^m \delta_i^2 - \frac{1}{2}\sum_{i=1}^m \delta'_{ri}[\lambda_i]\varphi_i - \frac{1}{2}\sum_{i=1}^m \delta'_{\tilde{r}i}[\tilde{\lambda}_i]\tilde{\varphi}_i - \frac{1}{2}\sum_{i=1}^m \delta'_{Yi}\delta_i \end{bmatrix}$$

$$\hat{\sigma}_1(\hat{r}) = \begin{bmatrix} \varphi_1\lambda_1 \\ \tilde{\varphi}_1\tilde{\lambda}_1 \\ \delta_1 \end{bmatrix}, \quad \hat{\sigma}_2(\hat{r}) = \begin{bmatrix} \varphi_2\lambda_2 \\ \tilde{\varphi}_2\tilde{\lambda}_2 \\ \delta_2 \end{bmatrix}, \quad \ldots \quad , \quad \hat{\sigma}_m(\hat{r}) = \begin{bmatrix} \varphi_m\lambda_m \\ \tilde{\varphi}_m\tilde{\lambda}_m \\ \delta_m \end{bmatrix}.$$

Denote by $\mathcal{L}$ the Lie algebra generated by the vector fields $\hat{\mu}(\hat{r}), \hat{\sigma}_1(\hat{r}), \ldots, \hat{\sigma}_m(\hat{r})$

$$\mathcal{L} = \{\hat{\mu}(\hat{r}), \hat{\sigma}_1(\hat{r}), \ldots, \hat{\sigma}_m(\hat{r})\}_{LA} = \{\hat{\mu}(\hat{r}), \hat{\sigma}(\hat{r}))\}_{LA}.$$

It is clear that the Lie algebra is not at all as simple as in the case of deterministic volatilities. In fact, it is not easy to handle even in the case of a scalar Wiener process. This is because each vector δ_i cannot be replaced by means of Gaussian elimination (dividing by the same non-zero function of r, $\tilde{r}$ and S) by a vector which would not depend on r, $\tilde{r}$ and S. However, we will show that in this general case it is still possible to provide **sufficient** conditions for the Lie algebra $\mathcal{L}$ to be finite dimensional. There exists, however, a simple case where we are actually able to provide **both sufficient and necessary** conditions for the Lie algebra, generated by the drift and volatility vector fields, to be finite. We present this simple case in the next section.

2.5.1 A scalar driving Wiener Process: necessary and sufficient conditions

In this section we consider a simple case where domestic and foreign forward rate equations and the exchange rate equation are driven by the same scalar Wiener process W. We assume that the domestic forward rate volatility is of the form

$$\sigma(\tilde{r}, r, x) = \varphi(r, \tilde{r})\lambda(x) \tag{2.136}$$

and the foreign forward rate volatility is of the form

$$\tilde{\sigma}(\tilde{r}, r, x) = \varphi(r, \tilde{r})\tilde{\lambda}(x), \tag{2.137}$$

where

$$\varphi : \mathcal{H} \times \tilde{\mathcal{H}} \to R.$$

Thus, we assume that the domestic and foreign vector fields σ and $\tilde{\sigma}$ have constant directions λ and $\tilde{\lambda}$ that are different for different countries. The length of the fields, however, is determined by any smooth functional, φ, of the two (domestic and foreign) entire forward rate curves. The functional φ is the same for both countries. Assume also that the exchange rate volatility takes the form

$$\delta(\tilde{r}, r, S) = \delta\varphi(r, \tilde{r}). \tag{2.138}$$

Again, we change variables $Y = \ln S$; then the dynamics of Y are

$$dY_t = (\mathbf{B}r_t - \mathbf{B}\tilde{r}_t - \frac{1}{2}\delta^2\varphi^2(r_t, \tilde{r}_t))dt + \delta\varphi(r_t, \tilde{r}_t)dW_t. \tag{2.139}$$

In this case we write the Stratonovich dynamics of $\hat{r}$ as follows:

$$\begin{aligned}
dr &= \left\{\mathbf{F}r + \varphi^2(r,\tilde{r})D - \frac{1}{2}\varphi'_r(r,\tilde{r})[\lambda]\varphi(r,\tilde{r})\lambda - \frac{1}{2}\varphi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}]\varphi(r,\tilde{r})\lambda\right\} dt \\
&+ \varphi(r,\tilde{r})\lambda \circ dW, \\
d\tilde{r} &= \left\{\mathbf{F}\tilde{r} + \varphi^2(r,\tilde{r})\tilde{D} - \frac{1}{2}\varphi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}]\varphi(r,\tilde{r})\tilde{\lambda} - \frac{1}{2}\varphi'_r(r,\tilde{r})[\lambda]\varphi(r,\tilde{r})\tilde{\lambda}\right. \\
&- \left.\varphi^2(r,\tilde{r})\tilde{\lambda}\delta\right\} dt + \varphi(r,\tilde{r})\tilde{\lambda} \circ dW, \\
dY &= \left\{\mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\delta^2\varphi^2(r,\tilde{r}) - \frac{1}{2}\delta\varphi'_r(r,\tilde{r})[\lambda]\varphi(r,\tilde{r}) - \frac{1}{2}\delta\varphi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}]\varphi(r,\tilde{r})\right\} dt \\
&+ \delta\varphi(r,\tilde{r}) \circ dW,
\end{aligned} \tag{2.140}$$

where $\varphi'_r(r,\tilde{r})[\lambda]$ and $\varphi'_{\tilde{r}}(r,\tilde{r})[\lambda]$ denote the Frechet derivatives operating on the vector λ and $\varphi'_r(r,\tilde{r})[\tilde{\lambda}]$ and $\varphi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}]$ denote the Frechet derivatives operating on the vector $\tilde{\lambda}$.

$$D(x) = \lambda(x)\int_0^x \lambda(s)ds \tag{2.141}$$

$$\tilde{D}(x) = \tilde{\lambda}(x)\int_0^x \tilde{\lambda}(s)ds. \tag{2.142}$$

Our task is to determine when the Lie algebra generated by

$$\hat{\mu}(\hat{r}) = \begin{bmatrix} \mathbf{F}r + \varphi^2(r,\tilde{r})D - \frac{1}{2}\varphi'_r(r,\tilde{r})[\lambda]\varphi(r,\tilde{r})\lambda - \frac{1}{2}\varphi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}]\varphi(r,\tilde{r})\lambda \\ \mathbf{F}\tilde{r} + \varphi^2(r,\tilde{r})\tilde{D} - \frac{1}{2}\varphi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}]\varphi(r,\tilde{r})\tilde{\lambda} - \frac{1}{2}\varphi'_r(r,\tilde{r})[\lambda]\varphi(r,\tilde{r})\tilde{\lambda} - \varphi^2(r,\tilde{r})\tilde{\lambda}\delta \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\delta^2\varphi^2(r,\tilde{r}) - \frac{1}{2}\delta\varphi'_r(r,\tilde{r})[\lambda]\varphi(r,\tilde{r}) - \frac{1}{2}\delta\varphi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}]\varphi(r,\tilde{r}) \end{bmatrix}$$

$$\hat{\sigma}(\hat{r}) = \begin{bmatrix} \varphi(r,\tilde{r})\lambda \\ \varphi(r,\tilde{r})\tilde{\lambda} \\ \delta\varphi(r,\tilde{r}) \end{bmatrix} \qquad (2.143)$$

is finite dimensional. Before we start computing the Lie algebra generated by these vector fields we make the following assumptions:

Assumption 2.5.3. *We assume that* $\varphi(r,\tilde{r}) \neq 0$ *for all* $(r,\tilde{r}) \in \mathcal{H} \times \tilde{\mathcal{H}}$.

Assumption 2.5.4. *Denote* $\Phi(r,\tilde{r}) = \varphi^2(r,\tilde{r})$. *We assume that for all* $(r,\tilde{r}) \in \mathcal{H} \times \tilde{\mathcal{H}}$

$$\Phi''_{r,r}(r,\tilde{r})[\lambda;\lambda] + 2\Phi''_{r,\tilde{r}}(r,\tilde{r})[\lambda;\tilde{\lambda}] + \Phi''_{\tilde{r},\tilde{r}}(\tilde{r},\tilde{r})[\tilde{\lambda};\tilde{\lambda}] \neq 0.$$

Under Assumption 2.5.3 the Lie algebra generated by vector fields (2.143) is in fact generated by the simpler system of vector fields $f_0 = \hat{\mu}(\hat{r})$ and $f_1 = \hat{\sigma}(\hat{r})$ (we use the Gaussian elimination and the fact that $\varphi'_r(r,\tilde{r})[\lambda]\varphi(r,\tilde{r})$, $\varphi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}]\varphi(r,\tilde{r})$ and $\varphi(r,\tilde{r})$ are scalar fields on $\mathcal{H} \times \tilde{\mathcal{H}}$), where

$$f_0 = \begin{bmatrix} \mathbf{F}r + \varphi^2(r,\tilde{r})D \\ \mathbf{F}\tilde{r} + \varphi^2(r,\tilde{r})\tilde{D} - \varphi^2(r,\tilde{r})\tilde{\lambda}\delta \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\delta^2\varphi^2(r,\tilde{r}) \end{bmatrix} \qquad f_1 = \begin{bmatrix} \lambda \\ \tilde{\lambda} \\ \delta \end{bmatrix}. \qquad (2.144)$$

We are now able to formulate the following *necessary and sufficient* conditions for $\lambda(x)$ and $\tilde{\lambda}(x)$ ensuring that the Lie algebra generated by vector fields f_0 and f_1 is finite dimensional and thus the whole system admits an FDR.

Proposition 2.5.5. *Assume that*

1. *The domestic and foreign forward rate volatilities are of the form (2.136)-(2.137) consequently, and the exchange rate volatility is of the form (2.138), and*
2. *Assumption 2.5.3 and Assumption 2.5.4 hold.*

The two-country interest rate model then has a finite dimensional realization if and only if $\lambda(x)$ and $\tilde{\lambda}(x)$ are QE. The function φ is allowed to be any smooth scalar field on $\mathcal{H} \times \tilde{\mathcal{H}}$.

Proof. The two-country model will have a generic FDR at the initial point $\hat{r}_0$ if and only if

$$dim\,\{\hat{\mu}, \hat{\sigma}\} < \infty$$

in the neighborhood of $\hat{r}_0$. We obtain the following matrices for the Frechet derivatives of vector fields $\hat{\mu}$ and $\hat{\sigma}$

$$\hat{\mu}'(\hat{r}_t) = \begin{bmatrix} \mathbf{F} + \Phi'_r(r,\tilde{r})D & \Phi'_{\tilde{r}}(r,\tilde{r})D & 0 \\ \Phi'_r(r,\tilde{r})(\tilde{D} - \delta\tilde{\lambda}) & \mathbf{F} + \Phi'_{\tilde{r}}(r,\tilde{r})(\tilde{D} - \delta\tilde{\lambda}) & 0 \\ \mathbf{B} - \frac{1}{2}\delta^2\Phi'_r(r,\tilde{r}) & -\mathbf{B} - \frac{1}{2}\delta^2\Phi'_{\tilde{r}}(r,\tilde{r}) & 0 \end{bmatrix}, \quad \hat{\sigma}'(\hat{r}_t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Lie bracket between the two fields is equal to

$$f_2 = [\hat{\mu}, \hat{\sigma}] = \begin{bmatrix} \mathbf{F}\lambda + \Phi'_r(r,\tilde{r})[\lambda]D + \Phi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}]D \\ \mathbf{F}\tilde{\lambda} + \Phi'_r(r,\tilde{r})[\lambda](\tilde{D} - \delta\tilde{\lambda}) + \Phi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}](\tilde{D} - \delta\tilde{\lambda}) \\ \lambda(0) - \tilde{\lambda}(0) - \frac{1}{2}\delta^2\Phi'_r(r,\tilde{r})[\lambda] - \frac{1}{2}\delta^2\Phi'_{\tilde{r}}(r,\tilde{r})[\tilde{\lambda}] \end{bmatrix}. \tag{2.145}$$

In the same way we obtain that

$$\begin{aligned} f_3 &= [[\hat{\mu}, \hat{\sigma}], \hat{\sigma}] \\ &= \begin{bmatrix} \Phi''_{r,r}(r,\tilde{r})[\lambda;\lambda]D + 2\Phi''_{r,\tilde{r}}(r,\tilde{r})[\lambda;\tilde{\lambda}]D + \Phi''_{\tilde{r},\tilde{r}}(\tilde{r},\tilde{r})[\tilde{\lambda};\tilde{\lambda}]D \\ \left\{\Phi''_{r,r}(r,\tilde{r})[\lambda;\lambda] + 2\Phi''_{r,\tilde{r}}(r,\tilde{r})[\lambda;\tilde{\lambda}] + \Phi''_{\tilde{r},\tilde{r}}(\tilde{r},\tilde{r})[\tilde{\lambda};\tilde{\lambda}]\right\}\left\{\tilde{D} - \delta\tilde{\lambda}\right\} \\ -\frac{1}{2}\delta^2\Phi''_{r,r}(r,\tilde{r})[\lambda;\lambda] - \delta^2\Phi''_{r,\tilde{r}}(r,\tilde{r})[\lambda;\tilde{\lambda}] - \frac{1}{2}\delta^2\Phi''_{\tilde{r},\tilde{r}}(\tilde{r},\tilde{r})[\tilde{\lambda};\tilde{\lambda}] \end{bmatrix}. \end{aligned}$$

Taking into account Assumption 2.5.4 and performing Gaussian elimination (we divide vector field f_3 by $\Phi''_{r,r}(r,\tilde{r})[\lambda;\lambda] + 2\Phi''_{r,\tilde{r}}(r,\tilde{r})[\lambda;\tilde{\lambda}] + \Phi''_{\tilde{r},\tilde{r}}(\tilde{r},\tilde{r})[\tilde{\lambda};\tilde{\lambda}] \neq 0$ and subtract $\varphi^2 f_3$ from f_0) we see that the Lie algebra is generated, in fact, by the system of much simpler vector fields

$$\begin{aligned} f_0(r) &= \begin{bmatrix} \mathbf{F}r \\ \mathbf{F}\tilde{r} \\ \mathbf{B}r - \mathbf{B}\tilde{r} \end{bmatrix} \\ f_1(r) &= \begin{bmatrix} \lambda \\ \tilde{\lambda} \\ \delta \end{bmatrix} \\ f_2(r) &= \begin{bmatrix} \mathbf{F}\lambda \\ \mathbf{F}\tilde{\lambda} \\ \mathbf{B}\lambda - \mathbf{B}\tilde{\lambda} \end{bmatrix} \\ f_3(r) &= \begin{bmatrix} D \\ \tilde{D} - \delta\tilde{\lambda} \\ -\frac{1}{2}\delta^2 \end{bmatrix}. \end{aligned} \tag{2.146}$$

Since all the fields except for the first are constant in the first two arguments, it is easy to calculate the Lie brackets. Also, it is easy to see that all the brackets between the vector fields f_i and f_j, $i, j \geq 1$ are equal to zero and only the brackets with f_0 are non-zero. After computing the Lie brackets we find, in fact, that

$$\{\hat{\mu}, \hat{\sigma}\}_{LA} = span \left\{ \begin{bmatrix} \mathbf{F}r \\ \mathbf{F}\tilde{r} \\ \mathbf{B}r - \mathbf{B}\tilde{r} \end{bmatrix}, \begin{bmatrix} \lambda \\ \tilde{\lambda} \\ \delta \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{F}^n\lambda \\ \mathbf{F}^n\tilde{\lambda} \\ \mathbf{B}\mathbf{F}^{n-1}\lambda - \mathbf{B}\mathbf{F}^{n-1}\tilde{\lambda} \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} D \\ \tilde{D} - \delta\tilde{\lambda} \\ -\frac{1}{2}\delta^2 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{F}^n D \\ \mathbf{F}^n\tilde{D} - \delta\mathbf{F}^n\tilde{\lambda} \\ \mathbf{B}\mathbf{F}^{n-1}D - \mathbf{B}\mathbf{F}^{n-1}\tilde{D} + \delta\mathbf{B}\mathbf{F}^{n-1}\tilde{\lambda} \end{bmatrix}, n = 1, 2 \ldots \right\}. \tag{2.147}$$

We note that a necessary condition for the Lie algebra generated by these vector fields to be finite dimensional is that the vector space spanned by

$$\left\{ \begin{bmatrix} \mathbf{F}r \\ \mathbf{F}\tilde{r} \\ \mathbf{B}r - \mathbf{B}\tilde{r} \end{bmatrix}, \begin{bmatrix} \lambda \\ \tilde{\lambda} \\ \delta \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{F}^n\lambda \\ \mathbf{F}^n\tilde{\lambda} \\ \mathbf{B}\mathbf{F}^{n-1}\lambda - \mathbf{B}\mathbf{F}^{n-1}\tilde{\lambda} \end{bmatrix} \right\} \tag{2.148}$$

is finite dimensional. According to Proposition 2.4.2, this occurs if and only if both $\lambda(x)$ and $\tilde{\lambda}(x)$ are quasi-exponential functions.

If $\lambda(x)$ and $\tilde{\lambda}(x)$ are QE, it follows that $D(x)$ and $\tilde{D}(x) - \delta\tilde{\lambda}(x)$ are QE by Lemma 2.8.13. As we already proved in Lemma 2.8.14, there exists a minimum degree annihilator polynomial $P(\mathbf{F})$ of degree $q = 2n$ (where n is the degree of the minimum annihilator polynomial for λ and $\tilde{\lambda}$), such that

$$P(\mathbf{F})D = P(\mathbf{F})(\tilde{D} - \delta\tilde{\lambda}) = 0. \tag{2.149}$$

Using Proposition 2.4.2 we realize that the Lie algebra spanned by

$$\left\{ \begin{bmatrix} D \\ \tilde{D} - \delta\tilde{\lambda} \\ -\frac{1}{2}\delta^2 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{F}^n D \\ \mathbf{F}^n \tilde{D} - \delta \mathbf{F}^n \tilde{\lambda} \\ \mathbf{B}\mathbf{F}^{n-1} D - \mathbf{B}\mathbf{F}^{n-1}\tilde{D} + \delta \mathbf{B}\mathbf{F}^{n-1}\tilde{\lambda} \end{bmatrix}, n = 1, 2 \ldots \right\}. \quad (2.150)$$

is finite dimensional and vector $u_{q+1} = \begin{bmatrix} \mathbf{F}^{q+1} D \\ \mathbf{F}^{q+1}(\tilde{D} - \lambda\delta) \\ \mathbf{B}\mathbf{F}^q D - \mathbf{B}\mathbf{F}^q \tilde{D} + \delta \mathbf{B}\mathbf{F}^q \tilde{\lambda} \end{bmatrix}$ can be expressed as the linear combination of vectors

$$u_1 = \begin{bmatrix} \mathbf{F}D \\ \mathbf{F}(\tilde{D} - \delta\tilde{\lambda}) \\ \mathbf{B}D - \mathbf{B}\tilde{D} + \delta\mathbf{B}\lambda \end{bmatrix}, u_2 = \begin{bmatrix} \mathbf{F}^2 D \\ \mathbf{F}^2(\tilde{D} - \delta\tilde{\lambda}) \\ \mathbf{B}\mathbf{F}D - \mathbf{B}\mathbf{F}\tilde{D} + \delta\mathbf{B}\mathbf{F}\tilde{\lambda} \end{bmatrix}, \ldots,$$

$$\ldots, u_q = \begin{bmatrix} \mathbf{F}^q D \\ \mathbf{F}^q(\tilde{D} - \delta\tilde{\lambda}) \\ \mathbf{B}\mathbf{F}^{q-1} D - \mathbf{B}\mathbf{F}^{q-1}\tilde{D} + \delta\mathbf{B}\mathbf{F}^{q-1}\tilde{\lambda} \end{bmatrix}$$

and the proof is completed. ■

2.5.2 General orthogonal noise model. Sufficient conditions.

Before moving to the general constant direction volatility model, we consider the simpler orthogonal noise model, where the set of Wiener processes W_i, $i \in M$ driving the domestic forward rate equation is uncorrelated with the set of Wiener processes W_i, $i \in N$ driving the foreign forward rate equation; both sets are uncorrelated with the set of Wiener processes W_i, $i \in K$ driving the exchange rate equation. In this case, because of independency, the Stratonovich dynamics (2.135) of $\hat{r}$ will be simplified as follows:

$$
\begin{aligned}
dr &= \left\{ \mathbf{F}r + \sum_{i\in M} \varphi_i^2(r,\tilde{r},Y)D_i - \frac{1}{2}\sum_{i\in M} \varphi'_{ri}(r,\tilde{r},Y)[\lambda_i]\varphi_i(r,\tilde{r},Y)\lambda_i \right\} dt \\
&+ \sum_{i\in M} \varphi_i(r,\tilde{r},Y)\lambda_i \circ dW_i, \\
d\tilde{r} &= \left\{ \mathbf{F}\tilde{r} + \sum_{i\in N} \tilde{\varphi}_i^2(r,\tilde{r},Y)\tilde{D}_i - \frac{1}{2}\sum_{i\in N} \tilde{\varphi}'_{\tilde{r}i}(r,\tilde{r},Y)[\tilde{\lambda}_i]\tilde{\varphi}_i(r,\tilde{r},Y)\tilde{\lambda}_i \right\} dt \\
&+ \sum_{i\in N} \tilde{\varphi}_i(r,\tilde{r},Y)\tilde{\lambda}_i \circ dW_i, \\
dY &= \left\{ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\sum_{i\in K} \delta_i^2(r,\tilde{r},Y) - \frac{1}{2}\sum_{i\in K} \delta'_{Yi}(r,\tilde{r},Y)\delta_i(r,\tilde{r},Y) \right\} dt \\
&+ \sum_{i\in K} \delta_i(r,\tilde{r},Y) \circ dW_i. \qquad (2.151)
\end{aligned}
$$

Thus, our task is to determine whether the Lie algebra generated by the following vector fields is finite dimensional:

$$
\hat{\mu}(\hat{r}) = \begin{bmatrix} \mathbf{F}r + \sum_{i\in M} \varphi_i^2(r,\tilde{r},Y)D_i - \frac{1}{2}\sum_{i\in M} \varphi'_{ri}(r,\tilde{r},Y)[\lambda_i]\varphi_i(r,\tilde{r},Y)\lambda_i \\ \mathbf{F}\tilde{r} + \sum_{i\in N} \tilde{\varphi}_i^2(r,\tilde{r},Y)\tilde{D}_i - \frac{1}{2}\sum_{i\in N} \tilde{\varphi}'_{\tilde{r}i}(r,\tilde{r},Y)[\tilde{\lambda}_i]\tilde{\varphi}_i(r,\tilde{r},Y)\tilde{\lambda}_i \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\sum_{i\in K} \delta_i^2(r,\tilde{r},Y) - \frac{1}{2}\sum_{i\in K} \delta'_{Yi}(r,\tilde{r},Y)\delta_i(r,\tilde{r},Y) \end{bmatrix},
$$

$$
\hat{\sigma}_i(\hat{r}) = \begin{bmatrix} \varphi_i(r,\tilde{r},Y)\lambda_i \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\sigma}_j(\hat{r}) = \begin{bmatrix} 0 \\ \tilde{\varphi}_j(r,\tilde{r},Y)\tilde{\lambda}_j \\ 0 \end{bmatrix}, \quad \hat{\sigma}_k(\hat{r}) = \begin{bmatrix} 0 \\ 0 \\ \delta_k(r,\tilde{r},Y) \end{bmatrix},
$$

$$
i \in M, \quad j \in N, \quad k \in K.
$$

To proceed with computing the Lie algebra generated by $\hat{\mu}(\hat{r})$, $\hat{\sigma}_i(\hat{r})$, $i \in M$, $\hat{\sigma}_j(\hat{r})$, $j \in N$ and $\hat{\sigma}_k(\hat{r})$, $k \in K$ we make the following assumption:

Assumption 2.5.6. *We assume that* $\varphi_i(r,\tilde{r},Y) \neq 0$, $\tilde{\varphi}_i(r,\tilde{r},Y) \neq 0$ *and* $\delta_i(r,\tilde{r},Y) \neq 0$ *for all* $(r,\tilde{r},Y) \in \mathcal{H} \times \tilde{\mathcal{H}} \times R_+$ *and for all* $i \in M \cup N \cup K$.

Under Assumption 2.5.6 we once again use Lemma 3.166 to see that the Lie algebra

$$\mathcal{L} = \{\hat{\mu}(\hat{r}), \hat{\sigma}_i(\hat{r}), i \in M \cup N \cup K\}_{LA} = \{\hat{\mu}(\hat{r}), \hat{\sigma}(\hat{r})\}_{LA}$$

is generated by a much simpler system of vector fields:

$$f_0 = \begin{bmatrix} \mathbf{F}r + \sum_{i\in M} \varphi_i^2(r,\tilde{r},Y)D_i \\ \mathbf{F}\tilde{r} + \sum_{i\in N} \tilde{\varphi}_i^2(r,\tilde{r},Y)\tilde{D}_i \\ \mathbf{B}r - \mathbf{B}\tilde{r} \end{bmatrix}, \quad f_i = \begin{bmatrix} \lambda_i \\ 0 \\ 0 \end{bmatrix}, \quad f_j = \begin{bmatrix} 0 \\ \tilde{\lambda}_j \\ 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$i \in M, \quad j \in N. \tag{2.152}$$

Although the system of vector fields we arrived at by means of Gaussian elimination is much simpler than the initial system of vector fields, once we start computing the Lie brackets we very soon realize that the resulting Lie algebra is still very difficult to analyze. However, the situation is not hopeless. Denote by $\mathcal{L}_1$ the Lie algebra generated by the following system of vector fields:

$$\tilde{f}_0 = \begin{bmatrix} \mathbf{F}r \\ \mathbf{F}\tilde{r} \\ \mathbf{B}r - \mathbf{B}\tilde{r} \end{bmatrix}, \quad f_i = \begin{bmatrix} \lambda_i \\ 0 \\ 0 \end{bmatrix}, \quad f_j = \begin{bmatrix} 0 \\ \tilde{\lambda}_j \\ 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$g_i = \begin{bmatrix} D_i \\ 0 \\ 0 \end{bmatrix}, \quad g_j = \begin{bmatrix} 0 \\ \tilde{D}_j \\ 0 \end{bmatrix}, \qquad i \in M, \quad j \in N.$$

Now we can see that the Lie algebra $\mathcal{L}$

$$\mathcal{L} = \{f_0, f_i, i \in M, f_j, i \in N, h\}_{LA} \tag{2.153}$$

is in fact included in the larger Lie algebra $\mathcal{L}_1$

$$\mathcal{L}_1 = \left\{\tilde{f}_0, f_i, g_i, i \in M, f_j, g_j, j \in N, h\right\}_{LA}, \tag{2.154}$$

since the generators of $\mathcal{L}$ are included among the generators of $\mathcal{L}_1$. It is clear then that if the larger Lie algebra is finite dimensional then the Lie algebra $\mathcal{L}$ will also have finite dimension. Larger Lie algebra $\mathcal{L}_1$ has a much simpler structure than $\mathcal{L}$. By studying $\mathcal{L}_1$ we can then easily obtain sufficient conditions for the existence of an FDR.

Existence of an FDR

The following proposition gives us necessary and sufficient conditions for the larger Lie algebra $\mathcal{L}_1$ (and thus sufficient conditions for the general orthogonal noise model (2.151)) to be realized by a finite dimensional state space model.

Proposition 2.5.7. *The Lie algebra $\mathcal{L}_1$ is finite dimensional if and only if each component of $\lambda(x)$ and $\tilde{\lambda}(x)$ is quasi-exponential. Thus a sufficient condition for the two-country general orthogonal noise model to have a finite dimensional realization is that all the components of $\lambda(x)$ and $\tilde{\lambda}(x)$ are quasi-exponential. The functions φ_i, $i \in M$, $\tilde{\varphi}_j$, $j \in N$ and δ_k, $k \in K$ are allowed to be any arbitrary smooth scalar fields on $\mathcal{H} \times \tilde{\mathcal{H}} \times R_+$.*

Proof. We start by computing the Lie brackets for the larger Lie algebra $\mathcal{L}_1$. It is easy to see that the only non-zero Lie brackets will result from taking brackets between vector field $\tilde{f}_0$ and all other vector fields. Thus, computing the Lie brackets we see that for $i \in M$, $\quad j \in N$

$$\mathcal{L}_1 = span\left\{ \begin{bmatrix} \mathbf{F}r \\ \mathbf{F}\tilde{r} \\ \mathbf{B}r - \mathbf{B}\tilde{r} \end{bmatrix}, \begin{bmatrix} \mathbf{F}^n\lambda_i \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{F}^n\tilde{\lambda}_j \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \mathbf{F}^n D_i \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{F}^n\tilde{D}_j \\ 0 \end{bmatrix}, n \geq 0 \right\}.$$

It is clear that a necessary condition for the relevant Lie algebra to be finite dimensional is that the vector space spanned by the subspace

$$\{\mathbf{F}\lambda_i, \quad n = 1, 2, \ldots, \quad i \in M\} \tag{2.155}$$

and the vector space spanned by subspace

$$\left\{\mathbf{F}\tilde{\lambda}_j, \quad n = 1, 2, \ldots, \quad j \in N\right\} \tag{2.156}$$

are finite dimensional. This is satisfied if and only if λ_i, $\tilde{\lambda}_j$, $i \in M$, $j \in N$ are QE. According to Lemma 2.8.13 D_i, $\tilde{D}_j$ are QE as well, which means that $\mathcal{L}_1$ is finite dimensional. Since this obviously leads to the finite dimensionality of the smaller algebra $\mathcal{L}$ we conclude that QE λ_i, $\tilde{\lambda}_j$, $i \in M$, $j \in N$ are sufficient for the two-country general orthogonal noise model to have a finite dimensional realization. ∎

2.5.3 General model. Sufficient conditions

Existence of an FDR

In this section we consider the general two-country model (2.135) where each equation of the system is allowed to be driven by the m-dimensional Wiener process W. In order to make predictions concerning the existence of an FDR in this model, we intend to study the dimension of the Lie algebra generated by the following vector fields:

$$\hat{\mu}(\hat{r}) = \begin{bmatrix} \mathbf{F}r + \sum_i^m \varphi_i^2 D_i - \frac{1}{2}\varphi_r'[\lambda]\varphi\lambda - \frac{1}{2}\varphi_{\tilde{r}}'[\tilde{\lambda}]\tilde{\varphi}\lambda - \frac{1}{2}\varphi_Y'\delta\lambda \\ \mathbf{F}\tilde{r} + \sum_{i=1}^m \tilde{\varphi}_i^2 \tilde{D}_i - \frac{1}{2}\tilde{\varphi}_{\tilde{r}}'[\tilde{\lambda}]\tilde{\varphi}\tilde{\lambda} - \frac{1}{2}\tilde{\varphi}_r'[\lambda]\varphi\tilde{\lambda} - \frac{1}{2}\tilde{\varphi}_Y'\tilde{\lambda}\delta - \tilde{\varphi}\delta\tilde{\lambda} \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\sum_{i=1}^m \delta_i^2 - \frac{1}{2}\delta_r'[\lambda]\varphi - \frac{1}{2}\delta_{\tilde{r}}'[\tilde{\lambda}]\tilde{\varphi} - \frac{1}{2}\delta_Y'\delta \end{bmatrix},$$

$$\hat{\sigma}_1(\hat{r}) = \begin{bmatrix} \varphi_1\lambda_1 \\ \tilde{\varphi}_1\tilde{\lambda}_1 \\ \delta_1 \end{bmatrix}, \quad \hat{\sigma}_2(\hat{r}) = \begin{bmatrix} \varphi_2\lambda_2 \\ \tilde{\varphi}_2\tilde{\lambda}_2 \\ \delta_2 \end{bmatrix}, \quad \ldots \quad, \quad \hat{\sigma}_m(\hat{r}) = \begin{bmatrix} \varphi_m\lambda_m \\ \tilde{\varphi}_m\tilde{\lambda}_m \\ \delta_m \end{bmatrix}.$$

We can see that the Lie algebra generated by these fields can no longer be reduced by means of Gaussian elimination to the simpler system of vector fields as in (2.152). Denote by $\mathcal{L}$ the initial Lie algebra generated by these fields. Define the Lie algebra $\mathcal{L}_1$ as the Lie algebra generated by the following vector fields (for $i = 1, \ldots, m$):

$$f_0 = \begin{bmatrix} \mathbf{F}r + \sum_{i=1}^m \varphi_i^2 D_i \\ \mathbf{F}\tilde{r} + \sum_{i=1}^m \tilde{\varphi}_i^2 \tilde{D}_i \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\sum_{i=1}^m \delta_i^2 \end{bmatrix}, \quad f_i = \begin{bmatrix} \lambda_i \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{f}_i = \begin{bmatrix} 0 \\ \tilde{\lambda}_i \\ 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We note now that the Lie algebra $\mathcal{L}$ is included in the larger Lie algebra $\mathcal{L}_1$. The vector fields generating the Lie algebra $\mathcal{L}_1$ can be considered as a system of the vector fields for the general orthogonal noise model. Thus, as pointed out in the previous section, the Lie algebra $\mathcal{L}_1$ can itself be included into even larger Lie algebra $\mathcal{L}_2$ generated by the following vector fields for $i = 1, \ldots, m$:

$$\tilde{f}_0 = \begin{bmatrix} \mathbf{F}r \\ \mathbf{F}\tilde{r} \\ \mathbf{B}r - \mathbf{B}\tilde{r} \end{bmatrix}, \quad f_i = \begin{bmatrix} \lambda_i \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{f}_i = \begin{bmatrix} 0 \\ \tilde{\lambda}_i \\ 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad g_i = \begin{bmatrix} D_i \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{g}_i = \begin{bmatrix} 0 \\ \tilde{D}_i \\ 0 \end{bmatrix}.$$

Thus, we obtain the following inclusion

$$\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2, \tag{2.157}$$

where

$$\mathcal{L} = \{\hat{\mu}, \hat{\sigma}\}_{LA}$$

$$\mathcal{L}_1 = \left\{ f_0, h, f_i, \tilde{f}_i, i = 1, \dots m \right\}_{LA}$$

$$\mathcal{L}_2 = \left\{ \tilde{f}_0, h, f_i, \tilde{f}_i, g_i, \tilde{g}_i, i = 1, \dots, m \right\}_{LA}.$$

Using Proposition 2.5.7 we immediately obtain the following result specifying the necessary and sufficient conditions for the larger Lie algebra $\mathcal{L}_2$ to be finite dimensional and thus the sufficient condition for the Lie algebra $\mathcal{L}$ to be finite dimensional.

Proposition 2.5.8. *The Lie algebra $\mathcal{L}_2$ if finite dimensional if and only if each component of $\lambda(x)$ and $\tilde{\lambda}(x)$ is quasi-exponential. Thus a sufficient condition for the two-country general constant direction volatility model to have a finite dimensional realization is that all the components of $\lambda(x)$ and $\tilde{\lambda}(x)$ are quasi-exponential. The functions φ_i, $\tilde{\varphi}_i$ and δ_i, $i = 1, \dots, m$ are allowed to be any arbitrary smooth scalar fields on $\mathcal{H} \times \tilde{\mathcal{H}} \times R_+$.*

Proof. Follows from Proposition 2.5.7 and inclusion (2.157). ■

Construction of an FDR

From Subsection 2.5.3 we know that the general constant direction volatility model admits finite dimensional realization if and only if $\lambda(x)$ and $\tilde{\lambda}(x)$ are QE. However, we were not able to determine the *minimal* dimension of the realization. Thus, we can only construct *non-minimal* finite dimensional realization. We use the same technique as in the simplest case, and given the simplicity of the generating vector fields, the operators $\exp\{\hat{\mu}t\}$ and $\exp\{\hat{\sigma}t\}$ are easy to find:

$$e^{\hat{\mu}t}\hat{r}_0 = \begin{bmatrix} r_0(x+t) \\ \tilde{r}_0(x+t) \\ y_0 + \int_0^t (r_s(0) - \tilde{r}_s(0))ds \end{bmatrix} \tag{2.158}$$

$$\begin{aligned} e^{f_i^k t}\hat{r}_0 &= \begin{bmatrix} r_0 + \mathbf{F}^k\lambda_i t \\ 0 \\ 0 \end{bmatrix}, \quad e^{\tilde{f}_i^k t}\hat{r}_0 = \begin{bmatrix} 0 \\ r_0 + \mathbf{F}^k\tilde{\lambda}_i t \\ 0 \end{bmatrix}, \\ e^{g_i^k t}\hat{r}_0 &= \begin{bmatrix} r_0 + \mathbf{F}^k D_i t \\ 0 \\ 0 \end{bmatrix}, \quad e^{\tilde{g}_i^k t}\hat{r}_0 = \begin{bmatrix} 0 \\ r_0 + \mathbf{F}^k\tilde{D}_i t \\ 0 \end{bmatrix}. \end{aligned} \tag{2.159}$$

Assume that the finite dimensional realization exists and λ_i, D_i, $\tilde{\lambda}_i$ and $\tilde{D}_i$ satisfy the following ODE:

$$\mathbf{F}^{n_i}\lambda_i = \sum_{k=0}^{n_i-1} c_i^k \mathbf{F}^k \lambda_i, \qquad \mathbf{F}^{n_i}\tilde{\lambda}_i = \sum_{k=0}^{\tilde{n}_i-1} \tilde{c}_i^k \mathbf{F}^k \tilde{\lambda}_i \tag{2.160}$$

$$\mathbf{F}^{q_i}D_i = \sum_{k=0}^{q_i-1} d_i^k \mathbf{F}^k D_i \qquad \mathbf{F}^{\tilde{q}_i}\tilde{D}_i = \sum_{k=0}^{\tilde{q}_i-1} \tilde{d}_i^k \mathbf{F}^k \tilde{D}_i. \tag{2.161}$$

Then we can construct the invariant manifold as follows:

$$\hat{G}(z^0, z_{ik}^1, z_{ij}^2, z_{il}^3, z_{ip}^4, z^5) = \prod (e^{f_i^k z_{ik}^1})(e^{g_i^k z_{ik}^2})(e^{\tilde{f}_i^j z_{ij}^3})(e^{\tilde{g}_i^j z_{ij}^4})(e^{h z^5}) e^{\hat{\mu} z^0}\hat{r}_0,$$

where $k = 0, 1, \ldots, n_i - 1$; $j = 0, 1, \ldots, q_i - 1$; $l = 0, 1, \ldots, \tilde{n}_i - 1$; $p = 0, 1, \ldots, \tilde{q}_i - 1$, $i = 1, \ldots, m$, $q_i = 2n_i$. This brings us to the forward rate curve parametrization

$$\begin{aligned} &\hat{G}(z^0, z_{ik}^1, z_{ij}^2, z_{il}^3, z_{ip}^4, z^5) \\ = &\begin{bmatrix} G_1(z^0, z_{ik}^1, z_{ij}^2, k = 0, 1, \ldots, n_i - 1; \quad j = 0, 1, \ldots, q_i - 1; \quad i = 1, \ldots, m) \\ G_2(z^0, z_{ik}^3, z_{ij}^4, k = 0, 1, \ldots, \tilde{n}_i - 1; \quad j = 0, 1, \ldots, \tilde{q}_i - 1; \quad i = 1, \ldots, m) \\ G_3(z^0, z^5) \end{bmatrix}, \end{aligned}$$

where

$$
\begin{aligned}
& G_1(z^0, z^1_{ik}, z^2_{ij}, k = 0, 1, \ldots, n_i - 1; j = 0, 1, \ldots, q_i - 1; i = 1, \ldots, m) \\
= \; & r_0(x + z^0) + \sum_{i=0}^{m} \sum_{k=0}^{n_i-1} \mathbf{F}^k \lambda_i(x) z^1_{ik} + \sum_{i=0}^{m} \sum_{j=0}^{q_i-1} \mathbf{F}^j D_i(x) z^2_{ij} \\
& G_2(z^0, z^3_{ik}, z^4_{ij}, k = 0, 1, \ldots, \tilde{n}_i - 1; j = 0, 1, \ldots, \tilde{q}_i - 1; i = 1, \ldots, m) \\
= \; & \tilde{r}_0(x + z^0) + \sum_{i=0}^{m} \sum_{k=0}^{\tilde{n}_i-1} \mathbf{F}^k \tilde{\lambda}_i(x) z^3_{ik} + \sum_{i=0}^{m} \sum_{j=0}^{\tilde{q}_i-1} \mathbf{F}^j \tilde{D}_i(x) z^4_{ij} \\
& G_3(z, z^5) = y_0 + \int_0^{z^0} (r_s(0) - \tilde{r}_s(0)) ds + z^5 \qquad (2.162)
\end{aligned}
$$

We can now infer the Stratonovich dynamics of the vector state process Z

$$dZ_t = a(Z_t)dt + b(Z_t) \circ dW_t$$

from the equations

$$\hat{G}_\star a = \hat{\mu}, \quad \hat{G}_\star b = \hat{\sigma},$$

which can be rewritten as follows:

$$
\begin{aligned}
G_1'(Z)a(Z) \; = \; & \frac{\partial}{\partial x} r_0(x + z^0) a_0 + \sum_{i=0}^{m} \sum_{k=0}^{n_i-1} \mathbf{F}^k \lambda_i(x) a^1_{ik} + \sum_{i=0}^{m} \sum_{j=0}^{q_i-1} \mathbf{F}^j D_i(x) a^2_{ij} \\
= \; & \frac{\partial}{\partial x} r_0(x + Z^0) + \sum_{i=0}^{m} \sum_{k=0}^{n_i-1} \mathbf{F}^{k+1} \lambda_i(x) Z^1_{ik} + \sum_{i=0}^{m} \sum_{j=0}^{q_i-1} \mathbf{F}^{j+1} D_i(x) Z^2_{ij} \\
+ \; & \sum_{i}^{m} \varphi_i^2 D_i - \frac{1}{2} \varphi_r'[\lambda] \varphi \lambda - \frac{1}{2} \varphi_{\tilde{r}}'[\tilde{\lambda}] \tilde{\varphi} \lambda - \frac{1}{2} \varphi_Y' \delta \lambda \\
G_2'(Z)a(Z) \; = \; & \frac{\partial}{\partial x} \tilde{r}_0(x + z^0) a_0 + \sum_{i=0}^{m} \sum_{k=0}^{\tilde{n}_i-1} \mathbf{F}^k \tilde{\lambda}_i(x) a^3_{ik} + \sum_{i=0}^{m} \sum_{j=0}^{\tilde{q}_i-1} \mathbf{F}^j \tilde{D}_i(x) a^4_{ij} \\
= \; & \frac{\partial}{\partial x} \tilde{r}_0(x + Z^0) + \sum_{i=0}^{m} \sum_{k=0}^{\tilde{n}_i-1} \mathbf{F}^{k+1} \tilde{\lambda}_i(x) Z^3_{ik} + \sum_{i=0}^{m} \sum_{j=0}^{\tilde{q}_i-1} \mathbf{F}^{j+1} \tilde{D}_i(x) Z^4_{ij} \\
+ \; & \sum_{i=1}^{m} \tilde{\varphi}_i^2 \tilde{D}_i - \frac{1}{2} \tilde{\varphi}_{\tilde{r}}'[\tilde{\lambda}] \tilde{\varphi} \tilde{\lambda} - \frac{1}{2} \tilde{\varphi}_r'[\lambda] \varphi \tilde{\lambda} - \frac{1}{2} \tilde{\varphi}_Y' \tilde{\lambda} \delta - \tilde{\varphi} \delta \tilde{\lambda}
\end{aligned}
$$

.

Substituting equations (2.160) in (2.163) we obtain the following solution for vector $a(Z)$:

$$
\begin{aligned}
a_0 &= 1 \\
a^1_{i,0} &= c^0_i Z^1_{i,n_i-1} - \frac{1}{2}\varphi'_{ri}(\hat{G}(Z))[\lambda_i]\varphi_i(\hat{G}(Z)) - \frac{1}{2}\varphi'_{\tilde{r}i}(\hat{G}(Z))[\tilde{\lambda}_i]\tilde{\varphi}_i(\hat{G}(Z)) \\
&- \frac{1}{2}\varphi'_{Yi}(\hat{G}(Z))\delta_i \\
a^1_{ik} &= Z^1_{i,k-1} + c^k_i Z^1_{i,n_i-1} \\
a^2_{i,0} &= d^0_i Z^2_{i,q_i-1} + \varphi^2_i(G(Z)) \\
a^2_{i,j} &= Z^2_{i,j-1} + d^j_i Z^2_{i,q_i-1} \\
a^3_{i,0} &= \tilde{c}^0_i Z^3_{i,\tilde{n}_i-1} - \frac{1}{2}\tilde{\varphi}'_{\tilde{r}i}(\hat{G}(Z))[\tilde{\lambda}_i]\tilde{\varphi}_i(\hat{G}(Z)) - \frac{1}{2}\tilde{\varphi}'_{ri}(\hat{G}(Z))[\lambda_i]\varphi_i(\hat{G}(Z)) \\
&- \frac{1}{2}\tilde{\varphi}'_{Yi}(\hat{G}(Z))\delta_i - \tilde{\varphi}_i(\hat{G}(Z))\delta_i \\
a^3_{ik} &= Z^3_{i,k-1} + \tilde{c}^k_i Z^3_{i,\tilde{n}_i-1} \\
a^4_{i,0} &= \tilde{d}^0_i Z^4_{i,\tilde{q}_i-1} + \tilde{\varphi}^2_i(G(Z)) \\
a^4_{i,j} &= Z^4_{i,j-1} + \tilde{d}^j_i Z^4_{i,\tilde{q}_i-1}
\end{aligned}
$$

.

Vector $b(z)$ can be found from the following system:

$$
\begin{aligned}
G'_1(z)b^i(z) &= \frac{\partial}{\partial x}r_0(x+z^0)b_0 + \sum_{i=0}^{m}\sum_{k=0}^{n_i-1}\mathbf{F}^k\lambda_i(x)b^1_{ik} + \sum_{i=0}^{m}\sum_{j=0}^{q_i-1}\mathbf{F}^j D_i(x)b^2_{ij} = \varphi_i\lambda_i \\
G'_2(z)b^i(z) &= \frac{\partial}{\partial x}\tilde{r}_0(x+z^0)b_0 + \sum_{i=0}^{m}\sum_{k=0}^{\tilde{n}_i-1}\mathbf{F}^k\tilde{\lambda}_i(x)b^3_{ik} + \sum_{i=0}^{m}\sum_{j=0}^{\tilde{q}_i-1}\mathbf{F}^j \tilde{D}_i(x)b^4_{ij} = \tilde{\varphi}_i\tilde{\lambda}_i
\end{aligned}
$$

$$
\begin{aligned}
b_0 &= 0 \\
b^1_{i0} &= \varphi_i(z), \\
b^1_{ik} &= 0, \quad k = 1,\ldots,n_i-1, \\
b^2_{ij} &= 0, \\
b^3_{i0} &= \tilde{\varphi}_i(z), \\
b^3_{ik} &= 0, \quad k = 1,\ldots,\tilde{n}_i-1, \\
b^4_{ij} &= 0
\end{aligned}
$$

Finally, comparing the dynamics of Y and Z^5 we are able to derive the dynamics of Z^5. The Stratonovich dynamics of the state variables are given by

$$
\begin{cases}
dZ_0 &= dt \\
dZ^1_{i,0} &= \Big\{ c_i^0 Z^1_{i,n_i-1} - \frac{1}{2}\varphi'_{ri}(\hat{G}(Z))[\lambda_i]\varphi_i(\hat{G}(Z)) - \frac{1}{2}\varphi'_{\tilde{r}i}(\hat{G}(Z))[\tilde{\lambda}_i]\tilde{\varphi}_i(\hat{G}(Z)) \\
&- \frac{1}{2}\varphi'_{Si}(\hat{G}(Z))\delta_i \Big\} dt + \varphi_i(\hat{G}(Z)) \circ dW_t \\
dZ^1_{ik} &= \left\{ Z^1_{i,k-1} + c_i^k Z^1_{i,n_i-1} \right\} dt, \quad k = 1, \dots, n_i - 1 \\
dZ^2_{i,0} &= \Big\{ d_i^0 Z^2_{i,q_i-1} + \varphi_i^2(\hat{G}(Z)) \Big\} dt \\
dZ^2_{i,j} &= \left\{ Z^2_{i,j-1} + d_i^j Z^2_{i,q_i-1} \right\} dt, \quad j = 1, \dots, q_i - 1 \\
dZ^3_{i,0} &= \{ \tilde{c}_i^0 Z^3_{i,\tilde{n}_i-1} - \frac{1}{2}\tilde{\varphi}'_{\tilde{r}i}(\hat{G}(Z))[\tilde{\lambda}_i]\tilde{\varphi}_i(\hat{G}(Z)) - \frac{1}{2}\tilde{\varphi}'_{ri}(\hat{G}(Z))[\lambda_i]\tilde{\varphi}_i(\hat{G}(Z)) \\
&- \frac{1}{2}\tilde{\varphi}'_{Si}(\hat{G}(Z))\delta_i - \tilde{\varphi}_i(\hat{G}(Z))\delta_i \} dt + \tilde{\varphi}_i(\hat{G}(Z)) \circ dW_t \\
dZ^3_{ik} &= \left\{ Z^3_{i,k-1} + \tilde{c}_i^k Z^3_{i,\tilde{n}_i-1} \right\} dt, \quad k = 1, \dots, \tilde{n}_i - 1 \\
dZ^4_{i,0} &= \Big\{ \tilde{d}_i^0 Z^4_{i,\tilde{q}_i-1} + \tilde{\varphi}_i^2(\hat{G}(Z)) \Big\} dt \\
dZ^4_{i,j} &= \Big\{ Z^4_{i,j-1} + \tilde{d}_i^j Z^4_{i,\tilde{q}_i-1} \Big\} dt, \quad j = 1, \dots, \tilde{q}_i - 1 \\
dZ_5 &= \Big\{ -\frac{1}{2}\delta^2\varphi^2(\hat{G}(Z)) - \frac{1}{2}\delta\varphi'_r(\hat{G}(Z))[\lambda]\varphi(\hat{G}(Z)) - \frac{1}{2}\delta\varphi'_{\tilde{r}}(\hat{G}(Z))[\tilde{\lambda}]\varphi(\hat{G}(Z)) \Big\} dt \\
&+ \delta\varphi(\hat{G}(Z)) \circ dW
\end{cases}
.
$$

Thus, we proved the following proposition.

