Credit Risk & Forward Price Models

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To Nuno

Abstract and Structure

This thesis consists of three distinct parts. Part I introduces the *basic concepts* and the notion of *general quadratic term structures* (GQTS) essential in some of the following chapters. Part II focuses on *credit risk models* and Part III studies *forward price term structure models* using both the classical and the geometrical approach.

Part I is organized as follows.

Chapter 1 is divided in two main sections. The first section presents some of the fundamental concepts which are a pre-requisite to the papers that follow. All of the concepts and results are well known and hence the section can be regarded as an introduction to notation and the basic principles of arbitrage theory. The second part of the chapter is of a more technical nature and its purpose is to summarize some key results on point processes or differential geometry that will be used later in the thesis.

For finite dimensional factor models, Chapter 2 studies GQTS. These term structures include, as special cases, the affine term structures and Gaussian quadratic term structures previously studied in the literature. We show, however, that there are other, non-Gaussian, quadratic term structures and derive sufficient conditions for the existence of these *general quadratic term structures* for zero-coupon bond prices.

On Part II we focus on credit risk models.

In Chapter 3 we propose a reduced form model for default that allows us to derive closed-form solutions for all the key ingredients in credit risk modeling: risk-free bond prices, defaultable bond prices (with and without stochastic recovery) and probabilities of survival. We show that all these quantities can be represented in general exponential quadratic forms, despite the fact that the intensity of default is allowed to jump producing shot-noise effects. In addition, we show how to price defaultable digital puts, CDSs and options on defaultable bonds.

Further on, we study a model for portfolio credit risk that considers both firm-specific and systematic risk. The model generalizes the attempt of Duffie and Gârleanu (2001). We find that the model produces realistic default correlation and clustering effects. Next, we show how to price CDOs, options on CDOs and how to incorporate the link to currently proposed credit indices.

In Chapter 4 we start by presenting a reduced-form multiple default type of model and derive abstract results on the influence of a state variable X on credit spreads when both the intensity and the loss quota distribution are driven by X. The aim is to apply the results to a real life situation, namely, to the influence of macroeconomic risks on the term structure of credit spreads.

There is increasing support in the empirical literature for the proposition that both the probability of default (PD) and the loss given default (LGD) are correlated and driven by macroeconomic variables. Paradoxically, there has been very little effort, from the theoretical literature, to develop credit risk models that would take this into account. One explanation might be the additional complexity this leads to, even for the "treatable" default intensity models.

The goal of this paper is to develop the theoretical framework necessary to deal with this situation and, through numerical simulation, understand the impact of macroeconomic factors on the term structure of credit spreads. In the proposed setup, periods of economic depression are both periods of higher default intensity and lower recovery, producing a *business cycle effect.* Furthermore, we allow for the possibility of an index volatility that depends negatively on the index level and show that, when we include this realistic feature, the impacts on the credit spread term structure are emphasized.

Part III studies forward price term structure models.

Forward prices differ from futures prices in stochastic interest rate settings and become an interesting object of study in their own right.

Forward prices with different maturities are martingales under different forward measures. This mathematical property implies that the term structure of forward prices is always linked to the term structure of bond prices, and this dependence makes forward price term structure models relatively harder to handle.

For finite dimensional factor models, Chapter 5 applies the concept of GQTS to the term structure of forward prices. We show how the forward price term structure equation depends on the term structure of bond prices. We then exploit this connection and show that even in quadratic short rate settings we can have affine term structures for forward prices.

Finally, we show how the study of futures prices is naturally embedded in the study of forward prices, that the difference between the two term structures may be deterministic in some (non-trivial) stochastic interest rate settings.

In Chapter 6 we study a fairly general Wiener driven model for the term structure of forward prices.

The model, under a fixed martingale measure, $\mathbb{Q}$, is described by using two infinite dimensional stochastic differential equations (SDEs). The first system is a standard HJM model for (forward) interest rates, driven by a multidimensional Wiener process W. The second system is an infinite SDE for the term structure of forward prices on some specified underlying asset driven by the same W. Since the zero coupon bond volatilities will enter into the drift part of the SDE for these forward prices, the interest rate system is needed as input to the forward price system.

Given this setup, we use the Lie algebra methodology of Björk et al. to investigate under what conditions, on the volatility structure of the forward prices and/or interest rates, the inherently (doubly) infinite dimensional SDE for forward prices can be realized by a finite dimensional Markovian state space model.

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One of the most important persons in my life since I arrived in Sweden in 2001 has been, without question, my supervisor, Tomas Björk. He has taught me invaluable technical skills and guided me through these years with an enormous amount of patience and encouragement. He was able to be tough when needed; I will never forget all the times I had to answer the question "what exactly do you mean?" over and over again, or the times when I had to get up and "prove it" on the board in his office, nor how many times I had to rewrite my first paper. But I will also never forget what he told me when I had to continue on my own, with fewer meetings and more time working out my own ideas with my co-authors, and I was scared: "*Now you are out there on a limb, and no one can do this for you. Of course it's hard, of course it's scary, of course you will fall a few times, but I will always be there like a safety net to make sure you don't get hurt*". Most importantly, he was able to be a great supervisor and, at the same time, one of my best friends. Now that I really do have to let go of the supervisor, knowing I will always be able to keep the friend, is my only consolation.

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MAF/ISCTE, the Stochastic Finance 2004 Conference, the Workshop on Stochastic Analysis and Applications in Finance in Leipzig, the Quantitative Methods in Finance 2005 Conference, and seminars at the Finance Department at SSE, the Department of Mathematics of Universidade Nova de Lisboa, the Management Department at ISEG, the Econometric Department of Erasmus University and the Department of Mathematics of Universidade de Coimbra. At these events I also had the opportunity to meet many remarkable young researchers: John Aquilina, Aytac Ilhan, Mikael Monoyios, João Pedro Nunes, Evangelia Petrou and Maria Siopacha are among those I know I will always stay in touch with.

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There is a saying: *Distance is to love as wind is to fire; it extinguishes the small and enkindles the great.* These years gave me the opportunity to realize that I also have fantastic, loving friends in Portugal, who love me a lot and with whom I spent some time writing e-mails and chatting on the web. Keeping in touch with them, and the past, and knowing they were waiting for me in the future, gave me strength to go on. For them I wrote my *Adventures in Sweden* diary, which helped me joke about my own difficulties and enjoy my life here much more (they know who they are). Among them I would just like to mention those who visited me in Sweden and, in particular, Pedro Sousa, who came more than five times, helped me move from apartment to apartment (ultimately becoming an expert at IKEA furniture) and with whom I did more traveling than with anyone else. But also Carlos Figueiredo (Cabé), Jorge Guimarães, Ana Lacerda, Ana Nunes and Miguel Pereira came to visit and gave me the great pleasure of their company.

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Stockholm

November 2005

Raquel M. Gaspar

Contents

Abbreviations

ATS	:	affine term structure
CDS	:	credit default swap
CDO	:	collateralized debt obligation
CRM	:	credit risk model
DDP	:	default digital put
DSMPP	:	doubly stochastic marked point process
GQTS	:	general quadratic term structure
FDR	:	finite dimensional realization
FtDS	:	first-to-default swap
FPM	:	forward price model
HJM	:	Heath-Jarrow-Morton
LGD	:	loss given default
MPP	:	marked point process
ODE	:	ordinary differential equation
PD	:	probability of default
PDE	:	partial differential equation
PQTS	:	pure quadratic term structures
QE	:	quasi-exponential
QTS	:	quadratic term structure
SDE	:	stochastic differential equation
SPV	:	special purpose vehicle
TS	:	term structure

a.s.	:	almost surely
iff	:	if and only if
l.h.s.	:	left-hand-side
r.h.s.	:	right-hand-side
s.t.	:	such that
u.c.	:	unit of count
w.l.o.g	:	without loss of generality
w.r.t.	:	with respect to

Notation

W	:	Wiener process
J	:	jump process
N	:	counting process
t	:	running time
T	:	maturity time
x	:	time to maturity
Z, X	:	state variables
μ, α	:	drift component on factor dynamics
σ, γ	:	volatility component on factor dynamics
H, G	:	term structure functions
r	:	instantaneous short rate of interest
I	:	market index
λ, μ	:	intensities
$p(t,T)$	:	price at time t of a zero-coupon risk-free bond price with maturity T
$\bar{p}(t,T)$	:	price at time t of a zero-coupon defaultable bond price with maturity T
$f(t,T)$	:	forward price at time t for a contract with maturity T
$F(t,T)$	:	futures price at time t for a contract with maturity T

Formulation of the GQTS setup

Underlying quadratic processes:

$$r(t, Z_t) = Z_t^\top Q(t) Z_t + g(t)^\top Z_t + f(t)$$

$$\eta(t, Z_t) = Z_t^\top \mathsf{Q}(t) Z_t + \mathsf{g}^\top(t) Z_t + \mathsf{f}(t)$$

Factor dynamics:

$$dZ(t) = \alpha(t, Z_t)dt + \sigma(t, Z_t)dW(t)$$

for Z m-dimensional and W n-dimensional

$$\begin{aligned}
\alpha(t,z) &= \mathbf{d}(t) + \mathbf{E}(t)z \\
\sigma(t,z)\sigma^\top(t,z) &= \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t) z_u + \sum_{u,k=1}^{m} z_u \mathbf{g_{uk}}(t) z_k.
\end{aligned}$$

and we define

$$\mathbf{K}(t) = \begin{pmatrix} \mathbf{k_1}(t) \\ \mathbf{k_2}(t) \\ \vdots \\ \mathbf{k_m}(t) \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{g_{11}}(t) & \mathbf{g_{12}}(t) & \cdots & \mathbf{g_{1m}}(t) \\ \mathbf{g_{21}}(t) & \mathbf{g_{22}}(t) & \cdots & \mathbf{g_{2m}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g_{m1}}(t) & \mathbf{g_{m2}}(t) & \cdots & \mathbf{g_{mm}}(t) \end{pmatrix}.$$

Part I

Basic Concepts

Chapter 1

Preliminaries

This chapter is divided in two main sections. The first section presents some of the fundamental concepts which are a pre-requisite to the papers that follow. All of the concepts and results are well known and hence the section can be regarded as an introduction to notation and the basic principles of arbitrage theory. The second part of the chapter is of a more technical nature and its purpose is to summarize some key results on point processes or differential geometry that will be used later in the thesis.

1.1 Concepts

1.1.1 Basic Setup

Formal proofs for the results presented in this section can be found in standard textbooks on arbitrage theory. We refer to Björk (2004a), and whenever concerning defaultable assets to Lando (2004) or Schönbucher (2003).

In each chapter of this thesis we consider a set of financial assets, whose price processes evolve stochastically over time. This price processes are sometimes driven only by Wiener processes other times by point processes as well. The assets considered can be ordinary bonds or stocks, but they can also be (financial) derivatives traded on the market.

A subset of all existent assets will be assumed to be already in the market (basic assets), others, the derivatives, are contracts defined in terms of the assets already present at the market. Most models consider as pre-existent what is know as *the bank account*, usually denoted by B and whose instantaneous rate of return is the interest rate r,

$$\begin{cases} dB(t) &= r(t)dt \\ B(0) &= 1 \end{cases} \qquad B(t) = \exp\left\{\int_t^T r(s)ds\right\} .$$

To find prices and hedging for these derivatives has been one of the main goals of arbitrage theory and the fundamental idea is that arbitrage portfolios should not exist in efficient markets. Roughly speaking, an arbitrage portfolio can be built at zero cost, will have positive value with positive probability and will never have a negative value, at any future time. In ideal markets (complete markets) the absence of arbitrage requirement will be enough to uniquely price the derivatives.

Consider the basic assets $S_0, S_1, \cdots, S_n$.

Definition 1.1. A portfolio is a locally bounded predictable process

$$h(t) = [h_0(t), h_1(t), \cdots, h_n(t)] \ .$$

The value process V associated with a portfolio h is defined by $V(t,h) = \sum_{i=0}^{n} h_i(t)S_i(t)$.
A portfolio strategy is said to be *self-financing* if it satisfies

$$dV(t,h) = \sum_{i=0}^{n} h_i(t)dS_i(t) \ .$$

Definition 1.2. An *arbitrage portfolio* is a self-financing portfolio h for which the associated value process satisfies

1. $V(0,h) = 0$;
2. $\mathbb{P}[V(T,h) \geq 0] = 1$ and $\mathbb{P}[V(T,h) > 0] > 0$
3. It is $\mathbb{Q}^0$-admissible for some martingale measure $\mathbb{Q}^0$.

The market is said to be *arbitrage-free* if no arbitrage portfolios exist for any $T \in \mathbb{R}_+$.

The "$\mathbb{Q}^0$-admissible" statement has to do with a needed integrability condition. To be more concrete, we need to introduce some definitions.

We start with the notion of filtration or information flow.

Definition 1.3. The *filtration* or *information flow* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ such that

$$\forall t \geq s \geq 0, \qquad \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} .$$

$\mathcal{F}_t$ can, thus, be interpreted as the information known at time t, which increase with time.

The next result on iterated expectations is frequently used.

Result 1.4. *Let $\mathcal{F}_t \subseteq \mathcal{F}_T$ and let X be an integrable random variable. Then $Z = \mathbb{E}[X|\mathcal{F}_t]$ is an integrable random variable and*

$$\mathbb{E}[Z|F_t] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_T]|\mathcal{F}_t] .$$

A probability space equipped with a filtration is called a *filtered probability space* and is denoted $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\right)$.

Definition 1.5. Consider a given filtration $(\mathcal{F}_t)_{t\geq 0}$ on some probability space and a variable X on the same space. We say the process X is *adapted* to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if $X_t \in \mathcal{F}_t$ for all $t \geq 0$.

Definition 1.6. The process M is a $\mathcal{F}_t$*-martingale* if

1. M is $(\mathcal{F}_t)_{t\geq 0}$ adapted.
2. $M_t \in L^1$ for each t
3. For every s and t, s.t. $0 \leq s \leq t$ it holds that

$$M_s = \mathbb{E}[X_t|\mathcal{F}_s], \qquad \mathbb{P} - a.s.$$

Definition 1.7. A *martingale measure* is a probability measure $\mathbb{Q}^0$ for which the following holds

(i) $\mathbb{Q}^0 \sim \mathbb{P}$.

(ii) The normalized price processes $Z_i^0(t) = \dfrac{S_i(t)}{S_0(t)}$ are martingales.

Where $S_0(t)$ is some strictly positive process called *the numeraire.*

Finally, we get to the definition of admissibility.

Definition 1.8. Fix a martingale measure $\mathbb{Q}^0$. A self-financing portfolio h is said to be $\mathbb{Q}^0$-*admissible* if the associated normalized process

$$\bar{V}(t,h) = \sum_{i=0}^{n} h_i(t) \frac{S_i(t)}{S_0(t)} ,$$

is a $\mathbb{Q}^0$-martingale, for some strictly positive numeraire process $S_0(t)$.

The general pricing formula is given in the next result

Result 1.9. *Fix a martingale measure $\mathbb{Q}^0$. An arbitrage free price process π of a T-contingent claim $\mathcal{X}$ is given by*

$$\pi(t,\mathcal{X}) = S_0(t)\mathbb{E}_t^0 \left[\frac{\mathcal{X}}{S_0(T)} \right] ,$$

where $\mathbb{E}_t^0 [\cdot]$ denotes expectation under the $\mathbb{Q}^0$-measure, condition on $\mathcal{F}_t$.

Whenever one chooses the bank account $B(t)$ as the numeraire, the martingale measure is denoted simply $\mathbb{Q}$ and is called the *risk-neutral martingale measure.*[1] This is by far the most common choice of martingale measure in the literature.

Result 1.10. *The price process π of a contingent T- claim $\mathcal{X}$ is given by*

$$\pi(t,\mathcal{X}) = \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r(s)ds} \mathcal{X} \right] .$$

where $\mathbb{E}_t^{\mathbb{Q}} [\cdot] = \mathbb{E}^{\mathbb{Q}} [\cdot | \mathcal{F}_t]$.

We can now present a mathematical description of the market and key assets we will be usually dealing with.

In the standard setup we consider a financial market living on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\geq 0})$ where $\mathbb{Q}$ is the risk-neutral martingale measure. The probability space will typically carry a multidimensional Wiener process W and sometimes an additional point process μ to model defaults, we delay the concrete characterization of the point processes to each of the chapters. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated both by W and μ, i.e., $\mathcal{F}_t = \sigma \{W, \mu\}$.

[1]It is standard to use $\mathbb{Q}$ to denote the risk-neutral martingale measure and $\mathbb{Q}^0$ to denote measures under some other numeraire.

1.1.2 Default-free Bonds

At any time t there are default-free zero-coupon bonds for all maturities $T > t$. The notional of the zero-coupon bonds is, w.l.o.g., normalized to 1. The price at time t of a bond with maturity T is denoted $p(t,T)$. Using Result 1.10 we immediately get that the zero-coupon bond with maturity T is given by

$$p(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\right].$$

The continously compounded default-free forward rate over the period $[S,T]$ contracted at time t is defined (for $t \leq S \leq T$) by

$$f(t,S,T) = \frac{1}{T-S}\left[\ln p(t,S) - \ln p(t,T)\right] .$$

The continuously compounded instantaneous default-free forward rate at time t for $T > t$ is defined as

$$f(t,T) = -\frac{\partial}{\partial T}\ln p(t,T) .$$

The instantaneous default-free short rate $r(t)$, the default-free discount factor $\beta(t)$ and the default-free bank account $B(t)$ are defined by

$$r(t) = f(t,t) , \qquad \beta(t) = \exp\left\{-\int_0^t r(s)ds\right\} , \qquad B(t) = \frac{1}{\beta(t)} .$$

We note that for each fixed maturity T the bond price $p(t,T)$ is a strictly positive process, and, thus a good candidate as numeraire. Whenever one chooses the zero-coupon bond price with maturity T, $p(t,T)$, as the numeraire, $\mathbb{Q}^T$ is what is called the *T-forward measure* and we have the following pricing result. Note that for each different maturity we have a different numeraire and, thus, a different martingale measure.

Result 1.11. *The price process π of a contingent T- claim $\mathcal{X}$ is given by*

$$\pi(t,\mathcal{X}) = p(t,T)\mathbb{E}_t^T\left[\mathcal{X}\right].$$

where $\mathbb{E}_t^T[\cdot]$ denotes expectation under the $\mathbb{Q}^T$-measure conditional on $\mathcal{F}_t$.

The relation between bond prices, forward interest rates and short interest rates, draws a connection in terms of their dynamics.

Result 1.12. *Consider the following dynamics for $p(t,T)$, $f(t,T)$ and $r(t)$.*

$$\begin{aligned} dp(t,T) &= p(t,T)m(t,T) + p(t,T)v(t,T)dW_t && (1.1)\\ df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW_t && (1.2)\\ dr(t) &= a(t)dt + b(t)dW_t , && (1.3) \end{aligned}$$

where the processes (t,T),$v(t,T)$,$\alpha(t,T)$,$\sigma(t,T)$,$a(t)$ *and* $b(t)$ *are adapted processes (and* T *is seen as a parameter).*

Then,

1. *If* $p(t,T)$ *satisfies (1.1), then for the forward rate dynamics are those in (1.2) with*

$$
\begin{aligned}
\alpha(t,T) &= \frac{\partial v}{\partial T}(t,T)v^\top(t,T) - \frac{\partial m}{\partial T}(t,T) \ , \\
\sigma(t,T) &= -\frac{\partial v}{\partial T}(t,T) \ .
\end{aligned}
$$

   where $\cdot^\top$ *denotes transpose.*

2. *If* $f(t,T)$ *satisfies (1.2) then the short rate dynamics are those in (1.3) with*

$$
\begin{aligned}
a(t) &= \frac{\partial f}{\partial T}(t,t) + \alpha(t,t) \\
b(t) &= \sigma(t,t).
\end{aligned}
$$

3. *If* $f(t,T)$ *satisfies (1.2) then* $p(t,T)$ *satisfies (1.1) with*

$$
\begin{aligned}
m(t,T) &= r(t) - \int_t^T \alpha(t,s)ds + \frac{1}{2}\|v(t,T)\|^2 \\
v(t,T) &= -\int_t^T \sigma(t,s)ds
\end{aligned}
$$

   where $\|\cdot\|$ *denotes the Euclidean norm.*

The interest rate models can be divided into two categories: *short rate models* and *HJM type of models.*

In short rate models, the short rate r is assumed to be the only state variable. Typically one models its dynamics directly under the risk-neutral measure $\mathbb{Q}$. Most models in the literature assume it is a Wiener driven process

$$
\begin{cases}
dr_t &= a(t,r_t)dt + b(t,r_t)dW_t \\
r_0 &= r^0
\end{cases} .
$$

One of the advantages of short rate modeling is that, using Kolmogorov's backward equation, pricing and hedging can be done in a PDE framework. The most well-know PDE is the so-called term structure equation for zero-coupon bond prices.

Result 1.13. *Suppose that the market is free of arbitrage. Then the price of a zero-coupon bond price is given by* $p(t;T) = H(t, r_t, T)$ *where*

$$\begin{cases} \dfrac{\partial H}{\partial t}(t,r,T) + \mu(t,r)\dfrac{\partial H}{\partial r}(t,r,T) + \dfrac{1}{2}\sigma(t,r)\dfrac{\partial^2 H}{\partial r^2}(t,r,T) & = \; rH(t,r,T) \\ H(T,r,T) & = \; 1 \end{cases}$$

where T *is seems as a parameter and the mapping* $T \longrightarrow H(t,r,T)$ *is called the term structure of bond prices at time* t.

For the above PDE to have a nice solution, we need to impose some structure on the bond prices. Example of a case where it is easy to solve is the class of *general quadratic term structures* (GQTS) that will be introduced in Chapter 2, which include as special cases the well-known affine term structures (ATS) and Gaussian quadratic term structures (Gaussian-QTS).

The biggest disadvantage of short rate models is that it is difficult to fit the model to the observed term structure and volatilities. The procedure of fitting the model to the observed term structure is known as *inversion of the yield curve.* Given the bond prices obtained in a short rate model, the task to match the initial term structure produced by the model, $p(0,T)$, to the market observed initial term structure $p^*(0,T)$.

An alternative to short rate models was introduce by D., Jarrow, and Morton (1992), and became known as the *HJM framework.* This framework avoids the problems associated with inverting the yield curve and obtaining reasonable forward rate volatilities, by modeling directly the entire forward rate curve $f(t,T)$. Again it is common to do the modeling directly under the risk-neutral measure $\mathbb{Q}$. One has the flexibility of specifying the forward rate volatilities and the initial term structure observed in the market $f^*(t,T)$ is taken as an input.

$$\begin{cases} df(t,T) & = \; \alpha(t,T)dt + \sigma(t,T)dW_t \\ f(0,T) & = \; f^*(0,T) \end{cases} .$$

What can be seen as a drawback is that in a HJM-type of model one needs to be careful not to introduce arbitrage opportunities into the model. In short rate models this is never a concern, since the term structure equation derivation guarantees absence of arbitrage.

To understand better what this means we note that an immediate consequence of modeling the forward rate curve is that we also model the bond price dynamics (recall Result 1.12 –3.). Since the normalized price process must be a $\mathbb{Q}$-martingale, we need to guarantee

$$\begin{aligned} r(t) - \int_t^T \alpha(t,s)ds + \frac{1}{2}\|v(t,T)\|^2 &= r(t) \\ \int_t^T \alpha(t,s)ds &= \frac{1}{2} \underbrace{\|v(t,T)\|^2}_{-\int_t^T \|\sigma(t,s)\|^2 ds} . \end{aligned}$$

Differentiating both sides w.r.t. T we get the *HJM drift condition*

$$\alpha(t,T) = v(t,T) \int_t^T v^\top(t,s)ds .$$

A true disadvantage of HJM models is that, in general the short rate, will not be Markovian (recall Result 1.12 –2.) and this leads to complications in pricing.

We finish this section by introducing what is known as the *Musiela parameterization.*

In practical applications it is more natural to use time *to* maturity instead of time *of* maturity, to parameterize bonds and forward rates. Up to now we denoted running time by t and time of maturity by T, then $x = T - t$ is the time to maturity. Using this parameterization we can write $r(t,x) = f(t, t+x)$ and the dynamics of the forward rates become[2]

$$dr(t,x) = \{\mathbf{F}r(t,x) + D(t,x)\}\, dt + \tilde{\sigma}(t,x)dW_t$$

for $\tilde{\sigma}(t,x) = \sigma(t,t+x)$, $D(t,x) = \tilde{\sigma}(t,x) \int_0^x \tilde{\sigma}^\top(t,s)ds$ and $\mathbf{F} = \frac{\partial}{\partial x}$.

1.1.3 Forward and Futures

So far we have used the expression "price at time t" of a contingent claim to refer to the amount of money one would have to pay at time t to become the holder of the contract at time t. There are, however, contracts which specify that the payment should be done differently. *Forward* and *futures* are examples of such contracts.

[2]This notation is standard in the literature and results in no confusion since the short rate, previously denoted by r does not depend on maturity. Still, when using the Musiela parameterization the short rate is usually denote by R and we have $R(t) = r(t,0)$.

Definition 1.14. Let $\mathcal{X}$ be a contingent T-claim. A *forward contract* on $\mathcal{X}$ contracted at time t, with delivery date T, and a forward price $f(t,T)$ is defined by the following payment scheme[3]:

- The holder of the forward contract receives, at time T, the stochastic amount $\mathcal{X}$ from the underwriter.
- The holder of the contract pays, at time T, the amount $f(t,T)$ to the underwriter.
- The forward price is determined at time t in such a way that the value of the contract is zero at this time.

We note all payments are due only at time T, though the forward price is determined at time t. Forward contracts are typically *over-the-counter* (OTC) instruments. We also note that even if the value of a specific forward contract equals zero at time t, it will usually have a non-zero value at any other point in time in $[t,T]$.

To determine the forward price we use Result 1.10 and the fact that at time t the value of the contract must be zero,

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}(\mathcal{X} - f(t,T))\right] &= 0 \\
p(t,T)f(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathcal{X}\right] \\
f(t,T) &= \frac{1}{p(t,T)}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathcal{X}\right] = \mathbb{E}_t^T\left[\mathcal{X}\right].
\end{aligned}$$

From the last expression we see that the forward price $f(t,T)$ is a $\mathbb{Q}^T$-martingale.

A futures contract is very much like a forward contract in the sense that it is an agreement between two parties to buy or sell a certain claim at a specified time T in the future. The principal different between them lies on the ways the payment is settled.

Definition 1.15. Let $\mathcal{X}$ be a contingent T-claim. A *futures contract* on $\mathcal{X}$ with time of delivery T, is a financial asset with the following properties.

- At every point in time $0 \leq t \leq T$, there exists in the market the object $F(t,T)$, known as the *futures price* for $\mathcal{X}$ at t, with delivery T.

[3] A work of caution is due in terms of notation. $f(t,T)$ is used for both the instantaneous forward rate when dealing in interest rates, and for forward prices. These objects are not to be confused, and in each of the chapters it will be made clear what we will be dealing with.

- A the time T of delivery, the holder of the contract pays $F(T,T)$ and receives the claim $\mathcal{X}$.
- During an arbitrary time interval $(s,t]$ the holder of the contract receives the amount $F(t,T) - F(s,T)$.
- The spot price, at any point t prior to delivery, of entering a futures contract is by definition zero.

Futures contracts, as opposed to forward contracts, are traded in a standardized manner at exchanges. We also note that the name "futures price" is somehow misleading as, from the above payment scheme, we realize no one actually pays that amount of money. A more adequate name for $F(t,T)$ would be "futures quotation", still futures price is broadly used both in the academic literature as in the market, so we stick to that.

A very well-known result for futures prices is that they are $\mathbb{Q}$-martingales.

Result 1.16. *Let $\mathcal{X}$ be a given contingent T-claim. Then the following holds for a fixed martingale measure $\mathbb{Q}$,*

$$F(t,T) = \mathbb{E}_t^{\mathbb{Q}}[\mathcal{X}] \ .$$

Finally we recall that the $\mathbb{Q}$-measure and the $\mathbb{Q}^T$-measure are equivalent in deterministic interest rate settings.

Result 1.17. *If the short rate is deterministic, then the forward and the futures price process coincide,*

$$f(t,T) = F(t,T) = \mathbb{E}_t^{\mathbb{Q}}[\mathcal{X}] \ .$$

1.1.4 Credit Risk Modeling

"It is a good time to be a financial-disaster writer. Scarcely a day goes by without a warning of some dire calamity."

Roger Lowenstein, June 2005, The New York Times.

Up to now, the possibility that market agents may not honor their obligations has been ignored. Default risk is exactly the risk that an obligator does not honor his payment obligations. This can easily be overcome by modeling *default events.* One part of this thesis will focus on *credit risk models* and studies such events.

In reality, these default events do happen and it is not difficult to find extremely well known examples. Only in recent years we have the cases of Long-term Capital Management (LTCM) (1998), Enron (2001), Sabena (2001), WorldCom (2002), United Airlines (2002), Parmalat SpA (2003) and the even more recently the downgrading of General Motor's credit rating to junk bonds (May 2005) and the ongoing scandal of the American International Group (AIG) still under investigation and that has originated the fastest decrease in a firm's market value since WorldCom. Besides these, of course many more less spectacular defaults occur daily in financial markets, all in all giving a clear justification to model defaults.

In modeling default, one typically considers a firm or a set of firms that is allowed to default. In addition, the firm has issued corporate bonds that are traded on the market. Defaults are supposed to be rare and unpredictable. Arrivals of defaults, losses given default or correlation of defaults across firms are objects of study in their own right in the credit risk literature.

Traditionally, credit risk models have been divided into two main categories: *structural models* and *reduced-form models.*

In structural models, shares and corporate bonds are seen as derivatives on the firm's assets. Limited liability gives the shareholders the option to abandon the firm. Bond holders can be, thus, interpreted as having a short position in this put option. Conversely, one can see equity as a call option on the value of the firm, with strike equal to the notional of the outstanding debt. Assuming geometric Brownian motion for the firm's value, results in a Black-Scholes type formula for equity. The structural approach has the following advantages: (i) it provides a relationship between different securities of the same firm, which allows to price products like convertible bonds, collateralized loans, etc.; (ii) default correlation between different firms can be modeled realistically; (iii) it has a fundamental orientation; (iv) it is well suited to explore other questions related to the firm's capital structure (corporate finance). Its disadvantages are: (i) it relies on the assumption of observability of the firm's value for calibration or fitting purposes; (ii) corporate bonds are not inputs but outputs; (iii) all data are rarely available; (iv) it is often complex and inflexible from the analytical point of view when some of the basic assumptions are relaxed; (v) its gives unrealistic short-term spreads.

Reduced-form models are mathematical models of default that tend to be more realistic and tractable and, thus, extremely useful for pricing and hedging. These models make use of *jump processes* whose key properties are summarized in Section 1.2.1. They easily incorporate the unpredictability of defaults and the fact that they cause discontinuous price changes. The main advantages of this approach are: (i) the models tend to be much more tractable even when we choose non-trivial intensities of default; (ii) credit spreads can be modeled directly; (iii) it can easily be fitted to observed market data; (iv) it is suitable for pricing credit derivatives. Its main disadvantages are: (i) it has little explanatory value; (ii) default correlation is potentially difficult to model.

In Chapter 3 we propose an intensity model of default that may be able to overcome the correlation difficulties.

Recently, there has been a merge between these two categories, giving us fundamental reasons to use reduced-form models. Indeed, Duffie and Lando (2001) show that the difference between the tractable reduced-form approach and the economically intuitive structural approach becomes moot when one includes realistic frictions in structural models. Assumptions of imperfect information about the asset or the liability structure, or introduction of jumps in the asset value process lead to unpredictability in defaults and the structural model has an equivalent reduced-form formulation.

"Intensity models are fully consistent with a structural approach in models where bond holders have incomplete information about the value of the firm."

Darell Duffie and David Lando, 2001.

Defaultable bonds

We now introduce the notation used when studying corporate bonds, which is defined in parallel to that of the risk-free bond market.

We assume that at any time t there are defaultable zero-coupon bonds for all maturities $T \leq T^*$ (T^* is the biggest maturity and is assumed to be finite [4]).

In some models, given default, the bondholder receives no payment; these are called *zero recovery models*. In other models, it is assumed that the bondholder will recover some fraction $0 < R < 1$ of what he was entitled to. The fraction he actually looses $(1 - R)$ is called *loss quota* and is usually denoted by q. Different assumptions on how this loss quota is defined lead to different *recovery models*.

The default time is typically denoted by τ. [5]

[4] This is a relistic technical assumption since one cannot find, in the market, corporate bonds with maturities higher than, say, 50 years.

[5] If more than one default times are to be considered, we use $\tau_1, \tau_2, \cdots$.

The price at time t of a zero-coupon defaultable bond with maturity T and zero recovery is denoted $\bar{p}_o(t,T)$. As before, w.l.o.g., its notional is normalized to 1. The payoff at time T of this bond is, thus, $\mathbf{1}_{\{\tau>T\}}$. That is, it pays one unit of account if the default has not occurred until T and nothing otherwise.

The continuously compounded defaultable forward rate over the period $[S,T]$ contracted at time t is defined (for $t \leq S \leq T$)

$$\bar{f}_o(t,S,T) = \frac{1}{T-S}\left[\ln \bar{p}_o(t,S) - \ln \bar{p}_o(t,T)\right] \ .$$

The continuously compounded instantaneous default-free forward rate at time t for $T > t$ is defined as

$$\bar{f}_o(t,T) = -\frac{\partial}{\partial T}\ln \bar{p}_o(t,T) \ .$$

The instantaneous defaultable short rate $\bar{r}(t)$, the defaultable discount factor $\bar{\beta}(t)$ and the defaultable bank account $\bar{B}_o(t)$ are defined by

$$\bar{r}_o(t) = \bar{f}_o(t,t) \ , \qquad \bar{\beta}_o(t) = \exp\left\{-\int_0^t \bar{r}_o(s)ds\right\} \ , \qquad \bar{B}_o(t) = \mathbf{1}_{\{t<\tau\}}\frac{1}{\bar{\beta}_o(t)} \ .$$

When recovery is considered we use the notation $\bar{p}(t,T)$ for the defaultable bond prices and $\bar{f}(t,S,T)$, $\bar{f}(t,T)$, $\bar{r}(t)$, $\bar{\beta}(t)$ and $\bar{B}(t)$ as above with $\bar{p}_o(t,T)$ replaced by $\bar{p}(t,T)$.

Using Result 1.10 it is easy to get the next formulas.

Result 1.18. *The price at time t of a zero-coupon bond given no default up to t is given by the following,*

- *if zero-recovery,*

$$\bar{p}_o(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbf{1}_{\{\tau>T\}}\right]$$

- *if recovery $R = (1-q)$*

$$\bar{p}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left[\mathbf{1}_{\{\tau>T\}} + (1-q)\mathbf{1}_{\{\tau\leq T\}}\right]\right]$$

The implied survival $\mathbb{Q}$-probability in the interval $(t,T]$, given no default up to time t is given by

$$\mathbb{Q}_S(t,T) = \mathbb{Q}(\tau > T) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau>T\}}\right] .$$

Credit derivatives

The popularity of credit risk models led to a boom in the use of what are called *credit derivatives.* Loosely speaking, a credit derivative is a contingent claim whose payoff is (totally or partially) dependent on the occurrence of *credit events* such as bankruptcy, failure to pay, restructuring, ratings downgrade, changes in credit spread, etc.

We will now be more specific and describe some of the most well-known credit derivatives and, at the same time, we will also introduce some market jargon.

Any credit event is defined with respect to a *reference credit* (or *name*) and some *reference credit asset(s)* issued by the reference credit. A reference credit may be a firm, while reference credit assets may be, for example, corporate bonds. If a credit event has occurred, the *default payment* has to be made by one of the counter parties. Besides the default payment, a credit derivative can give further payoffs that are not default contingent.

Buying a credit derivative typically means *buying credit protection*, which is economically equivalent to *shorting credit risk.* Hence *selling* credit protection means going *long* the credit risk. Alternatively, one may speak of *protection buyers/sellers* as the payers/receivers of the premium.

In addition to the traditional uses in terms of hedging, speculation and arbitrage, other uses of credit derivatives are the reduction of regulatory capital and an unfunded way to diversify revenue. We now define four of the most popular credit derivatives. We now defined four of the most popular credit derivatives.

Definition 1.19. A *default digital put* (DDP) transfers default risk between two parties A and B. Typically the payoffs consist of:

- The protection seller B has the *short position* and agrees to pay to A 1 u.c., at default time, if default occurs before the maturity of the contract T.
- For this, A who has the *long position* pays an up front fee, called *price* and denoted π_{DDP}, to B.

For fully defining a DDP we also need to know:

- the underlying the reference credit and his reference credit assets,
- the exact definition of the credit event that is to be insured,
- the maturity date.

Definition 1.20. A *credit default swap* (CDS) transfers some default risk between two parties A and B. Typically the payoffs consist of:

- The protection seller B has the *short position* and agrees to pay the default payment
$$\text{notional} \times (1 - R)$$
to A if a default has happened.
- For this, A, who has the *long position*, pays a periodic fee $\bar{s}$ to B at $T_1, T_2, \cdots, T_{N^*}$ (until the maturity T_{N^*} of the CDS or until default, whichever comes first).

For fully defining a CDS we also need to know:

- the underlying the reference credit and his reference credit assets,
- the exact definition of the credit event that is to be insured,
- the notional of the CDS,
- the start of the CDS and the start of protection (if not equal to the start of the CDS, it is called a *forward start CDS*),
- the maturity date T_{N^*},
- the credit default spread (typically established so that at the start of the CDS the value of both legs is the same),
- frequency for the spread payments,
- the payment at the credit even, and its settlement.

Besides credit derivatives on individual firms, portfolio credit spreads are also extremely popular. The underlying is a portfolio of defaultable assets and instead of reference credit it is usual to use the term *name* to refer to each of the firms who has issued assets in that portfolio.

The aim of these products is either to increase the credit quality of a portfolio, or to provide insurance against default correlations.

An example in the first category is the *first-to-default swap* (FtDS) that provides insurance against the first default of a portfolio (formal description appears bellow).

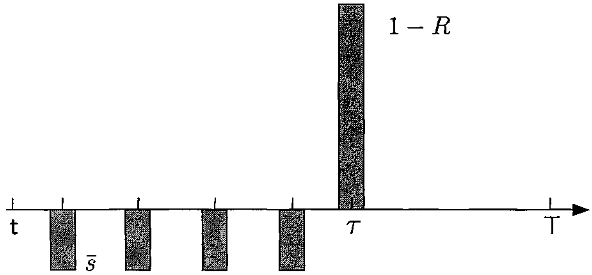


Figure 1.1: Cash flows for a CDS. Default occurs at τ before the option expires. The payoff is the difference of the par value of the bond to the bond's price at default, the recovery R. The default swap spread, $\bar{s}$, is paid regularly at times $T_1, \ldots$ until default.

Individual defaults in a large portfolio are expected to be diversified and, thus, do not require hedging. However, the clustering of defaults or joint defaults are not and that is why these products are also known as *default correlation products.* The most well known example of such products are *collateralized debt obligations* (CDOs)

Definition 1.21. A *first-to-default swap* (FtDS) transfers the first default risk on a portfolio between two parties A and B. Typically the payoffs consist of:

- The protection seller B has the *short position* and agrees to pay the default payment
$$M^{\mathrm{FtD}} \times (1 - R)$$
to A if a default has happened.

- For this, A who has the *long position* pays a periodic fee $\bar{s}^{\mathrm{FtD}}$ to B (until the maturity of the FDS or until the first default, whichever comes first).

For fully defining a FtDS we also need to know:

- the underlying names, reference credit assets and its respective nominals,

- the exact definition of the credit event that is to be insured,

- the start of the FtDS and the start of protection (if not equal to the start of the FDS, it is called a *forward start FtDS*),

- the maturity date,

- the spread $\bar{s}^{\text{FtD}}$ (typically established so that at the start of the FtDS the value of both legs is the same),
- frequency for the spread payments,
- the payment at the credit even, and its settlement.

The most popular instrument of all has been the collateralized debt obligations (CDOs). These products are based on a portfolio of defaultable assets, *cash-flow CDOs* are based on portfolios of corporate bonds, while *synthetic CDOs* are based on portfolios of CDSs.

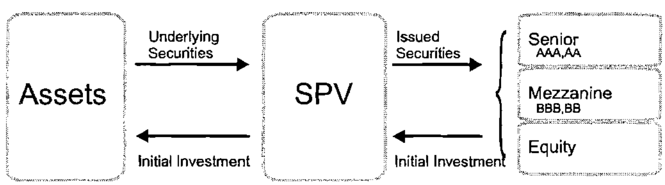
"Over recent years, CDOs, i.e. pools of credit exposure marketed to investors in the form of tranched securities became an important part of the global fixed income market."

Bank of International Settlements, 2005.

Definition 1.22. In a *collateralized debt obligation* (CDO) the goal is to transfer the portfolio to a special purpose vehicle (SPV) who then issues notes on different credit *tranches* of the portfolio.

- There exist at least three tranches, in descendent order of seniority: an *equity tranche* (or first loss), a *mezzanine tranche* and a *senior tranche.* It is, however, common to find CDOs with several mezzanine tranches with different seniorities.
- If during the life of the CDO one (of several) of the names default, all recovery payments are reinvested in default-free securities.
- At the maturity of the CDO, the portfolio is liquidated and the proceeds distributed to the tranches, according to their seniority ranking.

The following picture clarifies the structure of a CDO.



The pricing of portfolio derivatives – FtDS, CDOs tranches, etc. – are the most challenging questions of credit risk derivatives.

1.2 Mathematical Background

1.2.1 Jump processes

Formal proof to all the results in this section can be found in Cont and Tankov (2004).

The *characteristic function* of a random variable is the Fourier transform of its distribution. Many probabilistic properties of random variables correspond to analytical properties of their characteristic function, making this concept very useful for studying random variables. The characteristic of the random variable X, is the function ϕ_X defined by

$$\phi_X(u) = \mathbb{E}\left[e^{iuX}\right] \forall u \in \mathbb{R} \ .$$

The following properties link the characteristic function to the moments of a distribution.

Result 1.23.

1. *ϕ_X has n continuous derivatives at $u = 0$ if and only if $\mathbb{E}\left[|X|^k\right] < \infty$.*

2. *X possesses finite moments of all orders if and only if $u \to \phi_X(u)$ is C^∞ at $u = 0$.*

3. *Whenever its n-th moment exist the moments of X, m are related to the derivatives of ϕ_X by:*

$$\forall k = 1, \cdots, n \qquad m(k) = \mathbb{E}\left[X^k\right] = \frac{1}{i^k}\frac{\partial^k \phi_X}{\partial u^k}(0) \ .$$

The *Laplace transform* of a random variable X is the function φ_X defined by

$$\forall u \in \mathbb{R}, \qquad \varphi(u) = \mathbb{E}\left[e^{-uX}\right] \ .$$

Contrarily to the characteristic function, which is always well-defined, the Laplace transform is not always defined (as the integral from the above expectation may not converge for some value of u). When φ_X is well-defined, it can be formally related to the characteristic function ϕ_X by:

$$\varphi_X(u) = \phi_X(iu).$$

We will be particularly interested in Poisson and exponentially distributed random variables.

Poisson and exponentially distributed random variables

Definition 1.24. A stochastic variable X is said to be *Poisson distributed* with parameter ν if

$$\forall\, k \in \mathbb{N} \qquad \mathbb{P}(X = k) = e^{-\nu}\frac{\nu^k}{k!}$$

Result 1.25. *A stochastic variable X is Poisson distributed with parameter ν if and only if its characteristic distribution is given by*

$$\phi_X(u) = \exp\left[\nu\left(e^{iu} - 1\right)\right] \ .$$

Its Laplace transform is, thus, given by

$$\varphi_X(u) = \exp\left[\nu\left(e^{-u} - 1\right)\right] \ .$$

One interesting property of the Poisson distribution is the stability under convolution: if X_1 and X_2 are *independent* Poisson variables with parameters ν_1 and ν_2, then X_1+X_2 also follows a Poisson law with parameter $\nu_1 + \nu_2$. In particular this leads to the following consequence: for any integer n, a Poisson random variable X with parameter ν can be expressed as a sum of n independent (Poisson) random variables X_i, with parameter $\frac{\nu}{n}$.

Definition 1.26. A stochastic variable Y is said to be *exponentially distributed* with parameter λ if

$$\mathbb{P}(Y = y) = \lambda e^{-\lambda y}\mathbf{1}_{\{y \geq 0\}}$$

The distribution function of Y is then given by

$$\forall y \in [0, \infty), \qquad F_Y(y) = \mathbb{P}(Y \leq y) = 1 - \exp\{-\lambda y\}\,,$$

which is an invertible function and its inverse is

$$\forall u \in [0, 1], \qquad F_Y^{-1}(u) = -\frac{1}{\lambda}\ln(1-u) \ .$$

A simple consequence is that if U is uniformly distributed on $[0, 1]$ then $-\frac{1}{\lambda}\ln U$ is exponentially distributed with parameter λ.

Result 1.27. *A stochastic variable Y is exponentially distributed with parameter λ if and only if its characteristic distribution is given by*

$$\phi_Y(u) = \frac{\lambda}{\lambda - iu} \ .$$

Its Laplace transform is, thus, given by

$$\varphi_Y(u) = \frac{\lambda}{\lambda + u} .$$

We will often have to deal with events happening at random times. A *random time* is nothing else than a positive random variable $\tau \geq 0$ which represents the time at which some event is going to take place. Given an information flow $(\mathcal{F}_t)_{t\geq 0}$, a natural question is whether given the information in $\mathcal{F}_t$ one can determine whether the event has happened $\{\tau \leq t\}$ or not $\{\tau > t\}$. If the answer is "yes", the random time τ is called a non anticipating random time or stopping time.

Definition 1.28. τ is a non anticipating random time or $(\mathcal{F}_t)_{t\geq 0}$*-stopping time* if

$$\forall t \geq 0, \qquad \{\tau \leq t\} \in \mathcal{F}_t .$$

The most common assumption about random times is to consider that the events τ are exponentially distributed (with parameter λ).

One of the important properties of the exponential distribution is that it has *absence of memory.* That is,

$$\forall t, s > 0, \quad \mathbb{P}(\tau > t + s | \tau > t) = \frac{\int_{t+s}^{\infty} \lambda e^{-\lambda y} dy}{\int_t^{\infty} \lambda e^{-\lambda y} dy} = \mathbb{P}(\tau > s) .$$

In words, for the random time τ, the distribution of $\tau - t$ knowing $\tau > t$ is the same as the distribution of τ itself. In fact, the exponential distribution is the only distribution with this property.

Result 1.29. *Let $\tau \geq 0$ be a (nonzero) random variable s.t. $\mathbb{P}(\tau > t+s|\tau > t) = \mathbb{P}(\tau > s)$. Then τ has a exponential distribution.*

The reason why Poisson variables are of interest has to do with the following fact.

Result 1.30. *If $(\tau_i)_{i\geq 0}$ are independent exponentially distributed random variables with parameter λ then, for any $t > 0$, the random variable*

$$N_t = \inf \left\{ n \geq 1, \ \sum_{i=1}^{n} \tau_i > t \right\}$$

follows a Poisson distribution with parameter λt.

The Poisson process is therefore defined as a *counting* process: it counts the number of random times τ_i which occur between 0 and t. The following properties of a Poisson process are easy to deduce from what has just been said.

Result 1.31. *Let $(N_t)_{t\geq 0}$ be a Poisson process.*

1. *For any $t>0$, N_t is almost surely finite.*

2. *For any ω, the sample path $t \longrightarrow N_t(\omega)$ is piecewise constant and increases by jumps of size 1.*

3. *The sample paths $t \longrightarrow N_t$ are right continuous with left limit (*cadlag*).*

4. *For any $t>0$, $N_{t-}=N_t$ with probability 1.*

5. *(N_t) is continuous in probability:*

$$\forall t > 0, \qquad s \longrightarrow t \quad \Rightarrow \quad N_s \xrightarrow{\mathbb{P}} N_t \ .$$

6. *The distribution of (N_t) is given by $\forall n \in \mathbb{N}$,* $\qquad \mathbb{P}\left[N_t = n\right] = e^{\lambda t}\dfrac{(\lambda t)^n}{n!}$

7. *Its characteristic function is $\phi_{N_t}(u) = \exp\left[(\lambda t)\left(e^{iu}-1\right)\right]$.*

8. *(N_t) has independent increments.*

9. *The increments of N are homogeneous: for any $t>s$, $N_t - N_s$ has the same distribution of N_{t-s}.*

10. *(N_t) has the Markov property:*

$$\forall t > s, \quad \mathbb{E}\left[f(N_t)|N_u, u \leq s\right] = \mathbb{E}\left[f(N_t)|N_s\right] \ .$$

In addition the following result tells us that the only counting processes with independent stationary increments are Poisson processes.

Result 1.32. *Let $(N_t)_{t\geq 0}$ be a counting process with stationary independent increments. Then $(N_t)_{t\geq 0}$ is a Poisson process.*

Suppose now we are told that exactly one event has taken place by time t and we are asked to determine the distribution of the time at which the event occurred. Since the Poisson process possesses stationary and independent increments, it seems reasonable that each interval in $[0,t]$ of equal length should have the same probability of containing the event. In other words, the time of the event should be uniformly distributed over $[0,t]$. Indeed, we have

$$\begin{aligned}
\mathbb{P}[\tau < s|N(t)] &= \frac{\mathbb{P}[\tau > s, N(t)=1]}{\mathbb{P}[N(t)=1]} = \frac{\mathbb{P}[\tau \in [0,s), \tau \notin [s,t)]}{\mathbb{P}[N(t)=1]} \\
&= \frac{\mathbb{P}[\tau \in [0,s)]\mathbb{P}[\tau \notin [s,t)]}{P[N(t)=1]} = \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda} t e^{-\lambda t} = \frac{s}{t}
\end{aligned}$$

This result can easily be generalized to several events $\tau_1, \tau_2, \cdots$ by introducing the concept of *order statistics.*

Definition 1.33. Let $U_1, U_2, \cdots, U_n$ be n random variables. We say that $U_{1:n}, U_{2,n}, \cdots, U_{n:n}$ are the *order statistics* corresponding to $U_1, U_2, \cdots, U_n$ if $U_{k:n}$ is the k-th smallest value among $U_1, U_2, \cdots, U_n$.

If the U_i's are i.i.d. continuous variables with probability density f, then the joint density of the order statistics $U_{1:n}, U_{2:n}, \cdots, U_{n:n}$ is given by

$$f(u_1, u_2, \cdots, u_n) = n! \prod_{i=1}^{n} f(u_i), \qquad u_1 < u_2 < \cdots < u_n \ .$$

In particular if U_i's are uniformly distributed over $(0, t)$, then it follows that

$$f(u_1, u_2, \cdots, u_n) = \frac{n!}{t^n}, \qquad u_1 < u_2 < \cdots < u_n \ .$$

The next result will be very useful.

Result 1.34. *Given $N(t) = n$, the n arrival times $\tau_1, \tau_2, \cdots, \tau_n$ have the same distribution as the order statistics corresponding to the n independent random variables uniformly distributed on the interval* $(0, t)$.

We now go on an define the *centered* version of the Poisson process N_t by

$$\tilde{N}_t = N_t - v(t)$$

where $v(t) = \lambda t$ and in called the *compensator* of N_t and $\tilde{N}$ has the martingale property:

$$\forall t < s, \qquad \mathbb{E}\left[\tilde{N}_t | \tilde{N}_s\right] = \tilde{N}_s \ .$$

$\left(\tilde{N}_t\right)_{t\geq 0}$ is called a *compensated* Poisson process. We note that the compensated Poisson process is no longer integer valued, so it is not a counting process. And its rescaled version $\frac{\tilde{N}_t}{\lambda}$ has the same first moments as a standard Wiener process:

$$\mathbb{E}\left[\frac{\tilde{N}_t}{\lambda}\right] = 0 \qquad \mathrm{Var}\left[\frac{\tilde{N}_t}{\lambda}\right] = t \ .$$

The jump times $\tau_1, \tau_2, \cdots$ form a random configuration of points in $\mathbb{R}_+$ and the Poisson process N_t counts the number of such points in the interval $[0, t]$. This counting procedure defines a *measure* μ on $\mathbb{R}_+$.

Result 1.35. *For any measurable set $A \subset \mathbb{R}_+$ let*

$$\mu(\omega, A) = \#\{i \geq 1, \tau_i(\omega) \in A\} \ .$$

Then $\mu(\omega, \cdot)$ is a positive, integer valued measure and $\mu(A)$ is finite with probability 1 for any bounded set A. Furthermore, the intensity λ of the Poisson process determines the average value of the random measure $\mathbb{E}[\mu(A)] = \lambda|A|$ where $|A| = \int_A dx$ is the Lebesgue measure of A.

The Poisson process may then be expressed in the following way,

$$N_t(\omega) = \mu(\omega, [0,t]) = \int_0^t \mu(\omega, ds) \ .$$

Since this measure μ depends on ω we call it a *random measure.*

μ as above is a random counting measure on $\mathbb{R}_+$.

Inhomogeneous Poisson Processes

One way to generalize the above setup is to consider *inhomogeneous Poisson processes.*

Definition 1.36. Take a non-negative deterministic function $\lambda(t)$ such that $\Lambda(t) = \int_0^t \lambda(s)ds < \infty$ for all t. An *inhomogeneous Poisson process* with intensity $\lambda(t)$, is a non-decreasing, integer-valued process with initial value $N(0) = 0$ whose increments are independent and satisfy

$$Y_t(\omega) = \int_0^t \mu(\omega, ds) \tag{1.4}$$

where μ is a Poisson random measure on $\mathbb{R}_+$ with intensity $\nu(A) = \int_A \lambda(s)ds$, thus

$$\mathbb{P}[\mu(A) = k] = e^{-\nu(A)} \frac{\nu(A)^k}{k!} \qquad k \in \mathbb{N}.$$

A useful result is that the inhomogeneous Poisson process can also be represented as (it has the same law as) a time changed Poisson process.

Result 1.37. *The process Y in (1.4) has the same distribution as*

$$Y_t'(\omega) = N_{\Lambda(t)}(\omega) = \inf\left\{ n \geq 1, \ \sum_{i=1}^n \tau_i > \Lambda(t) \right\},$$

where $(N_t)_{t\geq 0}$ is a standard Poisson process and we take τ_is to be independent exponentially distributed variables with parameter 1.

Marked Point Processes

Another way to extended the standard Poisson setup to a more general setting, is to replace $\mathbb{R}_+$ by $\mathbb{R}_+ \times E$ and the Lebesgue measure by any Radon measure ν on $\mathbb{R}_+ \times E$.

Definition 1.38. A *Radon measure* on $(\mathbb{R}_+ \times E, (\mathcal{B}_+ \otimes \mathcal{E}))$ is a measure ν such that for every compact measurable set $A \in \mathcal{B}_+ \otimes \mathcal{E}$, $\nu(A) < \infty$.

Using the new structure we could then define what is known as a *Poisson random measure.*

$$\begin{aligned} \mu : \Omega \times (\mathcal{B}_+ \otimes \mathcal{E}) &\rightarrow \mathbb{N} \\ (\omega, A) &\rightarrow \mu(\omega, A) , \end{aligned}$$

such that $\mu(\omega, \cdot)$ is an integer-valued Radon measure on $\mathbb{R}_+ \times E$ a.s., and for any bounded measurable $A \in \mathcal{B}_+ \otimes \mathcal{E}$, $\mu(A) < \infty$ is a Poisson random variable with parameter $\nu(A)$

$$\forall k \in \mathbb{N}, \qquad \mathbb{P}(\mu(A) = k) = e^{-\nu(A)} \frac{(\nu(A))^k}{k!} .$$

In the same way as before we can define the compensated Poisson process by

$$\tilde{\mu}(A) = \mu(A) - \nu(A) \qquad\qquad A \in \mathcal{B}_+ \otimes \mathcal{E} .$$

The next results allows to construct, given a Randon measure ν, a Poisson random measure with intensity ν.

Result 1.39. *For any Radon measure ν on $\mathbb{R}_+ \times E$, there exists a Poisson random measure μ on $\mathbb{R}_+ \times E$ with intensity ν.*

We also note that a Poisson random measure on $R_+ \times E$ can be represented as a counting measure associated with a random sequence of points $(\tau_i, q_i)_{i \geq 1}$ with $\tau_i \in \mathbb{R}_+$ and $q_i \in E$, i.e,

$$\forall A \in \mathcal{B}_+ \otimes \mathcal{E}, \qquad \mu(\omega, A) = \sum_{i \geq 1} \mathbf{1}_{\{A\}}(\tau_i, q_i) .$$

μ is thus a sum of Dirac masses located at the random points $(\tau_i, q_i)_{i \geq 1}$,

$$\mu(\omega, \cdot) = \sum_{i \geq 1} \delta(\tau_i(\omega), q_i(\omega)) . \tag{1.5}$$

Using the representation in (1.5) one can define other measures with more complex dependence properties.

This is the case of the so called *marked point processes*, where $(q_i)_{i \geq 1}$ are interpreted as the *marks* associated with the events occurring at the random times $(\tau_i)_{i \geq 1}$.

Definition 1.40. A *marked point process* (MPP) on $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\geq 0})$ is a sequence (τ_i, q_i) where

1. $(\tau_i)_{i\geq 1}$ is a sequence of non anticipating random times with $\tau_i \to \infty$ a.s. when $i \to \infty$.
2. $(q_i)_{i\geq 1}$ is a sequence of random variables taking values in some space E.
3. The value of q_i is revealed at τ_i: q_i is $\mathcal{F}_{\tau_i}$ measurable.

Marked point processes are not Poisson random variables (they allow for arbitrary distributions and dependent structures). The reason is that, though any Poisson random variable can be represented as (1.5), $(\tau_i)_{i\geq 1}$ does not necessarily verify $\tau_i \to \infty$ as $i \to \infty$[6], so not all Poisson random measures can be represented by MPP, only those with $\nu([0,T] \times E) < \infty$.

Doubly-Stochastic Processes

The so-called Cox processes or doubly stochastic Poisson processes are a generalizations of the Poisson process allowing for *stochastic* intensity.

Consider a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$, concerning the background information, and a strictly non negative process $(\lambda_t)_{t\geq 0}$ such that $\Lambda(t) = \int_0^t \lambda(s)ds$ is adapted to $\mathcal{F}_t$. Let $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$.

If conditional on the background information $\mathcal{F}$, μ is a Poisson random measure on $\mathbb{R}_+$ with intensity measure $\nu(A) = \int_A \lambda(s)ds$ for all measurable sets $A \in \mathcal{B}_+$.

Then

$$Y_t(\omega) = \int_0^t \mu(\omega, ds)$$

is a $\mathcal{F}_t$ - *Cox process.*

As before, but this time as long as we are conditioning on $\mathcal{F}$, the conditional process can also be represented (it has the same distribution as) as a time changed Poisson process:

$$Y_t'(\omega)|\mathcal{F} = N_{\Lambda(t)}(\omega) = \inf\left\{n \geq 1, \sum_{i=1}^{n} \tau_i > \Lambda(t)\right\}, \tag{1.6}$$

where $(N_t)_{t\geq 0}$ is a standard Poisson process and we take τ_is to be independent exponentially distributed variables with parameter 1.

[6]This condition guarantees that the number of events occurring on $[0,T]$ is a.s. finite.

In credit risk models we are many times particularly interested in the *first* default time, τ_1. Specially on those cases, it is sometimes useful to reverse engineer the Cox processes and use the property in (1.6) to explicitly construct the first default time as

$$\tau_1 := \inf\left\{t \geq 0 : \int_0^t \lambda(u)du = E_1\right\}$$

where E_1 is a exponentially distributed random variable with parameter 1 and $(\lambda_t)_{t\geq 0}$ is a non-negative process adapted to $\mathcal{F}_t$.[7] Then, typically one assumes that the information on the default state

$$\mathcal{H}_t = \sigma\left\{1_{\{\tau_i \geq s\}}, 0 \leq s \leq t, \forall i\right\}$$

is independent of $\mathcal{F}$ and $\mathcal{G}_t = \mathcal{F} \vee \mathcal{H}_t$ is the full information set.

The same way we generalize Poisson processes to allow for the possibility of stochastic intensities, one can also generalize Marked Point Processes to allow for stochastic compensators, giving rise to what is known as doubly stochastic marked point processes (DSMPP).

In finance DSMPP are useful to model events (τ_i, q_i), where both the timing of the events and its marks are driven by (possibility dependent) stochastic variables. In Chapter 4 DSMPP are applied to credit risk issues.

1.2.2 Basics from differential geometry

To study term structures from a geometrical point of view, we need to introduce some basic concepts from differential geometry. We refer to the Björk (2004b) for a complete overview on the geometry studies concerning term structures, there you can find reference to the original papers and proofs.

The first thing one must realize is that, concepts and intuitions of ordinary differential geometry cannot be applied to stochastic Itô calculus. To be able to apply the differential geometry intuition we need to use of what is known as *Stratonovich integrals* (instead of Itô integrals).

Definition 1.41. For given semi martingales X and Y, the *Stratonovich integral* of X with respect to Y, $\int_0^t X_s \circ dY_s$, is defined as

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_t. \tag{1.7}$$

[7]This construction strategy can also be extended to several defaults but that is unnecessary for the purpose of this thesis.

where the first term on the r.h.s. is the Itô integral and the quadratic variation process $\langle X, Y\rangle$ can be computed via

$$d\langle X, Y\rangle = dX_t dY_t$$

with the usual multiplication rules: $dW \cdot dt = dt \cdot dt = 0$, $dW \cdot dW = dt$.

The Stratonovich formulation is geometrically more convenient because the Itô formula, in Stratonovich calculus, takes the form of the standard *chain rule* in ordinary calculus.

Result 1.42. *Assume that a function $F(t, y)$ is smooth. Then we have*

$$dF(t, Y_t) = \frac{\partial F}{\partial t}(t, Y_t)dt + \frac{\partial F}{\partial y}(t, Y_t) \circ dY_t. \tag{1.8}$$

We now present the needed geometric concepts. The presentation follows Björk and Svensson (2001) it will be in terms of a general real Hilbert Space Y and we denote by y an element of Y.

Consider a real Hilbert space Y.

By an n-dimensional *distribution* we mean a mapping F, which to each $y \in Y$ associates an n-dimensional subspace $F(y) \subseteq Y$.

A mapping (*vector field*) $f : Y \to Y$, is said to *lie in* F if $f(y) \in F(y)$ for every $y \in Y$.

A collection $f_1, \dots, f_n$ of vector fields lying in F *generates* (or spans) F if

$$\text{span}\,\{f_1(y), \dots, f_n(y)\} = F(y) \quad \text{for every } y \in Y,$$

where "span" denotes the linear hull over the real field.

The distribution is *smooth* if, for every $y \in Y$, there exist smooth vector fields $f_1, \dots, f_n$ spanning F. A vector field is smooth if it belongs to C^∞.

If F and G are distributions and $G(y) \subseteq F(y)$ for all y we say that F *contains* G, and we write $G \subseteq F$. The *dimension* of a distribution F is defined pointwise as $\dim F(y)$.

Let f and g be smooth vector fields on Y. Their *Lie bracket* is the vector field $[f, g]$, defined by

$$[f, g] = f'g - g'f,$$

where f' denotes the Frechet derivative of f at y, and similarly for g'. We will sometimes write $f'[g]$ instead of $f'g$ to emphasize that the Frechet derivative is operating on g.

A distribution F is called *involutive* if for all smooth vector fields f and g lying in F on Y, their lie bracket also lies in F, i.e.

$$f, g \in F \qquad \Rightarrow \qquad [f, g] \in F$$

for all $y \in Y$.

We are now ready to define the concept of a Lie algebra which will play a central role in what follows.

Definition 1.43. Let F be a smooth distribution on Y. The *Lie algebra* generated by F, denoted by $\{F\}_{LA}$ or by $\mathcal{L}\{F\}$, is defined as the minimal (under inclusion) involutive distribution containing F.

If, for example, the distribution F is spanned by the vector fields $f_1, \ldots, f_n$ then, to construct the Lie algebra $\{f_1, \ldots, f_n\}_{LA}$, you simply form all possible brackets, and brackets of brackets, etc. of the fields $f_1, \ldots, f_n$, and adjoin these to the original distribution until the dimension of the distribution is no longer increased.

When one tries to compute a concrete Lie algebra the following observations are often very useful. Taken together, they basically say that, when computing a Lie algebra, you are allowed to perform Gaussian elimination.

Result 1.44. *Take the vector fields $f_1, \ldots, f_k$ as given. It then holds that the Lie algebra $\{f_1, \ldots, f_k\}_{LA}$ remains unchanged under the following operations.*

- *The vector field f_i may be replaced by αf_i, where α is any smooth nonzero scalar field.*
- *The vector field f_i may be replaced by*

$$f_i + \sum_{j \neq i} \alpha_j f_j,$$

where $\alpha_1, \ldots, \alpha_k$ are any smooth scalar fields.

Let F be a distribution and let $\varphi : V \to W$ be a diffeomorphism between the open subset V and W on Y. Then we can define a new distribution $\varphi_\star F$ on Y by

$$(\varphi_\star F)(\varphi) := \varphi' F.$$

We now define an useful operator on our Hilbert space Y.

Definition 1.45. Let f be a smooth vector field on Y, and let y be a fixed point in Y. Consider the ODE

$$\begin{cases} \dfrac{dy_t}{dt} &= f(y_t) \\ y_0 &= y \end{cases}$$

We denote the solution y_t as $e^{ft}y$.

Finally, and for future reference, we define a particular type of functions –the quasi-exponential functions – that will turn out useful.

Definition 1.46. A *quasi-exponential* (or QE) function is by definition any function of the form

$$f(x) = \sum_u e^{\lambda_u x} + \sum_j e^{\alpha_j x}\left[p_j(x)\cos(w_j x) + q_j(x)\sin(w_j x)\right]$$

where λ_u,α_j,w_j are real numbers, whereas p_j and q_j are real polynomials.

Important properties of QE functions are given in Lemma 1.47.

Result 1.47. *The following holds for quasi-exponential functions*

- *A function is QE if and only if it is a component of the solution of a vector valued linear ODE with constant coefficients.*
- *A function is QE if and only if it can be written as $f(x) = ce^{Ax}b$. Where c is a row vector, A is a square matrix and b is a column vector.*
- *If f is QE, then f' is QE.*
- *If f is QE, then its primitive function is QE.*
- *If f and g are QE, then fg is QE.*

Chapter 2

General Quadratic Term Structures

"At that time [1970's], the notion of partial differential equations was very, very strange on Wall Street."

Robert C. Merton, *Derivative Strategies*, 1998.

For finite dimensional factor models, this chapter studies *general quadratic term structures*. These term structures include as special cases the affine term structures and the Gaussian quadratic term structures, previously studied in the literature. We show, however, that there are other, non-Gaussian, quadratic term structures and derive sufficient conditions for the existence of these *general quadratic term structures* for zero-coupon bond prices.

2.1 Introduction

The term structure of zero-coupon bond prices has been the main object of study in the term structure literature. Both affine term structures (ATS) and Gaussian quadratic term structure models (Gaussian QTS) have been exploited: Vasiček (1977), Cox, Ingersoll, and Ross (1985), Brown (1993) and Duffie and Kan (1996) on the ATS side, and Longstaff (1989), Beaglehole and Tenney (1991), Jamshidian (1996), Boyle and Tian (1999) and Gombani and Runggaldier (2001) on the Gaussian-QTS side, are among the most important. For survey studies and almost exhaustive lists of references on the subject, see Ahn, Dittmar, and Gallant (2002), Dai and Singleton (2003), Leippold and Wu (2002) or Rebonato (2003).

The existing results on Gaussian-QTS do not include the results on ATS as a special case. Indeed, all studies impose a *deterministic* volatility for the state space variable, which is not a requirement of the ATS literature. In this chapter, we will extend the existing results on quadratic term structures (QTS) of zero-coupon bond prices, studying what we define as *general quadratic term structures* (GQTS). By studying GQTS, we show that QTS have not been studied in full generality and how it is possible to extend the previous literature to include non-Gaussian quadratic term structures.

The generalization results from an *a priori* distinction between types of factors: those with, at most, a linear impact on the term structure of bond prices (*linear factors*) and those that may have a quadratic impact (*quadratic factors*); and from allowing more flexible factor dynamics for the first type of factors.

Our results will coincide with the results of the previous literature whenever there is (or we assume that there is) only one type of factors. When all factors are *quadratic factors* (or assumed to be so), we show that indeed a Gaussian setting for the factor dynamics is the most general setting, and we recover the results on Gaussian-QTS. In that sense, one can argue, that the previous literature is on *pure* quadratic term structures. When all factors are *linear factors*, we recover the results on ATS.

The main contributions of this chapter are as follows:

In Section 2.2, we present the general framework for zero-coupon bond prices (from now on simply referred to as *bond prices*) and their term structure dynamics. We model the entire term structures using the Heath-Jarrow-Morton approach. We define exactly what we mean by *general quadratic term structures* (GQTS) and show how they include both ATS and the Gaussian-QTS previously studied in the literature as special cases.

In Section 2.3, we derive sufficient conditions for GQTS of *bond prices*. We show that the generalization to non-Gaussian settings depends on an *a priori* classification of factors. Our main concern is in distinguishing factors that may have a quadratic impact on a GQTS from those that have at most a linear impact. We argue that, for *quadratic factors*, we need a deterministic volatility structure, but for *linear factors*, we can allow for a more general (stochastic) functional form. Thus, by imposing a deterministic volatility structure for the (entire) state variable, the previous Gaussian-QTS literature has either imposed an (unnecessary) ad hoc assumption or implicitly assumed that all factors are *quadratic*, and in that sense, has studied only pure quadratic term structures (PQTS).

Section 2.4 studies the implications of this more general volatility structure in terms of possible correlations between *linear* and *quadratic* factors.

Section 2.5 explains why the generalization technique introduced in the study of GQTS cannot be applied to higher-order term structures. By doing this, we recover the well-known result of Filipović (2002) that term structures of order *higher* than two are not consistent with the no-arbitrage assumption (except for pathological cases).

In Section 2.6, we illustrate the applicability of the theoretical results presented by explicitly computing the term structure of bond prices for two concrete models, one of which is non-Gaussian. Finally, Section 2.7 concludes the chapter summarizing the main results and pointing out directions for future research.

2.2 General Setting

Our main goal is the study of *general quadratic term structures* (GQTS) of bond prices, when those term structures can be expressed as functions of a finite dimensional state process Z.

Thus, we consider an m-dimensional (column) vector of factors Z,

$$Z(t) = \begin{pmatrix} Z_1(t) \\ Z_2(t) \\ \vdots \\ Z_m(t) \end{pmatrix}$$

and that the price, at time t, of a (zero-coupon) bond with maturity T, $p(t,T)$, is given by some deterministic function, H, so that we have

$$p(t,T) = H(t, Z(t), T).$$

In the above function, it is natural to view t and $Z(t)$ as variables and T (the maturity of the price) as a parameter. For a fixed t and z, the mapping $T \longrightarrow H(t,z,T)$, is the term structure of bond prices.

We now establish exactly what we mean by a general quadratic term structure (GQTS).

Definition 2.1. The term structure H is said to be *generally quadratic* if it has the form

$$\ln H(t,z,T) = A(t,T) + B^{\top}(t,T)z + z^{\top}C(t,T)z \tag{2.1}$$

where $\cdot^\top$ stands for transpose, $C_{(m\times m)}$ is symmetric and not necessarily different from $\mathbf{0}$, $B_{(m\times 1)}$ and $A_{(1\times 1)}$ are matrices of deterministic and smooth functions.

Note that requiring C to be symmetric is not restrictive. Any non-symmetric quadratic form can be rewritten in an equivalent symmetric way with the advantage that the symmetric representation is unique. Since later on we will be interested in determining C, the uniqueness property will be useful.

Based on Definition 2.1 we identify a few special cases of a GQTS.

Definition 2.2. General quadratic term structures as in (2.1) have the following special cases.

- *Pure quadratic term structures* (PQTS)
  Whenever all $Z_i \in Z$ show up in the quadratic term $z^\top C(t,T)z$ at least for some t, T, i.e. when
  $\forall\, i, \quad \exists\, t, T \quad \text{s.t.} \quad C_i(t,T) \neq \mathbf{0}$.
  In this case all factors in Z have a quadratic impact[1], so there will be only *quadratic factors.*

- *Mixed Quadratic term structures* (QTS)
  Whenever there exist a Z such that, Z_i never shows up in the quadratic term $z^\top C(t,T)z$, but Z_j does at least for some t, T, i.e.,
  $\exists\, i \quad \text{s.t.} \quad C_i(t,T) = \mathbf{0} \quad \forall\, t, T \quad \textit{and} \quad \exists\, t, T \text{ and } j \quad \text{s.t.} \quad C_j(t,T) \neq \mathbf{0}$.
  In this case there will be factors having a quadratic impact and factors having at most a linear impact on the term structure. So, there will be both *linear factors* and *quadratic factors.*

- *Affine term structures* (ATS) Whenever none of the factors $Z_i \in Z$ ever shows up on the quadratic term $z^\top C(t,T)z$, i.e.,
  $\forall\, t, T \quad C(t,T) = \mathbf{0}$.
  In that case all factors are *linear factors* and have at most a linear impact.

In the above definition, and from now on, "$\cdot_i$" stands for the i-th component/row of a vector/matrix.

[1] We note that, in this sense, a factor Z_i has a *quadratic impact* both if z_i^2 turn up in the term structure or if $z_i z_j$ does for some j. In other words, a *quadratic factor* will be any factor showing up in $z^\top C(t,T)z$.

From the above definition, it is clear that GQTS include non-pure quadratic terms structures (QTS).

The market also gives us a boundary condition, which for normalized bond prices (when we take the notional value to be one) is given by $H(T, Z(T), T) = 1$, or equivalently, $A(T,T) = 0$, $B(T,T) = \mathbf{0}$ and $C(T,T) = \mathbf{0}$.

Before we go on with the analysis, let us set the scene.

Assumption 2.3. *We assume that zero-coupon bond prices are of the form,*

$$p(t,T) = H(t, Z(t), T) \ , \tag{2.2}$$

where H is a smooth function with the boundary condition

$$H(T, z, T) = 1. \tag{2.3}$$

We will also consider that our m-dimensional factor model, under the martingale measure $\mathbb{Q}$, is driven by an n-dimensional Wiener process W. Assumption 2.4 sets some notation about the dynamics of the factors Z under the martingale measure $\mathbb{Q}$.

Assumption 2.4. *The dynamics of Z, under the $\mathbb{Q}$-measure, are given by*

$$dZ(t) = \alpha(t, Z(t))dt + \sigma(t, Z(t))dW(t) \tag{2.4}$$

where $\alpha(t,z)$ is a $m \times 1$ vector and $\sigma(t,z)$ is a $m \times n$ matrix, and W is a n-dimensional Wiener process.

Note that, by considering a n-dimensional Wiener process, we implicitly take W to be a column vector of n *independent* scalar Wiener processes. Recalling, however, that we can transform any system with correlated Wiener processes into an equivalent system with uncorrelated ones[2], the assumption of independence between the elements of W is not restrictive in any sense.

Remark 2.5. *It is always possible to transform a system of* correlated *Wiener processes*

$$dZ(t) = (\cdots)dt + \hat{\sigma}(t, Z(t))d\bar{W}(t) \tag{2.5}$$

for $\bar{W}$ d-dimensional with possibly correlated elements, into a system of independent *ones*

$$dZ(t) = (\cdots)dt + \sigma(t, Z(t))dW(t) \tag{2.6}$$

for W n-dimensional with independent elements.

[2]For a textbook treatment of this result see, for instance, Björk (2004a). A proof is also provided in Appendix A.

Furthermore, the following relation describes the connection the two Wiener processes

$$\bar{W} = \delta W$$

where δ *is a* $(d \times n)$ *matrix of deterministic constants such that the length of its rows is one* ($\|(\delta)_i\| = 1$) *and* $\delta\delta^\top = \rho$, *for* ρ *the correlation matrix of the elements in* $\bar{W}$. *Thus,*

$$\sigma(t, Z(t)) = \hat{\sigma}(t, Z(t))\delta. \tag{2.7}$$

The example below illustrates this in a very simple case.

Example 2.6. *Consider the following dynamics of two factors* Z_1 *and* Z_2*:*

$$d\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = \left(\cdots\right) dt + \underbrace{\begin{pmatrix} \sigma_1(t, Z(t)) & 0 \\ 0 & \sigma_2(t, Z(t)) \end{pmatrix}}_{\hat{\sigma}(t,Z(t))} d\begin{pmatrix} \bar{W}_1(t) \\ \bar{W}_2(t) \end{pmatrix}$$

with $d\bar{W}_1(t)d\bar{W}_2(t) = \rho dt$.

In this case $\delta = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$. *The system can, thus, be rewritten as*

$$d\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = \left(\cdots\right) dt + \underbrace{\begin{pmatrix} \sigma_1(t, Z(t)) & 0 \\ \rho\sigma_2(t, Z(t)) & \sigma_2(t, Z(t))\sqrt{1-\rho^2} \end{pmatrix}}_{\sigma(t,Z(t))} d\begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}$$

where we have $dW_1(t)dW_2(t) = 0$.

For a better understanding of some future results, it is important to stress now that, in our formulation, two correlated factors will be driven by *at least* one common scalar Wiener process. In Example 2.6, both Z_1 and Z_2 are driven by W_1 (the common Wiener process in this case).

Having defined the exact setup we can go on with the analysis. Applying the Itô formula to equation (2.2) and using the dynamics for the factors in (2.4) we can find the dynamics of the zero-coupon bond prices under the martingale measure $\mathbb{Q}$. Lemma 2.7 gives these dynamics.

Lemma 2.7. *Assume that the dynamics of Z are as in (2.4), then, if the zero-coupon bond prices are given by (2.2), their $\mathbb{Q}$-dynamics are described by*

$$dp(t,T) = \left\{ \frac{\partial H}{\partial t} + \sum_{i=1}^{m} \frac{\partial H}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H}{\partial z_i \partial z_j}\sigma_i\sigma_j^\top \right\} dt + \sum_{i=1}^{m} \frac{\partial H}{\partial z_i}\sigma_i dW(t)$$

All partial derivatives should be evaluated at $(t, Z(t), T)$, and all α_i and σ_i at $(t, Z(t))$.

Using the above Lemma and the fact that bonds are traded assets, we have the risk-free short rate r as the bond prices' local rate of return, under the martingale measure $\mathbb{Q}$. Hence, we recover the following standard term structure equation for bond prices.

Result 2.8. *Suppose the zero-coupon bond prices are given by (2.2) and Assumption 2.4 holds. Then H satisfies the following differential equation*

$$\begin{cases} \dfrac{\partial H}{\partial t} + \displaystyle\sum_{i=1}^{m} \frac{\partial H}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H}{\partial z_i \partial z_j}\sigma_i\sigma_j^\top & = \quad rH \\ \qquad\qquad\qquad\qquad\qquad\qquad H(T,z,T) & = \quad 1 \end{cases} \tag{2.8}$$

All partial derivatives should be evaluated at (t, z, T), and all α_i and σ_i at (t, z).

One of our goals is to obtain *sufficient conditions* for existence of a GQTS for bond prices. As in any term structure study, these will be conditions on the functional form of the short rate and on the factor dynamics. Let us establish some notation for the short rate, defining what we call a generally quadratic short rate (GQSR) setting.

Definition 2.9. A *general quadratic short rate* setting is defined by a short rate, r, with the following functional quadratic form

$$r(t, Z(t)) = Z^\top(t)Q(t)Z(t) + g^\top(t)Z(t) + f(t) \tag{2.9}$$

where $Q(t)_{(m\times m)}$, symmetric[3], and not necessarily different from $\mathbf{0}$, $g(t)_{(m\times 1)}$ and $f(t)_{1\times 1}$ are matrices of smooth and deterministic functions.

The next preliminary Lemma tells us that, whenever we have a GQTS for bond prices, we also have a GQSR for the short rate. Or, to put it differently, that a GQSR is a *necessary condition* for a GQTS of bond prices.

[3]Recall, from the previous section arguments, that the symmetry assumption both for C in (2.1) or for Q in (2.9) is not restrictive in any way.

Lemma 2.10. *If we have a GQTS for bond prices, then we have GQSR for the short rate, and the following connection exists between the functional matrices in (2.9) and (2.1),*

$$Q(t) = -\frac{\partial C}{\partial T}(t,t) \quad g(t) = -\frac{\partial B}{\partial T}(t,t) \quad f(t) = -\frac{\partial A}{\partial T}(t,t).$$

Proof. Recall from (2.2) that we have $p(t,T) = H(t, Z(t), T)$. Then we can use the relation between bond prices $p(t,T)$ and instantaneous forward interest rates, $f_r(t,T)$, to conclude

$$f_r(t,T) = -\frac{\partial \ln H}{\partial T}(t, Z(t), T).$$

Hence for GQTS of bond prices as in (2.1) we have

$$f_r(t,T) = -\left(\frac{\partial A}{\partial T}(t,T) + \frac{\partial B^\top}{\partial T}(t,T)Z(t) + Z^\top(t)\frac{\partial C}{\partial T}(t,T)Z(t)\right).$$

Since we also have $r(t) = f_r(t,t)$, then

$$r(t) = -\frac{\partial A}{\partial T}(t,t) - \frac{\partial B^\top}{\partial T}(t,t)Z(t) - Z^\top(t)\frac{\partial C}{\partial T}(t,t)Z(t),$$

and the result follows from the comparison of this expression and (2.9). ■

We will soon show that having a GQSR is also one of the *sufficient conditions* for a GQTS of bond prices, but not the only one. As one would guess, the others are on the functional form of $\alpha(t,z)$ and $\sigma(t,z)\sigma^\top(t,z)$ in the factor dynamics (2.4).

Before we can present our main result, we need one more definition.

Definition 2.11. The vector of factors Z is said to have *general quadratic $\mathbb{Q}$-dynamics* if $\alpha(t,z)$ and $\sigma(t,z)$ in (2.4) are such that

$$\alpha(t,z) = \mathbf{d}(t) + \mathbf{E}(t)z \tag{2.10}$$

$$\sigma(t,z)\sigma^\top(t,z) = \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t)z_u + \sum_{u,k=1}^{m} z_u \mathbf{g_{uk}}(t) z_k \tag{2.11}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$, $\mathbf{k_u}$ and $\mathbf{g_{uk}}$ for $\mathbf{u}, \mathbf{k} = \mathbf{1}, \cdots, \mathbf{m}$ are matrices of deterministic smooth functions.

We also define for future reference

$$\mathbf{K}(t) = \begin{pmatrix} \mathbf{k_1}(t) \\ \mathbf{k_2}(t) \\ \vdots \\ \mathbf{k_m}(t) \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{g_{11}}(t) & \mathbf{g_{12}}(t) & \cdots & \mathbf{g_{1m}}(t) \\ \mathbf{g_{21}}(t) & \mathbf{g_{22}}(t) & \cdots & \mathbf{g_{2m}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g_{m1}}(t) & \mathbf{g_{m2}}(t) & \cdots & \mathbf{g_{mm}}(t) \end{pmatrix}. \tag{2.12}$$

2.3 GQTS of bond prices

In a setting with GQSR and Z with general quadratic $\mathbb{Q}$-dynamics, we can classify the factors as follows.

2.3.1 Classification of Factors

Definition 2.12. Given a GQSR as in (2.9) and the general quadratic $\mathbb{Q}$-dynamics for Z (so that (2.10) and (2.11) hold for $\alpha(t,z)$ and $\sigma(t,z)$ in (2.4)).

- Z_i is a $Z^{(q)}$*-factor* if it satisfies *at least one* of the following requirements:
  - (i) it has a quadratic impact on the short rate of interest $r(t)$, i.e., there exists t such that $Q_i(t) \neq \mathbf{0}$;
  - (ii) it has a quadratic impact on the functional form of the matrix $\sigma(t,z)\sigma^{\top}(t,z)$, i.e., there exist k and t such that $\mathbf{g_{ik}}(t) \neq \mathbf{0}$;
  - (iii) it affects the drift term of the factors satisfying (i) or (ii), i.e., for Z_j satisfying (i) or (ii) we have $\mathbf{E}_{ji}(t) \neq 0$, at least for some t.[4]
- Z_i is a $Z^{(l)}$*-factor* if it does not satisfy (i)-(iii).

The reader may wonder about the motivation for this classification of factors, and may guess that it should somehow be related to different impacts on bond prices term structures. After presenting the main result on GQTS for bond prices, we will be able to give that motivation and to show that, indeed, this classification has to do with the impact that the factors may end up having on the bond prices term structure.

Before that, however, we want to show the implications of Definition 2.12 in terms of the forms of the matrices Q, $\mathbf{E}$ and $\mathbf{G}$ and to stress that the forms (2.13) bellow are *not* the result of any assumption, rather they hold *by definition.*

Remark 2.13. *We note that given Definition 2.12, it is always possible to reorder the vector of factors Z and its correspondent value vector, so that we have*

$$Z = \begin{pmatrix} Z^{(q)} \\ Z^{(l)} \end{pmatrix} \qquad z = \begin{pmatrix} z^{(q)} \\ z^{(l)} \end{pmatrix}.$$

[4]We see that in the most general drift dynamics, that is, when the drift of all factors depend linearly in all other factors, as long as there is one $Z^{(q)}$-factor, all factors would also be of the type $Z^{(q)}$. So, one can say that a *necessary condition* for existence of more than one type of factors is that the drift of factors satisfying (i) and (ii) in Definition 2.12 does not depend on all remaining factors.

Furthermore, with this reordering of factors we have, by definition, *the following forms for Q in (2.9) and for* $\mathbf{E}$ *and* $\mathbf{G}$ *in (2.10) and (2.12), respectively*

$$Q(t) = \begin{pmatrix} Q^{(qq)}(t) & 0 \\ 0 & 0 \end{pmatrix} \quad \mathbf{E}(t) = \begin{pmatrix} \mathbf{E}^{(\mathbf{qq})}(t) & \mathbf{0} \\ \mathbf{E}^{(\mathbf{lq})}(t) & \mathbf{E}^{(\mathbf{ll})}(t) \end{pmatrix} \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{G}^{(\mathbf{qq})}(t) & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.13}$$

2.3.2 Main result

We can now present the main result on GQTS of bond prices.

Theorem 2.14. *Suppose that Assumption 2.3 and 2.4 are in force. Furthermore suppose that we are in a GQSR, so that (2.9) holds and that Z has a general quadratic $\mathbb{Q}$-dynamics, (i.e., that α and σ from the factor dynamics (2.4), satisfy (2.10)-(2.11)). Finally assume that the factors are reordered according to Definition 2.12, and that the following restrictions apply to* $\mathbf{K}$ *and* $\mathbf{G}$ *in (2.12),*

$$\mathbf{k_u}(t) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{k}_\mathbf{u}^{(\mathbf{ll})}(t) \end{pmatrix} \quad \forall\, t, \forall\, \mathbf{u} \tag{2.14}$$

$$\mathbf{g_{uk}}(t) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{g}_\mathbf{uk}^{(\mathbf{ll})}(t) \end{pmatrix} \quad \forall\, t \textit{ and } \forall\, \mathbf{u},\mathbf{k} \textit{ s.t. } x_u, x_k \in \mathbf{z}^{(\mathbf{q})}. \tag{2.15}$$

Then the term structure of bond prices is generally quadratic, i.e. H from (2.2) can be written in the form (2.1) where A, B and C solve the following system of ordinary differential equations,

$$\begin{cases} \dfrac{\partial A}{\partial t} + \mathbf{d}^\top(t)B + \dfrac{1}{2}B^\top \mathbf{k_0}(t)B + \operatorname{tr}\{C\mathbf{k_0}(t)\} \;=\; f(t) \\ A(T,T) \;=\; 0 \end{cases} \tag{2.16}$$

$$\begin{cases} \dfrac{\partial B}{\partial t} + \mathbf{E}^\top(t)B + 2C\mathbf{d}(t) + \dfrac{1}{2}\bar{B}^\top \mathbf{K}(t)B + 2C\mathbf{k_0}(t)B \;=\; g(t) \\ B(T,T) \;=\; 0 \end{cases} \tag{2.17}$$

$$\begin{cases} \dfrac{\partial C}{\partial t} + C\mathbf{E}(t) + \mathbf{E}^\top(t)C + 2C\mathbf{k_0}(t)C + \dfrac{1}{2}\bar{B}^\top \mathbf{G}(t)\bar{B} \;=\; Q(t) \\ C(T,T) \;=\; 0 \end{cases} \tag{2.18}$$

where C has the special form $C = \begin{pmatrix} C^{(qq)} & 0 \\ 0 & 0 \end{pmatrix}$ and A, B and $C^{(qq)}$ should be evaluated at (t, T). $\mathbf{E}$, $\mathbf{d}$, $\mathbf{k_0}$, $\mathbf{K}$, $\mathbf{G}$ are the same as in (2.10)-(2.12), and

$$\bar{B} = \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B \end{pmatrix}. \tag{2.19}$$

Proof. First of all, given the forms (2.13) of the matrices Q, $\mathbf{E}$ and $\mathbf{G}$, it follows that $C^{(ll)}(t,T) = \mathbf{0}$ and $C^{(ql)}(t,T) = C^{(lq)\top}(t,T) = \mathbf{0}$ for all t, T always solve (2.18). This proves the last statement of the Theorem.

The main result follows from the fact that $\ln H(t,z,T) = A(t,T) + B^\top(t,T)z + z^\top C(t,T)z$ for A, B and C satisfying (2.16)-(2.18), solves the PDE (2.8) (term structure equation) which characterizes the bond prices in this setting.

(i) We start by showing that the term structure equation (2.8), is equivalent to the PDE

$$\begin{cases} \frac{\partial A}{\partial t} + \frac{\partial B}{\partial t} z + z^\top \frac{\partial C}{\partial t} z + \sum_{i=1}^{m} \left[(B)_i + (C)_i z + (C^\top)_i z \right] \alpha_i \\ +\frac{1}{2} \sum_{i,j=1}^{m} \left[(B)_i + (C)_i z + (C^\top)_i z \right] \sigma_i \sigma_j^\top \left[(B)_j + (C)_j z + (C^\top)_j z \right] \\ \qquad +\frac{1}{2} \sum_{i,j=1}^{m} ((C)_{ij} + (C)_{ji}) \sigma_i \sigma_j^\top \quad = \quad r \\ \exp\left\{ A(T,T) + B(T,T)^\top z + z^\top C(t,T) z \right\} \quad = \quad 1 \,. \end{cases} \tag{2.20}$$

For $\ln H(t,z,T) = A(t,T) + B(t,T)^\top z + z^\top C(t,T)z$, this follows from

$$\begin{aligned} \frac{\partial H}{\partial t} &= \left[\frac{\partial A}{\partial t} + \frac{\partial B}{\partial t}^\top z + z^\top \frac{\partial C}{\partial t} z \right] H \\ \frac{\partial H}{\partial z_i} &= \left[B_i + C_i z + C^\top{}_i z \right] H \\ \frac{\partial^2 H}{\partial z_i \partial z_j} &= \left[C_{ij} + C_{ji} \right] H + \left[B_i + C_i z + C^\top{}_i z \right] \left[B_j + C_j z + C^\top{}_j z \right] H \,. \end{aligned}$$

(ii) It remains to show that the system of ODE (2.16)-(2.18) solves the PDE (2.20). To see this, we substitute r, α and $\sigma\sigma^\top$ using (2.9), (2.10) and (2.11), respectively, and apply the restrictions (2.14)-(2.15). The PDE (2.20) becomes, then, a separable equation. In addition, by making use of the fact that $C^{(ll)}(t,T) = \mathbf{0}$ and $C^{(ql)}(t,T) = C^{(lq)\top}(t,T) = \mathbf{0}$ for all t, T are always a solution to (2.18), we can guarantee that all terms of order higher than two disappear. Hence, equation (2.20) becomes separable up to quadratic terms of z. Identification of terms, and the use of the assumption on the symmetry of C (which helps to simplify the expressions), leads to the system of ODEs (2.16)-(2.18). ∎

Verbally spoken, the result of Theorem 2.14 is twofold: First, it states sufficient conditions for GQTS of bond prices and second, it provides an explicit way to compute them. Because these sufficient conditions are somehow hidden in previous definitions and the intuition lost in the formulas we restated them, intuitively, as follows. It is

sufficient for a GQTS of bond prices

- to have an interest rate model that guarantees a GQSR (as in (2.9));
- to have Z-factors with general quadratic $\mathbb{Q}$-dynamics (as in (2.10)-(2.11)) *restricted* to guarantee deterministic $\sigma^{(q)}\sigma^{(q)\top}$, $\sigma^{(q)}\sigma^{(l)\top}$ (and hence, by symmetry also $\sigma^{(l)}\sigma^{(q)\top}$). That is, that the volatility structure of $Z^{(q)}$-factors, or of any factors correlated with them, is deterministic.

Using these sufficient conditions (and the implications of Definition 2.12 *per se*), the next Remark restates in a more visual way the first part of Theorem 2.14.

Remark 2.15. *Assume that a short rate model can be described by*

$$r(t, Z(t)) = Z^\top(t) \begin{pmatrix} Q^{(qq)}(t) & 0 \\ & \\ 0 & 0 \end{pmatrix} Z(t) + g^\top(t)Z(t) + f(t),$$

and that the factors Z have $\mathbb{Q}$-dynamics given by

$$dZ(t) = \alpha(t, z)dt + \sigma(t, Z(t))dW(t),$$

where α and $\sigma\sigma^\top$ satisfy

$$\begin{aligned}
\alpha(t,z) &= \mathbf{d}(t) + \begin{pmatrix} \mathbf{E}^{(\mathbf{qq})}(t) & 0 \\ & \\ \mathbf{E}^{(\mathbf{lq})}(t) & \mathbf{E}^{(\mathbf{ll})}(t) \end{pmatrix} z \\
\sigma(t,z)\sigma(t,z)^\top &= \begin{pmatrix} \mathbf{k}_\mathbf{0}^{(\mathbf{qq})}(t) & \mathbf{k}_\mathbf{0}^{(\mathbf{ql})}(t) \\ & \\ \mathbf{k}_\mathbf{0}^{(\mathbf{lq})}(t) & \mathbf{k}_\mathbf{0}^{(\mathbf{ll})}(t) + \sum_{i=1}^{m} \mathbf{k}_\mathbf{u}^{(\mathbf{ll})}(t) z_u + \sum_{z_u, z_k \in \mathbf{z}^{(\mathbf{q})}} z_u \mathbf{g}_{\mathbf{uk}}^{(\mathbf{ll})}(t) z_k \end{pmatrix}
\end{aligned} \tag{2.21}$$

Then we have a GQTS for bond prices as in (2.1).

2.3.3 Understanding the $Z^{(q)}$ and $Z^{(l)}$ factors

Now we can understand better the classification of factors in Definition 2.12. Let us first formalize the kind of impact a factor can have in a GQTS.

Definition 2.16. A factor Z_i is called *quadratic* and is said to have a *quadratic impact* on a GQTS

$$\ln H(t,z,T) = A(t,T) + B(t,T)z + z^\top C(t,T)z$$

if

$$\exists\, t,T \text{ s.t. } C_i(t,T) \neq \mathbf{0}. \tag{2.22}$$

A factor Z_j is called *linear* and is said to have at most a *linear impact* on a GQTS if it does not satisfy (2.22), i.e. if for all t,T $C_i(t,T) = \mathbf{0}$.

The impact of $Z^{(l)}$-factors is an immediate consequence of Theorem 2.14 that follows from $C^{(ll)}(t,T) = \mathbf{0}$ and $C^{(ql)}(t,T) = C^{(lq)\top}(t,T) = \mathbf{0}$ for all t,T.

Corollary 2.17. **(Linear factors)** *The $Z^{(l)}$-factors are linear factors in a GQTS of bond prices.*

It is, thus, particularly comforting to note that since $\sigma\sigma^\top$ can only depend quadraticly on $Z^{(q)}$- factors (check (2.21)), in models where there are only $Z^{(l)}$- factors the quadratic term of $\sigma\sigma^\top$ disappears and we recover the well known result of a linear $\sigma\sigma^\top$ for ATS.

It would now be interesting to show that the $Z^{(q)}$-factors actually have a quadratic impact. As we will soon see this seems to be the case for almost all models, but to show that for any $Z_i \in Z^{(q)}$, $C_i(t,T) \neq \mathbf{0}$ for some t,T is not a trivial task. Non-arbitrage arguments give us, however an easy partial answer[5].

Lemma 2.18. *A factor $Z_i \in Z^{(q)}$ for which $Q_i(t) \neq 0$ at least for some t has a quadratic impact in the bond prices term structure.*

Proof. It follows immediately from Lemma 2.10 that $Q_{ij}(t) \neq 0 \Rightarrow C_{ij}(t,T) \neq 0,\ \forall\, T$. ∎

For those $Z^{(q)}$ *factors* that do not affect quadraticly the short rate ($Z_i \in Z^{(q)}$ with $Q_i(t) = \mathbf{0}$ for all t) the answer is, however, less trivial. They are the factors that satisfy only (ii) or (iii) in Definition 2.12 and their concrete impact can only be accessed by studying the solution of the ODE (2.18). Definition 2.12 guarantees that for all such

[5]We write C_i instead of $(C)_i$ and, as before, it stands for the ith-row of the matrix C.

$Z_i \in Z^{(q)}$ and at least for some t, T, the ODE $C_i(t,T)$ is a Riccati equation with at least one non-zero independent term and that thus, in general, $C_i(t,T) \neq \mathbf{0}$. However, under some pathological situations when a factor Z_i affects quadraticly the volatility for many linear factors, and/or when it affects the drift of many quadratic factors, it could (in theory) be that the many non-zero independent terms on the Riccati equation for $C_i(t,T)$ would cancel each other. In practice this situation is only likely to occur by including in the model redundant factors or pathological constructions. The next assumption imposes the needed regularity condition on the model that guarantee that any $Z^{(q)}$-factor will, indeed have a quadratic impact.

Assumption 2.19. *For any i, k, t, T such that*

$$E_{ji}(t)C_{jk}(t,T) \neq 0 \quad \textit{for some } j$$

or

$$B_u(t,T)\mathbf{g}_{\mathbf{ik}_{uu}}(t)B_u(t,T) \neq 0 \quad \textit{for some } u$$

the following regularity condition[6] *holds:*

$$\sum_j E_{ji}(t)C_{jk}(t,T) + \sum_{u,v} B_u(t,T)\mathbf{g}_{\mathbf{ik}_{uv}}(t)B_v(t,T) \neq 0.$$

Proposition 2.20. **(Quadratic factors)** *As long as we exclude from the analysis any redundant factors and under the regularity condition of Assumption 2.19, the $Z^{(q)}$-factors are quadratic factors in a GQTS of bond prices.*

Proof. Lemma 2.18 proves that any $Z_i \in Z^{(q)}$ such that $Q_i(t) \neq \mathbf{0}$ at least for some t has a quadratic impact. It remains to show that for all $Z_i \in Z^{(q)}$ for which $Q_i(t) = \mathbf{0}$ for all t (all remaining $Z^{(q)}$-factors), we also have, at least for some t, T, $C_i(t,T) \neq \mathbf{0}$ as a solution to (2.18). It follows from Definition 2.12 that, as long as we do not consider the redundant factors, for such Z_i there will always be a k, t, T such that $E_{ji}(t)C_{jk}(t,T) \neq 0$ and/or $B_u(t,T)\mathbf{g}_{\mathbf{ik}_{uu}}(t)B_u(t,T) \neq 0$ for some u.[7] The regularity condition of Assumption 2.19 then guarantees that for that k, t, T, $C_{ik}(t,T)$ solves a Riccati equation with a non-zero independent term and that, thus $C_{ik}(t,T) \neq 0$. This guarantees the quadratic impact of Z_i according to Definition 2.16. ∎

[6]This regularity condition is only needed to guarantee that all $Z^{(q)}$*-factors* have a quadratic impact, one could just ignore it and think of the $Z^{(q)}$*-factors* as the factors that will, *in principle*, have a quadratic impact.

[7]If Z_u is a redundant (linear) factor $\mathbf{g}_{\mathbf{ik}_{uu}}(t) \neq 0$ does not guarantee $B_u(t,T)\mathbf{g}_{\mathbf{ik}_{uu}}(t)B_u(t,T) \neq 0$ at least some T and hence, for this pathological case, the quadratic impact of Z_i cannot be guaranteed.

2.3.4 Actually solving the system of ODEs

To actually solve the system of ODEs in (2.16)-(2.18) is a challenging task, especially because equations (2.17)-(2.18) are interrelated matrix Riccati equations[8] for B and C.

The good news is that, given the very special form of the matrices Q, $\mathbf{E}$ and $\mathbf{G}$ in (2.13) and the fact that we have $C^{(ll)}(t,T) = \mathbf{0}$ and $C^{(ql)}(t,T) = C^{(ql)\top}(t,T) = \mathbf{0}$ for all t, T, they turn out to be two *independent* Riccati equations and the strategy to solve them is as follows:

- Note that it is possible to split the vector equation (2.17) for B into two vector equations for $B^{(q)}$ and $B^{(l)}$, using $B = \begin{pmatrix} B^{(q)} \\ B^{(l)} \end{pmatrix}$ and $g = \begin{pmatrix} g^{(q)} \\ g^{(l)} \end{pmatrix}$.

  Moreover, replacing B by $B = \begin{pmatrix} B^{(q)} \\ B^{(l)} \end{pmatrix}$, C by $C = \begin{pmatrix} C^{(qq)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, Q by $Q = \begin{pmatrix} Q^{(qq)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ and simplifying, we can also write the matrix equation (2.18) for C in terms of $B^{(q)}$, $B^{(l)}$ and $C^{(qq)}$.

  Doing this we get

$$\begin{cases} \dfrac{\partial B^{(q)}}{\partial t} + \mathbf{E}^{(\mathrm{qq})^\top}(t) B^{(q)} + \mathbf{E}^{(\mathrm{lq})^\top}(t) B^{(l)} + \dfrac{1}{2}\bar{B}^{(l)\top}\mathbf{K}^{(\mathrm{q})}(t) B^{(l)} & \\ +2C^{(qq)}\mathbf{d}^{(\mathrm{qq})}(t) + 2\left[C^{(qq)}\mathbf{k}_0^{(\mathrm{qq})}(t)B^{(q)} + C^{(qq)}\mathbf{k}_0^{(\mathrm{lq})}(t)B^{(l)}\right] = g^{(q)}(t) & \\ B^{(q)}(T,T) = 0 & \end{cases} \tag{2.23}$$

$$\begin{cases} \dfrac{\partial B^{(l)}}{\partial t} + \mathbf{E}^{(\mathrm{ll})^\top}(t) B^{(l)} + \dfrac{1}{2}\bar{B}^{(l)\top}\tilde{\mathbf{K}}^{(\mathrm{ll})}(t) B^{(l)} = g^{(l)}(t) \\ B^{(l)}(T,T) = 0 \end{cases} \tag{2.24}$$

[8] For and interesting note on the importance of Riccati equations in Mathematical Finance, see Boyle, Tian, and Guan (2002).

$$
\begin{cases}
\dfrac{\partial C^{(qq)}}{\partial t} + 2C^{(qq)}\mathbf{k}_0^{(\mathbf{qq})}(t)C^{(qq)} + C^{(qq)}\mathbf{E}^{(\mathbf{qq})} + \mathbf{E}^{(\mathbf{qq})\top}C^{(qq)} & \\
\qquad\qquad +\tilde{\bar{B}}^{(l)\top}\tilde{\mathbf{G}}^{(\mathbf{ll})}(t)\tilde{\bar{B}}^{(l)} = Q^{(qq)}(t) & (2.25) \\
\qquad\qquad C^{(qq)}(T,T) = 0 &
\end{cases}
$$

where $\bar{B}^{(l)} = \begin{pmatrix} B^{(l)} & 0 & \cdots & 0 \\ 0 & B^{(l)} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B^{(l)} \end{pmatrix}$ and has dimension $q \times l.q$ for q the number of $Z^{(q)}$- factors and l the number of $Z^{(l)}$ - factors, and where we take

$$
\tilde{\mathbf{K}}^{(\mathbf{ll})} = \begin{pmatrix} \mathbf{k}_1^{(\mathbf{ll})} \\ \mathbf{k}_2^{(\mathbf{ll})} \\ \vdots \\ \mathbf{k}_\mathbf{m}^{(\mathbf{ll})} \end{pmatrix} \qquad \tilde{\mathbf{G}}^{(\mathbf{ll})} = \begin{pmatrix} \mathbf{g}_{11}^{(\mathbf{ll})}(t) & \mathbf{g}_{12}^{(\mathbf{ll})} & \cdots & \mathbf{g}_{1\mathbf{q}}^{(\mathbf{ll})} \\ \mathbf{g}_{21}^{(\mathbf{ll})}(t) & \mathbf{g}_{22}^{(\mathbf{ll})}(t) & \cdots & \mathbf{g}_{2\mathbf{q}}^{(\mathbf{ll})} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_{\mathbf{q}1}^{(\mathbf{ll})} & \mathbf{g}_{\mathbf{q}2}^{(\mathbf{ll})} & \cdots & \mathbf{g}_{\mathbf{qq}}^{(\mathbf{ll})} \end{pmatrix}.
$$

- Note that the ODE (2.24) for $B^{(l)}$ only depends on $B^{(l)}$ itself. That the ODE (2.25) depends on $C^{(qq)}$ and $B^{(l)}$ but not on $B^{(q)}$. And, finally, that the ODE (2.23) depend on all three functions $B^{(q)}$, $B^{(l)}$ and $C^{(qq)}$.

Given this, the next steps are obvious.

1. Solve equation (2.24) and get the solution for $B^{(l)}$.
2. Use the solution for $B^{(l)}$ to solve (2.25) and get the solution for $C^{(qq)}$.
3. Finally, use both solutions for $B^{(l)}$ and $C^{(qq)}$ to solve for (2.23) and get the solution for $B^{(q)}$.
4. This is equivalent to solve the ODEs (2.17)-(2.18). We can then insert the solutions for B and C into equation (2.16) and simply integrate to obtain A.

2.3.5 Important special cases

The most important special case of a GQTS of bond prices are the *Gaussian-QTS* and the *ATS* previously studied in the literature. In any of these cases the classification of factors becomes irrelevant.

Gaussian Quadratic Term Structures

In a Gaussian-QTS model all the factors have deterministic volatility,

$$\sigma(t,z)\sigma^{\top}(t,z) = \mathbf{k_0}(t) \quad \forall t,$$

and, thus, the volatility condition of Theorem 2.14 is always satisfied. Moreover, by reading the conditions of Theorem 2.14, we realize that the need to classify factors had to do with the need to check a special volatility structure, which is different depending on the type of factors only in the non-deterministic case (recall equation (2.11) and the conditions (2.14)-(2.15)). Given this, in a completely Gaussian setting, the volatility conditions are satisfied by definition and we can restate Theorem 2.14 without any allusion to the classification of factors.

Corollary 2.21. **(Gaussian-QTS)** *Suppose that Assumption 2.3 and 2.4 are in force. Furthermore suppose that we are in a GQSR setting, so that (2.9) and $Q(t) \neq 0$ for some t. Finally, assume that α and σ from the factor dynamics (2.4) are of the following form:*

$$\begin{aligned} \alpha(t,z) &= \mathbf{d}(t) + \mathbf{E}(t)z \\ \sigma(t,z)\sigma^{\top}(t,z) &= \mathbf{k_0}(t) \end{aligned}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$ are matrices of deterministic smooth functions.

Then the term structure of bond prices is generally quadratic, i.e. H from (2.2) can be written on the form (2.1). A, B and C in (2.1) solve the following system of ordinary differential equations.

$$\left\{ \begin{aligned} \frac{\partial A}{\partial t} + \mathbf{d}^{\top}(t)B + \frac{1}{2}B^{\top}\mathbf{k_0}(t)B + \operatorname{tr}\{(C)\mathbf{k_0}(t)\} &= f(t) \\ A(T,T) &= 0 \end{aligned} \right. \tag{2.26}$$

$$\left\{ \begin{aligned} \frac{\partial B}{\partial t} + \mathbf{E}^{\top}(t)B + 2C\mathbf{d}(t) + 2C\mathbf{k_0}(t)B &= g(t) \\ B(T,T) &= 0 \end{aligned} \right. \tag{2.27}$$

$$\left\{ \begin{aligned} \frac{\partial C}{\partial t} + C\mathbf{E}(t) + \mathbf{E}^{\top}(t)C + 2C\mathbf{k_0}(t)C &= Q(t) \\ C(T,T) &= 0 \end{aligned} \right. \tag{2.28}$$

The Gaussian structure of the model stops to be an *assumption* and becomes a *needed condition* when all the factors are of the type $Z^{(q)}$. So, for pure quadratic term structures (PQTS), the system of ODEs to be solved is also (2.26)-(2.28), not because we have an ad doc assumption, but because otherwise $\sigma\sigma^\top$ would not have the required form.