Proposition 2.5.9. *Given the initial domestic and foreign forward rate curves r_0 and $\tilde{r}_0$ and exchange rate S_0, the forward rate system, generated by volatilities defined in (2.126), (2.127) and (2.128), has a finite dimensional realization given by*

$$\hat{r}_t = \hat{G}(Z_t, x),$$

where $\hat{G}$ is defined as in (2.162). The Ito dynamics of the state variables Z are *given by*

$$\begin{cases} dZ_0 &= dt \\ dZ^1_{i,0} &= c^0_i Z^1_{i,n_i-1} dt + \varphi_i(\hat{G}(Z))dW_t \\ dZ^1_{ik} &= \left\{Z^1_{i,k-1} + c^k_i Z^1_{i,n_i-1}\right\} dt, \quad k = 1,\dots,n_i - 1 \\ dZ^2_{i,0} &= \left\{d^0_i Z^2_{i,q_i-1} + \varphi^2_i(\hat{G}(Z))\right\} dt \\ dZ^2_{i,j} &= \left\{Z^2_{i,j-1} + d^j_i Z^2_{i,q_i-1}\right\} dt, \quad j = 1,\dots,q_i - 1 \\ dZ^3_{i,0} &= \tilde{c}^0_i Z^3_{i,\tilde{n}_i-1} dt + \tilde{\varphi}_i(\hat{G}(Z))dW_t \\ dZ^3_{ik} &= \left\{Z^3_{i,k-1} + \tilde{c}^k_i Z^3_{i,\tilde{n}_i-1}\right\} dt, \quad k = 1,\dots,\tilde{n}_i - 1 \\ dZ^4_{i,0} &= \left\{\tilde{d}^0_i Z^4_{i,\tilde{q}_i-1} + \tilde{\varphi}^2_i(\hat{G}(Z))\right\} dt \\ dZ^4_{i,j} &= \left\{Z^4_{i,j-1} + \tilde{d}^j_i Z^4_{i,\tilde{q}_i-1}\right\} dt, \quad j = 1,\dots,\tilde{q}_i - 1 \\ dZ_5 &= \delta\varphi(\hat{G}(Z))dW, \end{cases}$$

where c^k_i, $\tilde{c}^k_i$, d^k_i *and* $\tilde{d}^k_i$ *are defined in (2.160).*

2.6 Existence of an FDR in a system with forward exchange rate

In the previous sections we studied the existence of an FDR in an economy described by the dynamics of domestic and foreign forward rates as well as spot exchange rates. However, we might want to model the whole term structure of exchange rates and then the natural problem would be to understand when there exists an FDR in the economy specified by the domestic and foreign forward rate dynamics as well as the dynamics of the *forward exchange rate*.

Definition 2.6.1. We define *forward exchange rate*, $F(t,T)$, contracted at time t, as units of domestic currency that will be paid per unit of foreign currency at time T.

Taking out a forward contract for foreign exchange means agreeing to buy foreign exchange at an agreed rate in the future. The existence of the forward market leads to a considerable amount of speculation. The forward exchange rate should be determined so as no arbitrage opportunities are introduced on the international market. Assume that we agree to buy foreign currency at time T at a rate of K units of domestic currency per unit of foreign currency. The price of such a contract equals

$$\Pi(t) = \mathbb{E}^{Q^d}\left[e^{\int_t^T R^d(u)du}[S(T) - K]\middle| \mathcal{F}_t\right]. \tag{2.163}$$

The price of such a contract on the initial date should be equal to zero if the price for one unit of forward currency in the future is set to the forward exchange rate; thus, the forward exchange rate, $F(t,T)$, should be equal

$$F(t,T) = \mathbb{E}^{T^d}\left[S(T)|\,\mathcal{F}_t\right], \tag{2.164}$$

where the expectation is computed under the domestic forward martingale measure.

Assume that an investor sells domestic bond $p(t,T)$, converts the proceeds to foreign currency, and buys $\frac{p(t,T)}{S(t)\tilde{p}(t,T)}$ foreign bonds. The foreign bonds yield 1 unit of foreign currency at time T, which can be converted back to domestic currency at the agreed time t in the forward contract rate, $F(t,T)$. Thus, the amount of domestic currency the investor obtains in the end is $\frac{p(t,T)F(t,T)}{S(t)\tilde{p}(t,T)}$. On the other hand, at time T we need to repay 1 unit of domestic currency. The strategy should yield zero so as not to introduce arbitrage on the market; thus,

$$F(t,T) = \frac{S(t)\tilde{p}(t,T)}{p(t,T)}. \tag{2.165}$$

Hence, the forward exchange rate can be expressed in terms of the spot exchange rate and domestic and foreign bond prices, which implies that the necessary and sufficient conditions for the existence of an FDR for all the objects on the righthand side are also the necessary and sufficient conditions for the existence of an FDR for $F(t,T)$, $p(t,T)$ and $\tilde{p}(t,T)$.

2.7 Conclusion

The purpose of the paper is to understand when the inherently infinite dimensional domestic and foreign forward rate processes can be realized by means of a Markovian finite dimensional state space model. Since under the martingale measure the forward rate dynamics is completely specified by its volatility we proceed by considering several different volatility specifications. Our findings for different types of volatilities can be summarized as follows.

We find that the necessary and sufficient conditions for constant domestic and foreign volatilities (volatilities that do not depend on the whole forward rate curves) and constant exchange rate volatility are that both forward rate volatilities are quasi-exponential. In this case, the exchange rate volatility does not restrict the results in any way. Once we have chosen quasi-exponential domestic and foreign volatilities, we automatically produce a model that admits an FDR. We can, however, reduce the number of factors by the number of Wiener processes driving the forward rate system if we make some specific assumptions concerning the dependence between forward rate volatilities and exchange rate volatility. Here we may choose the forward rate volatilities freely from the class of quasi-exponential functions. However, given a fixed choice of these volatilities, the choice of exchange rate volatility is restricted and determined by the choice of forward rate volatilities.

The case of constant direction domestic and foreign forward rate volatilities $\varphi(\hat{r})\lambda(x)$ and $\varphi(\hat{r})\tilde{\lambda}(x)$ is more complicated. We are able to provide necessary and sufficient conditions for the existence of an FDR only in a simple case where the model is driven by one Wiener process and where the lengths of all the vector fields (forward rate volatilities) $\varphi(r,\tilde{r})$ are the same and the exchange rate volatility takes the form $\delta\varphi(r,\tilde{r})$. It turns out that the necessary and sufficient conditions for the existence of an FDR are that the constant directions of the volatilities, $\lambda(x)$ and $\tilde{\lambda}(x)$ are quasi-exponential functions.

In the more general case when the system is allowed to be driven by an m-dimensional Wiener process and the length of the vector fields is allowed to be different, we can provide *sufficient* conditions only. The sufficient condition for the general two-country model to admit an FDR is that the constant directions of the volatilities are quasi-exponential.

We also study the two-country model specified by the domestic and foreign forward rate dynamics and the forward exchange rate dynamics. Since no arbitrage condition implies that the forward exchange rate can be expressed as a function of domestic and foreign bond prices and the spot exchange rate, the necessary and sufficient conditions for the existence of an FDR for all the objects on the righthand side are also the necessary and sufficient conditions for the existence of an FDR for $F(t,T)$, $p(t,T)$ and $\tilde{p}(t,T)$.

2.8 Appendix

2.8.1 The space

The analysis becomes more complicated given that, in general, operator $\mathbf{F} = \frac{\partial}{\partial x}$ is not bounded. This means that we cannot guarantee that the strong solution for the forward rate equation exists for any choice of the initial point r_0. Thus, to be able to proceed we need to restrict ourselves to the subspace where the operator **F** is in fact bounded. Hence, we define the following sufficiently regular subspace H_γ.

Definition 2.8.1. Consider a fixed real number $\gamma > 0$. The space H_γ is defined as the space of all infinitely differentiable functions

$$r : R_+ \to R$$

satisfying the norm condition $\|r\|_\gamma < \infty$. Here the norm is defined as

$$\|r\|_\gamma^2 = \sum_{n=0}^{\infty} 2^{-n} \int_0^\infty \left(\frac{\partial^n r}{\partial x^n}(x) \right)^2 e^{-\gamma x} dx.$$

The following Proposition can be proved (see Björk and Svensson (2001)).

Proposition 2.8.2. *The space H is a Hilbert space*[5], *i.e. it is complete. Furthermore, every function in this space is, in fact, real analytic and can thus be uniquely extended to a holomorphic function in the entire complex plane.*

Given all the above assumptions, the boundeness of the operator **F** implies that both the drift and the diffusion of the forward rate equations are smooth vector fields on H and thus the strong solution exists for every initial point $r_0 \in H$.

2.8.2 Basic concepts in differential geometry

Consider a real Hilbert space $\mathcal{H}$. We give the following definitions.

Definition 2.8.3. n-dimensional **distribution** is a mapping F, which to each $\hat{r} \in \hat{\mathcal{H}}$ associates an n-dimensional subspace $F(\hat{r}) \subseteq \hat{\mathcal{H}}$.

A vector field $f : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ is said to lie in F if $f(\hat{r}) \in F(\hat{r})$ for every $\hat{r} \in \hat{\mathcal{H}}$. A collection $f_1, \ldots, f_n$ of vector fields lying in F generates (or spans) F if *span* $\{f_1(\hat{r}), \ldots, f_n(\hat{r})\} = F(\hat{r})$ for every $\hat{r} \in \hat{\mathcal{H}}$, where *span* denotes the linear hull over the real field.

Definition 2.8.4. The distribution F is **smooth** if to every $\hat{r} \in \hat{\mathcal{H}}$ there exist smooth vector fields $f_1, \ldots, f_n$ spanning F.

A vector field is smooth if it belongs to C^∞. Assume now that we have two distributions F and G. If $G(\hat{r}) \subseteq F(\hat{r})$ for all $\hat{r}$, then F **contains** G.

The dimension of a distribution F is defined pointwise as $\dim F(\hat{r})$. Let now f and g be two smooth vector fields on U.

[5] We will suppress the subindex γ.

Definition 2.8.5. The **Lie bracket** of the two vector fields f and g is the vector field

$$[f, g](\hat{r}) = f'(\hat{r})g(\hat{r}) - g'(\hat{r})f(\hat{r}),$$

where $f'(\hat{r})$ denoted Frechet derivative of f at $\hat{r}$. The Frechet derivative is operating on g (the same applies to $g'(\hat{r})f(\hat{r})$).

Definition 2.8.6. A distribution F is called *involutive* if for all smooth vector fields f and g lying in F on U, their Lie bracket also lies in F, i.e.

$$[f, g](\hat{r}) \in F(\hat{r}), \quad \forall \hat{r} \in \hat{H}.$$

The following concept of Lie algebra will be used extensively further on in the paper.

Definition 2.8.7. Let F be a smooth distribution on $\hat{\mathcal{H}}$. The **Lie algebra** generated by F, denoted by $\{F\}_{LA}$ or by $\mathcal{L}\{F\}$, is defined as the minimal (under inclusion) involutive distribution containing F.

Let F be a distribution and let $\varphi : V \to W$ be a diffeomorphism between the open sets V and W on X. Then we can define a new distribution $\varphi_\star F$ on W by

$$\varphi_\star F(\varphi(x)) = \varphi'(x)F(x). \tag{2.166}$$

For any smooth vector field $f \in C^\infty(U, X)$ the field $\varphi_\star f$ is defined analogously. It is straightforward to verify that

$$\varphi_\star[f, g] = [\varphi_\star f, \varphi_\star g]. \tag{2.167}$$

This implies that if F is generated by $f_1, \ldots, f_n$, then $\varphi_\star F$ is generated by $\varphi_\star f_1, \ldots, \varphi_\star f_n$, and that F is involutive if and only if $\varphi_\star F$ is involutive.

2.8.3 Existence of a finite dimensional realization

Let $\hat{\mu}$, σ_1, ..., σ_m be smooth vector fields on $\hat{H}$. For a given initial point $\hat{r}_0 \in \hat{H}$ consider the following system of SDE on H.

$$\begin{cases} d\hat{r}_t &= \hat{\mu}(\hat{r}_t)dt + \hat{\sigma}(\hat{r}_t) \circ dW_t, \\ \hat{r}_0 &= \hat{r}^0, \end{cases} \tag{2.168}$$

where W_t is an m-dimensional Wiener process and $\circ$ denotes the Stratonovich integral. In this paper $\hat{r}_t$ is the extended vector process $(r_t, \tilde{r}_t, Y_t)^\star$. The following result gives the general necessary and sufficient conditions for the existence of an FDR.

Theorem 2.8.8. **(Björk and Svensson)** *Consider the SDE in (2.168) and assume that the dimension of the Lie algebra* $\{\hat{\mu}, \sigma_1, \ldots, \sigma_m\}_{LA}$ *is constant near the initial point* $\hat{r}_0 \in \hat{H}$. *Then (2.168) possesses an FDR if and only if*

$$dim\,\{\hat{\mu}, \sigma_1, \ldots, \sigma_m\}_{LA} < \infty$$

at the neighborhood of $\hat{r}^0$.

The following theorem gives us a parameterization of the forward rate curves produced by the model. To state this theorem we will need the following definition.

Definition 2.8.9. Let f be a smooth vector field on $\hat{H}$, and let $\hat{r}$ be a fixed point in $\hat{H}$. Consider the ODE

$$\begin{cases} \dfrac{\partial \hat{r}_t}{\partial t} &= f(\hat{r}_t), \\ \hat{r}_0 &= \hat{r}. \end{cases} \qquad (2.169)$$

We denote the solution $\hat{r}_t$ as $\hat{r}_t = e^{ft}\hat{r}$.

We can now state the second theorem.

Theorem 2.8.10. **(Björk and Svensson)** *Assume that the Lie algebra* $\{\hat{\mu}, \hat{\sigma}\}_{LA}$ *is spanned by the smooth vector fields* $\hat{f}_1, \ldots, \hat{f}_d$. *Then , for the initial point* $\hat{r}^0$, *all forward rate curves produced by the model will belong to the manifold* $\hat{\mathcal{G}} \in \hat{\mathcal{H}}$, *which can be parameterized as* $\hat{\mathcal{G}} = Im[\hat{G}]$, *where*

$$\hat{G}(z_1, \ldots, z_d) = e^{f_d z_d} \ldots e^{f_1 z_1}\hat{r}^0,$$

and where the operator $e^{f_i z_i}$ *is given in Definition 2.8.9.*

The manifold $\hat{\mathcal{G}}$ in the above theorem is obviously invariant under the forward rate dynamics. It will be therefore be referred to as the **invariant manifold**.

2.8.4 Quasi-exponential functions - some useful results

Lemma 2.8.11. *Take the following vector fields* f_1, f_2 *and* f_k *as given. The Lie algebra* $\mathcal{L} = \{f_1, f_2, \ldots, f_k\}_{LA}$ *remains unchanged under the following operations.*

- *The vector field* $f_i(r)$ *may be replaced by* $\alpha(r)f_i(r)$, *where* α *is any smooth nonzero scalar field.*
- *The vector field* $f_i(r)$ *may be replaced by*

$$f_i(r) + \sum_{j \neq i} \alpha_j(r) f_j(r),$$

where α_j *is any smooth scalar field.*

As we will see later, the following types of functions will play an important role for the rest of the paper.

Definition 2.8.12. A **quasi-exponential** (or QE) function is by definition any function of the form

$$f(x) = \sum_i e^{\lambda_i x} + \sum_j e^{\alpha_j x}[p_j(x)\cos(w_j x) + q_j(x)\sin(w_j x)], \tag{2.170}$$

where λ_i, α_i and w_j are real numbers, whereas p_j and q_j are real polynomials.

Lemma 2.8.13. *The following holds for the quasi-exponential polynomials:*

- *A function is QE if and only if it is a component of the solution of a vector valued linear ODE with constant coefficients.*
- *A function is QE if and only if it can be written as $f(x) = ce^{Ax}b$, where c is a row vector, A is a square matrix and b is a column vector.*
- *If f is QE, then f' is QE.*
- *If f is QE, then its primitive function is QE.*
- *If f and g are QE, then fg is QE.*

For use below we need the following simple result on QE functions.

Lemma 2.8.14. *The following two implications hold.*

- *Assume that two real valued functions f and g satisfy the same linear ODE with constant coefficients, i.e. they satisfy an equation (without prescribed boundary data) of the form*

$$P(\mathbf{F})f = 0, \tag{2.171}$$
$$P(\mathbf{F})g = 0, \tag{2.172}$$

  where P is a polynomial. Then f and g must be QE.

- *Consider two arbitrarily chosen QE functions f and g. Then there will exist a (not uniquely defined) polynomial P such that f and g satisfy:*

$$P(\mathbf{F})f = 0, \tag{2.173}$$
$$P(\mathbf{F})g = 0, \tag{2.174}$$

  i.e. f and g satisfy **the same** *ODE.*

Proof. In one direction the proof is trivial. Indeed, if $P(\mathbf{F})f = P(\mathbf{F})g = 0$, then according to Lemma 2.8.14, both f and g are QE since they are solutions of linear ODEs with constant coefficients.

Assume now that both f and g are QE. Then there exist two (possibly different) polynomials R and Q, such that $R(\mathbf{F})f = 0$ and $Q(\mathbf{F})g = 0$. Now construct the polynomial P as the product of the polynomials R and Q, then

$$P(\mathbf{F})f = Q(\mathbf{F})R(\mathbf{F})f = R(\mathbf{F})Q(\mathbf{F})g = P(\mathbf{F})g = 0,$$

which means that both f and g satisfy the same ODE with real coefficients specified by the polynomial P. ■

In the second statement above, there is of course no unicity of the polynomial P, since any polynomial of the form QP will do as well. For fixed f and g there will, however, exist a unique polynomial P of minimal degree.

Corollary 2.8.15. *Assume that f and g are QE functions, then there exists a unique polynomial of minimum degree with the property $P(\mathbf{F})f = P(\mathbf{F})g = 0$.*

Proof. Since f and g are QE, then there exist two different polynomials P and Q, such that $P(\mathbf{F})f = 0$ and $Q(\mathbf{F})g = 0$. We can always rewrite P and Q as $P(\mathbf{F}) = \tilde{P}(\mathbf{F})\tilde{M}(\mathbf{F})$ and $Q(\mathbf{F}) = \tilde{Q}(\mathbf{F})\tilde{M}(\mathbf{F})$, where

$$\tilde{M}(\mathbf{F}) = \prod_{j=0}^{l}(\mathbf{F} - \beta_j)^{n_j},$$

where $\{\beta_j\}_{j=1}^{l}$ is the set of common roots and n_j is the minimal multiplicity of each common root β_j. ($M = 1$ and $P = \tilde{P}$ and $Q = \tilde{Q}$ in case the two polynomials do not have common roots.) The minimal polynomial M can then be constructed as $M(\mathbf{F}) = \tilde{P}(\mathbf{F})\tilde{M}(\mathbf{F})\tilde{Q}(\mathbf{F})$. ■

Chapter 3

Joint Term Structures of Nominal and Real Interest Rates in a Two-Country Setting

The aim of this paper is to create a general affine framework that would allow the pricing of nominal and inflation-linked bonds in an international setting and that would help to explain term structure movements in terms of macroeconomic shocks, exchange rates or policy-related factors. The contribution of this paper is that it enlarges the multifactor affine Daffie-Kan framework to model affine joint domestic and foreign nominal and real term structures, domestic and foreign price levels, and nominal and real exchange rates. We assume that the bond prices are influenced by the multivariate factor whose price process is driven by both a multidimensional Wiener process and a general marked point process. We provide necessary and sufficient conditions on the factor dynamics, domestic and foreign short rates and/or exchange rate and price level dynamics in order to obtain jointly affine domestic and foreign real and nominal term structures and real exchange rate. We investigate if the general affine framework is consistent with the empirical evidence supporting the purchasing power parity (PPP) hypothesis; that is, we find necessary and sufficient conditions for the real exchange rate in the affine framework to be mean stationary. As compared to previous studies we include in the set of observables inflation-linked domestic and foreign bonds as well as nominal exchange rate and price levels. Thus we estimate the international ATS model for the joint domestic and foreign nominal and inflation-linked bonds, including nominal exchange rate and price levels in the set of observables. As a practical application of this international model we construct a "real exchange rate" option and demonstrate how to price it. This derivative can be introduced and used by investors to hedge exchange rate and inflation risks.[1]

[1]I gratefully acknowledge financial support from the Jan Wallander and Tom Hedelius Foundation. I thank Tomas Björk and Sveriges Riksbank for helpful comments.

3.1 Introduction.

Along with the general equilibrium models for the term structures of interest rates developed by Cox, Ingersoll, and Ross (1985), affine term structure models based on the no arbitrage approach have received considerable attention in the finance literature. These models are attractive mainly due to their analytical tractability which simplifies both pricing and estimation procedures. The main idea behind affine term structure models (ATSMs) is that bond prices are driven by a finite set of underlying Markovian factors that follow some exogenously specified diffusion processes. The idea then is to find necessary and sufficient conditions on the factor dynamics so that the bond prices take some predetermined functional form, i.e. are exponentially affine in the factors.

The most general no arbitrage multifactor model of the affine term structure models of interest rates was first described in Duffie and Kan (1996)[2]. Duffie and Kan (1996) also give an economic interpretation of those factors; in fact, they consider yields of zero-coupon bonds of various fixed maturities as factors. The yields follow the multivariate Markov diffusion process with "stochastic volatility".

The development of international markets made clear the need for tractable bond pricing models of internationally integrated markets. The extension is important since the factors driving the yield curves are to a large extent documented as being international. Thus, there is a clear need to take these factors into account when pricing or hedging financial derivatives (i.e. foreign exchange rate options). Typically we require the models of an international interest rate market to be flexible enough to capture all the realistic features of the integrated market. However, we would also like the model to be tractable enough to be able to produce closed-form solutions for bond prices which would, of course, favor estimation procedures. The contagion literature seems to suggest high correlations between different markets, implying that we might not need a large number of factors to capture the realistic behavior of bond prices.

Modeling exchange rate movements as diffusion processes started in the early 1980s with Biger and Hull (1983). Since then a number of models (Amin and Jarrow (1991), Frachot (1996), Ahn (1997)) have extended the original simple model to consider the joint dynamics for nominal exchange rate and domestic and foreign term structures. Backus, Foresi, and Telmer (2001) provide an extension to the general Duffie and Kan (1996) framework by constructing a two-country affine model and exploring the forward premium anomaly. They analyze whether in a discrete time setting affine term structure models are consistent with the

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[2]For the general review of affine term structure models see e.g Piazzesi (2003), Rebonato (2003) or Dai and Singleton (2000)

empirical evidence of the forward premium anomaly. Han and Hammond (2003) suggest a way to reconcile the forward premium anomaly with a pure square root economy. Finally, Mosburger and Schneider (2005) analyze which specification fits the domestic and foreign countries' yields and exchange rate dynamics exploring several different specifications and a more advanced estimation technique.

After Wednesday 29 January 1997, when the United States Treasury issued its first Treasury Inflation-Protected Securities (TIPS), and many other countries started issuing inflation-linked bonds, a clear need arose to develop a joint affine framework that would allow the pricing of both the domestic and foreign nominal bonds and inflation-linked bonds, and enable efficient estimation procedures for model parameters. Indeed, it becomes difficult to explain term structure movements in terms of outside factors connected to monetary policy if we only consider a simple nominal one-country model. Globalization suggests that the actions of central banks in different countries are very likely to influence term structures of other countries as well. Thus we need to introduce in our model some factors responsible for inflation or central bank actions in both countries.

Except for the general equilibrium models, most existing arbitrage free models for inflation-linked bond pricing are built on a one-country market only, such as *e.g.* Jarrow and Yildirim (2003). A study by Dillen (1997) suggests a three-factor model where the state variables are the world real interest rate, the real exchange rate and the inflation rate, and they are driven by a three-dimensional Ornstein-Uhlenbeck process. Dillen (1997) captures the effects on the term structure originating from exchange rate fluctuations and actions by central banks. He also considers the influence of regime shifts, such as changes in the target inflation rate and devaluations, by allowing the factor dynamics to be driven by a Poisson process. Another extension was proposed by Brennan and Xia (2004), who consider a pure Gaussian term structure model and explore the forward premium anomaly in this setting. Dewachter, Lyrio, and Maes (2001) estimate a joint model for the term structure of interest rates and the macroeconomy.

The aim of this paper is to create a general affine framework that would allow the pricing of nominal and inflation-linked bonds in an international setting and that would help to explain term structure movements in terms of exchange rates, macroeconomic shocks or policy-related factors. The paper creates the affine framework for pricing securities on international markets by enlarging the multifactor affine class Daffie-Kan type models to incorporate the dynamics of domestic and foreign price levels and thus the real exchange rate between the two countries. Thus, the contribution of this paper is that it extends a multifactor two-country economy setting so that it jointly models, in a tractable way, domestic and foreign term structures, nominal exchange rates, domestic and foreign price levels, domestic and foreign real term structures as well as the real exchange rate between the two countries. In particular, we answer the question of whether the affine framework is consistent with the empirical evidence supporting the purchasing power parity (PPP) hypothesis, i.e., under what conditions is the real exchange rate asymptotically mean stationary in the jointly affine framework?

Although there is a number of studies considering the term structures driven by the jump processes (e.g., Björk, Kabanov, and Runggaldier (1995), Eberlein and Raible (1999) and Piazzesi (2005)) very few two-country affine term structure models are considering factors following jump-diffusion dynamics and except for Dillen (1997), who considers the Poisson process, none of the models include jump processes. However, if we want to take into account effects of macroeconomic shocks and policies in international markets we might want to consider jump-diffusion factor dynamics. Thus, another contribution of the paper is that we assume that bond prices are driven by the multivariate factor whose price process is driven by a standard multidimensional Wiener process and the marked point process with an intensity that is allowed to depend on the same factors. This specification is mostly important for modeling the developing countries market. In particular, we provide the sufficient conditions on the factor dynamics, domestic and foreign short rates and/or exchange rate volatilities (alternatively, the form of the exchange rate), and factor jump size in order to obtain affine domestic and foreign term structures. We also provide necessary conditions on the factor dynamics and the foreign term structure for the domestic term structure and the exchange rate to be affine. We provide sufficient conditions for the real exchange rate, nominal and real domestic term structures, and nominal exchange rate to be jointly affine.

We do not give an economic interpretation of the factors assuming that they are unobservable (*latent*). The natural task is then to back out the dynamics of those factors from the observable variables. As we specified above, considerable work has been done concerning the estimation of the latent factors driving a single country nominal term structure. However, little has been done regarding the estimation of two-country ATSMs and to the best of our knowledge the international CIR ATS model for joint domestic and foreign nominal and inflation-linked bonds has never been estimated. Also, it seems that in the estimation of two-country ATSMs all other studies lose information about the unobservable factors not including in the set of observables (and thus not including in the estimation), real domestic and foreign yields, (and real exchange rate).

To estimate the dynamics of the latent factors in a two-country model, Han and Hammond (2003) and Dewachter and Maes (2001) used the Kalman filter technique. Han and Hammond (2003) included the exchange rate as one of the state variables. Affine exchange rate and affine price levels would simplify the estimation procedure since as compared to Han and Hammond (2003), the exact functional form of the exchange rate is known; that is, in this case we can find a function from state variables to observables.

As a practical application of the general international model presented here, we construct a European type "real exchange" call option, and show how to value it. This derivative can be introduced and used by investors to hedge not only exchange rate risk or inflation risk, but also joint exchange rate and inflation risks.

The outline of the paper is as follows.

1. In Section 1 we define the multifactor two-country economy and give the the basic definitions and results on foreign exchange theory and measure transformation.

2. In Section 2 we describe a two-country "nominal" economy and provide the sufficient conditions that we need to impose on the factor dynamics, domestic and foreign short rates and/or exchange rate volatilities (alternatively, the form of the exchange rate) in order to obtain exponentially affine domestic and foreign term structures. In addition we provide the necessary conditions on the factor dynamics and the foreign term structure for the domestic term structure and the exchange rate to be affine. Then, we assume that both the domestic and foreign term structures are affine and show that this implies affine factor dynamics under domestic and foreign martingale measures and derive the implications for the exchange rate dynamics.

3. In Section 3 we introduce inflation-protected securities and explain the analogy between Sections 2 and 3.

4. Section 4 introduces the enlarged two-country factor economy and gives the necessary and sufficient conditions for real exchange rate, nominal and real domestic term structures, and nominal exchange rate to be jointly affine drawing implications for real exchange rate dynamics. To enlarge the framework we extend the general affine theory for the framework of real and nominal zero-coupon bonds. Then, joining the affine "nominal economy" with the inflation-protected bonds economy, we obtain the enlarged model. In this section we also investigate if the affine framework is consistent with the empirical evidence supporting the purchasing power parity (PPP) hypothesis, that is, under what conditions is the real exchange rate asymptotically mean stationary in the jointly affine framework? We construct the European type "real exchange" derivatives and explain why they can be useful. We also show how to price the security introduced.

5. Section 5 estimates the nominal two-country model and the joint nominal-real affine international model. In the most general of those specifications we include nominal exchange rate and US price level in the estimation procedure.

3.2 Two-country economy

We consider a financial market in a two-country economy where all the objects in the economy are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $\mathbf{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ and P is the objective probability measure. The probability space carries both a multidimensional Wiener process W and a marked point process (MPP), $\mu(dt, dy)$, on the mark space $(\mathbf{R}, \mathcal{B})$. The predictable σ-algebra is denoted by $\mathcal{P}$, and we define $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}$. The filtration $\mathbf{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ is, in fact, generated by W and μ, i.e. $\mathcal{F}_t = \mathcal{F}_t^{\mu} \vee \mathcal{F}_t^{W}$. We assume that the point process $\mu(dt, dy)$ has a P-compensator of the form $\nu^P(dt, dy)$ which admits intensity λ^P, that is $\nu^P(dt, dy) = \lambda^P(t, dy)dt$. Denote the compensated process $\tilde{\mu}(dt, dy) = \mu(dt, dq) - \lambda^P(t, dy)dt$.

Consider a two-country economy where the price dynamics of the standard domestic and foreign risk-free assets, B_t^d and B_t^f, are given by

$$dB_t^d = B_t^d R_t^d dt, \quad \text{and} \tag{3.1}$$
$$dB_t^f = B_t^f R_t^f dt, \tag{3.2}$$

where R_t^d and R_t^f are domestic and foreign stochastic short rates.

Definition 3.2.1. *A* **domestic zero coupon bond** *with maturity date T (T-bond) is a derivative which gives the holder a* **face value** *of* 1 *domestic unit on date T. The price at time t of a domestic bond with maturity date T is denoted p(t,T).*

Definition 3.2.2. *A* **foreign zero coupon bond** *with maturity date T (T-bond) is a derivative which gives the holder a* **face value** *of* 1 *foreign unit on date T. The price at time t of a foreign bond with maturity date T is denoted $p^f(t,T)$.*

Definition 3.2.3. *The continuously compounded* **domestic zero coupon yield**, *$y(t,T)$, is given by*

$$y(t,T) = -\frac{\log p(t,T)}{T-t}, \tag{3.3}$$

where $p(t,T)$ is the price of the domestic zero coupon bond. For a fixed t, the function $T \rightarrow y(t,T)$ is called the (zero coupon) domestic yield curve.

The continuously compounded **foreign zero coupon yield**, $y^f(t,T)$, *is given by*

$$y^f(t,T) = -\frac{\log p^f(t,T)}{T-t}, \tag{3.4}$$

where $p^f(t,T)$ *is the price of the domestic zero coupon bond. For a fixed t, the function* $T \rightarrow y^f(t,T)$ *is called the (zero coupon) foreign yield curve.*

We assume that there is a market for domestic and foreign T-bonds for every maturity T. We assume that market is the arbitrage free, that is, there exist domestic and foreign martingale measures Q^d and Q^f. Assume that the Q^d and Q^f compensators are $\nu(dt,dy)$ and $\nu^f(dt,dy)$, admit intensities, $\lambda(t,dy)$ and $\lambda^f(t,dy)$, respectively.

The bond prices are, of course, expressed in different currencies. The exchange rate at time t, $S(t)$, is quoted as units of domestic currency per unit of foreign currency:

$$S(t) = \frac{\text{domestic currency}}{\text{foreign currency}}.$$

3.2.1 General results on foreign exchange theory and measure transformation

Assume that the exchange rate process exhibits jumps. For example, it is natural to expect that the exchange rate responds to surprise events, shocks or policy changes in the economy. Assume that the dynamics of the exchange rate under the domestic martingale measure are given by

$$dS(t) = S(t)\alpha_S(t)dt + S(t)\sigma_S(t)dW_t + S(t-)\int_{\mathbf{R}} \delta_S(t,y)\mu(dt,dy), \tag{3.5}$$

where $\alpha_S(t)$ and $\sigma_S(t)$ are any adapted processes, and $\delta_S(t,y)$ is a predictable process. Suppose that our task is to determine the domestic asset price dynamics under the foreign martingale measure. The answer to this problem is given by the Girsanov theorem. Assume that there is a domestic T-claim $Z(t)$ and a foreign T-claim $Z^f(t)$ in the economy. Denote the domestic price of the foreign asset by $\tilde{Z}^f(t)$, then

$$\tilde{Z}^f(t) = S(t)Z^f(t).$$

We define the likelihood process L by

$$\frac{dQ^f}{dQ^d} = L(t), \quad \text{on } \mathcal{F}_t; \quad 0 \leq t \leq T. \tag{3.6}$$

According to risk-neutral valuation, the price of the foreign asset at time $t = 0$ (we take $t = 0$ without loss of generality) is equal to

$$Z^f(0) = \mathbb{E}^{Q^f}\left[e^{-\int_0^T R^f(s)ds} Z^f(T)\middle| \mathcal{F}_0\right],$$

which is equivalent to

$$Z^f(0) = \mathbb{E}_0^{Q^d}\left[e^{-\int_0^T R^f(s)ds} Z^f(T)L(T)\right]. \tag{3.7}$$

Since the price of the foreign asset expressed in the domestic currency is equal to $\tilde{Z}^f(t) = S(t)Z^f(t)$, we can consider $\tilde{Z}^f(t)$ as the asset traded in the domestic market and hence according to the risk-neutral valuation we obtain that

$$S(0)Z^f(0) = \mathbb{E}_0^{Q^d}\left[e^{-\int_0^T R^d(s)ds} Z^f(T)S(T)\right],$$

which leads to

$$Z^f(0) = \mathbb{E}_0^{Q^d}\left[e^{-\int_0^T R^d(s)ds} Z^f(T)\frac{S(T)}{S(0)}\right]. \tag{3.8}$$

Equations (3.7) and (3.8) hold for all possible integrable T-claims; thus, comparing equations (3.7) and (3.8) we see that

$$L(T)e^{-\int_0^T R^f(s)ds} = e^{-\int_0^T R^d(s)ds}\frac{S(T)}{S(0)}$$

and the following should hold for any t

$$L(t) = \frac{S(t)}{S(0)}e^{-\int_0^t (R^d(s)-R^f(s))ds} = \frac{S(t)B^f(t)}{S(0)B^d(t)}, \quad 0 \leq t \leq T. \tag{3.9}$$

Hence the likelihood process has the following dynamics

$$dL(t) = L(t)\left\{R^f(t) - R^d(t) + \alpha_S(t)\right\}dt + \sigma_S(t)L(t)dW_t + L(t-)\int_{\mathbf{R}}\delta_S(t,y)\mu(dt,dy) \tag{3.10}$$

Since $L(t)$ is a martingale under the domestic martingale measure Q^d, the dynamics of $L(t)$ under Q^d must have zero drift

$$dL(t) = \sigma_S(t)L(t)dW_t + L(t-)\int_{\mathbf{R}}\delta_S(t,y)\tilde{\mu}(dt,dy), \tag{3.11}$$

which means that under the domestic martingale measure the drift of the exchange rate process equals

$$\alpha_S(t) = R^d(t) - R^f(t) - \int_{\mathbf{R}} \delta_S(t,y)\lambda(t,dy). \tag{3.12}$$

More formally we have the following result: the Girsanov measure transformation theorem for marked point processes (see e.g. Jacod and Shiryaev (1987)):

Theorem 3.2.4. *Assume that Q^d is a domestic martingale measure and Q^f is a foreign martingale measure. The exchange rate $S(t)$ has dynamics as in (3.5). Then*

1. *The Wiener processes under domestic and foreign martingale measures are connected via*
$$dW(t) = dW^f(t) + \sigma_S(t)dt,$$
where σ_S is allowed to be stochastic.

2. *The Q^f intensity of the MPP can be expressed as*
$$\lambda^f(t,dy) = (1+\delta_S(t,y))\lambda(t,dy). \tag{3.13}$$

We will also need the following result:

Lemma 3.2.5. *There exists a conditional jump distribution $M(t,dy)$, $M(t,Y) = 1$, such that Q^d almost surely is*

$$\nu(dt,dy) = M(t,dy)\bar{\nu}(dt) = M(t,dy)\nu(dt,Y). \tag{3.14}$$

Proof. See Last and Brandt (1995). ■

Lemma 3.2.6. *Assume that $M(t,dy)$ is a conditional jump distribution for Q^d intensity $\lambda(t,dy)$, that is,*

$$\lambda(t,dy) = M(t,dy)\lambda(t,Y) = M(t,dy)\tilde{\lambda}(t),$$

where $\tilde{\lambda}(t) = \lambda(t,Y)$, then

- *The Q^f intensity, $\tilde{\lambda}^f(t)$, relates to the Q^d intensity $\tilde{\lambda}(t)$, by*
$$\tilde{\lambda}^f(t) = \tilde{\lambda}(t)\int_{\mathbf{R}}(1+\delta_S(t,y))M(t,dy) \tag{3.15}$$

- *The conditional jump size distribution under Q^f, $M^f(t,dy)$, relates to the conditional jump size distribution under Q^d, $M(t,dy)$, by*
$$M^f(t,dy) = \frac{M(t,dy)(1+\delta_S(t,y))}{\int_{\mathbf{R}}(1+\delta_S(t,y))M(t,dy)}. \tag{3.16}$$

Proof. The predictable compensator takes the form

$$\nu(dt, dy) = \lambda(t, dy)dt = \tilde{\lambda}(t)M(t, dy)dt, \tag{3.17}$$

where

$$\tilde{\lambda}(t) = \lambda(t, Y), \quad M(t, dy) = \frac{\lambda(t, dy)}{\lambda(t, Y)}. \tag{3.18}$$

From the Girsanov theorem we obtain that under Q^f

$$\lambda^f(t, dy) = \lambda(t, dy)(1 + \delta_S(t, y)) \tag{3.19}$$

$$\begin{aligned}
\tilde{\lambda}^f(t) &= \int_{\mathbf{R}} (1 + \delta_S(t, y))\lambda(t, dy) = \tilde{\lambda}(t) \int_{\mathbf{R}} (1 + \delta_S(t, y))M(t, dy) \\
M^f(t, dy) &= \frac{\lambda(t, dy)(1 + \delta_S(t, y))}{\int_{\mathbf{R}}(1 + \delta_S(t, y))\lambda(t, dy)} = \frac{M(t, dy)(1 + \delta_S(t, y))}{\int_{\mathbf{R}}(1 + \delta_S(t, y))M(t, dy)}.
\end{aligned}$$

■

Corollary 3.2.7. *If the jump size of the exchange rate is deterministic, that is,* $\delta_S(t, \omega, y) = \delta_S(t)$, *then*

1. *The Q^f intensity, $\tilde{\lambda}^f(t)$, relates to the Q^d intensity $\tilde{\lambda}(t)$, by*

$$\tilde{\lambda}^f(t) = \tilde{\lambda}(t)(1 + \delta_S(t)) \tag{3.20}$$

2. *The conditional jump size distribution under Q^f, $M^f(t, dy)$, is equal to the conditional jump size distribution under Q^d, $M(t, dy)$.*

Proof. Follows directly from Lemma 3.2.6. If $\delta_S(t)$ is deterministic we see that

$$\begin{aligned}
\tilde{\lambda}^f(t) &= (1 + \delta_S(t))\lambda(t, Y) = (1 + \delta_S(t))\tilde{\lambda}(t) \\
M^f(t, dy) &= \frac{(1 + \delta_S(t))\lambda(t, dy)}{\int_{\mathbf{R}}(1 + \delta_S(t))\lambda(t, dy)} = \frac{\lambda(t, dy)}{\int_{\mathbf{R}} \lambda(t, dy)} = M(t, dy).
\end{aligned}$$

■

3.2.2 The Two-Country Multi-factor Economy: basic concepts

The main goal of this section is to model the joint dynamics of the domestic and foreign term structures and the nominal exchange rate in a two-country multifactor economy (the results can be easily generalized for the multi-country case). In particular, under what conditions on the factor dynamics are all the objects in the economy affine?

Assume that all the objects in the economy are driven by an *a priori* given n-dimensional factor price process X_t. It is natural to assume that the factor price process (and thus domestic and foreign interest rates) has a jump-diffusion dynamics reflecting for example the possibility of shocks as a result of instability in the domestic economy or surprise policy changes. We assume that the objects in the foreign economy are driven by the same multidimensional factor process exhibiting jumps. The general specification allows us to model global factors that affect both countries and local factors affecting only one of the countries.

We make the following assumptions concerning the factor dynamics and the dynamics of financial assets.

Assumption 3.2.8.

1. *There exists an a priori given factor process X with the following dynamics specified under a domestic martingale measure Q^d*

$$dX_t = \alpha(t, X_t)dt + \sigma(t, X_t)dW_t + \int_{\mathbf{R}} \gamma(t, X_t, y)\mu(dt, dy), \tag{3.21}$$

   where $\alpha(t, X_t)$ is an n-dimensional column vector. $\sigma(t, X_t)$ is an $n \times d$ matrix, W_t is a d-dimensional Q^d Wiener process and $\gamma(t, X_t, y)$ is an n-dimensional column vector process and by

$$\int_{\mathbf{R}} \gamma(t, X_t, y)\mu(dt, dy)$$

   we denote the column vector

$$\begin{bmatrix} \int_{\mathbf{R}} \gamma_1(t, X_t, y)\mu(dt, dy) \\ \vdots \\ \int_{\mathbf{R}} \gamma_n(t, X_t, y)\mu(dt, dy) \end{bmatrix}.$$

2. *The compensator $\nu(dt, dy)$ of the MPP $\mu(dt, dy)$ admits a predictable Q^d intensity*

$$\lambda(\omega, t, dy) = \lambda(t, X_{t-}(\omega), dy), \tag{3.22}$$

   where for each t and x, $\lambda(t, x, dy)$ is a deterministic measure on Y. According to Lemma 3.2.5 we can rewrite the intensity as

$$\lambda(t, X_{t-}, dy) = M(t, X_{t-}, dy)\lambda(t, X_{t-}, Y) = M(t, X_{t-}, dy)\tilde{\lambda}(t, X_{t-}), \tag{3.23}$$

   where $M(t, X_{t-}, dy)$ is a conditional jump distribution, $M(t, X_{t-}, Y) = 1$.

3. *Assume that for every maturity T, prices of domestic and foreign T-bonds have the form*

$$p(t,T) = F(t,X_t,T), \tag{3.24}$$
$$p^f(t,T) = G(t,X_t,T), \tag{3.25}$$

where F and G are two smooth functions of the entries t, T and X_t^i.

4. *Assume that the dynamics of the exchange rate under the domestic martingale measure are given by*

$$dS_t = S_t\alpha_S(t,X_t)dt + S_t\sigma_S(t,X_t)dW_t + S_{t-}\int_{\mathbf{R}}\delta_S(t,X_t,y)\mu(dt,dy), \tag{3.26}$$

where $\alpha_S(t,X_t)$ is a scalar function, $\sigma_S(t,X_t)$ is a $1\times d$ row vector, $\delta_S(t,X_t,y)$ is a scalar function and W_t is as above.

Definition 3.2.9. *If the term structure $\{p(t,T);\quad 0 \le t \le T, T \ge 0\}$ has the form*

$$p(t,T) = F(t,X(t),T), \tag{3.27}$$

where F has the form

$$F(t,X_t,T) = e^{A(t,T)-B(t,T)X_t} \tag{3.28}$$

and where A is a deterministic scalar function and B is a deterministic row vector function, then the model is said to possess an **affine term structure**.

Definition 3.2.10. *We will say that the short rate is affine if it is an affine function of factors*

$$R(t,X_t) = R_1(t)X_t + R_2(t), \tag{3.29}$$

where $R_1(t)$ is a deterministic row vector function of t only and $R_2(t)$ is a deterministic function of time t.

Remark 3.2.11. *Affine term structure implies affine short rate. Indeed, if the term structure is of the form (3.28), then the short rate can be found as*

$$R^d(t,x) = -\left.\frac{\partial \ln F(t,x,T)}{\partial T}\right|_{T=t} = -A_T(t,t) + B_T(t,t)x,$$

which implies that the short rate is indeed affine in the factors since it is of the form (3.29).

Definition 3.2.12. *If the exchange rate has the form*

$$S(t,X_t) = e^{P(t)X_t+Q(t)}, \tag{3.30}$$

where Q is a deterministic function of t only and P is a deterministic row vector function of t, then the model for the exchange rate is said to be **affine**.

Definition 3.2.13. *We will say that the factor dynamics (3.21) are* **affine** *if the drift vector* $\mu(t,x)$*, the covariance matrix* $\sigma(t,x)\sigma^\star(t,x)$ *and the intensity,* $\lambda(t,x,dy)$*, of the marked point process are affine componentwise and the jump size,* $\gamma(t,x,y)$*, of the process does not depend on* x*.*

3.2.3 Interpretation of the Factors

How should the factors $X = (X_1, X_2, \dots, X_n)$ be chosen? It is always convenient, of course, to assume that the term structures are influenced by some set of **observable** factors, since then we can find the bond prices at time t substituting the observable values of factors X_i at time t in the bond pricing equation. For example, Duffie and Kan (1996) suggest an *affine yield factor model* which assumes that the factors of the model are the yields $X = (X_1, X_2, \dots, X_n)$ of zero-coupon bonds of n different fixed maturities, $T_1, T_2, \dots, T_n$. These kinds of factors are indeed observable from the current yield curve and can thus be defined as

$$X_i = -\frac{\log p(t, X_t, T_i)}{T_i - t}, \quad i = 1, \dots, n. \tag{3.31}$$

It is quite unrealistic to expect that the prices of foreign bonds can be expressed in terms of domestic yields only. In the enlarged economy we could choose, for example, several purely domestic nominal and real yields, as well as several foreign nominal and real yields as factors. We could include domestic and foreign price levels in the sets of factors as well. Instead, we will assume that the bond prices and exchange rates are driven by the set of unobservable *latent* factors since in this case we do not have to assume *a priori* which factors influence term structures. The dynamics of the factors will be possible to estimate from the observable domestic and foreign bond prices, domestic and foreign inflation-protected bond prices as well as exchange rate.

3.2.4 Affine Term Structure Factor Model with Jumps: Sufficient Conditions

The main question we consider in this section is: what are the sufficient conditions that we need to impose on the factor dynamics, the domestic and foreign short rates and/or exchange rate volatilities (alternatively, the functional form of the exchange rate) in order to obtain affine domestic and foreign term structures?

We consider the factor dynamics under the domestic martingale measure Q^d given by (3.21).

$$dX_t = \alpha(t, X_t)dt + \sigma(t, X_t)dW_t + \int_{\mathbf{R}} \gamma(t, X_t, y)\mu(dt, dy),$$

Notations. As we specified earlier, the factor dynamics are said to be affine if the drift vector $\alpha(t,x)$, the covariance matrix $\sigma(t,x)\sigma^\star(t,x)$ and the intensity, $\lambda(t,x,dy)$, of the marked point process are affine componentwise and the jump size does not depend on x. We choose the following notations for the affine objects:

$$\begin{aligned} \alpha(t,x) &= \alpha^1(t) + \alpha^2(t)x \\ \sigma(t,x)\sigma^\star(t,x) &= d^1(t) + \sum_{i=1}^{n} d_i^2(t)x_i \\ \sigma(t,x)\sigma_S^\star(t,x) &= \delta^1(t) + \delta^2(t)x \\ \lambda(t,x,dy) &= \lambda^1(t,dy) + \lambda^2(t,dy)x, \end{aligned} \tag{3.32}$$

where $\alpha^1(t)$, $\delta^1(t)$ are column vectors and $\alpha^2(t)$, $d^1(t)$, $d_k^2(t)$, $\delta^2(t)$ are matrices depending on time t only, $\lambda^1(t,dy)$ is a scalar, $\lambda^2(t,dy)$ is a row vector. We assume that domestic and foreign short rates are affine and can be written as

$$\begin{aligned} R^d(t,x) &= R^1(t) + R^2(t)x \\ R^f(t,x) &= R_f^1(t) + R_f^2(t)x \end{aligned} \tag{3.33}$$

where $R^1(t)$, $R_f^1(t)$ are scalars and $R^2(t)$, $R_f^2(t)$ are row vectors depending on time t only.

The following proposition provides sufficient conditions for the factor dynamics, the domestic and foreign short rates and the exchange rate volatility which ensure that the domestic and foreign term structures are affine.

Proposition 3.2.14. (Sufficient conditions I: Affine domestic and foreign term structures)

Assume the following:

1. *The drift, $\alpha(t,x)$, of the factor process under the Q^d measure, matrices $\sigma(t,x)\sigma^\star(t,x)$, $\sigma(t,x)\sigma_S^\star(t,x)$, the domestic short rate, $R^d(t,x)$, the foreign short rate, $R^f(t,x)$, and the Q^d intensity $\lambda(t,x,dy)$, are affine in factor x.*

2. *The solutions to differential equations (3.38), (3.39) and (3.46), (3.45) stated below exist.*

3. *The jump size, $\gamma(t, y)$, of the factor process does not depend on factor X_t.*

4. *The jump size, $\delta_S(t, y)$, of the exchange rate process does not depend on factor X_t.*

Then the domestic and foreign term structures $F(t, x, T)$ and $G(t, x, T)$ are affine in factor X_t.

Proof. I. In the first part of the proof we show that under the assumptions of Proposition 3.2.14 the domestic term structure is affine. This result is known from Duffie and Kan (1996), but we provide it for the completeness of the arguments.

By Assumption 3.2.8 for every maturity T, the price of the T-bond has the form

$$p(t, T) = F(t, X_t, T).$$

According to risk-neutral valuation, the price process of the domestic bond can be expressed as

$$F(t, x, T) = \mathbb{E}^{Q^d}\left[\exp\left\{-\int_t^T R^d(s, X_s)ds\right\}\middle|\mathcal{F}_t\right].$$

We know that under the domestic martingale measure Q^d the factor price process follows

$$dX_t = \alpha(t, X_t)dt + \sigma(t, X_t)dW_t + \int_{\mathbf{R}} \gamma(t, y)\mu(dt, dy);$$

thus the term structure will be determined by the following PDE

$$\begin{aligned} F_t(x, t) + F_x(x, t)\alpha(t, x) + \frac{1}{2}tr\left[\sigma^\star(t, x)\mathbf{H}\sigma(t, x)\right] \\ + \int_{\mathbf{R}} \left\{F(t, x + \gamma(t, y)) - F(t, x)\right\}\mu(dt, dy) = R^d(t, x)F(x, t), \end{aligned} \tag{3.34}$$

where F_x denotes the row vector $\left(\frac{\partial F}{\partial x_i}\right)_{i=1}^n$ and $\mathbf{H}$ denotes the Hessian matrix $\mathbf{H}_{ij} = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{i,j=1}^n$.

Consider now the ansatz solution to PDE (3.34) given by

$$F(t, x, T) = \exp\left\{A(t, T) - B(t, T)x\right\}. \tag{3.35}$$

Substituting ansatz solution (3.35) in PDE (3.34) we obtain that

$$A_t(t,T) - B_t(t,T)x - B(t,T)\alpha(t,x) + \frac{1}{2}B(t,T)\sigma(t,x)\sigma^\star(t,x)B^\star(t,T)$$
$$+ \int_{\mathbf{R}} \left\{e^{B(t,T)\gamma(t,y)} - 1\right\} \lambda(t,x,dy) = R^d(t,x). \qquad (3.36)$$

Since by assumption α, $\sigma\sigma^\star$, R^d and λ are affine in x and using the notations above can be written as (3.32), we can collect all terms in x_i for every i and equate the terms in brackets to zero since (3.36) should hold for every x. We introduce the notation

$$I^i(t,Z(t)) = \int_{\mathbf{R}} \left\{e^{Z(t)\gamma(t,y)} - 1\right\} \lambda^i(t,dy), \quad i = 1,2, \qquad (3.37)$$

where $I^2(t,Z(t))$ is a column vector

$$I^2(t,Z(t)) = (I_1^2(t,Z(t)), I_2^2(t,Z(t)), \dots, I_n^2(t,Z(t)))^\star,$$

and thus obtain the following differential equations for $A(t,T)$ and $B(t,T)$

$$\begin{cases} \dfrac{\partial B_k}{\partial t}(t,T) &= -B(t,T)\alpha_k^2(t) + \frac{1}{2}B(t,T)d_k^2(t)B^\star(t,T) - I_k^2(t,B(t,T)) - R_k^2(t), \\ B_k(T,T) &= 0, \qquad k = 1,\dots,n, \end{cases} \qquad (3.38)$$

$$\begin{cases} \dfrac{\partial A}{\partial t}(t,T) &= B(t,T)\alpha^1(t) - \frac{1}{2}B(t,T)d^1(t)B^\star(t,T) - I^1(t,B(t,T)) - R^1(t), \\ A(T,T) &= 0. \end{cases} \qquad (3.39)$$

Once the solution to system (3.38) exists and is unique, we obtain the solution for A and substitute both solutions A and B in (3.38)-(3.39) in the ansatz solution (3.35). The obtained $F(t,x)$ is the solution to PDE (3.34). Indeed, the solution to (3.34) is unique and thus the solution is indeed defined by $F(t,x) = \exp\{A(t,T) - B(t,T)x\}$.

II. In the second part of the proof we show that under the assumptions of Proposition 3.2.14 the results proved in the first part imply that the foreign term structure is affine. Under the foreign martingale measure Q^f we obtain the following dynamics for the factor:

$$\begin{aligned} dX_t &= \alpha(t, X_t)dt + \sigma(t, X_t)\left\{dW_t^f + \sigma_S^\star(t, X_t)dt\right\} + \int_{\mathbf{R}} \gamma(t, y)\mu(dt, dy) \\ &= \{\alpha(t, X_t) + \sigma(t, X_t)\sigma_S^\star(t, X_t)\}\, dt + \sigma(t, X_t)dW_t^f + \int_{\mathbf{R}} \gamma(t, y)\mu(dt, dy). \end{aligned}$$

Since α and $\sigma\sigma_S^\star$ are affine by the assumption, the drift of the factor process under the foreign martingale measure

$$\beta(t, X_t) = \alpha(t, X_t) + \sigma(t, X_t)\sigma_S^\star(t, X_t) \tag{3.40}$$

is affine as well. Applying the Girsanov theorem we also have that under the foreign martingale measure

$$\lambda^f(t, X_t, dy) = (1 + \delta_S(t, y))\lambda(t, X_t, dy). \tag{3.41}$$

Since we assumed that Q^d intensity $\lambda(t, x, dy)$ is affine and the jump of the exchange rate process $\delta_S(t, y)$ does not depend on the factor, Q^f intensity $\lambda^f(t, x, dy)$ is affine as well. We apply the same reasoning as in the first part of Proposition 3.2.14 for the Q^f factor dynamics with the drift $\beta(t, x)$, the same covariance matrix $\sigma(t, x)\sigma^\star(t, x)$, foreign short rate $R^f(t, x)$ and intensity under the foreign martingale measure $\lambda^f(t, x, dy)$. We write down an ansatz solution for the foreign term structure

$$G(t, x, T) = \exp\{C(t, T) - D(t, T)x\} \tag{3.42}$$

and conclude that the foreign term structure is also affine once the solution to the differential equation for $D(t, T)$ exists. Introduce the following notation

$$J^i(t, Z(t)) = \int_{\mathbf{R}} \left\{e^{Z(t)\gamma(t,y)} - 1\right\} \lambda^{fi}(t, dy), \quad i = 1, 2, \tag{3.43}$$

which given the form of λ under the foreign martingale measure can be written as

$$J^i(t, Z(t)) = \int_{\mathbf{R}} \left\{e^{Z(t)\gamma(t,y)} - 1\right\} (1 + \delta_S(t, y))\lambda^i(t, dy), \quad i = 1, 2, \tag{3.44}$$

then $D(t, T)$ satisfies the following system of differential equations

$$\left\{\begin{aligned} \frac{\partial D_k}{\partial t}(t, T) &= D(t, T)\{\alpha_k^2(t) + \delta_k^2(t)\} + \tfrac{1}{2}D(t, T)d_k^2(t)D^\star(t, T) + J_k^2(t, D(t, T)) \\ &- R_{fk}^2(t), \\ D_k(T, T) &= 0, \qquad k = 1, \ldots, n, \end{aligned}\right. \tag{3.45}$$

Then we obtain the solution for C(t,T) by solving the following differential equation

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,T) &= D(t,T)\{\alpha^1(t)+\delta^1(t)\}+\frac{1}{2}D(t,T)d^1(t)D^\star(t,T)+J^1(t,D(t,T)) \\ &- R_f^1(t), \\ C(T,T) &= 0, \end{cases} \tag{3.46}$$

which completes the proof. ■

The following proposition also presents the sufficient conditions on the factor and the exchange rate dynamics to ensure that the domestic and the foreign term structures are affine.

Proposition 3.2.15. (Sufficient conditions II: Affine foreign term structure) *Assume that*

1. *The drift, $\alpha(t,x)$, of the factor process under the Q^d measure, the covariance matrix $\sigma(t,x)\sigma^\star(t,x)$, the domestic short rate $R^d(t,x)$ and the Q^d intensity $\lambda(t,x,dy)$ are affine in factor x.*

2. *The exchange rate is affine and takes the form*

$$S(t,X_t)=e^{P(t)X_t+Q(t)}.$$

Then

1. *The domestic term structure $F(t,x,T)$ is affine.*

2. *The foreign short rate $R^f(t,x)$ and Q^f intensity, $\lambda^f(t,x,dy)$, of the MPP are affine.*

3. *The foreign term structure $G(t,x,T)$ is affine.*

Proof. Following the arguments of Proposition 3.2.14 we see that since $\alpha(t,x)$, $\sigma(t,x)\sigma^\star(t,x)$, $R^d(t,x)$ and $\lambda(t,x,dy)$ are affine, the domestic term structure is affine if solutions to differential equations (3.38)and (3.39) exist and are unique.

The exchange rate dynamics for the exchange rate process are

$$\begin{aligned} dS_t &= S_t\left\{P'(t)X_t + Q'(t) + P(t)\alpha(t,X_t) + \frac{1}{2}P(t)\sigma(t,X_t)\sigma^\star(t,X_t)P^\star(t)\right. \\ &+ \left.\int_{\mathbf{R}}\left\{e^{P(t)\gamma(t,y)} - 1\right\}\lambda(t,X_t,dy)\right\}dt \\ &+ S_tP(t)\sigma(t,X_t)dW_t + S_{t-}\int_{\mathbf{R}}\left\{e^{P(t)\gamma(t,y)} - 1\right\}\tilde{\mu}(dt,dy), \end{aligned} \tag{3.47}$$

from which we explicitly find that the volatility of the exchange rate process is given by

$$\sigma_S(t,x) = P(t)\sigma(t,x).$$

We can also note that

$$\sigma(t,x)\sigma_S^\star(t,x) = \sigma(t,x)\sigma^\star(t,x)P^\star(t),$$

which implies that $\sigma(t,x)\sigma_S^\star(t,x)$ is affine since by assumption $\sigma(t,x)\sigma^\star(t,x)$ is affine. Thus, the drift of the factor price process under the foreign martingale measure is also affine. Indeed, applying the Girsanov theorem we have

$$\begin{aligned} dX_t &= \alpha(t,X_t)dt + \sigma(t,X_t)\left\{dW_t^f + \sigma_S^\star(t,X_t)dt\right\} + \int_{\mathbf{R}}\gamma(t,y)\mu(dt,dy) \\ &= \left\{\alpha(t,X_t) + \sigma(t,X_t)\sigma_S^\star(t,X_t)\right\}dt + \sigma(t,X_t)dW_t^f + \int_{\mathbf{R}}\gamma(t,y)\mu(dt,dy). \end{aligned}$$

The jump size of the exchange rate process equals

$$\delta_S(t,y) = e^{P(t)\gamma(t,y)} - 1 \tag{3.48}$$

and thus under the foreign martingale measure, the intensity of the MPP is

$$\lambda^f(t,X_t,dy) = \lambda(t,X_t,dy)e^{P(t)\gamma(t,y)} \tag{3.49}$$

and it follows that if $\lambda(t,x,dy)$ is affine then $\lambda^f(t,x,dy)$ is affine as well. Since we also know that under the domestic martingale measure the drift of the exchange rate process is equal to the difference between the short interest rates, the foreign short rate can be found as

$$\begin{aligned} R^f(t,X_t) &= R^d(t,X_t) - \left\{P(t)\alpha(t,X_t) - \frac{1}{2}P(t)\sigma(t,X_t)\sigma^\star(t,X_t)P^\star(t)\right\} \\ &- P'(t)X_t - Q'(t) - \int_{\mathbf{R}}\left\{e^{P(t)\gamma(t,y)} - 1\right\}\lambda(t,X_t,dy). \end{aligned} \tag{3.50}$$

Thus, the foreign short rate can be determined from (3.50) and turns out to be an affine function of factors as well. Turning to the arguments of Proposition 3.2.14 we see that since $\beta(t,x)$, $\sigma(t,x)\sigma^\star(t,x)$, $R^f(t,x)$ and $\lambda^f(t,x,dy)$ are affine, the foreign term structure is also affine. ■

To summarize, Propositions 3.2.14 and 3.2.15 above show how to model the joint factor and the exchange rate dynamics so as to ensure that the domestic and foreign term structures are affine. The propositions suggest the two following ways.

- Specify affine factor dynamics under the domestic martingale measure and an affine domestic short rate exogenously, and then model the affine exchange rate dynamics freely. This will ensure affine factor dynamics under the foreign martingale measure as well as an affine foreign short rate.

- In case we prefer to make as few assumptions as possible concerning the functional form of the exchange rate process, it is actually sufficient to specify, as we did above, affine factor dynamics under the domestic martingale measure and an affine domestic short rate exogenously. In this case we do not have to make assumptions concerning the functional form or the dynamics of the exchange rate. The only assumption we have to make is the assumption concerning the volatility of the exchange rate process, that is, we need to assume that $\sigma(t,x)\sigma_S^\star(t,x)$ is affine. This, however, goes at the cost of making the additional assumption about the foreign short rate functional form; in particular, we also need to assume that the foreign short rate $R^f(t,x)$ is affine.

The following section suggests the possible functional form of the exchange rate volatility consistent with the fact that $\sigma(t,x)\sigma_S^\star(t,x)$ is affine.

Exchange Rate Volatility

Sufficient conditions II shows that one way to obtain the affine domestic and foreign term structure is to model affine factor dynamics under the domestic martingale measure, affine domestic and foreign short rates and the exchange rate process volatility in such a way that $\sigma(x)\sigma_S^\star(x)$ is affine [3]. The natural question would be:

[3] We will omit the time-variable in this section.

Question: *Given that both $\sigma(x)\sigma^\star(x)$ and $\sigma(x)\sigma^\star_S(x)$ are affine, what are the implications for the exchange rate process volatility; that is, if we start modeling the exchange rate process directly, what would be the form of the exchange rate volatility?*

The result of this section is based on the result of Duffie and Kan (1996) stated below.

Proposition 3.2.16. *Assume that the factor volatility $\sigma(x)$ is $n \times d$-dimensional matrix, such that $\sigma(x)\sigma^\star(x)$ is affine in x, then under some nondegeneracy conditions and a possible reordering of indices, we can take*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{3.51}$$

where $\sigma(X_t)$ is of the form

$$\sigma(X_t) = \Sigma\mathbf{V}(X_t), \tag{3.52}$$

and matrix $\mathbf{V}$ is

$$\mathbf{V}(X_t) = \begin{pmatrix} \sqrt{v_1(X_t)} & 0 & \dots & 0 \\ 0 & \sqrt{v_2(X_t)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{v_d(X_t)} \end{pmatrix} \tag{3.53}$$

where $a \in \mathbf{R}^{n\times n}$, $b \in \mathbf{R}^n$ and $\Sigma \in \mathbf{R}^{n\times d}$, and

$$v_i(x) = \alpha_i + \beta_i x, \tag{3.54}$$

where for each i, α_i is a scalar and $\beta_i \in \mathbf{R}^n$.

On the basis of this proposition we draw the following implications concerning both the volatility structure of the factor and the exchange rate. From proposition 3.2.16 it follows directly that the volatility matrix of the factor with affine dynamics has to have the form (3.52), which means that the volatility matrix will essentially take the form

$$\sigma(X_t) = \begin{pmatrix} \sigma_{11}\sqrt{v_1(X_t)} & \sigma_{12}\sqrt{v_2(X_t)} & \dots & \sigma_{1d}\sqrt{v_d(X_t)} \\ \sigma_{21}\sqrt{v_1(X_t)} & \sigma_{22}\sqrt{v_2(X_t)} & \dots & \sigma_{2d}\sqrt{v_d(X_t)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1}\sqrt{v_1(X_t)} & \sigma_{n2}\sqrt{v_2(X_t)} & \dots & \sigma_{nd}\sqrt{v_d(X_t)} \end{pmatrix}. \tag{3.55}$$

The following result specifies the form of the exchange rate volatility matrix.

Proposition 3.2.17. *Assume that the factor volatility $\sigma(x)$ is $n \times d$-dimensional matrix, such that $\sigma(x)\sigma^\star(x)$ is affine in x, and $\sigma(x)\sigma_S^\star(x)$ is affine. Then the exchange rate volatility vector is of the form*

$$\sigma_S^\star(x) = \begin{pmatrix} \frac{\delta_1(x)}{\sqrt{v_1(x)}} \\ \vdots \\ \frac{\delta_d(x)}{\sqrt{v_d(x)}} \end{pmatrix}, \tag{3.56}$$

for some affine functions $\delta_1(x)$, $\delta_2(x)$,..., $\delta_d(x)$.

Proof. From Proposition 3.2.16 we know that $\sigma(x) = \Sigma\mathbf{V}(x)$. Since $\sigma(x)\sigma_S^\star(x)$ is affine, we can write

$$\sigma(x)\sigma_S^\star(x) = \Sigma\mathbf{V}(x)\sigma_S^\star(x) = \tilde{\delta}(x), \tag{3.57}$$

where $\tilde{\delta}(x) = (\tilde{\delta}_1(x), \tilde{\delta}_2(x), \ldots, \tilde{\delta}_d(x))^\star$ is a column vector of some affine functions $\tilde{\delta}_i(x)$. Thus, we notice that since Σ does not depend on x, $\Sigma\mathbf{V}(x)\sigma_S^\star(x)$ is affine for any Σ iff $\mathbf{V}(x)\sigma_S^\star(x)$ is affine. This leads to

$$\sigma_S^\star(x) = \mathbf{V}^{-1}(x)\delta(x), \tag{3.58}$$

for some vector of affine functions $\delta_1(x)$, $\delta_2(x)$,..., $\delta_d(x)$. Given that

$$\mathbf{V}^{-1}(X_t) = \begin{pmatrix} \frac{1}{\sqrt{v_1(X_t)}} & 0 & \ldots & 0 \\ 0 & \frac{1}{\sqrt{v_2(X_t)}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{\sqrt{v_d(X_t)}} \end{pmatrix},$$

we can rewrite (3.58) as (3.56). ■

The proposition shows that in fact if we choose the square root form for exchange rate volatility $\sigma_S(x)$, it should have the same structure as the factor volatility, that is

$$\sigma_S^\star(x) = \begin{pmatrix} \sigma_{S1}\sqrt{v_1(x)} \\ \sigma_{S2}\sqrt{v_2(x)} \\ \vdots \\ \sigma_{Sd}\sqrt{v_d(x)} \end{pmatrix}. \tag{3.59}$$

3.2.5 Affine Term Structure Factor Model with Jumps: Necessary Conditions

Necessary conditions I: Affine domestic term structure and affine exchange rate

In the previous Section we derive the conditions on the factor and the exchange rate dynamics which would guarantee affine domestic and foreign term structures in the economy. In this section we address the opposite question. Assume that we start by modelling affine domestic term structure. What are the implications for the factor dynamics? We would also like to know what would be the implications for the foreign term structure if we model domestic term structure and the exchange rate jointly affine. We will start with simpler case when the factor dynamics is driven by Wiener process as well as Poisson process with intensity $\lambda(t, X_t)$

$$dX_t = \alpha(t, X_t)dt + \sigma(t, X_t)dW_t + \gamma(t, X_t)dN_t. \tag{3.60}$$

We will need the following lemma to prove some of the results.

Lemma 3.2.18. *Assume that for any choice of T the following object is affine, that is,*

$$I(t, x, B(t,T)) = \left\{e^{-B(t,T)\gamma(t,x)} - 1\right\} \lambda(t, x) = g_1(t, T)x + g_2(t, T), \tag{3.61}$$

for some functions $g_1(t,T)$ and $g_1(t,T)$. We assume that $B(T,T) = 0$ and $B'_T(T,T) \neq 0$. Then $\gamma(t,x)$ does not depend on x and

$$\lambda(t, x) = h_1(t)x + h_2(t),$$

for some functions $h_1(t)$ and $h_2(t)$.

Proof. Differentiating (3.61) w.r.t. T, we obtain that

$$-B'(t,T)\gamma(t,x)e^{-B(t,T)\gamma(t,x)}\lambda(t,x) = g_1'(t,T)x + g_2'(t,T). \tag{3.62}$$

We evaluate the derivative at the point $t = T$ and find that

$$I_1(T,x) = \gamma(T,x)\lambda(T,x) = \frac{g_1'(T,T)x + g_2'(T,T)}{-B'(T,T)}, \tag{3.63}$$

and thus is affine. Differentiating (3.61) with respect to T second time, we obtain

$$-B''(t,T)\gamma(t,x)e^{-B(t,T)\gamma(t,x)}\lambda(t,x)+(B'(t,T))^2\gamma^2(t,x)\lambda(t,x) = g_1''(t,T)x+g_2''(t,T). \tag{3.64}$$

Evaluating it at the point $t = T$, we obtain

$$-B''(T,T)\gamma(T,x)\lambda(T,x) + (B'(T,T))^2\gamma^2(T,x)\lambda(T,x) = g_1''(T,T)x + g_2''(T,T). \tag{3.65}$$

Since $\gamma(T,x)\lambda(T,x)$ is affine, we conclude that $\gamma^2(T,x)\lambda(T,x)$ is affine as well. Continuing to differentiate w.r.t. T, we obtain that

$$I_n(T,x) = \gamma^n(T,x)\lambda(T,x) = g_{1n}(T,T)x + g_{2n}(T,T), \quad \forall n, \quad n \geq 1. \tag{3.66}$$

Equations (3.66) should hold for any choice of T, and thus, we conclude that $I_n(t,x)$ is affine for all n. Then we can write

$$\begin{aligned} \gamma(t,x)\lambda(t,x) &= g_{11}(t)x + g_{21}(t) \\ \gamma^2(t,x)\lambda(t,x) &= g_{12}(t)x + g_{22}(t) \\ \gamma^3(t,x)\lambda(t,x) &= g_{13}(t)x + g_{23}(t) \end{aligned}$$

We can find γ dividing third equation by second, and second by first. Equating those two γ, we find that

$$(g_{12}(t)x + g_{22}(t))^2 = (g_{11}(t)x + g_{21}(t))(g_{13}(t)x + g_{23}(t)),$$

from which we conclude that

$$g_{11}(t)x + g_{21}(t) = k(t)(g_{12}(t)x + g_{22}(t)),$$

which implies that $\gamma(t,x)$ does not depend on x and is a function of time only $\gamma(t,x) = \gamma(t)$. We then conclude that $\lambda(t,x) = xh_1(t) + h_2(t)$. ■

We now can derive some necessary conditions for the case when factor dynamics has the form (3.60). We start with the case when the factor dynamics is driven by Poisson process only.

Lemma 3.2.19. *Assume that*

1. *The factor process is driven by the jump Poisson process with intensity* $\lambda(t, X_t)$
$$dX_t = \gamma(t, X_t)dN_t. \tag{3.67}$$

2. *The domestic term structure is affine in* X *and given by*
$$F(t, X_t, T) = e^{A(t,T) - B(t,T)X_t}, \tag{3.68}$$
where $B_T'(T,T) \neq 0$.

Then the folowing holds

1. *The intensity of the Poisson process, $\lambda(t,x)$, is affine.*
2. *The jump size, $\gamma(t,x)$ does not depend on x.*

Proof. Substituting $F(x) = e^{A-Bx}$ in the PDE (3.34) we obtain the following equation

$$A_t(t,T) - B_t(t,T)x + \left\{e^{-B(t,T)\gamma(t,x)} - 1\right\}\lambda(t,x) = R^d(t,x). \tag{3.69}$$

We find that

$$\begin{aligned} I(t,x,B(t,T)) &= \left\{e^{-B(t,T)\gamma(t,x)} - 1\right\}\lambda(t,x) = R^d(t,x) - A_t(t,T) + B_t(t,T)x \\ &= g(t,T,x). \end{aligned} \tag{3.70}$$

Since affine term structure implies affine short rate, $R^d(t,x)$, we find that $g(t,T,x)$ is affine. We can now apply Lemma 3.2.19 to obtain that $\gamma(t,x)$ does not depend on x and $\lambda(t,x)$ is affine. ■

In the case when the factor dynamics is more general, we need to make more restrictive assumptions, in particular, we assume that α, σ, γ and the intensity of the Poisson process, λ, do not depend on time explicitly, that is, the is of the form:

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t + \gamma(X_t)dN_t. \tag{3.71}$$

Proposition 3.2.20. (Necessary conditions I: Affine term structure.)

Assume that

1. *The term structure is affine in X and given by*

$$F(t,X_t,T) = e^{A(t,T)-B(t,T)X_t}. \tag{3.72}$$

2. *The drift α, variance σ, jump size γ and the intensity of the Poisson process, λ do not depend on time explicitly.*

3. *There exist maturity dates T_1^i, T_2^i, ..., $T_{N_i}^i$, $i = 1,\dots,4$, where $N_1 = N_3 = N_4 = n$ and $N_2 = \frac{n^2+3n}{2}$, such that the following $N_i \times N_i$ matrices $K^i(T_1^i,\dots,T_{N_i}^i)$ is nonsingular.*

$$K^i(T_1,\dots,T_{N_i}) = \begin{pmatrix} k^i(T_1) \\ \vdots \\ k^i(T_{N_i}) \end{pmatrix}, \tag{3.73}$$

where $1 \times N_i$ *vectors* $k^i(T_i)$ *are defined as follows*

$$
\begin{aligned}
k^1(T_i) &= \{B_1'(T_i,T_i), B_2'(T_i,T_i), \dots, B_n'(T_i,T_i)\}, \\
k^2(T_i) &= \Big\{(B_1'(T_i,T_i))^2, (B_2'(T_i,T_i))^2, \dots, (B_n'(T_i,T_i))^2 \\
&\quad (B_1'(T_i,T_i))^2, B_1'(T_i,T_i)B_2'(T_i,T_i), \dots, B_1'(T_i,T_i)B_n'(T_i,T_i) \\
&\quad \frac{1}{2}(B_2'(T_i,T_i))^2, B_2'(T_i,T_i)B_3'(T_i,T_i), \dots, B_2'(T_i,T_i)B_n'(T_i,T_i), \dots, \\
&\quad B_{n-1}'(T_i,T_i)B_n'(T_i,T_i), \frac{1}{2}(B_n'(T_i,T_i))^2\Big\}, \\
k^3(T_i) &= \left\{(B_1'(T_i,T_i))^3, (B_2'(T_i,T_i))^3, \dots, (B_n'(T_i,T_i))^3\right\}, \\
k^4(T_i) &= \left\{(B_1'(T_i,T_i))^4, (B_2'(T_i,T_i))^4, \dots, (B_n'(T_i,T_i))^4\right\}, \qquad (3.74)
\end{aligned}
$$

where $B'(t,T)$ *denotes derivative with respect to* T *and* $B'(T,T) \neq 0$.

Then the folowing holds

1. *The factor drift vector* $\alpha(x)$ *and the covariance matrix* $\sigma(x)\sigma^\star(x)$ *are affine.*
2. *The intensity of the Poisson process,* $\lambda(x)$*, is affine.*
3. *The jump size is constant,* $\gamma(x) = \gamma$.

Proof. Substituting $F(x) = e^{A-Bx}$ in the PDE (3.34) we obtain the following equation

$$
\begin{aligned}
A_t(t,T) &- B_t(t,T)x - B(t,T)\alpha(x) + \frac{1}{2}B(t,T)\sigma(x)\sigma^\star(x)B^\star(t,T) \\
&+ \left\{e^{-B(t,T)\gamma(x)} - 1\right\}\lambda(x) = R^d(t,x). \qquad (3.75)
\end{aligned}
$$

Rearrange the terms in the equation above (3.75) and denote

$$
g(t,x,T) = R^d(t,x) - A_t(t,T) + B_t(t,T)x. \qquad (3.76)
$$

Then $g(t,x,T)$ is affine by construction since as it was shown before (see Remark 3.2.11), affine domestic term structure implies affine domestic short rate. We differentiate (3.75) with respect to maturity T:

$$
\begin{aligned}
&- B'(t,T)\alpha(x) + \frac{1}{2}B'(t,T)\sigma(x)\sigma^\star(x)B^\star(t,T) + \frac{1}{2}B(t,T)\sigma(x)\sigma^\star(x)(B^\star(t,T))' \\
&- B'(t,T)\left\{e^{-B(t,T)\gamma(x)} - 1\right\}\gamma(x)\lambda(x) = g'(t,x,T), \qquad (3.77)
\end{aligned}
$$

where B' denotes the derivative of $B(t,T)$ w.r.t T. We evaluate this expression at the point T and recalling that $B(T,T)=0$, we obtain

$$-B'(T,T)\alpha(x) - B'(T,T)\gamma(x)\lambda(x) = g'(T,x,T). \tag{3.78}$$

Denote

$$H^1(x) = \{\alpha_1(x)+\gamma_1(x)\lambda_1(x), \quad \alpha_2(x)+\gamma_2(x)\lambda_2(x), \quad \ldots, \quad \alpha_n(x)+\gamma_n(x)\lambda_n(x)\}$$

Rearrange (3.77) to get

$$g'(T,x,T) = k^1(T)H^1(x), \tag{3.79}$$

where

$$k^1(T_i) = \{B_1'(T_i,T_i), B_2'(T_i,T_i), \ldots, B_n'(T_i,T_i)\} \tag{3.80}$$

We choose n different maturities T_i and construct a matrix $K^1(T_1,T_2,\ldots,T_n)$ taking n vectors $\{k^1(T_i)\}_{i=1}^n$ as its rows

$$K^1(T_1,\ldots,T_n) = \begin{pmatrix} k^1(T_1) \\ \vdots \\ k^1(T_n) \end{pmatrix}. \tag{3.81}$$

By assumptions, the n maturities T_i can be chosen in such a way that the $n\times n$ matrix $K^1(T_1,\ldots,T_n)$ is not singular, thus the solution for $H^1(x)$ can be found as

$$H^1(x) = K^1(T_1,\ldots,T_n)^{-1}G(x,T_1,\ldots,T_n), \tag{3.82}$$

where

$$G(x,T_1,\ldots,T_n) = (-g'(T_1,x,T_1),\ldots,-g'(T_n,x,T_n))^\star. \tag{3.83}$$

Thus, if we are given A, B and thus K^1 and G, we can determine $\alpha(x)+\gamma(x)\lambda(x)$. Since G is affine from (3.76) and since K^1 does not depend upon x, H^1 will be affine.

Differentiate (3.77) second time w.r.t T we obtain

$$\begin{aligned} &- B''(t,T)\alpha(x) + \frac{1}{2}B''(t,T)\sigma(x)\sigma^\star(x)B^\star(t,T) + B'(t,T)\sigma(x)\sigma^\star(x)(B^\star(t,T))' \\ &+ \frac{1}{2}B(t,T)\sigma(x)\sigma^\star(x)(B^\star(t,T))'' - B''(t,T)\gamma(x)\lambda(x)e^{-B(t,T)\gamma(x)} \\ &+ (B'(t,T))^2\gamma^2(x)\lambda(x)e^{-B(t,T)\gamma(x)} = g''(t,x,T). \end{aligned} \tag{3.84}$$

Evaluating (3.84) at the point $t=T$ we obtain

$$\begin{aligned} &- B''(T,T)\{\alpha(x)+\gamma(x)\lambda(x)\} + B'(T,T)\sigma(x)\sigma^\star(x)(B^\star(T,T))' \\ &+ (B'(T,T))^2\gamma^2(x)\lambda(x) = g''(T,x,T). \end{aligned} \tag{3.85}$$

Define

$$g^1(T,x) = g''(T,x,T) - B''(t,T)\left\{\alpha(x) + \gamma(x)\lambda(x)\right\}.$$

Since we already know that $\alpha(x) + \gamma(x)\lambda(x)$ is affine, $g^1(x,T)$ is affine as well. We choose N_2 different maturities T_i and construct a matrix $K^2(T_1, T_2, \ldots, T_{N_2})$ taking N_2 vectors $\{k^2(T_i)\}_{i=1}^{N_2}$, where $N_2 = \frac{n^2+3n}{2}$ as its rows

$$K^2(T_1,\ldots,T_{N_2}) = \begin{pmatrix} k^2(T_1) \\ \vdots \\ k^2(T_{N_2}) \end{pmatrix}, \tag{3.86}$$

where

$$\begin{aligned} k^2(T_i) &= \{(B_1'(T_i,T_i))^2, B_1'(T_i,T_i)B_2'(T_i,T_i), \ldots, B_1'(T_i,T_i)B_n'(T_i,T_i) \\ &\quad \frac{1}{2}(B_2'(T_i,T_i))^2, B_2'(T_i,T_i)B_3'(T_i,T_i), \ldots, B_2'(T_i,T_i)B_n'(T_i,T_i), \ldots, \\ &\quad B_{n-1}'(T_i,T_i)B_n'(T_i,T_i), \frac{1}{2}(B_n'(T_i,T_i))^2, \\ &\quad (B_1'(T_i,T_i))^2, (B_2'(T_i,T_i))^2, \ldots, (B_n'(T_i,T_i))^2\} \end{aligned} \tag{3.87}$$

Since the matrix $K^2(T_1,\ldots,T_{N_2})$ is invertible, we find

$$H^2(x) = K^2(T_1,\ldots,T_{N_2})^{-1}G^1(x,T_1,\ldots,T_{N_2}), \tag{3.88}$$

where

$$G^1(x,T_1,\ldots,T_{N_2}) = (g^1(x,T_1),\ldots,g^1(x,T_{N_2}))^\star \tag{3.89}$$

and

$$H^2(x) = \left\{\epsilon_{11}(x),\epsilon_{12}(x),\ldots,\epsilon_{nn}(x),\gamma_1^2(x)\lambda_1(x),\quad \gamma_2^2(x)\lambda_2(x),\quad \ldots,\quad \gamma_n^2(x)\lambda_n(x)\right\},$$

where $\epsilon_{ij}(x) = \sigma_i(x)\sigma_j^\star(x)$ and $i \le j$. Thus, we conclude that $\sigma(x)\sigma^\star(x)$ and $\gamma^2(x)\lambda(x)$ are affine. Differentiating (3.77) once again w.r.t T, and evaluating it at the point $t = T$, we obtain:

$$\begin{aligned} &- B'''(T,T)\left\{\alpha(x)+\gamma(x)\lambda(x)\right\} + \frac{3}{2}B''(T,T)\sigma(x)\sigma^\star(x)(B^\star(T,T))' \\ &+ \frac{3}{2}B'(T,T)\sigma(x)\sigma^\star(x)(B^\star(T,T))'' + 3B''(T,T)B'(T,T)\gamma^2(x)\lambda(x) \\ &+ (B'(T,T)^3)\gamma^3(x)\lambda(x) = g'''(T,x,T). \end{aligned} \tag{3.90}$$

We define

$$\begin{aligned} g^2(x,T) &= g'''(T,x,T) + B'''(T,T)\left\{\alpha(x)+\gamma(x)\lambda(x)\right\} \\ &- \frac{3}{2}B''(T,T)\sigma(x)\sigma^\star(x)(B^\star(T,T))' + \frac{3}{2}B'(T,T)\sigma(x)\sigma^\star(x)(B^\star(T,T))'' \\ &+ 3B''(T,T)B'(T,T)\gamma^2(x)\lambda(x). \end{aligned}$$

and rewrite (3.90) as follows

$$(B'(T,T))^3\gamma^3(x)\lambda(x) = g^2(x,T). \tag{3.91}$$

We notice that $g^2(x,T)$ is an affine function and define

$$H^3(x) = \left\{\gamma_1^3(x)\lambda_1(x), \quad \gamma_2^3(x)\lambda_2(x), \quad \dots, \quad \gamma_n^3(x)\lambda_n(x)\right\}.$$

Rearrange (3.91) to get

$$g^2(x,T) = k^3(T)H^3(x), \tag{3.92}$$

where

$$k^3(T_i) = \left\{(B_1'(T_i,T_i))^3, (B_2'(T_i,T_i))^3, \dots, (B_n'(T_i,T_i))^3\right\}. \tag{3.93}$$

We choose n different maturities T_i and construct a matrix $K^2(T_1, T_2, \dots, T_n)$ taking n vectors $\{k^3(T_i)\}_{i=1}^n$ as its rows

$$K^3(T_1,\dots,T_n) = \begin{pmatrix} k^3(T_1) \\ \vdots \\ k^3(T_n) \end{pmatrix}. \tag{3.94}$$

We find then

$$H^3(x) = K^3(T_1,\dots,T_n)^{-1}G^2(x,T_1,\dots,T_n), \tag{3.95}$$

where

$$G^2(x,T_1,\dots,T_n) = (g^2(x,T_1),\dots,g^2(x,T_n))^\star. \tag{3.96}$$

Using the same strategy, we find that if there exist n maturities $T_1,\dots,T_n$, such that $n \times n$ matrix constructed from rows

$$k^4(T_i) = \left\{(B_1'(T_i,T_i))^4, (B_2'(T_i,T_i))^4, \dots, (B_n'(T_i,T_i))^4\right\} \tag{3.97}$$

is invertible, then $\gamma^4(x)\lambda(x)$ is affine. Using the arguments of Lemma 3.61 we conclude that γ is constant and $\lambda(x)$ is affine. Since we know that $\alpha(x)+\gamma(x)\lambda(x)$ is affine, we conclude that $\alpha(x)$ is affine as well and we conclude that the factor dynamics is affine. ■

We are ready now to consider a two-country economy. The following proposition considers a jump-diffusion dynamics of the type (3.71). We show that if we choose the domestic term structure and the exchange rate to be affine, this would imply the affine foreign term structure. We can then prove the following proposition:

Proposition 3.2.21. (Necessary conditions II: Affine domestic term structure and an affine exchange rate.)

Assume that

1. *The exchange rate S_t is affine in X and given by*

$$S(t, X_t) = e^{P(t)X_t + Q(t)}. \tag{3.98}$$

2. *Assumptions of the proposition (3.2.20) hold.*

Then

1. *The domestic and foreign short rates, $R^d(t,x)$ and $R^f(t,x)$, are affine functions of the factor x.*

2. *The factor Q^d-drift vector $\alpha(x)$ and the covariance matrix $\sigma(x)\sigma^\star(x)$ are affine.*

3. *The foreign term structure $G(t,x,T)$ is affine in x.*

4. *The Q^d-intensity of the Poisson process, $\lambda(x)$, is affine.*

5. *The Q^f-intensity of the Poisson process, $\lambda^f(x)$, is affine.*

Proof. From Proposition 3.2.20 we know that in the model with the factor dynamics of the form (3.71) affine domestic term structure implies affine factor dynamics, that is, affine $\alpha(x)$, $\sigma(x)\sigma^\star(x)$ and $\lambda(x)$, and constant jump size γ. According to Remark 3.2.11 we also conclude that domestic short rate is affine. We can then apply directly Proposition 3.2.15 with the mark space $\mathbf{R}$ containing a single point, denoted by y_0. Hence, $\mathbf{R} = \{\mathbf{y_0}\}$, the measure $\lambda(x, dy)$ is merely a point mass $\lambda(x, y_0)$ at x_0, and the jump function δ a real number $\delta(y_0)$. ■

To summarise, the Proposition 3.229 shows that if we know that the factor follows Wiener-Poisson dynamics, affine domestic term structure and affine exchange rate imply affine factor dynamics and affine foreign term structure.

Assume now that the factor price process is driven not just by a simple Poisson process, but also by a marked point process $\mu(dt, dx)$:

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbf{R}} \gamma(X_t, y)\mu(dt, dy) \tag{3.99}$$

As in the previous example we assume that α, σ, γ and intensity λ do not depend on time explicitly. We focus first on a one-country term structure and prove the following proposition.

Proposition 3.2.22. *Assume that the factor dynamics is as in (3.99) and assumptions of Proposition 3.2.20 hold, then* $\alpha(x)$, $\sigma(x)\sigma^\star(x)$ *and*

$$I(n,x) = \int_{\mathbf{R}} \gamma^n(x,y)\lambda(x,dy) \tag{3.100}$$

are affine for all $n \geq 1$.

Proof. Substituting $F(x) = e^{A-Bx}$ in the PDE (3.34) we obtain the following equation

$$\begin{aligned} A_t(t,T) \quad &- \quad B_t(t,T)x - B(t,T)\alpha(x) + \frac{1}{2}B(t,T)\sigma(x)\sigma^\star(x)B^\star(t,T) \\ &+ \quad \int_{\mathbf{R}} \left\{e^{-B(t,T)\gamma(x,y)} - 1\right\} \lambda(x,dy) = R^d(t,x). \end{aligned} \tag{3.101}$$

Differentiating (3.101) w.r.t T, we obtain

$$\begin{aligned} &- \quad B'(t,T)\alpha(x) + \frac{1}{2}B'(t,T)\sigma(x)\sigma^\star(x)B^\star(t,T) + \frac{1}{2}B(t,T)\sigma(x)\sigma^\star(x)(B^\star(t,T))' \\ &- \quad B'(t,T)\int_{\mathbf{R}} e^{-B(t,T)\gamma(x,y)}\gamma(x,y)\lambda(x,dy) = g'(t,x,T), \end{aligned} \tag{3.102}$$

where B' denotes the derivative of $B(t,T)$ w.r.t T. We evaluate this expression at the point T and recalling that $B(T,T) = 0$, we obtain

$$-B'(T,T)\alpha(x) - B'(T,T)\int_{\mathbf{R}} \gamma(x,y)\lambda(x,dy) = g'(T,x,T), \tag{3.103}$$

from which we conclude that (using the reasoning from Proposition 3.2.20)

$$\alpha(x) + \int_{\mathbf{R}} \gamma(x,y)\lambda(x,dy) \tag{3.104}$$

is affine. Using the same technique as in Proposition 3.2.20 we obtain that

$$\alpha(x), \quad \sigma(x)\sigma^\star(x), \quad \int_{\mathbf{R}} \gamma^n(x,y)\lambda(x,dy)$$

are affine $\forall\ n \geq 1$. ∎

If we allow α, σ, γ and intensity λ to be dependent on time explicitly, to obtain the following necessary conditions, we need to make more restrictive assumptions on the form of the intensity of the marked point process.

Proposition 3.2.23. (Necessary conditions II. Affine domestic term structure and an affine exchange rate.)

Assume that:

1. *The domestic term structure is affine in X and given by*

$$F(t, X_t, T) = e^{A(t,T)-B(t,T)X_t}. \tag{3.105}$$

2. *The exchange rate S_t is affine in X and given by*

$$S(t, X_t) = e^{P(t)X_t+Q(t)}. \tag{3.106}$$

3. *The jump size of the factor process $\gamma(t, y)$ does not depend on the factor X.*

4. *The conditional jump size distribution $M(t, dy)$ does not depend on the factor X.*

5. *There exist maturity dates T_1, T_2, ..., T_N, where $N = \frac{n^2+5n}{2}$, such that the following $N \times N$ matrix $K^d(T_1, \ldots, T_N)$ is nonsingular.*

$$K^d(T_1, \ldots, T_N) = \begin{pmatrix} k^d(T_1) \\ \vdots \\ k^d(T_N) \end{pmatrix}, \tag{3.107}$$

where $1 \times N$ vector $k^d(T_i)$ is defined as follows

$$\begin{aligned} k^d(t, T_i) &= \{B_1(t, T_i), B_2(t, T_i), \ldots, B_n(t, T_i), \ldots, \\ &\quad \frac{1}{2}B_1^2(t, T_i), B_1(t, T_i)B_2(t, T_i), \ldots, B_1(t, T_i)B_n(t, T_i) \\ &\quad \frac{1}{2}B_2^2(t, T_i), B_2(t, T_i)B_3(t, T_i), \ldots, B_2(t, T_i)B_n(t, T_i), \ldots, \\ &\quad B_{n-1}(t, T_i)B_n(t, T_i), \frac{1}{2}B_n^2(t, T_i), \int_{\mathbf{R}} \left\{e^{-B(t,T_i)\gamma(t,y)} - 1\right\} M(t, dy)\right\}. \end{aligned} \tag{3.108}$$

Then the folowing holds:

1. *The domestic and foreign short rates, $R^d(t, x)$ and $R^f(t, x)$, are affine functions of the factor x.*

2. *The factor Q^d drift vector $\alpha(t, x)$ and the covariance matrix $\sigma(t, x)\sigma^\star(t, x)$ are affine.*

3. *The foreign term structure $G(t, x, T)$ is affine in X.*

4. *The Q^d intensity of the marked point process, $\lambda(t, x, dy)$, is affine.*

5. *The Q^f intensity of the marked point process, $\lambda^f(t, x, dy)$, is affine.*

Proof. I. We first prove that an affine domestic term structure implies affine dynamics of factors under the domestic martingale measure Q^d.

Substituting $F(x) = e^{A-Bx}$ in the PDE (3.34) we obtain the following equation

$$\begin{aligned} A_t(t,T) &- B_t(t,T)x - B(t,T)\alpha(t,x) + \frac{1}{2}B(t,T)\sigma(t,x)\sigma^\star(t,x)B^\star(t,T) \\ &+ \int_{\mathbf{R}} \left\{e^{-B(t,T)\gamma(t,y)} - 1\right\} \lambda(t,x,dy) = R^d(t,x). \end{aligned} \tag{3.109}$$

Using the notations of the previous section, we can rewrite (3.109) as

$$\begin{aligned} A_t(t,T) &- B_t(t,T)x - B(t,T)\alpha(t,x) + \frac{1}{2}B(t,T)\sigma(t,x)\sigma^\star(t,x)B^\star(t,T) \\ &+ I(t,x,B(t,T)) = R^d(x). \end{aligned} \tag{3.110}$$

Rearrange the terms in the equation above (3.110) and denote

$$g^d(t,x,T) = R^d(t,x) - A_t(t,T) + B_t(t,T)x. \tag{3.111}$$

Then $g^d(t,x,T)$ is affine by construction since as it was shown before (see Remark 3.2.11), affine domestic term structure implies affine domestic short rate. Denote also

$$H^d(t,x) = \left\{\alpha_1(t,x), \ldots, \alpha_n(t,x), \epsilon_{11}(t,x), \epsilon_{12}(t,x), \ldots, \epsilon_{nn}(t,x), \tilde{\lambda}(t,x)\right\},$$

where $\epsilon_{ij}(t,x) = \sigma_i(t,x)\sigma_j^\star(t,x)$ and $i \le j$.

Rearrange (3.110) to get

$$g^d(t,x,T) = k^d(t,T)H^d(t,x), \tag{3.112}$$

where

$$\begin{aligned} k^d(t,T_i) = &\{B_1(t,T_i), B_2(t,T_i), \ldots, B_n(t,T_i), \ldots, \\ &\frac{1}{2}B_1^2(t,T_i), B_1(t,T_i)B_2(t,T_i), \ldots, B_1(t,T_i)B_n(t,T_i) \\ &\frac{1}{2}B_2^2(t,T_i), B_2(t,T_i)B_3(t,T_i), \ldots, B_2(t,T_i)B_n(t,T_i), \ldots, \\ &B_{n-1}(t,T_i)B_n(t,T_i), \frac{1}{2}B_n^2(t,T_i), \int_{\mathbf{R}} \left\{e^{-B(t,T_i)\gamma(t,y)} - 1\right\} M(t,dy)\right\}. \end{aligned}$$

We choose N different maturities T_i and construct a matrix $K(T_1, T_2, \dots, T_N)$ (3.73) taking N vectors $\{k^d(T_i)\}_{i=1}^N$ as its rows

$$K^d(T_1, \dots, T_N) = \begin{pmatrix} k^d(T_1) \\ \vdots \\ k^d(T_N) \end{pmatrix}. \tag{3.113}$$

If the N maturities T_i, where $N = \frac{n^2+5n}{2}$, can be chosen in such a way that the $N \times N$ matrix $K^d(T_1, \dots, T_N)$ is not singular, the solution for $H^d(x)$ in (3.112) can be found as

$$H^d(x) = K^d(T_1, \dots, T_N)^{-1} G^d(t, x, T_1, \dots, T_N), \tag{3.114}$$

where

$$G^d(t, x, T_1, \dots, T_N) = (g^d(t, x, T_1), \dots, g^d(t, x, T_N))^\star. \tag{3.115}$$

Thus, if we are given A, B and thus K^d and G^d, we can determine $H^d(t, x)$, that is, the drift $\alpha(t, x)$, covariance $\epsilon(t, x)$ and $\tilde{\lambda}(t, x)$. Since G^d is affine from (3.111) and since K^d does not depend upon x, H^d will be affine and thus also α, ϵ and $\tilde{\lambda}$ will be affine in X. Affine $\tilde{\lambda}$ implies the affine intensity $\lambda(t, x, dy)$ since $M(t, dy)$ does not depend on the factor.

II. Since α, ϵ and $\tilde{\lambda}$ are affine, the result of the proposition follows now directly from Proposition 3.2.15. ■

From the simple Poisson case it seems that the assumption that $\gamma(t, y)$ does not depend on x is needed. In case if $\gamma(t, x, y)$ depends on the factor x, there is no hope to obtain affine factor model since the factor x will appear in the exponential term and we will not be able to obtain equation which is linear in factor, which means we will not be able to separate the terms. Thus, one way out is to assume that $\gamma(t, x, y) = \gamma(t, y)$ does not depend on factor while still allowing the compensator $\lambda(t, x, dy)$ to be dependent on x through intensity λ of the counting process of the arrivals of the marked point process. Hence, we assume that $\tilde{\lambda}(t, x)$ depends on x while the conditional distribution $M(t, dy)$ does not, and factor jump size is not allowed to depend on the factor itself.

In fact, it is sufficient to assume that the Q^d-intensity is separable in x and y, that is, the intensity takes the form

$$\lambda(t, x, dy) = M_1(t, x) M_2(t, dy).$$

Then we can easily see that the conditional jump density $M(t, dy) = \frac{M_2(t, dy)}{M_2(t, Y)}$ does not depend on x and the proof holds.

Necessary conditions III: affine domestic and foreign term structures

In Proposition 3.2.25 we assume that both the domestic and foreign term structures are affine and show that this implies affine factor dynamics under domestic and foreign martingale measures. We investigate whether the exchange rate process can be derived uniquely and endogenously and if not, what can we actually say about the exchange rate dynamics if the domestic and foreign term structures are given and have affine functional form? Thus, we give restrictions on the nominal exchange rate dynamics which are imposed by the affine term structures' functional form.

Assume that we are given two pairs of the coefficients $A(t,T)$, $B(t,T)$ and $C(t,T)$, $D(t,T)$. We also make the following assumptions concerning the domestic and foreign intensities.

Assumption 3.2.24. *Intensities $\lambda^f(t,x,dy)$ and $\lambda(t,x,dy)$ are equivalent measures on $\mathcal{B}$, and the Girsanov kernel*

$$h(t,y) = \frac{\lambda^f(t,x,dy)}{\lambda(t,x,dy)}$$

does not depend on x. That is, there exists a function $h(t,y)$, $h(t,y) \neq 0$ for any t and y, such that $\lambda^f(t,x,dy)$ and $\lambda(t,x,dy)$ are connected via

$$\lambda^f(t,x,dy) = h(t,y)\lambda(t,x,dy). \tag{3.116}$$

Now we are ready to formulate the following proposition.

Proposition 3.2.25. (Necessary conditions III: domestic and foreign affine term structures)

Assume that

1. *The domestic term structure is affine in X and given by*

$$F(t,X_t,T) = e^{A(t,T)-B(t,T)X_t}. \tag{3.117}$$

2. *The foreign term structure is affine in X and given by*

$$G(t,X_t,T) = e^{C(t,T)-D(t,T)X_t}. \tag{3.118}$$

3. *There exist maturity dates T_1, T_2, ..., T_N, where $N = \frac{n^2+5n}{2}$, such that the following $N \times N$ matrix $K^d(T_1,\ldots,T_N)$ defined in Proposition 3.2.20 is nonsingular.*

4. *There exist maturity dates* $T_1^\star$, $T_2^\star$, ..., $T_N^\star$, *where* $N = \frac{n^2+5n}{2}$, *such that the* $N \times N$ *matrix* $K^f(T_1^\star, \ldots, T_N^\star)$ *defined in the foreign economy analoguosly to* K^d *as in Proposition 3.2.20) is nonsingular.*

5. *The jump size of the factor process* $\gamma(t, y)$ *does not depend on factor* X.

6. *The conditional jump size distribution* $M(t, dy)$ *does not depend on factor* X.

7. *The conditional jump size distribution* $M^f(t, dy)$ *does not depend on factor* X.

Then the following holds:

1. *Domestic and foreign short rates* $R^d(t, x)$ *and* $R^f(t, x)$ *are affine functions of* x.