The previous literature on quadratic term structures can, thus, be seen as *either* a literature on Gaussian-QTS, where a non-needed add hoc assumption is introduced, *or* as a literature on PQTS that has not considered the possibility that some factors may have only a linear impact on the term structure. In either case the results are those of Corollary 2.21 and are obtained from Theorem 2.14 by setting $\mathbf{K}(t) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t. In Section 2.6 we give examples of both pure (thus Gaussian) and non-pure QTS.

Affine Term Structures

From Corollary 2.17 we know that if there are only $Z^{(l)}$-factors we have an ATS of bond prices.

Thus, *sufficient conditions* to guarantee ATS of bond prices are: an affine setting of interest rates *and* a linear matrix $\sigma(t,z)\sigma^\top(t,z)$. Otherwise there would be at least one $Z^{(q)}$-factor (for one of the reasons (i) or (ii) in Definition 2.12) having a quadratic impact in bond prices term structure. So, $Q(t) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t, guarantee an ATS and Theorem 2.14 give us the right result when we include this fact. Also in this case Theorem 2.14 can be restated without any allusion to the classification of factors (there is only one type of factor, anyway).

Corollary 2.22. **(ATS)** *Suppose that Assumption 2.3 and 2.4 are in force. Furthermore suppose that we are in an affine interest rate setting, that is (2.9) hold with $Q(t) = \mathbf{0}$. Finally assume that α and σ from the factor dynamics (2.4) are of the following form:*

$$
\begin{aligned}
\alpha(t,z) &= \mathbf{d}(t) + \mathbf{E}(t)z \\
\sigma(t,z)\sigma^\top(t,z) &= \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t) z_u
\end{aligned}
$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$ and $\mathbf{k_u}$ for all $\mathbf{u}$, are matrices of deterministic smooth functions.

Then the term structure of bond prices is affine, i.e. H from (2.2) can be written on the form (2.1) with $C = \mathbf{0}$ and A, B solve the following system of ordinary differential equations.

$$
\left\{
\begin{aligned}
\frac{\partial A}{\partial t} + \mathbf{d}^\top(t)B + \frac{1}{2}B^\top \mathbf{k_0}(t)B + \operatorname{tr}\{C\}\mathbf{k_0}(t)\} &= f(t) \\
A(T,T) &= 0
\end{aligned}
\right.
\tag{2.29}
$$

$$\begin{cases} \dfrac{\partial B}{\partial t} + \mathbf{E}^\top(t)B + \dfrac{1}{2}\bar{B}^\top \mathbf{K}(t)B &= g(t) \\ B(T,T) &= 0 \end{cases} \tag{2.30}$$

where $\mathbf{K}$, $\bar{B}$ *are defined as in (2.12) and (2.19) respectively.*

2.4 On the factors volatility structure

From a careful reading of the proof of Theorem 2.14, one can realize that whenever there are both types of factors, we could, in principle, allow the drift α of $Z^{(l)}$-factors to depend also quadraticly on some of the factors z, since the PDE (2.20) would still be separable in a way that would not compromise the existence of solution. Concretely, a drift of the form,

$$\alpha(t,z) = \mathbf{d}(t) + \begin{pmatrix} \mathbf{E}^{(\mathrm{qq})}(t) & 0 \\ \mathbf{E}^{(\mathrm{lq})}(t) & \mathbf{E}^{(\mathrm{ll})}(t) \end{pmatrix} \begin{pmatrix} z^{(q)} \\ z^{(l)} \end{pmatrix} + \begin{pmatrix} 0 & z^{(q)\top} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^{(\mathrm{ll})}(t) \end{pmatrix} \begin{pmatrix} 0 \\ z^{(q)} \end{pmatrix}$$

could, theoretically, be considered.

This possibility is however ruled out in Definition 2.11 because, by allowing a quadratic drift, we *cannot* guarantee existence of a unique global solution [9] for the SDE (2.4), since there would be *no* K such that

$$||\alpha(t,x) - \alpha(t,y)|| \leq K||x-y||.$$

Therefore, the most general form for the drift term α is that it is linear in z (as in (2.10)).

We can, nonetheless, allow for quadratic $\sigma(t,z)\sigma^\top(t,z)$ (as in (2.11)), since this essentially means that $\sigma(t,z)$ is a linear function of z, and, therefore, we still can find a K such that

$$||\sigma(t,x) - \sigma(t,y)|| \leq K||x-y||.$$

Interesting questions to pose at this point are:

1. What do the conditions on $\sigma(t,z)\sigma^\top(t,z)$ really imply for the structure of $\sigma(t,z)$ itself?
2. What is the equivalent to a given form of $\sigma(t,z)$ in terms of volatility $\hat{\sigma}(t,z)$ of a model with correlated[10] Wiener processes?

[9] For a textbook treatment of solutions of SDEs see, for instance, Björk (2004a).

[10] Recall Remark 2.5 and Example 2.6.

In previous literature little effort has been spent in trying to answer the above questions, the only exception is Duffie and Kan (1996), that answers the *first* for affine $\sigma\sigma^\top$. The next two Propositions answer *both* questions for a quadratic $\sigma\sigma^\top$, that is when

$$\sigma(t,z)\sigma^\top(t,z) = \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t) z_u + \sum_{u,k=1}^{m} z_u \mathbf{g_{uk}}(t) z_k. \tag{2.31}$$

Proposition 2.23. *Taking $\sigma(t,z)^\top\sigma(t,z)$ to be of the form (2.31) is, under non degeneracy conditions and possible reordering of indices, the same as taking $\sigma(t,z)$ in (2.4) to be of the form*

$$\sigma(t,z) = \Sigma(t)U(t,z), \tag{2.32}$$

where $\Sigma(t)$ is a $(m \times n)$ deterministic matrix and $U(t,z)$ is a $(n \times n)$ matrix with the specific from

$$U(t,z) = \begin{pmatrix} \sqrt{u_1(t,z)} & 0 & \cdots & 0 \\ 0 & \sqrt{u_2(t,z)} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & \sqrt{u_n(t,z)} \end{pmatrix} \tag{2.33}$$

where

$$u_i(t,z) = e_i(t) + f_i(t)z + z^\top g_i(t) z \tag{2.34}$$

with e_i a scalar deterministic functions and f_i a deterministic row-vector functions for all $i \in \{1,\ldots,n\}$.

Proof. This is a straight forward generalization of a similar result from Duffie and Kan (1996) on the implications for the matrix $\sigma(t,z)$ of requiring a linear functional form for $\sigma\sigma^\top$ (the well-known ATS condition). ∎

Proposition 2.23 shows that each column of the matrix $\sigma(t,z)$ can only depend on the square root of one particular quadratic combination of factors; otherwise, $\sigma(t,z)\sigma^\top(t,z)$ would have elements of the form $\sqrt{u_i}\sqrt{u_j}$ which, for $u_i \neq u_j$, would not be linear in z. That is, the matrix $\sigma(t,z)$ in (2.4) needs to be of the following form

$$\sigma(t,z) = \begin{pmatrix} s_{11}\sqrt{u_1(t,z)} & \cdots & s_{1n}\sqrt{u_n(t,z)} \\ s_{21}\sqrt{u_1(t,z)} & \cdots & s_{2n}\sqrt{u_n(t,z)} \\ \vdots & & \vdots \\ s_{m1}\sqrt{u_1(t,z)} & \cdots & s_{mn}\sqrt{u_n(t,z)} \end{pmatrix}. \tag{2.35}$$

The implications from (2.32)-(2.34) are quite strong since by letting each column of $\sigma(t,z)$ depend at most on one particular square root function, it implies that the term associated with each of the n elements of the multi-dimensional Wiener process (since each column of σ multiply a different element of W) can also depend at most on one particular square root function.

This is just an algebraic fact that, however, together with the fact that the matrix $\sigma(t,z)$ is the volatility when we consider a multi-dimensional Wiener process (i.e. independent scalar Wiener processes) may turn out to be quite restrictive.[11] Note that given a model with correlated Wiener processes, the volatility of the equivalent model with independent Wiener process will incorporate the correlations between the original scalar Wiener processes in $\sigma(t,z)$ and so restrictions of this matrix lead to restrictions on possible correlations on the original scalar Wiener processes[12].

Taking these two observations simultaneously, it is easy to see that correlations can only be allowed between factors driven by the same u_i function.

We take a second look at Example 2.6.

Example 2.6 (cont.) The following original dynamics of two factors Z_1 and Z_2:

$$d\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = \left(\dots\right) dt + \underbrace{\begin{pmatrix} \sigma_1\sqrt{Z_1(t)} & 0 \\ 0 & \sigma_2\sqrt{Z_2(t)} \end{pmatrix}}_{\bar{\sigma}(t,Z(t))} d\begin{pmatrix} \bar{W}_1(t) \\ \bar{W}_2(t) \end{pmatrix} \tag{2.36}$$

with $d\bar{W}_1(t)d\bar{W}_2(t) = \rho dt$, has the following transformed form

$$d\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = \left(\dots\right) dt + \underbrace{\begin{pmatrix} \sigma_1\sqrt{Z_1(t)} & 0 \\ \rho\sigma_2\sqrt{Z_2(t)} & \sigma_2\sqrt{Z_2(t)}\sqrt{1-\rho^2} \end{pmatrix}}_{\sigma(t,Z(t))} d\begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix} \tag{2.37}$$

with $dW_1(t)dW_2(t) = 0$.

Since $\sigma(t, Z(t))$ in (2.37) does not have the form (2.35) for any $\rho \neq 0$, we immediately see that the model (2.36) when we allow for correlation between W_1 and W_2 is *not* in accordance with (2.31).

When the two Wiener processes are not correlated (i.e. if $d\bar{W}_1(t)d\bar{W}_2(t) = 0$), then we have the form (2.32)-(2.34) with $g_i = \mathbf{0}$ for all i, consequently the matrix $\sigma(t,z)$ is linear in z and in accordance with (2.31).

[11] Recall the comment just made after Example 2.6 (in Section 2.2), that any two correlated factors, in the assumed formulation must be driven by at least one common (scalar) Wiener process.

[12] Recall equation (2.7) in Remark 2.5 and note that the correlations will appear in $\sigma(t,z)$ trough the matrix δ.

Proposition 2.24 tells us what are the conditions in terms of the volatility structure $\hat{\sigma}(t,z)$ of a possibly correlated system that would guarantee a multidimensional representation with a volatility structure $\sigma(t,z)$ of the form (2.35).

Proposition 2.24. *Under non degeneracy conditions and possible reordering of terms we have a transformed matrix $\sigma(t,z)\sigma^\top(t,z)$ quadratic in z (of the form (2.31)), if and only if,*

- *the volatility structure of the factors $\hat{\sigma}(t,z)$ in a system with correlated Wiener processes (as in equation (2.5)) can be written as*

$$\hat{\sigma}(t,z) = \hat{\Sigma}(t)\hat{U}(t,z) \tag{2.38}$$

  where $\hat{\Sigma}(t)$ is a deterministic $(m \times d)$ matrix and $\hat{U}(t,z)$ a $(d \times d)$ matrix with the same form as (2.33)-(2.34),
  and

- *for any possible i and j, we only allow for correlations among the scalar wiener processes $\bar{W}_i$ and $\bar{W}_j$ in $\bar{W} = \begin{pmatrix} \bar{W}_1 \\ \vdots \\ \bar{W}_d \end{pmatrix}$ if we have $u_i(t,z) = u_j(t,z)$.*

Proof. It follows from Proposition 2.23 and the equivalence between the representation (2.5) and (2.6) in Remark 2.5. ∎

To understand the idea behind Proposition 2.24, note that, since $U(t,z)$ in (2.38) is of the same form as in (2.32), the difference between $\hat{\sigma}(t,z)$ and $\sigma(t,z)$ comes essentially from the matrices $\hat{\Sigma}(t)$ and $\Sigma(t)$. $\hat{\Sigma}(t)$ tend to be a better behaved diagonal matrix as $\Sigma(t)$ has to include possible correlations in off-diagonal cells. Since the basic structure is maintained when going from the original to the transformed dynamics of the factors Z, the very specific form in (2.35) only allows for very restrictive correlations among the elements of originally correlated Wiener processes $\bar{W}$.

Example 2.25. *A five-factor model with possibly correlated Wiener processes*

$$dZ(t) = (\cdots)dt + \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2\sqrt{Z_1(t)} & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4\sqrt{Z_1^2(t)+Z_3(t)} & 0 \\ 0 & 0 & 0 & 0 & \sigma_5\sqrt{Z_1(t)} \end{pmatrix} d \begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \\ \bar{W}_3 \\ \bar{W}_4 \\ \bar{W}_5 \end{pmatrix}$$

has a volatility structure $\hat{\sigma}(t,z)$ that can be rewritten as

$$\hat{\sigma}(t,z) = \underbrace{\begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \end{pmatrix}}_{\hat{\Sigma}} \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \sqrt{z_1} & 0 & \cdots & \vdots \\ \vdots & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \sqrt{z_1^2 + z_3} & 0 \\ 0 & \cdots & \cdots & 0 & \sqrt{z_1} \end{pmatrix} \tag{2.39}$$

From (2.39) we see that the only correlations we can allow for, according to Proposition 2.24, are between $\bar{W}_1$ and $\bar{W}_3$ on the one hand, and between $\bar{W}_2$ and $\bar{W}_5$ on the other.

The two propositions above study the form of σ (and $\hat{\sigma}$) when we want to guarantee a general quadratic form of $\sigma\sigma^\top$ as in (2.31). We know, however, from the previous analysis, that the form (2.31) does not guarantee a GQTS and only a restricted version of it does. Considering the needed restrictions we now analyze

$$\sigma(t,z)\sigma^\top(t,z) = \begin{pmatrix} \mathbf{k}_\mathbf{0}^{(\mathbf{qq})}(t) & \mathbf{k}_\mathbf{0}^{(\mathbf{ql})}(t) \\ \\ \mathbf{k}_\mathbf{0}^{(\mathbf{lq})}(t) & \mathbf{k}_\mathbf{0}^{(\mathbf{ll})}(t) + \sum_{i=1}^{m} \mathbf{k}_\mathbf{u}^{(\mathbf{ll})}(t) z_u + \sum_{z_u, z_k \in \mathbf{z}^{(\mathbf{q})}} z_u \mathbf{g}_{\mathbf{uk}}^{(\mathbf{ll})}(t) z_k \end{pmatrix}. \tag{2.40}$$

The needed conditions on σ (and $\hat{\sigma}$) are presented in the next Corollary.

Corollary 2.26. *Given the classification of factors in Definition 2.12, imposing the structure (2.40) on $\sigma(t,z)\sigma^\top(t,z)$ is equivalent to require, under non degeneracy conditions, that:*

- *the volatility structure, $\sigma(t,z)$ in (2.4), is of the following form*

$$\sigma(t,z) = \begin{pmatrix} \Sigma_A^{(q)}(t) & 0 \\ \\ \Sigma_A^{(l)}(t) & \Sigma_B^{(l)}(t) \end{pmatrix} \begin{pmatrix} I & 0 \\ \\ 0 & U(t,z) \end{pmatrix}$$

for some division of the elements in the multidimensional Wiener process into $W = \begin{pmatrix} W_A \\ W_B \end{pmatrix}$ and where $\Sigma_{\cdot}^{(\cdot)}$ are deterministic matrices and $U(t,z)$ has the same form as in (2.33)-(2.34).

or equivalently,

- *the volatility structure of the original model with possibly correlated Wiener processes, $\hat{\sigma}(t,z)$ in (2.5),*
  - *can be written as*

$$\hat{\sigma}(t,z)=\begin{pmatrix}\hat{\Sigma}_A^{(q)}(t) & 0\\ \hat{\Sigma}_A^{(l)}(t) & \hat{\Sigma}_B^{(l)}(t)\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & \hat{U}(t,z)\end{pmatrix}$$

  for some division of $\bar{W}=\begin{pmatrix}\bar{W}_A\\ \bar{W}_B\end{pmatrix}$ and where $\hat{\Sigma}^{(\cdot)}_{\cdot}$ are deterministic matrices and $\hat{U}(t,z)$ has the same form as in (2.33)-(2.34)
  and
  - *for any i and j we only allow for correlations among the scalar Wiener processes $\bar{W}_i$ and $\bar{W}_j$ in $\bar{W}=\begin{pmatrix}\bar{W}_1\\ \vdots\\ \bar{W}_m\end{pmatrix}$ if both belong to the group $\bar{W}_A$, or if we have $u_i(t,z)=u_j(t,z)$.*

Proof. It follows immediately from Propositions 2.23 and 2.24. ■

The previous results help us check, by inspection, if a model satisfies the volatility sufficient condition for a GQTS.

Remark 2.27. *An important consequence of the results above is that* any *factor (even a linear factor) that is correlated with a quadratic factor must also have a deterministic volatility structure.*

2.5 On higher order term structures

After studying GQTS, which include all term structures up to the order two, one may wonder about higher order term structures, say, general cubic term structures for bond prices (GCTS). These GCTS would include all term structures up to the order three, so they would include the GQTS previously studied as well as cubic term structures (CTS).

GCTS are, however, not nice objects of study. Without getting too technical, we note that when we leave the quadratic setting, the second-order derivatives $\dfrac{\partial^2 \ln H}{\partial z_i \partial z_j}$ are of a higher order than $\ln H$ itself, which makes almost impossible (it is possible only in pathological/lucky cases) that the term structure equation will have a solution. In other words, CTS are not compatible with no-arbitrage.

This is an interesting result also proved by Filipović (2002) using different techniques[13] and gives extra motivation for GQTS (i.e., non Gaussian-QTS) studied in this chapter. In some sense, this is as far as one can hope to go without introducing arbitrage into the model.

2.6 Applications

In this section, we consider two-factor models that generate GQTS of bond prices. The models are naive as their purpose is rather to illustrate the techniques earlier described, than to focus on realism. In the first model, the two factors will have a quadratic impact[14], while on the second model, we have a true GQTS, with one linear factor and one quadratic factor. For both cases, we explicitly compute the term structure of bond prices. Some of the results rely on the solution of scalar Riccati equations. In Appendix B one can find useful computations on these type of equations.

2.6.1 PQTS

Consider the following model for the short interest rate

$$\begin{aligned} dZ_1(t) &= [\beta_1(t) - \alpha_1 Z_1(t)]\, dt + \sigma_1 dW_1(t) \\ dZ_2(t) &= [\beta_2(t) - \alpha_2 Z_2(t)]\, dt + \sigma_2 dW_2(t) \\ dW_1(t)dW_2(t) &= 0 \end{aligned}$$

$$r(t) = \frac{1}{2}\left[Z_1^2(t) + Z_2^2(t)\right]$$

where α_1, α_2, σ_1 and σ_2 are deterministic constants and $\beta_1(\cdot)$ and $\beta_2(\cdot)$ deterministic functions of time.

[13]Filipović (2002) proves, under certain regularity conditions, that if one represents the forward rate as a time-separable polynomial function of a diffusion state vector, that the maximal consistent order of the polynomial is *two*. *Consistency* in this context, as discussed in Björk and Christensen (1999), means that the interest rate model will produce forward curves belonging to the parameterized family.

[14]This model is also presented in Jamshidian (1996) as an illustration of a Gaussian-QTS. There the task of obtaining the explicit solution is left to the reader.

We have a GQSR with

$$Q(t) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad g(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad f(t) = 0.$$

Both factors are $Z^{(q)}$-factors (according to Definition 2.12) and we have a general quadratic $\mathbb{Q}$-dynamics for the factors, i.e., (2.10) and (2.11) hold with

$$\begin{aligned} \mathbf{d}(t) &= \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}, \quad \mathbf{E}(t) = \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} \\ \mathbf{k_0}(t) &= \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}, \quad \mathbf{k_u}(t) = \mathbf{0}, \quad \mathbf{g_{uk}}(t) = \mathbf{0}. \end{aligned}$$

Hence, we are under the conditions for a PQTS for bond prices, and to obtain A, B and C we need to solve the system (2.26)-(2.28).

For the model above C solves,

$$\left\{ \begin{aligned} \frac{\partial C}{\partial t} + 2 \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} C + 2C^\top \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} C &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\ C(T,T) &= 0 \end{aligned} \right. .$$

We can immediately see that $C^{(12)} = C^{(21)} = 0$ is part of the solution, so we can solve the simpler ODE

$$\left\{ \begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} C^{(11)} & 0 \\ 0 & C^{(22)} \end{pmatrix} + 2 \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} \begin{pmatrix} C^{(11)} & 0 \\ 0 & C^{(22)} \end{pmatrix} & \\ +2 \begin{pmatrix} C^{(11)} & 0 \\ 0 & C^{(22)} \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} C^{(11)} & 0 \\ 0 & C^{(22)} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \\ \begin{pmatrix} C^{(11)} & 0 \\ 0 & C^{(22)} \end{pmatrix} (T,T) &= 0 \end{aligned} \right.$$

and we just need to solve two scalar Riccati equations.

The final solution of C is then,

$$\begin{aligned} C(t,T) &= \begin{pmatrix} C^{(11)}(t,T) & 0 \\ 0 & C^{(22)}(t,T) \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{1 - e^{2\gamma_1(T-t)}}{2(\alpha_1+\gamma_1)\left(e^{2\gamma_1(T-t)} - 1\right) + 4\gamma_1} & 0 \\ 0 & \dfrac{1 - e^{2\gamma_2(T-t)}}{2(\alpha_2+\gamma_1)\left(e^{2\gamma_2(T-t)} - 1\right) + 4\gamma_1} \end{pmatrix} \end{aligned}$$

and $\gamma_i = \sqrt{\alpha_i^2 + \sigma_i^2}$ for $i = 1, 2$.

With these solutions we can go on and solve the ODE for B and A.

$$\begin{cases} \dfrac{\partial B}{\partial t} + \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} B + 2C \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + 2C \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} B & = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ B(T,T) & = 0 \end{cases}$$

which simplifies to

$$\begin{cases} \dfrac{\partial B}{\partial t} + \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} B + 2 \begin{pmatrix} C^{(11)}(t,T) & 0 \\ 0 & C^{(22)}(t,T) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} & \\ \qquad +2 \begin{pmatrix} C^{(11)}(t,T) & 0 \\ 0 & C^{(22)}(t,T) \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} B & = g(t) \\ B(T,T) & = 0 \end{cases}$$

and the solution is given by

$$B^{(i)}(t,T) = 2 \int_t^T e^{\int_t^s \left(\alpha_i - 2\sigma_i^2 C^{(ii)}(u,T)\right) du} \beta_i(s) C^{(ii)}(s,T) ds \qquad i = 1, 2$$

Finally we can obtain A by simple integration of (2.26), that is

$$A(t,T) = \sum_{i=1}^{2} \left\{ \int_t^T \beta_i(s) B^{(i)}(s,T) ds + \frac{1}{2}\sigma_i^2 \int_t^T \left[B^{(i)}(s,T\right]^2 ds + \sigma_i^2 \int_t^T C^{(ii)}(s,T) ds \right\}.$$

2.6.2 A naive non-pure QTS

Consider the following two-factor model

$$\begin{aligned} dZ_1(t) &= \beta_1(t)dt + \sigma_1 dW_1(t) \\ dZ_2(t) &= \beta_2(t)dt + \sqrt{Z_2(t)} dW_2(t) \\ dW_1(t)dW_2(t) &= 0 \end{aligned}$$

$$r(t) = Z_1^2(t) + Z_2(t)$$

where σ_1 is a deterministic constant and $\beta_1(\cdot)$, $\beta_2(\cdot)$ are deterministic functions of time.

We have a GQSR with

$$Q(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad g(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f(t) = 0.$$

And, from Definition 2.12, we can conclude that Z_1 is a $Z^{(q)}$-factor and Z_2 is a $Z^{(l)}$-factor.

We also see that both the drift and the volatility conditions are satisfied with

$$\begin{aligned}
\mathbf{d}(t) &= \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}, \quad \mathbf{E}(t) = \mathbf{0} \\
\mathbf{k_0}(t) &= \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{k_1}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{k_2}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
\mathbf{g_{uk}}(t) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for } u, k = 1, 2 .
\end{aligned}$$

Hence bond prices have a QTS and A, B and C solve

$$\begin{cases} \dfrac{\partial A}{\partial t} + \begin{pmatrix} \beta_1(t) & \beta_2(t) \end{pmatrix} B + \dfrac{1}{2} B^\top \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix} B + \operatorname{tr}\left\{ C \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix} \right\} = 0 \\ A(T,T) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial B}{\partial t} + 2C \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \dfrac{1}{2} \bar{B}^\top \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} B + 2C \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ B(T,T) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial C}{\partial t} + 2C \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix} C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ C(T,T) = 0 \end{cases}$$

It is easy to see that in this case $C^{(11)}$ solves the scalar Riccati equation

$$\begin{cases} \dfrac{\partial C^{(11)}}{\partial t} + 2\sigma_1^2 \left(C^{(11)}\right)^2 = 1 \\ C^{(11)}(T,T) = 0 \end{cases}$$

whose solution is given by

$$C^{(11)}(t,T) = \frac{1 - e^{2\gamma(T-t)}}{\gamma\left(e^{2\gamma(T-t)} - 1\right) + 2\gamma} \qquad \text{with} \quad \gamma = \sqrt{2}\sigma_1 .$$

And for the remaining cells in C,(i.e. for $(ij) \neq (11)$), $C^{(ij)}(t,T) = 0$.

Each entry of B then solves

$$\begin{cases} \dfrac{\partial B^{(1)}}{\partial t} + 2C^{(11)}\beta_1(t) + 2C^{(11)}\sigma_1^2 B^{(1)} & = & 0 \\ B^{(1)}(T,T) & = & 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial B^{(2)}}{\partial t} + \dfrac{1}{2}\left(B^{(2)}\right)^2 & = & 1 \\ B^{(2)}(T,T) & = & 0 \end{cases}$$

We can solve first for $B^{(2)}$ and then for $B^{(1)}$ to get

$$\begin{aligned} B^{(1)}(t,T) &= \int_t^T e^{\sigma_1^2 \int_t^s 2C(11)(u,T)du} \left[2C^{(11)}(s,T)\beta_1(s)\right] ds \\ &= \int_t^T \beta_1(s) \frac{e^{\frac{\sigma_1^2\gamma}{Q}(s-t)}\left(1 - e^{2\gamma(T-s)}\right)\left[\gamma(e^{2\gamma(T-s)} - 1) + 2\gamma\right]^{\frac{\sigma_1^2}{Q}-1}}{\left[\gamma(e^{2\gamma(T-t)} - 1) + 2\gamma\right]^{\frac{\sigma_1^2}{Q}}} ds \\ B^{(2)}(t,T) &= \frac{\sqrt{2}\left(1 - e^{\sqrt{2}(T-t)}\right)}{\left(e^{\sqrt{2}(T-t)} - 1\right) + 2} \end{aligned}$$

Finally one easily sees that A is given by

$$A(t,T) = \sum_{i=1}^{2}\left\{\int_t^T \beta_i(s)B^{(i)}(s,T)ds\right\} + \frac{1}{2}\sigma_1^2 \int_t^T \left(B^{(1)}(s,T)\right)^2 ds + \sigma_1^2 \int_t^T C^{(11)}(s,T)ds$$

More complex GQTS of bond prices can also be computed.

2.7 Concluding remarks

This chapter investigates the term structure of *bond prices* when we assume that these prices are functions of a finite dimensional state process. It generalizes previous studies by considering non-Gaussian quadratic term structures.

This generalization relies on the *a priori* separation of factors into *quadratic* and *linear* factors. The ***Generalized Quadratic Term Structures*** (GQTS) have, as special cases, the affine and the Gaussian-quadratic term structures.

We show that unless all factors are of the quadratic type, the requirement of a deterministic volatility structure is not necessary for all factors. We devote some effort to understanding all the implications that different volatility conditions for different types of factors have in terms or their possible correlations. Finally, we explain why this generalization technique fails when it comes to higher order term structures recovering the well-known result that term structures of order higher than two are not consistent with the no-arbitrage assumption.

In terms of further research, it is worth mentioning that although the classification of factors is dependent on the concrete prices we are studying, the ideas presented here can be applied to other term structures (up to a different classification of factors).

In Chapter 5, for instance, the same concept of GQTS is applied to forward and futures prices.

A Appendix : Equivalence of Wiener systems

In this appendix we give a proof of Remark 2.5. We show that we can transform any system with correlated Wiener processes into an equivalent system with uncorrelated ones, so this assumption of independence between the elements of W is not restrictive in any sense.

Proof of Remark2.5

Proof. Let us consider the d-dimensional correlated Winer processes $\bar{W}_1, \ldots, \bar{W}_d$. Let furthermore the correlation (deterministic square symmetric matrix) be given,

$$\rho = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1d} \\ \rho_{12} & 1 & \cdots & \rho_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1d} & \rho_{2d} & \cdots & 1 \end{pmatrix}.$$

Define a deterministic matrix δ, with d-rows and as many columns as necessary to solve $\rho_{ij} = \delta_i \delta_j^\top$, where δ_i denotes the ith–row.

Denote n the (minimal) number of columns of δ. Then, we know $\|\delta_i\| = 1$.

It remains to show that, for the n-dimensional W, defined by $\bar{W} = \delta W$, we have $W_1, W_2, \ldots, W_n$ independent (scalar) Winer processes. It is straightforward to check that each of them follow a standard $N(0,1)$ normal distribution (it follows from $\|\delta_i\| = 1$ and $\mathbb{E}\left[\bar{W}_i\right] = 0$ for all $i = 1, \ldots, d$).

To show the absence of covariances, first we note that by assumption we have $\mathbb{E}\left[d\bar{W}_i d\bar{W}_j\right] = \rho_{ij}$, then we realize

$$\begin{aligned} \mathbb{E}\left[d\bar{W}_i d\bar{W}_j\right] &= \mathbb{E}\left[\sum_{k=1}^{n} \delta_{ik} dW_k \cdot \sum_{l=1}^{n} \delta_{jl} dW_l\right] \\ &= \sum_{kl} \delta_{ik}\delta_{jl} \mathbb{E}\left[dW_k dW_l\right] \\ &= \underbrace{\delta_i {\delta_j}^\top}_{\rho_{ij}} + \sum_{kl,\ k\neq l} \delta_{ik}\delta_{jl} \mathbb{E}\left[dW_k dW_l\right] \qquad \Leftrightarrow \qquad \mathbb{E}\left[dW_k dW_l\right] = 0 \text{ for } k \neq l. \end{aligned}$$

Finally, $\sigma(t, Z(t)) = \hat{\sigma}(t, Z(t))\delta$ is a consequence of the equivalence between (2.5) and (2.6).

∎

B Appendix : Scalar Riccati Equations

This appendix summarizes some useful computations related to scalar Riccati equations.

Lemma 2.28. *Consider the following Ricatti ODE where Z is a deterministic function of two variables (t,T), $\dot{Z}$ denotes $\frac{\partial Z}{\partial t}$ and $g, f, Q \in \mathbb{R}$*

$$\begin{cases} \dot{Z} = f + gZ + QZ^2 \\ Z(T,T) = 0 \end{cases} . \tag{2.41}$$

The solution to the above ODE is given by

$$Z(t,T) = \frac{2f\left[1 - e^{\psi x}\right]}{(\psi + g)\left[e^{\psi x} - 1\right] + 2\psi} \tag{2.42}$$

where we use $x = T - t$ and $\psi = \sqrt{g^2 - 4Qf}$.

Proof. Let us set $Z(t,T) = \frac{X(t)}{Y(t)}$ where $Y(t) \neq 0 \quad \forall t \in [t,T]$.

We, thus, have $\dot{Z} = \frac{\dot{X}Y - \dot{Y}Y}{Y^2}$ and it is easy to verify that the Riccati ODE (2.41) is equivalent to the following homogeneous system of linear ODEs

$$\begin{cases} \dot{X} &= gX + fy \\ \dot{Y} &= -QX \end{cases} = \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \underbrace{\begin{bmatrix} g & f \\ -Q & 0 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix}$$

with the boundary conditions $X(T) = 0$ and w.l.o.g. $Y(T) = 1$.

Suppose we can make a transformation of the variables so that the transformed system is diagonal

$$\begin{bmatrix} X \\ Y \end{bmatrix} = B \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}$$

then

$$\begin{bmatrix} \dot{\tilde{X}} \\ \dot{\tilde{Y}} \end{bmatrix} = B \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = BA \begin{bmatrix} X \\ Y \end{bmatrix} = BAB^{-1} \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}$$

so, if we can find B such that BAB^{-1} is diagonal, then we have

$$\begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} X \\ Y \end{bmatrix} = B^{-1} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} .$$

where λ_1, λ_2 are the diagonal terms of BAB^{-1}.

From linear algebra we know that B^{-1} is the matrix of the eigenvectors of A and the diagonal of BAB^{-1} are the corresponding eigenvalues.

Computing the eigenvalues of A we get

$$\begin{vmatrix} g-\lambda & f \\ -Q & -\lambda \end{vmatrix} = 0 \qquad \lambda^2 - \lambda g + Qf = 0 \qquad \lambda = \frac{g \pm \sqrt{g^2 - 4Qf}}{2} = \frac{g \pm \psi}{2}$$

where we have denoted $\psi := \sqrt{g^2 - 4Qf}$.

The eigenvectors can then we obtained by

$$\left\{ \begin{array}{rcl} (g-\lambda_1)X + fY &=& 0 \\ -QX - \lambda_2 Y &=& 0 \end{array} \right. \qquad \Rightarrow \qquad \left\{ \begin{bmatrix} -\frac{f}{g-\lambda_1} \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{\lambda_2}{Q} \\ 1 \end{bmatrix} \right\}.$$

Using the expressions $\lambda_1 = \dfrac{g+\psi}{2}$ and $\lambda_2 = \dfrac{g-\psi}{2}$ we can rewrite the eigenvectors as

$$\left\{ \begin{bmatrix} -\frac{2f}{g-\psi} \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{g-\psi}{2Q} \\ 1 \end{bmatrix} \right\}.$$

We finally we have

$$\begin{aligned} \begin{bmatrix} X \\ Y \end{bmatrix} &= \begin{bmatrix} -\frac{2f}{g-\psi} & -\frac{g-\psi}{2Q} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2f}{g-\psi} e^{\lambda_1 t} C_1 - \frac{g-\psi}{2Q} e^{\lambda_2 t} C_2 \\ e^{\lambda_1 t} C_1 + e^{\lambda_2 t} C_2 \end{bmatrix} \end{aligned} \tag{2.43}$$

where C_1 and C_2 are integration constants that can be determined by the boundary conditions

$$\begin{bmatrix} -\frac{2f}{g-\psi} e^{\lambda_1 T} C_1 - \frac{g-\psi}{2Q} e^{\lambda_2 T} C_2 \\ e^{\lambda_1 T} C_1 + e^{\lambda_2 T} C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving boundary system we get

$$C_1 = \frac{(g-\psi)^2}{(g-\psi)^2 - 4fQ} e^{-\lambda_1 T} \qquad C_2 = \frac{-4fQ}{(g-\psi)^2 - 4fQ} e^{-\lambda_2 T}$$

Using these expressions in (2.43) we get

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -\frac{2f}{g-\psi}e^{\lambda_1 t}\left(\frac{(g-\psi)^2}{(g-\psi)^2-4fQ}e^{-\lambda_1 T}\right) - \frac{g-\psi}{2Q}e^{\lambda_2 t}\left(\frac{-4fQ}{(g-\psi)^2-4fQ}e^{-\lambda_2 T}\right) \\ e^{\lambda_1 t}\left(\frac{(g-\psi)^2}{(g-\psi)^2-4fQ}e^{-\lambda_1 T}\right) + e^{\lambda_2 t}\left(\frac{-4fQ}{(g-\psi)^2-4fQ}e^{-\lambda_2 T}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2f(g-\psi)[e^{-\lambda_1 x}+e^{-\lambda_2 x}]}{(g-\psi)^2-4fg} \\ \frac{(g-\psi)^2 e^{-\lambda_1 x}-4fQe^{-\lambda_2 x}}{(g-\psi)^2-4fg} \end{bmatrix}$$

Using $Z = \frac{X}{Y}$ and noting that $\lambda_1 - \lambda_2 = \psi$ and $g^2 - \psi^2 = (g-\psi)(g+\psi) = +4fQ$, we can simplify to reach the result

$$\begin{aligned} Z(t,T) &= \frac{\frac{2f(g-\psi)[e^{-\lambda_1 x}+e^{-\lambda_2 x}]}{(g-\psi)^2-4fg}}{\frac{(g-\psi)^2 e^{-\lambda_1 x}-4fQe^{-\lambda_2 x}}{(g-\psi)^2-4fg}} = \frac{2f(g-\psi)\left[e^{(\lambda_1-\lambda_2)x}\right]}{(g-\psi)^2-4fQe^{(\lambda_1-\lambda_2)x}} \\ &= \frac{2f\left[e^{\psi x}-1\right]}{(g-\psi)-(g+\psi)e^{\psi x}} = -\frac{2f\left[e^{\psi x}-1\right]}{-(g+\psi)\left[e^{\psi x}-1\right]-2\psi} \end{aligned}$$

■

Lemma 2.29. *Let $\psi = \sqrt{g^2-4Qf}$. The following results holds for Z solving the ODE in (2.41):*

$$(i) \quad \int_t^s Z(u,T)du = -\frac{1}{Q}\ln\left(\frac{e^{-(g+\psi)\frac{s-t}{2}}\left[(g+\psi)\left[e^{\psi(T-t)}-1\right]+2\psi\right]}{(g+\psi)\left[e^{\psi(T-s)}-1\right]+2\psi}\right) \qquad (2.44)$$

$$(ii) \quad e^{-\int_t^s Z(u,T)du}Z(s,T) = \frac{2fe^{\frac{2f}{(\psi-g)}(s-t)}\left[(g+\psi)[e^{\psi(T-t)}-1]+2\psi\right]^{1/Q}\left[1-e^{\psi(T-s)}\right]}{\left[(g+\psi)[e^{\psi(T-s)-1]+2\psi}\right]^{1/Q+1}}$$

Proof. (i)

$$\int_t^s Z(u,T)du = \int_t^s \frac{2f\left[1-e^{\psi(T-u)}\right]}{(\psi+g)\left[e^{\psi(T-u)}-1\right]+2\psi}du$$

We now do a change of variable

$$y = \left[e^{\psi(T-u)}-1\right] \quad \Rightarrow \quad u = \frac{\ln(y+1)-\psi T}{\psi} \quad \Rightarrow \quad u\prime = \frac{1}{\psi}\frac{1}{y+1}$$

and get

$$\int \frac{2f\left[1-e^{\psi(T-u)}\right]}{(\psi+g)\left[e^{\psi(T-u)}-1\right]+2\psi}du = -2f\frac{1}{\psi}\int \frac{y}{(\psi+g)y+2\psi}\cdot\frac{1}{y+1}du \qquad (2.45)$$

Finally, we note that

$$\frac{2fy}{(\psi+g)y+2\psi}\frac{1}{y+1} = -\frac{2\psi}{(g-\psi)\left[(g+\psi)y+2\psi\right]} + \frac{1}{(g-\psi)(y+1)},$$

and using $(g-\psi)(g+\psi) = +4fQ$ we have

$$\begin{aligned}
(2.45) &= -\frac{2f}{\psi}\left\{-\int \frac{2\psi}{(g-\psi)\left[(g+\psi)y+2\psi\right]} + \int \frac{1}{(g-\psi)(y+1)}\right\} \\
&= -\frac{2f}{\psi}\left\{-\frac{2\psi}{(g-\psi)(g+\psi)} \ln\left[(g+\psi)y+2\psi\right] + \frac{1}{(g-\psi)}\ln[y+1]\right\} \\
&= \frac{1}{Q}\ln\left[(g+\psi)y+2\psi\right] - \frac{2f(g+\psi)}{\psi(g-\psi)(g+\psi)}\ln[y+1] \\
&= \frac{1}{Q}\ln\left[(g+\psi)y+2\psi\right] - \frac{(g+\psi)}{2\psi Q}\ln[y+1] \\
&= -\frac{1}{Q}\ln\left(\frac{[y+1]^{\frac{(g+\psi)}{2\psi}}}{(g+\psi)y+2\psi}\right)
\end{aligned}$$

Substituting back $y = \left[e^{\psi(T-u)} - 1\right]$ we get

$$(2.45) = -\frac{1}{Q}\ln\left(\frac{\left(e^{\psi(T-u)}\right)^{\frac{(g+\psi)}{2\psi}}}{(g+\psi)[e^{\psi(T-u)}-1]+2\psi}\right) = -\frac{1}{Q}\ln\left(\frac{e^{(g+\psi)\frac{(T-u)}{2}}}{(g+\psi)\left[e^{\psi(T-u)}-1\right]+2\psi}\right)$$

So,

$$\begin{aligned}
\int_t^s Z(u,T)du &= -\frac{1}{Q}\ln\left(\frac{e^{(g+\psi)\frac{(T-u)}{2}}}{(g+\psi)\left[e^{\psi(T-u)}-1\right]+2\psi}\right)\Bigg|_t^s \\
&= -\frac{1}{Q}\ln\left(\frac{\frac{e^{(g+\psi)\frac{(T-s)}{2}}}{(g+\psi)\left[e^{\psi(T-s)}-1\right]+2\psi}}{\frac{e^{(g+\psi)\frac{(T-t)}{2}}}{(g+\psi)\left[e^{\psi(T-t)}-1\right]+2\psi}}\right)
\end{aligned}$$

(ii) Using equations (2.42) and (2.44) we have

$$\begin{aligned}
e^{-\int_t^s Z(u,T)du}Z(s,T) &= e^{-\left\{-\frac{1}{Q}\ln\left(\frac{e^{-(g+\psi)\frac{s-t}{2}}\left[(g+\psi)\left[e^{\psi(T-t)}-1\right]+2\psi\right]}{(g+\psi)\left[e^{\psi(T-s)}-1\right]+2\psi}\right)\right\}}\times \\
&\quad\times\frac{2f\left[1-e^{\psi(T-s)}\right]}{(\psi+g)\left[e^{\psi(T-s)}-1\right]+2\psi} \\
&= \left[\frac{e^{-(g+\psi)\frac{s-t}{2}}\left[(g+\psi)\left[e^{\psi(T-t)}-1\right]+2\psi\right]}{(g+\psi)\left[e^{\psi(T-s)}-1\right]+2\psi}\right]^{1/Q}\frac{2f\left[1-e^{\psi(T-s)}\right]}{(\psi+g)\left[e^{\psi(T-s)}-1\right]+2\psi} \\
&= \frac{e^{-\frac{(g+\psi)}{2Q}(s-t)}\left[(g+\psi)[e^{\psi(T-t)}-1]+2\psi\right]^{1/Q}\left[1-e^{\psi(T-s)}\right]}{2f\left[(g+\psi)[e^{\psi(T-s)-1}]+2\psi\right]^{1/Q+1}}
\end{aligned}$$

The results follow from the use of $(g-\psi)(g+\psi) = +4fQ$. ∎

II

Part II

Credit Risk Models

Chapter 3

Quadratic shot-noise CRM

"The sciences [...] mainly make models. By a model is meant a mathematical construct which, with addiction of certain verbal interpretations, describes observed phenomena. The justification of such mathematical construct is solely and precisely that it is expected to work."

John von Neumann, 1944.

We propose a reduced form model for default that allows us to derive closed-form solutions to all the key ingredients in credit risk modeling: risk-free bond prices, defaultable bond prices (with and without stochastic recovery) and survival probabilities. We show that all these quantities can be represented in general exponential quadratic forms, despite the fact that the intensity is allowed to jump producing shot-noise effects. In addition, we show how to price defaultable digital puts, CDSs and options on defaultable bonds.

Further on, we study a model for portfolio credit risk where we consider both firm specific and systematic risks. The model generalizes the attempt from Duffie and Gârleanu (2001). We find that the model produces realistic default correlation and clustering of defaults. Then, we show how to price first-to-default swaps, CDOs, and draw the link to currently proposed credit indices.

3.1 Introduction

Most of the quadratic term structure (QTS) literature has focused on analyzing risk-free bond prices and considering only quadratic Gaussian models. Exceptions are the study in the previous chapter and Chen, Filipović, and Poor (2004). Chapter 2 introduces the so-called general quadratic term structure (GQTS) models for risk-free bond prices, which through an a priori classification of factors include both the affine term structure (ATS) models and the Gaussian-QTS models as special cases.

Instead, Chen, Filipović, and Poor (2004) study the traditional Gaussian-QTS, for both risk-free and defaultable bonds. In this chapter we use the concept of GQTS and augment it with a special type of jump-process, called shot-noise processes.

In this chapter, we use the concept of GQTS and augment it with a special type of jump process called a shot noise process. As special cases, we do not only include the models mentioned above, but also some jump-diffusion models like, for instance, the one in Duffie and Gârleanu (2001). While quadratic models naturally arise in intensity-based models, as the default intensity needs to be a positive process, the shot noise component makes it possible to obtain a suitable dynamic dependence structure for a market with a large number of defaultable entities. Needless to say, capturing dynamic dependencies is one of the most important points for modeling CDOs. Using the shot noise process solves a basic problem in Duffie and Gârleanu (2001), namely that the mean reversion speed of the diffusion part and the jump part is the same. In addition, shot noise processes induce an interesting behavior in the process, which will result in clustering of defaults. It is this feature that seems very promising to capture the complex dependencies of defaults in a portfolio with several defaultable assets.

As already mentioned, we consider an intensity-based approach to modeling default. This approach has always been very popular and has recently been justified by strong fundamental motivations. Indeed, Duffie and Lando (2001) show that the difference between the reduced-form approach and the economically more intuitive structural approach becomes irrelevant when one includes frictions in the structural model, such as imperfect information about the asset or the liability structure. Moreover, Collin-Dufrense, Goldstein, and Hugonnier (2004) proved that the price of a defaultable security is always the expectation of future discounted cash-flows, even when the "no-jump condition" is violated[1]. Thus, there are good reasons to expect closed-form solutions

[1] When the no-jump condition holds, the traditional risk-neutral measure can be used, basically by adding to the discount rate a term which reflects the default risk. If the no-jump condition does

for key ingredients of credit risk.

For a survey study on reduced form credit risk models we refer to Schmidt and Stute (2004).

The main goals of this chapter are:

- To adapt the GQTS setup to model default risk using an intensity-based credit risk modeling approach *a la* Jarrow, Lando, and Turnbull (1997), Lando (1998) and Duffie and Singleton (1999) and to obtain closed-form solutions for all key ingredients of credit risk modeling: risk-free bonds, defaultable bonds, probabilities of default, etc.. This way we extend the Gaussian-QTS of defaultable bonds considered by Chen, Filipović, and Poor (2004);

- To use shot noise processes to give a dynamic description of default dependence that is able to produce a high default correlation and contagion effects;

- To obtain, under some simplifying assumptions, a model providing closed-form solutions for the pricing of CDOs and other portfolio credit risk derivatives.

The chapter is organized as follows. In Section 3.2 we review the basic setup of GQTS and present the main result on risk-free bond prices. In Section 3.3 we present the model for the defaultable bond market. Considering uncertainty effects, we find a motivation for shot noise effects in the credit spreads. Next, the theoretical framework is given and we derive survival probabilities, defaultable bond prices, defaultable digital payoffs and show how to use these building blocks to price less trivial credit derivatives. We conclude the section by considering various recovery assumptions. In Section 3.4 we deal with portfolio credit risk issues. Special emphasis is given to default correlations and clustering effects implied by the framework we consider. The subsection 3.4.3 is devoted to pricing CDOs, while the rest of the section deals with other portfolio credit derivatives such as first-to-default swaps and options on credit indices. Section 3.5 illustrates the theoretical results by considering an easy three-factor model. Section 3.6 concludes the chapter and discusses future research.

not hold, using a new measure (the survival measure) allows roughly the same to be done. The survival measure, also used in Schönbucher (2000) and Eberlein, Kluge, and Schönbucher (2005), is the measure that puts zero probability on those paths for which default occurs prior to maturity. As such, this measure is only absolutely continuous with respect to the risk-neutral probability and not equivalent to it.

3.2 Risk-free Bond Market

For the risk-free bond market we use the general quadratic term structures setup studied in Chapter 2. Consider a finite set of time-dependent factors described by a $\mathbb{R}^m$-valued stochastic process $(Z_t)_{t\geq 0}$. The zero-coupon bond prices are assumed to depend on these factors by

$$p(t,T) = H(t,T,Z_t), \tag{3.1}$$

where H is a smooth function with the boundary condition $H(T,T,z) = 1$. In a general quadratic setting, H will turn out to have a quadratic form.

We propose the following dynamics for Z:

$$dZ_t = \alpha(t,Z_t)dt + \sigma(t,Z_t)dW_t, \tag{3.2}$$

where W is a m-dimensional Wiener Process, and it generates the filtration $\left(\mathcal{F}_t^W\right)_{t\geq 0}$. The drift and volatility terms, α, σ, shall have the following form:

$$\alpha(t,z) = d(t) + E(t)z \tag{3.3}$$

$$\sigma(t,z)\sigma^\top(t,z) = k_0(t) + \sum_{i=1}^{m} k_i(t)z_i + \sum_{i,j=1}^{m} z_i\, g_{ij}(t) z_j. \tag{3.4}$$

Here, z_i is the i-th component of z. The deterministic and smooth functions d, k_0, k_i, g_{ij} for $i,j = 1,\cdots,m$ take values in $\mathbb{R}^m$ while E takes values in $\mathbb{R}^{m\times m}$ and $\cdot^\top$ denotes the transpose.

Also for the short rate we assume a quadratic form.

Assumption 3.1. *Assume that the risk-free short rate $(r_t)_{t\geq 0}$ is given by*

$$r(t,Z_t) = Z_t^\top Q(t) Z_t + g^\top(t) Z_t + f(t). \tag{3.5}$$

Here, Q, g and f are deterministic and smooth functions with values in $\mathbb{R}^{m\times m}$, $\mathbb{R}^m$ and $\mathbb{R}$, respectively. Moreover, $Q(t)$ is assumed to be symmetric[2] *for all t.*

In Chapter 2 it has been shown how to identify factors *a priori* from their impact on the drift α, volatility σ or the functional form of the short rate. We, thus, classify the components of Z in the following two groups.

[2]The symmetry assumption is not restrictive. Any non-symmetric quadratic form can be rewritten in an equivalent symmetric way with the advantage that the symmetric representation is unique.

Definition 3.2. (Classification of risk-free factors)

- Z_i is a *risk-free quadratic-factor* if it satisfies *at least one* of the following requirements:

  (i) it has a quadratic impact on the short rate of interest $r(t)$, i.e., there exists t such that $Q_i(t) \neq 0$;

  (ii) it has a quadratic impact on the functional form of the matrix $\sigma(t,z)\sigma^\top(t,z)$, i.e., there exist k and t such that $g_{ik}(t) \neq 0$;

  (iii) it affects the drift term of the factors satisfying (i) or (ii), i.e., for Z_j satisfying (i) or (ii) we have $E_{ji}(t) \neq 0$, at least for some t.

- Z_i is a *risk-free linear-factor*, if it does not satisfy any of (i)-(iii).

We write $i \in Z^{(q)}$, if Z_i is a risk-free quadratic factor and $i \in Z^{(l)}$ if it is a risk-free linear factor.

The above classification immediately yields that Q, E and G have a certain form. To access this easily we introduce the following notation. We say a function $\mathcal{Q}$ has only *quadratic factors*, if its symbolic representation is of the form

$$\mathcal{Q}(t) = \begin{pmatrix} \mathcal{Q}^{(qq)}(t) & 0 \\ & \\ 0 & 0 \end{pmatrix}, \quad \text{for all } t. \tag{3.6}$$

With this notation we have that Q and G have only quadratic factors, while for E

$$E(t) = \begin{pmatrix} E^{(qq)}(t) & 0 \\ & \\ E^{(lq)}(t) & E^{(ll)}(t) \end{pmatrix}.$$

From Chapter 2 it is known that, provided the factors have been reordered as $Z = \left[Z^{(q)}, \ Z^{(l)}\right]^\top$, the following conditions are sufficient for existence of a GQTS for risk-free bond prices. In this chapter we assume that these conditions hold.

Assumption 3.3. *Assume that for k_i and g_{ij} in (3.4) the following holds:*

$$k_i = \begin{pmatrix} 0 & 0 \\ & \\ 0 & k_i^{(ll)} \end{pmatrix} \forall\ i \text{ and} \qquad g_{ij} = \begin{pmatrix} 0 & 0 \\ & \\ 0 & g_{ij}^{(ll)} \end{pmatrix} \forall\ i,j \ s.t. \ Z_i, Z_j \in Z^{(q)}.$$

The value of the bond-price can be determined by use of the Feynman-Kac formula in terms of certain ODEs. In our quadratic approach this will always lead to the following system of Riccati ODEs.

Definition 3.4 (Basic ODE System). Denote $\mathcal{T} := \{(t,T) \in \mathbb{R}^2 : 0 \le t \le T\}$ and consider functions A, B and C on $\mathcal{T}$ with values in $\mathbb{R}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, respectively. For functions ϕ_1 and ϕ_2, ϕ_3 on $\mathbb{R}^+$ with values in $\mathbb{R}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, respectively, we say that $(A, B, C, \phi_1, \phi_2, \phi_3)$ solves the *basic ODE system* if

$$\begin{aligned} \frac{\partial A}{\partial t} + d^\top(t)B + \frac{1}{2}B^\top k_0(t)B + \operatorname{tr}\{Ck_0(t)\} &= \phi_1(t) \\ \frac{\partial B}{\partial t} + E^\top(t)B + 2Cd(t) + \frac{1}{2}\tilde{B}^\top K(t)B + 2Ck_0(t)B &= \phi_2(t) \\ \frac{\partial C}{\partial t} + CE(t) + E^\top(t)C + 2Ck_0(t)C + \frac{1}{2}\tilde{B}^\top G(t)\tilde{B} &= \phi_3(t) \end{aligned}$$

subject to the boundary conditions $A(T,T) = 0, B(T,T) = 0, C(T,T) = 0$. A, B and C should always be evaluated at (t,T). E, d, k_0, are the functions from the above definitions (recall (3.3)-(3.4)) while

$$\tilde{B} := \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B \end{pmatrix}, \; K(t) = \begin{pmatrix} k_1(t) \\ \vdots \\ k_m(t) \end{pmatrix}, \; G(t) = \begin{pmatrix} g_{11}(t) & \cdots & g_{1m}(t) \\ \vdots & \ddots & \vdots \\ g_{m1}(t) & \cdots & g_{mm}(t) \end{pmatrix}, \tag{3.7}$$

where we have $\tilde{B}, K \in \mathbb{R}^{m^2\times m}$ and $G \in \mathbb{R}^{m^2\times m^2}$.