2. *The factors drift vector* $\alpha(t, x)$ *and the covariance matrix* $\sigma(t, x)\sigma^\star(t, x)$ *and* $\sigma(t, x)\sigma_S^\star(t, x)$ *are affine.*

3. *The* Q^d *intensity,* $\lambda(t, x, dy)$, *and* Q^f*-intensity,* $\lambda^f(t, x, dy)$, *are affine.*

Proof. Using the results from Proposition 3.2.20 we conclude that $\alpha(t, x)$, $\sigma(t, x)\sigma^\star(t, x)$ and $\tilde{\lambda}(t, x)$ are affine. In the same way applying the results from Proposition 3.2.20 to the foreign economy and foreign term structure we also find that $\beta(x)$, $\sigma(t, x)\sigma^\star(t, x)$ and $\tilde{\lambda}^f(t, x)$are affine. Since $\alpha(t, x)$ and $\beta(t, x)$ are affine, $\sigma(t, x)\sigma_S^\star(t, x)$ can be found as

$$\sigma(t, x)\sigma_S^\star(t, x) = \beta(t, x) - \alpha(t, x) \tag{3.119}$$

and is affine as well. Since $\tilde{\lambda}^f(t, x)$ and $\tilde{\lambda}(t, x)$ are affine, and $M(t, dy)$ and $M^f(t, dy)$ do not depend on x, we find that intensities $\lambda(t, x, dy) = \tilde{\lambda}(x)M(t, dy)$ and $\lambda^f(t, x, dy) = \tilde{\lambda}^f(x)M^f(t, dy)$ are also affine. ∎

As follows from Proposition 3.2.20 if we are given an affine term structure (a pair of coefficients $A(t, T)$ and $B(t, T)$), we conclude that the underlying factor dynamics are affine. Assume that the factor price process does not exhibit jumps ($\delta_S(t, y) \neq 0$). Even then, however, we cannot determine the dynamics exactly (except for the simple case when the factor is one-dimensional), since we can only determine the drift and covariance matrix $\sigma(t, x)\sigma^\star(t, x)$ (but not volatility $\sigma(t, x)$ itself). The distribution of factor X_t, however, will be unique.

Assume that the factor process has jump-diffusion dynamics and the term structures are affine. To be able to conclude whether the underlying factor dynamics are affine, we need to know the factor jump size $\gamma(t,y)$ and the conditional jump size distribution $M(t,dy)$, since they enter matrix $K(T_1,\ldots,T_N)$ as parameters. If matrix $K(T_1,\ldots,T_N)$ is invertible, we can determine the drift, intensity and covariance matrix of the factor process.

Necessary conditions II and III shows that if we are given the two pairs of coefficients determining the domestic and foreign term structures, we can draw the following implications concerning the exchange rate dynamics.

- Assume that the factor price process **does not** exhibit jumps. Then if we are given the two pairs of coefficients determining the domestic and foreign term structures, we can determine drifts, $\alpha(t,x)$ and $\beta(t,x)$, of the factor price process under the domestic and foreign martingale measures as well as covariance $\sigma(t,x)\sigma^\star(t,x)$.

- Since we know that the difference between the drifts is equal to $\sigma(t,x)\sigma_S^\star(t,x)$, we can always determine the term $\sigma(t,x)\sigma_S^\star(t,x)$. However, we cannot determine $\sigma_S(t,x)$ since $\sigma(t,x)$ is not necessarily invertible. Thus, from Proposition 3.2.25 it is clear that given that the domestic and foreign term structures are affine we cannot derive the exchange rate dynamics uniquely. In fact, there are several possible solutions for the exchange rate volatility consistent with (3.119) and thus several exchange rate processes are consistent with the domestic and affine foreign term structures specified. In case matrix $\sigma(t,x)$ is $n \times n$ and invertible, we know that the exchange rate volatility could be determined uniquely if $\sigma(x)$ is known. However, in general, given A and B, we can only determine $\sigma(t,x)\sigma^\star(t,x)$.

- Assume now that the factor price process exhibits jumps and the jump size is stochastic. In this case if $K(T_1,\ldots,T_N)$ is invertible, we can conclude that the drifts, covariances $\sigma(t,x)\sigma^\star(t,x)$ and $\sigma(t,x)\sigma_S^\star(t,x)$ and intensities under domestic and foreign martingale measures are affine.

3.2.6 Affine exchange rate dynamics

Does the fact that the domestic and foreign term structures are affine imply that the exchange rate is affine as well? In particular, if we are modeling affine term structures in both the domestic and foreign markets can we assume that the exchange rate is also affine as an immediate consequence of the assumptions?

Necessary conditions I and II show that if the domestic and foreign term structures are given ($A(t,T)$ and $B(t,T)$, $C(t,T)$ and $D(t,T)$ are given) we cannot derive the exchange rate process uniquely. The following proposition specifies under what conditions the exchange rate appears to be exponentially affine. However, even under those assumptions we cannot conclude that the exchange rate is unique. The following proposition 3.2.26 shows that the fact that the domestic and foreign term structures are both affine does not necessarily imply the affine exchange rate.

Proposition 3.2.26. (Affine exchange rate dynamics: Wiener process case)

If under the assumptions of Proposition 3.2.25 the pairs of coefficients $[A(t,T), B(t,T)]$ and $[C(t,T), D(t,T)]$ are such that

1. *there exists a deterministic vector function, $P(t)$, satisfying*[4]

$$\sigma(t,x)\sigma^\star(t,x)P^\star(t) = \beta(t,x) - \alpha(t,x) \quad \textit{and} \tag{3.120}$$

2. *vector $P(t)$ satisfies the following Riccati equation componentwise*

$$P'(t) + \alpha^2(t)P(t) + \frac{1}{2}P(t)d_k^2(t)P^\star(t) = (B_T)(t,t) + (D_T)(t,t) \tag{3.121}$$

then an affine exchange rate, $S(t,x)$, exists.

Proof. Define $P(t)$ as a solution to matrix equation

$$\sigma(t,x)\sigma^\star(t,x)P^\star(t) = \beta(t,x) - \alpha(t,x). \tag{3.122}$$

By assumption $P(t)$ does not depend on factor x. Define the volatility of the exchange rate process $\sigma_S(t,x)$ by

$$\sigma_S(t,x) = P(t)\sigma(t,x), \tag{3.123}$$

where $\sigma(x)$ is one of the admissible volatility matrices and $P(t)$ is one of the solutions to (3.120). Then $\sigma(t,x)\sigma_S^\star(t,x) = \beta(t,x) - \alpha(t,x)$ is satisfied. Define function $Q(t)$ as follows:

$$\begin{aligned} Q'(t) &= -A_T(t,t) + B_T(t,t)x + C_T(t,t) - D_T(t,t)x - P'(t)x - P(t)\alpha(x) \\ &- \frac{1}{2}P^\star(t)\sigma(x)\sigma^\star(x)P(t) \end{aligned} \tag{3.124}$$

[4]According to Proposition 3.2.20 both $\sigma(t,x)\sigma^\star(t,x)$ and drifts $\beta(t,x)$ and $\alpha(t,x)$ can be determined given the domestic and foreign term structures.

By assumption (3.124) does not depend on factor x since (3.121) is satisfied. Define the drift of the exchange rate process under the domestic martingale measure as the difference between the domestic and foreign short rates

$$\alpha_S(t) = R^d(t,x) - R^f(t,x) = A_T(t,t) - B_T(t,t)x - C_T(t,t) + D_T(t,t)x. \quad (3.125)$$

Using (3.124) we rewrite the exchange rate drift as

$$\alpha_S(t) = R^d(t,x)-R^f(t,x) = Q'(t)+P'(t)x+P(t)\alpha(t,x)+\frac{1}{2}P^\star(t)\sigma(t,x)\sigma^\star(t,x)P(t). \quad (3.126)$$

Finally, we can write the exchange rate dynamics as follows:

$$\begin{aligned} dS_t &= S_t(R^d(t,x) - R^f(t,x))dt + S_t\sigma_S(t,x)dW_t \\ &= S_t(Q'(t) + P'(t)x + P(t)\alpha(t,x) + \frac{1}{2}P^\star(t)\sigma(t,x)\sigma^\star(t,x)P(t))dt \\ &+ S_tP(t)\sigma(t,x)dW_t. \end{aligned} \quad (3.127)$$

The solution to (3.127) exists and is given by

$$S(t) = e^{P(t)X_t+Q(t)}, \quad (3.128)$$

and we conclude that an affine exchange rate exists. However, since there exists a whole set of functions $Q(t)$ satisfying (3.124), the affine exchange rate is not unique. ∎

As an example, consider the following one-factor model where the domestic and foreign term structures are driven by only one factor - the domestic short rate. The example shows that the affine domestic and foreign term structures do not necessarily imply an affine exchange rate. However, if the assumptions of Proposition 3.2.26 are satisfied an affine exchange rate exists.

Example 3.2.27. *Assume that we only have a one-dimensional factor process and we use the domestic short rate as a factor. The domestic term structure is of the form*

$$F(t, R^d, T) = \exp\{A(t,T) - B(t,T)R^d\},$$

where

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}$$

$$A(t,T) = \frac{\{B(t,T) - T + t\}\{ab - \frac{1}{2}\sigma^2\}}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}.$$

The foreign term structure is also dependent on the domestic short rate as a factor, and assume that it takes the form

$$G(t, R^d, T) = \exp\{C(t,T) - D(t,T)R^d\},$$

where

$$D(t,T) = \frac{1-\alpha_1}{a}\left\{1 - e^{-a(T-t)}\right\}, \quad \alpha_1 \in R,$$

$$\begin{aligned} C(t,T) &= \left\{\frac{1-\alpha_1}{a}(b+\delta\sigma^2) - \alpha_2 - \frac{\sigma^2}{2a^2}(1-\alpha_1)^2\right\}(T-t) \\ &- \frac{1-\alpha_1}{a^3}(b+\delta\sigma^2)(1-e^{-a(T-t)}) + \frac{(1-\alpha_1)^2}{a^2}\frac{\sigma^2}{a}(1-e^{-a(T-t)}) \\ &- \frac{\sigma^2}{4a}(1-e^{-2a(T-t)})\frac{(1-\alpha_1)^2}{a^2}. \end{aligned}$$

From the form of the domestic and foreign term structures we find that the domestic short rate dynamics under the domestic martingale measure satisfies

$$dR^d = (b - aR^d)dt + \sigma dW_t.$$

The domestic short rate dynamics under the foreign martingale measure can then be written as

$$dR^d = (b - aR^d + \delta\sigma^2)dt + \sigma dW^f.$$

The foreign short rate is equal to

$$R^f(R^d) = (1-\alpha_1)R^d - \alpha_2,$$

which implies that under the Q^d martingale measure the drift of the exchange rate process equals

$$\alpha_S(R^d) = \alpha_1 R^d + \alpha_2 \tag{3.129}$$

and we can define volatility of the exchange rate as $\sigma_S(t) = P(t)\sigma$, where $P(t)$ is given by

$$P(t) = \frac{b - aR^d + \delta\sigma^2 + aR^d - \delta\sigma^2}{\sigma^2} = \delta. \tag{3.130}$$

Thus the dynamics of the exchange rate are given as follows:

$$dS = S(\alpha_1 R^d + \alpha_2)dt + S\sigma\delta dW_t. \tag{3.131}$$

Proposition 3.2.26 *then says that for the affine exchange rate to exist (3.121) should be satisfied. In our case (3.121) reads*

$$\alpha_1 = -\delta a.$$

Once this condition is satisfied, we find $Q'(t)$ as

$$Q'(t) = +\alpha_2 - \delta b - \frac{1}{2}\sigma^2\delta^2. \tag{3.132}$$

Then we see that there exists a set of affine exchange rates of the form

$$S(R^d) = \exp\{PR^d + Q(t)\},$$

where the exchange rate dynamics are

$$dS = S(P(b - aR^d) + \frac{1}{2}P^2\sigma^2)dt + SP\sigma dW_t.$$

In the previous section, we assumed directly that the domestic and foreign term structures are affine and showed that, in general, many exchange rate processes are compatible with those term structures. We showed that it is not possible to derive endogenously the dynamics of the exchange rates knowing only the pairs of the coefficients determining the term structures. We also showed that under some particular conditions an affine exchange rate consistent with the affine domestic and foreign bond prices exists.

3.3 Inflation-Protected Securities

Inflation-protected securities are one of the new developments in the US capital markets. On Wednesday 29 January 1997 the United States Treasury issued its first inflation-linked bonds. In the absence of default risk, inflation risk is one of the major risks for the nominal bondholder since the purchasing power of the fixed payments is reduced by inflation. The inflation-protected bond pays a specified face value at maturity just as a conventional bond does. Its face value, however, is adjusted by the inflation rate. To make the discussion more precise we introduce the following definitions.

Definition 3.3.1. Consumer Price Index (CPI) *is a base-year quantity weighted average of the prices of a basket of goods and services consumed within a country*

- a Laspeyres price index. We will call the fixed basket of goods and services a **CPI unit**.

$$CPI_t = \sum_j \kappa_j P_{jt}, \tag{3.133}$$

where P_{jt} is the price paid for the good or service j by consumers at date t and κ_j is a base period expenditure weight on good j.

Definition 3.3.2. *Given a fixed numeraire, a country's time t* **price level**, *$I(t)$, is defined as the domestic purchase price at time t, in terms of the numeraire, of a well-defined basket of commodities (CPI unit). (Usually the domestic currency is used as a numeraire.)*

Inflation is defined as the increase in the level of prices. The **inflation rate**, $\tilde{I}(t)$, is usually measured by the changes in Consumer Price Index (CPI)

$$\tilde{I}(t) = \frac{dI(t)}{I(t)}. \tag{3.134}$$

We can now give a formal definition of inflation-linked securities[5]. Inflation-linked securities allow future cash flows to be expressed in terms of their purchasing power. For inflation-indexed securities future cash flows are certain in terms of their future purchasing power.

Denote the prices of domestic and foreign real T-bonds for every maturity T as $p_r(t, T)$ and $p_r^f(t, T)$. Inflation-protected securities are indexed by adjusting the principal. The adjustment is carried out by multiplying by the ratio of the CPI value at a given date to the CPI value at the security issuance date. The fixed coupon rate is also adjusted by multiplying by the relevant CPI ratio. In some sense, investors get an option since at maturity they can get either the inflation adjusted principal or, in case of deflation, the initial par amount. However, if deflation is unlikely, the securities prices are calculated without taking the option into account. To calculate the coupon and principal payments the CPI is used with a two-month delay.

Definition 3.3.3. *A* **Treasury Inflation-Protected Security (TIPS)** *with maturity T is a security indexed to inflation, that is, security with its principal adjusted by inflation at T.*

In particular, using the risk-neutral valuation, the price of the inflation-protected bond (without an issue date adjustment) can be found as follows:

$$P_{TIPS}(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T R(s)ds} I(T) 1_{CPI} \middle| \mathcal{F}_t \right], \tag{3.135}$$

[5]For the description of inflation derivatives see e.g. Hughston (1998)

where $I(t)$ is a time t CPI inflation index, that is, currency per CPI unit.

Definition 3.3.4. *The* **real bond** *with maturity T is the bond that gives its holder one CPI unit at maturity T. Denote the price of the real bond with maturity T as $p_r(t,T)$. The price of the real bond is expressed in CPI units.*

The price of the real bond can be found as:

$$p_r(t,T) = \frac{\mathbb{E}^Q\left[e^{-\int_t^T R(s)ds} I(T) 1_{CPI} \Big| \mathcal{F}_t\right]}{I(t)}. \tag{3.136}$$

We are now ready to define nominal and **real** forward and short rates.

Definition 3.3.5. We define

- the instantaneous nominal forward rate with maturity T as

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T}, \tag{3.137}$$

- the instantaneous real forward rate with maturity T as

$$f(t,T) = -\frac{\partial \log p_r(t,T)}{\partial T}, \tag{3.138}$$

- the instantaneous nominal short rate with maturity T as

$$R(t) = f(t,t), \text{and} \tag{3.139}$$

- the instantaneous real short rate with maturity T as

$$R^r(t) = f^r(t,t). \tag{3.140}$$

We can construct a hypothetical **real risk free account** $B^r(t)$ by "rolling over" just-maturing real bonds. In other words, we consider a self-financing portfolio consisting at each moment t of the real bonds that mature in a moment $t + x$[6], which will evolve according to the following dynamics

$$dB^r(t) = R^r(t)B^r(t)dt. \tag{3.141}$$

Assume that the price level process $I(t)$ has the following dynamics under Q

$$dI(t) = I(t)\pi(t)dt + I(t)\sigma_I(t)dW_t + I_{t-}\int_{\mathbf{R}} \delta_I(t,y)\tilde{\mu}(dt,dy), \tag{3.142}$$

[6]For details see Björk (2004).

where $\pi(t)$, $\sigma_I(t)$ are adapted processes and $\delta_I(t,y)$ is a predictable process, $\tilde{\mu}(dt,dy)$ is a compensated MPP.

The following two possibilities are equivalent:

- investing 1 CPI unit in the real riskless account, and
- investing in a nominal asset with price process $\tilde{B}^r(t)$, where

$$\tilde{B}^r(t) = B^r(t)I(t).$$

The dynamics of $\tilde{B}^r(t)$ are given by

$$d\tilde{B}^r(t) = \tilde{B}^r(t)\left\{\pi(t) + R^r(t)\right\}dt + \tilde{B}^r(t)\sigma_I(t)dW_t + \tilde{B}^r(t-)\int_{\mathbf{R}}\delta_I(t,y)\tilde{\mu}(dt,dy).$$

Since we assume that the possibility to invest in real riskless account should not introduce arbitrage in the market, then we conclude that under the domestic martingale measure Q^d the dynamics of $\tilde{B}^r(t)$ are given by

$$d\tilde{B}^r(t) = \tilde{B}^r(t)R(t)dt + \tilde{B}^r(t)\sigma_I(t)dW_t + \tilde{B}^r(t-)\int_{\mathbf{R}}\delta_I(t,y)\tilde{\mu}(dt,dy).$$

which means that the following holds:

$$R^r(t) = R(t) - \pi(t) = R(t) - \mathbb{E}^Q\left[\left.\frac{dI(t)}{I(t)}\right|\mathcal{F}_t\right]. \tag{3.143}$$

Following Jarrow and Yildirim (2003) (see also Mercurio (2005)) in order to price Treasury Inflation-Protected Securities (TIPS) we use the methodology of pricing currency/exchange rate derivatives in the two-country economy, where we associate nominal money with domestic currency, real money with foreign currency and inflation with the spot exchange rate. Using the analogy, we define the new **real** martingale measure Q^r on $\mathcal{F}_T$ as follows

$$\frac{dQ^r}{dQ} = \frac{B^r(T)I(T)}{B(T)I(0)} = L_T. \tag{3.144}$$

We note that from the definition of the real interest rate (3.143) it follows that L_T is a martingale since

$$\begin{aligned} dL(t) &= L(t)\left\{\pi(t) + R^r(t) - R(t)\right\}dt + \sigma_I(t)L(t)dW_t + L(t-)\int_{\mathbf{R}}\delta_I(t,y)\mu(dt,dy) \\ &= \sigma_I(t)L(t)dW_t + L(t-)\int_{\mathbf{R}}\delta_I(t,y)\mu(dt,dy) \end{aligned} \tag{3.145}$$

and thus determines the process

$$L_t = \mathbb{E}^Q\left[\frac{B^r(T)I(T)}{B(T)I(0)}\bigg|\mathcal{F}_t\right] = \frac{B^r(t)I(t)}{B(t)I(0)} \quad \text{on} \quad \mathcal{F}_t, \quad 0 \leq t \leq T.$$

We note now that under the measure Q^r all the assets expressed in CPI units are martingales when discounted by the real risk free account and the price of the real bond can be found as

$$\begin{aligned}\mathbb{E}^{Q^r}\left[\frac{1_{CPI}}{B^r(T)}\bigg|\mathcal{F}_0\right] &= \mathbb{E}^{Q^r}\left[\frac{p_r(T,T)}{B^r(T)}\bigg|\mathcal{F}_0\right] = \mathbb{E}^Q\left[\frac{p_r(T,T)}{B^r(T)}\frac{B^r(T)I(T)}{B(T)I(0)}\bigg|\mathcal{F}_0\right] \\ &= \frac{1}{I(0)}\mathbb{E}^Q\left[\frac{p_r(T,T)I(T)}{B(T)}\bigg|\mathcal{F}_0\right] = \frac{p_r(0,T)I(0)}{I(0)} = p_r(0,T)\end{aligned} \quad (3.146)$$

since $\frac{p_r(T,T)I(T)}{B(T)}$ is a martingale under the domestic martingale measure.

Changing the measure from Q to Q^r according to (3.144), we see that

$$\begin{aligned}P_{TIPS}(0,T) &= \mathbb{E}^Q\left[\frac{I(T)1_{CPI}}{B(T)}\bigg|\mathcal{F}_0\right] = \mathbb{E}^Q\left[\frac{L(T)I(0)1_{CPI}}{B^r(T)}\bigg|\mathcal{F}_0\right] \\ &= I(0)\mathbb{E}^{Q^r}\left[\frac{1_{CPI}}{B^r(T)}\bigg|\mathcal{F}_0\right] = I(0)p_r(0,T),\end{aligned} \quad (3.147)$$

where the conditional expectation is calculated under the **real martingale measure** Q^r.

The model suggested by Jarrow and Yildirim (2003) for pricing the inflation-protected securities has the simplicity and also the power to obtain closed-form solutions for inflation-protected term structures and options on the inflation index. The empirical fit of this kind of model, however, can be quite poor since it is important to take into account the macroeconomic conditions or surprise events when pricing inflation-protected securities. The model is thus unsatisfactory for the following reasons:

- It is important to take into account extreme (e.g. policy-related) changes in inflation, resulting, for example, from changes in the target inflation rate, and interest rates. We extend the model of Jarrow and Yildirim (2003) by introducing a marked point process.

- It is also important to take into account the effect of the exchange rate on inflation and interest rates. For example, a weakening of the domestic currency means that the price of foreign goods rises in the domestic currency contributing to higher inflation (which also raises interest rates). To account for this we include in the model the market for foreign nominal and real bonds as well as the exchange rate between the countries[7].

- We might also want to take into account the correlation with the real exchange rate while pricing TIPS, since the changes in the real exchange rate might influence the countries' price levels.

3.4 Two-country economy: real exchange rates

The study of real exchange rates is one of the most intensively researched areas in international macroeconomics. In this paper we do not aim to create a general equilibrium macroeconomic model that would be able to replicate the observed behaviour of real exchange rates. Instead we intend to exploit the arbitrage-free approach to derivatives pricing in order to construct a framework that would allow us to take into account the effects of macroeconomic shocks and policies when pricing inflation-protected and exchange rate derivatives. Thus, we need to enlarge the two-country nominal framework to model the joint dynamics of nominal exchange rates, (domestic and foreign) inflation, (domestic and foreign) term structures and real exchange rate. We believe that the shortcoming of the parsimonious framework is that the prices of the nominal exchange rate derivatives and inflation-protected securities are found irrespective of the general macroeconomic situation in the two-country world and do not take into account correlations with macroeconomic variables. The risk resulting from the correlation between the nominal exchange rate and other macroeconomic variables must also be priced.

Some attempts have been made to create a pricing model in an international economy. Dillen (1997) includes the real exchange rate in the arbitrage-free bond pricing models describing the three-factor model where the state variables are the world real interest rate, the real exchange rate and the inflation rate following a three-dimensional Ornstein-Uhlenbeck process. Dillen (1997) captures the effects on the term structure originating from exchange rate fluctuations and actions by a central bank. He also considers the influence of regime shifts such as changes in the target inflation rate and devaluations. See also Brennan and Xia (2004), who explore the forward premium anomaly in an international Gaussian model. Dewachter, Lyrio, and Maes (2001) estimate a joint model for the term structure of interest rates and the macroeconomy.

We create a general international framework by enlarging the Duffie-Kan class affine multifactor models incorporating the dynamics of domestic and foreign

[7]To introduce this extension in the Jarrow and Yildirim framework we could include foreign nominal and real forward rate dynamics.

price levels and thus real exchange rate dynamics. In this case, as indicated earlier, we do not specify the dynamics of the observable factors explicitly but rather assume that the object dynamics are driven by a set of unobservable latent factors. We assume that all the objects in the economy are driven by the same multidimensional factor process. We can, however, divide the whole set of factors into local and common factors influencing bond prices in both countries. One might want to specify the local factors influencing the price level process or the real exchange rate process only. We will concentrate on the class of affine term structure models since they will allow us to obtain the closed-form solutions for the nominal and real term structures, real exchange rate and related derivatives.

The empirically documented fact is that real exchange rates are highly volatile as compared to other economic variables and many researchers have struggled to come up with a macroeconomic theory that produces excess volatility of real exchange rates. Some economists believe that real shocks to technology and preferences could account for this excess volatility of exchange rates. Thus, to obtain the model that would better fit the true dynamics of real exchange rates we include jumps in the driving factor dynamics. We consider affine models consistent with the factor dynamics.

The affine theory of the nominal exchange rate and inflation-protected securities can be extended so as to model the joint affine dynamics of the real exchange rate, the exchange rate and inflation dynamics in addition to the nominal exchange rate and the domestic and foreign term structures.

The real exchange rate is defined as the nominal exchange rate that takes into account inflation differentials between the two countries.

Definition 3.4.1. *The real exchange rate, r, is defined as the nominal exchange rate adjusted by the ratio of the foreign and domestic price levels:*

$$r(t) = S(t)\frac{I^f(t)}{I(t)}. \tag{3.148}$$

The real exchange rate is important in that it can be used as an indicator of the competitiveness of the countries' foreign trade. A rise in the real exchange rate, or real depreciation, would mean that the prices of foreign goods expressed in domestic currency increase as compared to the prices of domestic goods. Thus, domestic goods would be relatively cheaper than foreign goods and consumers would probably prefer to switch to domestic goods. Conversely, if the real exchange rate appreciated, this would make domestic goods relatively more expensive and they would lose their competitiveness.

3.4.1 Enlarged two-country factor economy

To study the joint dynamics of the nominal exchange rate, the domestic and foreign term structures and the real exchange rate we first need to construct a pricing model for the inflation-protected securities specifying domestic and foreign price level dynamics so that we can adjust the nominal exchange rate for the inflation differentials between the countries.

All the dynamics specified in the nominal exchange rate section will be called **nominal dynamics** and the model described in the first section will be called the **nominal** model. Thus, for the nominal variables we will be using the same notations as in the first section. The martingale measures Q^d and Q^f are said to be the nominal martingale measures. We enlarge this nominal framework by adding both countries' price level dynamics. The measures under which the dynamics of the real domestic and foreign variables discounted by the real bank account are martingales will be called domestic and foreign real martingale measures and are denoted by Q_r^d and Q_r^d.

Assume that the whole economy is driven by the multifactor price process X. The factor dynamics, nominal exchange rate dynamics and the functional form of the domestic and foreign nominal bonds are assumed to be as in Assumption 3.2.8. In addition we make the following assumptions:

Assumption 3.4.2.

1. *Assume that for every maturity T, the prices of the domestic and foreign* **real** *T-bonds have the form*

$$p_r(t,T) = F^r(t, X_t, T) \tag{3.149}$$
$$p_r^f(t,T) = G^r(t, X_t, T), \tag{3.150}$$

   where F^r and G^r are two smooth functions of the entries t, T and X_i.

2. *Assume that the dynamics of the domestic CPI inflation index is*

$$dI_t = I_t\alpha_I(t, X_t)dt + I_t\sigma_I(t, X_t)dW_t + I_{t-}\int_X \delta_I(t, y, X_{t-})\mu(dt, dy), \tag{3.151}$$

   where $\alpha_I(t, X_t)$ is a scalar-adapted process, $\sigma_I(t, X_t)$ is an $1 \times d$ adapted row vector process, $\delta_I(t, X_t, y)$ is a scalar-predicted process.

3. *Assume that the dynamics of the foreign CPI^f inflation index is*

$$dI_t^f = I_t^f\alpha_{If}(t, X_t)dt + I_t^f\sigma_{If}(t, X_t)dW_t + I_{t-}^f\int_X \delta_{If}(t, y, X_{t-})\mu(dt, dy), \tag{3.152}$$

where $\alpha_{If}(t, X_t)$ is a scalar adapted process, $\sigma_{If}(t, X_t)$ is an $1 \times d$ adapted row vector process, and $\delta_{If}(t, X_t, y)$ is a scalar predicted process.

Assumption 3.4.2 and definition (3.4.1) tell us that the real exchange rate can be expressed as the function of factor X as well:

$$r(t, X_t) = S(t, X_t)\frac{I^f(t, X_t)}{I(t, X_t)}. \tag{3.153}$$

We can thus derive the Q^d dynamics of the real exchange rate using the Itô formula.

3.4.2 Affine real exchange rate and TIPS dynamics: sufficient conditions

Jarrow and Yildirim (2003) show that nominal prices can be associated with domestic prices, real prices can be associated with foreign prices and the CPI index can be associated with the spot exchange rate. As much as the methodology for the currency/exchange rate derivatives can be used for the pricing of the inflation-protected securities, all the results (necessary and sufficient conditions) derived for the joint interest rates and exchange rate dynamics can be applied for the inflation-protected securities as well. We thus immediately obtain the following sufficient conditions on the factor dynamics, covariances and short rates for the nominal and real term structures to be jointly affine.

Proposition 3.4.3. (Sufficient conditions I: affine nominal and real term structures)

Assume that

1. *The drift of the factor process, $\alpha(t, x)$, under the Q^d measure, covariance matrices*

$$\sigma(t, x)\sigma^\star(t, x), \quad \sigma(t, x)\sigma_S^\star(t, x), \quad \sigma(t, x)\sigma_I^\star(t, x), \quad \sigma(t, x)\sigma_{If}^\star(t, x),$$

   the domestic and foreign short rates, $R^d(t, x)$ and $R^f(t, x)$, the domestic and foreign real short rates, $R^r(t, x)$ and $R^{rf}(t, x)$, and the intensity of the marked point process under the domestic martingale measure, $\lambda(t, x, dy)$, are affine in x.

2. *The jump sizes of the factor process, γ, the exchange rate, δ_S, domestic and foreign inflation, δ_I and δ_{If}, do not depend on factor X_t.*

Then

1. *Domestic and foreign term structures* $F(t, X_t, T)$ *and* $G(t, X_t, T)$ *are affine in* X_t.

2. *Domestic and foreign real term structures* $F^r(t, X_t, T)$ *and* $G^r(t, X_t, T)$ *are affine in* X_t.

Proof. The proof is analogous to the proof of Proposition 3.2.14 for the nominal foreign and real domestic prices. Measure transformation from Q^d to Q^{rf}, for example, will change the drift to

$$\tilde{\beta}(t, X_t) = \alpha(t, X_t) + \sigma(t, X_t)\sigma_S^\star(t, X_t) + \sigma(t, X_t)\sigma_{If}^\star(t, X_t). \tag{3.154}$$

The Q^{rf} intensity can then be found as

$$\lambda^{rf}(t, X_t, dy) = \lambda(t, X_t, dy)(1 + \delta_S(t, y))(1 + \delta_{If}(t, y)). \tag{3.155}$$

Thus under the assumptions of Proposition 3.4.3 the drift $\tilde{\beta}(t, x)$, covariance $\sigma(t, x)\sigma^\star(t, x)$ and Q^{rf} intensity, $\lambda^{rf}(t, x, dy)$, of the factor process are affine. We then use Proposition 3.2.14 to conclude that $G^r(t, X_t, T)$ is affine. ■

Proposition 3.4.4. (Sufficient conditions II: affine TIPS term structure and affine real exchange rate dynamics)

Assume that

1. *The drift,* $\alpha(t, x)$, *of the factor process under the* Q^d *measure, the covariance matrices* $\sigma(t, x)\sigma^\star(t, x)$, *the domestic short rate,* $R^d(t, x)$, *the intensity* $\lambda(t, x, dy)$ *are affine in* x.

2. *The nominal exchange rate price process is affine in the factors and takes the form*
$$S(t, X_t) = e^{P(t)X_t + Q(t)}. \tag{3.156}$$

3. *The domestic CPI process is affine in the factors and takes the form*
$$I(t, X_t) = e^{P_I(t)X_t + Q_I(t)}. \tag{3.157}$$

4. *The foreign CPI process is affine in the factors and takes the form*
$$I^f(t, X_t) = e^{P_{If}(t)X_t + Q_{If}(t)}. \tag{3.158}$$

Then

1. *The domestic and foreign nominal and real term structures,* $F(t, X_t, T)$, $G(t, X_t, T)$, $F^r(t, X_t, T)$ *and* $G^r(t, X_t, T)$ *are affine.*
2. *The foreign short rate,* $R^f(t, x)$, *the domestic and foreign real short rates,* $R^r(t, x)$ *and* $R^{rf}(t, x)$ *are affine. The* Q^f *intensity* $\lambda^f(x, dy)$, Q^r*-intensity* $\lambda^r(x, dy)$ *and* Q^{rf} *intensity* $\lambda^{rf}(x, dy)$ *are affine.*
3. *The domestic and foreign TIPS,* $P^{TIPS}(t, X_t)$ *and* $P_f^{TIPS}(t, X_t)$, *and the term structures are affine.*
4. *The real exchange rate,* $r(t, X_t)$, *is affine.*

Proof. Follows from Proposition 3.2.15, (3.146) and the real exchange rate definition. ■

Thus, in case $S(t, X_t)$, $I(t, X_t)$ and $I^f(t, X_t)$ are exponentially affine, and the real exchange rate and is also affine and takes the form

$$r(t, X_t) = \exp\left\{\left\{P(t) + P_{If}(t) - P_I(t)\right\} X_t + Q(t) + Q_{If}(t) - Q_I(t)\right\}. \quad (3.159)$$

Purchasing Power Parity

Purchasing power parity (PPP) is a theory stating that exchange rates between currencies are in equilibrium when their purchasing power is the same in each of the two countries. The PPP theory is a result of the extension of the Law of One Price (LOP) to international economy. The LOP says that in two separate markets - if there are no transportation costs or taxes - the prices of the same goods should be equal since otherwise an arbitrage opportunity exists. The PPP theory is simply the LOP applied in aggregate. Instead of comparing the prices of separate goods across the countries we compare aggregate price levels I and I^f, calculated as the prices of the CPI baskets. Thus, PPP can be stated as

$$I(t) = S(t)I^f(t).$$

According to the PPP definition, the real exchange rate should be equal to 1 in equilibrium assuming that there is an adjustment mechanism on the international market that brings the real exchange rate to its equilibrium level 1. For more detailed discussion of real exchange rate and PPP see e.g. Obstfeld and Rogoff (1996). In practice, however, we cannot expect the PPP to hold if the countries have different CPI baskets. Thus, the argument that the equilibrium real exchange rate is 1 is not always correct but may be equal to some other constant

K. Thus, it is accepted that one major implication of the PPP theory is that the real exchange rate is **mean stationary** (or even stationary) (see e.g. Baillie and McMahon (1989)).

Definition 3.4.5. *Stochastic process $y(t)$ is mean stationary if $Ey(t) = \mu \ \forall \ t$.*

Although many empirical studies test the mean stationarity hypothesis, see e.g. Holmes (2001), it seems there is still no agreement on whether the evidence is in line with the mean stationarity of the real exchange rate. It is clear that there is very little hope to obtain stationary real exchange rate in our framework (except for perhaps the most simple cases), thus, in this section we focus on the property of *assymptotic mean stationarity.* Given that we can not reject the PPP, the aim of the following section is to understand under what conditions our model is consistent with the (less strong) PPP hypotheses. Thus, the question we address here is:

Question: *How shall we model the exchange rate process $S(t)$, domestic and foreign price levels, $I(t)$ and $I^f(t)$ in the jointly affine factor economy so that the real exchange rate process $r(t)$ is asymptotically mean stationary, that is,*

$$\lim_{t\to\infty} \mathbb{E}^P\left[r(t)\right] = \lim_{t\to\infty} K(t) = K, \tag{3.160}$$

where K is constant, and P is the domestic objective probability measure.

If the real exchange rate is an exponentially affine function of the factors, it is possible to derive the necessary conditions for the real exchange rate to be asymptotically mean stationary using the machinery of the characteristic functions.

Real exchange rate mean stationarity in Gaussian/Compound Poisson models.

In this section we assume for simplicity that the driving factor is one dimensional and consider the simplified factor dynamics. We will focus on the *Gaussian/Compound Poisson* Q^d factor dynamics of the form

$$dX_t = \alpha(t, X_t)dt + \sigma(t)dW_t + \int_{\mathbf{R}} y\mu(dt, dy), \tag{3.161}$$

where α is an adapted affine function of factors,

$$\alpha(t, X_t) = \tilde{b}(t) - \tilde{a}(t)X_t,$$

volatility $\sigma(t)$ and intensity $\tilde{\lambda}(t, dy)$ do not depend on the factor and are deterministic functions.

According to Proposition 3.4.4, the factor dynamics (3.161) together with affine domestic short rate and intensity guarantee affine domestic term structure. Moreover, an affine nominal exchange rate, and domestic and foreign CPI ensure that foreign term structure, domestic and foreign real term structures, and the real exchange rate are affine. Assume that the nominal exchange rate, domestic and foreign CPI indexes are as in (3.156), (3.157) and (3.158), then, according to Proposition 3.4.4, the real exchange rate is also affine and takes the form

$$r(t, X_t) = e^{\left\{P(t)+P_{If}(t)-P_I(t)\right\}X_t+Q(t)+Q_{If}(t)-Q_I(t)} = e^{h(t)X_t+g(t)}, \qquad (3.162)$$

where

$$\begin{aligned} h(t) &= P(t) + P_{If}(t) - P_I(t), \\ g(t) &= Q(t) + Q_{If}(t) - Q_I(t). \end{aligned}$$

To provide necessary conditions for the mean stationarity of the real exchange rate we will need to calculate the expected value of the real exchange rate process under the objective probability measure P. However, since the *Gaussian/Compound Poisson* factor dynamics is given under the domestic martingale measure Q^d, to compute the expectation, we need to find the factor dynamics under P. To see the connections between the drifts and intensities under the different measures we recall the appropriate Girsanov theorem [8]. We recall that if we define the process $L(t)$ by

$$\begin{cases} \dfrac{dL(t)}{L(t-)} &= \zeta(t)dW(t) + \int_{\mathbf{R}} \phi(t) \left\{\mu(dt, dy) - \tilde{\lambda}(t, dy)dt\right\}, \\ L(0) &= 1. \end{cases}$$

and define the probability measure P as follows

$$\frac{dP}{dQ} = L(t) \quad \text{on} \quad \mathcal{F}_t \quad \forall t > 0.$$

Then:

1. We can write
$$dW(t) = dW_P(t) + \zeta(t)dt, \qquad (3.163)$$
where W_P is a P-Wiener process.

2. The predictable compensator under P is as follows
$$\lambda^P(t, dy) = (1 + \phi_t(y))\lambda(t, dy). \qquad (3.164)$$

[8] See, for example, Björk (2004) or Schönbucher (2003).

$\phi_t(y)$ and $\zeta(t)$ in (4.19) and (3.164) are the *market prices of risk* and, in principle, depend on the factor and thus, are stochastic.

To compute the mean of the real exchange rate, we need to make some assumptions concerning the market prices of risks.

Assumption 3.4.6. *Assume that:*

- *Market price of jump risk, $\phi(t, y)$, is a deterministic function of time*

$$\phi(t, y) = \phi(t).$$

- *Market price of risk, $\zeta(t)$, is affine, that is,*

$$\zeta(t, X_t) = \zeta_1(t) + \zeta_2(t)X_t. \tag{3.165}$$

We can write down the factor dynamics under the objective probability measure P:

$$dX_t = (b(t) - a(t)X_t)dt + \sigma(t)dW_t^P + \int_{\mathbf{R}} y\mu^P(dt, dy), \tag{3.166}$$

where

$$\begin{aligned} a(t, X_t) &= \tilde{a}(t) - \zeta_2(t)\sigma(t), & (3.167)\\ b(t, X_t) &= \tilde{b}(t) + \zeta_1(t)\sigma(t), & (3.168)\\ \lambda(t, dy) &= \tilde{\lambda}(t, dy)(1 + \phi(t)). & (3.169) \end{aligned}$$

In the rest of the section we will focus on the special class of jump processes - Lévy processes. The reason is that in this case the characteristic function is available in closed form and we will therefore be able to compute the real exchange rate expectation. We will need the following results:

Lemma 3.4.7. *Let $(X_t)_{t\geq 0}$ have the following dynamics*

$$dX_t = \gamma(t)dt + \sigma(t)dW_t + \int_{\mathbf{R}} y\mu(dt, dy),$$

then the characteristic function of X_t is given by

$$\mathbb{E}\left[e^{iuX_t}\right] = e^{\int_0^t \psi_s(u)ds}, \quad \textit{with} \tag{3.170}$$

$$\psi_s(u) = -\frac{1}{2}u^2\sigma^2(s) + iub(s) + \int_{\mathbf{R}} \left(e^{iuy} - 1\right)\lambda(s, dy), \tag{3.171}$$

where $\lambda(t, dy)$ is an intensity of the process $\mu^P(dt, dy)$.

Proof. See Cont and Tankov (2004). ■

Lemma 3.4.8. *Let $f : [0,T] \to R$ be left-continuous and $(Z_t)_{t\geq 0}$ be a Lévy process. Then*

$$\mathbb{E}\left[\exp\left(i\int_0^t f(s)dZ_s\right)\right] = \exp\left\{\int_0^t \psi_s(f(s))ds\right\}, \tag{3.172}$$

where $\varphi(u)$ is the characteristic exponent of Z as in (3.170).

Proof. Define a new variable Y_t as

$$Y_t = \int_0^t f(s)dZ_s. \tag{3.173}$$

Using Itô formula we can find the dynamics of Y_t:

$$\begin{aligned} dY_t &= f(t)dX_t = f(t)\gamma(t)dt + f(t)\sigma(t)dW_t + \int_{\mathbf{R}} yf(t)\mu(dt, dy) \\ &= \tilde{\gamma}(t)dt + \tilde{\sigma}(t)dW_t + \int_{\mathbf{R}} g(y)\mu(dt, dy), \end{aligned} \tag{3.174}$$

where

$$\tilde{\gamma}(t) = f(t)\gamma(t), \quad \tilde{\sigma}(t) = f(t)\sigma(t), \quad g(t,y) = yf(t).$$

We find that the characteristic function of Y_t, $\varphi_s(u)$, as

$$\begin{aligned} \varphi_s(1) &= -\frac{1}{2}\tilde{\sigma}^2(s) + i\tilde{b}(s) + \int_{\mathbf{R}} (e^{ig(s)} - 1)\lambda(s, dy) \\ &= -\frac{1}{2}f^2(s)\sigma^2(s) + if(s)b(s) + \int_{\mathbf{R}} \left(e^{if(s)y} - 1\right)\lambda(s, dy) = \psi_s(f(s)) \end{aligned} \tag{3.175}$$

We find then that

$$\mathbb{E}\left[\exp\left(i\int_0^t f(s)dZ_s\right)\right] = \exp\left\{\int_0^t \psi_s(f(s))ds\right\}.$$

■

With the help of the above two lemmas we will be able to prove the following proposition.

Proposition 3.4.9. (Necessary conditions.) *Assume that*

1. *Assumptions of Proposition 3.4.4 hold.*

2. *The factor dynamics are Gaussian/Compound Poisson as in (3.161), and* X_0 *is deterministic.*

3. *The real exchange rate* $r(t, X_t)$ *is asymptotically mean stationary.*

Then the following condition must be satisfied:

$$\begin{aligned}\lim_{t\to\infty}\theta(t) &= \lim_{t\to\infty}\exp\left\{g(t)+h(0)X_0E(t,0)+\frac{1}{2}\int_0^t E^2(t,s)h^2(s)\sigma^2(s)ds\right.\\&+\left.\int_0^t h(s)b(s)E(t,s)ds+\int_0^t\int_R \lambda(s,dy)\left\{e^{yh(t)E(t,s)}-1\right\}ds\right\}=K,\end{aligned} \tag{3.176}$$

where K *is a constant,*

$$E(t,s)=e^{\int_s^t q(u)du},$$

$$q(t)=-a(t)+\frac{h'(t)}{h(t)}. \tag{3.177}$$

Proof. We notice first that

$$\theta_t=\mathbb{E}^P\left[r(t,X_t)\right]=\mathbb{E}^P\left[e^{h(t)X_t+g(t)}\right]=\mathbb{E}^P\left[e^{iu\tilde{h}(t)X_t}\right]e^{g(t)}, \tag{3.178}$$

where i is an imaginary unit, $u\in R$ and $\tilde{h}(t)=\frac{h(t)}{iu}$. Denote the new process $Y_t=\tilde{h}(t)X_t$; then we see that

$$\tilde{\theta}_t=\mathbb{E}^P\left[e^{iuY_t}\right]$$

is a characteristic function of Y_t. We can easily find the dynamics of Y_t

$$\begin{aligned}dY_t &= \left(\tilde{h}(t)(b(t)-a(t)X_t)+X_t\tilde{h}'(t)\right)dt+\tilde{h}(t)\sigma(t)dW_t^P+\int_{\mathbf{R}}\tilde{h}(t)y\mu^P(dt,dy)\\&= \left(-a(t)\tilde{h}(t)+\tilde{h}'(t)\right)X_tdt+\tilde{h}(t)b(t)dt+\tilde{h}(t)\sigma(t)dW_t^P+\int_{\mathbf{R}}\tilde{h}(t)y\mu^P(dt,dy)\\&= q(t)Y_tdt+\tilde{h}(t)b(t)dt+\tilde{h}(t)\sigma(t)dW_t^P+\int_{\mathbf{R}}\tilde{h}(t)y\mu^P(dt,dy),\end{aligned}$$

where

$$q(t)=-a(t)+\frac{\tilde{h}'(t)}{\tilde{h}(t)}.$$

We can solve for Y_t to find

$$Y_t=Y_0e^{\int_0^t q(u)du}+\int_0^t e^{\int_s^t q(u)du}\tilde{h}(s)dZ_s, \tag{3.179}$$

where

$$dZ_t = b(t)dt + \sigma(t)dW_t^P + \int_{\mathbf{R}} y\mu^P(dt, dy). \tag{3.180}$$

Using Lemma 3.4.8 we find that

$$\tilde{\theta}_t = \mathbb{E}^P\left[e^{iuY_t}\right] = \exp\left\{iuY_0 e^{\int_0^t q(u)du} + \int_0^t \psi_s(u\tilde{h}(s)e^{\int_s^t q(u)du})ds\right\}, \tag{3.181}$$

where $\psi_s(u)$ can be found using Lemma 3.4.7

$$\psi_s(u) = -\frac{1}{2}u^2\sigma^2(s) + iub(s) + \int_{\mathbf{R}}\left(e^{iuy} - 1\right)\lambda(s, dy). \tag{3.182}$$

Substituting (3.182) in (3.181) and recalling (3.178) we obtain that the mean of the real exchange rate is equal to

$$\begin{aligned}\theta(t) &= \exp\left\{h(0)X_0 e^{\int_0^t q(u)du} + g(t) + \int_0^t h(s)b(s)e^{\int_s^t q(u)du}\right.\\ &+ \left.\frac{1}{2}\int_0^t e^{2\int_s^t q(u)du}h^2(s)\sigma^2(s)ds + \int_0^t\int_R \lambda(s, dy)\left\{e^{yh(t)e^{\int_s^t q(u)du}} - 1\right\}ds\right\}.\end{aligned} \tag{3.183}$$

We obtain the result by denoting

$$E(t, s) = e^{\int_s^t q(u)du}.$$

Thus since the real exchange rate process is asymptotically mean stationary, θ_t tends to some constant K as $t \to \infty$, which is exactly (3.176). ■

We are now going to give a simple example of a factor dynamics that is consistent with the asymptotic mean stationarity of the real exchange rate assumption.

Example 3.4.10. Vasicek/Compound Poisson model *Consider the following factor dynamics*

$$dX_t = (b - aX_t)dt + \sigma dW_t^P + \int_{\mathbf{R}} y\mu^P(dt, dy), \tag{3.184}$$

and the real exchange rate of the affine form

$$r(t, X_t) = e^{hX_t + g(t)},$$

where

$$h(t) = h = P(t) + P_{If}(t) - P_I(t)$$

does not depend on t. We also assume that intensity $\lambda(dy)$ does not depend on t and there exists the limit

$$\lim_{t\to\infty} g(t) = g.$$

In this case we obtain that

$$\begin{aligned} \theta_t &= \exp\left\{hX_0e^{-at} + \frac{\sigma^2h^2}{4a}\left(1-e^{-2at}\right) + \frac{hb}{a}\left(1-e^{-at}\right) + g(t)\right. \\ &+ \left.\int_0^t \int_R \lambda(dy)\left\{e^{yhe^{-a(t-s)}} - 1\right\} ds\right\}. \end{aligned} \tag{3.185}$$

Denote

$$z = yhe^{a(t-s)}, \quad \xi = \frac{e^{(t-s)a}}{h}. \tag{3.186}$$

We can then rewrite the double integral in (3.185) as

$$\int_0^t \int_R \lambda(dy)\left\{e^{yhe^{-a(t-s)}} - 1\right\} ds = \int_{\mathbf{R}} (e^z - 1)\int_{\frac{1}{h}}^{\frac{e^{at}}{h}} \lambda(\xi dz)\frac{d\xi}{a\xi}. \tag{3.187}$$

Then (see Cont and Tankov (2004)) if the following condition is satisfied

$$\int_{|y|\geq\frac{1}{h}} \ln|y|\lambda(dy) < \infty, \tag{3.188}$$

then the following limit exists $\forall B$

$$\zeta(B) = \int_{\frac{1}{h}}^{\infty} \lambda(\xi B)d\frac{d\xi}{a\xi}$$

and thus ζ is a measure. It is easy to see now that

$$\begin{aligned} \theta_t &= \exp\left\{hX_0e^{-at} + \frac{\sigma^2h^2}{4a}\left(1-e^{-2at}\right) + \frac{hb}{a}\left(1-e^{-at}\right) + g(t)\right. \\ &+ \left.\int_{\mathbf{R}} (e^z - 1)\int_{\frac{1}{h}}^{\frac{e^{at}}{h}} \lambda(\xi dz)\frac{d\xi}{a\xi}\right\} \end{aligned}$$

tends to the limit

$$\theta_t = \exp\left\{\frac{\sigma^2h^2}{4a} + \frac{hb}{a} + g + \int_{\frac{1}{h}}^{\infty} \lambda_s(\xi dz)d\frac{d\xi}{a\xi}\right\},$$

which exists $\forall B$.

Example 3.4.11. *It is easy to find examples when the real exchange rate in the jointly affine economy is not mean stationary. For example, let us consider extended Vasicek model (without jumps), with $a(t) = a < 0$ constant, but where $b(t)$ is a function of time. Then, in any model such that*

$$\int_0^t hb(s)e^{-a(t-s)}ds \to \infty, \quad t \to \infty,$$

the real exchange rate is not mean stationary. The function $b(t) = kt$, where $k > 0$ is one possible example. Indeed,

$$\exp\left\{\frac{kt}{a} - \frac{k}{a^2}(1 - e^{-at})\right\} \to \infty, \quad t \to \infty. \tag{3.189}$$

Examples above show that if we model the Gaussian factor dynamics arbitrary, so that the condition (3.176) does not hold, we will never be able to find a good fit for the jointly affine factor model in case if PPP holds in this economy. The reason for that is that we create a model which is already at the modeling stage violates the PPP hypothesis, and thus, initially specifies the factor dynamics which would never allow the real exchange rate in the economy to be mean stationary.

Thus, we are naturally interested in the following question: how to model the factor dynamics, exchange rate and domestic and foreign price levels for the real exchange rate to be assymptotically mean stationary?

The examples above encourage us to give some sufficient conditions which ensure that the affine model will produce the real exchange rate which is assymptotically mean stationary. Now we can formulate the following proposition.

Proposition 3.4.12. (Sufficient conditions.) *Assume that*

1. *Assumptions of Proposition 3.4.4 hold.*
2. *The factor dynamics is Gaussian/Compound Poisson as in (3.161), X_0 is deterministic.*
3. *The function $h(t)$ is exponential*

$$h(t) = P(t) + P_{If}(t) - P_I(t) = C_1 e^{Ct}, \quad C \leq 0, C_1 > 0.$$

4. *Functions $\sigma(t)$ and $b(t)$ are of the form*

$$\sum_j e^{\beta_j t} p_j(t), \tag{3.190}$$

where for each j $p_j(t)$ is the polynomial, and $\beta_j < 0$; and $\alpha(t) = \alpha < 0$ is a constant.

5. *There exists the limit*

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} Q(t) + Q_{If}(t) - Q_I(t) = g.$$

6. *The intensity $\lambda(dy)$ does not depend on t and the following condition is satisfied:*

$$\int_{|y|\geq\frac{1}{h}} \ln|y|\lambda(dy) < \infty. \tag{3.191}$$

Then the real exchange rate is asymptotically mean stationary.

Proof. Since the assumptions of Proposition 3.4.4 hold, the real exchange rate is affine and takes the form

$$r(t, X_t) = e^{h(t)X_t+g(t)}$$

for some functions $h(t)$ and $g(t)$. We can find the expected value of the real exchange rate according to (3.183)

$$\begin{aligned}\theta(t) \quad &= \quad \exp\left\{h(0)X_0e^{\int_0^t q(u)du} + \frac{1}{2}\int_0^t e^{2\int_s^t q(u)du}h^2(s)\sigma^2(s)\,ds\right.\\ &+ \left.\int_0^t h(s)b(s)e^{\int_s^t q(u)du}ds + g(t) + \int_0^t\int_R \lambda(s,dy)\left\{e^{yh(t)e^{\int_s^t q(u)du}} - 1\right\}ds\right\}\end{aligned} \tag{3.192}$$

Using the assumptions of the Proposition we find that

$$q(t) = -a(t) + \frac{h'(t)}{h(t)} = -(a-C).$$

It follows then that

$$e^{2\int_s^t q(u)du}h^2(s)\sigma^2(s) = C_1^2e^{-2(a-C)(t-s)}e^{2Cs}\sigma^2(s) = C_1^2e^{-2a(t-s)+2Ct}\sigma^2(s) \tag{3.193}$$

and also that

$$h(s)b(s)e^{\int_s^t q(u)du} = C_1b(s)e^{-(a-C)(t-s)+Ct}. \tag{3.194}$$

Integrating (3.194), we obtain that

$$\int_0^t h(s)b(s)e^{\int_s^t q(u)du}ds = C_1e^{Ct}P(t), \tag{3.195}$$

where $P(t)$ is a function of type (3.190). We notice that

$$C_1e^{Ct}P(t) \to 0, t \to \infty \tag{3.196}$$

since $C < 0$ and $P(t) \to 0$ as $t \to \infty$. The same holds for (3.193). We can rewrite the double integral in (3.192) as

$$\int_0^t \int_R \lambda(dy) \left\{ e^{yhe^{-(a-C)(t-s)}} - 1 \right\} ds = \int_{\mathbf{R}} (e^z - 1) \int_{\frac{1}{h}}^{\frac{e^{(a-C)t}}{h}} \lambda(\xi dz) \frac{d\xi}{(a-C)\xi}. \tag{3.197}$$

If the following condition is satisfied

$$\int_{|y| \geq \frac{1}{h}} \ln |y| \lambda(dy) < \infty, \tag{3.198}$$

then there exists the limit $\forall B$

$$\zeta(B) = \int_{\frac{1}{h}}^{\infty} \lambda(\xi B) d\frac{d\xi}{\xi}$$

and thus ζ is a measure. Remembering that there exists the limit $\lim_{t\to\infty} g(t) = g$, we can easily see that as $t \to \infty$ there also exists a finite limit for the whole expression (3.192), which means that the real exchange rate is asymptotically mean stationary. ■

Mean stationarity of the real exchange rate CIR models.

Assume now that the factor dynamics follow the square root, CIR process

$$dX_t = (b - aX_t)dt + \sigma\sqrt{X_t}dW_t, \quad b \geq \frac{1}{2}\sigma^2. \tag{3.199}$$

We assume that a, b and σ are positive constants. We also assume that $\sigma^2 \leq 2a$, since this condition guarantees the positivity of the factor and thus of the volatility of the factor process. Assume as before that the exchange rate is of affine form

$$r(t, X_t) = e^{hX_t + g(t)}, \tag{3.200}$$

where

$$h = P(t) + P_{If}(t) - P_I(t)$$

does not depend on t. Using characteristic functions it is possible to prove the following proposition.

Proposition 3.4.13. *Assume that the factor price process has dynamics (3.199) and the real exchange rate is affine and of the form (3.200). Then the necessary condition for asymptotic mean stationarity of the real exchange rate is that*

$$P(t) + P_{If}(t) - P_I(t) = h > 0. \tag{3.201}$$

Proof. As in the previous example we note that

$$\theta_t = \mathbb{E}^P\left[r(t, X_t)\right] = \mathbb{E}^P\left[e^{hX_t + g(t)}\right] = \mathbb{E}^P\left[e^{hX_t}\right] e^{g(t)}. \tag{3.202}$$

We find that the expectation $\xi_t(h) = \mathbb{E}^P\left[e^{hX_t}\right]$. We start by computing

$$f(t, h, x) = \mathbb{E}^P\left[e^{hX_T} \middle| \mathcal{F}_t\right]. \tag{3.203}$$

Applying Itô formula to $f(t, h, X_t)$ we obtain

$$\frac{\partial f}{\partial t} + (a - bX_t)h\frac{\partial f}{\partial x} + \frac{1}{2}h^2\sigma^2\frac{\partial^2 f}{\partial x^2} = 0. \tag{3.204}$$

We note that $f(x, u, T) = e^{hx}$ and guess the solution in the form

$$f(x, u, t) = \exp C(T - t) + hD(T - t) + iux. \tag{3.205}$$

Substituting this solution to (3.204) we find $C(s)$ and $D(s)$ as solutions to the following differential equations

$$\begin{cases} D'(s) &= \frac{1}{2}h^2\sigma^2 D^2(s) + (h^2\sigma^2 - hb)D(s) + \frac{1}{2}h^2\sigma^2 - hb \\ D(0) &= 0, \end{cases}$$

$$\begin{cases} C'(s) &= D(s)ah + ah \\ C(0) &= 0. \end{cases}$$

We compute the characteristic function as

$$\begin{aligned} f(t, h, x) &= \mathbb{E}^P\left[\mathbb{E}^P\left[e^{hX_T} \middle| \mathcal{F}_t\right] \middle| \right] = \exp\left\{C(t) + hD(t) + hx\right\} \\ &= \left|1 + \frac{h\sigma^2}{2b - h\sigma^2}\frac{1}{e^{hbt}}\right|^{-\frac{2a}{h\sigma^2}} \left|\frac{2b}{2b - h\sigma^2}\right|^{\frac{2a}{h\sigma^2}} \exp\left\{\frac{1}{\left(1 - \frac{h\sigma^2}{2b}\right)e^{hbt} + \frac{h\sigma^2}{2b}}\right\}. \end{aligned} \tag{3.206}$$

We recall now that $b > 0$. Then if $h > 0$ we have that $e^{-hbt} \to 0$ if $t \to \infty$ and thus $\xi_t(h) \to C$, where C is a constant. If, however, $h < 0$, then $e^{-hbt} \to \infty$ if $t \to \infty$ and $\xi_t(h)$ converges to some constant limit only if $a < 0$, which contradicts our assumptions. Thus the following condition

$$P(t) + P_{If}(t) - P_I(t) = h > 0$$

is the necessary condition for the real exchange rate to be mean stationary. ∎

Pricing real exchange derivatives

As an application of the framework described earlier, we suggest a method to price **real exchange derivatives** in the jointly affine factor economy that would allow us to hedge both inflation and exchange rate risks simultaneously. Consider the following situation: Domestic company XX produces and exports toy pigs. The production of toy pigs requires domestic materials that are costly to import due to high transportation costs. Assume now that due to a macroeconomic shock the real exchange rate appreciates and our goods become less competitive than the foreign goods. It also becomes more expensive to manufacture toy pigs due to increased input costs, and thus foreign toy pig producers become more competitive and demand switches to the foreign country. Thus, there is a possibility that the domestic firms will not make sufficient profits to enable them to continue operations. The evidence shows that it can take up to five years for the real exchange rate to return to its equilibrium level. This means that to become more competitive on the market, domestic producers might want to reduce their manufacturing costs. If foreign imports are expensive and domestic producers are forced to continue using domestic inputs, they cannot reduce manufacturing costs and consumer prices. This means that it might be a good idea to hedge this type of risk (real appreciation risk), which basically includes two types of risk - inflation and nominal exchange rate risk. To do this we would like to introduce the following derivative - **European real exchange option**. Similar derivatives have previously been considered in the international finance literature, but to the best of our knowledge, no pricing models based on an arbitrage argument have been developed.

Since we are considering the prices of inputs needed for production we will consider the producer-based price index **Production Price Index** instead of the consumer-based price indices while using the technique employed in the previous section. To convert the option payoff into domestic currency we assume that each option is written on one PPI unit. Thus, at maturity time T the holder of the real exchange option gives us the maximum of the two amounts: the difference between the cost of the K PPI units at home and the cost of the PPI^f unit abroad and zero (due to PPP we could essentially take $K = 1$). Thus, the payoff of the option is

$$\Phi(S(T), I^f(T), I(T)) = \max[KI(T) - S(T)I^f(T), 0]. \tag{3.207}$$

Note, that since the option is written on PPI units, both $KI(T)$ and $S(T)I^f(T)$ are expressed in the domestic currency. To hedge its "competitiveness risk" the firm can buy the European real exchange option. Assume that the toy pig producer estimated the cost of production inputs as M domestic PPI units. In the equilibrium (according to the PPP theory) this will be exactly equal to the price of M foreign PPI units. If the domestic goods appreciate, the price of one domestic PPI unit will be higher than the price of one foreign PPI^f unit and thus domestic production becomes more expensive in real terms by the approximate difference $M(I(T) - S(T)I^f(T))$. Thus, if the domestic firm had bought the "real exchange" option, although its production becomes more expensive than that of foreign firms by this amount, it can now decrease the final consumer price of its goods (to be more competitive on the market) since the option payoff will compensate for the profit loss due to the price decrease.

Thus, according to the risk neutral valuation, the price of the derivative can be written as

$$C(t,T) = \mathbb{E}_t^{Q^d}\left[e^{-\int_t^T R^d(s)ds}\max[KI(T) - S(T)I^f(T), 0]\right]. \tag{3.208}$$

To price this derivative we will first solve the more general problem. Assume that we have two traded assets in the economy, and we need to price the *exchange option* with the payoff

$$\mathcal{X} = \max[S_1(T) - S_2(T), 0]. \tag{3.209}$$

Proposition 3.4.14. *Assume that S_1 and S_2 are traded assets. The price of the exchange option with payoff (3.209) is given by*

$$\Pi(0,\mathcal{X}) = S_1(0)Q^{S1}(S_1(T) \geq S_2(T)) - S_2(0)Q^{S2}(S_1(T) \geq S_2(T)), \tag{3.210}$$

where Q^{S1} denotes the martingale measure for the numeraire process $S_1(t)$ and Q^{S2} denotes the martingale measure for the numeraire process $S_2(t)$.

Proof. We write the payoff of the option (3.209) as

$$\mathcal{X} = [S_1(T) - S_2(T)] \cdot I\{S_1(T) \geq S_2(T)\}, \tag{3.211}$$

where I is the indicator function

$$I\{S_1(T) \geq S_2(T)\} = \begin{cases} 1, & \text{if} \quad S_1(T) \geq S_2(T), \\ 0, & \text{if} \quad S_1(T) < S_2(T). \end{cases}$$

According to the risk neurtal valuation we can find the price of the exchange option as (Q is the martingale measure, taking the bank account as numeraire)

$$\begin{aligned} \Pi(0,\mathcal{X}) &= \mathbb{E}_t^Q\left[e^{-\int_t^T R(s)ds}[S_1(T) - S_2(T)]I\{S_1(T) \geq S_2(T)\}\right] \\ &= \mathbb{E}_t^Q\left[e^{-\int_t^T R(s)ds}S_1(T)I\{S_1(T) \geq S_2(T)\}\right] \\ &- \mathbb{E}_t^Q\left[e^{-\int_t^T R(s)ds}S_2(T)I\{S_1(T) \geq S_2(T)\}\right]. \end{aligned} \tag{3.212}$$

We rewrite (3.212) using the change of measure technique (see, for example, Geman, Karoui, and Rochet (1995), Schroder (1999) for details). We use the measure Q^{S1} taking S_1 as numeraire in the first term, and use the measure Q^{S2} taking S_2 as numeraire in the second term to rewrite (3.212) as the general exchange option pricing formula (3.210). ■

Using Proposition 3.4.14 we can easily find the price of the real exchange rate option (3.207).

Proposition 3.4.15. *Assume that I and I_f are domestic and foreign price indices, and S is the exchange rate. The price of the real exchange option with payoff (3.207) is given by*

$$\Pi(t,\Phi) = KP_{TIPS}(t,T)Q^{TIPS}\{r(T) \leq K\} - S(t)P^f_{TIPS}(t,T)\tilde{Q}^{TIPS}\{r(T) \leq K\}, \tag{3.213}$$

where Q^{TIPS} denotes the martingale measure for the numeraire process $P_{TIPS}(t,T)$ and $\tilde{Q}^{TIPS}$ denotes the martingale measure for the numeraire process

$$S(t)P^f_{TIPS}(t,T),$$

and $r(T)$ is the real exchange rate at maturity date T.

Proof. We can rewrite payoff (3.207) as follows:

$$\begin{aligned}\Phi(S(T), I^f(T), I(T)) &= \max[KI(T) - S(T)I^f(T), 0] \\ &= \max[KI(T)P_r(T,T) - S(T)I^f(T)P^f_r(T,T), 0].\end{aligned} \tag{3.214}$$

We can now use the domestic $P_{TIPS}(t,T) = I(t)p_r(t,T)$ bond as a numeraire. Denote by Q^{TIPS} the martingale measure for the numeraire process $P_{TIPS}(t,T)$. Transform the foreign traded asset $P^f_{TIPS}(t,T)$ into domestic terms

$$\tilde{P}^f_{TIPS}(t,T) = S(t)P^f_{TIPS}(t,T). \tag{3.215}$$

We can now interpret $\tilde{P}^f_{TIPS}(t,T)$ as the price of the domestically traded asset; thus, we can take this asset as a numeraire. Denote by $\tilde{Q}^{TIPS}$ the martingale measure resulting from this change of numeraire. We can now apply the results from Proposition 3.4.14 to obtain the pricing formula (3.213). ■

To obtain a formula that can be used in practice, we need to compute the probabilities above. Using the standard results, we guess that we will be able to compute the price in the closed form if the factor price process does not exhibit jumps and the volatility is deterministic. Thus we make the following assumption.

Assumption 3.4.16. *Assume that the volatility of the factor process X_t is deterministic, that is, $\sigma(t, X_t) = \sigma(t)$ and the process does not exhibit jumps*[9].

Consider the new process $Z(t)$ defined as

$$Z(t) = \frac{S(t)I^f(t)P_r^f(t,T)}{I(t)P_r(t,T)}. \tag{3.216}$$

The process $Z(t)$ is a martingale under the Q^{TIPS} measure; thus, its price process has zero drift. Under the assumptions of Proposition 3.4.4 we infer that the exchange rate, domestic and foreign PPI^f indices and domestic and foreign real bonds and real exchange rate $r(t, X_t)$ are affine in the factors. Assuming that the domestic and foreign PPI indices are of the form (3.157) and (3.158) correspondingly, we can write the $Z(t)$ price process as

$$\begin{aligned} dZ(t) &= Z(t)\left\{P(t) + P_{If}(t) - P_I(t) - \bar{D}(t,T) + \bar{B}(t,T)\right\}\sigma(t)dW^{TIPS}(t) \\ &= Z(t)\sigma^Z(t,T)dW^{TIPS}(t). \end{aligned} \tag{3.217}$$

In this case, the solution to (3.217) can be found as

$$Z(t) = \frac{S(t)P^f_{TIPS}(t,T)}{P_{TIPS}(t,T)} \exp\left\{-\frac{1}{2}\int_t^T \|\sigma^Z\|^2(s)ds + \int_t^T \sigma^Z(s)dW^{TIPS}(s)\right\}. \tag{3.218}$$

Using the fact that the stochastic integral has a Gaussian distribution with zero mean and variance

$$\Sigma^2(T) = \int_t^T \|\sigma^Z\|^2(s)ds$$

we obtain that the probability can be found as

$$Q^{TIPS}\{r(T) \leq K\} = N(d_1), \tag{3.219}$$

where

$$d_1 = \frac{\ln \frac{KP_{TIPS}(t,T)}{S(t)P^f_{TIPS}(t,T)} - \frac{1}{2}\Sigma^2(T)}{\sqrt{\Sigma^2(T)}}. \tag{3.220}$$

[9]It is also possible to obtain the closed-form solution in case the factor price process follows CIR square root dynamics of the type

$$dx = (\alpha - \beta x)dt + \sigma\sqrt{x}dW.$$

Consider a new process $\tilde{Z}(t)$ defined as

$$\tilde{Z}(t) = \frac{I(t)P_r(t,T)}{S(t)I^f(t)P_r^f(t,T)}. \tag{3.221}$$

Thus, our task would be to find the probability

$$\tilde{Q}^{TIPS}(r(T) \leq K) = \tilde{Q}^{TIPS}(\tilde{Z}(T) \leq K). \tag{3.222}$$

The process $\tilde{Z}(t)$ is a martingale under the $\tilde{Q}^{TIPS}$ measure and thus its price process has zero drift. Under the assumptions of Proposition 3.4.4 we infer that the exchange rate, domestic and foreign PPI indices and domestic and foreign real bonds and real exchange rate $r(t, X_t)$ are affine in factors. Assuming that the domestic and foreign PPI indices are of the form (3.157) and (3.158) correspondingly, we can write the $Z(t)$ price process as

$$\begin{aligned} d\tilde{Z}(t) &= -\tilde{Z}(t)\left\{P(t) + P_{I^f}(t) - P_I(t) - \bar{D}(t,T) + \bar{B}(t,T)\right\}\sigma(t)d\tilde{W}^{TIPS}(t) \\ &= -\tilde{Z}(t)\sigma^Z(t,T)d\tilde{W}^{TIPS}(t). \end{aligned}$$

Now we can calculate the probability analogously to the first part and finally we obtain that

$$\tilde{Q}^{TIPS}(r(T) \leq K) = N(d_2), \tag{3.223}$$

where $d_2 = d_1 - \sqrt{\Sigma^2(T)}$. Thus, we obtain the following explicit formula for the real exchange option

$$KP_{TIPS}(t,T)N(d_1) - S(t)P^f_{TIPS}(t,T)N(d_2), \tag{3.224}$$

where d_1 and d_2 are determined as above.

3.5 Estimation.

3.5.1 Methodology.

A great deal of work has been done concerning the estimation of latent factors driving a single country nominal term structure. Little, however, has been done regarding the estimation of two-country ATSMs.

If we would be able to observe yields of some maturities exactly (without measurement errors), then we would be able to infer the factor dynamics by inverting the yield curve. The result will of course depend on the choice of maturities made. Other studies assume that the yields cannot be measured exactly, thus, we cannot infer the factor dynamics by inverting the yield curve any longer. Instead, we *estimate* the factors from the model which now can be seen as a combination of a term structure model and the model for the measurement errors.

To estimate the real-nominal two-country term structure model we use Kalman filter methodology used for the estimation of unobservable state variable dynamics in one-country models. The Kalman filter recursively computes the expectation of the unobservable state vector given the noisy observations of the state variable. Based on the assumed factor dynamics and the term structures observed in the past, we predict the value of the state vector in the next period, and update it given the observable new term structure on the market. The Kalman filter gives the linear prediction based on both sources of information. It creates the predicted estimate by weighting the information about the term structure and the linear prediction so that the source with the higher covariance matrix gets lower weight. As a by-product, the filter provides the fitting errors and their covariance matrices allowing us to construct the quasi-maximum likelihood. The values of the parameters are estimated by maximizing the likelihood function.

More formally, in a one-country case, affine models are viewed as state space systems with an *observation equation* which links observables (yields with time to maturity $\tau = T - t$) to the state vector X_t

$$Y_t(\tau) = A(\tau) + B(\tau)X_t + \epsilon_t(\tau), \tag{3.225}$$

(where the error ϵ_t comes from the fact that the yields are assumed to be imperfectly observed due to, for example, measurement errors and interpolation), and *state equations* (dynamics of the state variables)

$$dX_t = \mu(X_t)dt + \Sigma\sqrt{X_t}dW_t. \tag{3.226}$$

The linear function $A(\tau) + B(\tau)X_t$ gives the theoretical value of the yield of maturity τ given the vector X_t. The standard Ricatti equations provide cross-equation restrictions.

Unfortunately, the exact filtering of non-linear state space models is computationally infeasible (see Lund (1997)), so we exploit approximate filtering techniques as described, for example, in Lund (1997). Instead, most of the studies employ Kalman filter combined with quasi maximum likelihood estimation (QMLE). To estimate the model we will use Iterative Extended Kalman Filter algorithm (IEKF), which allows us to estimate the models of the following type:

$$Y_k = Z(X_k, \theta) + v_k, \tag{3.227}$$

$$X_{k+1} = F_{k0}(\theta) + F_{k1}(\theta)X_k + u_k, \tag{3.228}$$

where θ is a vector of parameters, v_k is allowed to have an arbitrary zero mean distribution $D(0, S_k(\theta))$, and u_k is allowed to have the same zero mean distribution $D(0, Q_k(\theta))$, but with a different covariance matrix, $k = 1, \dots, N$. As usual, the recursions of the IEKF method consist of prediction and update steps.

Although it is impossible to find the exact distribution of $X(t + h)$ conditional on $X(t)$ (except for special cases) as is shown by Duan and Simonato (1999), it is possible to compute the conditional mean and variance of the unobserved state variables over a discrete time interval of length h, $m(X_t, \theta, h) = \mathbb{E}\left[X(t+h)|\, X(t)\right]$ and $Q(X_t, \theta, h) = \mathrm{Var}\left[X(t+h)|\, X(t)\right]$, as affine functions of the factors. In this case the Kalman filter can still be applied to obtain approximations for the first and second moments of the model. See Appendix 3.8.1 for the description of the exact Kalman filter procedure.

Among other attempts to estimate diffusion processes are Brandt and Santa-Clara (2002) where they apply simulated likelihood estimation technique to estimation of exchange rate dynamics in incomplete markets and a nonparametric approach by Gallant and Tauchen (1992).

3.5.2 The model specification

Most of the studies (see, for example, Han and Hammond (2003), Dewachter and Maes (2001)), employ three-factor models for estimating the unobservable factor dynamics. Usually, three-factor models fit domestic and foreign yields quite well, whereas four-factor models exhibit too many spikes. Dewachter and Maes (2001) give strong evidence that the three factor international model possesses high explanatory power and do fairly good job in describing the joint market dynamics through time and maturity. The three-factor models are usually constructed with two local factors separately influencing domestic and foreign term structures (domestic and foreign factors), and one common factor influencing both term structures at the same time. We will also use three-factor specification for the international bond market modeling.

In the more general model, where we include domestic and foreign inflation-protected bonds in the international market, we will clearly need more factors. We assume that as in the nominal two-country model the domestic term structure is influenced by the domestic local factor X_1 and common factor X_2, and the foreign term structure is influenced by the foreign local factor X_3 and common factor X_2. Moreover, it is natural to assume that there exists a "real" domestic factor, X_4, influencing the domestic real term structure only, and "real" foreign factor, X_5, influencing the real foreign term structure only. We also assume that the domestic real term structure is influenced by the domestic local factor X_1, and the foreign real term structure is influenced by the common factor X_2 and the foreign local factor X_3. We consider two different five-factor models. First of all, we consider a five-factor term structure model with the factors following simplified CIR dynamics (Model I).

$$\begin{aligned} dX_1(t) &= a_1(b_1 - X_1(t))dt + \sigma_{11}\sqrt{X_1(t)}dW_1, \\ dX_2(t) &= a_2(b_2 - X_2(t))dt + \sigma_{22}\sqrt{X_2(t)}dW_2, \\ dX_3(t) &= a_3(b_3 - X_3(t))dt + \sigma_{33}\sqrt{X_3(t)}dW_3, \\ dX_4(t) &= a_4(b_4 - X_4(t))dt + \sigma_{44}\sqrt{X_4(t)}dW_4, \\ dX_5(t) &= a_5(b_5 - X_5(t))dt + \sigma_{55}\sqrt{X_5(t)}dW_5, \end{aligned} \tag{3.229}$$

where W_i are the independent Wiener processes. Following the assumptions above we model that

1. the US short rate depends on US local factor X_1 and common factor X_2

$$R^d(X) = R_1^d X_1 + R_2^d X_2,$$

2. the UK short rate depends on UK local factor X_3 and common factor X_2

$$R^f(X) = R_1^f X_2 + R_2^f X_3,$$

3. the US real short rate depends on US local factor X_3 and US real factor X_4

$$R^{dr}(X) = R_1^{dr} X_1 + R_2^{dr} X_4,$$

4. the UK real short rate depends on UK local factor X_3, common factor X_2 and UK real factor X_5

$$R^{fr}(X) = R_1^{fr} X_2 + R_2^{fr} X_3 + R_3^{fr} X_5.$$

Moreover, we need to estimate an additional ten parameters: λ_1, λ_2 - domestic market prices of risk, λ_1^f, λ_2^f, λ_3^f - foreign market prices of risk, λ_1^r, λ_4^r - domestic real market prices of risk, λ_2^{rf}, λ_3^{rf} and λ_5^{rf} - foreign real market prices of risk. All the remaining parameters for market prices of risk are restricted to zero.

The shortcoming of this model is that all the factors are uncorrelated. Since the data show that some of the US real yields are negatively correlated with the nominal US and UK yields, we might expect the US real yields to be governed by factors, that are negatively correlated with the set of all other factors. We thus introduce and estimate Model II, where four factors follow CIR dynamics and one factor can become negative and is allowed to be negatively (or positively) correlated with the other factors.

$$
\begin{aligned}
dX_1(t) &= a_1(b_1 - X_1(t))dt + \sigma_{11}\sqrt{X_1(t)}dW_1, \\
dX_2(t) &= a_2(b_2 - X_2(t))dt + \sigma_{22}\sqrt{X_2(t)}dW_2, \\
dX_3(t) &= a_3(b_3 - X_3(t))dt + \sigma_{33}\sqrt{X_3(t)}dW_3, \\
dX_4(t) &= -a_4X_4(t)dt + \rho_{41}\sqrt{X_1(t)}dW_1 + \rho_{43}\sqrt{X_3(t)}dW_3 \\
&+ \sqrt{1 + X_1(t) + X_3(t)}dW_4, \\
dX_5(t) &= a_5(b_5 - X_5(t))dt + \sigma_{55}\sqrt{X_5(t)}dW_5,
\end{aligned} \tag{3.230}
$$

where W_i are the independent Wiener processes. We note that factors $X_1(t)$ and $X_4(t)$ as well as $X_3(t)$ and $X_4(t)$ can be positively or negatively correlated depending on the sign of ρ_{41} and ρ_{43}. If ρ_{41}, $\rho_{43} < 0$ then $X_1(t)$ and $X_3(t)$ are negatively correlated with $X_4(t)$. Moreover, we note that, as compared to the previous model, factor $X_4(t)$ can take negative values. Following the assumptions above we model that

1. the US short rate depends on US local factor X_1 and common factor X_2
$$R^d(X) = R_1^d X_1 + R_2^d X_2,$$
2. the UK short rate depends on UK local factor X_3 and common factor X_2
$$R^f(X) = R_1^f X_2 + R_2^f X_3,$$
3. the US real short rate depends on US local factor X_1 and UK local factor X_3 as well US real factor X_4.
$$R^{dr}(X) = R_1^{dr} X_1 + R_2^{dr} X_3 + R_2^{dr} X_4.$$
Since we modeled the US real factor depending on the factor X_4, we note that the UK real short rate can be negatively correlated with the US nominal short rate and the UK nominal short rate.
4. the foreign short rate depends on foreign local factor X_3, common factor X_2 and foreign real factor X_5
$$R^{fr}(X) = R_1^{fr} X_2 + R_2^{fr} X_3 + R_3^{fr} X_5.$$

As in Model I, we estimate an additional ten parameters: λ_1, λ_2 - domestic market prices of risk, λ_1^f, λ_2^f, λ_3^f - foreign market prices of risk, λ_1^r, λ_4^r - domestic real market prices of risk, λ_2^{rf}, λ_3^{rf} and λ_5^{rf} - foreign real market prices of risk. All the remaining parameters for market prices of risk are restricted to zero.

3.5.3 Data

We use UK-US data to estimate the model, and for estimation we need data consisting of zero-coupon bond yields. Most studies use zero-coupon bond yields estimated from government coupon paying bonds. Recently, some studies started using yields as implied by LIBOR and swap rates. We use UK nominal yields extracted from UK government bonds for the set of maturities 0.5, 1, 2, 4, 5 and 6 years and US nominal yields extracted from US government bonds for the set of maturities 0.5, 1, 2, 3, 4 and 5 years[10]. The inflation-linked UK and US bond markets allow us to estimate the real rates which would imply the observed bond prices. We include in our sample three UK real yields with maturities 2, 3 and 4 years and three US real yields with maturities 5, 7 and 10 years. In total we construct 17 series of nominal and real bond yields, where we use thirty monthly observations for the period 25 November 2002 to 25 January 2006. Since we were looking for a period with observations for all the 17 yields we had to restrict ourselves to quite a short panel.

Figure 3.1 shows the time-series evolution of one- and five-year US and UK nominal yields. In the US, both yields rose between 1998 and 2000, and then decreased until the end of 2003. From 2003 they increased slowly again to their 2000 levels. Nominal UK yields decrease more rapidly over the entire interval from 1998 to 2006, except for 1998, when they dropped significantly and then rose to the previous level.

Figure 3.2 shows the time-series evolution of five- and seven-year US real yields and three- and four-year UK real yields. The five-year US real yield increases slightly from 2003 to 2004 from the level of approximately 1.5% to 2%. The seven-year US real yield decreases slightly during the same period and is more volatile. The real three- and four-year UK yields increasing between 1998 and 2001 and drop significantly from 2.5% to 1.5% with the lowest yield value being even less than 1%.

From the observed yields we can derive the average term structures for both countries (Figure 3.3). The average UK and US term structures are both upward sloping in the period of interest.

Besides the time-series observations of the real and nominal yields, we observe the time-series evolution of the exchange rate between US and UK (Figure 3.4). The US dollar appreciates during the period from 1998 to 2001, when the exchange

[10]We use the data on the nominal and real UK yields provided on the webpage of Bank of England http://www.bankofengland.co.uk/statistics/yieldcurve/index.htm. US nominal and real yields data is taken from the United States Department of the Treasury webpage http://www.treasury.gov/

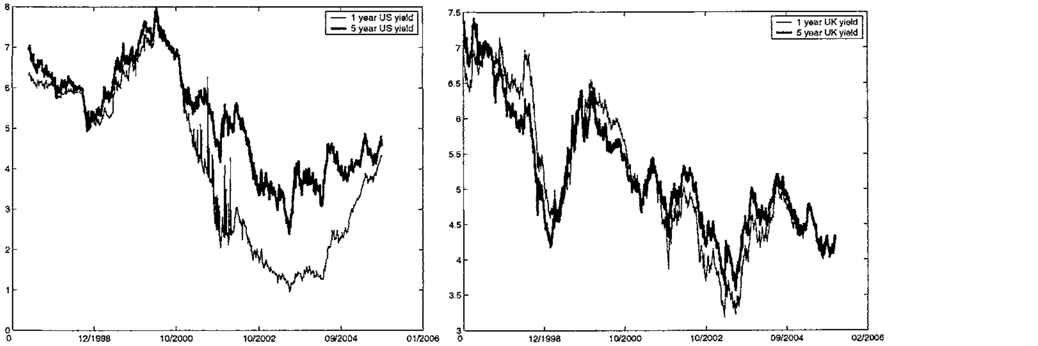

Figure 3.1: Time-series graphs of the one- and five-year US and UK nominal yields, July 1998 - February 2006.

rate is approximately 1.4 dollars per UK pound. In 2001 the US dollar begins depreciating and the exchange rate reaches the level of 1.9 dollars per UK pound.

The TIPS are indexed to the non-seasonally adjusted US City Average All Items Consumer Price Index for All Urban Consumers (CPI-U). We have monthly observations for CPI-U from January 1950 to February 2006. The CPI index is reported with a two- month lag, which allows us to calculate an approximate, but not exact real return. The time-series observations of the CPI-U index are indicated in Figure 3.4. It is easy to see that the price level increased steadily from November 2002 to July 2005. The base year for the CPI-U series is 1950.

We also report correlations between different yields: correlations between domestic yields, correlations between domestic and foreign yields, and correlations between foreign yields. The correlation table (see Table 3.1) shows that domestic nominal yields are highly positively correlated, as are foreign nominal yields, with an average correlation coefficient of 0.85. The within-country real yield correlations in both countries are even higher, up to 0.98. The cross-country correlations between the nominal domestic and the nominal foreign yields are also significantly high and positive (although the correlation coefficients at around 0.6 are lower than on separate markets). The cross-country correlations between the real yields are also positive, although the correlation coefficients are not high (around 0.2 on average). We also note that although UK real yields are highly positively correlated with UK nominal yields, some of the US real yields are negatively correlated with the US nominal yields (looking at yields of different maturities, however). US nominal and UK real yields are positively correlated while some of the UK nominal yields are negatively correlated with US real yields. We also

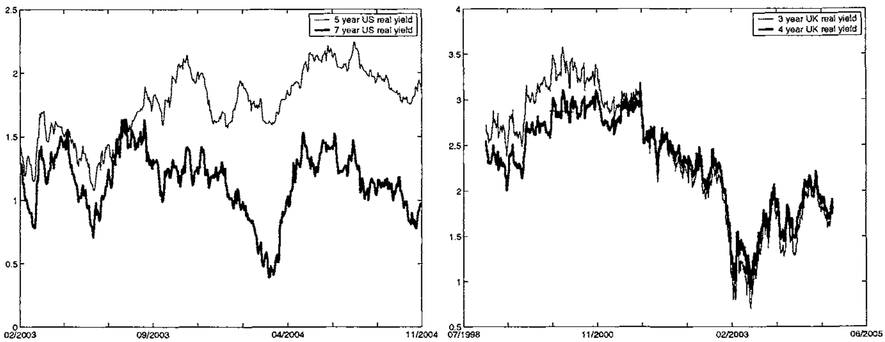

Figure 3.2: Time-series graphs of the five- and seven- year US real yields, February 2003-November 2004, and 3 and 4 year UK real yields, July 1998 - February 2006.

observe negative correlation coefficients of US real yields with US CPI index and the exchange rate.

3.5.4 Estimation results

Nominal two-country model

Parameters of the three-factor model consisting of the first three factors of (3.229) have been estimated using an approximate Kalman filter and quasi-maximum likelihood. The results of the estimation are shown in the Table 3.2.

The estimation shows that the foreign factor is slowly mean reverting while the common and domestic local factors revert more quickly to the mean. We can also see that local factors are more volatile than the common factor. Parameters b_i can be seen as proxies for the instantaneous interest rates, and our estimates seem to be of reasonable magnitude. In the brackets we have the standard deviations, which are quite low and imply that our estimates are significant. The actual and fitted two- and four-year US and UK yields are plotted in Figure 3.6. We note that the three-factor model demonstrates quite a good fit, but it appears that the fit for the UK bond market is not as good as for the US bond market.

In this section we estimate the factor dynamics only on the basis of the observations of UK and US nominal yields, without taking into account the exchange rate between the two countries. However, once we have fitted the yields, and with the model in mind, we can draw some implications concerning the volatilities of the exchange rate process (3.26), since we know that UK and US drifts are connected

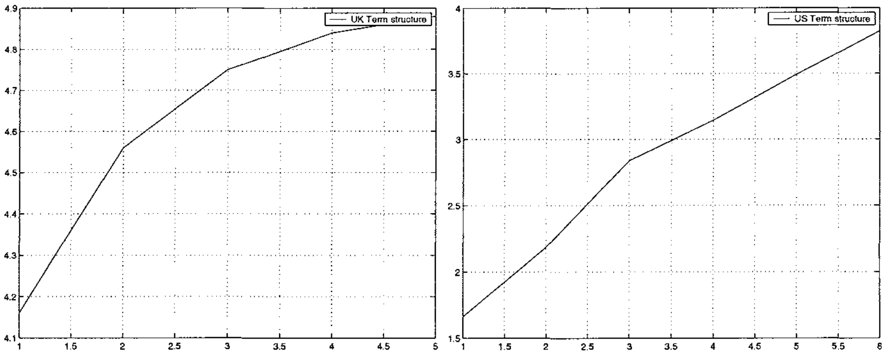

Figure 3.3: UK and US average term structures

via (3.40). Thus, the exchange rate volatility vector can be estimated as

$$\sigma_S = (0,3603; \quad 0,4186; \quad 2,193). \tag{3.231}$$

We will compare the implied volatilities with the estimates for the case when the exchange rate is included in the estimation. We note also that we restricted domestic market price λ_3 to be equal to zero to reduce the number of parameters to estimate.

Including the exchange rate in the estimation procedure.

In this section we include the exchange rate dynamics in the estimation. Unfortunately, we cannot directly include it in the set of observables since the exchange rate is not a specified function of factors. To overcome this complication, Dewachter and Maes (2001) derive the discrete time implications of the continuous time dynamics. They show that the exchange rate return over a discrete time interval and the conditional variance are affine functions of the factors. This allows us to add the observed exchange rate changes to the set of the observed yields. Han and Hammond (2003) suggest incorporating the exchange rate dynamics in the estimation procedure considering it as another state variable since although the exchange rate is observable it differs from observables in the model in that it is not a specified function of state variables. Another attempt has been made by Mosburger and Schneider (2005), who study the performance of the ATSM driven by a set of state, local and common variables. They estimate international ATSMs on the UK-US data taking into account the joint transition density of yields and exchange rate without assuming normality.

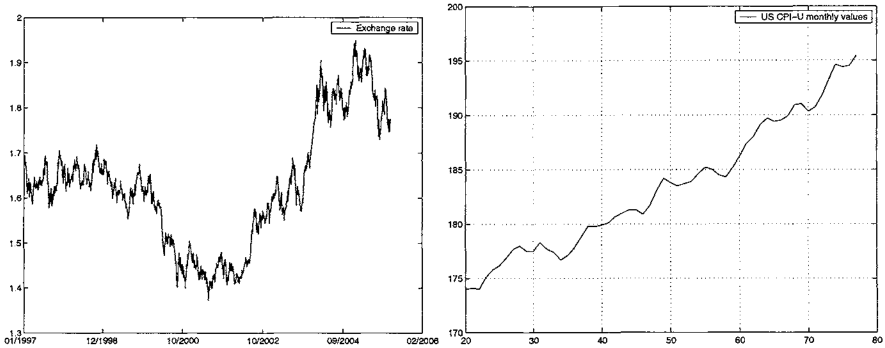

Figure 3.4: Exchange rate expressed in USD per UK Pound and Monthly CPI-U Index Levels, November 2002 to July 2005.

In contrast to the studies mentioned above, we assume that the exchange rate has an affine functional form as in (3.30). The affine exchange rate (and real exchange rate) would allow us to simplify the estimation procedure since, as compared to Han and Hammond (2003), we can include the exchange rate as one of the observable variables (the exact functional form is known). Thus, we can include nominal (and real) exchange rates directly in the set of observables. According to Proposition 3.2.15, if we specify affine factor dynamics under the domestic martingale measure and an affine domestic short rate exogenously, and then model the affine exchange rate dynamics freely, this will ensure affine factor dynamics under the foreign martingale measure as well as an affine foreign short rate. Thus, in this case

$$\ln S(t, X_t) = PX_t + Q, \tag{3.232}$$

where we parameterize the exchange rate by P_1, P_2, P_3 and Q. Given the parameters of the factor dynamics under Q^d and exchange rate parameters, we find foreign short rate as in (3.50) and the foreign market price of risk from (3.122). Thus, we estimate a slightly different set of parameters, which includes P_1, P_2, P_3 and Q and the domestic market price of risk parameters, but excludes the foreign market price of risk and foreign short rate parameter since they will be automatically determined by the estimated set. Further, we assume that the nominal exchange rate is observed imperfectly, so in addition to the previous section we estimate $s_e x$, the error in the nominal exchange rate observation.

Parameters of the three-factor model consisting of the first three factors of (3.229) have been estimated using an approximate Kalman filter and quasi-maximum likelihood. The results of the estimation are presented in Table 3.3. Including the US-UK exchange rate does not change the coefficients of the model very much,

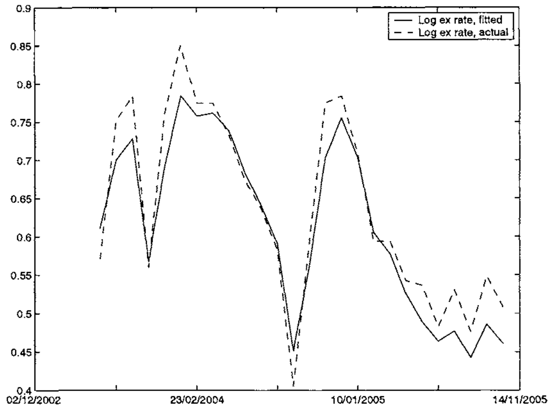

Figure 3.5: Fitted and actual log exchange rate dynamics, November 2002 to July 2005.

except that the common factor has an even higher mean reverting speed and both US local and common factors become much more volatile. The estimated errors are of the same order, and we are also able to estimate the measurement error for the exchange rate, which at around 0,000652 is higher than for yields. Given the vector

$$P = (0,527; \quad 0,705; \quad 1,051) \tag{3.233}$$

we can find the volatilities of the exchange rate according to $\sigma_S = P\sigma$, hence,

$$\sigma_S = (0,1648; \quad 0,1197; \quad 0,2417). \tag{3.234}$$

We note that these estimates are quite different from those that do not include the exchange rate; we observe much smaller volatilities than in (3.231).

Including the monthly log change of the exchange rate in the dynamics and imposing bond factors to fit predictions of exchange rate changes and/or volatility, Dewachter and Maes (2001) find that the three-factor model is unable to fit the volatile exchange rate series. This is quite a common finding in the literature and can be explained by the fact the exchange rate dynamics are influenced by other factors than those influencing term structures in both countries. Han and Hammond (2003) were able to fit the exchange rate by adding one additional factor influencing the exchange rate but not the term structures. It would appear that the model with the affine exchange rate provides a better fit for the exchange rates, as shown in Figure 3.5.

It is also clear that the actual log exchange rate is in fact more volatile then the predicted log exchange rate dynamics. Since by estimating the parameters of the exchange rate we also determine some parameters for the UK term structures, we can observe that the model does not fit the UK yields as well as the previous model.

It also seems that all other studies lose some information about the unobservable factors not included in the set of observables (and thus not included in the estimation), inflation-protected domestic and foreign bonds, and real exchange rates as this would allow us to take into account possible correlations with real exchange rates and inflation.

General two-country five-factor model

To fit the market consisting of the UK and US nominal and real yields we include two additional factors in the model, X_4 and X_5, which we call UK and US real factors. Parameters of the five-factor model consisting of all the factors in (3.229) have been estimated using an approximate Kalman filter and quasi-maximum likelihood. To do this, we need to express $m(X_t, \theta, h)$ and $Q(X_t, \theta, h)$ as affine functions of the factors. This is done in the Appendix 3.8.1.

The results of the estimation are presented in Table 3.4 and Table 3.5. The estimation shows that the UK real factor is relatively more mean reverting than the US real factors, and its level of mean reversion is approximately that of the common factor. The US and UK local factors are significantly more slowly mean reverting than the common factor. We can also note that real factors are much more volatile than the common and local nominal factors. Parameters b_i can be seen as proxies for the instantaneous interest rates, but our estimates seem to be of reasonable magnitude: the coefficients for the real factors are around 0.02. The UK and US nominal short rate coefficients are lower than in the previous estimations and are around 0.5. The real short rate coefficients are around 0.05.

The actual and fitted two- and four-year US and UK yields are plotted in Figure 3.8 and Figure 3.9. We can see that the five-factor model demonstrates, in general, quite a good fit; however, the fit for the US real market appears to be quite poor. We observe that the predicted US real yields are less volatile than the US actual real yields. Neither could we obtain a good fit for the UK nominal term structure. Thus, we do not seem to have chosen the best specification for the nominal-real two-country market. This may perhaps be explained by the correlation structure. The model specification in fact restricts the correlation structure significantly since in CIR models all the factors are positively correlated, and there seem to be some negative correlation coefficients in the correlation table. Thus, we could try to improve the model's fit by replacing one of the real CIR factor dynamics with Gaussian dynamics; for this reason we estimate Model II. The results of the estimation are presented in Table 3.6 and Table 3.7.

We estimate the factor dynamics (Model II) based only on the observations of UK and US nominal and real yields and without taking into account the exchange rate and CPI index between the two countries. However, once we have fitted the yields, and with the model in mind, we can draw some implications concerning the volatilities of the exchange rate and CPI index processes, (3.26) and (3.151), since we know that UK and US drifts are connected via (3.40) and (3.154). Thus, the exchange rate volatility vector can be estimated as

$$\sigma_S = (3,95; \quad 0,276; \quad 2,36; \quad 0; \quad 0). \tag{3.235}$$

The US CPI index volatility vector can be estimated as

$$\sigma_I = (4,04; \quad 3,431; \quad 0; \quad 1,99; \quad 0) \tag{3.236}$$

and the UK CPI index volatility vector can be estimated as

$$\sigma_{If} = (1,92; \quad 2,96; \quad 0,247; \quad 0; \quad 1,58). \tag{3.237}$$

We will compare the two models and the implied volatilities of the exchange rate, and UK and US CPI indices. We see that as compared to Model I, Model II shows that the UK real factor estimate is much slower mean reverting, and the US local factor is faster mean reverting. The volatilities of the domestic local factor and UK real factors are significantly lower than in the first model, while the volatility of the common factor is higher. The coefficients for the US nominal and the UK real short rates are higher but we see that the coefficients of the US real short rate are lower than in the first model. In addition, we estimate the correlations between the domestic real factor and the US and UK local nominal factors to be around -0.4. Comparing the actual and fitted yields for both models, we see a much better fit for the US real yields, providing higher volatility to the fitted yields, but the fit for the UK real yields are not as good.

We will also compare the implied volatilities with the estimates for the case when the exchange rate is included in the estimation.

General two-country 5-factor model, including exchange rate and US CPI index

According to Proposition 3.2.15 we specify affine factor dynamics under the domestic martingale measure and an affine domestic short rate exogenously. We also model affine exchange rate and affine CPI US index, which will ensure that the factor dynamics is affine also under the foreign and domestic real martingale measures as well as affine foreign and domestic real short rates. Thus, we assume that the exchange rate takes the form

$$\ln S(t, X_t) = P_1 X_{1t} + P_2 X_{2t} + P_3 X_{3t} + Q, \tag{3.238}$$

and that the CPI US price index takes the form

$$\ln I(t, X_t) = P_1^I X_{1t} + P_2^I X_{3t} + P_3^I X_{4t} + Q^I. \tag{3.239}$$

Given the parameters of the factor dynamics, exchange rate and the CPI price index, we find foreign and domestic real short rates, and foreign and and domestic real market prices of risk as in (3.154). We assume also that nominal exchange rate and CPI US index are observable with an error, thus, in addition to the previous section we also estimate s_{ex} and s_{inf}, the volatilities of the measurement errors. The results of the estimation are presented in Table 3.8.

Given the vector

$$P = (0,7039; \quad 0,5998; \quad 0,7998; \quad 0; \quad 0) \tag{3.240}$$

we can find vector of volatilities of the exchange rate as $\sigma_S = P\sigma$, hence,

$$\sigma_S = (0,311; \quad 0,048; \quad 0,080; \quad 0; \quad 0). \tag{3.241}$$

We notice that those estimates are quite different from the ones implied by the estimation where the exchange rate is not included in the estimation, that is, we observe much smaller volatilities as compared to (3.231). Also, given the vector

$$P^I = (0,8003; \quad 0; \quad 0,8001; \quad 0,8514; \quad 0) \tag{3.242}$$

we can find vector of volatilities of the CPI US price index as $\sigma_I = P^I\sigma$, hence,

$$\sigma_I = (0,3537; \quad 0; \quad 0,0639; \quad 0,0851; \quad 0), \tag{3.243}$$

which is also much lower than the previously estimated parameters. From the pictures we can see that unfortunately the model is unable to fit well the jointly affine real and nominal yields when both the affine exchange rate and CPI US price index are included in the estimation procedure. Thus, we can conclude that other factors than the factors influencing the real and nominal term structures are influencing the nominal exchange rate and the CPI price index. It seems that including in the estimation procedure the US CPI index does not really improve the fit of the model.

3.6 Future research

A number of questions remain for future research. It would be interesting to see how the failure of the model to fit the real exchange rate dynamics even on the modelling stage, affects the model's fit to the nominal and real domestic and foreign yields. We can pick two diffrent models above (one which does not produce mean stationary real exchange rates and one being able to produce mean stationary real exchange rate) and consider a counrty where the real exchange rate is mean stationary. We can compare those two model's performance on the data consisting of the nominal and real domestic and foreign yields.

We leave for the future research the task of estimating the model in case the factor dynamics exhibit jumps. Even though by just inspecting the US and UK yields' dynamics we can not reject the hypothesis that the factor dynamics exhibits jumps, modeling the jump-diffusion dynamics would be of course more important for the developing countries. For example, it would be interesting to compare the fit of the jump-diffusion model to the Russian-US bond market, since the RUR/US exchange rate exhibit jums.

3.7 Conclusion

The paper suggests a jointly affine model of domestic and foreign nominal and real yields, nominal exchange rate, domestic and foreign price levels. The model produces closed form affine solutions for nominal and also inflation protected bonds on the international market, which allows to simplify the pricing and estimation procedures.

Another contribution of the paper is that we allow the factor dynamics to be driven by Wiener process as well as marked point process. This specification is most important for developing countries where the economic crisis can cause jumps of different size in the dynamics of the exchange rate and price level. We give necessary and sufficient conditions for the nominal and the inflation-linked bonds to be affine in the international market.

We also give some necessary and sufficient conditions for the real exchange rate to be asymptotically mean stationary in the jointly affine framework, that is, we try to rule out those models which are inconsistent with the empirical evidence supporting the Purchasing Power Parity (PPP) hypothesis.

As a practical application of the presented general international model we construct a European type "real exchange" call option, and show how to price it. This derivative can be introduced and used by investors to hedge not just their exchange rate risk or just inflation risk, but the joint exchange rate and inflation risks.