We recall that the risk-free zero-coupon bond prices are given by

$$p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t^W\right],$$

and that only in special cases can we obtain the bond prices in closed-form.

As proven in Theorem 2.14 the general quadratic case is one of those special cases and the zero-coupon bond prices can be easily obtained from solving the basic ODE system.

Result 3.5. *Suppose that Assumptions 3.1 holds. Furthermore assume Assumption 3.3 is verified when the factors Z are reordered as $Z = \left[Z^{(q)}, \;\; Z^{(l)}\right]^\top$.*

Then, the term structure of risk-free zero-coupon bond prices is given by

$$p(t,T) = \exp\left[A(t,T) + B^\top(t,T)Z_t + Z_t^\top C(t,T)Z_t\right]$$

where (A, B, C, f, g, Q) solves the basic ODE from Definition 3.4. Recall that f, g and Q were given in Equation (3.5). Furthermore, C has only quadratic factors in the sense of (3.6).

3.3 Defaultable bond market

In this section we present the defaultable bond market. Before we present the actual model, we revise some general results needed for the valuation of defaultable bonds and related derivatives.

3.3.1 Doubly Stochastic Random Times

The results summarized in this section are all well-known, and may be found in any of the following books: McNeil, Frey, and Embrechts (2005), Lando (2004), Bielecki and Rutkowski (2002) or Schönbucher (2003).

We take the approach of explicitly constructing the doubly-stochastic random time τ, which will represent a single default, while the obtained results also hold in more general cases.

Definition 3.6 (Setup). Consider a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. On this probability space there exists:

- a filtration $(\mathcal{F}_t)_{t\geq 0}$,
- a strictly nonnegative process $(\mu_t)_{t\geq 0}$ adapted to $(\mathcal{F}_t)_{t\geq 0}$,
- a random variable E_1 which is exponentially distributed with parameter 1 which is independent of $\mathcal{F}_\infty$.

Then, $\int_0^t \mu_u \, du$ is an increasing, continuous process. We define the *default time* τ as

$$\tau := \inf\{t \geq 0 : \int_0^t \mu_u \, du = E_1\}. \tag{3.8}$$

The *information on the default state* is denoted $\mathcal{H}_t := \sigma(\mathbf{1}_{\{\tau > s\}} : 0 \leq s \leq t)$ and the *total information* by $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$.

From the independence of μ and E_1 and under the assumption that E_1 is exponentially distributed, we directly obtain

Lemma 3.7. *For the random time τ, constructed in* (3.8), *it holds that*

$$\mathbf{1}_{\{\tau>t\}}\mathbb{Q}\big(\tau > T|\mathcal{F}_T \vee \mathcal{H}_t\big) = \mathbf{1}_{\{\tau>t\}} \exp\Big(-\int_t^T \mu_u\,du\Big).$$

Proof. The essence of the proof is to use independency of E_1 and $\mathcal{F}_T$. First, observe that $\{\tau > T\} = \{\int_0^T \mu_u\,du < E_1\}$. So we have that on $\{\tau > t\}$

$$\mathbb{Q}(\tau > T|\mathcal{F}_T \vee \mathcal{H}_t) = \frac{\mathbb{Q}\Big(\int_0^T \mu_u\,du < E_1, \int_0^t \mu_u\,du < E_1|\mathcal{F}_T\Big)}{\mathbb{Q}\big(\int_0^t \mu_u\,du < E_1|\mathcal{F}_T\big)}.$$

As E_1 is exponentially distributed and independent from $\mathcal{F}_T$ we obtain

$$\mathbb{Q}\Big(E_1 > \int_0^t \mu_u\,du|\mathcal{F}_T\Big) = \exp\Big(-\int_0^t \mu_u\,du\Big),$$

and a similar result for the nominator. Observing, that the probability is zero on $\{\tau \le t\}$, the conclusion follows. ∎

The valuation of defaultable claims will base on the following two results

Theorem 3.8. *For a $\mathcal{F}$-adapted process $(X_t)_{t\ge0}$ and the random time τ, constructed in* (3.8), *it holds that*

$$\begin{aligned}
&(i) \qquad \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\Big(X_T\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t\Big) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\Big(X_T e^{-\int_t^T \mu_u\,du}|\mathcal{F}_t\Big),\\
&(ii) \qquad \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\Big(X_\tau\mathbf{1}_{\{t<\tau\le T\}}|\mathcal{G}_t\Big) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\Big(\int_t^T \Big[X_s\mu_s e^{-\int_t^s \mu_u\,du}\Big]\,ds|\mathcal{F}_t\Big).
\end{aligned}$$

Proof. We first prove (i). Using the definition of τ,

$$\mathbb{E}^{\mathbb{Q}}(X_T\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t) \;=\; \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\big[X_T\mathbb{E}^{\mathbb{Q}}(\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_T \vee \mathcal{H}_t)|\mathcal{G}_t\big].$$

Now, Lemma 3.7 can be applied to obtain the inner probability. Finally, we use that $X_T e^{-\int_t^T \mu_u\,du}$ is $\mathcal{F}_T$-measurable and hence independent of E_1. In $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ the σ-algebra $\mathcal{H}_t$ contains additional information on E_1, but using independency this can be dropped, such that $\mathcal{G}_t$ can be replaced by $\mathcal{F}_t$.

For (ii), we give an intuitive argument[3]. Assume that $h : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function. Let $\tilde{\mathcal{F}}_T = \mathcal{F}_T \vee \mathcal{H}_t$, then,

$$\mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\big(h(\tau)\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_T\big) = \int_t^T h(s) f_{\tilde{\mathcal{F}}_T}(s)\,ds, \tag{3.9}$$

[3]A formal proof can be found in Bielecki and Rutkowski (2002, Prop. 8.2.1).

where $f_{\tilde{\mathcal{F}}_t}(s)$ is the conditional density of τ given $\mathcal{G}_t$. f is derived with Lemma 3.7,

$$f_{\tilde{\mathcal{F}}_T}(s) = \lambda(s) \exp\Big(- \int_t^s \mu_u \, du \Big), \qquad \text{for } s \in (t, T].$$

Finally, on $\{\tau > t\}$,

$$\mathbb{E}^{\mathbb{Q}}\Big(X_\tau \mathbf{1}_{\{t<\tau\leq T\}} \big| \mathcal{G}_t \Big) = \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\Big(X_\tau \mathbf{1}_{\{t<\tau\leq T\}} \big| \tilde{\mathcal{F}}_T \Big) \big| \mathcal{G}_t \Big].$$

In the inner expectation X is measurable, such that we can apply (3.9) with X replacing h and obtain

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T X_s f_{\tilde{\mathcal{F}}_T}(s) \, ds \big| \mathcal{G}_t \Big].$$

As previously, by independence, we can replace $\mathcal{G}_t$ by $\mathcal{F}_t$. ■

3.3.2 Incomplete Information results in Shot-Noise Effects

Besides the already given motivation, in this section we look at a certain scenario which gives rise to shot-noise effects in intensities.

We consider the situation arising after the Enron accounting debacle[4] in 2001. As became clear to the investors that accounting manipulations hit the disastrous financial situation of Enron, a big trouble in credit markets arouse. Of course, investors were questioning how serious was the impact on other companies and if there had been other manipulations.

Seen from a mathematical viewpoint, investors who want to estimate the default intensity of a company, say A that might also be in difficulties face the following situation. Assume it is reasonable to consider two cases only, the more general case can be treated similarly: First, the case where the company A is also in difficulties, represented by a high default intensity μ_H and second, the case where it is not, represented by a much smaller μ_L. Denote the probability for the first case by p.

For a certain time, there is no new information to the investors except that the company did not default. If it defaults, there is no need anymore to worry about default intensity. Behaving rational, the investors would seek to determine the true default intensity by conditional expectation,

$$\mathbb{E}\big(\mu \big| \tau > t\big). \tag{3.10}$$

[4]See, e.g. http://en.wikipedia.org/wiki/Enron for a short starter.

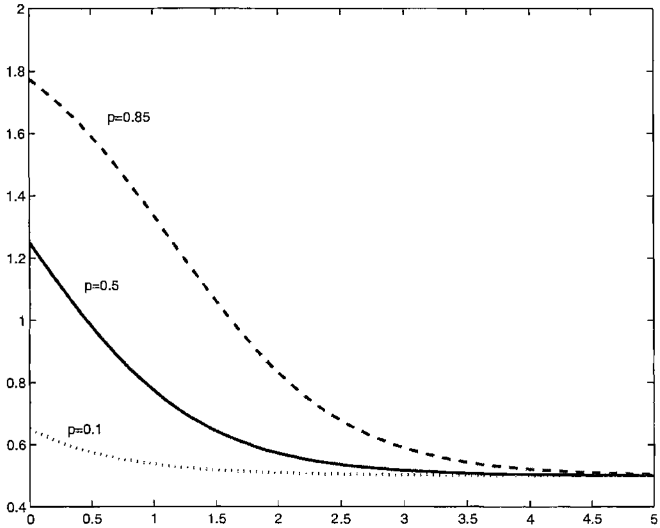


Figure 3.1: The Graph shows the conditional expectation (3.10) for several choices of p. On the x-axis we plot time in years. The values of μ_H and μ_L are 2.5 and 0.5, respectively.

The default intensity μ is the random variable which takes the values μ_H, μ_L with probability p and $1-p$, respectively.

In a first step we compute

$$\mathbb{P}\big(\mu = \mu_H | \tau > t\big) = \frac{\mathbb{P}(\mu = \mu_H, \tau > t)}{\mathbb{P}(\tau > t)} = \frac{pe^{-\mu_H t}}{pe^{-\mu_H t} + (1-p)e^{-\mu_L t}}.$$

This yields, that the conditional expectation equals

$$(3.10) = \frac{\mu_H pe^{-\mu_H t} + \mu_L(1-p)e^{-\mu_L t}}{pe^{-\mu_H t} + (1-p)e^{-\mu_L t}}.$$

We plot the expectation in Figure 3.1. The result is quite intuitive.

First, it is clear that the expectation is between μ_H and μ_L and starts at $\bar{\mu} := p\mu_H + (1-p)\mu_L$, i.e. the average if $p = 0.5$. Second, if p is big enough the graph is not descending rapidly at the beginning, because of the high probability that the riskier case is true. Otherwise the function declines rapidly and converges to μ_L for larger t.

A word of caution is now due. The above considerations refer to $\mathbb{P}$ expectations, while the whole setup of this chapter is under some equivalent martingale measure $\mathbb{Q}$.

Following the argumentation in Elliott and Madan (1998) we argue, that it is reasonable to assume that shot-noise processes under $\mathbb{P}$ should also be shot-noise processes under $\mathbb{Q}$, just with different parameters. However, this is certainly not true for all martingale measures, but at least for some. Hence, the intensity under $\mathbb{Q}$ will also be a process with shot-noise effects. This assumption necessarily corresponds to an assumption on the market prices of risk. A thorough study of this would be far beyond the scope of this chapter and will be treated elsewhere.

Remark 3.9. *In order to use the incomplete information argument as motivation for introducing shot-noise effects we implicitly assume that the market price of jump risk is such that the shot-noise behavior of the intensity holds both under the measures $\mathbb{P}$ and $\mathbb{Q}$.*

We now propose a quadratic model for the default events that includes shot-noise effects. We mainly seek for explicit expressions to all key elements. We start by presenting the setup for default. Then we compute explicitly all key building block as well as the price of some credit derivatives.

3.3.3 Default events

In this section we will propose the model which will drive the default process. As already mentioned, we will combine a quadratic model with a shot-noise process. The shot-noise process will allow the default intensity to depend on past events, especially on its severity. Moreover, recent events will influence the intensity more than the distant past.

Assumption 3.10. *Consider as given a Wiener process W, a standard Poisson process $\tilde{N}$ with intensity l, both with respect to a common filtration*[5] *and an independent exponentially distributed variable with parameter 1, E_1. Denote the jumping times of $\tilde{N}$ by $\tilde{\tau}_i, i = 1, 2, \ldots$.*

The state-variable Z is driven by W with quadratic dynamics as in (3.2)-(3.4).[6]

[5] If W and $\tilde{N}$ are a Wiener and a Poisson process w.r.t. a common filtration, they are independent. This is because $W + \tilde{N}$ then is a process with independent increments, hence a Lévy process. It is well known that the continuous and the jump parts of a Lévy process are independent, see for instance Sato (1999, Theorem 19.2).

[6] Taking the same factors Z as for the risk-free process is no loss of generality.

Define the strictly positive processes (η), (J) *and* (μ) *as follows*

$$\begin{aligned}\eta(t, Z_t) &= Z_t^\top \mathsf{Q}(t) Z_t + \mathsf{g}^\top(t) Z_t + \mathsf{f}(t) &(3.11)\\ J_t &= \sum_{\tilde{\tau}_i \le t} Y_i h(t - \tilde{\tau}_i) &(3.12)\\ \mu_t &= \eta_t + J_t &(3.13)\end{aligned}$$

where, Q, g *and* f *are deterministic and smooth functions with values in* $\mathbb{R}^{m\times m}$, $\mathbb{R}^m$ *and* $\mathbb{R}$, *respectively. Moreover,* $\mathsf{Q}(t)$ *is assumed to be symmetric for all* t. J *is called a* shot-noise process, $Y_i, i = 1, 2, \ldots$ *are i.i.d. with distribution function* F_Y *and* h *is a differentiable function on* $\mathbb{R}^+$.

Furthermore, we assume that the default time τ *is given as in (3.8) with the intensity of the form (3.13).*

The filtrations dealt with in Section 3.3.1 were rather general. In the following definition we specify precisely their meaning in the considered setup. [7]

Definition 3.11 (Filtrations). The filtration $(\mathcal{F})$ describes the accumulated information from market factors Z and J, defined by $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^J = \sigma(Z_s, J_s : 0 \le s \le t)$. Instead, $(\mathcal{H})$ represents information on the default state $\mathcal{H}_t := \sigma\left(1_{\{\tau > s\}} : 0 \le s \le t\right)$. The total information to market participants is $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$.

Recall the incomplete information argument described in Section 3.3.2. The shot-noise effect in the argument is very well captured[8] by the process (J) proposed in Assumption 3.10.The function h describes the declining, more precisely the declining from $\bar{\mu} - \mu_L$ to 0. The jump height represents the market view on $\bar{\mu}$, while in our considerations the function h is very general.[9] Figure 3.2 show us a possible realization of the process (J).

The reason to include a quadratic component in the intensity has to do with the intuition that the intensity should be driven by a predictable component (the quadratic part) as well as by an unpredictable component (the jump part).

[7]The intensity μ is adapted to the filtration $\mathcal{F}$. Given the independence between W and $\tilde{N}$ we have $\mathcal{F}^W$ and $\mathcal{F}^J$ independent on one another. So, for any process independent of J, conditioning on $\mathcal{F}$ is the same as conditioning on $\mathcal{F}_t^W$. Likewise, for any process independent of r and η (and so of W), conditioning on $\mathcal{F}$ is the same as conditioning on $\mathcal{F}^J$.

[8]Up to a market price of jump risk consideration. See Remark 3.9.

[9]As will be shown, to impose Markovianity, h needs to be of the form ae^{-bt} (see Proposition 3.15 below).

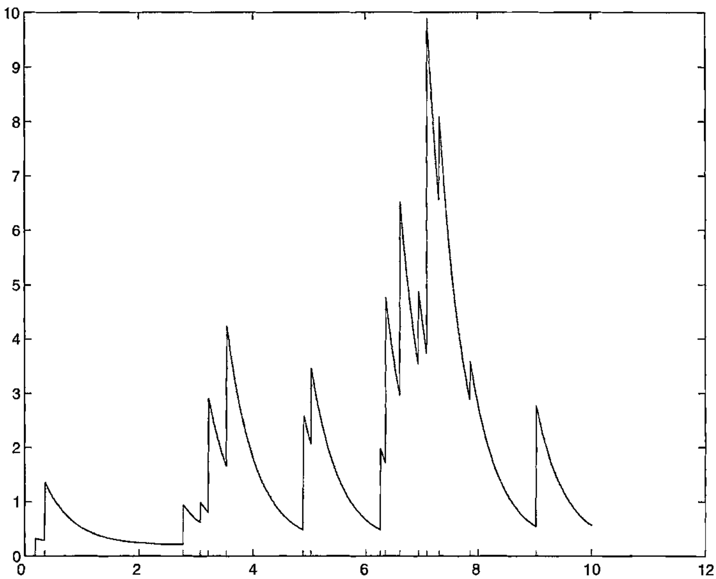


Figure 3.2: Possible realization of the process J with $h(x) = e^{-0.5x}$ and χ_2^2-distributed Y_i.

The following lemma will be essential to guarantee non-negativity of the default intensity.

Lemma 3.12. *Consider an arbitrary vector $Z \in \mathbb{R}^m$, a symmetric, nonnegative definite matrix $\mathsf{Q} \in \mathbb{R}^{m \times m}$ and $\mathsf{g} \in \mathbb{R}^m$ such that g lies in the subspace spanned by the columns of Q and $\mathsf{f} \in \mathbb{R}$. Let Z_* be the solution of $\mathsf{Q}Z = -\frac{1}{2}\mathsf{g}$. Then the polynomial of degree two*

$$Z^\top \mathsf{Q} Z + \mathsf{g}^\top Z + \mathsf{f} \tag{3.14}$$

is nonnegative, if and only if $Z_* + \mathsf{f} \geq 0$.

Proof. According to Harville (1997, Section 19.1) and letting $\mathsf{f} = 0$, Z_* is the minimum of the polynomial in Equation (3.14). Then $Z_* + \mathsf{f} \geq 0$ implies non-negativity of (3.14). ∎

If we have a linear factor, say Z^i, positivity follows if $Z^i \geq 0$ and $\mathsf{g}^i \geq 0$; or, alternatively from $Z^i \leq 0$ and $\mathsf{g}^i \leq 0$. After this, we can concentrate on the factors which have quadratic impact, which were denoted by $Z^{(q)}$. The respective part of Q and g were $\mathsf{Q}(qq)$ and $\mathsf{g}(q)$. With the aid of Lemma 3.12 we obtain the following.

Proposition 3.13. *Assume that $\mathsf{Q}^{(qq)}(t)$ is symmetric and nonnegative definite and $\mathsf{g}^{(q)}(t)$ lies in the subspace spanned by the columns of $\mathsf{Q}^{(qq)}(t)$, both for all $t \geq 0$. Denote by $Z_*(t)$ the solution of $\mathsf{Q}^{(qq)}(t)Z = -\frac{1}{2}\mathsf{g}^{(q)}(t)$. Then $\eta(t, Z_t)$ defined in (3.11) is positive, if*

1. *If Z^i is a linear factor then either $Z^i \geq 0$ and $\mathsf{g}^i \geq 0$ or $Z^i \leq 0$ and $\mathsf{g}^i \leq 0$*

2. *For all $t \geq 0$ it holds that $Z_*(t) + \mathsf{f}(t) \geq 0$.*

Using their impact on the drift α, on the volatility σ or on the funtional form of the intensity, we can provide an *intensity* classification of factors.

Definition 3.14. (Classification of intensity factors)

- Z_i is a *intensity quadratic-factor* if it satisfies *at least one* of the following requirements:
  - (i) it has a quadratic impact on (η), i.e. there exists some t such that $\mathsf{Q}_i(t) \neq 0$;
  - (ii) it has a quadratic impact on the functional form of the matrix $\sigma(t, z)\sigma^\top(t, z)$, i.e., there exist k and t such that $g_{ik}(t) \neq 0$;
  - (iii) it affects the drift term of the factors satisfying (i) or (ii), i.e., for Z_j satisfying (i) or (ii) we have $E_{ji}(t) \neq 0$ for some t.
- Z_i is a *intensity linear-factor* if it does not satisfy any of (i)-(iii).

As previously, we write in symbolic form $Z_i \in Z_\eta^{(q)}, Z_\eta^{(l)}$ for the quadratic intensity and linear intensity factors, respectively.

We use the symbolic notation $\bar{Z}^{(q)} = Z^{(q)} \cup Z_\eta^{(q)}$ and $\bar{Z}^{(l)} = Z^{(l)} \cap Z_\eta^{(l)}$, whenever the factors must be ordered according to both their impact on the risk-free short rate, r and on the quadratic part of the intensity, η.

In general, the considered shot-noise processes need not be Markovian. Anyway, from a computational point of view Markovianity is very important. There exists a clear classification, when the considered shot-noise process is Markovian or not.

Proposition 3.15. *Assume that for all $x \in [0, \infty)$ $h(x) \neq 0$. Then the process $(\mu_t)_{t\geq 0}$ is Markovian, if and only if h is of the form $h(t) = ae^{-bt}$.*

Proof. It is clear that for $b = 0$ the process is Markovian, so we need to consider the case where h is not constant.

Assume w.l.o.g. that $h(0) = 1$. As η is a Markovian process, we just have to look at J. To show that J is a Markov-process we calculate the conditional expectation. Consider $s < t$ and recall $\mathcal{F}_t^J := \sigma\{J_s : s \leq t\}$. Then

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[J_t|\mathcal{F}_s^J\right] &= \sum_{i=1}^{\tilde{N}_s} Y_i h(t-\tilde{\tau}_i) + \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=\tilde{N}_s+1}^{\tilde{N}_t} Y_i h(t-\tilde{\tau}_i)\,\Big|\,\mathcal{F}_s^J\right] \\ &= \sum_{i=1}^{\tilde{N}_s} Y_i h(t-\tilde{\tau}_i) + \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=\tilde{N}_s+1}^{\tilde{N}_t-\tilde{N}_s+\tilde{N}_s} Y_i h(t-\tilde{\tau}_i)\,\Big|\,\mathcal{F}_s^J\right]. \end{aligned} \tag{3.15}$$

In the last expectation, all terms are either measurable w.r.t. $\mathcal{F}_s^J$ or independent of $\mathcal{F}_s^J$. As the Y_i are identically distributed, we can shift the sum and obtain for the expectation

$$\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=j+1}^{\tilde{N}_t-\tilde{N}_s+j} Y_i h\big(t-\tilde{\tau}_i\big)\,\Big|\,\tilde{N}_s = j\right] = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{\tilde{N}_{t-s}} Y_i h\big(t-\tilde{\tau}_i\big)\right] =: f(s,t),$$

Hence Equation (3.15) equals

$$\sum_{i=1}^{\tilde{N}_s} Y_i h(t-\tilde{\tau}_i) + f(s,t). \tag{3.16}$$

As $f(s,t)$ is deterministic, necessary for Markovianity is that there exists a function $F(t,s,x)$, such that

$$\sum_{i=1}^{\tilde{N}_s} Y_i h(t-\tilde{\tau}_i) = F\left(t,s,J_s\right) = F\left(t,s,\sum_{i=1}^{\tilde{N}_s} Y_i h(s-\tilde{\tau}_i)\right), \tag{3.17}$$

so the first term in (3.16) can be represented as (measurable) function of J_s. We note that each Y_i is independent of all the other appearing terms. We will exploit this property to analyze the behavior of F.

Fix j and consider (3.17) on the set $\{\tilde{N}_t > j\}$. Taking the conditional expectation of (3.17) w.r.t. $Y_j = y$, we obtain

$$\mathbb{E}^{\mathbb{Q}}\Big(yh(t-\tilde{\tau}_j) + \sum_{i=1,i\neq j}^{\tilde{N}_s} Y_i h(t-\tilde{\tau}_i)\Big) = \mathbb{E}^{\mathbb{Q}}\Big(F\big(t,s,yh(s-\tilde{\tau}_j) + \sum_{i=1,i\neq j}^{\tilde{N}_s} Y_i h(s-\tilde{\tau}_i)\big)\Big).$$

Deriving w.r.t. y shows that

$$\mathbb{E}^{\mathbb{Q}}\big(h(t-\tilde{\tau}_j)\big) = \mathbb{E}^{\mathbb{Q}}\Big[F_x\Big(t,s,yh(s-\tilde{\tau}_j) + \sum_{i=1,i\neq j}^{\tilde{N}_s} Y_i h(s-\tilde{\tau}_i)\Big)h(s-\tilde{\tau}_j)\Big],$$

where we denoted the partial derivative of F w.r.t. x by F_x. As the l.h.s. does not depend on y, $F_x(t,s,x)$ must be constant in x, and we obtain that F must be of the form $\alpha(t,s) + \beta(t,s)x$.

Examining F on the set $\{\tilde{N}_t = 0\}$, we see that $\alpha(t,s)$ must necessarily be 0. In the next step we determine β. From Equation (3.17) we obtain for any i

$$h(t-\tilde{\tau}_i) = \beta(s,t)h(s-\tilde{\tau}_i).$$

Hence, $\beta(s,t) = h(t-y)/h(s-y)$ for any $y \geq 0$, and so $b(s,t) = h(t)/h(s)$. From this

$$\frac{h(t-y)}{h(s-y)} = \frac{h(t)}{h(s)}, \qquad \text{for all } t,s,y \geq 0.$$

By letting $s = y$ we obtain that $h(t-y) = h(0)h(t)/h(y)$ and so $h(t+y) = h(t)h(y)/h(0)$. We conclude $h'(y) = h'(0)h(y)/h(0)$. Therefore h is of the form ae^{-by}.

For the converse, note that for $h(y) = e^{-by}$

$$\sum_{i=1}^{\tilde{N}_t} Y_i h(t-\tilde{\tau}_i) = h(t)\sum_{i=1}^{\tilde{N}_t} Y_i h(-\tilde{\tau}_i),$$

and hence J is Markovian. ∎

3.3.4 Building Blocks

In this section we give closed-form analytical expressions to what is known as building blocks in credit risk models. We make extensive use of Theorem 3.8, and thus we ask the reader to recall the various filtrations mentioned.

Survival Probabilities

The survival probabilities under $\mathbb{Q}$ can explicitly be computed and are given in general quadratic form, which we will show in this section. First, observe that the survival probability will be denoted by $\mathbb{Q}_S$ and equals

$$
\begin{aligned}
\mathbb{Q}_S(t,T) &= \mathbb{Q}\left[\tau > T|\mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \mu_u du}\Big|\,\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\exp(-\int_t^T \eta_u + J_u\,du)\Big|\mathcal{F}_t\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\exp(-\int_t^T \eta_u\,du)\Big|\mathcal{F}_t^W\Big]\mathbb{E}^{\mathbb{Q}}\Big[\exp(-\int_t^T J_u\,du)\Big|\mathcal{F}_t^J\Big].
\end{aligned}
\tag{3.18}
$$

The first term can be computed using Result 3.5. We note that in the result (r) has to be replaced by (η). Therefore we have to assume a different reordering of factors.

Lemma 3.16. *Suppose Assumption 3.10 hold. Furthermore, assume Assumption 3.3 is verified when the factors Z are reordered as $Z = \left[Z_\eta^{(q)},\ \ Z_\eta^{(l)}\right]^\top$. Then,*

$$
\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T \eta_u\,du\Big)\Big|\mathcal{F}_t^W\Big] = \exp\Big[\mathcal{A}(t,T) + \mathcal{B}^\top(t,T)Z_t + Z_t^\top\mathcal{C}(t,T)Z_t\Big]. \tag{3.19}
$$

where $(\mathcal{A},\mathcal{B},\mathcal{C},\mathsf{f},\mathsf{g},\mathsf{Q})$ solve the basic ODE system in Definition 3.4. Recall that f, g and Q are given in Equation (3.11). Furthermore, $\mathcal{C}$ has only quadratic factors.

Next, we consider the second term in (3.18):

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\exp(-\int_t^T J_u\,du)\Big|\mathcal{F}_t^J\Big] \\
&\quad= \exp\Big(-\int_t^T \sum_{\tilde\tau_i\le t} Y_i h(u-\tilde\tau_i)\,du\Big)\mathbb{E}^{\mathbb{Q}}\Big[\exp\big(-\int_t^T \sum_{\tilde\tau_i\in(t,u]} Y_i h(u-\tilde\tau_i)\,du\big)\Big|\mathcal{F}_t^J\Big].
\end{aligned}
\tag{3.20}
$$

The first term on the l.h.s. denotes the measurable part. It depends on the history of J and it equals

$$
\begin{aligned}
\exp\Big(-\int_t^T \sum_{\tilde\tau_i\le t} Y_i h(u-\tilde\tau_i)\,du\Big) &= \exp\Big(-\sum_{\tilde\tau_i\le t} Y_i\big[H(T-\tilde\tau_i) - H(t-\tilde\tau_i)\big]\Big) \\
&= \exp\Big\{\tilde J_t - \tilde J(t,T)\Big\}
\end{aligned}
$$

where we have following notations:

$$
H(x) = \int_0^x h(u)\,du\ . \tag{3.21}
$$

and

$$\tilde{J}(t,T) = \sum_{\tilde{\tau}_i \leq t} Y_i H(T - \tilde{\tau}_i) \qquad \tilde{J}(t,t) = \tilde{J}_t. \tag{3.22}$$

Remark 3.17. *Luckily, in the Markovian case the above term simplifies considerably. By Proposition 3.15 we necessarily have that $h(x) = ae^{-bx}$ and w.l.o.g. we can assume that $a = 1$. Then,*

$$H(x) = \int_0^x h(u)\,du = \frac{1}{b}\Big(1 - e^{-bx}\Big).$$

Therefore,

$$\begin{aligned} H(T - \tilde{\tau}_i) - H(t - \tilde{\tau}_i) &= \frac{1}{b}\big[e^{-b(t-\tilde{\tau}_i)} - e^{-b(T-\tilde{\tau}_i)}\big] \\ &= \frac{1}{b}\big[e^{-b(t-\tilde{\tau}_i)} - e^{-b(t-\tilde{\tau}_i)-b(T-t)}\big] \\ &= h(t - \tilde{\tau}_i) \cdot H(T-t). \end{aligned}$$

which implies

$$\begin{aligned} \tilde{J}_t - \tilde{J}(t,T) &= -\sum_{\tilde{\tau}_i \leq t} Y_i \big[H(T - \tilde{\tau}_i) - H(t - \tilde{\tau}_i)\big]\Big] \\ &= -H(T-t) \sum_{\tilde{\tau}_i \leq t} Y_i h(t - \tau_i) = -H(T-t)\, J_t. \end{aligned} \tag{3.23}$$

Let us consider the remaining expectation (second term in (3.20)). First, recall that jumps occur with intensity l. We will use that a Poisson process has independent increments. Note that the number of jumps in $(t,u]$ is given by $\tilde{N}_u - \tilde{N}_t$, such that this term is independent of N_t. Then

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T \sum_{\tilde{\tau}_i \in (t,u]} Y_i h(u - \tilde{\tau}_i)\,du\Big)\Big|\mathcal{F}_t^J\Big] \\ &= \sum_{k=0}^{\infty} \mathbb{E}^{\mathbb{Q}}\Big[\mathbf{1}_{\{\tilde{N}_T - \tilde{N}_t = k\}} \exp\Big(-\sum_{\tilde{\tau}_i \in (t,T]} Y_i \int_t^T \mathbf{1}_{\{\tilde{\tau}_i \leq u\}} h(u - \tilde{\tau}_i)\,du\Big)\Big|\mathcal{F}_t^J\Big] \end{aligned} \tag{3.24}$$

It is well-known, that conditional on k jumps the jump times are distributed like the order statistics of uniform random variables over the interval, see for example Rolski, Schmidli, Schmidt, and Teugels (1999, p. 502). More precisely, denote by η_i, $i = 1, \ldots, k$ independent $U[0,1]$ random variables.

Define and set $x = T - t$. Then, the expectation in (3.24) equals e^{-lx} for $k = 0$ and for $k \geq 1$,

$$\begin{aligned} e^{-lx}\frac{(lx)^k}{k!}\,\mathbb{E}^{\mathbb{Q}}&\left[\exp\left(-\sum_{i=1}^{k} Y_i \int_{\tilde{\tau}_i}^{T} h(u-\tilde{\tau}_i)\,du\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=1}^{k} Y_i H(T-t-(T-t)\eta_{i:k})\right)\right]. \end{aligned}$$

As the Y_i are i.i.d. we can interchange the order of the sum. Denote by $\varphi_Y(\cdot)$ the Laplace transform of Y. Then

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=1}^{k} Y_i H\big(x(1-\eta_i)\big)\right)\right] = \left[\int_0^1 \varphi_Y\Big(H\big(xu\big)\Big)\,du\right]^k =: D(x)^k. \qquad (3.25)$$

The previous computations give the following lemma.

Lemma 3.18. *If $D(T-t)$ exists, then with $\tilde{J}$ as defined in* (3.22) *we have that*

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T J_u\,du\right)\Big|\mathcal{F}_t^J\right] = \exp\left[\tilde{J}_t - \tilde{J}(t,T) + (T-t)l\big(D(T-t)-1\big)\right]. \quad (3.26)$$

Summing up, we obtain the survival probabilities in the following form.

Proposition 3.19. *Denote by $x := T-t$ and consider $\mathcal{A}, \mathcal{B}, \mathcal{C}$ from Lemma 3.16 and $\tilde{J}$ as in (3.22). Then the survival probability on the interval (t,T), is given by*

$$\mathbb{Q}_S(t,T) = \exp\left[\tilde{J}_t - \tilde{J}(t,T) + \mathcal{A}(t,T) + xl\big(D(x)-1\big) + \mathcal{B}^\top(t,T)Z_t + Z_t^\top \mathcal{C}(t,T)Z_t\right].$$

Note that the exponent splits up into a deterministic part $\mathcal{A}(t,T) + x(lD(x) - l)$, a linear part $\mathcal{B}^\top(t,T)Z_t$ and a quadratic part $Z_t^\top\mathcal{C}(t,T)Z_t$ and the term $\tilde{J}_t - \tilde{J}(t,T)$, which is is affine in J in the Markovian case (recall (3.23)).

Defaultable bond prices with zero recovery

The price of a defaultable zero coupon bond under zero recovery, given by the risk-neutral expectation of its discounted payoff equals, on $\{\tau > t\}$,

$$\begin{aligned} \bar{p}_0(t,T) &= \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u du\Big)\mathbf{1}_{\{\tau>T\}}\Big|\mathcal{G}_t\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u + \mu_u du\Big)\Big|\mathcal{F}_t\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u + \eta_u + J_u du\Big)\Big|\mathcal{F}_t\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T J_u\Big)|\mathcal{F}_t^J\Big]\cdot\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u + \eta_u du\Big)\Big|\mathcal{F}_t^W\Big] \end{aligned}$$

The first expectation has been computed in (3.26). It remains to compute the second expectation. Once again we will use Result 3.5. To this, we need to replace r by $r+\eta$. Also, we have to consider the proper ordering.

Lemma 3.20. *Suppose Assumption 3.3 holds for Z, reordered as $Z = \left[\bar{Z}^{(q)}, \ \bar{Z}^{(l)}\right]$. Under Assumption 3.10 and, we have that*

$$\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u + \eta_u du\Big)\Big|\mathcal{F}_t\Big] = \exp\left[\bar{A}(t,T) + \bar{B}^\top(t,T)Z_t + Z_t^\top \bar{C}(t,T)Z_t\right]. \quad (3.27)$$

Here $(\bar{A}, \bar{B}, \bar{C}, f + \mathsf{f}, g + \mathsf{g}, Q + \mathsf{Q})$ solve the basic ODE in Definition 3.4. Furthermore, $\bar{C}$ has only quadratic factors.

Using the above computations we obtain the following formula in general quadratic form.

Proposition 3.21. *Denote by $x := T - t$, consider $\bar{A}, \bar{B}, \bar{C}$ from Lemma 3.20, $\tilde{J}$ from (3.22) and D from (3.25). Then, the price of a defaultable zero-coupon bond under zero recovery is*

$$\bar{p}_0(t,T) = \exp\left[\tilde{J}_t - \tilde{J}(t,T) + \bar{A}(t,T) + xl\big(D(x) - 1\big) + \bar{B}^\top(t,T)Z_t + Z_t^\top \bar{C}(t,T)Z_t\right]. \quad (3.28)$$

In particular we note that

$$\bar{p}_0(t,T) \neq p(t,T)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \mu_u du}\Big|\mathcal{F}_t\right].$$

The reason why this equation does not hold is the dependence of r and η on the same state variable Z. It is not caused by J, because J is independent of X and, thus, of these two processes.

Default digital payoffs

It is well known, that evaluating a payment at default time, typically involves computing the following expectation

$$e(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mu_T e^{-\int_t^T r_u + \mu_u du}|\mathcal{F}_t\right]$$

which can be interpreted as the price of a security which pays 1 under the assumption that default happens at time T.[10]

[10]Formally, if we denote the price of security that pays 1 unit of currency if default happens between $[T, T+\delta]$ by $e^*(t,T,T+\delta)$. Then, $e(t,T) = \lim_{\delta\to 0}\frac{1}{\delta}e^*(t,T,T+\delta)$.

Luckily, we will be able to use already computed expectations for this. It may be recalled that by φ_Y we denote the Laplace transform of Y and the functions D, H were defined in (3.25) and (3.21), respectively.

Before we actually compute $e(t,T)$ we introduce the notion of *interlinked ODE system.*

Definition 3.22 (Interlinked ODE system). Denote $\mathcal{T} = \{(t,T) \in \mathbb{R}^2 : 0 \le t \le T\}$ and consider functions a, b, c, B and C on $\mathcal{T}$ with values in $\mathbb{R}$, $\mathbb{R}^m$,$\mathbb{R}^{m\times m}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, respectively. For functions ϕ_1 and ϕ_2, ϕ_3 with values in $\mathbb{R}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, respectively, we say that $(a, b, c, B, C, \phi_1, \phi_2, \phi_3)$ solves the *interlinked ODE system* if it solves

$$\frac{\partial a}{\partial t} + d^\top(t)b + B^\top k_0(t)b + \operatorname{tr}\{ck_0(t)\} = 0 \tag{3.29}$$

$$\frac{\partial b}{\partial t} + E^\top(t)b + 2cd(t) + \frac{1}{2}\tilde{B}^\top k_0(t)b + 2ck_0(t)B + 2Ck_0(t)b = 0 \tag{3.30}$$

$$\frac{\partial c}{\partial t} + cE(t) + E^\top(t)c + 4Ck_0(t)c + \frac{1}{2}\tilde{B}^\top G(t)\tilde{b} = 0 \tag{3.31}$$

subject to the boundary conditions $a(T,T) = \phi_1(T)$, $b(T,T) = \phi_2(T)$, $c(T,T) = \phi_3(T)$. a, b, c and B, C should always be evaluated at (t,T). E, d, k_0, are the functions from (3.4) while $\tilde{B}, K \in \mathbb{R}^{m^2\times m}$ and $G \in \mathbb{R}^{m^2\times m^2}$ are as in (3.7).

Proposition 3.23. *Let x:=T-t. The term $e(t,T)$ computes to*

$$\begin{aligned} e(t,T) = \bar{p}_o(t,T) \cdot \Big\{ &\bar{a}(t,T) + \bar{b}^\top(t,T)Z_t + Z_t^\top \bar{c}(t,T)Z_t \\ &+ J(t,T) - l \cdot \Big[D(x)(1-x) - 1 + x\varphi_Y\big(H(x)\big)\Big]\Big\}, \end{aligned} \tag{3.32}$$

where [11]

$$J(t,T) := \sum_{\tilde{\tau}_i \le t} Y_i h(T - \tilde{\tau}_i), \tag{3.33}$$

and $(\bar{a}, \bar{b}, \bar{c}, \bar{B}, \bar{C}, \mathsf{f}, \mathsf{g}, \mathsf{Q})$ solve the interlinked ODE system of Definition 3.22 with $\bar{B}$,$\bar{C}$ are as in Lemma 3.20.

[11] This J notation is consistent with the use of J in (3.12), since we have $J(t,t) = J_t$.

Proof. We start by noting that

$$\begin{aligned} e(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\mu(T)e^{-\int_t^T r(u)+\mu(u)du}|\mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[(\eta(T)+J(T))\,e^{-\int_t^T r(u)+\eta(u)+J(u)du}|\mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\eta(T)e^{-\int_t^T r(u)+\eta(u)du}|\mathcal{F}_t^W\right]\underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T J(u)du}|\mathcal{F}_t^J\right]}_{II} \\ &\quad +\mathbb{E}^{\mathbb{Q}}\left[J(T)e^{-\int_t^T J(u)du}|\mathcal{F}_t^J\right]\underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)+\eta(u)du}|\mathcal{F}_t^W\right]}_{III} \end{aligned}$$

The expectations II and III have already been computed in Lemmas 3.18 and 3.20, respectively.

It remains to compute the expectations

$$IV = \mathbb{E}^{\mathbb{Q}}\left[\eta(T)e^{-\int_t^T r(u)+\eta(u)du}|\mathcal{F}_t^W\right] \qquad \text{and} \qquad V = \mathbb{E}^{\mathbb{Q}}\left[J(T)e^{-\int_t^T J(u)du}|\mathcal{F}_t^J\right].$$

From Lemma 3.54 in the appendix we know that

$$\begin{aligned} IV &= III\cdot\left(\bar{a}(t,T)+\bar{b}(t,T)^\top(t)Z_t+Z_t^\top\bar{c}(t,T)(t)Z_t\right) \\ V &= II\cdot\left\{\sum_{\tilde{\tau}_i\le t}Y_ih(T-\tilde{\tau}_i)-l\cdot\left[D(x)(1-x)-1+x\varphi_Y\big(H(x)\big)\right]\right\} \end{aligned}$$

To achieve the result, recall that $\bar{p}_o(t,T)=II\times II$ and observe that

$$\begin{aligned} e(t,T) &= IV\cdot II+III\cdot V \\ &= \bar{p}_o(t,T)\cdot\Big\{\bar{a}(t,T)+\bar{b}^\top(t,T)Z_t+Z_t^\top\bar{c}(t,T)Z_t \\ &\quad +\sum_{\tilde{\tau}_i\le t}Y_ih(T-\tilde{\tau}_i)-l\cdot\left[D(x)(1-x)-1+x\varphi_Y\big(H(x)\big)\right]\Big\}. \end{aligned}$$

∎

Remark 3.24. *In the Markovian case, $h(x)=ae^{-bx}$ and the above formula may be simplified to*

$$\sum_{\tilde{\tau}_i\le t}Y_ih(T-\tilde{\tau}_i)=\frac{h(T)}{h(t)}\sum_{\tilde{\tau}_i\le t}Y_ih(t-\tilde{\tau}_i)=\frac{h(T)}{h(t)}J_t.$$

We note that

$$e(t,T)=\mathbb{E}^{\mathbb{Q}}\left[\mu(T)e^{-\int_t^T r(u)+\mu(u)du}|\mathcal{G}_t\right]=\bar{p}_o(t,T)\bar{\mathbb{E}}^T\left[\mu(T)|\mathcal{G}_t\right]$$

thus, using (3.32), we obtain the following.

Corollary 3.25. *By $\bar{\mathbb{E}}^T$ we denote the expectation under the T-survival measure. Then*

$$\begin{aligned}\bar{\mathbb{E}}^T\Big(\mu(T)|\mathcal{F}_t\Big) &= \bar{a}(t,T) + \bar{b}^\top(t,T)Z_t + Z_t^\top\bar{c}(t,T)Z_t \\ &+ J(t,T) - l\cdot\Big[D(x)(1-x) - 1 + x\varphi_Y\big(H(x)\big)\Big].\end{aligned}$$

with $J(t,T)$ as in (3.33).

3.3.5 Incorporating positive Recovery

The expressions computed in the previous section mainly rely on the zero-recovery assumption. Of course, quantities like defaultable bonds typically have a positive recovery. In this section we will show how to extend the previous results to incorporate different recovery schemes.

We will consider two cases, recovery of treasury (RT) and recovery of market value (RMV). They differ in the interpretation of what is known as *loss quota q*. The exact meaning of q will be made clear in the descriptions below. Here we just point out that q is allowed to be some arbitrary random variable with values in $[0,1]$, as long as it is independent of everything else. [12]

Recall the definition of filtration $\mathcal{F}$ from Definition 3.11.

Definition 3.26. A *T-defaultable asset* is given by an $\mathcal{F}_T$-measurable random variable $\mathcal{X}$. At maturity T, the amount $\mathcal{X}$ is paid if no default happened until then. If a default happened before T, some recovery is paid.

Assumption 3.27. *The recovery of a T-defaultable asset $\mathcal{X}$ depends on the loss quota q, which is given by a random variable in the unit interval $[0,1]$ with distribution F_q. We assume that the loss quota q is independent of $\mathcal{G}_\infty$.*

Denote the expected value of q by $\bar{q} = \mathbb{E}^{\mathbb{Q}}[q]$.

Recovery of Treasury

In the recovery of treasury (RT) setup, the recovery of defaultable claims is expressed in terms of the market value of equivalent default-free assets. If a default happened

[12]The assumption of independence between default events and recovery has been standard in the literature. In Chapter 4 this assumption is relaxed to include realistic credit spreads features, at the cost of tractability. There, it is shown that one must rely on numerical simulations to price any defaultable asset. In this chapter we stick to the "traditional" assumption.

before maturity, the final payoff is reduced to a proportion, $(1-q)$ times the promised payoff. q is revealed at default, and the reduced payment, the recovery, is paid at maturity. It is assumed to be no more subject to default risk.

Under RT it is straightforward to price any defaultable assets based on prices of equivalent risk-free and defaultable zero-recovery assets. The equivalent risk-free asset has the same payoff as the defaultable asset, but it is not subject to default risk. The next proposition states the general pricing rule under recovery of treasury.

Proposition 3.28. *Consider a T-defaultable asset. Let $\bar{\pi}_o(t)$ be the price of this defaultable asset under zero recovery and $\pi(t)$ be the price of the equivalent risk-free asset. Assume, that the recovery is of type RT and Assumption 3.27 holds. Then, on $\{t < \tau\}$, the price of the defaultable asset at time t is given by*

$$\bar{\pi}_{RT}(t) = \bar{q}\bar{\pi}_o(t) + (1-\bar{q})\pi(t).$$

Proof. We are working on $\{\tau > t\}$. Then, by definition, RT yields

$$\bar{\pi}_{RT}(t) \;=\; \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\Big(\mathcal{X}\mathbf{1}_{\{\tau>T\}} + (1-q)\mathcal{X}\mathbf{1}_{\{\tau\leq T\}}\Big)\Big|\mathcal{G}_t\right].$$

If we condition on q we have by independence

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\Big((1-q)\mathcal{X} + q\mathcal{X}\mathbf{1}_{\{\tau>T\}}\Big)\Big|\mathcal{G}_t \vee q\right] \\
&\quad = (1-q)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\mathcal{X}\Big|\mathcal{F}_t\right] + q\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\mathcal{X}\mathbf{1}_{\{\tau>T\}}\Big|\mathcal{G}_t\right] \\
&\quad = (1-q)\pi(t) + q\bar{\pi}_o(t).
\end{aligned}$$

Besides q the above term is $\mathcal{G}_t$ measurable. As q is independent of $\mathcal{G}_\infty$, and $\pi(t)$, $\bar{\pi}_o(t)$ do not depend on q, we get

$$\bar{\pi}_{RT}(t) = \mathbb{E}^{\mathbb{Q}}[(1-q)]\pi(t) + \mathbb{E}^{\mathbb{Q}}[q]\bar{\pi}_o(t)\ ,$$

and the result follows. ∎

With this result we easily obtain zero-coupon defaultable bond prices under recovery of treasury.

Corollary 3.29. *Let $x := T-t$. Under RT, the price at time t of a zero-coupon bond maturing at T is*

$$\begin{aligned}
\bar{p}_{RT}(t,T) &= \bar{q}\exp\left[\tilde{J}_t - \tilde{J}(t,T) + xl\Big(D(x)-1\Big) + \bar{A}(t,T) + \bar{B}^\top(t,T)Z_t + Z_t^\top\bar{C}(t,T)Z_t\right] \\
&\quad + (1-\bar{q})\exp\left[A(t,T) + B^\top(t,T)Z_t + Z_t^\top C(t,T)Z_t\right].
\end{aligned}$$

where A, B, C are as in Result 3.5, $\bar{A}, \bar{B}, \bar{C}$ as in Lemma 3.20 and $\tilde{J}$ as defined in (3.22).

Recovery of market value

When we consider recovery of market value (RMV) we assume that if a default happens, then the recovery of the defaultable asset is $(1-q)$ times its pre-default value,[13]

$$(1-q)\bar{\pi}_{RMV}(\tau-). \tag{3.34}$$

The following result is a straightforward adaption to our setup of a well know result.[14]

Result 3.30. *Consider a T-defaultable asset $\mathcal{X}$ and assume that Assumption 3.27 is in force. Then the price of the defaultable asset under RMV equals*

$$\bar{\pi}_{RMV}(t) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\bar{q}\mu_s ds}\mathcal{X}|\mathcal{F}_t\right] + \mathbf{1}_{\{\tau\le t\}}e^{\int_\tau^t r_s\,ds}\,(1-q)\,\bar{\pi}_{RMV}(\tau-).$$

For a general payoff $\mathcal{X}$ there is not much more to say, but given a concrete situation more explicit formulas can be obtained. The next proposition gives the price of a defaultable zero-coupon bond under RMV in closed-form.

Proposition 3.31. *The price at time t of a zero-coupon bond maturing at T under RMV equals*

$$\begin{aligned}\bar{p}_{RMV}(t,T) &= \mathbf{1}_{\{\tau\le t\}}e^{\int_\tau^t r_s\,ds}(1-q)\,\bar{p}_{RMV}(\tau-,T)\\ &\quad+\mathbf{1}_{\{\tau>t\}}e^{\left\{\bar{J}_t-\bar{J}(t,T)+(T-t)l(D(\bar{q},T-t)-1)+\bar{A}(\bar{q},t,T)+\bar{B}^\top(\bar{q},t,T)Z_t+Z_t^\top\bar{C}(\bar{q},t,T)Z_t\right\}}\end{aligned}$$

where $(\bar{A},\bar{B},\bar{C},f+\bar{q}\,\mathrm{f},g+\bar{q}\,\mathrm{g},Q+\bar{q}\mathrm{Q})$ solves the basic ODE in 3.4 and we denote, with H from (3.21)*, and*

$$D(\bar{q},x) := \int_0^1 \varphi_Y(\bar{q}H(x(1-u)))du = \int_0^1 \varphi_Y(\bar{q}H(xu))du. \tag{3.35}$$

Proof. For a zero coupon bond price the payoff at maturity is $\mathcal{X}=1$. We apply Result 3.30 and we need to compute

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\bar{q}\mu_s ds}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\bar{q}\eta_s ds}|\mathcal{F}_t^W\right]\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T q\cdot J_s ds}|\mathcal{F}_t^J\right].$$

Note the similarities to the expectations computed in Lemmas 3.18 and 3.20. Following exactly the steps from the proofs while keeping track of the "$\bar{q}$" gives the result. Details may be found in the appendix. ■

[13]The RMV model is inspired by the recovery rules of OTC derivatives.

[14]Note that it is a consequence of Definition 3.11 that the process $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\bar{q}\mu_s ds}\mathcal{X}|\mathcal{F}_t\right]$ does not jump at τ. For the original results, compare Lando (2004) or Schönbucher (2003) to find an intuitive discretization of the result when q is assumed constant. The generalization to random q under Assumption 3.27 follows easily.

3.3.6 Pricing Credit Derivatives

In this section we price credit derivatives using the prices and key ingredients previously derived. Among others, we show how prices for credit default swaps (CDS) can be obtained. The CDS is the most liquid credit risky product, so pricing formulas are necessary for calibration to real data.

Default Digital put

We start by pricing what is known as a default digital put (DDP) with maturity T. A DDP pays off 1 exactly at default if default happens before or at T. Its value at time t (given no previous default) is

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r_u\,du}\mathbf{1}_{\{\tau<T\}}\Big|\mathcal{G}_t\Big] \\
&\quad= \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^s r_u+\mu_u\,du}\mu_s\,ds\Big|\mathcal{F}_t\Big] \\
&\quad= \int_t^T \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^s r_u+\mu_u\,du}\mu_s\Big|\mathcal{F}_t\Big]\,ds \\
&\quad= \int_t^T e(t,s)ds \\
&\quad= \int_t^T \bar{p}_o(t,s)\Big\{\bar{a}(t,s)+\bar{b}^\top(t,s)Z_t+Z_t^\top\bar{c}(t,s)Z_t \\
&\qquad\qquad + J(t,s) - l\cdot\Big[D(s-t)(1-s+t)-1+(s-t)\varphi_Y\big(H(s-t)\big)\Big]\Big\}\,ds,
\end{aligned}$$

where φ_Y is the Laplace transform of Y while a, b and c are solutions of (3.29)-(3.31), and $J(t,\cdot)$, D are defined in (3.33) and (3.25), respectively. The above integrals can easily be evaluated using the already obtained expressions of all ingredients.

Credit Default Swap

Definition 3.32. A *credit default swap* (CDS) consists of two legs, the fixed and the floating leg[15]. The fixed leg involves a regular *fee payment* and the floating leg offers a protection payment at default.

[15] The floating leg is also called the *default insurance.*

The CDS starts at some point T_0 and payments are done at the dates $T_1 < T_2 < \cdots < T_{N^*}$. At each T_n the following payments occur:

- *Fixed leg:* pays a fixed amount called *the spread,* $\bar{s}$, times the length of the interval, $\bar{s} \cdot (T_n - T_{n-1})$ if there was no default in $(T_{n-1} - T_n]$
- *Floating leg:* pays the difference between the nominal value and the recovery value if default occured in $(T_{n-1}, T_n]$. Typically the nominal value is normalized to 1 u.c. and the payment is equal to the loss quota q. Of course the loss quota is related to the recovery R by $q = (1 - R)$,

Initially, the spread $\bar{s}$ of the CDS is determined in such a way that the initial value of the CDS is zero. The spread remains fixed such that as time passes by the value of the CDS can become quite different from zero.

Typically $t = T_0$. Otherwise the CDS is called a forward-start-CDS, and the spread can be computed using similar methods. The value at time t of the fixed leg is

$$\bar{s} \sum_{n=1}^{N^*} (T_n - T_{n-1}) \bar{p}_o(t, T_n) \ .$$

To compute the floating leg, we need the value of 1 unit of money payed at T_n if default happens in $(T_{n-1}, T_n]$. This value is denoted by $e^*(t, T_{n-1}, T_n)$. Observe that e^* was not computed in the previous section, but is closely related to e as

$$e(t, T_n) = \lim_{T_{n-1} \to T_n} \frac{1}{T_n - T_{n-1}} e^*(t, T_{n-1}, T_n).$$

A basic difference to a risk-free swap appears in the above formulation: not all terms needed to compute the credit spread are liquidly traded in the market: The $\bar{p}_o(t, T_n)$ in this case. Of course, under the assumption of fixed recovery one could compute these from ordinary bond prices, but nevertheless it is not a priori clear what the right recovery assumption is.