Finally, we estimate the factor dynamics for several two-country models: 1) nominal three-factor affine model including the domestic and foreign nominal yields in the set of observables, 2) nominal three-factor affine model including yields as well as the exchange rate, 3) five-factor nominal/real model including the domestic and foreign nominal yields in the set of observables and 4) five-factor model including the exchange rate and the price level in addition to the yields. We show that models 1) and 3) produce quite a good fit to the observed term structure. We also show that in the model 2) we obtain a satisfactory fit to the term structures as well the nominal exchange rate without introducing additional factors (influencing only the exchange rate dynamics).

3.8 Appendix

3.8.1 Kalman filter algorithm

Let the $\mathcal{F}_t$ be the σ-algebra generated by the observations up to and including time t.

Denote

1. The conditional mean and variance of the unobserved state variables over a discrete time interval h (an *a priori* state estimate at step $t+h$ given the knowledge of the process prior to time t)

$$m(X_t, \theta, h) = \mathbb{E}\left[X(t+h) \mid X(t)\right] \tag{3.244}$$
$$Q(X_t, \theta, h) = \text{Var}\left[X(t+h) \mid X(t)\right] \tag{3.245}$$

2. The state estimate at step t, given $\mathcal{F}_t$ by $\hat{X}_t = \mathbb{E}\left[X(t) \mid \mathcal{F}_t\right]$

3. The estimate error covariance at step $t+h$, $P_{t+h|t} = \text{Var}\left[X(t+h) \mid \mathcal{F}_t\right]$

4. The estimate error covariance, $P_t = \text{Var}\left[X(t) \mid \mathcal{F}_t\right]$.

We specify the measurement and the transition equations as follows:

$$Y_t = \bar{A} + \bar{B}' X_t + v_t, \tag{3.246}$$
$$X_{t+h} = F_0 + F_1 X_t + u_{t+h}, \tag{3.247}$$

where $v_t - N(0, D)$ and $u_{t+h} - N(0, Q)$. Assume that N_1 is the number of nominal US yields, N_2 is the number of nominal UK yields, G_1 is the number of real US yields and G_2 is the number of real UK yields. The $N \times 1$ data vector Y_t is obtained by combining together $N = N_1 + N_2 + G_1 + G_2$ yields, exchange rate and US CPI index observations. We take the measurements one a month, that is, assume approximately that $h = 0,08$. $\bar{A}$ is $N \times 1$ and constructed by stacking together coefficients

$$\bar{A} = \left\{ \frac{A(\tau)}{\tau}, \frac{C(\tau)}{\tau}, \frac{A^f(\tau)}{\tau}, \frac{C^f(\tau)}{\tau}, Q, Q_I \right\},$$

and $\bar{B}$ is $N \times d$ and constructed by stacking together

$$\bar{B} = \left\{ \frac{B(\tau)}{\tau}, \frac{D(\tau)}{\tau}, \frac{B^f(\tau)}{\tau}, \frac{D^f(\tau)}{\tau}, P, P_I \right\},$$

where d is the number of factors. We also assume that the covariance matrix of the measurement error matrix is diagonal, $N \times N$, with parameters w_i, $i = 1, \ldots, N$. The exact filtering procedure is specified as follows. As initial conditions for factor expectation and variance we take

$$\hat{X}_0^i = b_i \tag{3.248}$$

$$\hat{P}_0^i = \frac{\sigma_i^2 b_i}{2a_i}. \tag{3.249}$$

At the prediction step we obtain

$$\hat{X}_{t+h|t} = F_0 + F_1 \hat{X}_t \tag{3.250}$$

$$\hat{P}_{t+h|t} = F_1 \hat{P}_t F_1' + Q_t, \tag{3.251}$$

where

$$F_1 = \begin{bmatrix} e^{-a_1 h} & 0 & 0 & 0 & 0 \\ 0 & e^{-a_2 h} & 0 & 0 & 0 \\ 0 & 0 & e^{-a_3 h} & 0 & 0 \\ 0 & 0 & 0 & e^{-a_4 h} & 0 \\ 0 & 0 & 0 & 0 & e^{-a_5 h} \end{bmatrix}, \quad F_0 = \begin{bmatrix} b_1(1 - e^{-a_1 h}) \\ b_2(1 - e^{-a_2 h}) \\ b_3(1 - e^{-a_3 h}) \\ b_4(1 - e^{-a_4 h}) \\ b_5(1 - e^{-a_5 h}) \end{bmatrix} \tag{3.252}$$

The updation step takes the form

$$\hat{X}_{t+h} = \hat{X}_{t+h|t} + K_{t+h} \Delta_{t+h} \tag{3.253}$$

$$\hat{P}_{t+h} = L_t \hat{P}_{t+h|t}, \tag{3.254}$$

where

$$K_{t+h} = \hat{P}_{t+h|t} \bar{B} V_{t+h}^{-1} \tag{3.255}$$

$$L_{t+h} = I_{5\times 5} - K_{t+h} \bar{B}', \tag{3.256}$$

and the likelihood contributors are

$$\Delta_{t+h} = Y_{t+h} - F_0 - F_1 \hat{X}_{t+h|t} \quad (3.257)$$

$$V_{t+h} = \bar{B}' \hat{P}_{t+h|t} \bar{B} + D. \quad (3.258)$$

and

$$-2 \ln L_t = \ln |V_t| + \Delta_t' V_t^{-1} \Delta_t. \quad (3.259)$$

We can express $Q(X_t, \theta, h)$ as follows

$$\begin{aligned}
Q(X_t, i, i, t) &= \frac{(1 - e^{-a(i)h})^2 \sigma^2(i) b(i)}{2a(i)} + \frac{(e^{-a(i)h} - e^{-2a(i)h}) X_{1t}) \sigma^2(i)}{a(i)} \\
Q(4, 4, t) &= \frac{(1 - e^{-a(4)h})}{2a(4)} + \frac{b(1)(1 + \rho_{41}^2)(1 - e^{-2a(4)h})}{4a(4)} \\
&- \frac{b(1)(1 + \rho_{41}^2)(e^{-a(1)h} - e^{-2a(4)h})}{(2(a(1) - 2a(4)))} \\
&+ \frac{X_{1t})(1 + \rho_{41}^2)(e^{-a(1)h} - e^{-2a(4)h})}{2(a(1) - 2a(4))} + \frac{b(3)(1 + \rho_{43}^2)(1 - e^{-2a(4)h})}{4a(4)} \\
&- \frac{b(5)(1 + \rho_{43}^2)(e^{-a(3)h} - e^{(-2a(4)h)})}{2(a(3) - 2a(4))} + \frac{X_{3t}(1 + \rho_{43}^2)(e^{-a(3)h} - e^{-2a(4)h})}{2(a(3) - 2a(4))} \\
Q(3, 4, t) &= \frac{(e^{-a(3)h} - e^{-(a(3)+a(4))h}) \sigma(3) \rho_{43} X_{t3}}{2a(4)} + \frac{\sigma(3) \rho_{43} b(3)((1 - e^{-(b(3)+b(4))h})}{2(a(3) + a(4))} \\
&- \frac{(e^{-b(3)h} - e^{-(b(3)+b(4))} h)}{a(4)} \\
Q(1, 4, t) &= \frac{(e^{-a(1)h} - e^{-(a(1)+a(4))h}) \sigma(1) \rho_{13} X_{1t}}{2a(4)} + \frac{\sigma(1) \rho_{13} b(1)((1 - e^{-(a(1)+a(4))h}}{2(a(1) + a(4))} \\
&- \frac{e^{-a(1)h} - e^{-(a(1)+a(4))h}}{a(4)} \\
Q(4, 3, t) &= Q(3, 4, t) \\
Q(1, 3, t) &= Q(3, 1, t)
\end{aligned}$$

3.8.2 Estimation results: graphs and tables

Implied versus actual yields: estimation results.

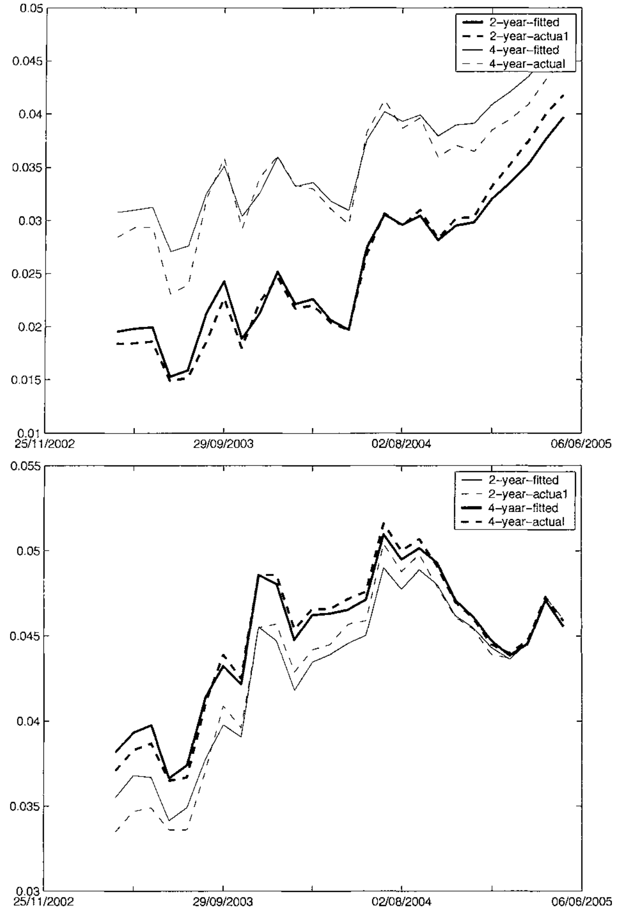

Figure 3.6: Actual versus fitted yields, two- and four-year US and UK yields, November 2002- July 2005.

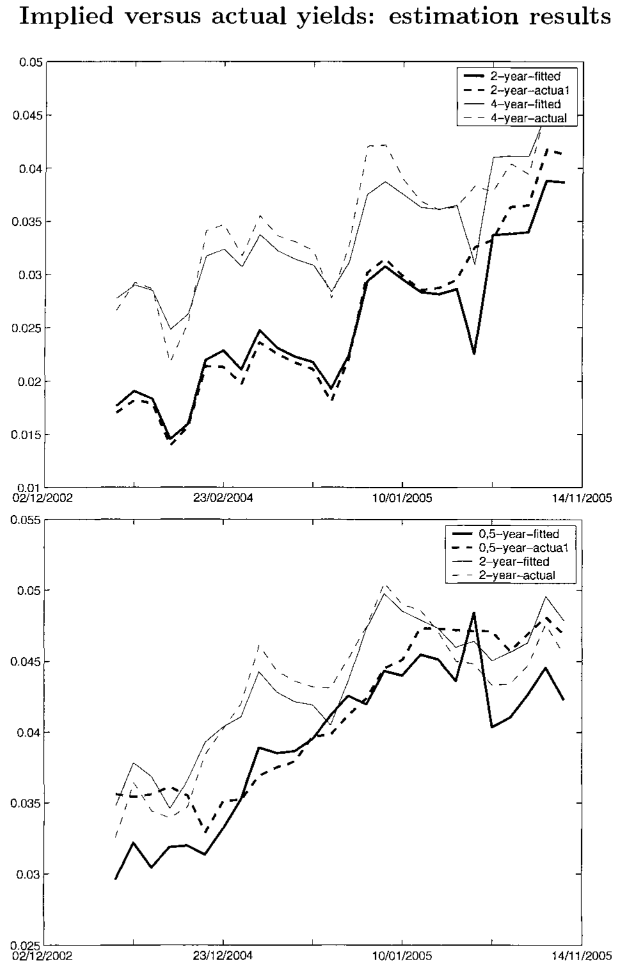

Figure 3.7: Actual versus fitted yields, two- and four-year US and UK yields, November 2002-July 2005, exchange rate included in the estimation.

Implied versus actual nominal yields: estimation results for the five-factor model

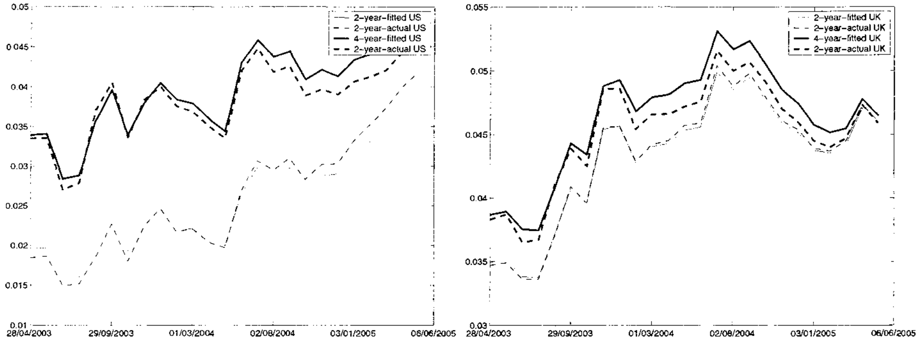

Figure 3.8: Actual versus fitted yields, two- and four-year US and UK nominal yields, November 2002-July 2005.

Implied versus actual real yields: estimation results for the five-factor model

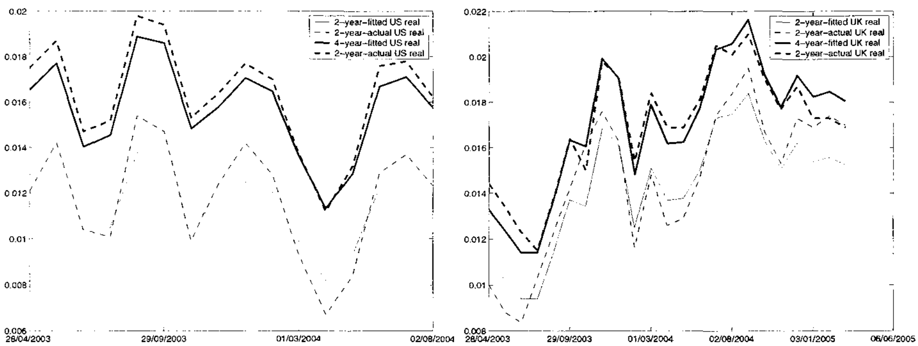

Figure 3.9: Actual versus fitted yields, two- and four-year US and UK real yields, November 2002-July 2005.

Implied versus actual nominal yields: estimation results for five-factor, Model II.

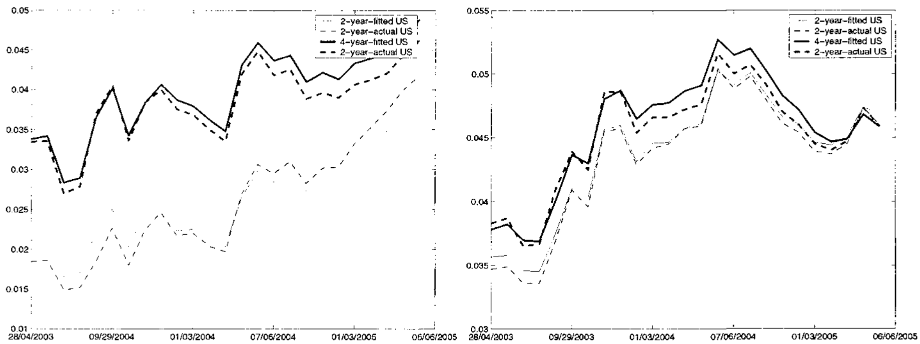

Figure 3.10: Actual versus fitted yields, two- and four-year US and UK nominal yields, November 2002-July 2005.

Implied versus actual real yields: estimation results for five-factor Model II.

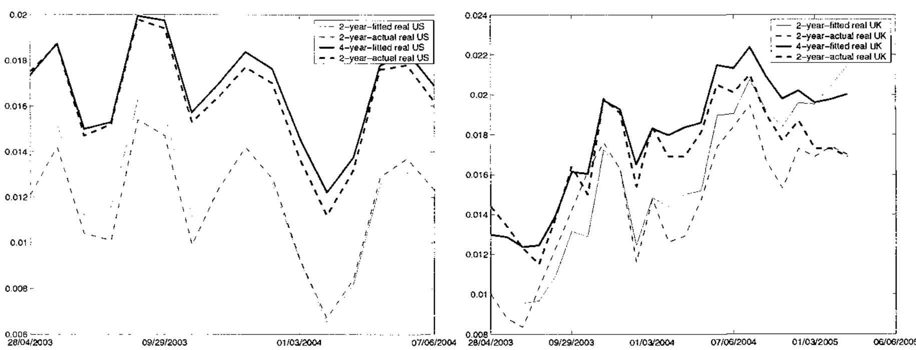

Figure 3.11: Actual versus fitted yields, two and four-year US and UK real yields, November 2002-July 2005.

Correlations between the yields

	$y^d(0.5)$	$y^d(1)$	$y^d(2)$	$y^d(4)$	$y^d(5)$	$y^d(6)$	$y^{fr}(2)$	$y^{fr}(3)$	$y^{fr}(4)$	$y^{dr}(5)$
$y^d(0.5)$	1.0000	0.9753	0.8092	0.6622	0.9903	0.5027	0.6376	0.5246	0.3455	-0.0484
$y^d(1)$	0.9753	1.0000	0.9141	0.8010	0.9379	0.6662	0.7051	0.6441	0.4889	0.0739
$y^d(2)$	0.8092	0.9141	1.0000	0.9744	0.7298	0.9099	0.7511	0.7570	0.6555	0.3648
$y^d(4)$	0.6622	0.8010	0.9744	1.0000	0.5674	0.9797	0.7252	0.7567	0.6886	0.5091
$y^d(5)$	0.9903	0.9379	0.7298	0.5674	1.0000	0.3973	0.5768	0.4304	0.2396	-0.1082
$y^d(6)$	0.5027	0.6662	0.9099	0.9797	0.3973	1.0000	0.6755	0.7323	0.6981	0.6102
$y^{fr}(2)$	0.6376	0.7051	0.7511	0.7252	0.5768	0.6755	1.0000	0.8639	0.7752	0.3071
$y^{fr}(3)$	0.5246	0.6441	0.7570	0.7567	0.4304	0.7323	0.8639	1.0000	0.9751	0.2568
$y^{fr}(4)$	0.3455	0.4889	0.6555	0.6886	0.2396	0.6981	0.7752	0.9751	1.0000	0.2696
$y^{dr}(5)$	-0.0484	0.0739	0.3648	0.5091	-0.1082	0.6102	0.3071	0.2568	0.2696	1.0000
$y^{dr}(7)$	-0.3033	-0.1781	0.1518	0.3338	-0.3610	0.4779	0.1508	0.1259	0.1846	0.9534
$y^{dr}(10)$	-0.4296	-0.3072	0.0326	0.2292	-0.4837	0.3915	0.0613	0.0519	0.1330	0.8861
$y^f(1)$	0.6755	0.7518	0.7309	0.6515	0.5998	0.5616	0.6473	0.8104	0.7737	-0.1338
$y^f(2)$	0.4442	0.5715	0.6662	0.6521	0.3443	0.6206	0.6253	0.8592	0.8828	0.0247
$y^f(3)$	0.2880	0.4365	0.5889	0.6114	0.1799	0.6140	0.5819	0.8445	0.9018	0.1113
$y^f(4)$	0.1744	0.3335	0.5205	0.5668	0.0632	0.5925	0.5384	0.8149	0.8949	0.1648
$y^f(5)$	0.0862	0.2513	0.4622	0.5260	-0.0262	0.5688	0.4987	0.7834	0.8799	0.2031
Ex.rate	0.6813	0.6650	0.4708	0.3178	0.6631	0.1712	0.3040	0.3706	0.2813	-0.5145
Infl	0.9114	0.8993	0.7398	0.5967	0.8882	0.4497	0.6195	0.5924	0.4582	-0.2271

	$y^{dr}(7)$	$y^{dr}(10)$	$y^f(1)$	$y^f(2)$	$y^f(3)$	$y^f(4)$	$y^f(5)$	Ex.rate	Infl
$y^d(0.5)$	-0.3033	-0.4296	0.6755	0.4442	0.2880	0.1744	0.0862	0.6813	0.9114
$y^d(1)$	-0.1781	-0.3072	0.7518	0.5715	0.4365	0.3335	0.2513	0.6650	0.8993
$y^d(2)$	0.1518	0.0326	0.7309	0.6662	0.5889	0.5205	0.4622	0.4708	0.7398
$y^d(4)$	0.3338	0.2292	0.6515	0.6521	0.6114	0.5668	0.5260	0.3178	0.5967
$y^d(5)$	-0.3610	-0.4837	0.5998	0.3443	0.1799	0.0632	-0.0262	0.6631	0.8882
$y^d(6)$	0.4779	0.3915	0.5616	0.6206	0.6140	0.5925	0.5688	0.1712	0.4497
$y^{fr}(2)$	0.1508	0.0613	0.6473	0.6253	0.5819	0.5384	0.4987	0.3040	0.6195
$y^{fr}(3)$	0.1259	0.0519	0.8104	0.8592	0.8445	0.8149	0.7834	0.3706	0.5924
$y^{fr}(4)$	0.1846	0.1330	0.7737	0.8828	0.9018	0.8949	0.8799	0.2813	0.4582
$y^{dr}(5)$	0.9534	0.8861	-0.1338	0.0247	0.1113	0.1648	0.2031	-0.5145	-0.2271
$y^{dr}(7)$	1.0000	0.9811	-0.2924	-0.0851	0.0347	0.1127	0.1706	-0.6739	-0.4218
$y^{dr}(10)$	0.9811	1.0000	-0.3649	-0.1370	-0.0036	0.0845	0.1503	-0.7151	-0.5111
$y^f(1)$	-0.2924	-0.3649	1.0000	0.9406	0.8598	0.7887	0.7264	0.7304	0.8233
$y^f(2)$	-0.0851	-0.1370	0.9406	1.0000	0.9815	0.9485	0.9122	0.5418	0.6067
$y^f(3)$	0.0347	-0.0036	0.8598	0.9815	1.0000	0.9913	0.9729	0.4061	0.4533
$y^f(4)$	0.1127	0.0845	0.7887	0.9485	0.9913	1.0000	0.9947	0.3069	0.3421
$y^f(5)$	0.1706	0.1503	0.7264	0.9122	0.9729	0.9947	1.0000	0.2282	0.2557
Ex.rate	-0.6739	-0.7151	0.7304	0.5418	0.4061	0.3069	0.2282	1.0000	0.7532
Infl	-0.4218	-0.5111	0.8233	0.6067	0.4533	0.3421	0.2557	0.7532	1.0000

Table 3.1: The correlation coefficients between the domestic and foreign nominal and real yields, exchange rate and price level, based on the observations from November 2002-July 2005. Almost all the coefficients are significant.

QML parameter estimates for the UK-US joint bond market

Parameters	Domestic factor X_1	Common factor X_2	Foreign factor X_3
a_i	0,21812 $(10^{-3} \cdot 0.0467)$	0,51861 $(10^{-3} \cdot 0,0720)$	0,06213 $(10^{-3} \cdot 0,0249)$
b_i	0,06064 $(10^{-3} \cdot 0,0246)$	0,04868 $(10^{-3} \cdot 0,0220)$	0,04906 $(10^{-3} \cdot 0,0222)$
σ_i	0,12744 $(10^{-3} \cdot 0,0357)$	0,07002 $(10^{-3} \cdot 0,0265)$	0,22998 $(10^{-3} \cdot 0,0479)$
R^d	1,1501 $(10^{-3} \cdot 0,1072)$	0,2000 $(10^{-3} \cdot 0,0447)$	
R^f		0,48469 $(10^{-3} \cdot 0,0696)$	1,5000 $(10^{-3} \cdot 0,1225)$
λ^d	-0,48110 $(10^{-3} \cdot 0,0693)$	-0,48630 $(10^{-3} \cdot 0,06973)$	
λ^f	-0,52705 $(10^{-3} \cdot 0,07259)$	-0,51560 $(10^{-3} \cdot 0,07180)$	-0,50435 $(10^{-3} \cdot 0,0710)$
s_1	0,00000230 $(10^{-3} \cdot 0,0001519)$		0,0000016 $(10^{-3} \cdot 0,000126)$
s_2	0,00000133 $(10^{-3} \cdot 0,0001156)$		0,00000077 $(10^{-3} \cdot 0,000088)$
s_3	0,00000104 $(10^{-3} \cdot 0,0001021)$		0,00000077 $(10^{-3} \cdot 0,000088)$
s_4	0,00000324 $(10^{-3} \cdot 0,000179)$		0,00000435 $(10^{-3} \cdot 0,000208)$
s_5	0,00000280 $(10^{-3} \cdot 0,000167)$		0,0000031 $(10^{-3} \cdot 0,000176)$
s_6	0,00000218 $(10^{-3} \cdot 0,000147)$		0,000000078 $(10^{-3} \cdot 0,000027)$

Table 3.2: In the table we present the parameter estimates for the three-factor model, based on 30 monthly data observations for 12 yield time series observations- 0.5, 1, 2, 4, 5, 6 years US yields and 0.5, 1, 2, 3, 4, 5 years UK yields, from November 2002-July 2005. Standard errors are in brackets.)

QML parameter estimates for the UK-US joint bond market with an affine exchange rate.

Parameters	Domestic factor X_1	Common factor X_2	Foreign factor X_3
a_i	0.20804 $(10^{-3} \cdot 0.045679)$	0.61865 $(10^{-3} \cdot 0.078645)$	0.06160 $(10^{-3} \cdot 0.024875)$
b_i	0.06042 $(10^{-3} \cdot 0.024756)$	0.04877 $(10^{-3} \cdot 0.022156)$	0.04936 $(10^{-3} \cdot 0.022185)$
σ_i	0.31283 $(10^{-3} \cdot 0.055934)$	0.16984 $(10^{-3} \cdot 0.041235)$	0.23095 $(10^{-3} \cdot 0.04801)$
R^d	1.01484 $(10^{-3} \cdot 0.100782)$	0.20023 $(10^{-3} \cdot 0.044778)$	
λ^d	-0.480971 $(10^{-3} \cdot 0.069368)$	-0.48616 $(10^{-3} \cdot 0.069735)$	
P	0.52719 $(10^{-3} \cdot 0.072602)$	0.705062 $(10^{-3} \cdot 0.083971)$	1.05117 $(10^{-3} \cdot 0.102508)$
Q		0.01266 $(10^{-3} \cdot 0.011251)$	
s_1	0.0000042 $(10^{-3} \cdot 0.00002)$		0.0000051 $(10^{-3} \cdot 0.0000228)$
s_2	0.0000045 $(10^{-3} \cdot 0.0000214)$		0.0000046 $(10^{-3} \cdot 0.0000216)$
s_3	0.000005 $(10^{-3} \cdot 0.0000224)$		0.0000046 $(10^{-3} \cdot 0.0000216)$
s_4	0,0000051 $(10^{-3} \cdot 0.0000226)$		0.0000048 $(10^{-3} \cdot 0.0000221)$
s_5	0.0000044 $(10^{-3} \cdot 0.0000210)$		0.00000515 $(10^{-3} \cdot 0.0000227)$
s_6	0.0000051 $(10^{-3} \cdot 0.0000227)$		0.00000515 $(10^{-3} \cdot 0.0000236)$
s_{ex}		0.000652 $(10^{-3} \cdot 0.0002551$	

Table 3.3: The parameter estimates for the three-factor model with affine exchange rate, based on the same 30 monthly data observations for 12 yields, from November 2002-July 2005. Standard errors are in brackets. P_i are Q are the parameters for the affine exchange rate, s_{ex} -error volatility.)

QML parameter estimates for the UK-US general bond market, 5 factor, I.

Param	US factor X_1	Common X_2	UK factor X_3	US real factor	UK real factor
a_i	0.0510 $(10^{-4} \cdot 0.0710)$	0.2616 $(10^{-4} \cdot 0.1617)$	0.0262 $(10^{-4} \cdot 0.0512)$	0.1205 $(10^{-4} \cdot 0.1097)$	0.3000 $(10^{-4} \cdot 0.1732)$
b_i	0.060 $(10^{-4} \cdot 0.0775)$	0.0483 $(10^{-4} \cdot 0.0696)$	0.0399 $(10^{-4} \cdot 0.0632)$	0.0199 $(10^{-4} \cdot 0.0447)$	0.0200 $(10^{-4} \cdot 0.0447)$
σ_i	0.3744 $(10^{-4} \cdot 0.1934)$	0.2699 $(10^{-4} \cdot 0.1643)$	0.0230 $(10^{-4} \cdot 0.0480)$	0.4999 $(10^{-4} \cdot 0.2236)$	0.5999 $(10^{-4} \cdot 0.2449)$
R^d	0.5999 $(10^{-4} \cdot 0.2449)$	0.500 $(10^{-4} \cdot 0.2236)$			
R^f		0.4699 $(10^{-4} \cdot 0.2168)$	0.6000 $(10^{-4} \cdot 0.2449)$		
R^{dr}	0.0500 $(10^{-4} \cdot 0.0707)$			0.0500 $(10^{-4} \cdot 0.0709)$	
R^{fr}		0.100 $(10^{-4} \cdot 0.100)$	0.0900 $(10^{-4} \cdot 0.100)$		0.0498 $(10^{-4} \cdot 0.0707)$
λ^f	-0.527 $(10^{-4} \cdot 0.2295)$	-0.8515 $(10^{-4} \cdot 0.2918)$	0.0543 $(10^{-4} \cdot 0.0710)$		
λ^d	-1.610 $(10^{-4} \cdot 0.4013)$	-0.9262 $(10^{-4} \cdot 0.3043)$			
λ^{dr}	-0.5014 $(10^{-4} \cdot 0.2239)$			0.999 $(10^{-4} \cdot 0.3163)$	
λ^{fr}		-0.050 $(10^{-4} \cdot 0.0707)$	0.0620 $(10^{-4} \cdot 0.3082)$		0.950 $(10^{-4} \cdot 0.0710)$
s_1	0.00000217 $(10^{-4} \cdot 0.000480)$		0.00000147 $(10^{-4} \cdot 0.000399)$		
s_2	0.00000102 $(10^{-4} \cdot 0.000368)$		0.000000231 $(10^{-3} \cdot 0.000288)$		
s_3	0.00000078 $(10^{-4} \cdot 0.000324)$		0.00000022 $(10^{-3} \cdot 0.0000286)$		

Table 3.4: The parameter estimates for the five-factor model based on the same 30 monthly data observations for 17 yields, from November 2002-July 2005. Standard errors are in brackets. s_i are the estimates for the volatility of the measurement errors for UK and US nominal yields.

QML parameter estimates for the UK-US general bond market (continued).

Param	US factor X_1	Common X_2	UK factor X_3	US real factor	UK real factor
s_4	0,00000211 $(10^{-4} \cdot 0.000589)$		0.00000325 $(10^{-4} \cdot 0.0000667)$		
s_5	0.00000316 $(10^{-4} \cdot 0.000526)$		0.0000029 $(10^{-4} \cdot 0.0000560)$		
s_6	0.0000324 $(10^{-4} \cdot 0.00409)$				
w_1				0.00000417 $(10^{-4} \cdot 0.0000715)$	
w_2				0.0000039 $(10^{-4} \cdot 0.0000720)$	
w_3				0.0000041 $(10^{-4} \cdot 0.0000716)$	
w_1					0.000004 $(10^{-4} \cdot 0.0000720)$
w_2					0.000004 $(10^{-4} \cdot 0.0000718)$
w_3					0.0000043 $(10^{-4} \cdot 0.0000712)$

Table 3.5: The parameter estimates for the five-factor model based on the same 30 monthly data observations for 17 yields, from November 2002-July 2005. Standard errors are in brackets. w_i are the estimates for the volatility of the measurement errors for UK and US real yields.

QML parameter estimates for the UK-US general bond market, 5 factor, II.

Param	US factor X_1	Common X_2	UK factor X_3	US real factor	UK real factor
a_i	0.1811 $(10^{-4} \cdot 0.1351)$	0.2616 $(10^{-4} \cdot 0.1677)$	0.0262 $(10^{-4} \cdot 0.0512)$	0.1205 $(10^{-4} \cdot 0.1195)$	0.05 $(10^{-4} \cdot 0.0707)$
b_i	0.0601 $(10^{-4} \cdot 0.0776)$	0.0484 $(10^{-4} \cdot 0.0695)$	0.040 $(10^{-4} \cdot 0.06391)$		0.020 $(10^{-4} \cdot 0.04479)$
σ_i	0.2744 $(10^{-4} \cdot 0.1665)$	0.3613 $(10^{-4} \cdot 0.1910)$	0.0230 $(10^{-4} \cdot 0.2236)$		0.200 $(10^{-4} \cdot 0.0543)$
R^d	0.6 $(10^{-4} \cdot 0.2465)$	0.6001 $(10^{-4} \cdot 0.2651)$			
R^f		0.5700 $(10^{-4} \cdot 0.2408)$	0.6001 $(10^{-4} \cdot 0.32429)$		
R^{dr}	0.03000 $(10^{-4} \cdot 0.0548)$			0.0401 $(10^{-4} \cdot 0.1409)$	
R^{fr}		0.2000 $(10^{-4} \cdot 0.1421)$	0.4000 $(10^{-4} \cdot 0.2004)$		0.8 $(10^{-4} \cdot 0.3685)$
λ^f	-0.3271 $(10^{-4} \cdot 0.1808)$	-1.4916 $(10^{-4} \cdot 0.3869)$	0.0504 $(10^{-4} \cdot 0.0710)$		
λ^d	-1.9110 $(10^{-4} \cdot 0.4390)$	-0.9263 $(10^{-4} \cdot 0.4432)$			
λ^{dr}	-0.5014 $(10^{-4} \cdot 0.2239)$			0.999 $(10^{-4} \cdot 0.3162)$	
λ^{fr}		-0.5015 $(10^{-4} \cdot 0.0707)$	-1 $(10^{-4} \cdot 0.3082)$		-0.05 $(10^{-4} \cdot 0.2862)$
s_1	0.00000346 $(10^{-4} \cdot 0.00233)$		0.00000158 $(10^{-4} \cdot 0.0000128)$		
s_2	0.00000257 $(10^{-4} \cdot 0.00249)$		0.000000775 $(10^{-3} \cdot 0.0000088)$		
s_3	0.000000844 $(10^{-4} \cdot 0.000507)$		0.000000775 $(10^{-3} \cdot 0.00000886)$		

Table 3.6: The parameter estimates for the five-factor Model II, based on the same 30 monthly data observations for 17 yields, from November 2002-July 2005. Standard errors are in brackets. s_i are the estimates for the volatility of the measurement errors for UK and US nominal yields.

QML parameter estimates for the UK-US general bond market (continued).

Param	US factor X_1	Common X_2	UK factor X_3	US real factor	UK real factor
s_4	0.00000258 $(10^{-4} \cdot 0.00141)$		0.00000435 $(10^{-4} \cdot 0.000020)$		
s_5	0.00000239 $(10^{-4} \cdot 0.0000534)$		0.00000312 $(10^{-4} \cdot 0.000017)$		
s_6	0.00000313 $(10^{-4} \cdot 0.00193)$				
w_1				0.000005 $(10^{-4} \cdot 0.0000223)$	
w_2				0.000005 $(10^{-4} \cdot 0.000022)$	
w_3				0.000005 $(10^{-4} \cdot 0.0000223)$	
w_1					0.0000048 $(10^{-4} \cdot 0.0000223)$
w_2					0.0000049 $(10^{-4} \cdot 0.0000224)$
w_3					0.000005 $(10^{-4} \cdot 0.0000223)$
ρ	-0.04 $(10^{-4} \cdot 0.2)$				-0.04 $(10^{-4} \cdot 0.2)$

Table 3.7: The parameter estimates for the five-factor model based on the same 30 monthly data observations for 17 yields, from November 2002-July 2005. Standard errors are in brackets. w_i are the estimates for the volatility of the measurement errors for UK and US real yields.

QML parameter estimates for the General UK-US bond market, 5 factor affine model.

Param	US factor X_1	Common X_2	UK factor X_3	Real US factor	Real UK factor
a_i	0.2989 $(10^{-3} \cdot 0.0421)$	0.0511 $(10^{-3} \cdot 0.0038)$	0.0119 $(10^{-3} \cdot 0.0013)$	0.1501 $(10^{-4} \cdot 0.0056)$	0.049 $(10^{-4} \cdot 0.0007)$
b_i	0.0605 $(10^{-3} \cdot 0.0173)$	0.0484 $(10^{-3} \cdot 0.0033)$	0.060 $(10^{-3} \cdot 0.0008)$	1.0011 $(10^{-3} \cdot 0.0464)$	0.020 $(10^{-3} \cdot 0.0004)$
σ_i	0.4422 $(10^{-3} \cdot 0.0811)$	0.0799 $(10^{-3} \cdot 0.0030)$	0.1000 $(10^{-3} \cdot 0.0011)$	0.2002 $(10^{-3} \cdot 0.0112)$	0.5000 $(10^{-3} \cdot 0.0022)$
R^d	0.1304 $(10^{-3} \cdot 0.0164)$	0.1502 $(10^{-3} \cdot 0.0083)$	0.001 $(10^{-3} \cdot 0.00011)$		
R^{fr}		0.0499 $(10^{-3} \cdot 0.0007)$	0.0199 $(10^{-3} \cdot 0.0004)$		0.0500 $(10^{-3} \cdot 0.0007)$
P	0.7039 $(10^{-4} \cdot 0.0163)$	0.5998 $(10^{-4} \cdot 0.0049)$	0.7998 $(10^{-4} \cdot 0.0062)$		
P_I	0.8003 $(10^{-3} \cdot 0.0124)$		0.8001 $(10^{-3} \cdot 0.0072)$	0.8514 $(10^{-3} \cdot 0.0565)$	
Q	4.45 $(10^{-3} \cdot 0.2491)$				
Q_I	0.448 $(10^{-3} \cdot 0.0805)$				
λ^d	-0.9110 $(10^{-3} \cdot 0.0882)$	-1.4999 $(10^{-3} \cdot 0.0043)$	-0.3999 $(10^{-3} \cdot 0.002)$	-0.2997 $(10^{-3} \cdot 0.0103)$	-0.3000 $(10^{-3} \cdot 0.00173)$
λ^{fr}		-0.2000 $(10^{-3} \cdot 0.0014)$	-0.5000 $(10^{-3} \cdot 0.0022)$		-0.5000 $(10^{-3} \cdot 0.0022)$
s_1	0.0000458 $(10^{-3} \cdot 0.00207)$		0.0000089 $(10^{-3} \cdot 0.000198)$		
s_2	0.0000245 $(10^{-3} \cdot 0.00162)$		0.0000007 $(10^{-3} \cdot 0.00130)$		
s_3	0.0000132 $(10^{-3} \cdot 0.000927)$		0.0000059 $(10^{-3} \cdot 0.000533)$		

Table 3.8: The parameter estimates for the five-factor model with affine exchange rate, US CPI index, based on the same 30 monthly data observations for 17 yields, from November 2002 to July 2005. Standard errors are in brackets. s_i are the estimates for the volatility of the measurement errors for nominal UK and US yields.

QML parameter estimates for the General UK-US bond market, 5 factor affine model.

Param	US factor X_1	Common X_2	UK factor X_3	Real US factor	Real UK factor
s_4	0.0000159 $(10^{-4} \cdot 0.000701)$		0.00000214 $(10^{-3} \cdot 0.000625)$		
s_5	0.0000534 $(10^{-4} \cdot 0.002316)$		0.00000664 $(10^{-3} \cdot 0.00039)$		
s_6	0.0000257 $(10^{-4} \cdot 0.000991)$				
w_1				0.000006 $(10^{-3} \cdot 0.000123)$	
w_2				0.0000117 $(10^{-3} \cdot 0.000277)$	
w_3				0.0000229 $(10^{-3} \cdot 0.000723)$	
w_1					0.0000119 $(10^{-3} \cdot 0.000683)$
w_2					0.00000599 $(10^{-3} \cdot 0.000350)$
w_3					0.0000095 $(10^{-3} \cdot 0.000473)$
s_{ex}			0.00023 $(10^{-3} \cdot 0.0090)$		
s_{inf}			0.0067 $(10^{-3} \cdot 0.2649)$		

Table 3.9: The parameter estimates for the five-factor model with affine exchange rate, US CPI index, based on the same 30 monthly data observations for 17 yields, from November 2002 to July 2005. Standard errors are in brackets. P_i are Q are the parameters for the affine exchange rate, P_I are Q_I are the parameters for the affine US CPI index, s_{ex} - exchange rate estimation error volatility, s_{inf} - US CPI index estimation error volatility.)

Part III

Credit Risk Modeling

Chapter 4

Correlation between intensity and recovery in credit risk models

We start by presenting a reduced-form multiple default type of model and derive abstract results on the influence of a state variable X on credit spreads, when both the intensity and the loss quota distribution are driven by X. The aim is to apply the results to a concrete real life situation, namely, to the influence of macroeconomic risks on credit spreads term structures. There has been increasing support in the empirical literature that both the probability of default (PD) and the loss given default (LGD) are correlated and driven by macroeconomic variables. Paradoxically, there has been very little effort from the theoretical literature to develop credit risk models that would include this possibility. A possible justification has to do with the increase in complexity this leads to, even for the "tractable" default intensity models. The goal of this paper is to develop the theoretical framework needed to handle this situation and, through numerical simulation, understand the impact on credit risk term structures of the macroeconomic risks. In the proposed model the state of the economy is modeled trough the dynamics of a market index, that enters directly on the functional form of both the intensity of default λ and the distribution of the loss quota q given default. Given this setup, we are able to characterize periods of economic depression by higher default intensity and low recovery, producing thus a *business cycle effect*. Furthermore, we allow for the possibility of an index volatility that depends negatively on the index level and show that, when we include this realistic feature, the impacts on the credit spread term structure are emphasized.[1]

[1]The paper is coauthored with Raquel Gaspar, ISEG, Universidade Téchica de Lisboa. We are indebted to Tomas Björk for fruitful discussions and constant motivation. We gratefully acknowledge helpful comments and suggestions from Phillip Schönbucher and Thorsten Schmidt.

4.1 Introduction

Recent empirical studies show that there is a significant systematic risk component in defaultable credit spreads. See Frye (2000a), Frye (2000b), Frye (2003), Altman, Resti, and Sironi (2004), Düllmann and Trapp (2000) or Elton and Gruber (2004). The model underlying the Basel II internal ratings-based capital calculation (see Basel Committee (2003) and Wilde (2001) measures credit portfolio losses only, that is, portfolio losses that are due to external influences and hence cannot be diversified away. This gives us an indication of what the main concerns are in practice and highlights the need for a realistic model of systematic risk. The purpose of this study is to present a reduced-form multiple default model[2], which analyzes the influence of macroeconomic risks on the term structure of credit spreads at the firm level.

The current theoretical literature considers models where only the default intensity, or equivalently, the *probability of default* (PD) is dependent on a state variable assuming that the *loss given default* (LGD) is either fixed or at least independent of default intensities. See Wilson (1997), Saunders (1999), JP Morgan (1997), Gordy (2000), or Schönbucher (2001). We take that analysis one step further and consider the situation in which the same state variable influences both PD and LGD, making these two quantities dependent on one another.

Both PD and LGD are key in accessing expected capital losses and measuring the exposure of portfolios of defaultable instruments to credit risk. In accessing capital at risk, it is extremely important not to ignore the interdependence between PD and LGD, since this would lead to underestimation of the true risk borne by portfolio holders.

Eventually, we are interested in the case where our state variable represents macroeconomic risks. As a proxy for macroeconomic conditions we consider a market index. It is well known that market uncertainty and its level are negatively correlated. See, for instance, Gaspar (2001) and a recent study by Giese (2005). That is, periods of recession (low index level) also tend to be periods of high uncertainty (high index volatility) reflecting some sort of market panic, while periods of economic boom are perceived as safe periods and with low uncertainty. In setting up the dynamics of the market index, we incorporate this realistic feature by allowing the local volatility of the index to depend negatively on its level.

In terms of the PD and LGD, we concretely take the default intensity and the recovery (given default) to depend on the market situation (the index level). With the PD dependence we try to account for the fact that during bad economic times

[2]For the description of reduced-form models see e.g. Duffie and Singleton (1999).

it is reasonable to expect more defaults, while with the LGD dependence we try to account for the fact that if the entire market is down, the market value of any firm's assets should be lower, and debt holders should recover less if a default occurs.

The main contributions of this study can be summarized as follows.

- We derive abstract results for a multiple default reduced-form model when the default events are modeled by a doubly stochastic marked point process (DSMPP), where both intensity and the mark's density depend on a state variable X.
- We propose a model for the influence of macroeconomic risks on credit spreads.
- Using a concrete model, we are able to simulate realistic behaviors of the term structure of credit spreads.

The rest of the paper is organized as follows.

- In Section 2 we concretize our reduced-form multiple default model and justify the modeling choices, which involve modeling all the variables under the risk-neutral measure $\mathbb{Q}$. We start by describing the setup for the default-free and defaultable bond market and we continue by deriving the abstract results. Finally, we discuss the relation between the risk-neutral assumptions and the objective $\mathbb{P}$-assumptions.
- In Section 3 we introduce the macroeconomic model, presenting the index dynamics and justifying the assumptions about the influence of such risks on the intensity and recovery processes using empirical facts. We derive qualitative results on the influence of the market index on credit spreads.
- In Section 4 we present a concrete model for the intensity and recovery dependence and simulate their impacts on credit spread term structures.
- Section 5 concludes, summarizing the main results and suggesting directions for future research.

4.2 The Setup and Abstract Results

We consider a financial market living on a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{0\leq t\leq T}\right)$ where $\mathbb{Q}$ is the risk-neutral probability measure.

The probability space carries a multidimensional Wiener process W and, in addition, a doubly stochastic marked point process (DSMPP), $\mu(dt, dq)$, on a measurable mark space $(E, \mathcal{E})$ to model the default events.

The filtration $(\mathcal{F}_t)_{0 \le t \le T}$ is generated by W and μ, i.e. $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$.

4.2.1 Default-free bond market

We assume the existence of a liquid market for default-free zero-coupon bonds, for every possible maturity T. We denote the price at time t of a default-free zero-coupon bond with maturity T by $p(t, T)$.

In order to describe the default-free bonds market we use Heath-Jarrow-Morton (HJM) framework, modeling directly, under the martingale measure $\mathbb{Q}$, the dynamics of the instantaneous forward rates, $f(t, T)$. There is a one-to-one correspondence between zero-coupon bond prices and forward rates, so by assuming existence of zero-coupon bond prices for all maturities we also guarantee existence of forward rates for all maturities. We recall the fundamental relation:

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} \qquad \Leftrightarrow \qquad p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}. \tag{4.1}$$

Assumption 4.2.1. (Default-free forward rates)

The dynamics of default free forward rates, under the martingale measure $\mathbb{Q}$, *are given by*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \tag{4.2}$$

with

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma^*(t,s)ds. \tag{4.3}$$

where $\sigma(\cdot, T)$ *is a row vector of regular enough adapted processes,* W *is a* $\mathbb{Q}$-*Wiener process.*

The default free short-rate is $r(t) = f(t,t)$ and the default free zero-coupon bond prices are denoted by $p(t, T)$. No arbitrage and the fundamental relation in (4.1) give use the bond prices dynamics as

$$\frac{dp(t,T)}{p(t,T)} = r(t)dt + \eta(t,T)dW(t),$$

where $\eta(t,T) = -\int_t^T \sigma(t,s)ds.$

4.2.2 Defaultable bonds market

In addition to the risk-free bond market mentioned above, we consider a defaultable bond market. We assume that each company on the market issues a continuum of bonds with maturities T.

Assumptions 4.2.6 below characterize the default events and the dependence of both the default intensity and the recovery rate distribution on an abstract stochastic state variable X.

Assumption 4.2.2. *There exist an underlying stochastic state variable X, whose dynamics under the risk neutral measure $\mathbb{Q}$ are given by*

$$dX_t = \alpha_X(t, X_t)dt + \sigma_X(t, X_t)dW_t, \tag{4.4}$$

where W, is the same as in (4.2).[3]

Concepts and Assumptions

In Assumption 4.2.6 we will define the basic multiple default setup. A multiple default setup is based on the observation that whenever the obligor defaults, the company is not liquidated but instead reorganized. The firm and its claims continue to live and operate. However, the face value of the claims is reduced by a fraction q. Behind this model is the intuition that, given a distress situation for the obligator's business, the debt holders are willing to accept the renegotiation of their claims (accepting to lose some fraction q of the face value of the claims) in order to avoid a process of bankruptcy, which is typically costly, and allowing the firm to continue operating.[4] It is possible that a whole sequence of defaults is taking place, every time the company reducing the face value of the debt and the bondholder accepting the conditions of the deal.

Definition 4.2.3. *The* **loss quota** *is the fraction by which the promised final payoff of the defaultable claim is reduced at each time of default. We denote the loss quota by q.*

Definition 4.2.4. *The* **remaining value**, *after all reductions in the face value of the defaultable claim due to defaults in the time interval $[0, t]$, is denoted $V(t)$.*

[3]This way we allow for the possibility that one of our factors may be related to the risk-free side of the economy. In that case, some consistency relation must hold between (4.2) and (4.4).

[4]This model mimics the effect of a rescue plan as it is described in many bankruptcy codes. The old claimants have to give up some of their claims in order to allow for rescue capital to be invested in the defaulted firm. They are not paid out in cash (this would drain the defaulted firm of valuable liquidity) but in new defaultable bonds of the same maturity. [Schönbucher (2003)]

Definition 4.2.5. *$\bar{p}(t,T)$ is, at time t, the* **price of a defaultable zero-coupon bond** *with maturity T. The payoff at time T of the bond is, thus, $V(T)$ the remaining part of the face value of the bond after all reductions due to defaults in the time interval $[0,T]$, i.e.,*

$$\bar{p}(T,T) = V(T).$$

Assumption 4.2.6.

1. *We assume that default happens at the following sequence of the stopping times $\tau_1 < \tau_2 < \dots$, where τ_i is the time of the i-th jump of our point process.*
2. *At each default time τ_i the jump size, q_i mark or loss quota, is drawn from the mark space $E = (0,1)$.*
3. *We assume that there is no total loss at default, i.e., the loss quota $q_i < 1$ for all $i = 1, 2, \cdots$.*
4. *We assume that both*
   (i) the arrivals of default times $(\tau_i)_{i\geq 1}$
   (ii) the distribution of the loss quotas given default $(q_i)_{i\geq 1}$
   depend upon our stochastic state process X.

Given that at each default time τ_i the final claim amount is reduced by a loss quota q_i to $(1-q_i)$ times what it was before, we obtain

$$V(t) = \prod_{\tau_i \leq t} (1 - q_i), \tag{4.5}$$

where q_i is the stochastic marker to the default time τ_i.

According to risk-neutral valuation, the price at time t, of the defaultable bond with maturity T equals to

$$\bar{p}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} V(T) \Big| \mathcal{F}_t \right].$$

or, equivalently,

$$\bar{p}(t,T) = p(t,T)\mathbb{E}^T\left[V(T) | \mathcal{F}_t \right] , \tag{4.6}$$

where $\mathbb{E}^T\left[\cdot | \mathcal{F}_t \right]$ denote conditional expectation under the T-forward measure.