The following proposition gives an expression in closed form.

Proposition 3.33. *We have the following*

$$e^*(t, T_{n-1}, T_n) = \bar{p}_o(t, T_{n-1}) e^{\alpha(t, T_{n-1}, T_n) + \beta^\top(t, T_{n-1}, T_n) Z_t + Z_t^\top \gamma(t, T_{n-1}, T_n) Z_t} - \bar{p}_o(t, T_n), \tag{3.36}$$

where α, β and γ are deterministic functions and solve the follwoing system of ODE

$$\begin{cases} \dfrac{\partial\alpha}{\partial t} + d^\top(t)\beta + \dfrac{1}{2}\beta^\top k_0(t)\beta + \operatorname{tr}\gamma k_0(t) + \beta^\top k_0(t)\bar{B} &= 0 \\ \alpha(T_{n-1}, T_{n-1}, T_n) &= A(T_{n-1}, T_n) \end{cases} \tag{3.37}$$

$$\begin{cases} \dfrac{\partial\beta}{\partial t} + E^\top(t)\beta + 2\gamma d(t) + \dfrac{1}{2}\tilde{\beta}^\top K(t)\beta + 2\gamma k_0(t)\beta & \\ \qquad +2\bar{C}k_0(t)\beta + 2\gamma k_0(t)\bar{B} + \tilde{\beta}^\top K(t)\bar{B} &= 0 \\ \beta(T_{n-1}, T_{n-1}, T_n) &= B(T_{n-1}, T_n) \end{cases} \tag{3.38}$$

$$\begin{cases} \dfrac{\partial\gamma}{\partial t} + \gamma E(t) + E^\top(t)\gamma + 2\gamma k_0(t)\gamma + \dfrac{1}{2}\tilde{\beta}^\top G(t)\tilde{\beta} & \\ \qquad +4\bar{C}k_0(t)\gamma + \tilde{\bar{B}}^\top G(t)\tilde{\beta} &= 0 \\ \gamma(T_{n-1}, T_{n-1}, T_n) &= C(T_{n-1}, T_n) \end{cases} \tag{3.39}$$

A, B and C are from Result 3.5, while $\bar{B}$ and $\bar{C}$ from Proposition 3.21. α, β, γ should be evaluated at (t, T_{n-1}, T_n) and $\bar{B}, \bar{C}$ at $(t, T-n-1)$.

Proof. We first note that, the expected discounted value of 1 paid at T_n if default happens in $(T_{n-1}, T_n]$ is given by

$$\begin{aligned} e^*(t, T_{n-1}, T_n) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_n} r_s ds}\left(\mathbf{1}_{\{\tau > T_{n-1}\}} - \mathbf{1}_{\{\tau > T_n\}}\right)\Big|\mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_n} r_s ds}\left(e^{-\int_t^{T_{n-1}} \mu_s ds} - e^{-\int_t^{T_n} \mu_s ds}\right)\Big|\mathcal{F}_t\right], \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_n} r_s ds} e^{-\int_t^{T_{n-1}} \mu_s ds}\Big|\mathcal{F}_t\right] - \bar{p}_o(t, T_n). \end{aligned}$$

It remains to compute the expectation. Note that

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_n} r_s ds} e^{-\int_t^{T_{n-1}} \mu_s ds}\Big|\mathcal{F}_t\right] \\ &\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_{n-1}} r_s + \mu_s ds} p(T_{n-1}, T_n)\Big|\mathcal{F}_t\right] \\ &\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_{n-1}} r_s + \eta_s ds} p(T_{n-1}, T_n)\Big|\mathcal{F}_t^W\right] \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_{n-1}} J_s ds}\Big|\mathcal{F}_t^J\right]. \end{aligned}$$

The last step is due to the independence between (J) and the other terms. The second expectation was computed in Lemma 3.18. Observe that $p(T_{n-1}, T_n)$ has the well-known form given in Result 3.5. Then, Lemma 3.53 allows us to derive the above expectation. We give the full details in the appendix, which show (3.36). ∎

For $T_{n-1} \to T_n$ we recover many well-known functions out of α, β and γ as shown in the following lemma.

Lemma 3.34. *For the triple $(\alpha, \bar{A}, a)$ we have the following relation:*

$$\lim_{\delta\to 0} \frac{\partial\alpha}{\partial\delta}(t,T,T+\delta) - \frac{\partial\bar{A}}{\partial\delta}(t,T+\delta) = a(t,T). \tag{3.40}$$

Here, α is as in Proposition 3.33, $\bar{A}$ as in Lemma 3.20 and a as in Result 3.5. The result also holds for $(\beta, \bar{B}, b)$ and $(\gamma, \bar{C}, c)$.

Proof. First, note that

$$\begin{aligned} e(t,T) &= \lim_{\delta\to 0} \frac{1}{\delta} e^*(t,T,T+\delta) \\ &= \lim_{\delta\to 0} \bar{p}_o(t,T)\left(\frac{\partial\alpha}{\partial\delta}(t,T,T+\delta) + \frac{\partial\beta}{\partial\delta}^\top(t,T,T+\delta)Z_t + Z_t^\top \frac{\partial\gamma}{\partial\delta}(t,T,T+\delta)Z_t\right) \\ &\quad - \lim_{\delta\to 0} \frac{\partial\bar{p}_o}{\partial\delta}(t,T+\delta) \end{aligned} \tag{3.41}$$

We use the representation of $\bar{p}_o(t,T+\delta)$ derived in Proposition 3.21 and obtain

$$\begin{aligned} \frac{\partial}{\partial\delta}\bar{p}_o(t,T,T+\delta) = &\Big\{ \frac{\partial}{\partial\delta}\bar{A}(t,T+\delta) + lD(T+\delta-t) + l(T+\delta-t)\frac{\partial D}{\partial\delta}(T+\delta-t) - l \\ &+ \frac{\partial}{\partial\delta}\bar{B}(t,T+\delta)Z_t - J_t + Z_t^\top\frac{\partial}{\partial\delta}\bar{C}(t,T+\delta)Z_t\Big\}\bar{p}(t,T+\delta) \end{aligned}$$

Thus,

$$\begin{aligned} (3.41) = &\bar{p}_o(t,T)\Big\{ \left(\lim_{\delta\to 0}\frac{\partial\alpha}{\partial\delta}(t,T,T+\delta)\right) + \left(\lim_{\delta\to 0}\frac{\partial\beta}{\partial\delta}(t,T,T+\delta)\right)^\top Z_t \\ &+ Z_t^\top\left(\lim_{\delta\to 0}\frac{\partial\gamma}{\partial\delta}(t,T,T+\delta)\right)Z_t - \left(\lim_{\delta\to 0}\frac{\partial\bar{A}}{\partial\delta}(t,T,T+\delta)\right) - lD(T-t) \\ &- l(T-t)\frac{\partial D}{\partial T}(T-t) + l - \left(\lim_{\delta\to 0}\frac{\partial\bar{B}}{\partial\delta}(t,T+\delta)\right)^\top Z_t + J_t \\ &- Z_t^\top\left(\lim_{\delta\to 0}\frac{\partial\bar{C}}{\partial\delta}(t,T+\delta)\right)Z_t\Big\}. \end{aligned}$$

and the result follows from $\frac{\partial D}{\partial T}(T-t) = [1-\varphi_Y(H(T-t))]$ comparing the above expression with (3.41). ∎

With the above results the value of the floating leg can be obtained in closed form:

$$q\sum_{n=1}^{N^*} e^*(t,T_{n-1},T_n).$$

Finally, the spread $\bar{s}$ that leads to equal value of both legs at time t is

$$\bar{s} = q \frac{\sum_{i=1}^{N^*} e^*(t, T_{n-1}, T_n)}{\sum_{i=1}^{N^*} (T_n - T_{n-1}) \bar{p}_o(t, T_n)}. \tag{3.42}$$

It is straightforward to generalize to random recovery, which is independent of all the other factors. Then, R simply has to be replaced by $\bar{R} = \mathbb{E}^{\mathbb{Q}}(R)$ in the above formulas.

Options on defaultable bonds

In this section we consider a put option on a zero-recovery defaultable bond. The payoff at maturity of a put option with maturity T written on a bond with maturity $T^* > T$ and with strike X is given by $\max(X - \bar{p}_o(T, T^*), 0)$. Here $\bar{p}_o(T, T^*)$ denotes the price of a zero-recovery bond, compare Proposition 3.21.

In the Markovian case we are able to deduce a quite concrete formula for European option prices. We define

$$\Delta_Z(Z_T, T, T^*) = \bar{A}(T, T^*) + (T^* - T)l[D(T^* - T) - l] + \bar{B}^\top(T, T^*)Z_T + Z_T^\top \bar{C}(T, T^*)Z_T$$

For motivation, take a put on a zero-recovery defaultable bond. The price of the put equals

$$\begin{aligned}
\mathrm{put}(t, T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}[X - \bar{p}_0(T, T^*)]\mathbf{1}_{\{\bar{p}_0(T,T^*)<X\}}\Big|\, \mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} X \mathbf{1}_{\{\bar{p}_0(T,T^*)<X\}}\Big|\, \mathcal{G}_t\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\bar{p}_0(T, T^*)\mathbf{1}_{\{\bar{p}_0(T,T^*)<X\}}\Big|\, \mathcal{G}_t\right].
\end{aligned}$$

We look carefully at the second expectation above. To this, we use the explicit form of $\bar{p}_0$ from Proposition 3.21,

$$\bar{p}_o(T, T^*) = \exp\left(\tilde{J}_T - \tilde{J}(T, T^*)\Delta_Z(Z_T, T, T^*)\right),$$

where Δ_Z is defined above and we note that it is of quadratic form. It is clear that if J is not Markovian, one has to look more closely at $\tilde{J}$ terms. For the Markovian case, however, we recall (3.23) and obtain that

$$\bar{p}_o(T, T^*) = \exp\Big(- H(T^* - T)\, J_T + \Delta_Z(Z_T, T, T^*)\Big).$$

In Appendix B we show how to determine the conditional distribution of J_T given $\mathcal{F}_t^J$, if not explicitly then by inverting the Laplace-transform. At this point of generality one can not get much further, but in concrete examples (i.e. for specific distributions of Y) it is possible to derive more detailed formulas.

For now, we denote the conditional density of J_T given $\mathcal{F}_t^J$ by $F_{J_T|J_t}$ and for every European claim with the payoff $X(Z_T, J_T)$ at T we can use independence of Z and J. Define

$$\tilde{X}(z, J_t) := \int X(z, j) F_{J_T|J_t}(dj).$$

The first step in evaluating a derivative is to compute $\tilde{X}$ on basis of J_t. With this, one can use in a second step the structure of the quadratic setup to derive the price of the derivative:

$$\mathbb{E}\Big(e^{-\int_t^T r_s\,ds} X(Z_T, J_T)\Big|\mathcal{F}_t\Big) = \mathbb{E}\Big(e^{-\int_t^T r_s\,ds} \tilde{X}(Z_T, J_t)\Big|Z_t, J_t\Big).$$

The remaining expectations can be computed numerically or using inverse Fourier–Laplace transform.[16]

We now go on with the analysis and consider several firms issuing default securities. This will allow us to address issues of portfolio credit risk.

3.4 Portfolio Credit Risk

3.4.1 Setup

To study portfolio credit risk we need to consider defaultable securities, from several firms $k = 1, \cdots, \bar{K}$ also called *names*. We denote the notional associated with each firm by M^k.

Each firm may default only once and its default time is denoted by T^k. The counting process counting *all* defaults is denoted by $N_t := \sum 1_{\{T^k \leq t\}}$. If a default of name k happens, we denote the loss quota by q^k.

We order the default times $T^1, \dots, T^{\bar{K}}$ and denote the outcome by $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_{\bar{K}}$.

[16]This technique was originally proposed by Duffie, Pan, and Singleton (2000), generalized by Heston (1993) and Leippold and Wu (2002). A clever step used in Eberlein and Raible (1999) improves the computational speed.

Furthermore, we need to know which company defaulted at τ_k, and we therefore define the *identity of the j-th default* by

$$a_j = k \qquad \text{if} \qquad \tau_j = T^k.$$

At time t we therefore know $a_1, \cdots, a_{N_t}$.

For modeling individual defaults we take a setup similar to the previous section, except that each firm's default intensity is now driven by firm specific as well as systematic risks, which are common to all firms. Assumption 3.35 formally states the new intensity form.

Assumption 3.35. *Set* $\mathbf{k} = \{1, \cdots, \bar{K}\}$. *Consider independent processes* μ^i *of the form quadratic* [17] *plus jump and identical in distribution, for* $i \in \mathbf{k} \cup \{c\}$, *i.e.,*

$$\mu_t^i = \eta_t^i + J_t^i \quad \text{and} \quad J_t^i = \sum_{\tilde{\tau}_j^i \le t} Y_j^i h^i(t - \tilde{\tau}_j^i), \qquad \eta_t^i = Z_t^\top \mathrm{Q}^i(t) Z_t + \mathrm{g}^i(t)^\top Z_t + \mathrm{f}^i(t)$$

The default intensity of each defaultable firm $k \in \mathbf{k}$ *is modeled as*[18]

$$\lambda_t^k = \mu_t^k + \epsilon^k \mu_t^c. \tag{3.43}$$

Furthermore, we assume that the risk-free short rate r *is independent of the firm specific intensity* μ^k *but not necessarily of the common intensity* μ^c.

The higher ϵ_i the bigger is the dependence of the common default risk driven by μ^c.

For intuition, take $\epsilon_i \equiv \epsilon$. Then, if μ^c jumps then suddenly the default risk of all the assets increase a lot and we will see numerous defaults. This can also be caused by a rise in the quadratic part to a high level, but then it is more or less predictable. The first effect causes some clustering similar to contagion effects, which means if one company defaults and others are closely related to this company, they are very likely to default also. The latter effect is more like a business cycle effect, so on bad days more companies default than on good days.

The formulas listed in the following remark are fundamental building blocks for the portfolio setup. They are more ore less straightforward generalizations of the results

[17]We note that to get indepence of μ^i we also need, in particular, independence of η^i. Given that we are dealing with the same Z state variables independence is achieved imposing, for a given i, that if we have $(Q^i)_j \neq 0$ or $(g^i)_j \neq 0$, then $(Q^i)_j = 0$, $(g^k)_j = 0$ for all $k \neq i$. In words, any element in Z can only appear in *one* η^i.

[18]When dealing with only one firm, as in Section 3.3, the distinction between firm specific and systematic risks becomes irrelevant. This distinction only makes sense in a portfolio context.

given in the previous sections. We give full details in Lemma 3.54 in the appendix. We also introduce a concise short hand notation for the different expressions which will be helpful in the computations to come.

Remark 3.36. *Set $x = T - t$ and $\mathbf{k} = \{1, \cdots, \bar{K}\}$. We take $\theta \in \mathbb{R}$ and $i = \mathbf{k} \cup \{c\}$. Furthermore, we introduce the short hand notation on the l.h.s:*

$$\begin{aligned}
S^i_\eta(\theta,t,T) = &\quad \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\eta^i_s ds}|\mathcal{F}^W_t\right] &= e^{\mathcal{A}^i(\theta,t,T)+\mathcal{B}^{i\top}(\theta,t,T)Z_t+Z_t^\top\mathcal{C}^i(\theta,t,T)Z_t}\\
S^i_J(\theta,t,T) = &\quad \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta J^i_s ds}|\mathcal{F}^J_t\right] &= e^{\theta(\tilde{J}_t-\tilde{J}(t,T))+l^i x[D^i(\theta,x)-1]}\\
\bar{S}^c_\eta(\theta,t,T) = &\quad \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\eta^c_s ds}|\mathcal{F}^W_t\right] &= e^{\bar{A}^c(\theta,t,T)+\bar{B}^{c\top}(\theta,t,T)Z_t+Z^\top\bar{C}^c(\theta,t,T)Z_t}
\end{aligned}$$

where $(\mathcal{A}^i, \mathcal{B}^i, \mathcal{C}^i, \theta\mathrm{Q}^i, \theta\mathrm{g}^i, \theta\mathrm{f}^i)$, $(\bar{A}^c, \bar{B}^c, \bar{C}^c, Q+\theta\mathrm{Q}^c, g+\theta\mathrm{g}^c, f+\theta\mathrm{f}^c)$ solve the basic ODE system of Definition 3.4, $D^i(\theta,x) = \int_0^1 \varphi[\theta H^i(x(1-u))]du$, $H^i(x) = \int_0^x h^i(u)du$, and $\tilde{J}^i$ is defined similarly to (3.22) (using H^i and Y^i).

$$\begin{aligned}
\Gamma^i_\eta(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta^i_T e^{-\int_t^T \theta\eta^i_s ds}|\mathcal{F}^W_t\right]\\
&= S^i_\eta(\theta,t,T)\cdot\left(a^i(\theta,t,T)+b^{i\top}(\theta,t,T)Z_t+Z_t^\top c^i(\theta,t,T)Z_t\right)\\
\Gamma^i_J(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta J^i_T e^{-\int_t^T \theta J^i_s ds}|\mathcal{F}^J_t\right]\\
&= S^i_J(\theta,t,T)\cdot\left[\theta J^i(t,T) - l^i\cdot\left(D^i(\theta,x)(1-x)-1+x\varphi^i\big(\theta H^i(x)\big)\right)\right]\\
\bar{\Gamma}^c(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta^c_T e^{-\int_t^T r_s+\theta\eta^c_s ds}|\mathcal{F}^W_t\right]\\
&= \bar{S}^c_\eta(\theta,t,T)\cdot\left(\bar{a}^c(\theta,t,T)+\bar{b}^{c\top}(\theta,t,T)Z_t+Z_t^\top\bar{c}^c(\theta,t,T)Z_t\right)
\end{aligned}$$

where $(a^i,\ b^i,\ c^i,\ \mathcal{B}^i,\ \mathcal{C}^i,\ \theta\mathrm{f}^i,\ \theta\mathrm{g}^i,\ \theta\mathrm{Q}^i)$, $(\bar{a}^c,\ \bar{b}^c, \bar{c}^c,\ \bar{B}^c,\ \bar{C}^c,\ \theta\mathrm{f}^c,\ \theta\mathrm{g}^c,\ \theta\mathrm{Q}^c)$ solve the interlinked system of Definition 3.22 and $J^i(t,T)$ is defined similarly to (3.33) (using h^i and Y^i).

Furthermore, we have

$$\begin{aligned}
S^i(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\mu^i_s ds}|\mathcal{F}^W_t\right] = S^i_\eta(\theta,t,T)\cdot S^i_J(\theta,t,T)\\
\bar{S}^c(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\mu^c_s ds}|\mathcal{F}^W_t\right] = \bar{S}^c_\eta(\theta,t,T)\cdot S^c_J(\theta,t,T)\\
\Gamma^i(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\mu^i e^{-\int_t^T \theta\mu^i_s ds}|\mathcal{F}^W_t\right] = \Gamma^i_\eta(\theta,t,T)S^i_J(\theta,t,T)+\Gamma^i_J(\theta,t,T)S^i_\eta(\theta,t,T)\\
\bar{\Gamma}^c(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\mu^c e^{-\int_t^T r_s+\theta\mu^c_s ds}|\mathcal{F}^W_t\right] = \bar{\Gamma}^c_\eta(\theta,t,T)S^c_J(\theta,t,T)+\Gamma^c_J(\theta,t,T)\bar{S}^c_\eta(\theta,t,T).
\end{aligned}$$

Finally, for $\theta = 1$ we use (t,T) instead of $(1,t,T)$ on the l.h.s. notation.

We keep the notation from the previous section but we have to add a superscript "$\cdot^k$" to be able to distinguish across firms. This way, $\mathbb{Q}_S^k$, denotes the survival probability of firm k, $\bar{p}^k(t,T)$ is the price of a T-defaultable bond issued by firm k, $e^k(t,T)$ can be interpreted as the price of a payoff of 1 u.c if the firm k defaults at T, while $e^{*k}(e,T_{n-1},T_n)$ is the price if you get 1 u.c. paid if the firm k defaults in $(T_{n-1},T_n]$.

In the next Lemma we derive the key building blocks using the new intensity (3.43).

Lemma 3.37. *Given Assumption 3.35 we have the following closed form solutions:*

$$\begin{aligned}
\mathbb{Q}_S^k(t,T) &= S^k(t,T)\cdot S^c(\epsilon^k,t,T)\\
\bar{p}_o^k(t,T) &= S^k(t,T)\cdot \bar{S}^c(\epsilon^k,t,T)\\
e^k(t,T) &= \Gamma^k(t,T)\cdot \bar{S}^c(\epsilon^k,t,T)+\bar{\Gamma}^c(\epsilon^k,t,T)\cdot S^k(t,T)\\
e^{*k}(t,T_{n-1},T_n) &= e^{\alpha^k(t,T_{n-1},T_n)+\beta^{k\top}(t,T_{n-1},T_n)Z_t+Z_t^\top\gamma^k(t,T_{n-1},T_n)Z_t}\cdot\bar{p}_o^k(t,T_{n-1})-\bar{p}_o^k(t,T_n)
\end{aligned}$$

where all the $S^\cdot$ and $\Gamma^\cdot$ are as in Remark 3.36 and α,β,γ are as in Proposition 3.33.

Proof. All results follow from the independence of μ^k and μ^c. Concretely, for $\mathbb{Q}_S^k(t,T)$ and $e^k(t,T)$ we have,

$$\begin{aligned}
\mathbb{Q}_S^k(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\lambda_s^k ds}\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\mu_s^k ds}\cdot e^{-\int_t^T\epsilon^k\mu_s^c ds}\middle|\mathcal{F}_t\right].
\end{aligned}$$

As μ^k and μ^c are independent we immediately obtain $\mathbb{Q}_S^k(t,T)=S^k(t,T)\cdot S^c(\epsilon^k,t,T)$. Similarly,

$$\begin{aligned}
e^k(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\lambda_T^k e^{-\int_t^T r_s+\lambda_s^k ds}\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\mu_T^k e^{-\int_t^T\mu_s^k ds}\cdot e^{-\int_t^T(r_s+\epsilon^k\mu_s^c)ds}\middle|\mathcal{F}_t\right]+\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\mu_s^k ds}\cdot\epsilon^k\mu_T^c e^{-\int_t^T(r_s+\epsilon^k\mu_s^c)ds}\middle|\mathcal{F}_t\right]\\
&= \Gamma^k(t,T)\cdot\bar{S}^c(\epsilon^k,t,T)+S^k(t,T)\cdot\bar{\Gamma}^c(\epsilon^k,t,T).
\end{aligned}$$

The same type of argument can be used to compute $p_o^k(t,T)$ and $e^{*k}(t,T_{n-1},T_n)$. ∎

3.4.2 Default correlation and Clustering

It is often argued that in the framework used here, where the default times are conditionally independent, the resulting default correlation is nothigh enough. However, already Duffie and Gârleanu (2001) showed that this is not the case. Especially through jumps or, more precisely, high peaks in the intensity a high default correlation is induced.

A problem showing up in the jump-diffusion setting of Duffie and Gârleanu (2001) is the right choice of the mean reversion speed which affects both the diffusion and the jump part.[19] In their model, big jumps are necessary to induce high default correlation. To avoid that the intensity stays on a very high level for a long time, the mean reversion speed must be quite high. On the other hand, such a high mean reversion speed gives unrealistic behavior for the diffusive part. In the framework presented here, this problem is solved, as the mean reversion speeds can be different.

The so-called default correlation is basically the correlation between the default indicators of two companies. Denote by $\mathbb{Q}_D^i$ the probability of company i defaulting in $(t,T]$ and by $\mathbb{Q}_D^{i,j}(t,T)$ the probability that companies i and j default in $(t,T]$. The default correlation is defined as

$$\rho^{i,j}(t,T) = \frac{\mathbb{Q}_D^{i,j}(t,T) - \mathbb{Q}_D^i(t,T)\mathbb{Q}_D^j(t,T)}{\sqrt{\mathbb{Q}_D^i(t,T)[1-\mathbb{Q}_D^i(t,T)]\mathbb{Q}_D^j(t,T)[1-\mathbb{Q}_D^j(t,T)]}}.$$

The default probabilities relate to the survival probabilities by $\mathbb{Q}_D^k(t,T) = 1 - \mathbb{Q}_S^k(t,T)$ where $\mathbb{Q}_S$ is given in Lemma 3.37.

Proposition 3.38. *Suppose Assumption 3.35 holds. Then, the default correlation of two different companies i and j is given by*

$$\rho^{i,j}(t,T) = \frac{S^i(t,T)S^j(t,T)\left[S^c(\epsilon^i+\epsilon^j,t,T) - S^c(\epsilon^i,t,T)S^c(\epsilon^j,t,T)\right]}{\sqrt{\mathbb{Q}_D^i(t,T)[1-\mathbb{Q}_D^i(t,T)]\mathbb{Q}_D^j(t,T)[1-\mathbb{Q}_D^j(t,T)]}}.$$

where $S^{\cdot}$ are as in Remark 3.36 and we recall $\mathbb{Q}_D^{\cdot} = 1 - \mathbb{Q}_S^{\cdot}$.

Proof. The probability of joint default of the firms i,j until time T given that none

[19] In their work the intensity is of the form

$$d\mu_t = \kappa\big(\theta - \mu_t\big)\,dt + \sigma\sqrt{\mu_t}dW_t + dJ_t,$$

where (W) is a Brownian motion and (J) is a pure jump process and thus a special case of a shot-noise process.

With this formulation the authors obtain bond prices in an affine form. A problem of this approach is to adjust κ in the right way. This is because κ controls the mean reversion speed of the diffusive as well as of the jump part.

has defaulted until t is given by

$$\begin{aligned}
\mathbb{Q}_D^{i,j}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T^i<T\}}\mathbf{1}_{\{T^j<T\}}\middle|\mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(1-e^{-\int_t^T \lambda_s^i ds}\right)\left(1-e^{-\int_t^T \lambda_s^j ds}\right)\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(1-e^{-\int_t^T \mu_s^i+\epsilon^i\mu_s^c ds}\right)\left(1-e^{-\int_t^T \mu_s^j+\epsilon^j\mu_s^c ds}\right)\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[1-e^{-\int_t^T \mu_s^i+\epsilon^i\mu_s^c ds}-e^{-\int_t^T \mu_s^j+\epsilon^j\mu_s^c ds}+e^{-\int_t^T \mu_s^i+\mu_s^j+(\epsilon^i+\epsilon^j)\mu_s^c ds}\middle|\mathcal{F}_t\right] \\
&= 1-\mathbb{Q}_S^i(t,T)-\mathbb{Q}_S^j(t,T)+\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \mu_s^i+\mu_s^j+(\epsilon^i+\epsilon^j)\mu_s^c ds}\middle|\mathcal{F}_t\right].
\end{aligned}$$

Again using independence of μ^i, μ^j and μ^c we obtain

$$\mathbb{Q}_D^{i,j}(t,T) = 1-\mathbb{Q}_S^i(t,T)-\mathbb{Q}_S^j(t,T)+S^i(t,T)S^j(t,T)S^c(\epsilon^i+\epsilon^j,t,T).$$

By the definition of $\rho^{i,j}$ the result follows from

$$\begin{aligned}
\mathbb{Q}_D^i(t,T)\mathbb{Q}_D^j(t,T) &= (1-\mathbb{Q}_S^i(t,T))(1-\mathbb{Q}_S^j(t,T)) \\
&= 1-\mathbb{Q}_S^i(t,T)-\mathbb{Q}_S^j(t,T)+\mathbb{Q}_S^i(t,T)\mathbb{Q}_S^j(t,T) \\
&= 1-\mathbb{Q}_S^i(t,T)-\mathbb{Q}_S^j(t,T)+S^i(t,T)S^c(\epsilon^i,t,T)S^j(t,T)S^c(\epsilon^j,t,T). \blacksquare
\end{aligned}$$

In Section 3.5 we simulate correlations and show that with our model we can get quite high correlation across firms. The high correlation also produces clustering of defaults, which looks like contagion.

3.4.3 Portfolio Credit Derivatives

To obtain concrete formulas, we will make a simplifying assumption about homogeneity of the portfolio. Assumptions like this are quite usual in the literature on CDOs. However, it should be pointed out that the following calculations go through in a similar fashion without these assumptions, but the expressions will get more involved. Nonetheless, typically, portfolio credit derivatives base on a portfolio of homogeneous credits, therefore the following assumption is quite plausible for practical purposes.

Assumption 3.39. *We consider a portfolio of homogeneous credits, i.e. the notionals are equal, $M^i = M$, the recoveries are assumed to be time-independent and i.i.d. $q^{a_j}(\tau_j) = q_j$ and the correlation factors are the same, $\epsilon^i = \epsilon$. Moreover, we assume that Assumption 3.35 holds, and that the processes J^k for all $k = 1\cdots\bar{K}$ therein are equivalent, i.e. they are based on the same set of parameters F_Y, h, l. On the other hand, J^c has the parameters F_Y^c, h^c, l^c.*

First-to-Default Swaps

A *first-to-default swap* (FtDS) is a contract which offers protection on the first default of a portfolio only. The two counterparties which exchange payments are named protection seller and protection buyer. Payments are due at fixed payment dates, say $t_1, \cdots, t_{N^*}$. Moreover, there is an initiation date $t_0 < t_1$. If t_0 is in the future, the FtDS is called *forward-start FtDS.* The FtDS is characterized by the so-called *first-to-default spread* s^{FtD} which is fixed at initiation of the contract.

- The *protection seller* pays at t_n, if the first default occurred in $(t_{n-1}, t_n]$ the *default payment.* Assume that name k is the one which defaulted first. Then the default payment equals $M^k \cdot q^k$. If no default happens until t_{N^*} the protection seller pays nothing.

- The *protection buyer* pays the spread s^{FtD}, until the maturity of the FtDS, t_{N^*}, or until the first default (whichever comes first).

As the protection buyer has only fixed payments, the payments due to him are also called *fixed leg*, while the payments of the protection seller are called *floating leg*.

The spread $\bar{s}^{\text{FtD}}$ is chosen in such a way that at initiation of the FtDS its value at t_0 equals zero. Note that there are no payments until t_1. If a default happens before t_0, the contract is worthless. To emphasize the dependence of the spread on the current time write $\bar{s}^{\text{FtD}}(t)$.

The following results rely on the distribution of the first default time, which is the minimum of all default times. The main result is Theorem 3.41

We will make use of Assumption 3.39 to ease exposition. Denote the probability that the first default, τ_1, occurs in $(t, T]$ by $\mathbb{Q}_S^{\text{FtD}}(t, T)$. The next lemma deals with properties of the first default time. Recall from Assumption 3.39 that ϵ the sensitivity of each intensity λ^i to the common part μ^c.

Lemma 3.40. *Suppose that Assumption 3.39 is in force. Consider a portfolio of $\bar{K}$ names and assume no default has occurred up to time t. Then, the survival probability of the first default is given by*

$$\mathbb{Q}_S^{FtD}(t, T) = 1_{\{\tau_1 > t\}}\, S^c(\epsilon \bar{K}, t, T) \cdot \prod_{k=1}^{\bar{K}} S^k(t, T).$$

Furthermore, the value of one unit of currency paid at T only if $\tau_1 > T$ is given by

$$\bar{p}^{FtD}(t,T) = \mathbf{1}_{\{\tau_1 > t\}}\, \bar{S}^c(\epsilon\bar{K}, t, T) \cdot \prod_{k=1}^{\bar{K}} S^k(t,T). \tag{3.44}$$

Proof. The result is trivial on $\{\tau_1 \le t\}$, so we consider $\{\tau_1 > t\}$ from now on. Then, by definition $\mathbb{Q}_S^{\mathrm{FtD}}(t,T) = \mathbb{Q}(\tau_1 > T|\mathcal{G}_t)$. We start by conditioning on $\mu^c_{[t,T]} \vee \mathcal{G}_t$. Recall that the default time of name k is denoted by T_k. Then,

$$\begin{aligned}\mathbb{Q}\left(\tau_1 > T\middle|\mu^c_{[t,T]} \vee \mathcal{G}_t\right) &= \mathbb{Q}\left(\min(T_1, T_2, \cdots, T_{\bar{K}}) > T\middle|\mu^c_{[t,T]} \vee \mathcal{G}_t\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T_1 > T, T_2 > T, \cdots, T_{\bar{K}} > T\}}\middle|\mu^c_{[t,T]} \vee \mathcal{G}_t)\right].\end{aligned} \tag{3.45}$$

As $T_1, \ldots, T_{\bar{K}}$ are independent, conditionally on $\mu^c_{[t,T]}$, we obtain

$$\begin{aligned}(3.45) &= \mathbb{E}^{\mathbb{Q}}\Bigg(\exp\Big[-\sum_{k=1}^{\bar{K}}\int_t^T \lambda_s^k ds\Big]\Big|\mu^c_{[t,T]} \vee \mathcal{F}_t\Bigg) \\ &= e^{-\bar{K}\epsilon\int_t^T \mu_s^c ds} \cdot \mathbb{E}^{\mathbb{Q}}\Bigg(\exp\Big[-\sum_{k=1}^{\bar{K}}\int_t^T (J_s^k + \eta_s^k) ds\Big]\Big|\mu^c_{[t,T]} \vee \mathcal{F}_t\Bigg)\end{aligned}$$

As $\eta^1, \ldots, \eta^{\bar{K}}, J^1, \ldots, J^{\bar{K}}$ are mutually independent, using the expressions given in Remark 3.36, we obtain

$$(3.45) = e^{-\epsilon\bar{K}\int_t^T \mu_s^c ds} \cdot \prod_{k=1}^{\bar{K}} S^k(t,T).$$

It may be recalled that $S^k = S_\eta^k S_J^k$. Thus,

$$\begin{aligned}\mathbb{Q}(\tau_1 > T|\mathcal{G}_t) &= \mathbb{E}^{\mathbb{Q}}\Bigg(e^{-\epsilon\bar{K}\int_t^T \mu_s^c ds} \cdot \prod_{k=1}^{\bar{K}} S^k(t,T)\Big|\mathcal{F}_t\Bigg) \\ &= S^c(\epsilon\bar{K}, t, T) \cdot \prod_{k=1}^{\bar{K}} S^k(t,T).\end{aligned}$$

Using the same methodology with the fact that r is independent of μ^k for all $k \in \mathbf{k}$ but not of μ^c determines $\bar{p}^{\mathrm{FtD}}(t,T)$. ∎

The spread of the FtDS is given by the following result. Recall that $\bar{p}^{\mathrm{FtD}}(t,T)$ was computed in (3.44).

Theorem 3.41. *Suppose Assumption 3.39 is in force. Consider a portfolio of $\bar{K}$ names and assume no default has occurred up to time t. Then, the spread of the FtDS is given by*

$$\bar{s}^{FtD}(t) = \bar{q}\frac{\sum_{n=1}^{N^*} e^{FtD*}(t, t_{n-1}, t_n)}{\sum_{n=1}^{N^*}(t_n - t_{n-1})\bar{p}^{FtD}(t, t_n)}$$

where

$$e^{FtD*}(t, t_{n-1}, t_n) = e^{\alpha^c(t,t_{n-1},t_n)+\beta^{c\top}(t,t_{n-1},t_n)Z_t+Z_t^\top\gamma^c(t,t_{n-1},t_n)Z_t} \cdot \bar{p}^{FtD}(t, t_{n-1}) - \bar{p}^{FtD}(t, t_n). \tag{3.46}$$

Here α, β, γ solve the system in (3.37)-(3.39). Furthermore, there α, β, γ must be evaluated at (t, T_{n-1}, T_n) while $\bar{B}, \bar{C}$ must be evaluated at $(\epsilon\bar{K}, t, t_{n-1})$.

Proof. For ease of notation we write s^{FtD} instead of $s^{\mathrm{FtD}}(t)$. The value at time t of the fixed leg of the FtDS follows from the results in the previous lemma:

$$\mathbb{E}^{\mathbb{Q}}\Big[\sum_{n=1}^{N^*} e^{-\int_t^{t_n} r_s\,ds}\, s^{\mathrm{FtD}}\, \mathbf{1}_{\{\tau>t_n\}}\big|\mathcal{G}_t\Big] = s^{\mathrm{FtD}}\sum_{n=1}^{N^*}(t_n - t_{n-1})\cdot \bar{p}^{FtD}(t, T_n).$$

For the pricing of the floating leg we need to compute

$$\begin{aligned} e^{\mathrm{FtD}*}(t, T_{n-1}, T_n) &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau_1\in(t_n,t_{n-1}]\}}\Big|\,\mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau_1>t_{n-1}\}}\Big|\,\mathcal{G}_t\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau_1>t_n\}}\Big|\,\mathcal{G}_t\right], \end{aligned}$$

where the second expectation equals $\bar{p}^{\mathrm{FtD}}(t, t_n)$. Furthermore,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau_1>t_{n-1}\}}\Big|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[p(t_{n-1}, t_n)\cdot e^{-\int_t^{t_{n-1}} r_s ds}\mathbf{1}_{\{\tau_1>t_{n-1}\}}\Big|\,\mathcal{F}_t\right]. \tag{3.47}$$

Following the steps from the previous lemma we can deduce the following. Alternatively, in the conditionally independent approach the default intensity of the minimum of the default times is simply the sum over all intensities. However, we get the following

$$(3.47) = \mathbb{E}^{\mathbb{Q}}\left[p(t_{n-1}, t_n)\cdot e^{-\int_t^{t_{n-1}}\left[r_s+\epsilon\bar{K}\mu_s^c+\sum_{k=1}^{\bar{K}}\mu_s^k\right]ds}\Big|\mathcal{F}_t\right]$$

We write short $\tilde{\mathcal{F}}_{t,t_{N^*}}$ for $\mathcal{F}_t \vee \sigma(\mu_s^c, r_s : t \le s \le t_{N^*})$. Conditioning on $\tilde{\mathcal{F}}$ we obtain

$$(3.47) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\Big(e^{-\sum_{k=1}^{\bar{K}}\int_t^{t_{n-1}}\mu_s^k ds}\big|\tilde{\mathcal{F}}_{t,t_{N^*}}\Big)\cdot p(t_{n-1}, t_n)e^{-\int_t^{t_{n-1}}\left(r_s+\epsilon\bar{K}\mu_s^c\right)ds}\Big|\mathcal{F}_t\right].$$

Let us consider the inner expectation more closely. By Assumption 3.27 we have that $\eta^1, \dots, \eta^{\bar{K}}$ are independent of μ^c and r, so that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\Big(e^{-\sum_{k=1}^{\bar{K}}\int_t^{t_{n-1}}\mu_s^k ds}\big|\tilde{\mathcal{F}}_{t,t_{N^*}}\Big) &= \mathbb{E}^{\mathbb{Q}}\Big(e^{-\sum_{k=1}^{\bar{K}}\int_t^{t_{n-1}}\eta_s^k ds}\big|\mathcal{F}_t\Big)\cdot\mathbb{E}^{\mathbb{Q}}\Big(e^{-\int_t^{t_{n-1}}\sum_{k=1}^{\bar{K}}J_s^k ds}\big|\mathcal{F}_t\Big) \\ &= \prod_{k=1}^{\bar{K}} S^k(t, t_{n-1}). \end{aligned}$$

It may be recalled that $S^k = S^k_\eta \cdot S^k_J$. To evaluate (3.47) we can proceed exactly as in Proposition 3.33. In analogy to Proposition 3.33 we obtain that

$$\mathbb{E}^{\mathbb{Q}}\Big[p(t_{n-1},t_n)e^{-\int_t^{t_{n-1}}\left(r_s+\epsilon\bar{K}\eta^c_s\right)ds}\Big|\mathcal{F}_t\Big] = \bar{S}^c_\eta(\epsilon\bar{K},t,t_{n-1})\cdot$$
$$\cdot\exp\Big(\alpha^c(t,t_{n-1},t_n)+\beta^{c\top}(t,t_{n-1},t_n)Z_t+Z_t^\top\gamma^c(t,t_{n-1},t_n)Z_t\Big).$$

The remaining part with J^c is given by S^c_J such that by $\bar{S}^c = \bar{S}^c_\eta \cdot S^c_J$ expression (3.47) equals

$$e^{\alpha^c(t,t_n,t_{n-1})+\beta^{c\top}(t,t_n,t_{n-1})Z_t+Z_t^\top\gamma^c(t,t_n,t_{n-1})Z_t}\cdot\underbrace{\bar{S}^c(\epsilon\bar{K},t,t_{n-1})\cdot\prod_{k=1}^{\bar{K}}S^k(t,t_{n-1})}_{\bar{p}^{\mathrm{FtD}}(t,t_{n-1})},$$

where α,β,γ are as stated in the theorem. ■

CDOs

We introduce the concept of Collateralized Debt Obligations (CDOs) as in Duffie and Gârleanu (2001). The aim of this section is to price the so-called *synthetic CDOs.*

A synthetic CDO is an asset-backed security whose underlying collateral is a portfolio of CDSs. A CDO allocates interest income and principal repayments from a collateral pool of CDSs to a prioritized collection of CDO securities, called *tranches.*

While there are many variations, a standard prioritization scheme is simple subordination: *senior* CDO notes are paid before *mezzanine* and *equity pice* is paid with the any residual cash-flow.

In addition to the general portfolio setup introduced in Section 3.4.1 we need to introduce some additional notation to describe the cash-flow of CDOs.

We consider a CDO with several tranches $i = 1,\cdots,\bar{I}$. In the case were we have senior, mezzanine and equity tranches only we would simply take $\bar{I} = 3$. The tranches are separated according to fixed barriers b_i. That is, b_1 separates tranche 1 from tranche 2, b_2 separated the tranche 2 from tranche 3, and so on - compare figure 3.3.

The loss at each default time τ_j is generally given by $M^{a_j}q^{a_j}(\tau_j)$.[20] However, under the homogeneity Assumption 3.39, it simplifies to $\xi_j := Mq_j$.

[20]We note that the loss at each default time would depend on the notional amount of the defaulted firm and on the recovery process of that firm evaluated at the default time.

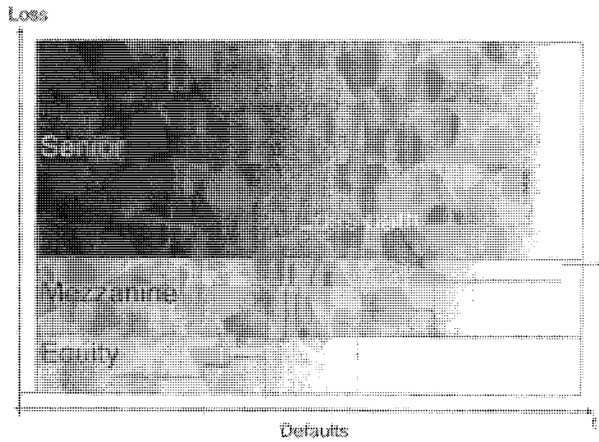


Figure 3.3: Tranches' losses in CDOs.

The *loss process* of the CDO is given by

$$L(t) := \sum_{j=1}^{N_t} \xi_j.$$

It describes the reduction in face value of the whole underlying portfolio due to according defaults. The *loss of tranche* i is given by

$$L^i(t) = \begin{cases} 0 & \text{if } \; L(t) < b^{i-1} \\ L(t) - b^{i-1} & \text{if } \; b^{i-1} \leq L(t) < b^i \\ b^i - b^{i-1} & \text{if } \; L(t) \geq b^i \end{cases} \tag{3.48}$$

Figure 3.3 illustrates the CDO setup with a possible loss path affecting various tranches.

We start by computing the *distribution of portfolios losses* under both the martingale measure and the T- forward measure. This will serve as building block for the pricing of CDOs.

In the following we will need the distribution of sums of losses, for which we define

$$\bar{F}_{q,k}(y) := \mathbb{Q}\Big(\sum_{j=1}^{k} q_j > y \Big). \tag{3.49}$$

Depending on the distribution of the losses, this could be in more ore less closed form. At this point we stick with this abbreviation.

Remark 3.42. *It is somehow a natural choice to model q with a beta-distribution. The beta-distribution is a flexible class of distributions which have support $[0,1]$. Unfortunately, the convolution is not given in closed form. However, $F_{q,k}$ can be obtained via inverting the Fourier-transform. There exist numerical algorithms to do this quite efficiently. Note also, that this has to be computed once and therefore does not affect speed of the valuation algorithm.*

Alternatively, one could use the uniform distribution, and obtain a closed form solution.

Given our setup we can always conclude for the unconditional distribution of the loss function L.

Unfortunately, for pricing and risk management it is necessary to consider L after some time passed by, and we therefore will be interested also in the *conditional* distribution of the loss function. In the general case there it is not possible to derive closed-form solutions.

Nontheless, we can require the processes $(\lambda_t^k)_{t\geq 0}$, $k = 1, \ldots, \bar{K}$ to be Markovian. We recall that this is equivalent to $h(t) = a\exp(-bt)$ by Proposition 3.15. Then and interesting question is if, using the Markovian property, we can conclude for the conditional distribution of L. Lemma 3.43 gives the answer.

Before, however, to be able to handle defaulted and non-defaulted companies in a concise way, we need to introduce some more notation.

Denote by $\mathcal{S}_t$ the set which contains the indices of assets not defaulted until t, the "survivors":

$$\mathcal{S}_t := \{1 \leq k \leq \bar{K} : T^k > t\}.$$

In the following Lemma we will fix the number of defaults in the interval $(t, T]$ and then sum over all possible combinations defaults.

We write $\sum\limits_{\mathbf{k}_n \in \mathcal{S}_t}$ for the sum over all sets $\mathbf{k}_n = \{k_1, \ldots, k_n\}$ of size n with pairwise different elements and $k_1, \ldots, k_n \in \mathcal{S}_t$. $\mathbf{k}_n$ represents the n companies which default in $(t, T]$.

Given $\mathbf{k}_n$, the companies *not* defaulting are denoted by

$$\mathcal{S}_t \backslash \mathbf{k}_n := \{1 \leq l \leq n : l \in \mathcal{S}_t, l \notin \mathbf{k}_n\}.$$

Furthermore, we write short $\{T^{\mathbf{k}_n} \in (t, T]\}$ for $\{T^{k_1} \in (t, T], \ldots, T^{k_n} \in (t, T]\}$.

Lemma 3.43. *Suppose the function $h(x)$ and $h^c(x)$ in Assumption 3.39 are of the form ae^{-bx}. Then the conditional distribution of the portfolio losses, L, is given by*

$$\mathbb{Q}(L_T \le x|\mathcal{G}_t) = \mathbf{1}_{\{T^{\mathcal{S}_t} > t\}} \sum_{n=0}^{\bar{K}-N_t} \cdot F_{q,n}\Big(\frac{x - L_t}{M}\Big) \times$$
$$\times \sum_{\mathbf{k}_n \in \mathcal{S}_t} \Big\{ S^c(\epsilon(\bar{K} - N_t - n), t, T) \prod_{k \in \mathcal{S}_t \setminus \mathbf{k}_n} S^k(t,T) - S^c(\epsilon(\bar{K} - N_t), t, T) \prod_{k \in \mathcal{S}_t} S^k(t,T) \Big\}$$

where $F_{q,n}(\cdot)$ is defined in (3.49) and S^k and S^c are of exponential quadratic form as defined in Remark 3.36.

Furthermore, if $t = 0$, the above expression gives the unconditional expectation and the functions $h(x)$, $h^c(x)$ need not have any special form.

Proof. The conditional distribution of L is given by

$$\mathbb{Q}(L_T \le x|\mathcal{G}_t) = \mathbb{Q}\big(L_T - L_t \le x - L_t|\mathcal{G}_t\big) = \mathbb{Q}\Big(\sum_{j=1}^{N_T - N_t} \xi_j \le x - L_t \big| \mathcal{G}_t \Big)$$
$$= \mathbb{Q}\Big(\sum_{j=1}^{N_T - N_t} q^j \le \frac{x - L_t}{M} |\mathcal{G}_t \Big) = F_{q, N_T - N_t}\Big(\frac{x - L_t}{M}\Big).$$

Recall that (N) is the counting process of *all* defaults. For the following, we first condition on μ^c. Then all individual defaults τ^i are independent and stem from independent Cox-processes with (also independent) intensities $(\lambda^k(t))_{t\ge 0}$, $k = 1, \ldots, \bar{K}$. Observe that $N_T - N_t$ is not independent from N_t[21]. But, it is not difficult to compute the conditional distribution. However, in contrast to the unconditional distribution, we need to distinguish which company defaults.

Using the Markovianity of the processes μ^k we need to determine

$$\mathbb{Q}\big(N_T - N_t = k|\mathcal{S}_t, N_t, \mu^c_{[t,T]}, \mathcal{F}_t\big). \tag{3.50}$$

We write $\tilde{\mathcal{F}}_t := \sigma(\mathcal{S}_t, N_t, \mu^c_{[t,T]}, \mathcal{F}_t)$. In the above probability we will have k companies defaulting in $(t, T]$. Summing over all possible indices was denoted by $\sum_{\mathbf{k}_n \in \mathcal{S}_t}$. Then,

$$(3.50) = \sum_{\mathbf{k}_n \in \mathcal{S}_t} \mathbb{Q}\big(T^{\mathbf{k}_n} \in (t,T]|\tilde{\mathcal{F}}_t\big)\, \mathbb{Q}\big(T^{\mathcal{S}_t \setminus \mathbf{k}_n} > T|\tilde{\mathcal{F}}_t\big).$$

[21]For example, if all companies default before t, hence $N_t = \bar{K}$ it follows that $N_T - N_t = 0$.

Note that the survival probability of asset k is given by

$$\mathbb{Q}(T^k > T|\tilde{\mathcal{F}}_t) = \mathbb{Q}(T^k > T|\mathbf{1}_{\{T^k>t\}}, \mu^c_{[t,T]}, \mathcal{F}_t)$$
$$= \mathbf{1}_{\{T^k>t\}} \exp\Big(- \epsilon \int_t^T \mu^c_s \, ds\Big) \underbrace{\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T \mu^k_s \, ds\Big)|\mathcal{F}_t\Big]}_{=S^k(t,T)}.$$

The expectation on the r.h.s. is of the exponential quadratic from as given by Remark 3.36. In the Markovian case, (3.23) can be used to simplify this even further. Furthermore, since, conditionally on μ^c the defaults occur independently, we have

$$\mathbb{Q}(T^{\mathbf{k}_n} > T|\tilde{\mathcal{F}}_t) = \mathbf{1}_{\{T^{\mathbf{k}_n}>t\}} \exp\Big(- n\epsilon \int_t^T \mu^c_s \, ds\Big) \prod_{k\in\mathbf{k}_n} S^k(t,T).$$

Note that S^k takes the form given in Remark 3.36 and can moreover be simplified according to Equation (3.23).

On $\{T^k > t\}$ we also have that $\mathbb{Q}(T^k \in (t,T]|\tilde{\mathcal{F}}_t) = 1 - \mathbb{Q}(T^k > T|\tilde{\mathcal{F}}_t)$. Hence,

$$\begin{aligned}
&\mathbb{Q}(N_T - N_t = n|\tilde{\mathcal{F}}_t) = \\
&= \sum_{\mathbf{k}_n\in\mathcal{S}_t} \Big\{1 - e^{-n\epsilon\int_t^T \mu^c_s\,ds} \prod_{k\in\mathbf{k}_n} S^k(t,T)\Big\} \cdot e^{-(\bar{K}-N_t-n)\epsilon\int_t^T \mu^c_s\,ds} \prod_{k\in\mathcal{S}_t\setminus\mathbf{k}_n} S^k(t,T) \\
&= \sum_{\mathbf{k}_n\in\mathcal{S}_t} \Big[e^{-(\bar{K}-N_t-n)\int_t^T \mu^c_s\,ds} \prod_{k\in\mathcal{S}_t\setminus\mathbf{k}_n} S^k(t,T) - e^{-\epsilon(\bar{K}-N_t)\int_t^T \mu^c_s\,ds} \prod_{k\in\mathcal{S}_t} S^k(t,T)\Big]
\end{aligned} \tag{3.51}$$

After we have done all calculation conditioned on μ^c we finally have to consider the unconditional expectation. This is, on $\{T^{\mathcal{S}_t} > t\}$,

$$\begin{aligned}
&\mathbb{Q}(N_T - N_t = n|\mathcal{S}_t, N_t, \mathcal{F}_t) = \\
&= \sum_{\mathbf{k}_n\in\mathcal{S}_t} \Big[S^c(\epsilon(\bar{K} - N_t - n), t, T) \prod_{k\in\mathcal{S}_t\setminus\mathbf{k}_n} S^k(t,T) - S^c(\epsilon(\bar{K} - N_t), t, T) \prod_{k\in\mathcal{S}_t} S^k(t,T)\Big].
\end{aligned}$$

■

Proposition 3.44. *Denote by $\mathbb{Q}^T$ the T-forward measure. With the above notation we have*

$$\mathbb{Q}^T\left(L_T \le x|\mathcal{G}_t\right) = \mathbf{1}_{\{T^{\mathcal{S}_t}>t\}} \frac{1}{p(t,T)} \sum_{n=0}^{\bar{K}-N_t} F_{q,n}\Big(\frac{x - L_t}{M}\Big) \times$$
$$\times \sum_{\mathbf{k}_n\in\mathcal{S}_t} \Big\{\bar{S}^c(\epsilon(\bar{K} - N_t - n), t, T) \prod_{k\in\mathcal{S}_t\setminus\mathbf{k}_n} S^k(t,T) - \bar{S}^c(\epsilon(\bar{K} - N_t), t, T) \prod_{k\in\mathcal{S}_t} S^k(t,T)\Big\}$$

$\bar{S}^c$,S^k from Remark 3.36 and $p(t,T)$ from Result 3.5 are of exponential quadratic form.

Proof. First, observe that

$$p(t,T)\mathbb{Q}^T\left(L_T \le x \middle| \mathcal{G}_t\right) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\mathbf{1}_{\{L_T\le x\}}\middle|\mathcal{G}_t\right).$$

We therefore just need to compute $\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\mathbf{1}_{\{L_T\le x\}}\middle|\mathcal{G}_t\right)$.