Definition 4.2.7. *We define the* **instantaneous defaultable forward rate**, $\bar{f}(t,T)$, *similarly to its risk-free equivalent*

$$\bar{f}(t,T) = -\frac{\partial}{\partial T} \ln \bar{p}(t,T). \tag{4.7}$$

The **defaultable short rate** *is defined as* $\bar{r}(t) = \bar{f}(t,t)$.

Using the above definition we also have

$$\bar{p}(t,T) = V(t) \exp\left\{-\int_t^T \bar{f}(t,s)ds\right\}, \tag{4.8}$$

where recall that $\bar{p}(t,t) = V(t)$ and $V(t)$ is given in (4.5).

Definition 4.2.8. *The* **short credit spread** $s(t)$ *is defined as the difference between the defaultable and non-defaultable short rates*

$$s(t) = \bar{r}(t) - r(t).$$

Definition 4.2.9. *The* **forward credit spread** $s(t,T)$ *is defined as the difference between the defaultable short rate and non-defaultable forward rates*

$$s(t,T) = \bar{f}(t,T) - f(t,T).$$

Existence of intensity

We start by giving the abstract definitions of a Marked *Poisson* Point Process, a Cox Process and a *Doubly Stochastic* Marked Poisson Process (DSMPP). To introduce the definitions we need first to define the following filtrations.

Notation 1. (Filtrations)

- We call the filtration generated by $W(t)$ the *background filtration* $\left(\mathcal{F}_t^W\right)_{t\geq 0}$, and

  $$\mathcal{G}^W = \bigvee_{t\geq 0} \mathcal{F}_t^W,$$

  is the information set containing all future and past background information.

  In our setup it will be assumed that all the default-free processes are adapted to $\left(\mathcal{F}_t^W\right)_{t\geq 0}$.

- The full filtration is reached by combining $\left(\mathcal{F}_t^W\right)_{t\geq 0}$ and the filtration $(\mathcal{F}_t^\mu)_{t\geq 0}$ which is generated by MPP μ

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu.$$

- We define the filtration generated by *all* the information concerning the background process X, and our only past information on our MPP μ

$$\mathcal{G}_t^W = \mathcal{G}^W \vee \mathcal{F}_t^\mu.$$

Definition 4.2.10. (DSMPP)

- *We call the Marked Point Process $\hat{\mu}$ an $\mathcal{F}_t^\mu$- Marked* Poisson *Process if there exists a deterministic measure $\hat{\nu}$ on $\mathbb{R}_+ \times E$ such that*

$$\mathbb{P}\left(\hat{\mu}((s,t] \times B) = k \,|\mathcal{F}_s^\mu\right) = \frac{(\hat{\nu}((s,t] \times B))^k}{k!} e^{-\hat{\nu}((s,t]\times B)}, \quad a.s., \quad B \in E.$$

- *We call a counting process $N = (T_n)$ adapted to right-continuous filtration a $\mathcal{G}_t^W$ - Cox Process if there is an $\mathcal{G}^W$-measurable random measure ν satisfying*

$$\mathbb{P}\left(N(s,t] = k \,\middle|\mathcal{G}_s^W\right) = \frac{(\nu((s,t]))^k}{k!} e^{-\nu((s,t])}, \quad a.s \quad k \in \mathbb{N}\ .$$

- *We call the Marked Point Process μ an $\mathcal{G}_t^W$- DSMPP if there exists a $\mathcal{G}^W$-measurable random measure ν on $\mathbb{R}_+ \times E$ such that*

$$\mathbb{P}\left(\mu((s,t] \times B) = k \,\middle|\mathcal{G}_s^W\right) = \frac{(\nu((s,t] \times B))^k}{k!} e^{-\nu((s,t]\times B)}, \quad a.s., \quad B \in E.$$

Our goal is to construct a Marked Point Process such that its compensator is allowed to depend on our stochastic state variable X (which in general means that its conditional default distribution as well as intensity are *both* allowed to depend on X_t), and conditional on the realization of the state variable it is $\mathcal{G}_t^W$-DSMPP. In other words, we want to prove the existence of MPP such that its compensator ν is allowed to depended on our stochastic state variable X, so we can write

$$\nu(dt, dq, \omega) = \nu(t, dq, X_t), \qquad \mathbb{Q} - a.s. \tag{4.9}$$

We define the compensated point process $\tilde{\mu}(dt, dq)$ as

$$\tilde{\mu}(dt, dq) = \mu(dt, dq) - \nu(t, dq, X_t).$$

The following Theorem shows that the DSMPP with compensator of the form (4.9) exists.

Theorem 4.2.11.

Assume that ν admits intensity and define $\nu(dt, dq, X_t) = M_t(dq, X_t)dt$, $\mathbb{Q} - a.s$ where $M_t(dq, x)$ is a deterministic measure on E for any fixed x and t.

Let $\hat{\nu}(dt, dq) = m_t(dq)dt$ be a deterministic compensator for some Marked Poisson Process $\hat{\mu}$. Assume that:

(i) $M(t, dq, x)$ is measurable w.r.t. $\mathcal{G}^W$

(ii) $M(t, dq, x)$ is absolutely continuous w.r.t. $m(t, dq)$ on $\mathcal{E}$, that is,

$$M_t(dq, x) << m_t(dq)$$

Then, there exists a $\mathcal{G}_t^W$-DSMPP μ, such that its compensator is of the form (4.9).

Proof. We fix $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0 \le t \le T}\right)$ and a Marked Point Process μ with the compensator

$$\hat{\nu}(dt, dq) = m_t(dq)dt$$

and, as before, $\mathcal{G}_t^W = \mathcal{G}^W \vee \mathcal{F}_t^\mu$.

Since $M_t(x, dq)$ is absolutely continuous w.r.t $m_t(dq)$ on $\mathcal{E}$, then according to the Radon-Nikodym Theorem for every t there exists a $\mathcal{E} \times \mathcal{G}^W$-measurable nonnegative function $\varphi_t(q, x)$, $\varphi : E \times R_+ \to R_+$, such that

$$M(t, x, A) = \int_A \varphi(t, q, x) m(t, dq), \quad \text{for all} \quad A \in \mathcal{E}$$

or

$$M(t, x, dq) = \varphi(t, q, x) m(t, dq).$$

We define the process L_t as

$$\begin{cases} dL_t &= L_{t-} \int_E \{\varphi(t, q, X_t) - 1\} \{\hat{\mu}(dt, dq) - m_t(dq)dt\} \\ L_0 &= 1. \end{cases}$$

We notice that $\varphi(t, q, X_t) \in \mathcal{G}_0^W$. Define the new measure on $\mathcal{G}_t^W$, $0 \le t \le T$ as

$$d\mathbb{Q} = L_t d\mathbb{P}$$

According to the Girsanov transformation the $\mathbb{Q}$-compensator of the new process is exactly

$$\nu(dt, dq) = \hat{\nu}(dt, dq)(1 + \varphi_t(q, X_t) - 1) = \varphi_t(q, X_t) m_t(dq)dt = M_t(dq, X_t)dt.$$

First, we would like to show that the $\mathbb{Q}$-distribution of ν is the same as the $\mathbb{P}$-distribution. We note that $\mathcal{G}_0^W = \mathcal{G}^W$ and that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_0^W} = L_0 = 1 ,$$

thus, $\mathbb{P} = \mathbb{Q}$ on $\mathcal{G}_0^W$.

Second, we would like to show that

$$\mathbb{P}\left(\mu((s,t]\times B) = k \,\middle|\, \mathcal{G}_s^W\right) = \frac{(\nu((s,t]\times B))^k}{k!} e^{-\nu((s,t]\times B)}, \quad \text{a.s.}, \quad B \in E. \quad (4.10)$$

We prove (4.10) using characteristic functions. Define the stochastic process

$$Y_t = \int_0^t \int_E q\hat{\mu}(dt, dq).$$

Changing the measure we obtain that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{iuY_t}\middle|\, \mathcal{G}_0^W\right] = \mathbb{E}^{\mathbb{P}}\left[L_t e^{iuY_t}\middle|\, \mathcal{G}_0^W\right].$$

Define $Z_t = L_t e^{iuY_t}$, then the dynamics of Z_t is

$$\begin{aligned}
dZ_t &= L_t \int_E \left\{e^{iu(Y_{t-}+q)} - e^{iuY_{t-}}\right\} \mu(dt, dq) \\
&\quad + L_{t-} e^{iuY_t} \int_E (\varphi(t,q,X_t) - 1)\left\{\hat{\mu}(dt,dq) - m_t(dq)dt\right\} \\
&\quad + \int_E L_{t-}(\varphi(t,q,X_t) - 1)\left\{e^{iu(Y_{t-}+q)} - e^{iuY_{t-}}\right\} \hat{\mu}(dt, dq) \\
&= \int_E Z_{t-}\varphi(t,q,X_t) m_t(dq)(e^{iuq} - 1)dt + \int_E Z_{t-}(e^{iuq} - 1)\varphi(t,q,X_t)\tilde{\mu}(dt,dq) \\
&\quad + \int_E Z_{t-}(\varphi(t,q,X_t) - 1)\tilde{\mu}(dt,dq)
\end{aligned}$$

where $\tilde{\mu}(dt, dq) = \hat{\mu}(dt, dq) - m_t(dq)$.

We notice also that $Z_0 = 1$, then

$$\begin{aligned}
Z_t &= 1 + \int_0^t \int_E Z_{s-}\varphi(s,q,X_s) m_t(dq)(e^{iuq} - 1)ds + \int_0^t \ldots \tilde{\mu}(ds, dq) \\
&= 1 + \int_0^t \int_E Z_{s-}(e^{iuq} - 1) M_s(dq, X_s) ds + \int_0^t \ldots \tilde{\mu}(ds, dq).
\end{aligned}$$

Denote $\xi_t = \mathbb{E}^{\mathbb{P}}\left[Z_t \middle| \mathcal{G}_0^W\right]$, then

$$\xi_t = 1 + \int_0^t \int_E \xi_s (e^{iuq} - 1) M_s(dq, X_s) ds.$$

thus since ξ_t does not depend on q and $M_s(dq, X_s)$ is $\mathcal{G}_0^W$-measurable

$$\xi_t = e^{\int_0^t \int_E (e^{iuq}-1) M_s(dq, X_s) ds},$$

Note that $\nu(dt, dq, X_t) = M_t(dq, X_t)dt$ is $\mathcal{G}^W$ measurable.

The result follows from the fact that the characteristic function of the process

$$\bar{Y}_t = \int_0^t \int_E q\bar{\mu}(dt, dq)$$

where $\bar{\mu}$ is a Market Poisson Process with compensator $\bar{\nu}(t, dq)$ is given by

$$\mathbb{E}\left[e^{iu\bar{Y}_t}\right] = \exp\left\{\int_0^t \int_E (e^{iuq} - 1)\bar{\nu}(t, dq)\right\}.$$

■

In our application we would like to model separately the dependence of the intensity process and of the loss quota distribution on our state variable X. This allows us, not only to include empirically observed facts in both quantities but it also makes the interpretation of the results more straightforward. To model the loss quota distribution, in the context of our DSMPP, we need the notion of a stochastic kernel.

Definition 4.2.12. *K is a* **stochastic kernel** *from $\mathbb{R}_+$ to E, i.e. from $(\mathbb{R}_+, \mathcal{B}_+)$ to $(E, \mathcal{E})$, if it is a mapping from $R_+ \times \mathcal{E}$ into R_+ such that:*

- *$K(\cdot, A)$ is measurable for all $A \in \mathcal{E}$*
- *$K(t, \cdot)$ is a measure on E $\forall t$*

If $K(t, E) = 1$, then the kernel is called a **probability distribution**.

In our case K will be stochastic only to the extent that X is stochastic. That is, conditional on the state variable realization we will have a deterministic loss quota distribution.

The ideal construction procedure can then be described as follows.

Remark 4.2.13. Construction procedure *We construct our DSMPP μ as follows.*

1. *We specify our Wiener driven stochastic state variable X.*

2. *We specify the intensity* $\lambda(t, X_t)$ *as a function of our state variable.*

3. *We specify the instantaneous conditional loss quota distribution as a function of the state variable* $K(t, dq, X_t)$.

4. *Finally, we construct the stochastic compensator* ν

$$\nu(dt, dq, X_t) = K(t, dq, X_t)\lambda(t, X_t)dt, \tag{4.11}$$

A compensator representation as in (4.11) has the additional advantage of satisfying a needed consistency requirement. Indeed, it allows us to value credit derivatives that do not depend on recovery in a consistent way with those that do depend upon recovery. Without the separation result we would have to model at once the whole compensator $M_t(dq, X_t)$, which is not very intuitive and we would not know how to derive in a consistent way key ingredients that depend exclusively upon $\lambda(t, X_t)$, like the implied survival probability, the price default digital payoffs or the price defaultable bonds with zero recovery.

Before we go on to the abstract results, we note that Theorem 4.2.11 suffices to guarantee the existence of a DSMPP μ with a compensator of the form (4.11), as we can simply set

$$M_t(dq, X_t) = K(t, dq, X_t)\lambda(t, X_t),$$

and notice that $M_t(dq, x)$ is a measure for fixed t and x.

4.2.3 Abstract results

In this section we derive the main results concerning the short and forward credit spreads given the setup above.

Proposition 4.2.14. *Consider a* T*-defaultable claim* $\mathcal{X}$. *For the purpose of computing expectations, and in particular its price at time* $t \leq T$

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T r s_d s} V(T)\mathcal{X}\right],$$

it is equivalent to use the following two dynamics for the remaining value process

(i)

$$\begin{aligned} \frac{dV(t)}{V(t-)} &= -\int_0^1 q\mu(dt, dq) \\ V(t) &= v, \end{aligned} \tag{4.12}$$

(ii)

$$\begin{aligned} \frac{dV(t)}{V(t-)} &= -q^e(t-, X_{t-})dN_t \qquad (4.13) \\ V(t) &= v \ . \end{aligned}$$

where μ is a DSMPP with compensator $\nu(t, X_t) = \lambda(t, X_t)K(t, dq, X_t)dt$, N is a Cox process with intensity $\lambda(t, X_t)$ and we define

$$q^e(t, X_t) = \int_0^1 K(t, dq, X_t) \ .$$

Proof. Using the V dynamics in *(i)* we get,

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}V(T)\mathcal{X}\Big|\mathcal{F}_t\right] = \\ &= V(t)\underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathcal{X}\Big|\mathcal{F}_t\right]}_{\pi(t,\mathcal{X})} - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T\int_0^1 qV_{s-}\mu(dq, ds)\mathcal{X}\Big|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T\int_0^1 qV_{s-}\mu(dq, ds)\mathcal{X}\Big|\mathcal{G}_t^W\right]\Big|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T\int_0^1 qV_{s-}\mu(dq, ds)\mathcal{X}\Big|\mathcal{G}_t^W\right]\Big|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}\left\{\int_0^1 qK(s, dq, X_s)\right\}\lambda(s, X_s)ds\mathcal{X}\Big|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s, X_s)\lambda(s, X_s)ds\mathcal{X}\Big|\mathcal{F}_t\right] \end{aligned}$$

Using the V dynamics in *(ii)* we get,

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}V(T)\mathcal{X}\Big|\mathcal{F}_t\right] = \\ &= V(t)\underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathcal{X}\Big|\mathcal{F}_t\right]}_{\pi(t,\mathcal{X})} - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s, X_s)dN(s)\mathcal{X}\Big|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s, X_s)dN(s)\mathcal{X}\Big|\mathcal{G}_t^W\right]\Big|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T V_{s-}q^e(s, X_s)dN(s)\mathcal{X}\Big|\mathcal{G}_t^W\right]\Big|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s, X_s)\lambda(s, X_s)ds\mathcal{X}\Big|\mathcal{F}_t\right] \end{aligned}$$

The results follows from comparing the final expressions on both cases. ■

Proposition 4.2.15. *Given Assumption 4.2.6, and under the martingale measure* $\mathbb{Q}$.

1. *The short credit spreads,* $s(t)$, *have the following functional form*

$$s(t) = \lambda(t, X_t) q^e(t, X_t) > 0 \tag{4.14}$$

   where

$$q^e(t, X_t) = \int_0^1 qK(t, dq, X_t)$$

   can be interpreted as the locally expected loss quota (which is positive for $q > 0$*).*

2. *Then the forward credit spread* $s(t, T)$ *takes the form*

$$s(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\{r(T) + \lambda(T, X_T) q^e(T, X_T)\} e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]} - f(t,T) \tag{4.15}$$

Proof. The time t price of the defaultable zero-coupon bond with maturity T is equal to

$$\bar{p}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} V(T) \middle| \mathcal{F}_t\right], \tag{4.16}$$

where $V(T)$ is the residual of the face value after multiple defaults up to time T.

Making use of Proposition 4.2.14, instead of $\frac{dV(t)}{V(t-)} = -\int_0^1 q\mu(dt, dq)$ with our DSMPP μ (these dynamics follow directly from (4.5)), we use

$$\frac{dV(t)}{V(t-)} = -q^e(t-, X_{t-}) dN_t$$

where N is the Cox process with intensity $\lambda(t, X_t)$.

For every fixed t, define $Z(u)$ as follows

$$Z(u) = e^{\int_t^u q^e(s,X_s)\lambda(s,X_s)ds} V(u).$$

We note that then the dynamics of $Z(u)$ take the form

$$dZ(u) = -Z_{u-} q^e(u-, X_{u-}) \{dN_u - \lambda(u, X_u) du\}, \qquad u \geq t, \quad t\text{-fixed}$$

and $Z(u)$ is a $\mathbb{Q}$-martingale conditional on the filtration $\mathcal{F}_t^W$. Thus,

$$\mathbb{E}^{\mathbb{Q}}\left[Z(T)|\mathcal{F}_t^W\right] = Z(t).$$

The price of a defaultable bond is then can be found as

$$\begin{aligned}
\bar{p}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}V(T)\Big|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds}Z(T)\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds}Z(T)\Big|\mathcal{G}_t^W\right]\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds}\mathbb{E}^{\mathbb{Q}}\left[Z(T)|\mathcal{G}_t^W\right]\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds}Z(t)\Big|\mathcal{F}_t\right] \\
&= V(t)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds}\Big|\mathcal{F}_t\right] \\
&= V(t)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \{r(s)+q^e(s,X_s)\lambda(s,X_s)\}ds}\Big|\mathcal{F}_t\right].
\end{aligned}$$

Using the basic relations between defaultable bond prices and defaultable forward rates in (4.7) we can, thus, write

$$\bar{f}(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\{r(T)+\lambda(T,X_T)q^e(T,X_T)\}\, e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}. \tag{4.17}$$

Finally using that $\bar{f}(t,t) = \bar{r}(t)$ in the above expression we obtain

$$\bar{r}(t,r_t,X_t) = r(t) + q^e(t,X_t)\lambda(t,X_t).$$

The result follow from $s(t) = \bar{r}(t) - r(t)$ and $s(t,T) = \bar{f}(t,T) - f(t,T)$. ∎

From the proof of the previous Proposition results also a nice formulation for the defaultable bond prices.

Corollary 4.2.16. *Under the conditions of Proposition 4.2.15 we have*

$$\bar{p}(t,T) = V(t)\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}_s ds}\right]. \tag{4.18}$$

It is obvious from (4.15) that the expression of the forward credit spread looks quite messy. This happens because we choose to present it under the martingale measure $\mathbb{Q}$. The next lemma give us two simpler representations at the cost of using forward measures.

Lemma 4.2.17. *The forward credit spread, $s(t,T)$, has the following representation, equivalent to (4.15)*

(i)

$$s(t,T) = -\frac{\partial}{\partial T} \ln\left(\mathbb{E}_t^T\left[V(T)\right]\right)$$

where $\mathbb{E}_t^T[\cdot]$ *stands for expectation under the* T*-forward measure.*

(ii)

$$s(t,T) = \bar{\mathbb{E}}_t^T\left[\bar{r}(T)\right] - \mathbb{E}_t^T\left[r(T)\right].$$

where $\bar{\mathbb{E}}_t^T\cdot$ *stands for expectation under the measure* $\bar{\mathbb{Q}}^T$ *where we have*

$$L^T = \frac{d\bar{\mathbb{Q}}^T}{d\mathbb{Q}} = \frac{V(0)e^{-\int_0^T r_s+\lambda(s,X_s)q^e(s,X_s)ds}}{\bar{p}(0,T)} \quad on\ \mathcal{F}_T\ .$$

Proof. To prove (i) we note that

$$\begin{aligned}\bar{p}(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}V(T)\right] \\ &= p(t,T)\mathbb{E}_t^T\left[V(T)\right].\end{aligned}$$

From the definition of the forward defaultable rate and the forward credit spread we also have

$$\begin{aligned}\bar{p}(t,T) &= V(t)e^{-\int_t^T \bar{f}(t,u)du} \\ &= V(t)e^{-\int_t^T f(t,u)du}e^{-\int_t^T s(t,u)du} = V(t)p(t,T)e^{-\int_t^T s(t,u)du}.\end{aligned}$$

Comparing we realize $V(t)e^{-\int_t^T s(t,u)du} = \mathbb{E}_t^T\left[V(T)\right]$, differentiating w.r.t. T and solving for $s(t,T)$ we get

$$s(t,T) = \frac{\partial}{\partial T}\ln\left(\frac{V(t)}{\mathbb{E}_t^T\left[V(T)\right]}\right) = \underbrace{\frac{\partial}{\partial T}\ln(V(t))}_{0} - \frac{\partial}{\partial T}\ln\left(\mathbb{E}_t^T\left[V(T)\right]\right).$$

To prove (ii) we start by noting that it is a well known fact that for the risk-free rates $f(t,T) = \mathbb{E}_t^T\left[r(T)\right]$.[5]

Thus, it remains to show that

$$\bar{f}(t,T) = \bar{\mathbb{E}}_t^T\left[\bar{r}(T)\right]\ .$$

Using $\bar{r}(t) = r(t) + q^e(t,X_t)\lambda(t,X_t)$, equation (4.17) becomes

$$\bar{f}(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\bar{r}(T)e^{-\int_t^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}\ .$$

[5]See Björk (2004) for further details.

Since $\frac{\bar{p}(t,T)}{V(t)} = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \{r(s)+q^e(s,X_s)\lambda(s,X_s)\}ds} \middle| \mathcal{F}_t\right]$ we have

$$\bar{f}(t,T)\frac{\bar{p}(t,T)}{V(t)} = \mathbb{E}_t^{\mathbb{Q}}\left[\bar{r}(T)e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right] .$$

For the r.h.s the following holds

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}\left[\bar{r}(T)e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right] = \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\bar{r}(T)\underbrace{\frac{V(0)e^{-\int_0^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}}{\bar{p}(0,T)}}_{L^T}\right]\frac{e^{\int_0^t \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\bar{p}(0,T)}{V(0)} \\
&= \bar{\mathbb{E}}_t^T r(T)\mathbb{E}_t^{\mathbb{Q}}\left[L^T\right]\frac{e^{\int_0^t \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\bar{p}(0,T)}{V(0)} \\
&= \bar{\mathbb{E}}_t^T r(T)\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right] \\
&= \frac{\bar{p}(t,T)}{V(t)}\bar{\mathbb{E}}_t^T\left[r(T)\right]
\end{aligned}$$

Where we define $L^T = \frac{V(0)e^{-\int_0^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}}{\bar{p}(0,T)}$ on $\mathcal{F}_T$.

Comparing with the l.h.s we get $\bar{f}(t,T) = \bar{\mathbb{E}}_t^T \bar{r}(T)$. ■

4.2.4 On the market price of jump risk

We now note that the setup has been defined under the martingale measure $\mathbb{Q}$. Now we specify the implicit assumption on the market price of jump risk that will allows us to extrapolate from our objective intuitions ($\mathbb{P}$-intuitions) when setting up a concrete model later on.

In order to see the connections between the intensities under the different measures we recall the appropriate Girsanov theorem [6].

Theorem 4.2.18. (Girsanov for DSMPP)
Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$ be a filtered probability space which supports $\mathbb{Q}$-Brownian motion $W(t)$ and a marked point process $\mu(dt, dq)$, where the marker q is drawn from the mark space $X = [0,1]$. The $\mathbb{Q}$-compensator is assumed to take the form $\nu(dt, dq) = K(t, dq)\lambda(t)dt$ under $\mathbb{Q}$. Here λ is the $\mathbb{Q}$-intensity of the arrivals of the point process and $K(t, dq)$ is the $\mathbb{Q}$-conditional distribution of the marker. Let

[6] See, for example, Björk (2004) or Schönbucher (2003).

h be a predictable process and $\phi_t(q)$, $\phi_t(q) > -1$ a predictable function with the properties

$$\int_0^t \|h\|^2 ds < \infty, \qquad \int_0^t \int_X |\phi_t(q)| K(s, dq) \lambda(s) ds < \infty.$$

Define the process $L(t)$ by

$$\begin{cases} \dfrac{dL(t)}{L(t-)} &= h(t)dW(t) + \int_X \phi(t) \left\{\mu(dt, dq) - \nu(dt, dq)\right\} \\ L(0) &= 1 \end{cases} .$$

Define the probability measure $\mathbb{P}$ as follows

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = L(t) \quad on \quad \mathcal{F}_t \quad \forall t > 0.$$

Then:

1. *We can write*
$$dW(t) = dW_P(t) + \zeta(t)dt, \tag{4.19}$$
where W_P is a P-Wiener process.

2. *The predictable compensator under $\mathbb{P}$ is as follows*
$$\nu^P(dt, dq) = (1 + \phi_t(q))\nu(dt, dq). \tag{4.20}$$

$\phi_t(q)$ in (4.20) is the *market price of jump risk* and, in principle, is stochastic $\phi_t(q, \omega, X)$.

The following Lemma shows that in the case when Girsanov kernels are deterministic the measure transformation does not affect K and gives us the relationship between the intensities of default under $\mathbb{P}$ and under $\mathbb{Q}$.

Lemma 4.2.19. *Assume that the market price of jump risk, ϕ in (4.20) is a deterministic function of time. That is*

$$\phi_t(q, \omega) = \phi(t) \ . \tag{4.21}$$

Then,

1. *The $\mathbb{Q}$-default intensity, λ, relates to the $\mathbb{P}$-default intensity λ^P, by*
$$\lambda^P(t, X) = \lambda(t, X)(1 + \phi(t)) \tag{4.22}$$

2. *The $\mathbb{Q}$-loss quota distribution, conditional on default, $K_t(dq)$, equals to the conditional on default loss quota distribution under $\mathbb{P}$, $K_t^P(dq)$.*

Proof. The predictable compensator takes the form

$$\nu(dt, dq) = M_t(dq)dt = \lambda(t)K(t, dq)dt,$$

where

$$\lambda(t) = M_t(E), \quad K(t, dq) = \frac{M_t(dq)}{M_t(E)}$$

From the Girsanov theorem we obtain that under $\mathbb{P}$

$$M_t^P(dq) = M_t(dq)(1 + \phi(t))$$

$$\lambda^P(t) = \int_E (1 + \phi(t))M_t(dq)$$

Thus is case if $\phi(t)$ is deterministic and does not depend on q we see that

$$\lambda_t^P = (1 + \phi(t))M_t(E) = (1 + \phi(t))\lambda(t) \tag{4.23}$$

$$K_t^P(dq) = \frac{(1 + \phi(t))M_t(dq)}{\int_E (1 + \phi(t))M_t(dq)} = \frac{M_t(dq)}{\int_E M_t(dq)} = K_t(dq). \tag{4.24}$$

■

We see that, in this case, the conditional distribution of the loss quota remains unchanged while intensity changes according to (4.23), i.e. multiplied by a deterministic function of time. This also means that once we have specified the influence of the state variable X under $\mathbb{Q}$ we have also specified its influence under the objective measure (up to some deterministic factor in the case of the intensity process λ).

Obviously it is discussable if assuming (4.22) is a very strong restriction or not. For this study we assume it is not so we can use our objective intuitions in setting up the applied model for macroeconomic risks. Nonetheless, the unhappy reader, can also interpret all our results as only $\mathbb{Q}$-results.

4.3 The Macroeconomic Risks

4.3.1 Setup

We model systematic risk of an economy by considering what we call a *market index*, and consider this market index both when it is the price of an important traded asset in the economy and when it is not the price of any traded asset.

An example of an *index* that is the price of a traded asset is, say, oil price. Indices, for instance, stock market indices or any other type of indices, on the other hand, are examples of *indices* that are not prices of traded assets. We consider the two possibilities at all times.

It is rather well known that market index volatility (for example S&P500) tends to increase when the market as a whole is depressed (low values of the index) and, conversely, volatility decreases when the market index is high (see Gaspar (2001)). In order to account for this fact, we make market index volatility dependent on the index level.

Assumption 4.3.1. (Market Index)
Under the martingale measure $\mathbb{Q}$, the market index I, satisfies the following stochastic differential equation (SDE)

- *When I is the price of a traded asset*

$$dI_t = r(t)I_t dt + \gamma(t, I_t)I_t dW(t)$$

- *When I is not the price of a traded asset*

$$dI_t = \zeta(t)I_t dt + \gamma(t, I_t)I_t dW(t),$$

where r is the short rate, γ is a row-vector, W is a $\mathbb{Q}$-Wiener process.

Furthermore, for each entry γ_i, the following holds

(i)

$$\frac{\partial \gamma_i}{\partial I}(t, I) < 0 \ . \tag{4.25}$$

The property *(i)* in (4.25) represents the empirically observed fact that periods when the market as a whole is depressed are periods of higher volatility, while booms are associated with low volatilities.

Since it would be wrong in our setup to consider I to be independent of the risk-free short rate r (at least under the risk-neutral measure $\mathbb{Q}$), and we cannot ignore its influence, we can also consider r as one of the factors.

We will now present explicitly the dependence between the default intensities and the loss quota distribution on the factors. The results from the abstract section can be used immediately with

$$X = \begin{bmatrix} I \\ r \end{bmatrix} .$$

Finally, it is also reasonable to assume that firms that are less sensitive to systematic risks will suffer less in terms of an increase in their default intensities than firms that are more sensitive. Hence, we introduce a measure of sensitivity to systematic risk, ϵ, $\epsilon \in [0, 1]$.

Assumption 4.3.2. (The default Intensity)
The intensity is a deterministic function of (t, r, I, ϵ). *Furthermore, we have*

(ii)

$$\lambda(t, r, I, 0) = \bar{\lambda} \quad \bar{\lambda} \in \mathbb{R}_+ \tag{4.26}$$

(iii)

$$\frac{\partial \lambda(t, r, I, \epsilon)}{\partial \epsilon} > 0 \ , \tag{4.27}$$

(iv)

$$\frac{\partial \lambda(t, r, I, \epsilon)}{\partial I} < 0 \ , \tag{4.28}$$

(v)

$$\frac{\partial \lambda(t, r, I, \epsilon)}{\partial r} > 0 \ . \tag{4.29}$$

Assumption 4.3.2 is based on the intuition that if a firm's financial situation is strong enough, it should not really matter if the economy is booming or if it is in recession. That is, firms that are financially solid should be much less sensitive to the business cycle than those in a less solid financial position. One can also regard the parameter ϵ as a measure of a firm's creditworthiness. Firms with high creditworthiness typically tend to be less sensitive to the business cycle influence than less creditworthy firms.

If this is so, then it makes sense to include properties *(ii)* and *(iii)* in (4.26) and (4.27), respectively.

Properties *(iv)* and *(v)* (eq (4.28),(4.29)) tell us that if the default intensity depends on macroeconomic variables, then the PD is higher during recessions or periods of high risk-free interest rates, and lower in booms or periods of low risk-free interest rates. The influence of the index value is related to the increase in uncertainty during periods of recession, while the influence of the short rate is related to the increase in the difficulty of refinancing existing debt when the cost of borrowing money is higher (if a firm can only borrow money at high costs the PD is higher).

Assumption 4.3.3. (Loss Quota)
The conditional distribution of loss quota is a deterministic function of (t, r, I). K is a stochastic kernel from $R_+ \times R_+ \times R_+ \to [0, 1]$ for any realization of (t, r, I)

We denote the cumulative distribution function of loss quota conditional on default as $\tilde{K}$

$$\tilde{K}(t, r, I, x) = \int_0^x K(t, r, I, dq), \qquad \int_0^1 K(t, r, I, dq) = 1, \quad \forall t, r, I$$

with the following properties

$$\begin{array}{lccc} \tilde{K}(t, r, I_1, x) \geq \tilde{K}(t, r, I_2, x), & \text{if} & I_1 \geq I_2, & \forall x \in R \ , \\ \tilde{K}(t, r_1, I, x) \leq \tilde{K}(t, r_2, I, x), & \text{if} & r_1 \geq r_2, & \forall x \in R \ . \end{array}$$

That is

(vi)

$$\frac{\partial \tilde{K}(t, r, I, x)}{\partial I} > 0 \ , \tag{4.30}$$

(vii)

$$\frac{\partial \tilde{K}(t, r, I, x)}{\partial r} \leq 0 \ . \tag{4.31}$$

For fixed (t, r), $\tilde{K}(t, r, I, x)$ stochastically dominates *all the conditional distributions with parameter $\underline{I}$, such that $\underline{I} \leq I$ and, for fixed (t, I) $\tilde{K}(t, r, I, x)$* stochastically dominates *all the conditional distributions with parameter $\bar{r}$, such that $\bar{r} \geq r$.*

Property *(vi)* in (4.30) can be justified by the following argument. Given that default has occurred and the debt holders are negotiating their loss quota so that the firm is able to continue operating (assumption underlying any multi-default model), it seems reasonable that if the firm's assets are worth less they are willing to give in more (higher loss quota) since they would not get that much in the event of bankruptcy. Moreover, bankruptcy costs tend to be higher in periods of recession, emphasizing this effect.

As to the risk-free interest rate, the argument for property *(vii)* in (4.31) is more difficult to support. Thus in our view, it is reasonable to assume $\frac{\partial \tilde{K}(t,r,I,x)}{\partial r} \leq 0$, which includes the possibility of no influence. However, if we assume the existence of an impact, then we argue that it should be in the direction of higher interest rates leading to a decrease in the probability of lower loss quotas. In periods of high interest rates, debt holders have better options to invest their money and are less willing to give in.

From the stochastic dominance assumption above we can now infer the impacts on the expected loss quota.

Lemma 4.3.4. *Given Assumption 4.3.3, the following relations hold for the expected value*

$$q^e(r,I) = \int_0^1 qK(r,I,dq),$$

(viii)

$$\frac{\partial q^e(r,I)}{\partial I} < 0 \ , \tag{4.32}$$

(ix)

$$\frac{\partial q^e(r,I)}{\partial r} \geq 0 \ . \tag{4.33}$$

Proof. The proof is similar to both properties and is a consequence of *(vi)⇒(viii)* and *(vii)⇒(ix)*. We show it for the index value case.

$$q^e(r,I) = \int_0^1 qK(r,I,dq) = \int_0^1 qd\tilde{K}(t,r,I,q).$$

Integrating by parts we get

$$\begin{aligned}
\int_0^1 qd\tilde{K}(t,r,I,q) &= q\tilde{K}(t,r,I,q)\big|_0^1 - \int_0^1 \tilde{K}(t,r,I,q)dq \\
&= q\underbrace{\tilde{K}(t,r,I,1)}_{1} - q\underbrace{\tilde{K}(t,r,I,0)}_{0} - \int_0^1 \tilde{K}(t,r,I,q)dq \\
&= q - \int_0^1 \tilde{K}(t,r,I,q)dq \ .
\end{aligned}$$

The results follows from differentiating this last expression w.r.t I and using the property (vi)

$$\frac{\partial q^e(r,I)}{\partial I} = -\int_0^1 \underbrace{\frac{\partial \tilde{K}(t,r,I,q)}{\partial I}}_{>0,\forall q} dq < 0 \ .$$

The same argument will work for the interest rate r. ∎

Remark 4.3.5. (Tractability)
We note that besides the above mentioned properties K, conditional on the state variable information, must be the distribution of a random variable taking values in $(0,1)$, and the intensity λ must be always positive. It is, thus, extremely hard to find a treatable model where these two facts together with properties (i)-(ix) are satisfied.

In particular, we have found that no model of affine or quadratic spreads[7] will verify all the above properties.

Given these tractability difficulties we go on with the analysis and draw *qualitative* results of the influence of the market index on credit spreads.

4.3.2 Short credit spreads

Using the abstract result on Proposition 4.2.15 and Assumptions 4.3.2 and 4.3.3 we can study the impacts of our macroeconomic variables $X = \begin{bmatrix} I \\ r \end{bmatrix}$ on the short spread $s(t)$.

Remark 4.3.6. *Given the results in Proposition 4.2.15, Assumption 4.3.2 and Lemma 4.3.4, the short credit spread can be rewritten as a function of (t, r, I, ϵ) and*

$$s(t, r, I, \epsilon) = \lambda(t, r, I, \epsilon) q^e(t, r, I) \ . \tag{4.34}$$

Furthermore we have

$$\begin{aligned} s(t,r,I,0) &= \bar{\lambda} q^e(t,r,I) \ , \qquad \frac{\partial s(t,r,I,\epsilon)}{\partial \epsilon} = \underbrace{\frac{\partial \lambda(t,r,I,\epsilon)}{\partial \epsilon}}_{>0} q^e(t,r,I) > 0 \ , \\ \frac{\partial s(t,r,I,\epsilon)}{\partial I} &= \underbrace{\frac{\partial \lambda(t,r,I,\epsilon)}{\partial I}}_{<0} q^e(t,r,I) + \lambda(t,r,I,\epsilon) \underbrace{\frac{\partial q^e(t,r,I)}{\partial I}}_{<0} < 0 \ , \\ \frac{\partial s(t,r,I,\epsilon)}{\partial r} &= \underbrace{\frac{\partial \lambda(t,r,I,\epsilon)}{\partial r}}_{>0} q^e(t,r,I) + \lambda(t,r,I,\epsilon) \underbrace{\frac{\partial q^e(t,r,I)}{\partial r}}_{\geq 0} > 0 \ . \end{aligned}$$

We note that given a concrete functional form for the intensity λ, and the loss quota distribution and, thus, q^e the above effects on the short spread can actually be quantified. Unfortunately, this is not going to be the situation when dealing with forward credit spreads.

[7]Outside the class of affine or quadratic spread models it is basically impossible to find closed-form solutions.

Before, however, we move on to forward credit spreads, we derive the dynamics by of the short credit spread under $\mathbb{Q}$.

Proposition 4.3.7. *The dynamics of the short credit spread under $\mathbb{Q}$ is, in short hand notation, as follows*

$$\begin{aligned} d\bar{s} &= \left(\frac{\partial s}{\partial t} + \frac{\partial s}{\partial r}a(t) + \frac{\partial s}{\partial I}\gamma(I_t)I_t + \frac{1}{2}\frac{\partial^2 s}{\partial r^2}b^2(t) + \frac{1}{2}\frac{\partial^2 s}{\partial I^2}\gamma^2(I_t)I_t^2 + \frac{\partial^2 s}{\partial I \partial r}b(t)\gamma(I_t)I_t\right) dt \\ &\quad +\sigma_s(t)dW_t \end{aligned}$$

where

$$\begin{aligned} a(t) &= \frac{\partial f(t,T)}{\partial T}|_{T=t} + \alpha(t,t) \\ b(t) &= \sigma(t,t) \\ \sigma_s(t) &= \frac{\partial s}{\partial r}b(t) + \frac{\partial s}{\partial I}\gamma(I_t)I_t \ . \end{aligned} \tag{4.35}$$

where $\alpha(t,T)$ and $\sigma(t,T)$ are the drift and the volatility of the forward risk-free interest rates in (4.2) and the usual HJM drift condition in (4.3) holds.

Proof. We apply Itô formula to equation (4.34).

In addition to the Itô formula we just need to deduce the dynamics of the risk-free short rate r, from the dynamics of the forward interest rate in (4.2). We recall $r(t) = f(t,t)$ and it is easy to show that [8]

$$dr(t) = \left(\frac{\partial f(t,T)}{\partial T}|_{T=t} + \alpha(t,t)\right) dt + \sigma(t,t)dW_t \ .$$

■

Without a concrete functional form of the intensity and loss quota distribution, there is not much one can say about the dynamics of the short spread. Except, perhaps that since $\frac{\partial s}{\partial I} < 0$ and $\frac{\partial s}{\partial r} \geq 0$ (recall Remark 4.3.6) we see that increases in the volatility of the index lead to decreases in the volatility of the short spread, while increases in the volatility of the short rate r lead to increases in its volatility. Consequences to the drift cannot be drawn from Assumptions 4.3.2 and 4.3.3 alone.

[8]For a proof of this result see Björk (2004).

Remark 4.3.8. **($\mathbb{P}$ considerations)**
The same qualitative relations would still hold under the objective probability measure $\mathbb{P}$ as long as the market price of jump risk is positive, which seems reasonable.

As to forward spreads $s(t, T)$, given that we could only obtain expressions in terms of expectations that have to be numerically evaluated, there little one can say.

In the next section we propose a way to model all the needed functions and show the simulation results.

4.4 A concrete model

In this section we illustrate the theoretical results previously derived using a "toy model".

We aim to highlight the importance of considering the dependence between recovery and intensity of default by showing the applicability of our results and that the impacts obtained are substantial, rather than to be as realistic as possible.

For that reason, when setting up our model, we do so as simply as possible with the advantage of obtaining more tractable formulas and a better understanding of what drives the simulation results.

The theoretical results apply, of course, to the general case outlined above and many more examples could have been considered.

In order to have a concrete model we need to:

- establish the dependence of the volatility of the index γ, on the index level;
- provide the intensity functional form for λ, in terms of (t, r, I, ϵ);
- decide on a distribution for the loss quota q for all possible (t, r, I).

In our toy model we take the risk-free rate, r, as constant and abstain from considerations about the term structure of risk-free interest rates. Although this is unrealistic, it is not harmful to our goal of understanding the impact on *spreads*.

I is assumed to be the price of a traded asset. To consider a non-traded asset, we would need further considerations on the market price of index risk.

For simplicity we also take all functions to be time homogenous; the extension to non-time homogeneous functions is straightforward.

Given these simplifications, to have a completely specified model to simulate we need to define a function $\gamma(I)$ for the index volatility, a function $\lambda(I,\epsilon)$ for the intensity, and a distribution function $K(dq,I)$ for the loss quota.

4.4.1 Choosing the market index volatility γ

We start by defining a concrete volatility for the index $\gamma(I)$. As stated in Assumption 4.3.1 we would like to the index volatility inversely related to the index level.

We start by defining a ratio which relates the current value of the index to its long-run trend value. Let us define

$$m(I) = \frac{\bar{I}}{I},$$

where $\bar{I}$ is a priori given and can be interpreted as the long-run trend value of the market index.

The above ratio measures how close or far away from the long-run trend value parameter, $\bar{I}$, the current value of the index I is. Intuitively, it seems reasonable to make the volatility dependent on some relative value of the index, instead on its absolute value. $\bar{I}$ will be assumed to grow at the risk-free rate over time.

Reasonable levels for $m(I)$ typically range from 0.7 and 1.3. We note that the higher the current level of the index the lower is $m(I)$, i.e.

$$\frac{\partial m}{\partial I} = -\frac{\bar{I}}{I^2} < 0 \ .$$

That is, a value of, say, $m = 0.7$ refers to a bull market while $m = 1.3$ refers to a bear market.

Based on the above ratio we now define the *volatility of index* as a function of our moneyness level m in the following way

$$\gamma(I) = \bar{\gamma}\,(m(I))^{\frac{1}{2}} \quad \forall I, \bar{\gamma} \in \mathbb{R}_+ \ . \tag{4.36}$$

In accordance with Assumption 4.3.1, the higher the current value of the index the lower is the index volatility γ,

$$\frac{\partial \gamma(I)}{\partial I} = \underbrace{\bar{\gamma}\frac{1}{2}}_{>0}\underbrace{[m(I)]^{-\frac{1}{2}}}_{>0}\underbrace{\frac{\partial m(I)}{\partial I}}_{<0} < 0 \quad \forall I > 0 \ .$$

Figure 4.1 shows us two possible paths for the index process, one assuming γ to be just a constant and the other where the index volatility depends on the index level as in (4.36).

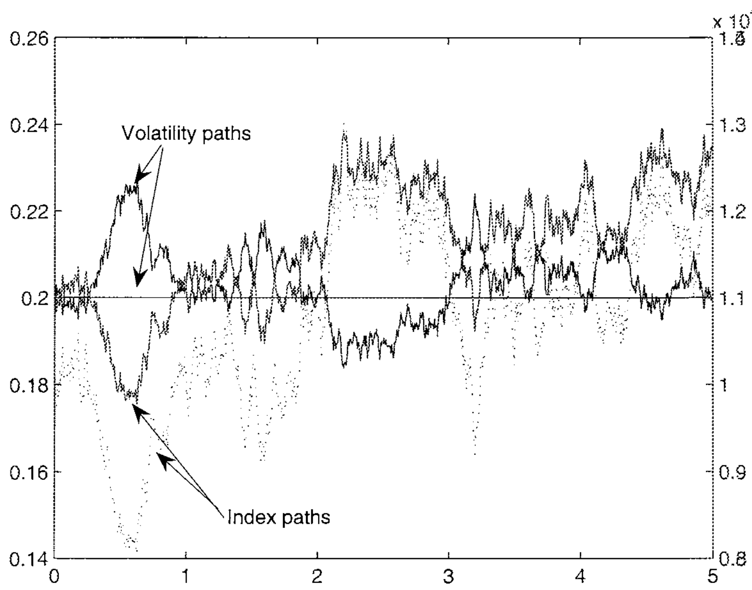

Figure 4.1: Two paths for the index level and volatility. The same noise was used for both cases, and we took $\bar{I} = 10000$ and $I_0 = 10000$. Case 1: constant volatility $\gamma = 0.2$, the index process is the full line. Case 2: stochastic volatility as in (4.36), the index process is the dotted line.

4.4.2 Choosing default intensities and the loss quota distribution

Default intensity

Having defined the index volatility we now define the intensity function

$$\lambda(I, \epsilon) = \bar{\lambda}\,[m(I)]^{\epsilon} = \frac{\bar{\lambda}}{\bar{\gamma}}\,[m(I)]^{\epsilon - \frac{1}{2}}\,\gamma(I) \quad \text{for } \bar{\lambda} \in \mathbb{R}+ \text{ and } \epsilon \in [0, 1] \ .$$

We note that with this modelization we can interpret the intensity function as a function of the index level or, if we prefer, as a function of the index volatility. One can argue that the intensity should not be affected by index level, but instead by its volatility since it is the volatility that represents the "risk". The above definition includes the two possibilities.

$$\begin{aligned} \frac{\partial \lambda}{\partial I} &= \underbrace{\bar{\lambda}\epsilon}_{>0} \underbrace{(m(I))^{\epsilon-1}}_{>0} \underbrace{\frac{\partial m(I)}{\partial I}}_{<0} < 0 \\ \frac{\partial \lambda}{\partial \gamma} &= \underbrace{\frac{\bar{\lambda}}{\bar{\gamma}}}_{>0} \underbrace{[m(I)]^{\epsilon-\frac{1}{2}}}_{>0} > 0. \end{aligned}$$

Figure 4.2 show the functions $\lambda(I)$ and $\gamma(I)$ for different values of $m(I)$.

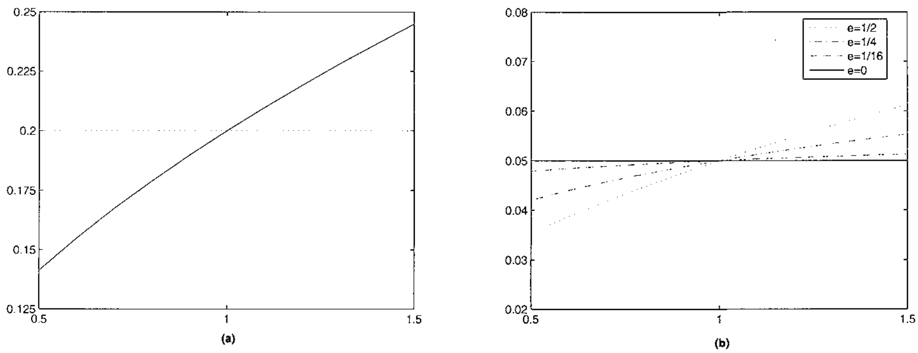

Figure 4.2: (a) : $\gamma(I)$ for different levels of $m(I)$ vs naive constant volatility $\bar{\gamma} = 0.2$. (b):$\lambda(I)$, for different levels of $m(I)$ and different $\epsilon = 0, 1/16, 1/4, 1/2$, $\bar{\lambda} = 0.05$.

Loss quota q

Finally, we need to decide on our loss quota distribution.

As before we will make use of our m ratio in setting up the distribution of the loss process which we will consider to belong to the Beta class. We start by recalling some basic properties of the beta distribution.

Remark 4.4.1. *The beta density function is given by*

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}\mathbf{1}_{(0,1)}(x)$$

where $a > 0$, $b > 0$ and $B(a, b)$ is the beta function:

$$B(a,b) \int_0^1 x^{a-1}(1-x)^{b-1} ds \ .$$

Furthermore, we have

$$\mathbb{E}\left[X\right] = \frac{a}{a+b} = \mu \qquad \operatorname{Var} X = \frac{ab}{(a+b+1)(a+b)^2} \qquad \mathbb{E}\left[(X-\mu)^r\right] = \frac{B(r+a,b)}{B(a,b)}$$

In our concrete application, for any fixed value of I, we choose

$$q \sim Beta\left(2m(I), 2\right) \quad \text{i.e.} \quad a = 2m(I) \quad \text{and} \quad b = 2\ , \tag{4.37}$$

which is consistent with the desired properties referred in Assumption 4.3.3.

Thus,

$$\tilde{K}(q, I) = \frac{1}{B(2m(I), 2)} \int_0^q x^{2m(I)-1}(1-x)dx\ .$$

Figure 4.3 shows us the loss quota density and its cumulative distribution function for three different values of $m(I)$: $m = 0.7$ representing a bull market, $m = 1$ for the case where the market is at its long run level, and $m = 1.3$ representing a bear market.

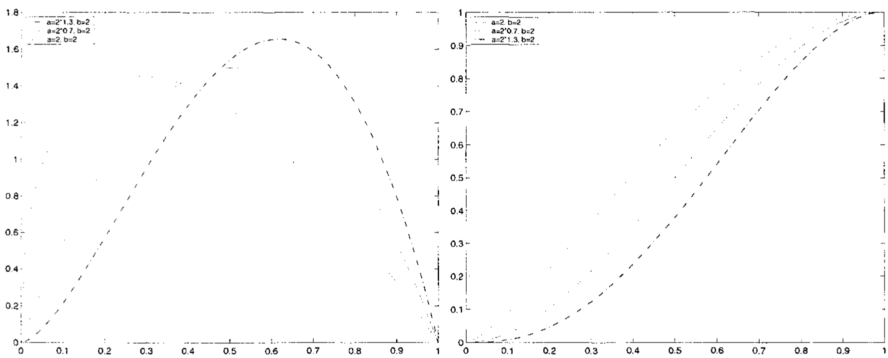

Figure 4.3: Density and Cumulative distribution functions of loss quota for $m = 1.3$,$m = 1$,$m = 0.7$

The next properties, follow from the properties of the Beta distribution:

- The expected loss is given by

$$q^e(I) = \mathbb{E}\left[q(I)\right] = \frac{m(I)}{1+m(I)}$$

- $\frac{\partial q^e(I)}{\partial I} = -\frac{\frac{\partial m(I)}{\partial I}}{(1+m(I))^2} < 0$

  Furthermore

  - if default occurs exactly at the long-run level the loss expected quota is exactly 1/2;
  - if default occurs when the index level is "high" ($m < 1$) one expects to recover more, expected loss quota decreases;
  - if default occurs when the index level is "low" ($m > 1$) one expects to recover less, expected loss quota increases.

Figure 4.4 bellow shows both possible realizations of the loss quota (drawn from the beta density with the appropriate mean for each m) (stars), the expected loss quota levels for different values of m (full line) in contrast with the naive approach of taking $\bar{q} = \frac{1}{2}$ (dotted line).

Before we go on we illustrate as well a possible relation between the intensity and the recovery process $(1-q)$ and the intensity λ. Figure 4.5 shows the scatter plot λ versus one possible recovery realization for different levels of the index.

4.4.3 Simulation Results

In our simulations we use the Monte Carlo method where the step size is equivalent to one trading day (we do 250 steps per year) and all simulations concern 5,000 paths. The same noise matrix is used for all scenarios and cases so that the values obtained can actually be compared (discretization errors would be in the same direction for all scenarios).

The spreads with zero maturity correspond to the short spread; all other maturities correspond to the forward spread.

Table 4.1 tells us the reference parameters, while Table 4.2 characterizes all possible scenarios.

Credit spreads

We start by looking into short spread dynamics.

Figure 4.6 presents three possible paths for the short spread under each scenario. Obviously, three paths are not representative in any sense; still we believe the intuition is good and we choose paths with different characteristics. In (a) the

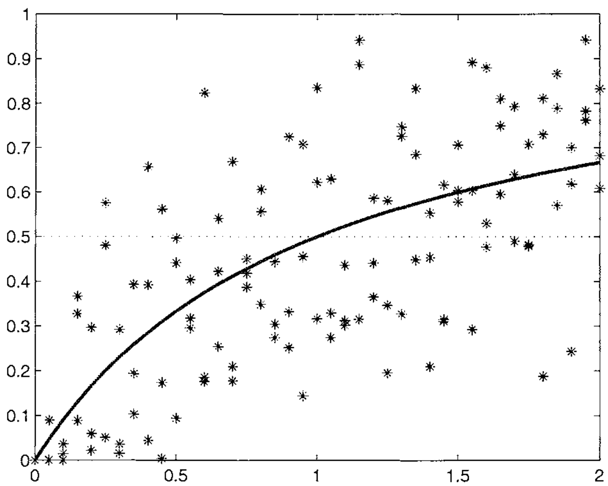

Figure 4.4: Loss quota possible realizations and expected value for different values of the "moneyness" m. Dotted line is the naive $q = \frac{1}{2}$.

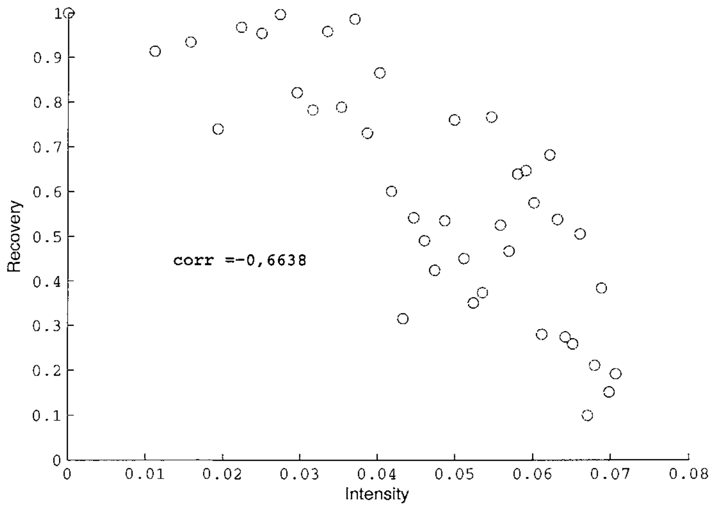

Figure 4.5: Scatter plot of intensity versus possible recovery realization for different values of m.

REFERENCE PARAMETERS

Maturities (T)	From days up to 5 years	
Risk-free interest rate	5%	
$m(I)$	Case A: bull market	0.7
	Case B: normal market	1.0
	Case C: bear market	1.3
Long-run index value	$10.000e^{0.5*T}$	
Fixed index volatility ($\bar{\gamma}$)	20%	
Fixed intensity value ($\bar{\lambda}$)	5%	
Fixed recovery value ($\bar{q} = \frac{1}{2}$)	50%	

Table 4.1: Reference values for the parameters in the model.

DIFFERENT SCENARIOS

Scenario	Index Volatility	Intensity	Recovery
(1)	F	F	F
(2)	S	F	F
(3)	F	F	S
(4)	S	F	S
(5)	F	S	F
(6)	S	S	F
(7)	F	S	S
(8)	S	S	S

Table 4.2: Basic reference scenarios for simulations. F= Fixed, S= Stochastic.

market index decreases over time, leading to an increase in the short spreads. In (b) we have a mixed path, and in (c) the index value increases in the end leading to a reduction in the short spreads. An analysis of this figure shows that allowing for some stochasticity either in the intensity process or in the expected loss quota leads to similar short spread dynamics and that it is the combined effect that really makes the difference.

In any of the three paths, if only one of the effects is considered, the short spreads do not oscillate more than 1% below or above the naive 2.5%, while for the combined effect the variation can be as large as 4% (in the case of path (a)) and quite often above 2%.

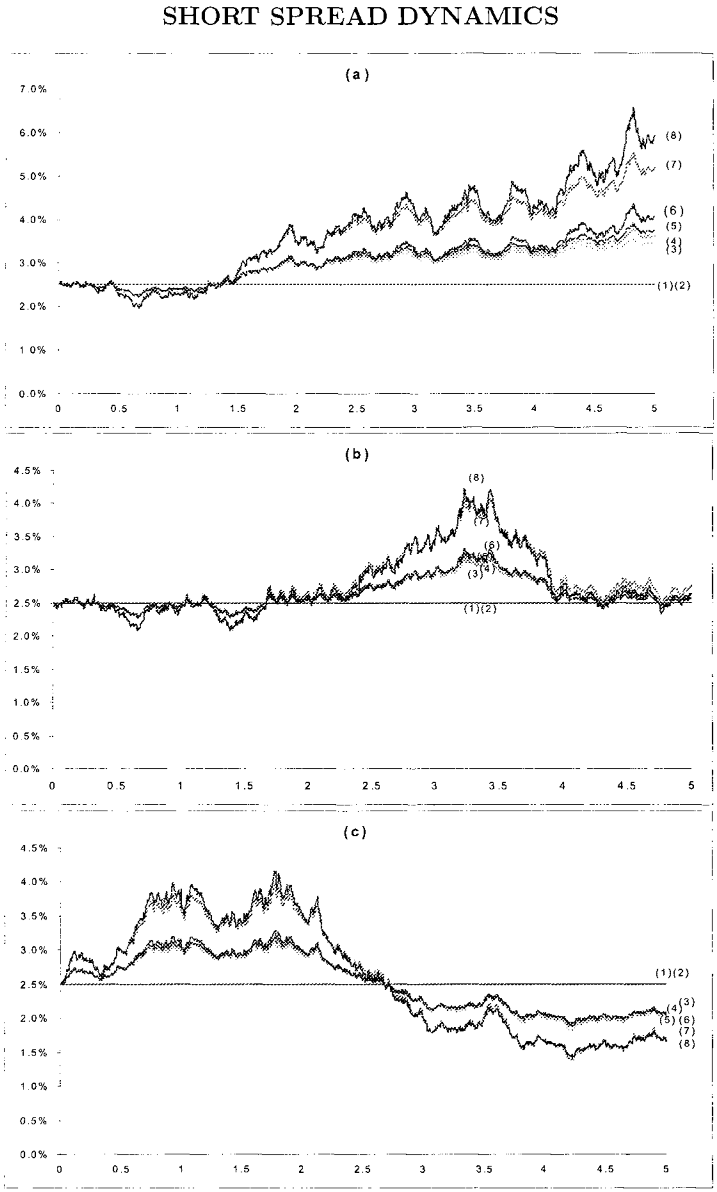

Figure 4.6: Three possible paths for the short spread, $s(t)$, dynamics.

Figure 4.7 gives us the term structure of credit spreads for three possible market situations: a bull market (where we took $m = 0.7$ for the initial point), a normal market (initially $m = 1$) and a bear market (at the beginning $m = 1.3$).

As expected, as long as the market index is allowed to have an influence either on the PDs or on the LGD, the forward spread TS is not flat. It naturally increases with maturity and its level is lower than 2.5% in bull markets and higher in bear markets. As in Figure 4.6 the effect of the index on the intensity alone or on recovery alone is of the same order of magnitude. Still, the intensity seems to have a more pronounced effect on the slope over time. Nonetheless, the sizable difference results from the combined effect, especially when to that we associate the impact of the negative relation between the index level and its volatility. Note that scenario (8) gives us the highest or the lowest term structure in all circumstances.

Table 4.3 presents figures for these spreads for different maturities.

Prices of credit securities

The pricing of many credit derivatives can be made by computing what is know as *key building blocks*. See for instance Schönbucher (2003).

Those building blocks are:

- The price of a zero-coupon defaultable bond with maturity t, under zero recovery,
$$\bar{p}_o(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)+\lambda(u)du}\right] .$$
- The price at time t of a zero-coupon defaultable bonds with maturity T under non-zero recovery, in our case,
$$\bar{p}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)+\lambda(u)q^e(u)du}\right] .$$
- The implied survival probabilities during the interval $[t,T]$
$$\text{prob}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)+\lambda(u)q^e(u)du}\right] .$$
- The price of a default digital payoff of 1 at default time, if default occurs in the interval $[t,T]$
$$\text{digital} = \int_t^T e(t,s)ds$$
where
$$e(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\lambda(T)e^{-\int_t^T r(u)+\lambda(u)du}\right] .$$

FORWARD SPREADS

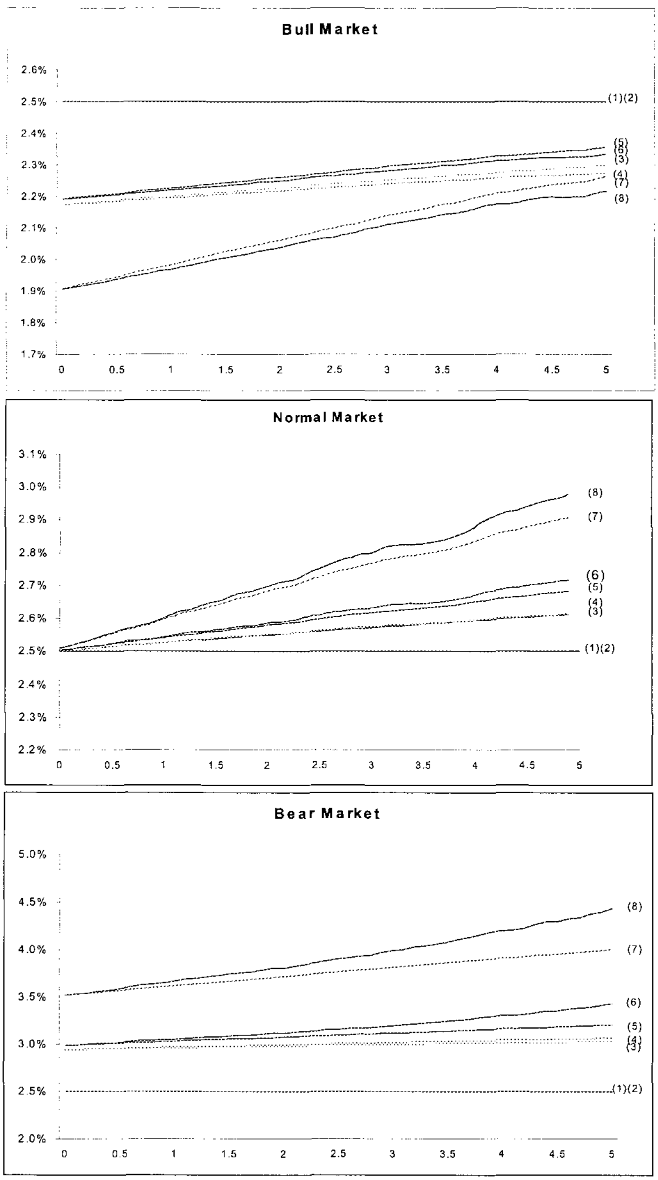

Figure 4.7: Forward spreads for all scenarios, under three possible market conditions.

(a) SPREADS

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.500%	2.174%	2.174%	2.192%	2.192%	1.906%	1.906%
0.1	2.500%	2.177%	2.176%	2.196%	2.195%	1.914%	1.912%
0.5	2.500%	2.188%	2.185%	2.209%	2.206%	1.945%	1.937%
1	2.500%	2.201%	2.196%	2.226%	2.220%	1.983%	1.968%
1.5	2.500%	2.215%	2.208%	2.245%	2.236%	2.025%	2.006%
2	2.500%	2.228%	2.219%	2.262%	2.251%	2.063%	2.040%
3	2.500%	2.253%	2.241%	2.296%	2.284%	2.140%	2.112%
5	2.500%	2.299%	2.275%	2.355%	2.334%	2.264%	2.217%

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5038%	2.5038%	2.5088%	2.5088%
0.5	2.5000%	2.5124%	2.5126%	2.5189%	2.5193%	2.5440%	2.5450%
1	2.5000%	2.5246%	2.5247%	2.5376%	2.5386%	2.5869%	2.5894%
1.5	2.5000%	2.5362%	2.5378%	2.5573%	2.5612%	2.6327%	2.6416%
2	2.5000%	2.5479%	2.5492%	2.5755%	2.5816%	2.6736%	2.6875%
3	2.5000%	2.5700%	2.5736%	2.6142%	2.6300%	2.7606%	2.7953%
5	2.5000%	2.6121%	2.6145%	2.6836%	2.7178%	2.9082%	2.9769%

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.9413%	2.9411%	2.9881%	2.9880%	3.5157%	3.5155%
0.1	2.5000%	2.9433%	2.9439%	2.9925%	2.9941%	3.5258%	3.5296%
0.5	2.5000%	2.9512%	2.9551%	3.0103%	3.0199%	3.5672%	3.5897%
1	2.5000%	2.9608%	2.9687%	3.0324%	3.0533%	3.6182%	3.6666%
1.5	2.5000%	2.9704%	2.9822%	3.0544%	3.0882%	3.6686%	3.7458%
2	2.5000%	2.9803%	2.9940%	3.0742%	3.1167%	3.7121%	3.8084%
3	2.5000%	2.9982%	3.0201%	3.1189%	3.1985%	3.8114%	3.9857%
5	2.5000%	3.0303%	3.0688%	3.2088%	3.4289%	4.0014%	4.4276%

(b) ZERO-COUPON DEFAULTABLE BONDS W/ RECOVERY

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9925	0.9928	0.9928	0.9928	0.9928	0.9930	0.9930
0.5	0.9631	0.9647	0.9647	0.9646	0.9646	0.9659	0.9659
1	0.9277	0.9306	0.9306	0.9304	0.9304	0.9328	0.9329
1.5	0.8935	0.8976	0.8977	0.8973	0.8974	0.9007	0.9009
2	0.8606	0.8658	0.8659	0.8654	0.8655	0.8696	0.8698
3	0.7985	0.8053	0.8055	0.8046	0.8048	0.8100	0.8104
5	0.6872	0.6963	0.6966	0.6949	0.6953	0.7013	0.7021

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9630	0.9630
1	0.9277	0.9275	0.9275	0.9275	0.9275	0.9272	0.9272
1.5	0.8935	0.8933	0.8933	0.8931	0.8931	0.8926	0.8926
2	0.8606	0.8602	0.8602	0.8600	0.8599	0.8591	0.8590
3	0.7984	0.7976	0.7975	0.7971	0.7970	0.7953	0.7950
5	0.6872	0.6852	0.6852	0.6840	0.6836	0.6800	0.6792

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9920	0.9920	0.9919	0.9919	0.9914	0.9914
0.5	0.9631	0.9609	0.9609	0.9607	0.9606	0.9581	0.9580
1	0.9276	0.9234	0.9234	0.9229	0.9228	0.9178	0.9176
1.5	0.8935	0.8874	0.8873	0.8865	0.8863	0.8789	0.8785
2	0.8606	0.8527	0.8526	0.8515	0.8511	0.8416	0.8407
3	0.7984	0.7872	0.7870	0.7853	0.7844	0.7710	0.7692
5	0.6872	0.6706	0.6700	0.6670	0.6644	0.6452	0.6400

Table 4.3: Credit Spreads for several maturities $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$. For $T = 0$ it is the short spread, for all others the forward spread.

Tables 4.3 (b) and 4.4 (a),(b),(c) show the values of all these key quantities for various scenarios and different possible maturities.

Concerning Table 4.3 (b), the first point that should be highlighted is that even for low maturities there is a difference in the prices produced by naive scenarios ((1) and (2)), scenarios where either the PD or the LGD is dependent on the index level ((3) and (4) for the LGD and (5) and (6) for the PD), and scenarios where we consider the combined effect. We note that for the bull and bear markets the difference in pricing is already evident for bonds with approximately one month to maturity ($T = 0.1$), and that with a maturity of 5 years the underpricing of the naive model can be of up to 5% in a bull market, and up to 10% in a bear market. In normal market situations, the difference between the scenarios are slope differences with an overall tendency of the naive model to overprice if in reality the market index affects the PD, the LGD or both.

In Table 4.4 the quantities presented do not depend on recovery (recall the building blocks equations), and therefore our scenarios are reduced to three possibilities: constant intensity (scenarios (1) (2) (3) (4)); intensity dependent on the index with constant volatility (scenarios (5) (7)); or intensity dependent on the index with stochastic volatility (scenarios (6) (8)). In Table 4.4 (a), we see that the naive model can also lead to underpricing in bull markets when we consider defaultable zero recovery bonds, and overpricing in bear markets. The dimension of these mispricings is similar to the mispricing in the previous table.

When looking into survival probabilities (Table4.4 (b)) the scenarios considering the index influence show less probability of surviving in bear markets than in bull markets, with a difference of up to 7% already the 5-year maturity. This indicates that using the naive model could lead to a possible overestimation of the true survival probabilities in bear markets, and underestimation in bull markets. As in most of the previous tables, the size of the errors tends to be considerably larger in bear markets than in bull markets. In the case of survival probability, overestimation can reach 5% in bear markets in contrast with up to 2% in bull markets. Finally, in Table 4.4 (c), we have the price of a digital payoff of 1 at default time. As before, the differences to the naive model are more drastic in bear markets particularly considering that a horizon of 5 years is quite short and that all the impacts are likely to increase for higher maturities, as we will see in the next section.

In Section 4.4.4 we run some maturity related robustness checks and draw the reader's attention to what we consider to be interesting aspects related to the sensitivity parameter ϵ (assumed fixed up to now) and to the possible tracking of term structures of credit spreads by market volatilities.

(a) ZERO-COUPON DEFAULTABLE BOND PRICES W/O RECOVERY

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	0.9899	0.9905	0.9905	0.1	0.9899	0.9899	0.9899	0.1	0.9899	0.9888	0.9888
0.5	0.9510	0.9539	0.9539	0.5	0.9510	0.9510	0.9510	0.5	0.9510	0.9463	0.9462
1	0.9047	0.9100	0.9100	1	0.9047	0.9043	0.9043	1	0.9047	0.8954	0.8953
1.5	0.8605	0.8679	0.8680	1.5	0.8605	0.8598	0.8598	1.5	0.8605	0.8472	0.8468
2	0.8186	0.8276	0.8278	2	0.8186	0.8173	0.8173	2	0.8186	0.8013	0.8006
3	0.7407	0.7522	0.7525	3	0.7407	0.7382	0.7380	3	0.7407	0.7165	0.7151
5	0.6064	0.6203	0.6209	5	0.6064	0.6010	0.6003	5	0.6064	0.5714	0.5674

(b) IMPLIED SURVIVAL PROBABILITIES

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	99.481%	99.545%	99.545%	0.1	99.481%	99.481%	99.481%	0.1	99.481%	99.380%	99.380%
0.5	97.511%	97.806%	97.808%	0.5	97.511%	97.502%	97.502%	0.5	97.511%	97.022%	97.018%
1	95.104%	95.660%	95.667%	1	95.104%	95.068%	95.067%	1	95.104%	94.135%	94.116%
1.5	92.756%	93.546%	93.559%	1.5	92.756%	92.677%	92.673%	1.5	92.756%	91.313%	91.270%
2	90.466%	91.463%	91.484%	2	90.466%	90.329%	90.321%	2	90.466%	88.559%	88.484%
3	86.054%	87.392%	87.432%	3	86.054%	85.765%	85.740%	3	86.054%	83.245%	83.078%
5	77.865%	79.641%	79.729%	5	77.865%	77.164%	77.080%	5	77.865%	73.372%	72.850%

(c) DEFAULT DIGITAL PAYOFFS

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	0.00495	0.00435	0.00435	0.1	0.00495	0.00496	0.00496	0.1	0.00495	0.00592	0.00592
0.5	0.02426	0.02141	0.02139	0.5	0.02426	0.02437	0.02437	0.5	0.02426	0.02904	0.02909
1	0.04733	0.04199	0.04193	1	0.04733	0.04771	0.04772	1	0.04733	0.05672	0.05691
1.5	0.06928	0.06178	0.06165	1.5	0.06928	0.07008	0.07012	1.5	0.06928	0.08309	0.08352
2	0.09016	0.08079	0.08059	2	0.09016	0.09150	0.09157	2	0.09016	0.10821	0.10893
3	0.12892	0.11658	0.11621	3	0.12892	0.13161	0.13184	3	0.12892	0.15489	0.15645
5	0.19572	0.17983	0.17908	5	0.19572	0.20179	0.20250	5	0.19571	0.23542	0.23993

Table 4.4: (a) Prices of zero-recovery, zero-coupon bond prices for several maturities (top table). (b) Implied survival probabilities in the interval $[t, T]$ for several maturities. (c) Price of a digital payoff at default if it occurs in the interval $[t, T]$. Three different market conditions: bull, normal and bear.

4.4.4 Additional Considerations

Higher maturities

In this section we extend the maturity horizon and analyze to what extent our results hold. Table 4.5 presents forward spreads and defaultable bonds for several maturities, while Table 4.6 presents zero recovery bond prices, implied survival probabilities, and the price of default digital payoff of 1. First, we can clearly see that for all the three scenarios survival probabilities decrease significantly for longer horizons T. If we extend the horizon from 5 to 15 years, in the stochastic volatility scenario the survival probability decreases by almost 40% for the bull market and up to 50% for the bear market. In our opinion, this is a realistic feature of our model since at the longer horizons when the market is in recession and firms are known to be sensitive to the fluctuations of the market, the probability of default is quite high.

Moreover, it is also interesting to note that in a bull market, although a stochastic volatility scenario yields higher survival probabilities at all the maturities, the difference in survival probabilities is much smaller at the higher maturities. In a bear market, on the other hand, survival probabilities are lower for the stochastic volatility case, and the difference in survival probabilities between the stochastic volatility and naive scenarios is more pronounced. In contrast to the bull market, the difference increases by approximately 5% when the investment horizon is extended from 5 to 15 years.

Overall, the results are essentially the same with one interesting additional result for the long end of the forward spreads term structure.

From Figure 4.8 we see that when we consider the dependence of both PD and LGD and the negative relation the index level and volatility (scenario (8)), the TS seems to converge faster to its long-run level. In fact, for maturities higher than 15 years the TS of this scenario are is relatively flat. Thus, the forward credit spreads are most sensitive to the influence of the market index at the relatively shorter maturities, and around the 15 years maturity, the credit spreads become relatively flatter and less sensitive to the market index, moving in fact closer to each other.

Ratings and different sensitivities

We now look into the parameter ϵ in

$$\lambda(I, \epsilon) = \bar{\lambda}\,[m(I)]^{\epsilon}\,,$$

which is a measure of the sensitivity of a firms PD to the market situation.

FORWARD SPREADS - Higher Maturities

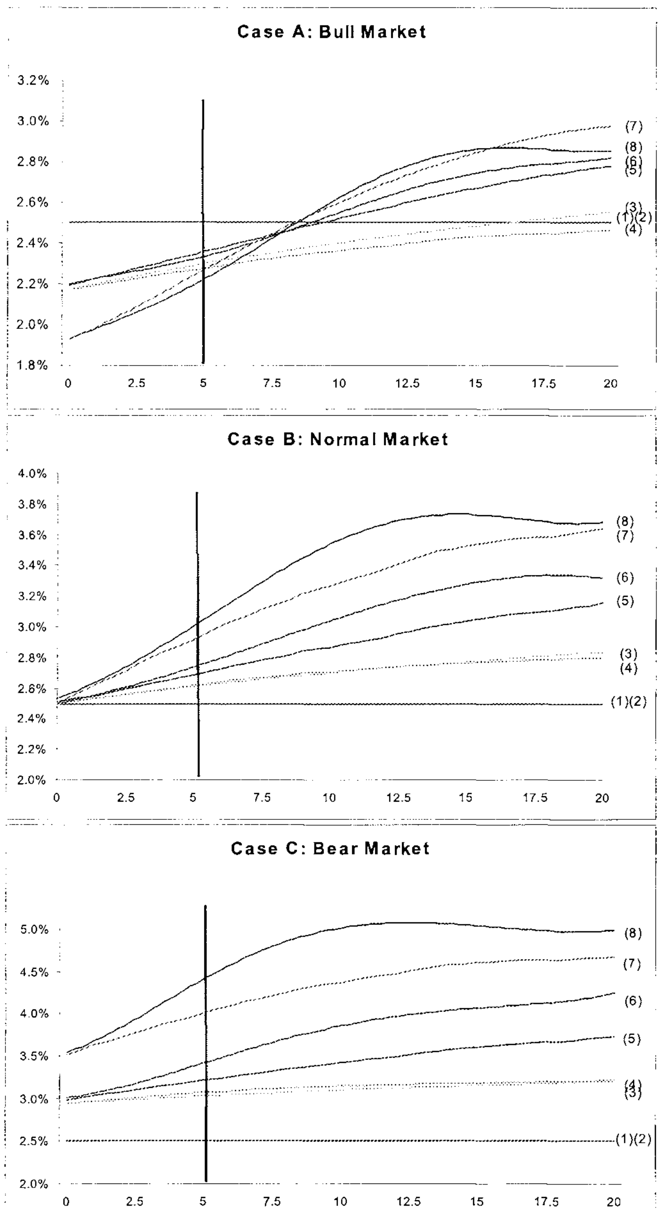

Figure 4.8: Forward spreads for all scenarios, under three possible market conditions and higher maturity values $T = 5, 6, 7, 8, 10, 12, 15, 10$.

(a) SPREADS

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.500%	2.299%	2.275%	2.355%	2.334%	2.264%	2.217%
6	2.5000%	2.3222%	2.2990%	2.3940%	2.3838%	2.3494%	2.3231%
7	2.5000%	2.3432%	2.3166%	2.4248%	2.4137%	2.4120%	2.3825%
8	2.5000%	2.3633%	2.3338%	2.4573%	2.4603%	2.4769%	2.4691%
10	2.5000%	2.4016%	2.3632%	2.5188%	2.5416%	2.5916%	2.6039%
12	2.5000%	2.4371%	2.3910%	2.5774%	2.6189%	2.6953%	2.7190%
15	2.5000%	2.4824%	2.4302%	2.6739%	2.7691%	2.8554%	2.8981%
20	2.5000%	2.5542%	2.4668%	2.7863%	2.8439%	2.9922%	2.8864%

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.5000%	2.6121%	2.6145%	2.6836%	2.7178%	2.9082%	2.9769%
6	2.5000%	2.6310%	2.6404%	2.7281%	2.8012%	3.0045%	3.1441%
7	2.5000%	2.6495%	2.6595%	2.7625%	2.8530%	3.0731%	3.2376%
8	2.5000%	2.6671%	2.6768%	2.7988%	2.9226%	3.1432%	3.3545%
10	2.5000%	2.7011%	2.7054%	2.8669%	3.0235%	3.2643%	3.4930%
12	2.5000%	2.7326%	2.7315%	2.9314%	3.1352%	3.3706%	3.6233%
15	2.5000%	2.7696%	2.7640%	3.0359%	3.2997%	3.5253%	3.7635%
20	2.5000%	2.8352%	2.7992%	3.1547%	3.3834%	3.6428%	3.7094%

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.5000%	3.0303%	3.0688%	3.2088%	3.4289%	4.0014%	4.4276%
6	2.5000%	3.0434%	3.0890%	3.2580%	3.5343%	4.0992%	4.6167%
7	2.5000%	3.0582%	3.1067%	3.2979%	3.6118%	4.1738%	4.7347%
8	2.5000%	3.0720%	3.1210%	3.3398%	3.7093%	4.2486%	4.8668%
10	2.5000%	3.0994%	3.1449%	3.4174%	3.8324%	4.3721%	4.9761%
12	2.5000%	3.1247%	3.1631%	3.4899%	3.9246%	4.4752%	5.0054%
15	2.5000%	3.1505%	3.1813%	3.6048%	4.1097%	4.6074%	5.0888%
20	2.5000%	3.2070%	3.2096%	3.7295%	4.2577%	4.6777%	4.9864%

(b) ZERO-COUPON DEFAULTABLE BONDS W/ RECOVERY

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	0.6872	0.6963	0.6966	0.6949	0.6953	0.7013	0.7021
6	0.6374	0.6470	0.6475	0.6454	0.6459	0.6518	0.6529
7	0.5913	0.6012	0.6019	0.5993	0.5998	0.6054	0.6066
8	0.5486	0.5586	0.5594	0.5563	0.5568	0.5619	0.5632
10	0.4722	0.4819	0.4829	0.4789	0.4792	0.4832	0.4843
12	0.4064	0.4155	0.4167	0.4118	0.4118	0.4147	0.4155
15	0.3245	0.3321	0.3336	0.3275	0.3268	0.3283	0.3285
20	0.2231	0.2281	0.2299	0.2226	0.2215	0.2209	0.2217

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	0.6872	0.6852	0.6852	0.6840	0.6836	0.6800	0.6792
6	0.6373	0.6347	0.6346	0.6330	0.6322	0.6276	0.6261
7	0.5913	0.5880	0.5878	0.5858	0.5846	0.5791	0.5768
8	0.5485	0.5446	0.5444	0.5419	0.5402	0.5340	0.5308
10	0.4721	0.4670	0.4668	0.4633	0.4605	0.4531	0.4483
12	0.4064	0.4003	0.4000	0.3956	0.3917	0.3836	0.3776
15	0.3245	0.3172	0.3170	0.3113	0.3060	0.2976	0.2907
20	0.2230	0.2147	0.2148	0.2077	0.2020	0.1938	0.1882

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	0.6872	0.6706	0.6700	0.6670	0.6644	0.6452	0.6400
6	0.6372	0.6185	0.6176	0.6138	0.6098	0.5888	0.5811
7	0.5911	0.5707	0.5696	0.5650	0.5597	0.5373	0.5274
8	0.5484	0.5264	0.5252	0.5199	0.5132	0.4900	0.4781
10	0.4720	0.4478	0.4463	0.4397	0.4305	0.4067	0.3918
12	0.4063	0.3808	0.3792	0.3713	0.3604	0.3368	0.3207
15	0.3244	0.2983	0.2967	0.2872	0.2748	0.2529	0.2369
20	0.2230	0.1982	0.1969	0.1862	0.1741	0.1561	0.1438

Table 4.5: (a) Credit Spreads and (b)Price of defaultable bond with recovery for several higher maturities $T = 5, 6, 7, 8, 10, 12, 15, 20$.

(a) ZERO-COUPON DEFAULTABLE BOND PRICES W/O RECOVERY

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
5	0.6064	0.6203	0.6209	5	0.6064	0.6010	0.6003	5	0.6064	0.5714	0.5674
6	0.5483	0.5624	0.5633	6	0.5483	0.5411	0.5400	6	0.5483	0.5090	0.5030
7	0.4962	0.5099	0.5109	7	0.4961	0.4874	0.4856	7	0.4961	0.4537	0.4461
8	0.4489	0.4621	0.4630	8	0.4489	0.4387	0.4363	8	0.4489	0.4041	0.3950
10	0.3676	0.3788	0.3795	10	0.3675	0.3549	0.3514	10	0.3675	0.3199	0.3090
12	0.3009	0.3100	0.3105	12	0.3009	0.2865	0.2823	12	0.3009	0.2528	0.2414
15	0.2229	0.2287	0.2287	15	0.2229	0.2071	0.2024	15	0.2229	0.1770	0.1663
20	0.1352	0.1370	0.1373	20	0.1352	0.1199	0.1165	20	0.1352	0.0973	0.0897

(b) IMPLIED SURVIVAL PROBABILITIES

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
5	77.865%	79.641%	79.729%	5	77.865%	77.164%	77.080%	5	77.865%	73.372%	72.850%
6	74.008%	75.912%	76.037%	6	74.008%	73.043%	72.885%	6	74.008%	68.712%	67.903%
7	70.398%	72.357%	72.493%	7	70.398%	69.161%	68.912%	7	70.398%	64.379%	63.300%
8	66.965%	68.931%	69.070%	8	66.965%	65.447%	65.094%	8	66.965%	60.280%	58.932%
10	60.592%	62.458%	62.572%	10	60.592%	58.505%	57.931%	10	60.592%	52.750%	50.953%
12	54.826%	56.491%	56.577%	12	54.826%	52.201%	51.430%	12	54.826%	46.071%	43.988%
15	47.189%	48.424%	48.412%	15	47.189%	43.839%	42.847%	15	47.189%	37.473%	35.204%
20	36.751%	37.242%	37.313%	20	36.751%	32.598%	31.660%	20	36.751%	26.447%	24.388%

(c) DEFAULT DIGITAL PAYOFFS

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
5	0.19572	0.17983	0.17908	5	0.19572	0.20179	0.20250	5	0.19572	0.23542	0.23993
6	0.21978	0.20407	0.20299	6	0.21978	0.22820	0.22960	6	0.21978	0.26456	0.27158
7	0.24522	0.22920	0.22805	7	0.24522	0.25560	0.25765	7	0.24522	0.29506	0.30397
8	0.26824	0.25224	0.25108	8	0.26824	0.28054	0.28331	8	0.26824	0.32250	0.33321
10	0.30792	0.29263	0.29166	10	0.30792	0.32378	0.32791	10	0.30792	0.36927	0.38268
12	0.34040	0.32631	0.32551	12	0.34040	0.35930	0.36456	12	0.34040	0.40680	0.42174
15	0.37843	0.36654	0.36624	15	0.37843	0.40093	0.40729	15	0.37843	0.44949	0.46528
20	0.42119	0.41237	0.41169	20	0.42119	0.44694	0.45306	20	0.42119	0.49454	0.50944

Table 4.6: (a) Prices of zero-recovery, zero-coupon bond prices for several maturities (top table). (b)Implied survival probabilities in the interval $[t, T]$ for several maturities. (c) Price of a digital payoff at default if it occurs in the interval $[t, T]$.

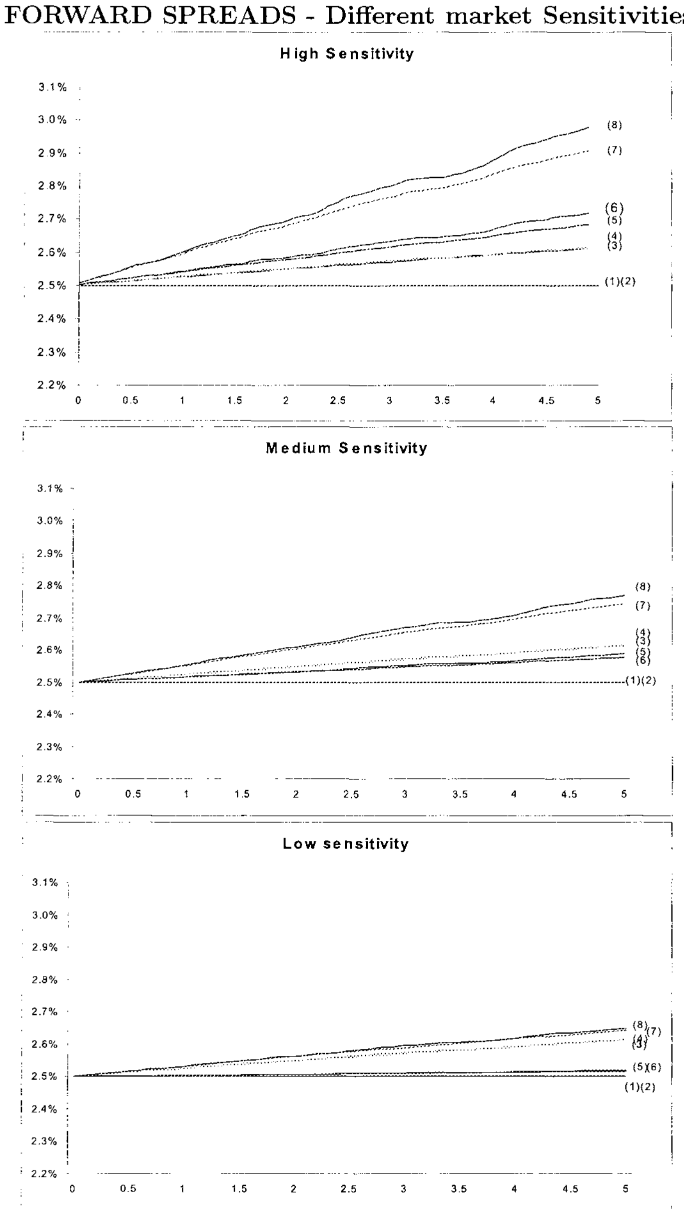

Figure 4.9: Forward spreads for all scenarios, under normal market conditions. For three different values of ϵ: high $\epsilon = \frac{1}{2}$, medium $\epsilon = \frac{1}{4}$ and low $\epsilon = \frac{1}{16}$.

(a) SPREADS

High

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5038%	2.5038%	2.5088%	2.5088%
0.5	2.5000%	2.5124%	2.5126%	2.5189%	2.5193%	2.5440%	2.5450%
1	2.5000%	2.5246%	2.5247%	2.5376%	2.5386%	2.5869%	2.5894%
1.5	2.5000%	2.5362%	2.5378%	2.5573%	2.5612%	2.6327%	2.6416%
2	2.5000%	2.5479%	2.5492%	2.5755%	2.5816%	2.6736%	2.6875%
3	2.5000%	2.5700%	2.5736%	2.6142%	2.6300%	2.7606%	2.7953%
5	2.5000%	2.6121%	2.6145%	2.6836%	2.7178%	2.9082%	2.9769%

Medium

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5016%	2.5016%	2.5054%	2.5053%
0.5	2.5000%	2.5124%	2.5126%	2.5079%	2.5080%	2.5266%	2.5271%
1	2.5000%	2.5246%	2.5247%	2.5157%	2.5160%	2.5525%	2.5535%
1.5	2.5000%	2.5362%	2.5378%	2.5236%	2.5252%	2.5792%	2.5837%
2	2.5000%	2.5479%	2.5492%	2.5313%	2.5335%	2.6038%	2.6099%
3	2.5000%	2.5700%	2.5736%	2.5471%	2.5527%	2.6544%	2.6697%
5	2.5000%	2.6121%	2.6145%	2.5768%	2.5879%	2.7433%	2.7701%

Low

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5003%	2.5003%	2.5032%	2.5032%
0.5	2.5000%	2.5124%	2.5126%	2.5017%	2.5017%	2.5156%	2.5159%
1	2.5000%	2.5246%	2.5247%	2.5033%	2.5034%	2.5310%	2.5313%
1.5	2.5000%	2.5362%	2.5378%	2.5050%	2.5053%	2.5460%	2.5482%
2	2.5000%	2.5479%	2.5492%	2.5067%	2.5070%	2.5606%	2.5629%
3	2.5000%	2.5700%	2.5736%	2.5100%	2.5110%	2.5891%	2.5951%
5	2.5000%	2.6121%	2.6145%	2.5165%	2.5184%	2.6419%	2.6491%

(b) ZERO-COUPON DEFAULTABLE BONDS W/ RECOVERY

High

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9630	0.9630
1	0.9277	0.9275	0.9275	0.9275	0.9275	0.9272	0.9272
1.5	0.8935	0.8933	0.8933	0.8931	0.8931	0.8926	0.8926
2	0.8606	0.8602	0.8602	0.8600	0.8599	0.8591	0.8590
3	0.7984	0.7976	0.7975	0.7971	0.7970	0.7953	0.7950
5	0.6872	0.6852	0.6852	0.6840	0.6836	0.6800	0.6792

Medium

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9630	0.9630
1	0.9277	0.9275	0.9275	0.9276	0.9276	0.9274	0.9274
1.5	0.8935	0.8933	0.8933	0.8934	0.8933	0.8930	0.8930
2	0.8606	0.8602	0.8602	0.8604	0.8603	0.8597	0.8597
3	0.7984	0.7976	0.7975	0.7979	0.7978	0.7966	0.7964
5	0.6872	0.6852	0.6852	0.6859	0.6858	0.6829	0.6826

Low

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9631	0.9631
1	0.9277	0.9275	0.9275	0.9276	0.9276	0.9275	0.9275
1.5	0.8935	0.8933	0.8933	0.8935	0.8935	0.8932	0.8932
2	0.8606	0.8602	0.8602	0.8606	0.8606	0.8601	0.8601
3	0.7984	0.7976	0.7975	0.7983	0.7983	0.7974	0.7973
5	0.6872	0.6852	0.6852	0.6869	0.6869	0.6847	0.6846

Table 4.7: (a) Credit Spreads and (b)Price of defaultable bond with recovery for several maturities and three different values of ϵ: high $\epsilon = \frac{1}{2}$, medium $\epsilon = \frac{1}{4}$ and low $\epsilon = \frac{1}{16}$.

(a) ZERO-COUPON DEFAULTABLE BOND PRICES W/O RECOVERY

High				Medium				Low			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	0.9899	0.9899	0.9899	0.1	0.9899	0.9899	0.9899	0.1	0.9899	0.9899	0.9899
0.5	0.9510	0.9510	0.9510	0.5	0.9510	0.9510	0.9510	0.5	0.9510	0.9510	0.9510
1	0.9047	0.9043	0.9043	1	0.9047	0.9045	0.9045	1	0.9047	0.9046	0.9046
1.5	0.8605	0.8598	0.8598	1.5	0.8605	0.8602	0.8602	1.5	0.8605	0.8605	0.8605
2	0.8186	0.8173	0.8173	2	0.8186	0.8181	0.8180	2	0.8186	0.8185	0.8185
3	0.7407	0.7382	0.7380	3	0.7407	0.7396	0.7396	3	0.7407	0.7405	0.7404
5	0.6064	0.6010	0.6003	5	0.6064	0.6041	0.6039	5	0.6064	0.6059	0.6059

(b) IMPLIED SURVIVAL PROBABILITIES

High				Medium				Low			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	99.481%	99.481%	99.481%	0.1	99.481%	99.481%	99.481%	0.1	99.481%	99.481%	99.481%
0.5	97.511%	97.502%	97.502%	0.5	97.511%	97.508%	97.508%	0.5	97.511%	97.511%	97.511%
1	95.104%	95.068%	95.067%	1	95.104%	95.089%	95.089%	1	95.104%	95.101%	95.101%
1.5	92.756%	92.677%	92.673%	1.5	92.756%	92.723%	92.722%	1.5	92.756%	92.749%	92.749%
2	90.466%	90.329%	90.321%	2	90.466%	90.409%	90.406%	2	90.466%	90.454%	90.453%
3	86.054%	85.765%	85.740%	3	86.054%	85.933%	85.924%	3	86.054%	86.028%	86.026%
5	77.865%	77.164%	77.080%	5	77.865%	77.568%	77.539%	5	77.865%	77.800%	77.795%

(c) DEFAULT DIGITAL PAYOFFS

High				Medium				Low			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	0.00495	0.00496	0.00496	0.1	0.00495	0.00495	0.00495	0.1	0.00495	0.00495	0.00495
0.5	0.02426	0.02437	0.02437	0.5	0.02426	0.02430	0.02430	0.5	0.02426	0.02427	0.02427
1	0.04733	0.04771	0.04772	1	0.04733	0.04749	0.04750	1	0.04733	0.04737	0.04737
1.5	0.06928	0.07008	0.07012	1.5	0.06928	0.06962	0.06963	1.5	0.06928	0.06935	0.06936
2	0.09016	0.09150	0.09157	2	0.09016	0.09072	0.09075	2	0.09016	0.09028	0.09029
3	0.12892	0.13161	0.13184	3	0.12892	0.13004	0.13013	3	0.12892	0.12916	0.12918
5	0.19572	0.20179	0.20250	5	0.19572	0.19828	0.19854	5	0.19572	0.19627	0.19632

Table 4.8: (a) Prices of zero-recovery, zero-coupon bond prices for several maturities (top table). (b)Implied survival probabilities in the interval $[t, T]$ for several maturities. (c) Price of a digital payoff at default if it occurs in the interval $[t, T]$. Three different values of ϵ: high $\epsilon = \frac{1}{2}$, medium $\epsilon = \frac{1}{4}$ and low $\epsilon = \frac{1}{16}$. Higher maturities and three different market conditions: bull, normal and bear.

The intuition comes from the fact the PD of firms with high creditworthiness should depend much less on market oscillation than that of less creditworthy firms that are more sensitive to business cycles. In this sense, different ϵ parameters could represent the term structure of firms with different credit ratings.

In the following we consider three different values for *epsilon*: high $\epsilon = 1/2$, medium $\epsilon = 1/4$ and low $\epsilon = 1/16$.[9]

Figure 4.9 and Tables 4.7 and 4.8 show the simulation results for the different ϵ values under normal market conditions. The key feature that results from considering different sensitivities is that the TS of less sensitive firms has a smaller TS slope. This is particularly obvious for scenarios (7) and (8) when the index influences both PD and LGD, and less obvious when it affects only one of them.

Thus, from a practical point of view, it is more important to take into account the correlation with the market index when considering a portfolio of securities with low credit ratings. We believe that the value of the portfolio will fluctuate significantly with the business cycle. The effect will be even more pronounced for firms with low credit ratings and especially in the case when we have stochastic index volatility dependent of the index level.

Using implied ATM volatilities as credit spread trackers

An interesting side effect of our model, when we take the index volatility to be stochastic and negatively related to the index level, is that short spread dynamics can be tracked quite well by observing the index volatility. See Figure 4.10 with three possible volatility paths and compare with the short spread evolution for the same paths in Figure 4.6.

If the (spot) volatility seems to be a good tracker of the short spread, then implied volatilities of options with longer maturities may be good trackers of the forward spread TS. This is all due to the negative correlation between the index level and its volatility.

Furthermore, it provides a fundamental reason for using implied volatilities of options on indices as predictors of the forward spread term structure and is in line with what seems to be common practice among traders who typically use at-the-money (ATM) volatility term structures in predicting forward credit spreads. For example, Collin-Dufresne, Goldstein, and Martin (2001) investigated the determinants of credit spread changes. They showed that credit spreads are mostly driven by a single common factor and that implied volatilities of index options

[9]The case of total insensitivity, or $\epsilon = 0$, is always considered since in scenarios (1)(2)(3)(4) $\lambda(I, \epsilon) = \bar{\lambda}$.

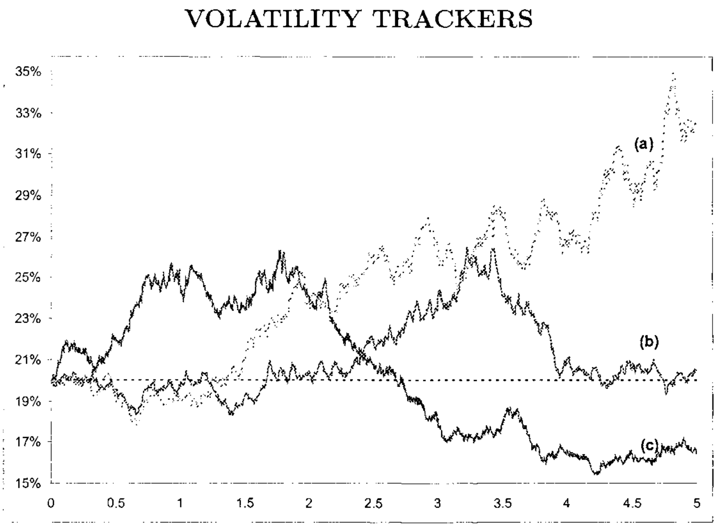

Figure 4.10: Volatility paths corresponding to the spot spread paths in Figure 4.6.

contain important information for credit spreads[10].

4.5 Conclusions and future research

We propose a concrete reduced-form model where both the PD and LGD are dependent on a macroeconomic index. Furthermore, during recessions both the PD and the LGD increase (the reverse happens during economic booms). Finally, recessions are periods of higher market volatility, while booms are characterized by low volatility.

Through simulation we show that many of the realistic features of our model depend on:

(i) the fact that *both* PD and LGD are driven by the same factors, and

[10]Recent papers (see e.g. Cremers, Dreissen, and Weinbaum (2004)) start using measures of volatility and skewness based also on individual stock options to explain credit spreads on corporate bonds. Implied volatilities of individual options are shown to contain important information for credit spreads. They showed that those implied volatilities improve on both implied volatilities of index options. However, in our framework we cannot model this feature since the reduced models do not allow us to model stock and corporate bonds together.

(ii) the negative relation of the index level and its volatility.

Unfortunately, these two aspects together make the model untractable from an analytical point of view. However, this may be the right price to pay given all the behaviors it captures. The main realistic features captured by our model can be summarized as follows:

- The difference between short spreads in bull versus bear markets can be up to three times more than the difference produced by models that consider the market influence in the PD or LGD only [mainly property (i)].
- Convergence to long-run levels is faster, originating flat TS for maturities higher than 15 years [properties (i) and (ii)].
- Market volatility tracks the short credit spread dynamics quite well, suggesting that the TS of ATM implied volatilities of index options may do the same for forward credit spreads [mainly property (ii)].

It seems clear that a model with both characteristic (i) and (ii) will not allow for closed-form solutions. We would, nonetheless, like to conclude the chapter on a positive note related to future research.

Given the tractability drawback, and since one must rely on numerical simulations, the ideal situation would be to model the intensity and the distribution of the loss quota as realistically as possible (this may involve different functional forms and a different market price of jump risk assumption). A study of the credit TS shapes observed in the market can help to define such functional forms. Calibration of our "toy model", or other more complex models, to market data seems to be another obvious next step. In addition to the single firm, this framework could also be extended to several firms and help deal with portfolio credit risk issues. For portfolio credit risk the interdependence between PD and LGD is likely to have a much more relevant impact than at the individual firm level. Indeed, portfolio losses depend upon both quantities, and the fact that periods when default is more likely may also be periods when recovery is lower suggests caution in using naive models to establish bank reserves and related precautionary measures. Finally, our last comment on volatility trackers may help to construct a bridge between equity and credit markets, and deserves further investigation.

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