To this, let $\tilde{\mathcal{G}}_t := \sigma(\mathcal{S}_t, N_t, \mu^c_{[t,T]}, \mathcal{G}_t)$ and recall r has common factors, i.e., conditional on μ^c it is known. We thus have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbf{1}_{\{L_T\le x\}}\middle|\mathcal{G}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{L_T\le x\}}\middle|\tilde{\mathcal{G}}_t\right]\middle|\mathcal{G}_t\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{Q}\left(L_T\le x|\tilde{\mathcal{G}}_t\right)\middle|\mathcal{G}_t\right]\end{aligned}$$

For the inner expectation we may use Equation (3.51) to obtain that the above equals

$$\mathbb{E}^{\mathbb{Q}}\Bigg\{e^{-\int_t^T r_s ds}\sum_{n=0}^{\bar{K}-N_t} F_{q,n}\left(\frac{x-L_t}{M}\right)$$
$$\cdot\sum_{k_n\in\mathcal{S}_t}\left[e^{-\epsilon(\bar{K}-N_t-n)\int_t^T\mu_s^c ds}\left(\prod_{k\in\mathcal{S}_t\setminus k_n} S^k(t,T)\right) - e^{-\epsilon(\bar{K}-N_t)\int_t^T\mu_s^c ds}\left(\prod_{k\in\mathcal{S}_t} S^k(t,T)\right)\right]\Bigg|\mathcal{G}_t\Bigg\}.$$

Recalling the notation defined in Remark 3.36 we have that

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}e^{-\epsilon(\bar{K}-N_t-n)\int_t^T\mu_s^c ds}\middle|\mathcal{G}_t\right) = \bar{S}^c(\epsilon(\bar{K}-N_t-n),t,T)$$

such that we obtain the given expression immediately. ∎

We can now focus on the *pricing of tranches of synthetic CDOs.* We make the following normalizations:

- The CDO offers notes on each tranche with par value 1.
- Interest is paid at times $t_1,\cdots,t_{N^*}$
- The value of the entire tranche at the time the CDO is issued (time zero) is

$$V^i(0) = b^i - b^{i-1}$$

- At each intermediate time $t_j < t_{N^*}$ we receive the coupon c^i (the coupon is, obviously, tranche dependent). The payment is on the remaining principal in the trance, so that the payments due at t_j are

$$\left(1-\frac{L^i(t_j)}{b^i-b^{i-1}}\right)c^i(t_j-t_{j-1})$$

- At maturity t_{N^*} the coupon plus the remain value of the tranche is paid:

$$\left(1 - \frac{L^i(t_{N^*})}{b^i - b^{i-1}}\right)\left(c^i(t_{N^*} - t_{N^*-1}) + 1\right).$$

The critical point here is the reinvestment of the recovery payment. Note that in reality, the default of an entity from the underlying pool leads to a non-payment of the future coupons. The recovery has to be re-invested at the current market level and possibly gets a lower coupon. In this section, we assume that these missing future coupons are included in the recovery. This means, that the recovery after default is the actual recovery minus financing cost of the future coupons (which also could be a gain if the market offers better conditions at default).

Define $c_j^i := c^i(t_j - t_{j-1})$ for $i < N^*$ and $c_{N^*}^i := c^i(t_{N^*} - t_{N^*-1}) + 1$ and denote the value of the tranche i at time t by $V^i(t)$. Then $V^i(t)$ is given by

$$\begin{aligned}
V^i(t) &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{N^*} e^{-\int_t^{t_j} r(u)du}\left(1 - \frac{L^i(t_j)}{b^i - b^{i-1}}\right)c_j^i \,\middle|\, \mathcal{G}_t\right] \\
&= \sum_{j=1}^{N^*} p(t,t_j)\left(1 - \frac{\mathbb{E}^{t_j}\left[L^i(t_j)|\,\mathcal{G}_t\right]}{b^i - b^{i-1}}\right)c_j^i,
\end{aligned}$$

where $\mathbb{E}^{t_j}\left[\cdot|\,\mathcal{G}_t\right]$ denotes conditional expectation under the t_j-forward measure.

The price of risk-free bonds have been computed in Result 3.5 and the only difficulty is in computing $\mathbb{E}^{t_j}\left[L^i(t_j)|\,\mathcal{G}_t\right]$. In literature on CDOs it is quite common to assume independence of interest rates and all processes related to the loss process. As we saw when computing the distribution of the portfolio losses, in our framework dealing with the T-forward measure requires only little additional effort, and so here we relax this assumption.

The next theorem gives $\mathbb{E}^T\left[L^i(T)|\,\mathcal{G}_t\right]$ for all $T > t$ in closed form and concludes the CDO analysis. It uses the notation introduced on page 112. We denote the density of the sum of k losses, similar to the distribution function defined in (3.49), by $f_{q,k}(\cdot)$.

Theorem 3.45. *Consider $t < T$. The conditional distribution function of L_{t_j} is defined by $F_{\mathcal{G}_t}^{L_T}(x) := \mathbb{Q}^T(L_{t_j} \leq x|\mathcal{G}_t)$. We have that its density is given by*

$$\begin{aligned}
f_{\mathcal{G}_t}^{L_T}(x) &= \mathbf{1}_{\{T^{\mathcal{S}_t} > t\}}\frac{1}{p(t,T)}\sum_{n=0}^{\bar{K}-N_t} \cdot f_{q,n}\Big(\frac{x - L_t}{M}\Big) \times \\
&\times \sum_{\mathrm{k}_n \in \mathcal{S}_t}\left\{\bar{S}^c(\epsilon(\bar{K} - N_t - n), t, T)\prod_{k \in \mathcal{S}_t \setminus \mathrm{k}_n} S^k(t,T) - \bar{S}^c(\epsilon(\bar{K} - N_t), t, T)\prod_{k \in \mathcal{S}_t} S^k(t,T)\right\}
\end{aligned}$$

and the conditional expected value of tranche i equals

$$\mathbb{E}^T(L_T^i|\mathcal{G}_t) = (b^i - b^{i-1})\Big[1 - F_{\mathcal{G}_t}^{L_T}(b^i)\Big] - b^{i-1}\Big[F_{\mathcal{G}_t}^{L_T}(b^i) - F_{\mathcal{G}_t}^{L_T}(b^{i-1})\Big] + \int_{b^{i-1}}^{b^i} x f_{\mathcal{G}_t}^{L_T}(x)\,dx.$$

Proof. By the definition of the tranche loss in (3.48) we have that

$$\begin{aligned}\mathbb{E}^T(L_T^i|\mathcal{G}_t) &= \mathbb{E}^T\big(L_T \mathbf{1}_{\{L_T\in(b^{i-1},b^i]\}}|\mathcal{G}_t\big) - b^{i-1}\mathbb{Q}^T\big(L_T \in (b^{i-1}, b^i]|\mathcal{G}_t\big)\\ &\quad + (b^i - b^{i-1})\mathbb{Q}^T\big(L_T > b^i|\mathcal{G}_t\big).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{Q}^T\big(L_T \in (b^{i-1}, b^i]|\mathcal{G}_t\big) &= \mathbb{Q}^T\big(L_T \le b^i|\mathcal{G}_t\big) - \mathbb{Q}^T\big(L_T \le b^{i-1}|\mathcal{G}_t\big)\\ \mathbb{E}^T\big(L_T \mathbf{1}_{\{L_T\in(b^{i-1},b^i]\}}|\mathcal{G}_t\big) &= \int_{b^{i-1}}^{b^i} x f_{\mathcal{G}_t}^T(x)\,dx.\end{aligned}$$

The result follows from the closed form for $\mathbb{Q}^T(L_{t_j} \le x|\mathcal{G}_t)$ computed in Proposition 3.44. ■

3.5 Illustration

In this section we illustrate the results derived in the previous sections with a concrete three-factor model.

3.5.1 The model

We take

$$Z = \begin{pmatrix} Z^1 \\ Z^2 \\ r \end{pmatrix}$$

as the state variable, and assume its $\mathbb{Q}$-dynamics given by

$$dZ_t^1 = \big[\beta_1(t) - \alpha_1 Z_t^1\big]\,dt + \sigma_1 dW_t^1 \tag{3.52}$$

$$dZ_t^2 = \big[\beta_2(t) - \alpha_2 Z_t^2\big]\,dt + \sigma_2 dW_t^2 \tag{3.53}$$

$$dr_t = \alpha_r\,[\beta_r - r_t]\,dt + \sigma_r\sqrt{r_t}dW_t^r \tag{3.54}$$

where α_i, σ_i, for $i = 1, 2, r$ and β_r are constants, while $\beta_1(\cdot)$, $\beta_2(\cdot)$ are deterministic functions of t and W^1, W^2 and W^r are independent $\mathbb{Q}$-Wiener processes.

We will analyze two firms, denoted 1 and 2. Each firm's intensity is driven by firm's specific as well as common factors in accordance with Assumption 3.35.

For each firm the intensity is given by

$$\lambda_t^k = \mu_t^k + \epsilon^k \mu_t^c, \quad \mu_t^k = \eta_t^k = \left(Z_t^k\right)^2, \quad k = 1,2\ \epsilon^1, \epsilon^2 \in \mathbb{R} \tag{3.55}$$

$$\mu^c = J^c + \delta r, \quad J_t^c = \sum_{\tilde{\tau}_i < t} Y_i h^c(t - \tilde{\tau}_i), \quad Y_i \sim \chi^2(2) \quad h^c(x) = e^{-bx},\ b \in \mathbb{R}_+ \, . \tag{3.56}$$

and the $\tilde{\tau}_i$ are the jumps of a Poisson process with intensity l^c.

We note that the firms specific terms do not have jumps and are purely quadratic terms, while the common factors depend linearly on the short rate and allow for jumps. The common jumps follow the shot-noise formulation in Assumption 3.35, the $\tilde{\tau}_i$'s stem from a standard Poisson distribution with constant parameter l^c and the Y_i's have a χ^2 distribution with two degrees of freedom.

Using the notation previously described we identify all the needed matrices:

drift as in (3.3): $$d(t) = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \alpha_r \beta_r \end{bmatrix} \qquad E(t) = \begin{bmatrix} -\alpha_1 & 0 & 0 \\ 0 & -\alpha_2 & 0 \\ 0 & 0 & -\alpha_r \end{bmatrix}$$

variance as in (3.4): $$k_0(t) = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad k_r(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_r^2 \end{bmatrix}$$

$$k_i(t) = g_{uj}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad i = 1,2; \quad uj = 1,2,r \, .$$

Furthermore we also have

$$\eta^k \;=\; (Z^k)^2 \quad \Rightarrow \quad (\mathsf{Q}^k)_{ij}(t) = \begin{cases} 1 & ij = kk \\ 0 & \text{otherwise} \end{cases}, \quad \mathsf{g}^k(t) = \mathbf{0}, \quad \mathsf{f}^k(t) = 0; \quad k = 1,2$$

$$r \;=\; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad Q(t) = \mathbf{0}, \quad g(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad f(t) = 0 \, .$$

3.5.2 Risk-free term structure

From (3.54) we recognize the CIR model for the short rate, and thus, we know that the risk-free bond prices have an ATS.

From Result 3.5 we have

$$p(t,T) = e^{A(t,T)+B^\top(t,T)Z_t+Z_t^\top C(t,T)Z_t}$$

and solving the basic system of ODEs we get

$$A(t,T) = \frac{2\alpha_r\beta_r}{\sigma_r^2}\ln\left(\frac{2\gamma_r e^{(\alpha_r+\gamma_r)\frac{x}{2}}}{(\gamma_r+\alpha_r)(e^{\gamma_r x}-1)+2\gamma_r}\right),\ B(t,T) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{2(e^{\gamma_r x}-1)}{(\gamma_r+\alpha_r)(e^{\gamma_r x}-1)+2\gamma_r}\end{bmatrix}$$

$C(t,T) = \mathbf{0}$ and above we use the notation $x = T-t$, $\gamma_r = \sqrt{\alpha_r^2+2\sigma_r^2}$.

Result 3.46. *The closed-formula solution for the risk-free bond prices is,*

$$p(t,T) = \left(\frac{2\gamma_r e^{(\alpha_r+\gamma_r)\frac{x}{2}}}{(\gamma_r+\alpha_r)(e^{\gamma_r x}-1)+2\gamma_r}\right)^{\frac{2\alpha_r\beta_r}{\sigma_r^2}} \times \exp\left\{\left(\frac{2(1-e^{\gamma x})}{(\gamma+\alpha_r)(e^{\gamma x}-1)+2\gamma}\right)r_t\right\}. \tag{3.57}$$

where $x = T-t$, $\gamma_r = \sqrt{\alpha_r^2+2\sigma_r^2}$.

3.5.3 Key building blocks for credit risk

We will now compute all the basic quantities in Remark 3.36. In what follows the expressions can get quite long so, whenever possible, we use $x := T-t$.

- $S^k(\theta,t,T)$:

We start by noting that the firm specific components have no jumps, so we have

$$S^k(\theta,t,T) = S^k_\eta(\theta,t,T) = \exp\left(\mathcal{A}^k(\theta,t,T) + \mathcal{B}^{k\top}(\theta,t,T)Z_t + Z_t^\top \mathcal{C}^k(\theta,t,T)Z_t\right)$$

where $(\mathcal{A}^k, \mathcal{B}^k, \mathcal{C}^k, \theta Q^k, 0, 0)$ solve the basic ODE system of Definition 3.4.

We also note that, due to independence between the three factors in Z, we can immediately conclude

$$\left(\mathcal{B}^k\right)_{ij}(\theta,t,T) = \begin{cases}\mathcal{B}^k(\theta,t,T) & i=k \\ 0 & \text{otherwise}\end{cases},\quad \left(\mathcal{C}^k\right)_{ij}(\theta,t,T) = \begin{cases}\mathcal{C}^k(\theta,t,T) & i=j=k \\ 0 & \text{otherwise}\end{cases}. \tag{3.58}$$

$\mathcal{B}^k$ and $\mathcal{C}^k$ on the r.h.s. are now *scalar* functions [22] that solve the scalar ODE system

$$\begin{cases} \dfrac{\partial \mathcal{B}^k}{\partial t} - \alpha_k \mathcal{B}^k + 2\mathcal{C}^k \beta_k + 2\sigma_k^2 \mathcal{C}^k \mathcal{B}^k = 0 \\ \mathcal{B}^k(\theta, T, T) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial \mathcal{C}^k}{\partial t} - 2\alpha_k \mathcal{C}^k + 2\sigma_k^2 \left(\mathcal{C}^k\right)^2 = \theta \\ \mathcal{C}^k(\theta, T, T) = 0 \end{cases}$$

whose solution is given by (3.59)-(3.60):

$$\mathcal{C}^k(\theta, t, T) = \frac{\theta\left[1 - e^{2\gamma_k x}\right]}{(\gamma_k + \alpha_k)\left[e^{2\gamma_k x} - 1\right] + 2\gamma_k}, \tag{3.59}$$

$$\begin{aligned} \mathcal{B}^k(t, T) &= \int_t^T \beta_k(s) e^{\int_t^s \alpha_k - 2\mathcal{C}(\theta, u, T) du} 2\mathcal{C}(\theta, s, T) ds \\ &= \left[2(\alpha_k + \gamma_k)(e^{2\gamma_k x} - 1) + 4\gamma_k\right]^{-1/\sigma_k^2} \times \qquad (3.60) \\ &\quad \times \int_t^T \left(\beta_k(s) \frac{4\theta e^{\left(\alpha_k + \frac{4\theta}{\gamma_k - \alpha_k}\right)(s-t)} \left(1 - e^{2\gamma_k (T-s)}\right)}{\left[2(\alpha_k + \gamma_k)(e^{2\gamma_k (T-s)} - 1) + 4\gamma_k\right]^{1 - \frac{1}{\sigma_k^2}}} \right) ds \end{aligned}$$

with $\gamma_k = \sqrt{\alpha_k^2 + 2\sigma_k^2 \theta}$.

Finally, the ODE for $\mathcal{A}^k$ reduces to

$$\begin{cases} \dfrac{\partial \mathcal{A}^k}{\partial t} + \beta^k(t)\mathcal{B}^k + \dfrac{1}{2}\sigma_k^2 \left(\mathcal{B}^k\right)^2 + \sigma_k^2 \mathcal{C}^k = 0 \\ \mathcal{A}(\theta, T, T) = 0 \end{cases}$$

and integrating we get

$$\begin{aligned} \mathcal{A}^k(\theta, t, T) &= -\int_t^T \beta_k(s)\mathcal{B}^k(\theta, s, T) + \frac{1}{2}\sigma_k^2 \left(\mathcal{B}^k(\theta, s, T)\right)^2 + \sigma_k^2 \mathcal{C}^k(\theta, s, T) ds \\ &= \frac{1}{2} \ln \left(\frac{2\gamma_k e^{(\gamma_k + \alpha_k)x}}{(\gamma_k + \alpha_k)(e^{2\gamma_k x} - 1) + 2\gamma_k} \right) - \int_t^T \beta_k(s)\mathcal{B}^k(\theta, s, T) + \frac{1}{2}\sigma_k^2 \left(\mathcal{B}^k(\theta, s, T)\right)^2 ds. \end{aligned}$$

for $\mathcal{B}^k$ as in (3.60).

[22] Even though we acknowledge that using the same notation on both the vector/matrix on the l.h.s and the scalar functions on the r.h.s of (3.58) may be misleading, we believe it is better than introducing more notation. Moreover, given the independence between our three factors, it should be clear at each point which entrance in a vector/matrix entrance is not zero and we will always be referring to that one.

Result 3.47. *So, for k=1,2, we have* $S^k(\theta,t,T) = S^k_\eta(\theta,t,T)$ *and*

$$S^k(\theta,t,T) = \sqrt{\frac{2\gamma_k e^{(\gamma_k+\alpha_k)x}}{(\gamma_k+\alpha_k)(e^{2\gamma_k x}-1)+2\gamma_k}} \exp\left\{\frac{\theta\left[1-e^{2\gamma_k x}\right]}{(\gamma_k+\alpha_k)\left[e^{2\gamma_k x}-1\right]+2\gamma_k}\left(Z^k_t\right)^2\right\}\times$$
$$\times\exp\left\{-\int_t^T\left(\beta_k(s)\mathcal{B}^k(\theta,s,T)+\frac{1}{2}\sigma_k^2\left(\mathcal{B}^k(\theta,s,T)\right)^2\right)ds+\mathcal{B}^k(\theta,t,T)Z^k_t\right\} \quad (3.61)$$

where $\mathcal{B}^k$ *is as in (3.60) and the expressions involving* $\mathcal{B}^k$ *must be evaluated numerically.*

- $S^c_\eta(\theta,t,T)$, $\bar{S}^c_\eta(\theta,t,T)$:

We note that $\eta^c_t = \delta r_t$. So,

$$\begin{aligned}
S^c_\eta(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\theta\eta^c_s ds}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\theta\delta r_s ds}\middle|\mathcal{F}_t\right]\\
&= \exp\left\{\mathcal{A}^c(\theta,t,T)+\mathcal{B}^{c\top}(\theta,t,T)Z_t+Z_t^\top\mathcal{C}^c(\theta,t,T)Z_t\right\}\\
\bar{S}^c_\eta(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\eta_s ds}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T(1+\theta\delta)r_s ds}\middle|\mathcal{F}_t\right]\\
&= \exp\left\{\bar{\mathcal{A}}^c(\theta,t,T)+\bar{\mathcal{B}}^{c\top}(\theta,t,T)Z_t+Z_t^\top\bar{\mathcal{C}}^c(\theta,t,T)Z_t\right\}\ .
\end{aligned}$$

Note that the above expectations are very similar to the expectation needed to compute for the risk-free bond prices.

Indeed, it is easy to show that both quantities can be obtained from $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\Delta r_s ds}\middle|\mathcal{F}_t\right]$ and we have

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\Delta r_s ds}\middle|\mathcal{F}_t\right] = \exp\left\{A(\Delta,t,T)+B^\top(\Delta,t,T)Z_t+Z_t^\top C(\Delta,t,T)Z_t\right\} \quad (3.62)$$

where

$$A(\Delta,t,T) = \frac{2\alpha_r\beta_r}{\sigma_r^2}\ln\left[\frac{2\hat{\gamma}_r e^{(\hat{\gamma}_r+\alpha_r)\frac{T-t}{2}}}{(\alpha_r+\hat{\gamma}_r)[e^{\hat{\gamma}_r(T-t)}-1]+2\hat{\gamma}_r}\right] \quad (3.63)$$

$$B(\Delta,t,T) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{2\Delta\left[e^{\hat{\gamma}_r(T-t)}-1\right]}{(\hat{\gamma}_r+\alpha_r)\left[e^{\hat{\gamma}_r(T-t)}-1\right]+2\hat{\gamma}_r}\end{bmatrix}, \quad (3.64)$$

$$C(\Delta,t,T) = \mathbf{0}. \quad (3.65)$$

and $\hat{\gamma}_r = \sqrt{\alpha_r^2+2\sigma_r^2\Delta}$.

We finally can obtain $\bar{\mathcal{A}}^c$, $\bar{\mathcal{B}}^c$, $\bar{\mathcal{C}}^c$ and $\mathcal{A}^c$, $\mathcal{B}^c$, $\mathcal{C}^c$ as follows:[23]

$$\begin{array}{lll} \bar{\mathcal{A}}^c(\theta,t,T) = A((1+\theta\delta),t,T) & \bar{\mathcal{B}}^c(\theta,t,T) = B((1+\theta\delta),t,T) & \bar{\mathcal{C}}^c(\theta,t,T) = \mathbf{0} \\ \mathcal{A}^c(\theta,t,T) = A(\theta\delta,t,T) & \mathcal{B}^c(\theta,t,T) = B(\theta\delta,t,T) & \mathcal{C}^c(\theta,t,T) = \mathbf{0} \end{array}$$

where A, B, C are as in (3.63)-(3.65).

Result 3.48. *We, thus, have:*

$$\begin{aligned} S^c_\eta(\theta,t,T) &= \left(\frac{2\gamma^c_\eta e^{(\gamma^c_\eta+\alpha_r)\frac{x}{2}}}{(\alpha_r+\gamma^c_\eta)[e^{\gamma^c_\eta x}-1]+2\gamma^c_\eta}\right)^{\frac{2\alpha_r\beta_r}{\sigma_r^2}} \times \\ &\quad \times \exp\left\{\left(\frac{2\theta\delta\left[1-e^{\gamma^c_\eta x}\right]}{(\gamma^c_\eta+\alpha_r)\left[e^{\gamma^c_\eta x}-1\right]+2\gamma^c_\eta}\right)r_t\right\} \\ \textit{where} \quad & \gamma^c_\eta = \sqrt{\alpha_r^2+2\sigma_r^2\theta\delta} \end{aligned} \tag{3.66}$$

$$\begin{aligned} \bar{S}^c_\eta(\theta,t,T) &= \left(\frac{2\bar{\gamma}^c_\eta e^{(\bar{\gamma}^c_\eta+\alpha_r)\frac{x}{2}}}{(\alpha_r+\bar{\gamma}^c_\eta)[e^{\bar{\gamma}^c_\eta x}-1]+2\bar{\gamma}^c_\eta}\right)^{\frac{2\alpha_r\beta_r}{\sigma_r^2}} \times \\ &\quad \times \exp\left\{\left(\frac{2(1+\theta\delta)\left[1-e^{\bar{\gamma}^c_\eta x)}\right]}{(\bar{\gamma}^c_\eta+\alpha_r)\left[e^{\bar{\gamma}^c_\eta x}-1\right]+2\bar{\gamma}^c_\eta}\right)r_t\right\} \\ \textit{where} \quad & \bar{\gamma}^c_\eta = \sqrt{\alpha_r^2+2\sigma_r^2(1+\theta\delta)} \end{aligned} \tag{3.67}$$

$\bullet S^c_J(\theta,t,T)$:

From Remark 3.36 and noting that in our model we are in the Markovian case, we get

$$S^c_J(\theta,t,T) = e^{-H^c(T-t)J_t+l^c x[D^c(\theta,x)-1]}$$

and the functions H^c and D^c can be obtained directly from $h^c(x) = e^{-bx}$ and the fact that Y^c_is are $\chi^2(2)$ distributed (recall equations (3.21) and (3.25)).

We start by computing

$$H^c(x) = \int_0^x e^{-bs} = -\frac{1}{b}\left[e^{-bx}-1\right] = \frac{1}{b}\left[1-e^{-bx}\right] . \tag{3.68}$$

[23]The alternative to these computations is to solve the basic ODE system of Definition 3.4 for $(\bar{\mathcal{A}}^c, \bar{\mathcal{B}}^c, \bar{\mathcal{C}}^c, 0, (1+\delta)g, 0)$ and $(\mathcal{A}^c, \mathcal{B}^c, \mathcal{C}^c, 0, \delta\theta, 0)$. Given the specificities of our model the above is faster.

To compute D^c we will make use of the Laplace transform of the $\chi^2(\nu)$ distribution.[24]

For $\nu = 2$,

$$\varphi^c(u) = \frac{1}{1+2u}, \tag{3.69}$$

and D^c will have a simple formula. However, for any choice of ν there is an explicit expression, just the formulas will get lengthier.

With $\nu = 2$ we find,

$$\begin{aligned} D(\theta, s) &= \int_0^1 \varphi^c(\theta H(s - su))\, du \\ &= \int_0^1 \Big(1 + 2\theta H\big(s(1-u)\big)\Big)^{-1} du \\ &= \int_0^1 \Big[1 + \frac{2\theta}{b}\Big(1 - e^{-bs(1-u)}\Big)\Big]^{-1} du. \end{aligned}$$

Integrating gives[26]

$$D^c(\theta, s) = \frac{1}{2+b\theta}\left[b + \frac{1}{s}\ln\left(1 + \frac{2\theta}{b}(1 - e^{-bs})\right)\right]. \tag{3.70}$$

Putting all the information together, we conclude the following.

Result 3.49.

$$S_J^c(\theta, t, T) = \left(1 + \frac{2}{b}e^{-b(T-t)}\right)^{\overline{\frac{l^c}{2+b\theta}}} \exp\left\{\frac{1}{b}\left[e^{-bx} - 1\right] J_t + l^c(T-t)\left[\frac{b}{2+b\theta} - 1\right]\right\}. \tag{3.71}$$

Furthermore, from Remark 3.36, we know

$$S^c(\theta, t, T) = S_\eta^c(\theta, t, T) S_J^c(\theta, t, T) \tag{3.72}$$

$$\bar{S}^c(\theta, t, T) = \bar{S}_\eta^c(\theta, t, T) S_J^c(\theta, t, T) \tag{3.73}$$

with S_η^c as in (3.66), $\bar{S}_\eta^c$ as in (3.67) and S_J^c as in (3.71).

[24]Recall that for $u \geq 0$ the Laplace transform of random variable which has χ^2 distribution with ν degrees of freedom, equals [25],

$$\varphi_{\chi^2_\nu}(u) = \mathbb{E}(e^{-u\chi^2_\nu}) = (1+2u)^{-\nu/2}.$$

[26]Note that the primitive of $(a + be^{cu})^{-1}$ is

$$\frac{u}{a} - \frac{1}{ac}\ln\left(a + be^{cu}\right).$$

• $\Gamma^k(\theta, t, T)$:

$$\Gamma^k(\theta,t,T) = \Gamma^k_\eta(\theta,t,T) = S^k_\eta(\theta,t,T)\exp\left(a^k(\theta,t,T) + b^{k\top}(\theta,t,T)Z_t + Z_t^\top c^k(\theta,t,T)Z_t\right)$$

where a^k, b^k and c^k solve the interlinked system of Definition 3.22. In our case, and for each fixed k, the system can be simplified since

$$(b^k)_i(\theta,t,T) = \begin{cases} b^k(\theta,t,T) & i=k \\ 0 & \text{otherwise} \end{cases} \qquad (c^k)_{ij}(\theta,t,T) = \begin{cases} c^k(\theta,t,T) & i=j=k \\ 0 & \text{otherwise} \end{cases}. \tag{3.74}$$

For a^k and b^k, c^k (on the l.h.s above) we get the scalar system of ODE

$$\begin{cases} \dfrac{\partial a^k}{\partial t} + \beta_k b^k + \sigma_k^2 \mathcal{B}^k b^k + \sigma_k^2 c &= 0 \\ a^k(\theta,T,T) &= 0 \end{cases}$$
$$\begin{cases} \dfrac{\partial b^k}{\partial t} - \alpha_k b^k + 2\beta_k c + 2\sigma_k^2 \mathcal{C}^k b^k + 2\sigma_k^2 \mathcal{B}^k c^k &= 0 \\ b^k(\theta,T,T) &= 0 \end{cases}$$
$$\begin{cases} \dfrac{\partial c^k}{\partial t} - 2\alpha_k c^k + 4\sigma_k^2 \mathcal{C}^k c^k &= 0 \\ c^k(\theta,T,T) &= \theta \end{cases}$$

where $\mathcal{B}^k$ and $\mathcal{C}^k$ are as in (3.59)-(3.60).

Solving first for c^k we get

$$\begin{aligned} c^k(\theta,t,T) &= \theta\exp\left\{-2\int_t^T \alpha_k - 2\sigma_k^2\mathcal{C}^k(s,T)ds\right\} \\ &= \frac{\theta\left[(\gamma_k+\alpha_k)(e^{\gamma_k x}-1)+2\gamma_k\right]}{\gamma_k e^{(\gamma_k+3\alpha_k)x}} \end{aligned} \tag{3.75}$$

where $\gamma_k = \sqrt{\alpha_k^2 + 2\sigma_k^2\theta}$.

Then b^k

$$b^k(\theta,t,T) = -2\int_t^T e^{\int_t^s \alpha_k - 2\sigma_k^2 \mathcal{C}^k(u,T)du}\left(\beta_k(s) - \sigma^2\mathcal{B}^k(s,T)\right)c^k(s,T)ds \tag{3.76}$$

where $\mathcal{B}^k$ and $\mathcal{C}^k$ are as in (3.59)-(3.60) and c^k as in (3.75). Finally

$$a^k(\theta,t,T) = -\int_t^T \beta_k(s) + \sigma_k^2\mathcal{B}^k(s,T)b + \sigma_k^2 c^k(s,T)ds. \tag{3.77}$$

with $\mathcal{B}$ as in (3.60) and c^k as in (3.75).

Result 3.50. *Concluding,*

$$\Gamma^k(\theta,t,T) = \Gamma^k_\eta(\theta,t,T) = S^k_\eta(\theta,t,T)\exp\left\{a^k(\theta,t,T) + b^k(\theta,t,T)Z^k_t + c^k(\theta,t,T)\left(Z^k_t\right)^2\right\} \tag{3.78}$$

with c^k as in (3.75) and b^k and a^k numerically evaluated using (3.76)-(3.77).

- $\Gamma^c_\eta(\theta,t,T)$, $\bar{\Gamma}^c_\eta(\theta,t,T)$:

Once again, we note that in our special case, to obtain $\Gamma^c_\eta(\theta,t,T)$ and $\bar{\Gamma}^c_\eta(\theta,t,T)$ we need to solve expressions of the type $\mathbb{E}^{\mathbb{Q}}\left[\theta r_T e^{-\int_t^T \Delta r_s ds}\middle| \mathcal{F}_t\right]$, which can be easily proven to be of the form

$$\mathbb{E}^{\mathbb{Q}}\left[\theta r_T e^{-\int_t^T \Delta r_s ds}\middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \Delta r_s ds}\middle| \mathcal{F}_t\right] e^{a(\Delta,t,T)+b(\Delta,t,T)r_t}.$$

The expectation on the r.h.s. has been previously computed (compare with equation (3.62)) and a, b solve a simplified scalar version of the interlinked ODE system in Definition 3.22

$$\begin{cases} \dfrac{\partial a}{\partial t} + \alpha_r\beta_r b &= 0 \\ a(\Delta,T,T) &= 0 \end{cases}$$
$$\begin{cases} \dfrac{\partial b}{\partial t} - \alpha_r b + \dfrac{1}{2}\sigma_k^2 Bb &= 0 \\ b(\Delta,T,T) &= \theta \end{cases}$$

where B can be obtained from (3.64).

The solution to the above system is given by

$$a(\Delta,t,T) = \int_t^T \alpha_r\beta_r b(\Delta,s,T)ds \tag{3.79}$$

$$\begin{aligned} b(\Delta,t,T) &= \theta e^{-\int_t^T \alpha_r - \frac{1}{2}\sigma_r^2 B(s,T)ds} \\ &= \frac{\theta\left[(\alpha_r+\hat{\gamma}_r)(e^{\hat{\gamma}_r x}-1)+2\hat{\gamma}_r\right]}{2\hat{\gamma}_r e^{(3\alpha_r+\hat{\gamma}_r)\frac{x}{2}}} \end{aligned} \tag{3.80}$$

where a must be evaluated numerically using b in (3.80) and $\hat{\gamma}_r = \sqrt{\alpha_r + 2\sigma_r^2\Delta}$

Using the above derived equations (3.79)-(3.80) we can finally obtain and

$$\begin{aligned} \Gamma^c_\eta(\theta,t,T) &= S^c_\eta(\theta,t,T)\exp\left\{a(\theta\delta,t,T) + b(\theta\delta,t,T)r_t\right\} \\ \bar{\Gamma}^c_\eta(\theta,t,T) &= \bar{S}^c_\eta(\theta,t,T)\exp\left\{a((1+\theta\delta),t,T) + b((1+\theta\delta),t,T)r_t\right\} \end{aligned}$$

Result 3.51.

$$\begin{aligned}\Gamma^c_\eta(\theta,t,T) &= S^c_\eta(\theta,t,T)\exp\left\{a(\theta\delta,t,T)+\left(\frac{\theta\left[(\alpha_r+\gamma^c_\eta)(e^{\gamma^c_\eta x}-1)+2\gamma^c_\eta\right]}{2\gamma^c_\eta e^{(3\alpha_r+\gamma^c_\eta)\frac{x}{2}}}\right)r_t\right\}\\ \textit{where} \quad & \gamma^c_\eta=\sqrt{\alpha_r^2+2\sigma_r^2\theta\delta}\end{aligned} \tag{3.81}$$

$$\begin{aligned}\bar\Gamma^c_\eta(\theta,t,T) &= \bar S^c_\eta(\theta,t,T)\exp\left\{a((1+\theta\delta),t,T)+\left(\frac{\theta\left[(\alpha_r+\bar\gamma^c_\eta)(e^{\bar\gamma^c_\eta x}-1)+2\bar\gamma^c_\eta\right]}{2\bar\gamma^c_\eta e^{(3\alpha_r+\bar\gamma^c_\eta)\frac{x}{2}}}\right)r_t\right\}\\ \textit{where} \quad & \gamma^c_\eta=\sqrt{\alpha_r^2+2\sigma_r^2(1+\theta\delta)}\end{aligned} \tag{3.82}$$

$S^c_\eta(\theta,t,T)$ and $\bar S^c_\eta(\theta,t,T)$ can be obtained from (3.66), (3.67) and a must be evaluated numerically using (3.79)-(3.80).

- $\Gamma^c_J(\theta,t,T)$:

$$\Gamma^c_J(\theta,t,T) = S^c_J(\theta,t,T)\left[\theta J^k(t,T)-l^c[D^c(\theta,x)(1-x)-1]+x\varphi^c(\theta H^c(x))\right]$$

$H^c(x)$ and $D^c(\theta,x)$ are as in (3.68), (3.70). While from the Laplace transform of the $\chi^2(2)$ in (3.69) we immediately obtain

$$\varphi^c(\theta H^c(x))=\frac{1}{1+\frac{2\theta}{b}(1-e^{-bx})}$$

Result 3.52. *Thus,*

$$\begin{aligned}\Gamma^c_J(\theta,t,T)=S^c_J(\theta,t,T)\Bigg\{&\theta J^c(t,T)+x\frac{1}{1+\frac{2\theta}{b}(1-e^{-bx})}\\ &-l^c\left[\frac{1}{2+b\theta}\left(b+\frac{1}{x}\ln\left(1+\frac{2\theta}{c}(1-e^{-bx})\right)\right)(1-x)-1\right]\Bigg\}\end{aligned} \tag{3.83}$$

where $J^c(t,T)=\sum_{\tilde\tau_i\le t}Y_i e^{-b(T-\tilde\tau_i)}$ thus known at time t, and $S^c_J(\theta,t,T)$ is as in (3.71). Furthermore, using Remark 3.36 we know

$$\Gamma^c(\theta,t,T) = \Gamma^c_\eta(\theta,t,T)S^c_J(\theta,t,T)+\Gamma^c_J(\theta,t,T)S^c_\eta(\theta,t,T) \tag{3.84}$$

$$\bar\Gamma^c(\theta,t,T) = \bar\Gamma^c_\eta(\theta,t,T)S^c_J(\theta,t,T)+\bar\Gamma^c_J(\theta,t,T)\bar S^c_\eta(\theta,t,T) \tag{3.85}$$

In Results 3.46 to 3.52, we have computed in closed-form (up to the numerical integration of some nasty integrals), all the needed ingredients to derive, in the context of our model, explicit expressions to the abstract results of the previous sections.

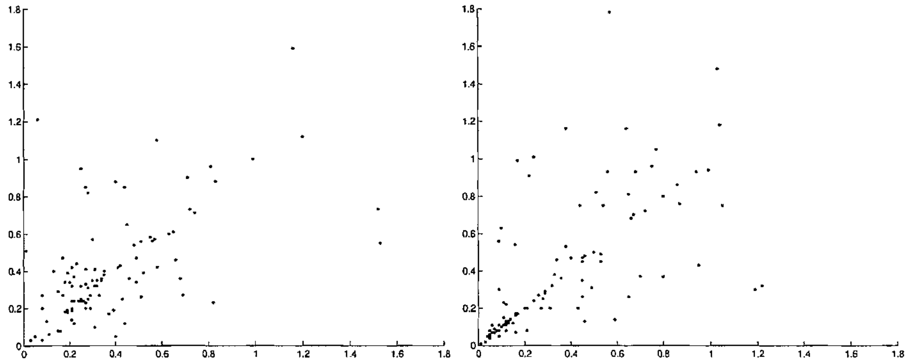


Figure 3.4: Simulated defaults of two companies according to the model in Section 3.5.1. Parameters are for $i = 1, 2$: $\beta_i = 1$, $\alpha_i = 0.5$, $\sigma_i = 0.2$, $l^c = 2$. The jumps are χ_2^2-distributed. The left picture has $\epsilon_i = 0.1$, the right $\epsilon_i = 0.5$.

For instance,

- Firm's k survival probability is given by

$$\mathbb{Q}_S^k(t,T) = S^k(t,T) \times S^c(\epsilon^k, t, T) \tag{3.86}$$

  with $S^k(t,T) = S^k(1,t,T)$ from (3.61), S^c from (3.72).

- Firm's k price of the zero-recovery defaultable bond is given by

$$\bar{p}_o^k(t,T) = S^k(t,T) \times \bar{S}^c(\epsilon^k, t, T) \tag{3.87}$$

  with $S^k(t,T) = S^k(1,t,T)$ from (3.61), $\bar{S}^c$ from (3.73).

- The price of 1 u.c. at default if firm k defaults in $(t,T]$ is give by

$$e^k(t,T) \;\; = \;\; \Gamma^k(t,T) \times \bar{S}^c(\epsilon^k, t, T) + \bar{\Gamma}^c(\epsilon^k, t, T) \times S^k(t,T)$$

  with $S^k(t,T) = S^k(1,t,T)$ from (3.61), $\Gamma^k(t,T) = \Gamma^k(1,t,T)$ from (3.78), $\bar{S}^c$ from (3.73) and $\bar{\Gamma}^c$ from (3.85).

- The default correlation between firm 1 and 2 is given by

$$\rho^{1,2}(t,T) = \frac{S^1(t,T)S^2(t,T)\left[S^c(\epsilon^1+\epsilon^2,t,T) - S^c(\epsilon^1,t,T)S^c(\epsilon^2,t,T)\right]}{\sqrt{\mathbb{Q}_D^1(t,T)[1-\mathbb{Q}_D^1(t,T)]\mathbb{Q}_D^2(t,T)[1-\mathbb{Q}_D^2(t,T)]}}$$

where for $k = 1, 2$ we have $S^k(t,T) = S^k(1,t,T)$ from (3.61), $\mathbb{Q}_D^k(t,T) = 1 - \mathbb{Q}_S^k(t,T)$, for $\mathbb{Q}_S^k$ from (3.86) and S^c as in (3.72).

Figure 3.4 shows simulated default times for different choices of ϵ_i. The left plot has $\epsilon_i = 0.1$ while the right plot has $\epsilon_i = 0.5$. Especially the plot on the r.h.s. shows a strong dependence of the two default times.

3.6 Conclusion

We have presented a class of reduced-form models for credit risk for which it is possible to compute, in closed-form, all relevant quantities in credit risk modeling both at the firm level and at the portfolio level. In addition we computed explicit formulas for pricing credit derivatives such as CDSs, CDOs and FtDS.

In our class of models we use intensities that have both a predictable component and an unpredictable component. The predictable component is of the general quadratic type as introduced in Chapterpaper1a, while the unpredictable component is modeled as a shot noise process. Quadratic models are particularly useful for modeling intensities since, by definition, they need to be strictly positive processes. Furthermore, it is also well known that it is the largest polynomial order one can deal with without introducing arbitrage (recall discussion in Section 2.5 in the previous chapter). The shot noise component is essential in producing realistic default correlation levels across firms.

The illustration section helps to clarify the use of the abstract results.

In our opinion, the class of models proposed in this chapter is particularly suited to fit real data and handle in closed-form portfolio issues. A natural step for future research is the calibration of a concrete model to market data. Another direction for research would be to extend the shot-noise component of the model to self-exciting processes, which would allow us to deal with features such as contagion. In terms of prices, it would be interesting to formalize and handle cash-flow CDOs, which have been much less studied in the literature and possess interesting embedded options, and to exploit the connection with currently proposed credit indices.

A Appendix : Technical details and Proofs

Some of the proofs make use of the following Lemmas.

Lemma 3.53. *For any deterministic function G of the state variable Z, and any function F, quadratic in the state variable Z, s.t.*

$$F(t,z) = \phi_1(t) + \phi_2^\top(t)z + z^\top \phi_3(t) z$$

the following property holds

$$\mathbb{E}^{\mathbb{Q}}\left[G(Z_T,T)e^{-\int_t^T F_s ds}|\mathcal{F}_t\right] = g(t,Z_t,T)e^{A(t,T)+B^\top(t,T)Z_t+Z_t^\top C(t,T)Z_t} \tag{3.88}$$

where $(A,B,C,\phi_1,\phi_2,\phi_3)$ solve the basic ODE system of Definition 3.4 and g solves the following PDE

$$\begin{cases} \dfrac{\partial g}{\partial t} + \sum_i \dfrac{\partial g}{\partial z_i}\alpha_i + \dfrac{1}{2}\sum_{ij}\left(\dfrac{\partial^2 g}{\partial z_i \partial z_j} + \dfrac{\partial g}{\partial z_i}\dfrac{\partial h}{\partial z_j} + \dfrac{\partial g}{\partial z_j}\dfrac{\partial h}{\partial z_i}\right)\sigma_i\sigma_j &= 0 \\ g(T,z,T) &= G(z,T) \end{cases}$$

Proof. Let $y(t,Z_t,T) = \mathbb{E}^{\mathbb{Q}}\left[G(T,Z_T,T)e^{-\int_t^T F_s ds}|\mathcal{F}_t\right]$. Then it must solve the PDE

$$\begin{cases} \dfrac{\partial y}{\partial t} + \sum_i \dfrac{\partial y}{\partial z}\alpha_i + \dfrac{1}{2}\sum_{ij}\dfrac{\partial^2 y}{\partial z_i \partial z_j}\sigma_i\sigma_j &= Fy \\ y(T,z,T) &= G(z,T) \end{cases} \tag{3.89}$$

where all partial derivatives should be evaluated at (t,T) and α and σ are the drift and diffusion of the state variable Z as defined in (3.4).

We start by doing some computations that turn out to be useful later on. If above expectation would be the form

$$y(t,z,T) = g(t,z,T)e^{A(t,T)+B^\top(t,T)z+z^\top C(t,T)z} = g(t,z,T)e^{h(t,z,T)}$$

where z is allowed to be multi-dimensional, we have the following partial derivatives

$$\begin{aligned} \frac{\partial y}{\partial t} &= \frac{\partial g}{\partial t}\cdot e^h + \frac{\partial h}{\partial t}\cdot g\cdot e^h = \frac{\partial g}{\partial t}\cdot e^h + \frac{\partial h}{\partial t}\cdot y \\ \frac{\partial y}{\partial z_i} &= \frac{\partial g}{\partial z_i}e^h + g\cdot\frac{\partial h}{\partial z_i}\cdot e^h = \frac{\partial g}{\partial z_i}e^h + \frac{\partial h}{\partial z_i}\cdot y \\ \frac{\partial^2 y}{\partial z_i\partial z_j} &= \left[\frac{\partial^2 g}{\partial z_i \partial z_j}\cdot e^h + \frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot e^h + \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}e^h + g\left(\frac{\partial^2 h}{\partial z_i\partial z_j}\cdot e^h + \frac{\partial h}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot e^h\right)\right] \\ &= \frac{\partial^2 g}{\partial z_i\partial z_j}\cdot e^h + \frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot e^h + \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}\cdot e^h + \frac{\partial^2 h}{\partial z_i \partial z_j}\cdot y + \frac{\partial h}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot y \end{aligned}$$

And if that is so the PDE (3.89) becomes

$$\begin{cases} \frac{\partial g}{\partial t}\cdot e^h + \frac{\partial h}{\partial t}\cdot y + \sum_i \left(\frac{\partial g}{\partial z_i}e^h + \frac{\partial h}{\partial z_i}\cdot y\right)\alpha_i + \\ \qquad +\frac{1}{2}\sum_{ij}\left(\frac{\partial^2 g}{\partial z_i \partial z_j}\cdot e^h + \frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot e^h + \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}\cdot e^h\right)\sigma_i\sigma_j \\ \qquad +\frac{1}{2}\sum_{ij}\left(\frac{\partial^2 h}{\partial z_i \partial z_j}\cdot y + \frac{\partial h}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot y\right)\sigma_i\sigma_j = Fy \\ y(T,z,T) = G(z,T) \end{cases}$$

which using separation of variables in y and e^h can be split into two PDEs, one for g and one for h.

$$\begin{cases} \frac{\partial h}{\partial t} + \sum_i \frac{\partial h}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{ij}\left(\frac{\partial^2 h}{\partial z_i \partial z_j} + \frac{\partial h}{\partial z_i}\frac{\partial h}{\partial z_j}\right)\sigma_i\sigma_j = F \\ h(T,z,T) = 0 \end{cases} \tag{3.90}$$

$$\begin{cases} \frac{\partial g}{\partial t} + \sum_i \frac{\partial g}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{ij}\left(\frac{\partial^2 g}{\partial z_i \partial z_j} + \frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j} + \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}\right)\sigma_i\sigma_j = 0 \\ g(T,z,T) = G(z,T) \end{cases} \tag{3.91}$$

To prove the result it remains to show that $h(t,z,T) = A(t,T) + B^\top(t,T)z + z^\top(t,T)z$ with A, B and C from the basic ODE system of Definition 3.4 solves the PDE (3.90). The result follows from

$$\frac{\partial h}{\partial t} = \frac{\partial \bar{A}}{\partial t} + \frac{\partial \bar{B}}{\partial t}^\top z + z^\top \frac{\partial \bar{C}}{\partial t} z \qquad \frac{\partial h}{\partial z_i} = \left(\bar{B}_i + 2\bar{C}_i z\right) \qquad \frac{\partial^2 h}{\partial z_i \partial z_j} = 2\bar{C}_{ij}$$

and the fact that the PDE (3.90) becomes a separable equation equivalent to the basic ODE system. ■

With the notation $\tilde{J}$ from Equation (3.22), $J(t,T)$ in (3.33) and $D(\theta,\cdot)$ from Remark 3.36 we have the following lemma.

Lemma 3.54. *Let $x = T - t$ and consider r as in (3.5), J as in (3.12) and η as in (3.11) and some constant $\theta \in \mathbb{R}$. For (ii) we require existence of $D(\theta, x)$ and for (v) also in some surrounding of x. Then,*

$$\begin{aligned}
(i) \quad S_\eta(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\eta_s ds}|\mathcal{F}_t^W\right] \\
&= \exp\Big(\mathcal{A}(\theta,t,T) + \mathcal{B}^\top(\theta,t,T)Z_t + Z_t^\top \mathcal{C}(\theta,t,T)Z_t\Big) \\
(ii) \quad S_J(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta J_s ds}|\mathcal{F}_t^J\right] \\
&= \exp\Big(\theta(\tilde{J}_t - \tilde{J}(t,T)) + lx[D(\theta,x) - 1]\Big) \\
(iii) \quad \bar{S}(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\eta_s ds}|\mathcal{F}_t^W\right] \\
&= \exp\Big(\bar{A}(\theta,t,T) + \bar{B}^\top(\theta,t,T)Z_t + Z^\top(t)\bar{C}(\theta,t,T)Z_t\Big)
\end{aligned}$$

$$\begin{aligned}
(iv) \quad \Gamma_\eta(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta_T e^{-\int_t^T \theta\eta_s ds}|\mathcal{F}_t^W\right] \\
&= \big(a(\theta,t,T) + b^\top(\theta,t,T)Z_t + Z_t^\top c(\theta,t,T)Z_t\big)\cdot S_\eta(\theta,t,T)
\end{aligned}$$

$$\begin{aligned}
(v) \quad \Gamma_J(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta J_T e^{-\int_t^T \theta J_s ds}|\mathcal{F}_t^J\right] \\
&= S_J(\theta,t,T)\cdot\Big\{\theta J(t,T) - l\cdot\Big[D(\theta,x)(1-x) - 1 + x\varphi_Y\big(\theta H(x)\big)\Big]\Big\}
\end{aligned}$$

$$\begin{aligned}
(vi) \quad \bar{\Gamma}(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta_T e^{-\int_t^T r_s+\theta\eta_s ds}|\mathcal{F}_t^W\right] \\
&= \big(\bar{a}(\theta,t,T) + \bar{b}^\top(\theta,t,T)Z_t + Z_t^\top \bar{c}(\theta,t,T)Z_t\big)\cdot \bar{S}_\eta(\theta,t,T)
\end{aligned}$$

where $(\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \theta\mathsf{f},\ \theta\mathsf{g},\ \theta\mathrm{Q})$ *and* $(\bar{A},\ \bar{B},\ \bar{C},\ f+\theta\mathsf{f},\ g+\theta\mathsf{g},\ Q+\theta\mathrm{Q})$ *solve the basic ODE system of Definition 3.4, while* $(a,\ b,\ c,\ \mathcal{B},\ \mathcal{C},\ \theta\mathsf{f},\ \theta\mathsf{g},\ \theta\mathrm{Q})$ *and* $(\bar{a},\ \bar{b},\ \bar{c},\ \bar{B},\ \bar{C},\ \theta\mathsf{f},\ \theta\mathsf{g},\ \theta\mathrm{Q})$ *solve the interlinked system of Definition (3.22).*
Furthermore,

$$\begin{aligned}
(vii) \quad S(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\mu_s ds}|\mathcal{F}_t\right] = S_\eta(\theta,t,T)\cdot S_J(\theta,t,T) \\
(viii) \quad \bar{S}(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\mu_s ds}|\mathcal{F}_t\right] = \bar{S}_\eta(\theta,t,T)\cdot S_J(\theta,t,T) \\
(ix) \quad \Gamma(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\mu_T e^{-\int_t^T \theta\mu_s ds}|\mathcal{F}_t\right] \\
&= \Gamma_\eta(\theta,t,T)\cdot S_J(\theta,t,T) + \Gamma_J(\theta,t,T)\cdot S_\eta(\theta,t,T) \\
(x) \quad \bar{\Gamma}(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\mu_T e^{-\int_t^T r_s+\theta\mu_s ds}|\mathcal{F}_t\right] \\
&= \bar{\Gamma}_\eta(\theta,t,T)\cdot S_J(\theta,t,T) + \Gamma_J(\theta,t,T)\cdot \bar{S}_\eta(\theta,t,T)
\end{aligned}$$

Proof. Properties $(vii)-(ix)$ follow from those in $(i)-(vi)$ by independence between $\mathcal{F}^W$ and $\mathcal{F}^J$ and the fact that $\mu = \eta + J$. Note also that (i) follows from (iii) as well as (iv) from (vi) when we take $f(t)=0, g(t)=0, Q(t)=0 \Rightarrow r_t = 0, \forall t$. Thus, it remains to prove (ii), (iii), (v), (vi). These four expectations, however, are quite similar to some expectation computed in the main text. They can be computed using the more ore less the same methodology as already laid out, being cautious with the constant "θ".

Proof of (ii). We basically mimic the proof of Lemma 3.18. Recall the notation of $\tilde{J}$ from (3.22) and set $x = T - t$. Then,

$$S_J(\theta,t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta J_u\,du}|\mathcal{F}_t^J\right]$$
$$= \exp\left\{\theta(\tilde{J}_t - \tilde{J}(t,T))\right\}\mathbb{E}^{\mathbb{Q}}\left[\exp\Big(-\sum_{\tilde{\tau}_i\in(t,T]}\theta Y_i\int_t^T \mathbf{1}_{\{\tilde{\tau}_i\le u\}}h(u-\tilde{\tau}_i)\,du\Big)\Big|\mathcal{F}_t^J\right].$$

The expectation equals

$$e^{-lx} + \sum_{k=1}^{\infty} e^{-lx}\frac{(lx)^k}{k!}\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=1}^{k} Y_i\theta H\big(x(1-\eta_i)\big)\right)\right] = e^{lx(D(\theta,x)-1)}, \tag{3.92}$$

as

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-Y_1\theta H\big(x(1-\eta_1)\big)\right)\right] = \int_0^1 \varphi_Y\Big(\theta H(xu)\Big)\,du = D(\theta,x).$$

Proof of (iii)

$$\begin{aligned}\bar{S}(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\eta_s ds}|\mathcal{F}_t^W\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \left(Z_s^\top(Q+\theta\mathrm{Q}^c(s))Z_s+(g+\theta\mathrm{g}^c(s))^\top Z_s+(f+\theta\mathrm{f}^c(s))\,ds\right)}|\mathcal{F}_t^W\right]\\ &= \exp\left\{\bar{A}(\theta,t,T)+\bar{B}^\top(\theta,t,T)Z_t + Z_t^\top\bar{C}(\theta,t,T)Z_t\right\}.\end{aligned}$$

Comparing with (3.27) and as $(\bar{A},\bar{B},\bar{C},f+\theta\mathrm{f},g+\theta\mathrm{g},Q+\theta\mathrm{Q})$ solve the basic system of ODEs from Definition 3.4 the result follows.

Proof of (v) Recall the notations for $\tilde{J}(t,T)$ and $J(t,T)$ introduced in (3.22) and (3.33), respectively. Proceeding similar to the proof of (ii) we split in a measurable and future part.

$$\Gamma_J(\theta,t,T) = \mathbb{E}^{\mathbb{Q}}\left[\theta\Big(\sum_{\tilde{\tau}_i\le t}Y_ih(T-\tilde{\tau}_i)+\sum_{\tilde{\tau}_i\in(t,T]}Y_ih(T-\tilde{\tau}_i)\Big)e^{-\int_t^T\theta J_s\,ds}\Big|\mathcal{F}_t^J\right]$$
$$= \theta J(t,T)S_J(\theta,t,T) + e^{\theta(\tilde{J}_t-\tilde{J}(t,T))}\mathbb{E}^{\mathbb{Q}}\left[\theta\sum_{\tilde{\tau}_i\in(t,T]}Y_ih(T-\tilde{\tau}_i)e^{-\theta\int_t^T\sum_{\tilde{\tau}_i\in(t,s]}Y_iH(s-\tilde{\tau}_i)\,ds}\Big|\mathcal{F}_t^J\right].$$

Next, we consider the expectation more closely. The idea is to consider $\tilde{D}$ and derive w.r.t. T. We define

$$\tilde{J}^t(s) := \sum_{\tilde{\tau}_i \in (t,s]} Y_i h(s - \tilde{\tau}_i).$$

Then, the above expectation can be stated in a form suitable for our derivation:

$$\mathbb{E}^{\mathbb{Q}}\left[\sum_{\tilde{\tau}_i \in (t,T]} Y_i h(T - \tilde{\tau}_i) e^{-\theta \int_t^T \sum_{\tilde{\tau}_i \in (t,s]} Y_i H(s-\tilde{\tau}_i)\, ds} \middle| \mathcal{F}_t^J \right] = \mathbb{E}^{\mathbb{Q}}\left[\tilde{J}^t(T) e^{-\theta \int_t^T \tilde{J}^t(s)\, ds} \middle| \mathcal{F}_t^J \right] \tag{3.93}$$

Note that H is continuous and recall (3.92). So, if $D(\theta, x)$ exists in a neighborhood of x, we can derive the following expression w.r.t. x and obtain

$$\begin{aligned}\frac{\partial}{\partial x} e^{lx\left(\tilde{D}(\theta,x)-1\right)} &= \frac{\partial}{\partial x} \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T \theta \tilde{J}^t(s)\, ds} \middle| \mathcal{F}_t^J \right) \\ &= -\mathbb{E}^{\mathbb{Q}}\left(\theta \tilde{J}_T^t \cdot e^{-\int_t^T \theta \tilde{J}^t(s)\, ds} \middle| \mathcal{F}_t^J \right).\end{aligned}$$

The last equation follows if D is bounded in a neighborhood of x. This follows if the Laplace transform is continuous around x. So we found a nice expression of (3.93). With

$$\begin{aligned}\frac{\partial}{\partial x} D(\theta, x) &= \int_0^1 \varphi_Y'\left(\theta H(xu)\right) \cdot \theta h(xu) \cdot u\, du \\ &= \varphi_Y\left(\theta H(xu)\right) u \Big|_0^1 - \int_0^1 \varphi_Y\left(\theta H(xu)\right) du \\ &= \varphi_Y\left(\theta H(x)\right) - D(\theta, x).\end{aligned}$$

we obtain

$$\frac{\partial}{\partial x} e^{lx\left(D(\theta,x)-1\right)} = e^{lx\left(D(\theta,x)-1\right)} \cdot l \cdot \left[D(\theta,x)(1-x) - 1 + x\varphi_Y\left(\theta H(x)\right) \right] \tag{3.94}$$

Noticing that $e^{\left\{\theta(\tilde{J}_t - \tilde{J}(t,T)\right\}} e^{\left\{lx(\tilde{D}(\theta,x)-1)\right\}} = S_J(\theta, t, T)$, we conclude.

Proof of (vi)

Let us denote

$$y(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\theta \eta_T e^{-\int_t^T r_u + \theta \eta_u du} | \mathcal{F}_t^W \right]$$

since both r and η are quadratic functions of our factors Z and setting $G(T, z) = \theta\eta(T, z)$ we are exactly under the conditions of Lemma 3.53. Thus we know

$$\mathbb{E}^{\mathbb{Q}}\left[G(Z_T, T) e^{-\int_t^T r_s + \mu_s ds} | \mathcal{G}_t \right] = g(t, Z_t, T) \underbrace{e^{\tilde{A}(t,T) + \tilde{B}^\top(t,T) Z_t + Z_t^\top \tilde{C}(t,T) Z_t}}_{\tilde{S}_\eta(\theta,t,T)} \tag{3.95}$$

and g solves the following PDE

$$\left\{\begin{array}{rcl} \frac{\partial g}{\partial t}+\sum_i \frac{\partial g}{\partial z_i}\alpha_i+\frac{1}{2}\sum_{ij}\left(\frac{\partial^2 g}{\partial z_i \partial z_j}+\frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j}+\frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}\right)\sigma_i\sigma_j & = & 0 \\ g(T,z,T) & = & \eta(T,z) \end{array}\right. \tag{3.96}$$

and it remains to show that

$$g(t,z,T)=\bar{a}(\theta,t,T)+\bar{b}^\top(\theta,t,T)z+z^\top\bar{c}(\theta,t,T)z \tag{3.97}$$

with $(\bar{a},\ \bar{b},\ \bar{c},\ \bar{B},\bar{C},f+\theta\mathsf{f},f+\theta\mathsf{f},f+\theta\mathsf{f})$ solving the interlinked ODE system of Definition 3.22. We now compute

$$\frac{\partial g}{\partial t}=\frac{\partial \bar{a}}{\partial t}+\frac{\partial \bar{b}}{\partial t}z+z^\top\frac{\partial \bar{c}}{\partial t}z\ , \qquad \frac{\partial g}{\partial z_i}=\bar{b}_i+2\bar{c}_i z\ , \qquad \frac{\partial^2 g}{\partial z_i\partial z_j}=2\bar{c}_{ij}.$$

Replacing in the PDE (3.96), all these partial derivatives and η, α and $\sigma\sigma^\top$ using equations (3.11) and (3.4), we get an equivalent PDE, which in vector notation becomes[27]

$$\left\{\begin{array}{rcl} \frac{\partial \bar{a}}{\partial t}+\frac{\partial \bar{b}}{\partial t}z+z^\top\frac{\partial \bar{c}}{\partial t}z+d^\top\bar{b}+\left(E^*\bar{b}\right)z+\left(2\bar{c}d\right)z+z^\top\left(\bar{c}E\right)z & & \\ +z^\top\left(E^*\bar{c}\right)z+\frac{1}{2}\left[\bar{B}^\top k_0 b+2\,\mathrm{tr}\left\{\bar{c}K_0\right\}+\left(\bar{\tilde{B}}^\top K\bar{b}\right)z\right] & & \\ +\frac{1}{2}\left[\left(2\bar{C}k_0\bar{b}+2\bar{c}k_0\bar{B}\right)z+z^\top\left(4\bar{C}k_0\bar{c}\right)z+z^\top\left(\bar{\tilde{B}}G\bar{\tilde{b}}\right)z\right] & = & 0 \\ g(T,z,T) & = & \theta\eta(T,z) \end{array}\right. \tag{3.98}$$

From the analysis of the PDE equation one soon realizes it is separable, in terms independent of z, linear in z and quadratic in z equivalent to the interlinked ODE system of Definition 3.22. To check the boundary conditions, note

$$\begin{array}{rcl} g(T,z,T) & = & \theta\eta(T,z) \\ & \Leftrightarrow & \\ \bar{a}(\theta,T,T)+\bar{b}^\top(\theta,T,T)z+z^\top\bar{c}(\theta,T,T)z & = & z^\top\mathsf{Q}(T)z+\mathsf{g}^\top(T)z+\mathsf{f}(T) \\ & \Downarrow & \\ \bar{a}(\theta,T,T)=\theta\mathsf{f}(T)\quad \bar{b}(\theta,T,T) & = & \theta\mathsf{g}(T)\quad \bar{c}(\theta,T,T)=\theta\mathsf{Q}(t). \end{array}$$

Finally, this implies

$$\bar{\Gamma}(\theta,t,T)=\left(\bar{a}(\theta,t,T)+\bar{b}^\top(\theta,t,T)z+z^\top\bar{c}(\theta,t,T)z\right)\cdot\bar{S}_\eta(\theta,t,T)\ .$$

■

[27] Terms of order higher than two are omitted from the equation since the final solution must set those terms equal to zero and they are hard to write in vector notation.

Proof of Proposition 3.33

Proof. (missing details)

It remains to show that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_1} r_s+\eta_s ds}p(T_1,T_2)|\mathcal{F}_t^W\right] = e^{\alpha(t,T_1,T_2)+\beta^\top(t,T_1,T_2)Z_t+Z_t^\top\gamma(t,T_1,T_2)Z_t} \cdot e^{\bar{A}(t,T_1)+\bar{B}^\top(t,T_1)Z_t+Z_t^\top\bar{C}(t,T_1)Z_t}$$

where α, β and γ are deterministic and solve the system stated in the Proposition.

This is the case because the expectation is under the conditions of Lemma 3.53 where the risk-free bond price $p(T_1,T_2)$ works like any other function known at time T_1 and concretely we know from Result 3.5 that

$$p(T_1,T_2) = \exp\left(A(T_1,T_2) + B^\top(T_1,T_2)Z_{T_1} + Z_{T_1}^\top C(T_1,T_2)\right)$$

thus,

$$G(T_1,Z_{T_1}) = \exp\left(A(T_1,T_2) + B^\top(T_1,T_2)Z_{T_1} + Z_{T_1}^\top C(T_1,T_2)\right).$$

and $y(t,z,T_1) = g(t,z,T_1)e^{h(t,z,T_1)}$.

As we see T_2 is just a parameter. To be easier to identify the role of T_2 we write $G(T_1,Z_{T_1},T_2)$, $g(t,Z_t,T_1,T_2)$ instead of just $G(T_1,Z_{T_1})$, $g(t,Z_t,T_1)$ but we should not forget that T_2 is just a parameter and will play no important role in the PDE we have to solve (note the boundary at T_1). Finally we know that g solves

$$\begin{cases} \dfrac{\partial g}{\partial t} + \sum_i \dfrac{\partial g}{\partial z_i}\alpha_i + \dfrac{1}{2}\sum_{ij}\left(\dfrac{\partial^2 g}{\partial z_i \partial z_j} + \dfrac{\partial g}{\partial z_i}\dfrac{\partial h}{\partial z_j} + \dfrac{\partial g}{\partial z_j}\dfrac{\partial h}{\partial z_i}\right)\sigma_i\sigma_j = 0 \\ g(T_1,z,T_1,T_2) = G(T_1,Z_{T_1},T_2) \end{cases}$$

Finally we note that for $g(T_1,z,T_2) = \exp\left(\alpha(T_1,T_2) + \beta^\top(T_1,T_2)Z_t + Z_t^\top\gamma(T_1,T_2)Z_t\right)$,

$$\begin{aligned} \frac{\partial g}{\partial t} &= \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \beta}{\partial t}z + z^\top\frac{\partial \gamma}{\partial t}z\right)g \\ \frac{\partial g}{\partial z_i} &= (\beta_i + 2\gamma_i z)\,g \\ \frac{\partial^2 g}{\partial z_i \partial z_j} &= \left[2\gamma_{ij} + (\beta_i + 2\gamma_i x)(\beta_j + 2\gamma_j x)\right]g \end{aligned}$$

Replacing these in the above PDE, as well as α_i and $\sigma_i\sigma_j^\top$ from (3.4), leaves us with a PDE which is separable and solves the required ODEs given in the proposition. ∎

B Appendix : Laplace transform of shot-noise processes

In this section we compute the conditional Laplace-transform for shot-noise processes. The conditional Fourier-transform follows similarly. We also comment on the conditional distribution function.

Recall $x = T - t$ and that φ_Y denoted the Laplace transform of the jump heights $Y_1, Y_2, \dots$.

Proposition 3.55. *If*

$$D(\theta, x) := \int_0^1 \varphi_Y\Big(\theta \cdot h(xu)\Big)\, du$$

exists for all $v \geq 0$, then the conditional Laplace transform of J equals

$$\begin{aligned}\varphi_{J_{t+x}|\mathcal{F}_t^J} &= \mathbb{E}^{\mathbb{Q}}\Big(e^{-\theta J_{t+x}}|\mathcal{F}_t^J\Big) \\ &= \exp\Big[lx\big(D(\theta,x)-1\big) - \theta \sum_{\tilde{\tau}_i \leq t} Y_i h(t+x-\tilde{\tau}_i)\Big].\end{aligned}$$

Proof. First, we distinguish the measurable part from the future part:

$$\mathbb{E}^{\mathbb{Q}}\Big(e^{-\theta J_{t+x}}|\mathcal{F}_t^J\Big) = e^{-\theta \sum_{\tilde{\tau}_i \leq t} Y_i h(t+x-\tilde{\tau}_i)} \cdot \mathbb{E}^{\mathbb{Q}}\Big[e^{-\theta \sum_{\tilde{\tau}_i \in (t,t+x]} Y_i h(T-\tilde{\tau}_i)}\Big|\mathcal{F}_t^J\Big].$$

Second, we compute the expectation proceeding similarly as in the proof or Lemma 3.18. To this, denote by $\eta_1, \eta_2, \dots$ i.i.d. $U[0,1]$ variables. Then

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\Big[e^{-\theta \sum_{\tilde{\tau}_i \in (t,t+x]} Y_i h(T-\tilde{\tau}_i)}\Big|\mathcal{F}_t^J\Big] \\ &\qquad = e^{-lx} + \sum_{k=1}^{\infty} e^{-lx}\frac{(lx)^k}{k!}\mathbb{E}^{\mathbb{Q}}\Big(e^{-\theta \sum_{i=1}^k Y_i h\left(x(1-\eta_i)\right)}\Big).\end{aligned}$$

The expectation can be computed using the Laplace transform of the Y_i, as

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big(e^{-\theta \sum_{i=1}^k Y_i h\left(x(1-\eta_i)\right)}\Big) &= \Big[\mathbb{E}^{\mathbb{Q}}\Big(e^{-\theta Y_1 h\left(x(1-\eta_1)\right)}\Big)\Big]^k \\ &= \Big[\int_0^1 \varphi_Y\Big(\theta h\big(xu\big)\Big)\, du\Big]^k = D(\theta,x)^k.\end{aligned}$$

Hence,

$$\mathbb{E}^{\mathbb{Q}}\Big[e^{-\theta \sum_{\tilde{\tau}_i \in (t,t+x]} Y_i h(T-\tilde{\tau}_i)}\Big|\mathcal{F}_t^J\Big] = \exp\Big[-lx + lx\, D(\theta,x)\Big]$$

and the conclusion follows. ∎

With the conditional Laplace transform at hand, one can use Raible's method to numerically derive prices for any European contingent claim, cf. Eberlein and Raible (1999).

Sometimes, it can be possible to obtain the distribution function quite explicitly. This is, for example, the case when the Y_i are normally distributed. Of course, in our framework, this is not suitable. But in the case of χ^2-distributions one can proceed similarly. Proceeding as above,

$$\mathbb{Q}\big(J_T \leq c|\mathcal{F}_t^J\big) = \mathbb{Q}\Big(\sum_{\tilde{\tau}_i \in (t,T]} Y_i h(T-\tilde{\tau}_i) \leq c - \sum_{\tilde{\tau}_i \leq t} Y_i h(T-\tilde{\tau}_i) \big| \mathcal{F}_t^J \Big)$$

and it becomes clear that for computing the conditional distribution one needs a nice expression for

$$\mathbb{Q}\Big(\sum_{i=1}^{k} Y_i h\big(x(1-\eta_i)\big) \leq c\Big).$$

Chapter 4

Including Macroeconomic Risks

"An approximate answer to the right problem is worth a good deal more than an exact answer to an approximate problem."

John Tukey, 1977.

We start by presenting a reduced-form multiple default type of model and derive abstract results on the influence of a state variable X on credit spreads when both the intensity and the loss quota distribution are driven by X. The aim is to apply the results to a real life situation, namely, to the influence of macroeconomic risks on the term structure of credit spreads.

There is increasing support in the empirical literature for the proposition that both the probability of default (PD) and the loss given default (LGD) are correlated and driven by macroeconomic variables. Paradoxically, there has been very little effort, from the theoretical literature, to develop credit risk models that would take this into account. One explanation might be the additional complexity this leads to, even for the "treatable" default intensity models.

The goal of this chapter is to develop the theoretical framework necessary to deal with this situation and, through numerical simulation, understand the impact of macroeconomic factors on the term structure of credit spreads. In the proposed setup, periods of economic depression are both periods of higher default intensity and lower recovery, producing a *business cycle effect.*

4.1 Introduction

Recent empirical studies show that there is a significant systematic risk component in defaultable credit spreads. See Frye (2000a), Frye (2000b), Frye (2003), Altman, Resti, and Sironi (2004), Düllmann and Trapp (2000) or Elton and Gruber (2004). The model underlying the Basel II internal ratings-based capital calculation (see Basel Committee (2003) and Wilde (2001) measures credit portfolio losses only, that is, portfolio losses that are due to external influences and hence cannot be diversified away. This gives us an indication of what the main concerns are in practice and highlights the need for a realistic model of systematic risk. The purpose of this study is to present a reduced-form multiple default model, which analyzes the influence of macroeconomic risks on the term structure of credit spreads at the firm level.

The current theoretical literature considers models where only the default intensity, or equivalently, the *probability of default* (PD) is dependent on a state variable assuming that the *loss given default* (LGD) is either fixed or at least independent of default intensities. See Wilson (1997), Saunders (1999), JP Morgan (1997), Gordy (2000), or Schönbucher (2001). We take that analysis one step further and consider the situation in which the same state variable influences both PD and LGD, making these two quantities dependent on one another.

Both PD and LGD are key in accessing expected capital losses and measuring the exposure of portfolios of defaultable instruments to credit risk. In accessing capital at risk, it is extremely important not to ignore the interdependence between PD and LGD, since this would lead to underestimation of the true risk borne by portfolio holders.

Eventually, we are interested in the case where our state variable represents macroeconomic risks. As a proxy for macroeconomic conditions we consider a market index. It is well known that market uncertainty and its level are negatively correlated. See, for instance, Gaspar (2001) and a recent study by Giese (2005). That is, periods of recession (low index level) also tend to be periods of high uncertainty (high index volatility) reflecting some sort of market panic, while periods of economic boom are perceived as safe periods and with low uncertainty. In setting up the dynamics of the market index, we incorporate this realistic feature by allowing the local volatility of the index to depend negatively on its level.

In terms of the PD and LGD, we concretely take the default intensity and the recovery (given default) to depend on the market situation (the index level). With the PD dependence we try to account for the fact that during bad economic times it is reasonable to expect more defaults, while with the LGD dependence we try to account for the fact that if the entire market is down, the market value of any firm's assets should be lower, and debt holders should recover less if a default occurs.

The main contributions of this study can be summarized as follows.

- We derive abstract results for a multiple default reduced-form model when the default events are modeled by a doubly stochastic marked point process (DSMPP), where both intensity and the mark's density depend on a state variable X.
- We propose a model for the influence of macroeconomic risks on credit spreads.
- Using a concrete model, we are able to simulate realistic behaviors of the term structure of credit spreads.

The rest of the chapter is organized as follows. In Section 4.2 we concretize our reduced-form multiple default model and justify the modeling choices, which involve modeling all the variables under the risk-neutral measure $\mathbb{Q}$. We start by describing the setup for the default-free and defaultable bond market and we continue by deriving the abstract results. Finally, we discuss the relation between the risk-neutral assumptions and the objective $\mathbb{P}$-assumptions.

In Section 4.3 we introduce the macroeconomic model, presenting the index dynamics and justifying the assumptions about the influence of such risks on the intensity and recovery processes using empirical facts. We derive qualitative results on the influence of the market index on credit spreads.

In Section 4.4 we present a concrete model for the intensity and recovery dependence and simulate their impacts on credit spread term structures.

Section 4.5 concludes, summarizing the main results and suggesting directions for future research.

4.2 The Setup and Abstract Results

We consider a financial market living on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{0\leq t\leq T})$ where $\mathbb{Q}$ is the risk-neutral probability measure.

The probability space carries a multidimensional Wiener process W and, in addition, a doubly stochastic marked point process (DSMPP), $\mu(dt, dq)$, on a measurable mark space $(E, \mathcal{E})$ to model the default events.

The filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ is generated by W and μ, i.e. $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$.

We, now, introduce the details.

4.2.1 Default-free bond market

We assume the existence of a liquid market for default-free zero-coupon bonds, for every possible maturity T. We denote the price at time t of a default-free zero-coupon bond with maturity T by $p(t,T)$.

In order to describe the default-free bonds market we use Heath-Jarrow-Morton (HJM) framework, modeling directly, under the martingale measure $\mathbb{Q}$, the dynamics of the instantaneous forward rates, $f(t,T)$. There is a one-to-one correspondence between zero-coupon bond prices and forward rates, so by assuming existence of zero-coupon bond prices for all maturities we also guarantee existence of forward rates for all maturities. We recall the fundamental relation:

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} \qquad \Leftrightarrow \qquad p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}. \tag{4.1}$$

Assumption 4.1. (Default-free forward rates)

The dynamics of default free forward rates, under the martingale measure $\mathbb{Q}$, are given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \tag{4.2}$$

with

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma^*(t,s)ds. \tag{4.3}$$

where $\sigma(\cdot,T)$ is a row vector of regular enough adapted processes, W is a $\mathbb{Q}$-Wiener process.

The default free short-rate is $r(t) = f(t,t)$ and the default free zero-coupon bond prices are denoted by $p(t,T)$. No arbitrage and the fundamental relation in (4.1) give use the bond prices dynamics as

$$\frac{dp(t,T)}{p(t,T)} = r(t)dt + \eta(t,T)dW(t)$$

where $\eta(t,T) = -\int_t^T \sigma(t,s)ds$.

4.2.2 Defaultable bonds market

In addition to the risk-free bond market mentioned above, we consider a defaultable bond market. We assume that each company on the market issues a continuum of bonds with maturities T.

Assumptions 4.6 below characterize the default events and the dependence of both the default intensity and the recovery rate distribution on an abstract stochastic state variable X.

Assumption 4.2. *There exist an underlying stochastic state variable X, whose dynamics under the risk neutral measure $\mathbb{Q}$ are given by*

$$dX_t = \alpha_X(t, X_t)dt + \sigma_X(t, X_t)dW_t \tag{4.4}$$

where W, is the same as in (4.2).[1]

Concepts and Assumptions

In Assumption 4.6 we will define the basic multiple default setup. A multiple default setup is based on the observation that whenever the obligor defaults, the company is not liquidated but instead reorganized. The firm and its claims continue to live and operate. However, the face value of the claims is reduced by a fraction q. Behind this model is the intuition that, given a distress situation for the obligator's business, the debt holders are willing to accept the renegotiation of their claims (accepting to lose some fraction q of the face value of the claims) in order to avoid a process of bankruptcy, which is typically costly, and allowing the firm to continue operating.[2] It is possible that a whole sequence of defaults is taking place, every time the company reducing the face value of the debt and the bondholder accepting the conditions of the deal.

Definition 4.3. The *loss quota* is the fraction by which the promised final payoff of the defaultable claim is reduced at each time of default. We denote the loss quota by q.

[1]This way we allow for the possibility that one of our factors may be related to the risk-free side of the economy. In that case, some consistency relation must hold between (4.2) and (4.4).

[2]This model mimics the effect of a rescue plan as it is described in many bankruptcy codes. The old claimants have to give up some of their claims in order to allow for rescue capital to be invested in the defaulted firm. They are not paid out in cash (this would drain the defaulted firm of valuable liquidity) but in new defaultable bonds of the same maturity. [Schönbucher (2003)]

Definition 4.4. The *remaining value*, after all reductions in the face value of the defaultable claim due to defaults in the time interval $[0,t]$, is denoted $V(t)$.

Definition 4.5. $\bar{p}(t,T)$ is, at time t, the *price of a defaultable zero-coupon bond* with maturity T. The payoff at time T of the bond is, thus, $V(T)$ the remaining part of the face value of the bond after all reductions due to defaults in the time interval $[0,T]$, i.e.,

$$\bar{p}(T,T) = V(T).$$

Assumption 4.6.

1. *We assume that default happens at the following sequence of the stopping times $\tau_1 < \tau_2 < \ldots$, where τ_i is the time of the i-th jump of our point process.*

2. *At each default time τ_i the jump size, q_i mark or loss quota, is drawn from the mark space $E = (0,1)$.*

3. *We assume that there is no total loss at default, i.e., the loss quota $q_i < 1$ for all $i = 1, 2, \cdots$.*

4. *We assume that both:*

   (i) the arrivals of default times $(\tau_i)_{i\geq 1}$

   (ii) the distribution of the loss quotas given default $(q_i)_{i\geq 1}$

   depend on our stochastic state process X.

Given that at each default time τ_i the final claim amount is reduced by a loss quota q_i to $(1-q_i)$ times what it was before, we obtain

$$V(t) = \prod_{\tau_i \leq t} (1 - q_i), \tag{4.5}$$

where q_i is the stochastic marker to the default time τ_i.

According to risk-neutral valuation, the price at time t, of the defaultable bond with maturity T equals to

$$\bar{p}(t,T) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} V(T) \middle| \mathcal{F}_t \right].$$

or, equivalently,

$$\bar{p}(t,T) = p(t,T)\mathbb{E}^T \left[V(T) \middle| \mathcal{F}_t \right] , \tag{4.6}$$

where $\mathbb{E}^T [\cdot | \mathcal{F}_t]$ denote conditional expectation under the T-forward measure.

Definition 4.7. We define the *instantaneous defaultable forward rate*, $\bar{f}(t,T)$, similarly to its risk-free equivalent

$$\bar{f}(t,T) = -\frac{\partial}{\partial T} \ln \bar{p}(t,T). \tag{4.7}$$

The *defaultable short rate* is defined as $\bar{r}(t) = \bar{f}(t,t)$.

Using the above definition we also have

$$\bar{p}(t,T) = V(t) \exp\left\{ -\int_t^T \bar{f}(t,s)ds \right\}, \tag{4.8}$$

where recall that $\bar{p}(t,t) = V(t)$ and $V(t)$ is given in (4.5).

Definition 4.8. The *short credit spread* $s(t)$ is defined as the difference between the defaultable and non-defaultable short rates

$$s(t) = \bar{r}(t) - r(t).$$

Definition 4.9. The *forward credit spread* $s(t,T)$ is defined as the difference between the defaultable short rate and non-defaultable forward rates

$$s(t,T) = \bar{f}(t,T) - f(t,T).$$

Existence of intensity

We start by giving the abstract definitions of a Marked *Poisson* Point Process, a Cox Process and our *Doubly Stochastic* Marked Poisson Process (DSMPP). To introduce the definitions we need first to define the following filtrations.

Notation 1. (Filtrations)

- We call the filtration generated by $W(t)$ the *background filtration* $\left(\mathcal{F}_t^W\right)_{t\geq 0}$, and

  $$\mathcal{G}^W = \bigvee_{t\geq 0} \mathcal{F}_t^W,$$

  is the information set containing all future and past background information.

  In our setup it will be assumed that all the default-free processes are adapted to $\left(\mathcal{F}_t^W\right)_{t\geq 0}$.

- The full filtration is reached by combining $\left(\mathcal{F}_t^W\right)_{t\geq 0}$ and the filtration $(\mathcal{F}_t^\mu)_{t\geq 0}$ which is generated by MPP μ

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu.$$

- We define the filtration generated by *all* the information concerning the background process X, and our only past information on our MPP μ

$$\mathcal{G}_t^W = \mathcal{G}^W \vee \mathcal{F}_t^\mu.$$

Definition 4.10. (DSMPP)

- We call the Marked Point Process $\hat{\mu}$ an $\mathcal{F}_t^\mu$- Marked *Poisson* Process if there exists a deterministic measure $\hat{\nu}$ on $\mathbb{R}_+ \times E$ such that

$$\mathbb{P}\left(\hat{\mu}((s,t] \times B) = k \,|\mathcal{F}_s^\mu\right) = \frac{(\hat{\nu}((s,t] \times B))^k}{k!} e^{-\hat{\nu}((s,t]\times B)}, \quad \text{a.s.,} \quad B \in E.$$

- We call a counting process $N = (T_n)$ adapted to right-continuous filtration a $\mathcal{G}_t^W$ - Cox Process if there is an $\mathcal{G}^W$-measurable random measure ν satisfying

$$\mathbb{P}\left(N(s,t] = k \,\middle|\mathcal{G}_s^W\right) = \frac{(\nu((s,t]))^k}{k!} e^{-\nu((s,t])}, \quad \text{a.s} \quad k \in \mathbb{N}\,.$$

- We call the Marked Point Process μ an $\mathcal{G}_t^W$- DSMPP if there exists a $\mathcal{G}^W$-measurable random measure ν on $\mathbb{R}_+ \times E$ such that

$$\mathbb{P}\left(\mu((s,t] \times B) = k \,\middle|\mathcal{G}_s^W\right) = \frac{(\nu((s,t] \times B))^k}{k!} e^{-\nu((s,t]\times B)}, \quad \text{a.s.,} \quad B \in E.$$

Our goal is to construct a Marked Point Process such that its compensator is allowed to depend on our stochastic state variable X (which in general means that its conditional default distribution as well as intensity are *both* allowed to depend on X_t), and conditional on the realization of the state variable it is $\mathcal{G}_t^W$-DSMPP. In other words, we want to prove the existence of MPP such that its compensator ν is allowed to depended on our stochastic state variable X, so we can write

$$\nu(dt, dq, \omega) = \nu(t, dq, X_t), \qquad \mathbb{Q} - a.s. \tag{4.9}$$

We define the compensated point process $\tilde{\mu}(dt, dq)$ as

$$\tilde{\mu}(dt, dq) = \mu(dt, dq) - \nu(t, dq, X_t).$$

The following Theorem shows that the DSMPP with compensator of the form (4.9) exists.

Theorem 4.11.

Assume that ν admits intensity and define $\nu(dt, dq, X_t) = M_t(dq, X_t)dt$, $\mathbb{Q}-a.s$ where $M_t(dq, x)$ is a deterministic measure on E for any fixed x and t.

Let $\hat{\nu}(dt, dq) = m_t(dq)dt$ be a deterministic compensator for some Marked Poisson Process $\hat{\mu}$. Assume that:

(i) $M(t, dq, x)$ is measurable w.r.t. $\mathcal{G}^W$

(ii) $M(t, dq, x)$ is absolutely continuous w.r.t. $m(t, dq)$ on $\mathcal{E}$, that is,

$$M_t(dq, x) << m_t(dq)$$

Then, there exists a $\mathcal{G}_t^W$-DSMPP μ, such that its compensator is of the form (4.9).

Proof. We fix $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\le t\le T})$ and a Marked Point Process μ with the compensator

$$\hat{\nu}(dt, dq) = m_t(dq)dt$$

and, as before, $\mathcal{G}_t^W = \mathcal{G}^W \vee \mathcal{F}_t^\mu$.

Since $M_t(x, dq)$ is absolutely continuous w.r.t $m_t(dq)$ on $\mathcal{E}$, then according to the Radon-Nikodym Theorem for every t there exists a $\mathcal{E} \times \mathcal{G}^W$-measurable nonnegative function $\varphi_t(q, x)$, $\varphi : E \times R_+ \to R_+$, such that

$$M(t, x, A) = \int_A \varphi(t, q, x)m(t, dq), \quad \text{for all} \quad A \in \mathcal{E}$$

or

$$M(t, x, dq) = \varphi(t, q, x)m(t, dq).$$

We define the process L_t as

$$\begin{cases} dL_t &= L_{t-} \int_E \{\varphi(t, q, X_t) - 1\} \{\hat{\mu}(dt, dq) - m_t(dq)dt\} \\ L_0 &= 1. \end{cases}$$

We notice that $\varphi(t, q, X_t) \in \mathcal{G}_0^W$. Define the new measure on $\mathcal{G}_t^W$, $0 \le t \le T$ as

$$d\mathbb{Q} = L_t d\mathbb{P}$$

According to the Girsanov transformation the $\mathbb{Q}$-compensator of the new process is exactly

$$\nu(dt, dq) = \hat{\nu}(dt, dq)(1 + \varphi_t(q, X_t) - 1) = \varphi_t(q, X_t)m_t(dq)dt = M_t(dq, X_t)dt.$$

First, we would like to show that the $\mathbb{Q}$-distribution of ν is the same as the $\mathbb{P}$-distribution. We note that $\mathcal{G}_0^W = \mathcal{G}^W$ and that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_0^W} = L_0 = 1 ,$$

thus, $\mathbb{P} = \mathbb{Q}$ on $\mathcal{G}_0^W$.

Second, we would like to show that

$$\mathbb{P}\left(\mu((s,t]\times B) = k \,\middle|\mathcal{G}_s^W\right) = \frac{(\nu((s,t]\times B))^k}{k!}e^{-\nu((s,t]\times B)}, \quad \text{a.s.,} \quad B \in E. \qquad (4.10)$$

We prove (4.10) using characteristic functions. Define the stochastic process

$$Y_t = \int_0^t \int_E q\hat{\mu}(dt, dq).$$

Changing the measure we obtain that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{iuY_t}\middle|\,\mathcal{G}_0^W\right] = \mathbb{E}^{\mathbb{P}}\left[L_t e^{iuY_t}\middle|\,\mathcal{G}_0^W\right].$$

Define $Z_t = L_t e^{iuY_t}$, then the dynamics of Z_t is

$$\begin{aligned} dZ_t &= L_t \int_E \left\{e^{iu(Y_{t-}+q)} - e^{iuY_{t-}}\right\}\mu(dt, dq) \\ &\quad + L_{t-}e^{iuY_t}\int_E (\varphi(t,q,X_t) - 1)\left\{\hat{\mu}(dt,dq) - m_t(dq)dt\right\} \\ &\quad + \int_E L_{t-}(\varphi(t,q,X_t) - 1)\left\{e^{iu(Y_{t-}+q)} - e^{iuY_{t-}}\right\}\hat{\mu}(dt,dq) \\ &= \int_E Z_{t-}\varphi(t,q,X_t)m_t(dq)(e^{iuq} - 1)dt + \int_E Z_{t-}(e^{iuq} - 1)\varphi(t,q,X_t)\tilde{\mu}(dt,dq) \\ &\quad + \int_E Z_{t-}(\varphi(t,q,X_t) - 1)\tilde{\mu}(dt,dq) \end{aligned}$$

where $\tilde{\mu} = \hat{\mu}(dt, dq) - m_t(dq)dt$.

We notice also that $Z_0 = 1$, then

$$\begin{aligned} Z_t &= 1 + \int_0^t \int_E Z_{s-}\varphi(s,q,X_s)m_t(dq)(e^{iuq} - 1)ds + \int_0^t \ldots \tilde{\mu}(ds,dq) \\ &= 1 + \int_0^t \int_E Z_{s-}(e^{iuq} - 1)M_s(dq, X_s)ds + \int_0^t \ldots \tilde{\mu}(ds,dq). \end{aligned}$$

Denote $\xi_t = \mathbb{E}^{\mathbb{P}}\left[Z_t\middle|\,\mathcal{G}_0^W\right]$, then

$$\xi_t = 1 + \int_0^t \int_E \xi_s(e^{iuq} - 1)M_s(dq, X_s)ds.$$

thus since ξ_t does not depend on q and $M_s(dq, X_s)$ is $\mathcal{G}_0^W$-measurable

$$\xi_t = e^{\int_0^t \int_E (e^{iuq}-1)M_s(dq,X_s)ds},$$

Note that $\nu(dt, dq, X_t) = M_t(dq, X_t)dt$ is $\mathcal{G}^W$ measurable.

The result follows from the fact that the characteristic function of the process

$$\bar{Y}_t = \int_0^t \int_E q\bar{\mu}(dt, dq)$$

where $\bar{\mu}$ is a Market Poisson Process with compensator $\bar{\nu}(t, dq)$ is given by

$$\mathbb{E}\left[e^{iu\bar{Y}_t}\right] = \exp\left\{\int_0^t \int_E (e^{iuq} - 1)\bar{\nu}(t, dq)\right\}.$$

∎

In our application we would like to model separately the dependence of the intensity process and of the loss quota distribution on our state variable X. This allows us, not only to include empirically observed facts in both quantities but it also makes the interpretation of the results more straightforward. To model the loss quota distribution, in the context of our DSMPP, we need the notion of a stochastic kernel.

Definition 4.12. K is a *stochastic kernel* from $\mathbb{R}_+$ to E, i.e. from $(\mathbb{R}_+, \mathcal{B}_+)$ to $(E, \mathcal{E})$, if it is a mapping from $R_+ \times \mathcal{E}$ into R_+ such that:

- $K(\cdot, A)$ is measurable for all $A \in \mathcal{E}$
- $K(t, \cdot)$ is a measure on E $\forall t$

If $K(t, E) = 1$, then the kernel is called a *probability distribution.*

In our case K will be stochastic only to the extent that X is stochastic. That is, conditional on the state variable realization we will have a deterministic loss quota distribution.

The ideal construction procedure can then be described as follows.

Remark 4.13. Construction procedure *We construct our DSMPP μ as follows.*

1. *We specify our Wiener driven stochastic state variable X.*
2. *We specify the intensity $\lambda(t, X_t)$ as a function of our state variable.*

3. *We specify the instantaneous conditional loss quota distribution as a function of the state variable* $K(t, dq, X_t)$.

4. *Finally, we construct the stochastic compensator* ν

$$\nu(dt, dq, X_t) = K(t, dq, X_t)\lambda(t, X_t)dt, \tag{4.11}$$

A compensator representation as in (4.11) has the additional advantage of satisfying a needed consistency requirement. Indeed, it allows us to value credit derivatives that do not depend on recovery in a consistent way with those that do depend upon recovery. Without the separation result we would have to model at once the whole compensator $M_t(dq, X_t)$, which is not very intuitive and we would not know how to derive in a consistent way key ingredients that depend exclusively upon $\lambda(t, X_t)$, like the implied survival probability, the price default digital payoffs or the price defaultable bonds with zero recovery.

Before we go on to the abstract results, we note that Theorem 4.11 suffices to guarantee the existence of a DSMPP μ with a compensator of the form (4.11), as we can simply set

$$M_t(dq, X_t) = K(t, dq, X_t)\lambda(t, X_t),$$

and notice that $M_t(dq, x)$ is a measure for fixed t and x.

4.2.3 Abstract results

In this section we derive the main results concerning the short and forward credit spreads given the setup above.

Proposition 4.14. *Consider a T-defaultable claim $\mathcal{X}$. For the purpose of computing expectations, and in particular its price at time $t \leq T$*

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T r_s ds} V(T)\mathcal{X}\right],$$

it is equivalent to use the following two dynamics for the remaining value process

(i)

$$\begin{aligned} \frac{dV(t)}{V(t-)} &= -\int_0^1 q\mu(dt, dq) \\ V(t) &= v, \end{aligned} \tag{4.12}$$

(ii)

$$\begin{aligned} \frac{dV(t)}{V(t-)} &= -q^e(t-, X_{t-})dN_t \qquad (4.13) \\ V(t) &= v \ . \end{aligned}$$

where μ is a DSMPP with compensator $\nu(t, X_t) = \lambda(t, X_t)K(t, dq, X_t)dt$, N is a Cox process with intensity $\lambda(t, X_t)$ and we define

$$q^e(t, X_t) = \int_0^1 K(t, dq, X_t) \ .$$

Proof. Using the V dynamics in *(i)* we get,

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}V(T)\mathcal{X}\middle|\mathcal{F}_t\right] = \\ &= V(t)\underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathcal{X}\middle|\mathcal{F}_t\right]}_{\pi(t,\mathcal{X})} - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T\int_0^1 qV_{s-}\mu(dq, ds)\mathcal{X}\middle|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T\int_0^1 qV_{s-}\mu(dq, ds)\mathcal{X}\middle|\mathcal{G}_t^W\right]\middle|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T\int_0^1 qV_{s-}\mu(dq, ds)\mathcal{X}\middle|\mathcal{G}_t^W\right]\middle|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}\left\{\int_0^1 qK(s, dq, X_s)\right\}\lambda(s, X_s)ds\mathcal{X}\middle|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s, X_s)\lambda(s, X_s)ds\mathcal{X}\middle|\mathcal{F}_t\right] \end{aligned}$$

Using the V dynamics in *(ii)* we get,

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}V(T)\mathcal{X}\middle|\mathcal{F}_t\right] = \\ &= V(t)\underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathcal{X}\middle|\mathcal{F}_t\right]}_{\pi(t,\mathcal{X})} - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s, X_s)dN(s)\mathcal{X}\middle|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s, X_s)dN(s)\mathcal{X}\middle|\mathcal{G}_t^W\right]\middle|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T V_{s-}q^e(s, X_s)dN(s)\mathcal{X}\middle|\mathcal{G}_t^W\right]\middle|\mathcal{F}_t\right] \\ &= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s, X_s)\lambda(s, X_s)ds\mathcal{X}\middle|\mathcal{F}_t\right] \end{aligned}$$

The results follow from comparing the final expressions on both cases. ■

Proposition 4.15. *Given Assumption 4.6, and under the martingale measure* $\mathbb{Q}$.

1. *The short credit spreads,* $s(t)$*, have the following functional form*

$$s(t) = \lambda(t, X_t) q^e(t, X_t) > 0 \tag{4.14}$$

where

$$q^e(t, X_t) = \int_0^1 qK(t, dq, X_t)$$

can be interpreted as the locally expected loss quota (which is positive for $q > 0$*).*

2. *Then the forward credit spread* $s(t,T)$ *takes the form*

$$s(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\{r(T) + \lambda(T, X_T) q^e(T, X_T)\} e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]} - f(t,T) \tag{4.15}$$

Proof. The time t price of the defaultable zero-coupon bond with maturity T is equal to

$$\bar{p}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} V(T) \middle| \mathcal{F}_t\right], \tag{4.16}$$

where $V(T)$ is the residual of the face value after multiple defaults up to time T.

Making use of Proposition 4.14, instead of $\frac{dV(t)}{V(t-)} = -\int_0^1 q\mu(dt, dq)$ with our DSMPP μ (these dynamics follow directly from (4.5)), we use

$$\frac{dV(t)}{V(t-)} = -q^e(t-, X_{t-}) dN_t$$

where N is the Cox process with intensity $\lambda(t, X_t)$.

For every fixed t, define $Z(u)$ as follows

$$Z(u) = e^{\int_t^u q^e(s,X_s)\lambda(s,X_s)ds} V(u).$$

We note that then the dynamics of $Z(u)$ take the form

$$dZ(u) = -Z_{u-} q^e(u-, X_{u-}) \{dN_u - \lambda(u, X_u) du\}, \qquad u \geq t, \quad t\text{-fixed}$$

and $Z(u)$ is a $\mathbb{Q}$-martingale conditional on the filtration $\mathcal{F}_t^W$. Thus,

$$\mathbb{E}^{\mathbb{Q}}\left[Z(T) | \mathcal{F}_t^W\right] = Z(t).$$

The price of a defaultable bond is then can be found as

$$\begin{aligned}
\bar{p}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}V(T)\Big|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds}Z(T)\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds}Z(T)\Big|\mathcal{G}_t^W\right]\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds}\mathbb{E}^{\mathbb{Q}}\left[Z(T)|\,\mathcal{G}_t^W\right]\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds}Z(t)\Big|\mathcal{F}_t\right] \\
&= V(t)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds}\Big|\mathcal{F}_t\right] \\
&= V(t)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \{r(s)+q^e(s,X_s)\lambda(s,X_s)\}ds}\Big|\mathcal{F}_t\right].
\end{aligned}$$

Using the basic relations between defaultable bond prices and defaultable forward rates in (4.7) we can, thus, write

$$\bar{f}(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\{r(T)+\lambda(T,X_T)q^e(T,X_T)\}\,e^{-\int_t^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}. \tag{4.17}$$

Finally using that $\bar{f}(t,t) = \bar{r}(t)$ in the above expression we obtain

$$\bar{r}(t,r_t,X_t) = r(t) + q^e(t,X_t)\lambda(t,X_t).$$

The result follow from $s(t) = \bar{r}(t) - r(t)$ and $s(t,T) = \bar{f}(t,T) - f(t,T)$. ■

From the proof of the previous Proposition results also a nice formulation for the defaultable bond prices.

Corollary 4.16. *Under the conditions of Proposition 4.15 we have*

$$\bar{p}(t,T) = V(t)\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}_s ds}\right]. \tag{4.18}$$

It is obvious from (4.15) that the expression of the forward credit spread looks quite messy. This happens because we choose to present it under the martingale measure $\mathbb{Q}$. The next lemma gives us two simpler representations at the cost of using forward measures.

Lemma 4.17. *The forward credit spread, $s(t,T)$, has the following representation, equivalent to (4.15)*

(i)

$$s(t,T) = -\frac{\partial}{\partial T}\ln\left(\mathbb{E}_t^T\left[V(T)\right]\right)$$

where $\mathbb{E}_t^T[\cdot]$ stands for expectation under the T-forward measure.

(ii)

$$s(t,T) = \bar{\mathbb{E}}_t^T\left[\bar{r}(T)\right] - \mathbb{E}_t^T\left[r(T)\right].$$

where $\bar{\mathbb{E}}_t^T\cdot$ *stands for expectation under the measure* $\bar{\mathbb{Q}}^T$ *where we have*

$$L^T = \frac{d\bar{\mathbb{Q}}^T}{d\mathbb{Q}} = \frac{V(0)e^{-\int_0^T r_s+\lambda(s,X_s)q^e(s,X_s)ds}}{\bar{p}(0,T)} \quad on\ \mathcal{F}_T\ .$$

Proof. To prove (i) we note that

$$\begin{aligned}\bar{p}(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}V(T)\right]\\ &= p(t,T)\mathbb{E}_t^T\left[V(T)\right].\end{aligned}$$

From the definition of the forward defaultable rate and the forward credit spread we also have

$$\begin{aligned}\bar{p}(t,T) &= V(t)e^{-\int_t^T \bar{f}(t,u)du}\\ &= V(t)e^{-\int_t^T f(t,u)du}e^{-\int_t^T s(t,u)du}\\ &= V(t)p(t,T)e^{-\int_t^T s(t,u)du}.\end{aligned}$$

Comparing we realize $V(t)e^{-\int_t^T s(t,u)du} = \mathbb{E}_t^T\left[V(T)\right]$, differentiating w.r.t. T and solving for $s(t,T)$ we get

$$s(t,T) = \frac{\partial}{\partial T}\ln\left(\frac{V(t)}{\mathbb{E}_t^T\left[V(T)\right]}\right) = \underbrace{\frac{\partial}{\partial T}\ln(V(t))}_{0} - \frac{\partial}{\partial T}\ln\left(\mathbb{E}_t^T\left[V(T)\right]\right).$$

To prove (ii) we start by noting that it is a well known fact that for the risk-free rates $f(t,T) = \mathbb{E}_t^T\left[r(T)\right]$.[3]

Thus, it remains to show that

$$\bar{f}(t,T) = \bar{\mathbb{E}}_t^T\left[\bar{r}(T)\right]\ .$$

Using $\bar{r}(t) = r(t) + q^e(t,X_t)\lambda(t,X_t)$, equation (4.17) becomes

$$\bar{f}(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\bar{r}(T)e^{-\int_t^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}\ .$$

Since $\frac{\bar{p}(t,T)}{V(t)} = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\{r(s)+q^e(s,X_s)\lambda(s,X_s)\}ds}\middle|\mathcal{F}_t\right]$ we have

$$\bar{f}(t,T)\frac{\bar{p}(t,T)}{V(t)} = \mathbb{E}_t^{\mathbb{Q}}\left[\bar{r}(T)e^{-\int_t^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]\ .$$

[3] See Björk (2004a) for further details.

For the r.h.s the following holds

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}\left[\bar{r}(T)e^{-\int_t^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right] = \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\bar{r}(T)\underbrace{\frac{V(0)e^{-\int_0^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}}{\bar{p}(0,T)}}_{L^T}\right]\frac{e^{\int_0^t\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\bar{p}(0,T)}{V(0)} \\
&= \bar{\mathbb{E}}_t^T r(T)\mathbb{E}_t^{\mathbb{Q}}\left[L^T\right]\frac{e^{\int_0^t\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\bar{p}(0,T)}{V(0)} \\
&= \bar{\mathbb{E}}_t^T r(T)\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right] = \frac{\bar{p}(t,T)}{V(t)}\bar{\mathbb{E}}_t^T\left[r(T)\right]
\end{aligned}$$

Where we define $L^T = \frac{V(0)e^{-\int_0^T\{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}}{\bar{p}(0,T)}$ on $\mathcal{F}_T$.

Comparing with the l.h.s we get $\bar{f}(t,T) = \bar{\mathbb{E}}_t^T\bar{r}(T)$. ∎

4.2.4 On the market price of jump risk

We now note that the setup has been defined under the martingale measure $\mathbb{Q}$. Now we specify the implicit assumption on the market price of jump risk that will allows us to extrapolate from our objective intuitions ($\mathbb{P}$-intuitions) when setting up a concrete model later on.

In order to see the connections between the intensities under the different measures we recall the appropriate Girsanov theorem [4].

Theorem 4.18. **(Girsanov for DSMPP)**
Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$ be a filtered probability space which supports $\mathbb{Q}$-Brownian motion $W(t)$ and a marked point process $\mu(dt, dq)$, where the marker q is drawn from the mark space $X = [0,1]$. The $\mathbb{Q}$-compensator is assumed to take the form $\nu(dt, dq) = K(t, dq)\lambda(t)dt$ under $\mathbb{Q}$. Here λ is the $\mathbb{Q}$-intensity of the arrivals of the point process and $K(t, dq)$ is the $\mathbb{Q}$-conditional distribution of the marker. Let h be a predictable process and $\phi_t(q)$, $\phi_t(q) > -1$ a predictable function with the properties

$$\int_0^t \|h\|^2 ds < \infty, \quad \int_0^t\int_X |\phi_t(q)|K(s,dq)\lambda(s)ds < \infty.$$

[4] See, for example, Björk (2004a) or Schönbucher (2003).

Define the process $L(t)$ by

$$\begin{cases} \dfrac{dL(t)}{L(t-)} &= h(t)dW(t) + \int_X \phi(t) \{\mu(dt,dq) - \nu(dt,dq)\} \\ L(0) &= 1 \end{cases} .$$

Define the probability measure $\mathbb{P}$ as follows

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = L(t) \quad on \quad \mathcal{F}_t \quad \forall t > 0.$$

Then:

1. *The process $W_{\mathbb{P}}$ is a Brownian motion and*

$$dW_P(t) = dW(t) - h(t)dt.$$

2. *The predictable compensator under $\mathbb{P}$ is as follows*

$$\nu^P(dt,dq) = (1 + \phi_t(q))\nu(dt,dq). \tag{4.19}$$

$\phi_t(q)$ in (4.19) is the *market price of jump risk* and, in principle, is stochastic $\phi_t(q,\omega,X)$.

The following Lemma shows that in the case when Girsanov kernels are deterministic the measure transformation does not affect K and gives us the relationship between the intensities of default under $\mathbb{P}$ and under $\mathbb{Q}$.

Lemma 4.19. *Assume that the market price of jump risk, ϕ in (4.19) is a deterministic function of time. That is*

$$\phi_t(q,\omega) = \phi(t) . \tag{4.20}$$

Then,

1. *The $\mathbb{Q}$-default intensity, λ, relates to the $\mathbb{P}$-default intensity λ^P, by*

$$\lambda^P(t,r,I) = \lambda(t,r,I)(1 + \phi(t)) \tag{4.21}$$

2. *The $\mathbb{Q}$-loss quota distribution, conditional on default, $K_t(dq)$, equals to the conditional on default loss quota distribution under $\mathbb{P}$, $K_t^P(dq)$.*

Proof. The predictable compensator takes the form

$$\nu(dt,dq) = M_t(dq)dt = \lambda(t)K(t,dq)dt,$$

where

$$\lambda(t) = M_t(E), \quad K(t, dq) = \frac{M_t(dq)}{M_t(E)}$$

From the Girsanov theorem we obtain that under $\mathbb{P}$

$$M_t^P(dq) = M_t(dq)(1 + \phi(t))$$

$$\lambda^P(t) = \int_E (1 + \phi(t))M_t(dq)$$

Thus is case if $\phi(t)$ is deterministic and does not depend on q we see that

$$\begin{aligned} \lambda_t^P &= (1 + \phi(t))M_t(E) = (1 + \phi(t))\lambda(t) && (4.22) \\ K_t^P(dq) &= \frac{(1 + \phi(t))M_t(dq)}{\int_E (1 + \phi(t))M_t(dq)} = \frac{M_t(dq)}{\int_E M_t(dq)} = K_t(dq). && (4.23) \end{aligned}$$

■

We see that, in this case, the conditional distribution of the loss quota remains unchanged while intensity changes according to (4.22), i.e. multiplied by a deterministic function of time. This also means that once we have specified the influence of the state variable X under $\mathbb{Q}$ we have also specified its influence under the objective measure (up to some deterministic factor in the case of the intensity process λ).

Obviously it is discussable if assuming (4.20) is a very strong restriction or not. For this study we assume it is not so we can use our objective intuitions in setting up the applied model for macroeconomic risks. Nonetheless, the unhappy reader, can also interpret all our results as only $\mathbb{Q}$-results.

4.3 The Macroeconomic Risks

4.3.1 Setup

We model systematic risk of an economy by considering what we call a *market index*, and consider this market index both when it is the price of an important traded asset in the economy and when it is not the price of any traded asset.

An example of an *index* that is the price of a traded asset is, say, oil price. Indices, for instance, stock market indices or any other type of indices, on the other hand, are examples of *indices* that are not prices of traded assets. We consider the two possibilities at all times.

It is rather well known that market index volatility (for example S&P500) tends to increase when the market as a whole is depressed (low values of the index) and, conversely, volatility decreases when the market index is high (see Gaspar (2001)). In order to account for this fact, we make market index volatility dependent on the index level.

Assumption 4.20. (Market Index)
Under the martingale measure $\mathbb{Q}$, the market index I, satisfies the following stochastic differential equation (SDE)

- *When I is the price of a traded asset*

$$dI_t = r(t)I_t dt + \gamma(t, I_t)I_t dW(t)$$

- *When I is not the price of a traded asset*

$$dI_t = \zeta(t)I_t dt + \gamma(t, I_t)I_t dW(t),$$

where r is the short rate, γ is a row-vector, W is a $\mathbb{Q}$-Wiener process.

Furthermore, for each entry γ_i, the following holds

(i)

$$\frac{\partial \gamma_i}{\partial I}(t, I) < 0 \ . \tag{4.24}$$

The property *(i)* in (4.24) represents the empirically observed fact that periods when the market as a whole is depressed are periods of higher volatility, while booms are associated with low volatilities.

Since it would be wrong in our setup to consider I to be independent of the risk-free short rate r (at least under the risk-neutral measure $\mathbb{Q}$), and we cannot ignore its influence, we can also consider r as one of the factors.

We will now present explicitly the dependence between the default intensities and the loss quota distribution on the factors. The results from the abstract section can be used immediately with $X = \begin{bmatrix} I \\ r \end{bmatrix}$.

Finally, it is also reasonable to assume that firms that are less sensitive to systematic risks will suffer less in terms of an increase in their default intensities than firms that are more sensitive. Hence, we introduce a measure of sensitivity to systematic risk, ϵ, $\epsilon \in [0, 1]$.

Assumption 4.21. (The default Intensity)
The intensity is a deterministic function of (t, r, I, ϵ). *Furthermore, we have*

(ii)
$$\lambda(t, r, I, 0) = \bar{\lambda} \quad \bar{\lambda} \in \mathbb{R}_+ \tag{4.25}$$

(iii)
$$\frac{\partial \lambda(t, r, I, \epsilon)}{\partial \epsilon} > 0 \ , \tag{4.26}$$

(iv)
$$\frac{\partial \lambda(t, r, I, \epsilon)}{\partial I} < 0 \ , \tag{4.27}$$

(v)
$$\frac{\partial \lambda(t, r, I, \epsilon)}{\partial r} > 0 \ . \tag{4.28}$$

Assumption 4.21 is based on the intuition that if a firm's financial situation is strong enough, it should not really matter if the economy is booming or if it is in recession. That is, firms that are financially solid should be much less sensitive to the business cycle than those in a less solid financial position. One can also regard the parameter ϵ as a measure of a firm's creditworthiness. Firms with high creditworthiness typically tend to be less sensitive to the business cycle influence than less creditworthy firms.

If this is so, then it makes sense to include properties *(ii)* and *(iii)* in (4.25) and (4.26), respectively.

Properties *(iv)* and *(v)* (eq (4.27),(4.28)) tell us that if the default intensity depends on macroeconomic variables, then the PD is higher during recessions or periods of high risk-free interest rates, and lower in booms or periods of low risk-free interest rates. The influence of the index value is related to the increase in uncertainty during periods of recession, while the influence of the short rate is related to the increase in the difficulty of refinancing existing debt when the cost of borrowing money is higher (if a firm can only borrow money at high costs the PD is higher).

Assumption 4.22. (Loss Quota)
The conditional distribution of loss quota is a deterministic function of (t, r, I). K *is a stochastic kernel from* $R_+ \times R_+ \times R_+ \to [0, 1]$ *for any realization of* (t, r, I)

We denote the cumulative distribution function of loss quota conditional on default as $\tilde{K}$

$$\tilde{K}(t, r, I, x) = \int_0^x K(t, r, I, dq), \qquad \int_0^1 K(t, r, I, dq) = 1, \quad \forall t, r, I$$

with the following properties

$$\begin{aligned} \tilde{K}(t,r,I_1,x) &\geq \tilde{K}(t,r,I_2,x), \quad \textit{if} \quad I_1 \geq I_2, \quad \forall x \in R\ , \\ \tilde{K}(t,r_1,I,x) &\leq \tilde{K}(t,r_2,I,x), \quad \textit{if} \quad r_1 \geq r_2, \quad \forall x \in R\ . \end{aligned}$$

That is

(vi)

$$\frac{\partial \tilde{K}(t,r,I,x)}{\partial I} > 0\ , \tag{4.29}$$

(vii)

$$\frac{\partial \tilde{K}(t,r,I,x)}{\partial r} \leq 0\ . \tag{4.30}$$

For fixed (t,r), $\tilde{K}(t,r,I,x)$ stochastically dominates *all the conditional distributions with parameter* $\underline{I}$*, such that* $\underline{I} \leq I$ *and, for fixed* (t,I) $\tilde{K}(t,r,I,x)$ stochastically dominates *all the conditional distributions with parameter* $\bar{r}$*, such that* $\bar{r} \geq r$.

Property *(vi)* in (4.29) can be justified by the following argument. Given that default has occurred and the debt holders are negotiating their loss quota so that the firm is able to continue operating (assumption underlying any multi-default model), it seems reasonable that if the firm's assets are worth less they are willing to give in more (higher loss quota) since they would not get that much in the event of bankruptcy. Moreover, bankruptcy costs tend to be higher in periods of recession, emphasizing this effect.

As to the risk-free interest rate, the argument for property *(vii)* in (4.30) is more difficult to support. Thus in our view, it is reasonable to assume $\frac{\partial \tilde{K}(t,r,I,x)}{\partial r} \leq 0$, which includes the possibility of no influence. However, if we assume the existence of an impact, then we argue that it should be in the direction of higher interest rates leading to a decrease in the probability of lower loss quotas. In periods of high interest rates, debt holders have better options to invest their money and are less willing to give in.

From the stochastic dominance assumption above, we can now infer the impacts on the expected loss quota.

Lemma 4.23. *Given Assumption 4.22, the following relations hold for the expected value*

$$q^e(r,I) = \int_0^1 qK(r,I,dq),$$

(viii)

$$\frac{\partial q^e(r,I)}{\partial I} < 0\ , \tag{4.31}$$

(ix)

$$\frac{\partial q^e(r,I)}{\partial r} \geq 0 \; . \tag{4.32}$$

Proof. The proof is similar to both properties and is a consequence of *(vi)⇒(viii)* and *(vii)⇒(ix)*. We show it for the index value case.

$$q^e(r,I) = \int_0^1 qK(r,I,dq) = \int_0^1 qd\tilde{K}(t,r,I,q).$$

Integrating by parts we get

$$\begin{aligned}
\int_0^1 qd\tilde{K}(t,r,I,q) &= q\tilde{K}(t,r,I,q)\big|_0^1 - \int_0^1 \tilde{K}(t,r,I,q)dq \\
&= q\underbrace{\tilde{K}(t,r,I,1)}_{1} - q\underbrace{\tilde{K}(t,r,I,0)}_{0} - \int_0^1 \tilde{K}(t,r,I,q)dq \\
&= q - \int_0^1 \tilde{K}(t,r,I,q)dq \; .
\end{aligned}$$

The results follows from differentiating this last expression w.r.t I and using the property (vi)

$$\frac{\partial q^e(r,I)}{\partial I} = -\int_0^1 \underbrace{\frac{\partial \tilde{K}(t,r,I,q)}{\partial I}}_{>0,\forall q} dq < 0 \; .$$

The same argument will work for the interest rate r. ∎

Remark 4.24. (Tractability)
We note that besides the above mentioned properties K, conditional on the state variable information, must be the distribution of a random variable taking values in $(0,1)$, and the intensity λ must be always positive. It is, thus, extremely hard to find a treatable model where these two facts together with properties (i)-(ix) are satisfied.

In particular, we have found that no model of affine or quadratic spreads[5] *will verify all the above properties.*

Given these tractability difficulties we go on with the analysis and draw *qualitative* results of the influence of the market index on credit spreads.

[5]Outside the class of affine or quadratic spread models it is basically impossible to find closed-form solutions.

4.3.2 Credit spreads term structures

Using the abstract result on Proposition 4.15 and Assumptions 4.21 and 4.22 we can study the impacts of our macroeconomic variables $X = \begin{bmatrix} I \\ r \end{bmatrix}$ on the short spread $s(t)$.

Remark 4.25. *Given the results in Proposition 4.15, Assumption 4.21 and Lemma 4.23, the short credit spread can be rewritten as a function of* (t, r, I, ϵ) *and*

$$s(t,r,I,\epsilon) = \lambda(t,r,I,\epsilon)q^e(t,r,I) \ . \tag{4.33}$$

Furthermore we have

$$\begin{aligned} s(t,r,I,0) &= \bar{\lambda}q^e(t,r,I) \ , \qquad \frac{\partial s(t,r,I,\epsilon)}{\partial \epsilon} = \underbrace{\frac{\partial \lambda(t,r,I,\epsilon)}{\partial \epsilon}}_{>0} q^e(t,r,I) > 0 \ , \\ \frac{\partial s(t,r,I,\epsilon)}{\partial I} &= \underbrace{\frac{\partial \lambda(t,r,I,\epsilon)}{\partial I}}_{<0} q^e(t,r,I) + \lambda(t,r,I,\epsilon) \underbrace{\frac{\partial q^e(t,r,I)}{\partial I}}_{<0} < 0 \ , \\ \frac{\partial s(t,r,I,\epsilon)}{\partial r} &= \underbrace{\frac{\partial \lambda(t,r,I,\epsilon)}{\partial r}}_{>0} q^e(t,r,I) + \lambda(t,r,I,\epsilon) \underbrace{\frac{\partial q^e(t,r,I)}{\partial r}}_{\geq 0} > 0 \ . \end{aligned}$$

We note that given a concrete functional form for the intensity λ, and the loss quota distribution and, thus, q^e the above effects on the short spread can actually be quantified. Unfortunately, this is not going to be the situation when dealing with forward credit spreads.

Before, however, we move on to forward credit spreads, we derive the dynamics by of the short credit spread under $\mathbb{Q}$.

Proposition 4.26. *The dynamics of the short credit spread under* $\mathbb{Q}$ *is, in short hand notation, as follows*

$$\begin{aligned} d\bar{s} &= \left(\frac{\partial s}{\partial t} + \frac{\partial s}{\partial r} a(t) + \frac{\partial s}{\partial I} \gamma(I_t) I_t + \frac{1}{2} \frac{\partial^2 s}{\partial r^2} b^2(t) + \frac{1}{2} \frac{\partial^2 s}{\partial I^2} \gamma^2(I_t) I_t^2 + \frac{\partial^2 s}{\partial I \partial r} b(t) \gamma(I_t) I_t \right) dt \\ &\quad + \sigma_s(t) dW_t \end{aligned}$$

where

$$\begin{aligned} a(t) &= \frac{\partial f(t,T)}{\partial T}\Big|_{T=t} + \alpha(t,t) \\ b(t) &= \sigma(t,t) \end{aligned}$$

$$\sigma_s(t) = \frac{\partial s}{\partial r} b(t) + \frac{\partial s}{\partial I} \gamma(I_t) I_t \ . \tag{4.34}$$

where $\alpha(t,T)$ and $\sigma(t,T)$ are the drift and the volatility of the forward risk-free interest rates in (4.2) and the usual HJM drift condition in (4.3) holds.

Proof. We apply Itô formula to equation (4.33).

In addition to the Itô formula we just need to deduce the dynamics of the risk-free short rate r, from the dynamics of the forward interest rate in (4.2). We recall $r(t) = f(t,t)$ and it is easy to show that [6]

$$dr(t) = \left(\frac{\partial f(t,T)}{\partial T}|_{T=t} + \alpha(t,t) \right) dt + \sigma(t,t) dW_t \ .$$

■

Without a concrete functional form of the intensity and loss quota distribution, there is not much one can say about the dynamics of the short spread. Except, perhaps that since $\frac{\partial s}{\partial I} < 0$ and $\frac{\partial s}{\partial r} \geq 0$ (recall Remark 4.25) we see that increases in the volatility of the index lead to decreases in the volatility of the short spread, while increases in the volatility of the short rate r lead to increases in its volatility. Consequences to the drift cannot be drawn from Assumptions 4.21 and 4.22 alone.

Remark 4.27. **($\mathbb{P}$ considerations)**
The same qualitative relations would still hold under the objective probability measure $\mathbb{P}$ as long as the market price of jump risk is positive, which seems reasonable.

As to forward spreads $s(t,T)$, given that we could only obtain expressions in terms of expectations that have to be numerically evaluated, there little one can say.

In the next section we propose a way to model all the needed functions and show the simulation results.

4.4 A concrete model

In this section we illustrate the theoretical results previously derived using a "toy model".

We aim to highlight the importance of considering the dependence between recovery and intensity of default by showing the applicability of our results and that the impacts obtained are substantial, rather than to be as realistic as possible.

[6]For a proof of this result see Björk (2004a).

For that reason, when setting up our model, we do so as simply as possible with the advantage of obtaining more tractable formulas and a better understanding of what drives the simulation results.

The theoretical results apply, of course, to the general case outlined above and many more examples could have been considered.

In order to have a concrete model we need to:

- establish the dependence of the volatility of the index γ, on the index level;
- provide the intensity functional form for λ, in terms of (t, r, I, ϵ);
- decide on a distribution for the loss quota q for all possible (t, r, I).

In our toy model we take the risk-free rate, r, as constant and abstain from considerations about the term structure of risk-free interest rates. Although this is unrealistic, it is not harmful to our goal of understanding the impact on *spreads*.

I is assumed to be the price of a traded asset. To consider a non-traded asset, we would need further considerations on the market price of index risk.

For simplicity we also take all functions to be time homogenous; the extension to non-time homogeneous functions is straightforward.

Given these simplifications, to have a completely specified model to simulate we need to define a function $\gamma(I)$ for the index volatility, a function $\lambda(I, \epsilon)$ for the intensity, and a distribution function $K(dq, I)$ for the loss quota.

4.4.1 Choosing the market index volatility γ

We start by defining a concrete volatility for the index $\gamma(I)$. As stated in Assumption 4.20, we would like to the index volatility inversely related to the index level.

We start by defining a ratio which relates the current value of the index to its long-run trend value. Let us define

$$m(I) = \frac{\bar{I}}{I},$$

where $\bar{I}$ is a priori given and can be interpreted as the long-run trend value of the market index.

The above ratio measures how close or far away from the long-run trend value parameter, $\bar{I}$, the current value of the index I is. Intuitively, it seems reasonable to make the volatility dependent on some relative value of the index, instead on its absolute value. $\bar{I}$ will be assumed to grow at the risk-free rate over time.

Reasonable levels for $m(I)$ typically range from 0.7 and 1.3. We note that the higher the current level of the index the lower is $m(I)$, i.e.

$$\frac{\partial m}{\partial I} = -\frac{\bar{I}}{I^2} < 0 \, .$$

That is, a value of, say, $m = 0.7$ refers to a bull market while $m = 1.3$ refers to a bear market.

Based on the above ratio we now define the *volatility of index* as a function of our ratio m in the following way

$$\gamma(I) = \bar{\gamma}\,(m(I))^{\frac{1}{2}} \quad \forall I, \bar{\gamma} \in \mathbb{R}_+ \, . \tag{4.35}$$

In accordance with Assumption 4.20, the higher the current value of the index the lower is the index volatility γ,

$$\frac{\partial \gamma(I)}{\partial I} = \underbrace{\bar{\gamma}\frac{1}{2}}_{>0}\,\underbrace{[m(I)]^{-\frac{1}{2}}}_{>0}\,\underbrace{\frac{\partial m(I)}{\partial I}}_{<0} < 0 \quad \forall I > 0 \, .$$

Figure 4.1 shows us two possible paths for the index process, one assuming γ to be just a constant and the other where the index volatility depends on the index level as in (4.35).

4.4.2 Choosing default intensities and the loss quota distribution

Default intensity

Having defined the index volatility we now define the intensity function

$$\lambda(I, \epsilon) = \bar{\lambda}\,[m(I)]^{\epsilon} = \frac{\bar{\lambda}}{\bar{\gamma}}\,[m(I)]^{\epsilon - \frac{1}{2}}\,\gamma(I) \quad \text{for } \bar{\lambda} \in \mathbb{R}+ \text{ and } \epsilon \in [0, 1] \, .$$

We note that, with this modellization, we can interpret the intensity function as a function of the index level or, if we prefer, as a function of the index volatility. One can argue that the intensity should not be affected by index level, but instead by its volatility since it is the volatility that represents the "risk". The above definition

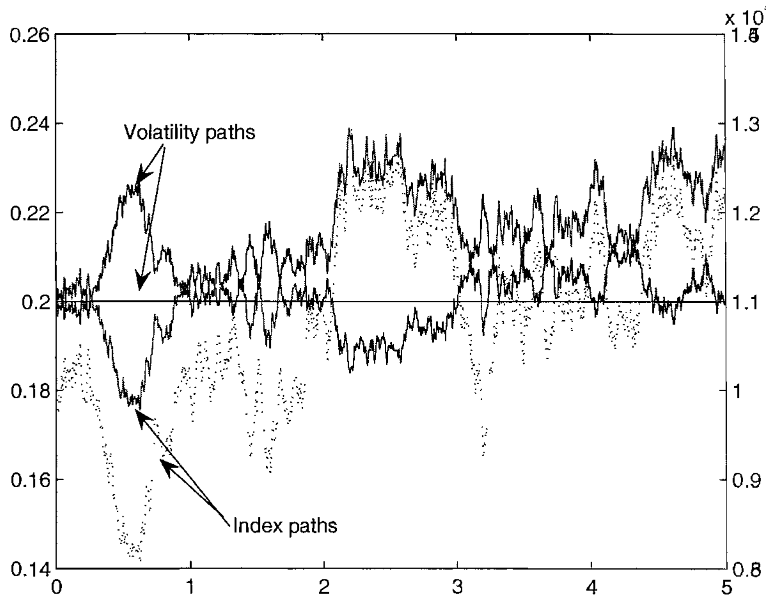


Figure 4.1: Two paths for the index level and volatility. The same noise was used for both cases, and we took $\bar{I} = 10000$ and $I_0 = 10000$. Case 1: constant volatility $\gamma = 0.2$, the index process is the full line. Case 2: stochastic volatility as in (4.35), the index process is the dotted line.

includes the two possibilities.

$$
\begin{aligned}
\frac{\partial \lambda}{\partial I} &= \underbrace{\bar{\lambda}\epsilon}_{>0} \underbrace{(m(I))^{\epsilon-1}}_{>0} \underbrace{\frac{\partial m(I)}{\partial I}}_{<0} < 0 \\
\frac{\partial \lambda}{\partial \gamma} &= \underbrace{\frac{\bar{\lambda}}{\bar{\gamma}}}_{>0} \underbrace{[m(I)]^{\epsilon-\frac{1}{2}}}_{>0} > 0.
\end{aligned}
$$

Figure 4.2 show the functions $\lambda(I)$ and $\gamma(I)$ for different values of $m(I)$.

Loss quota q

Finally, we need to decide on our loss quota distribution. As before the ratio m will help to model the dependence on the index level.

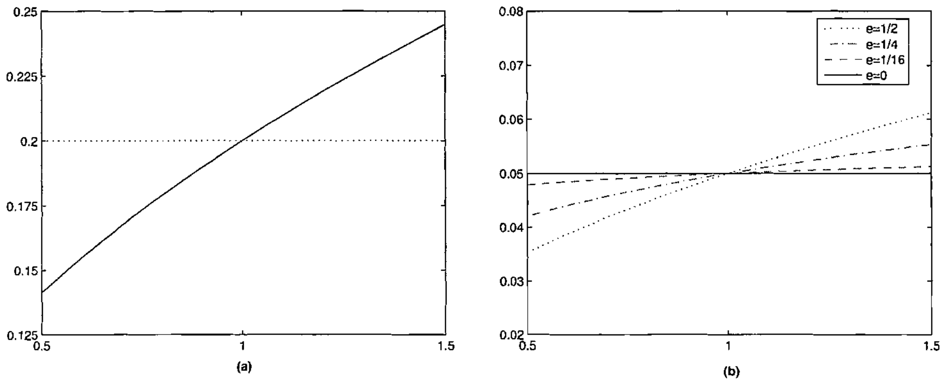


Figure 4.2: (a) : $\gamma(I)$ for different levels of $m(I)$ vs naive constant volatility $\bar{\gamma} = 0.2$. (b):$\lambda(I)$, for different levels of $m(I)$ and different $\epsilon = 0, 1/16, 1/4, 1/2$, $\bar{\lambda} = 0.05$.

We assume that the distribution of the loss process belongs to the Beta class and start by recalling some basic properties.

Remark 4.28. *The beta density function is given by*

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}\mathbf{1}_{(0,1)}(x)$$

where $a > 0$, $b > 0$ and $B(a,b)$ is the beta function:

$$B(a,b) \int_0^1 x^{a-1}(1-x)^{b-1} ds \ .$$

Furthermore, we have

$$\mathbb{E}[X] = \frac{a}{a+b} = \mu \qquad \mathrm{Var}\, X = \frac{ab}{(a+b+1)(a+b)^2} \qquad \mathbb{E}[(X-\mu)^r] = \frac{B(r+a,b)}{B(a,b)}$$

In our concrete application, for any fixed value of I, we choose

$$q \sim Beta\,(2m(I), 2) \quad \text{i.e.} \quad a = 2m(I) \quad \text{and} \quad b = 2\ , \tag{4.36}$$

which is consistent with the desired properties referred in Assumption 4.22.

Thus,

$$\tilde{K}(q, I) = \frac{1}{B(2m(I), 2)} \int_0^q x^{2m(I)-1}(1-x)dx \ .$$

Figure 4.3 shows us the loss quota density and its cumulative distribution function for three different values of $m(I)$: $m = 0.7$ representing a bull market, $m = 1$ for the case where the market is at its long run level, and $m = 1.3$ representing a bear market.

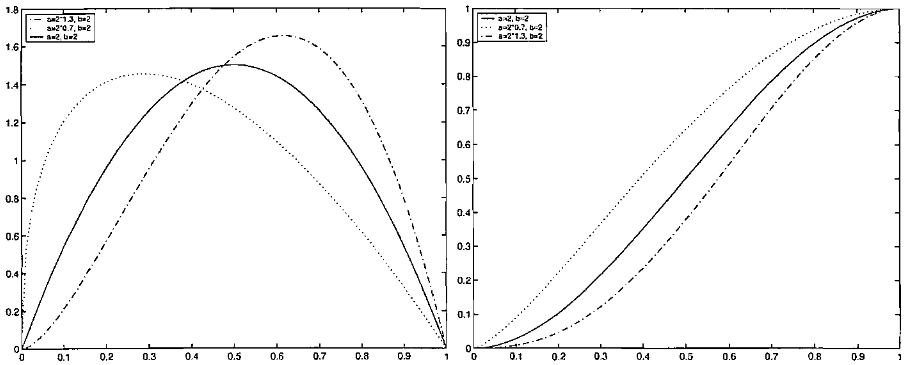


Figure 4.3: Density and Cumulative distribution functions of loss quota for $m = 1.3$,$m = 1$,$m = 0.7$

The next properties follow from the properties of the Beta distribution:

- The expected loss is given by

$$q^e(I) = \mathbb{E}\left[q(I)\right] = \frac{m(I)}{1 + m(I)}$$

- $\dfrac{\partial q^e(I)}{\partial I} = \dfrac{\frac{\partial m(I)}{\partial I}}{(1 + m(I))^2} < 0$
  Furthermore
  - if default occurs exactly at the long-run level the loss expected quota is exactly 1/2;
  - if default occurs when the index level is "high" ($m < 1$) one expects to recover more, expected loss quota decreases;
  - if default occurs when the index level is "low" ($m > 1$) one expects to recover less, expected loss quota increases.

Figure 4.4 bellow shows both possible realizations of the loss quota (drawn from the beta density with the appropriate mean for each m) (stars), the expected loss quota levels for different values of m (full line) in contrast with the naive approach of taking $\bar{q} = \frac{1}{2}$ (dotted line).

Before we go on we illustrate as well a possible relation between the intensity and the recovery process $(1 - q)$ and the intensity λ. Figure 4.5 shows the scatter plot λ versus one possible recovery realization for different levels of the index.

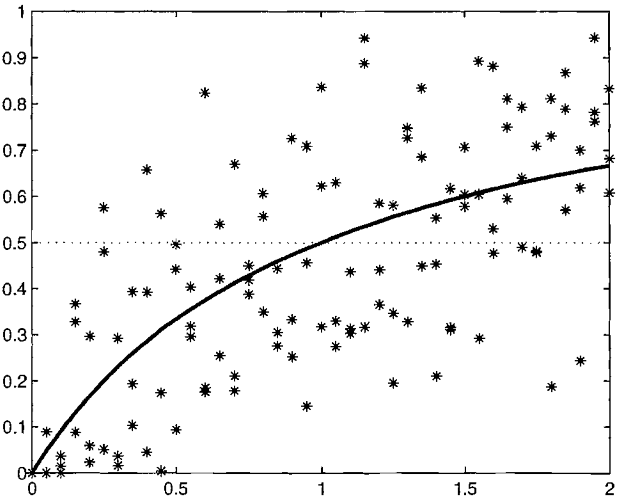


Figure 4.4: Loss quota possible realizations and expected value for different values of the ratio m. Dotted line is the naive $q = \frac{1}{2}$.

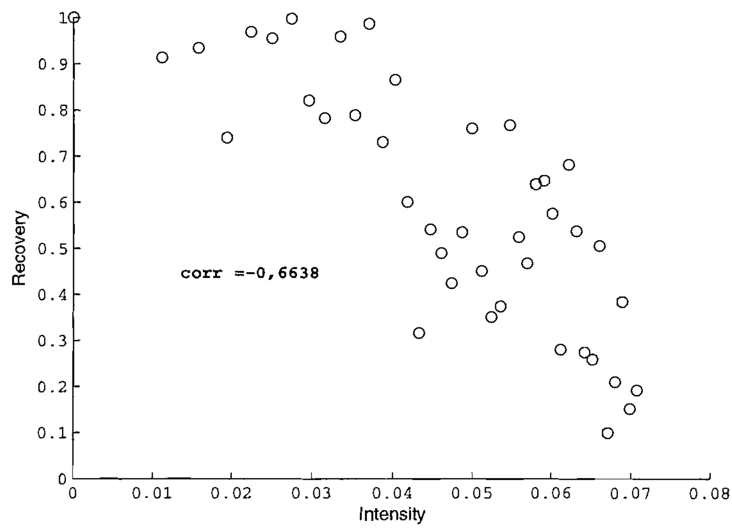


Figure 4.5: Scatter plot of intensity versus possible recovery realization for different values of m.

4.4.3 Simulation Results

In our simulations we use the Monte Carlo method where the step size is equivalent to one trading day (we do 250 steps per year) and all simulations concern 5,000 paths. The same noise matrix is used for all scenarios and cases so that the values obtained can actually be compared (discretization errors would be in the same direction for all scenarios).

The spreads with zero maturity correspond to the short spread; all other maturities correspond to the forward spread.

Table 4.1 tells us the reference parameters, while Table 4.2 characterizes all possible scenarios.

REFERENCE PARAMETERS

Maturities (T)	From days up to 5 years	
Risk-free interest rate (r)	5%	
$m(I)$	Case A: bull market	0.7
	Case B: normal market	1.0
	Case C: bear market	1.3
Long-run trend value	$10.000e^{r*T}$	
Fixed index volatility ($\bar{\gamma}$)	20%	
Fixed intensity value ($\bar{\lambda}$)	5%	
Fixed recovery value ($\bar{q} = \frac{1}{2}$)	50%	

Table 4.1: Reference values for the parameters in the model.

Credit spreads

We start by looking into short spread dynamics.

Figure 4.6 presents three possible paths for the short spread under each scenario. Obviously, three paths are not representative in any sense; still we believe the intuition is good and we choose paths with different characteristics. In (a) the market index decreases over time, leading to an increase in the short spreads. In (b) we have a mixed path, and in (c) the index value increases in the end leading to a reduction in the short spreads. An analysis of this figure shows that allowing for some stochasticity either

DIFFERENT SCENARIOS

Scenario	Index Volatility	Intensity	Recovery
(1)	F	F	F
(2)	S	F	F
(3)	F	F	S
(4)	S	F	S
(5)	F	S	F
(6)	S	S	F
(7)	F	S	S
(8)	S	S	S

Table 4.2: Reference scenarios for simulations. F= Fixed, S= Stochastic.

in the intensity process or in the expected loss quota leads to similar short spread dynamics and that it is the combined effect that really makes the difference.

In any of the three paths, if only one of the effects is considered, the short spreads do not oscillate more than 1% below or above the naive 2.5%, while for the combined effect the variation can be as large as 4% (in the case of path (a)) and quite often above 2%.

Figure 4.7 gives us the term structure of credit spreads for three possible market situations: a bull market (where we took $m = 0.7$ for the initial point), a normal market (initially $m = 1$) and a bear market (at the beginning $m = 1.3$).

As expected, as long as the market index is allowed to have an influence either on the PDs or on the LGD, the forward spread TS is not flat. It naturally increases with maturity and its level is lower than 2.5% in bull markets and higher in bear markets. As in Figure 4.6 the effect of the index on the intensity alone or on recovery alone is of the same order of magnitude. Still, the intensity seems to have a more pronounced effect on the slope over time. Nonetheless, the sizable difference results from the combined effect, especially when to that we associate the impact of the negative relation between the index level and its volatility. Note that scenario (8) gives us the highest or the lowest term structure in all circumstances.

Table 4.3 presents figures for these spreads for different maturities.

SHORT SPREAD DYNAMICS

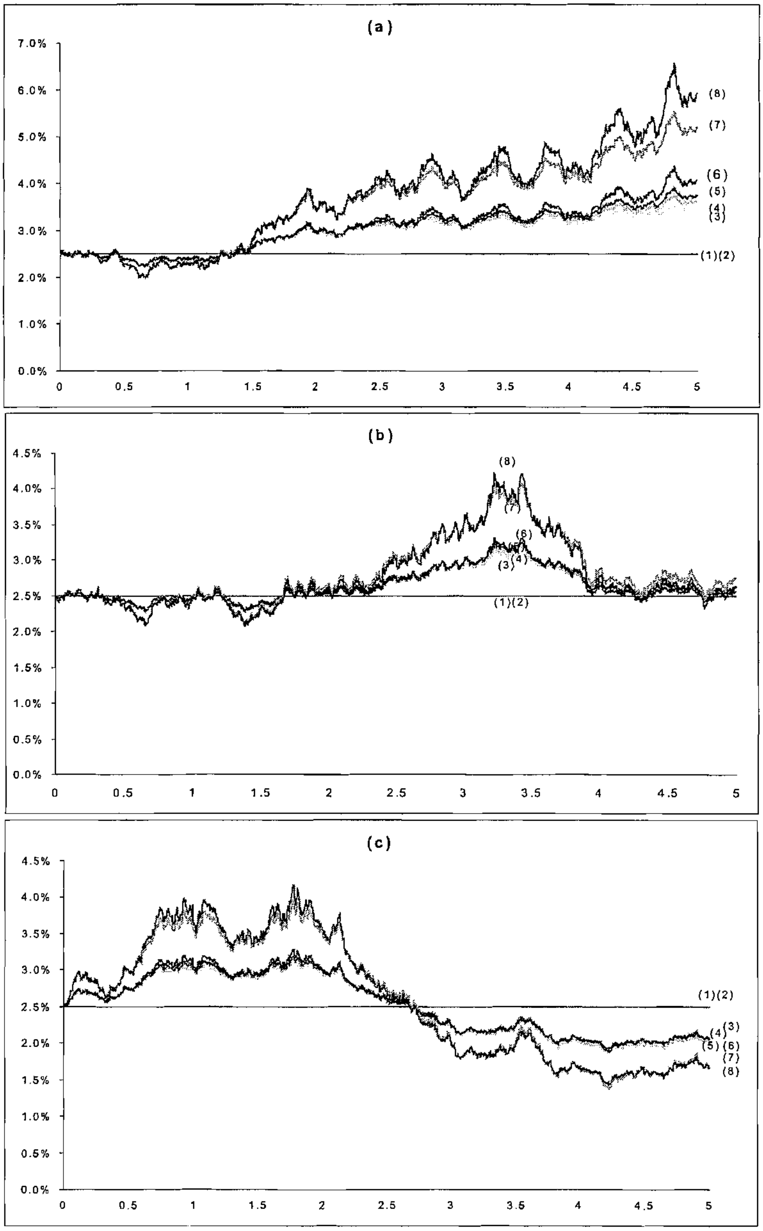


Figure 4.6: Three possible paths for the short spread, $s(t)$, dynamics.

FORWARD SPREADS

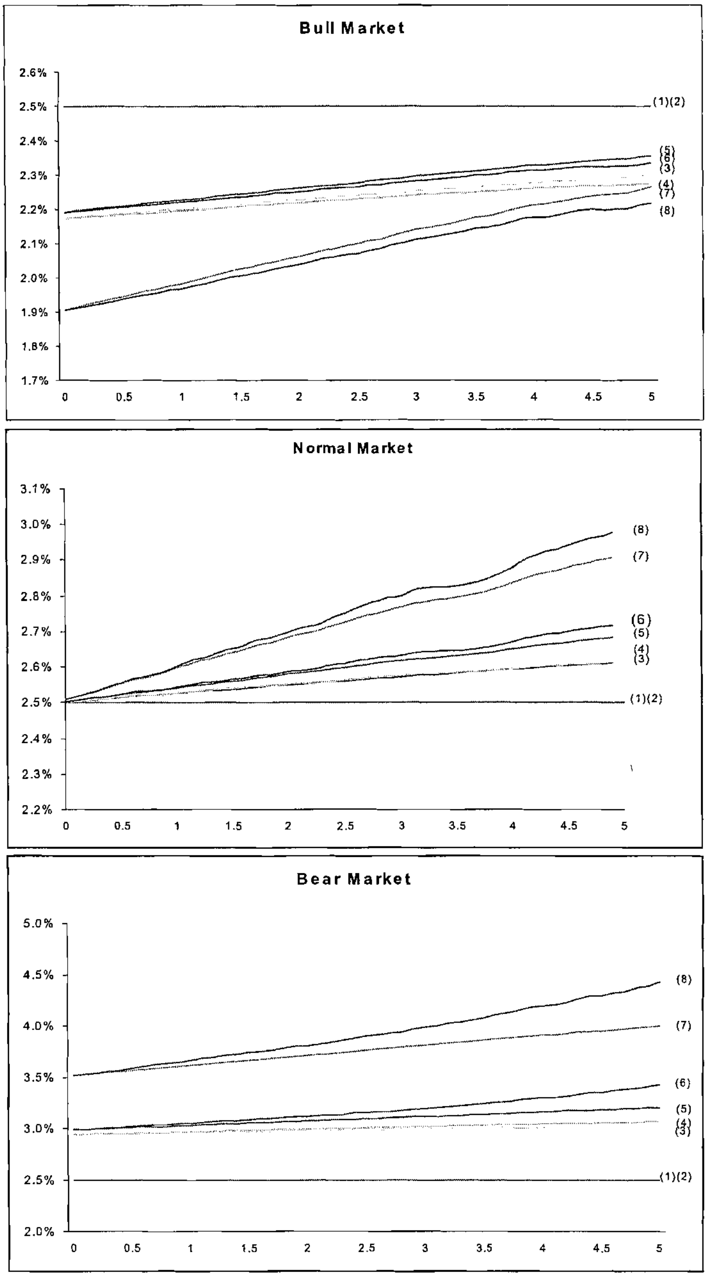


Figure 4.7: Forward spreads for all scenarios, under three possible market conditions.

(a) SPREADS

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.500%	2.174%	2.174%	2.192%	2.192%	1.906%	1.906%
0.1	2.500%	2.177%	2.176%	2.196%	2.195%	1.914%	1.912%
0.5	2.500%	2.188%	2.185%	2.209%	2.206%	1.945%	1.937%
1	2.500%	2.201%	2.196%	2.226%	2.220%	1.983%	1.968%
1.5	2.500%	2.215%	2.208%	2.245%	2.236%	2.025%	2.006%
2	2.500%	2.228%	2.219%	2.262%	2.251%	2.063%	2.040%
3	2.500%	2.253%	2.241%	2.296%	2.284%	2.140%	2.112%
5	2.500%	2.299%	2.275%	2.355%	2.334%	2.264%	2.217%

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5038%	2.5038%	2.5088%	2.5088%
0.5	2.5000%	2.5124%	2.5126%	2.5189%	2.5193%	2.5440%	2.5450%
1	2.5000%	2.5246%	2.5247%	2.5376%	2.5386%	2.5869%	2.5894%
1.5	2.5000%	2.5362%	2.5378%	2.5573%	2.5612%	2.6327%	2.6416%
2	2.5000%	2.5479%	2.5492%	2.5755%	2.5816%	2.6736%	2.6875%
3	2.5000%	2.5700%	2.5736%	2.6142%	2.6300%	2.7606%	2.7953%
5	2.5000%	2.6121%	2.6145%	2.6836%	2.7178%	2.9082%	2.9769%

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.9413%	2.9411%	2.9881%	2.9880%	3.5157%	3.5155%
0.1	2.5000%	2.9433%	2.9439%	2.9925%	2.9941%	3.5258%	3.5296%
0.5	2.5000%	2.9512%	2.9551%	3.0103%	3.0199%	3.5672%	3.5897%
1	2.5000%	2.9608%	2.9687%	3.0324%	3.0533%	3.6182%	3.6666%
1.5	2.5000%	2.9704%	2.9822%	3.0544%	3.0882%	3.6686%	3.7458%
2	2.5000%	2.9803%	2.9940%	3.0742%	3.1167%	3.7121%	3.8084%
3	2.5000%	2.9982%	3.0201%	3.1189%	3.1985%	3.8114%	3.9857%
5	2.5000%	3.0303%	3.0688%	3.2088%	3.4289%	4.0014%	4.4276%

(b) ZERO-COUPON DEFAULTABLE BONDS W/ RECOVERY

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9925	0.9928	0.9928	0.9928	0.9928	0.9930	0.9930
0.5	0.9631	0.9647	0.9647	0.9646	0.9646	0.9659	0.9659
1	0.9277	0.9306	0.9306	0.9304	0.9304	0.9328	0.9329
1.5	0.8935	0.8976	0.8977	0.8973	0.8974	0.9007	0.9009
2	0.8606	0.8658	0.8659	0.8654	0.8655	0.8696	0.8698
3	0.7985	0.8053	0.8055	0.8046	0.8048	0.8100	0.8104
5	0.6872	0.6963	0.6966	0.6949	0.6953	0.7013	0.7021

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9630	0.9630
1	0.9277	0.9275	0.9275	0.9275	0.9275	0.9272	0.9272
1.5	0.8935	0.8933	0.8933	0.8931	0.8931	0.8926	0.8926
2	0.8606	0.8602	0.8602	0.8600	0.8599	0.8591	0.8590
3	0.7984	0.7976	0.7975	0.7971	0.7970	0.7953	0.7950
5	0.6872	0.6852	0.6852	0.6840	0.6836	0.6800	0.6792

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9920	0.9920	0.9919	0.9919	0.9914	0.9914
0.5	0.9631	0.9609	0.9609	0.9607	0.9606	0.9581	0.9580
1	0.9276	0.9234	0.9234	0.9229	0.9228	0.9178	0.9176
1.5	0.8935	0.8874	0.8873	0.8865	0.8863	0.8789	0.8785
2	0.8606	0.8527	0.8526	0.8515	0.8511	0.8416	0.8407
3	0.7984	0.7872	0.7870	0.7853	0.7844	0.7710	0.7692
5	0.6872	0.6706	0.6700	0.6670	0.6644	0.6452	0.6400

Table 4.3: Credit Spreads for several maturities $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$. For $T = 0$ it is the short spread, for all others the forward spread.

Prices of credit securities

The pricing of many credit derivatives can be made by computing what is know as *key building blocks*, as the study in Chapter 3.

Those building blocks are:

- The price of a zero-coupon defaultable bond with maturity t, under zero recovery,

$$\bar{p}_o(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)+\lambda(u)du}\right] .$$

- The price at time t of a zero-coupon defaultable bonds with maturity T under non-zero recovery, in our case,

$$\bar{p}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)+\lambda(u)q^e(u)du}\right] .$$

- The implied survival probabilities during the interval $[t,T]$

$$\text{prob}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)+\lambda(u)q^e(u)du}\right] .$$

- The price of a default digital payoff of 1 at default time, if default occurs in the interval $[t,T]$

$$\text{digital} = \int_t^T e(t,s)ds$$

  where

$$e(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\lambda(T)e^{-\int_t^T r(u)+\lambda(u)du}\right] .$$

Tables 4.3 (b) and 4.4 (a),(b),(c) show the values of all these key quantities for various scenarios and different possible maturities.

Concerning Table 4.3 (b), the first point that should be highlighted is that even for low maturities there is a difference in the prices produced by naive scenarios ((1) and (2)), scenarios where either the PD or the LGD is dependent on the index level ((3) and (4) for the LGD and (5) and (6) for the PD), and scenarios where we consider the combined effect. We note that for the bull and bear markets the difference in pricing is already evident for bonds with approximately one month to maturity ($T = 0.1$), and that with a maturity of 5 years the underpricing of the naive model can be of up to 5% in a bull market, and up to 10% in a bear market. In normal market situations, the difference between the scenarios are slope differences with an overall tendency of the naive model to overprice if in reality the market index affects the PD, the LGD or both.

(a) ZERO-COUPON DEFAULTABLE BOND PRICES W/O RECOVERY

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	0.9899	0.9905	0.9905	0.1	0.9899	0.9899	0.9899	0.1	0.9899	0.9888	0.9888
0.5	0.9510	0.9539	0.9539	0.5	0.9510	0.9510	0.9510	0.5	0.9510	0.9463	0.9462
1	0.9047	0.9100	0.9100	1	0.9047	0.9043	0.9043	1	0.9047	0.8954	0.8953
1.5	0.8605	0.8679	0.8680	1.5	0.8605	0.8598	0.8598	1.5	0.8605	0.8472	0.8468
2	0.8186	0.8276	0.8278	2	0.8186	0.8173	0.8173	2	0.8186	0.8013	0.8006
3	0.7407	0.7522	0.7525	3	0.7407	0.7382	0.7380	3	0.7407	0.7165	0.7151
5	0.6064	0.6203	0.6209	5	0.6064	0.6010	0.6003	5	0.6064	0.5714	0.5674

(b) IMPLIED SURVIVAL PROBABILITIES

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	99.481%	99.545%	99.545%	0.1	99.481%	99.481%	99.481%	0.1	99.481%	99.380%	99.380%
0.5	97.511%	97.806%	97.808%	0.5	97.511%	97.502%	97.502%	0.5	97.511%	97.022%	97.018%
1	95.104%	95.660%	95.667%	1	95.104%	95.068%	95.067%	1	95.104%	94.135%	94.116%
1.5	92.756%	93.546%	93.559%	1.5	92.756%	92.677%	92.673%	1.5	92.756%	91.313%	91.270%
2	90.466%	91.463%	91.484%	2	90.466%	90.329%	90.321%	2	90.466%	88.559%	88.484%
3	86.054%	87.392%	87.432%	3	86.054%	85.765%	85.740%	3	86.054%	83.245%	83.078%
5	77.865%	79.641%	79.729%	5	77.865%	77.164%	77.080%	5	77.865%	73.372%	72.850%

(c) DEFAULT DIGITAL PAYOFFS

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	0.00495	0.00435	0.00435	0.1	0.00495	0.00496	0.00496	0.1	0.00495	0.00592	0.00592
0.5	0.02426	0.02141	0.02139	0.5	0.02426	0.02437	0.02437	0.5	0.02426	0.02904	0.02909
1	0.04733	0.04199	0.04193	1	0.04733	0.04771	0.04772	1	0.04733	0.05672	0.05691
1.5	0.06928	0.06178	0.06165	1.5	0.06928	0.07008	0.07012	1.5	0.06928	0.08309	0.08352
2	0.09016	0.08079	0.08059	2	0.09016	0.09150	0.09157	2	0.09016	0.10821	0.10893
3	0.12892	0.11658	0.11621	3	0.12892	0.13161	0.13184	3	0.12892	0.15489	0.15645
5	0.19572	0.17983	0.17908	5	0.19572	0.20179	0.20250	5	0.19571	0.23542	0.23993

Table 4.4: (a) Prices of zero-recovery, zero-coupon bond prices for several maturities (top table). (b) Implied survival probabilities in the interval $[t, T]$ for several maturities. (c) Price of a digital payoff at default if it occurs in the interval $[t, T]$. Three different market conditions: bull, normal and bear.

In Table 4.4 the quantities presented do not depend on recovery (recall the building blocks equations), and therefore our scenarios are reduced to three possibilities: constant intensity (scenarios (1) (2) (3) (4)); intensity dependent on the index with constant volatility (scenarios (5) (7)); or intensity dependent on the index with stochastic volatility (scenarios (6) (8)). In Table 4.4 (a), we see that the naive model can also lead to underpricing in bull markets when we consider defaultable zero recovery bonds, and overpricing in bear markets. The dimension of these mispricings is similar to the mispricing in the previous table.

When looking into survival probabilities (Table4.4 (b)) the scenarios considering the index influence show less probability of surviving in bear markets than in bull markets, with a difference of up to 7% already the 5-year maturity. This indicates that using the naive model could lead to a possible overestimation of the true survival probabilities in bear markets, and underestimation in bull markets. As in most of the previous tables, the size of the errors tends to be considerably larger in bear markets than in bull markets. In the case of survival probability, overestimation can reach 5% in bear markets in contrast with up to 2% in bull markets. Finally, in Table 4.4 (c), we have the price of a digital payoff of 1 at default time. As before, the differences to the naive model are more drastic in bear markets particularly considering that a horizon of 5 years is quite short and that all the impacts are likely to increase for higher maturities, as we will see in the next section.

In Section 4.4.4 we run some maturity related robustness checks and draw the reader's attention to what we consider to be interesting aspects related to the sensitivity parameter ϵ (assumed fixed up to now) and to the possible tracking of term structures of credit spreads by market volatilities.

4.4.4 Additional Considerations

Higher Maturities

In this section we extend the maturity horizon and analyze to what extent our results hold. Table 4.5 presents forward spreads and defaultable bonds for several maturities, while Table 4.6 presents zero recovery bond prices, implied survival probabilities, and the price of default digital payoff of 1. First, we can clearly see that for all the three scenarios survival probabilities decrease significantly for longer horizons T. If we extend the horizon from 5 to 15 years, in the stochastic volatility scenario the survival probability decreases by almost 40% for the bull market and up to 50% for the bear market. In our opinion, this is a realistic feature of our model since at the longer horizons when the market is in recession and firms are known to be sensitive to the fluctuations of the market, the probability of default is quite high.

Moreover, it is also interesting to note that in a bull market, although a stochastic volatility scenario yields higher survival probabilities at all the maturities, the difference in survival probabilities is much smaller at the higher maturities. In a bear market, on the other hand, survival probabilities are lower for the stochastic volatility case, and the difference in survival probabilities between the stochastic volatility and naive scenarios is more pronounced. In contrast to the bull market, the difference increases by approximately 5% when the investment horizon is extended from 5 to 15 years.

Overall, the results are essentially the same with one interesting additional result for the long end of the forward spreads term structure.

From Figure 4.8 we see that when we consider the dependence of both PD and LGD and the negative relation the index level and volatility (scenario (8)), the TS seems to converge faster to its long-run level. In fact, for maturities higher than 15 years the TS of this scenario are is relatively flat. Thus, the forward credit spreads are most sensitive to the influence of the market index at the relatively shorter maturities, and around the 15 years maturity, the credit spreads become relatively flatter and less sensitive to the market index, moving in fact closer to each other.

Ratings and different sensitivities

We now look into the parameter ϵ in

$$\lambda(I, \epsilon) = \bar{\lambda}\,[m(I)]^{\epsilon},$$

which is a measure of the sensitivity of a firms PD to the market situation.

The intuition comes from the fact the PD of firms with high creditworthiness should depend much less on market oscillation than that of less creditworthy firms that are more sensitive to business cycles. In this sense, different ϵ parameters could represent the term structure of firms with different credit ratings.

In the following we consider three different values for *epsilon*: high $\epsilon = 1/2$, medium $\epsilon = 1/4$ and low $\epsilon = 1/16$.[7]

[7]The case of total insensitivity, or $\epsilon = 0$, is always considered since in scenarios (1)(2)(3)(4) $\lambda(I, \epsilon) = \bar{\lambda}$.

FORWARD SPREADS - Higher Maturities

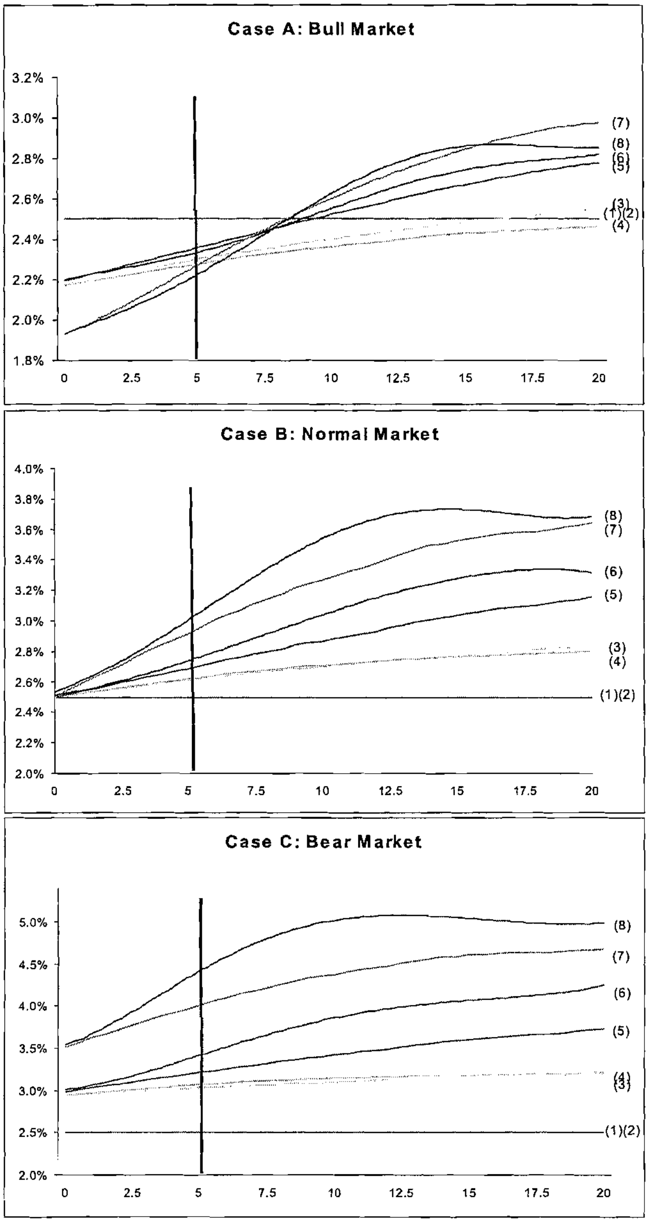


Figure 4.8: Forward spreads for all scenarios, under three possible market conditions and higher maturity values $T = 5, 6, 7, 8, 10, 12, 15, 10$.

(a) SPREADS

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.500%	2.299%	2.275%	2.355%	2.334%	2.264%	2.217%
6	2.5000%	2.3222%	2.2990%	2.3940%	2.3838%	2.3494%	2.3231%
7	2.5000%	2.3432%	2.3166%	2.4248%	2.4137%	2.4120%	2.3825%
8	2.5000%	2.3633%	2.3338%	2.4573%	2.4603%	2.4769%	2.4691%
10	2.5000%	2.4016%	2.3632%	2.5188%	2.5416%	2.5916%	2.6039%
12	2.5000%	2.4371%	2.3910%	2.5774%	2.6189%	2.6953%	2.7190%
15	2.5000%	2.4824%	2.4302%	2.6739%	2.7691%	2.8554%	2.8981%
20	2.5000%	2.5542%	2.4668%	2.7863%	2.8439%	2.9922%	2.8864%

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.5000%	2.6121%	2.6145%	2.6836%	2.7178%	2.9082%	2.9769%
6	2.5000%	2.6310%	2.6404%	2.7281%	2.8012%	3.0045%	3.1441%
7	2.5000%	2.6495%	2.6595%	2.7625%	2.8530%	3.0731%	3.2376%
8	2.5000%	2.6671%	2.6768%	2.7988%	2.9226%	3.1432%	3.3545%
10	2.5000%	2.7011%	2.7054%	2.8669%	3.0235%	3.2643%	3.4930%
12	2.5000%	2.7326%	2.7315%	2.9314%	3.1352%	3.3706%	3.6233%
15	2.5000%	2.7696%	2.7640%	3.0359%	3.2997%	3.5253%	3.7635%
20	2.5000%	2.8352%	2.7992%	3.1547%	3.3834%	3.6428%	3.7094%

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.5000%	3.0303%	3.0688%	3.2088%	3.4289%	4.0014%	4.4276%
6	2.5000%	3.0434%	3.0890%	3.2580%	3.5343%	4.0992%	4.6167%
7	2.5000%	3.0582%	3.1067%	3.2979%	3.6118%	4.1738%	4.7347%
8	2.5000%	3.0720%	3.1210%	3.3398%	3.7093%	4.2486%	4.8668%
10	2.5000%	3.0994%	3.1449%	3.4174%	3.8324%	4.3721%	4.9761%
12	2.5000%	3.1247%	3.1631%	3.4899%	3.9246%	4.4752%	5.0054%
15	2.5000%	3.1505%	3.1813%	3.6048%	4.1097%	4.6074%	5.0888%
20	2.5000%	3.2070%	3.2096%	3.7295%	4.2577%	4.6777%	4.9864%

(b) ZERO-COUPON DEFAULTABLE BONDS W/ RECOVERY

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	0.6872	0.6963	0.6966	0.6949	0.6953	0.7013	0.7021
6	0.6374	0.6470	0.6475	0.6454	0.6459	0.6518	0.6529
7	0.5913	0.6012	0.6019	0.5993	0.5998	0.6054	0.6066
8	0.5486	0.5586	0.5594	0.5563	0.5568	0.5619	0.5632
10	0.4722	0.4819	0.4829	0.4789	0.4792	0.4832	0.4843
12	0.4064	0.4155	0.4167	0.4118	0.4118	0.4147	0.4155
15	0.3245	0.3321	0.3336	0.3275	0.3268	0.3283	0.3285
20	0.2231	0.2281	0.2299	0.2226	0.2215	0.2209	0.2217

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	0.6872	0.6852	0.6852	0.6840	0.6836	0.6800	0.6792
6	0.6373	0.6347	0.6346	0.6330	0.6322	0.6276	0.6261
7	0.5913	0.5880	0.5878	0.5858	0.5846	0.5791	0.5768
8	0.5485	0.5446	0.5444	0.5419	0.5402	0.5340	0.5308
10	0.4721	0.4670	0.4668	0.4633	0.4605	0.4531	0.4483
12	0.4064	0.4003	0.4000	0.3956	0.3917	0.3836	0.3776
15	0.3245	0.3172	0.3170	0.3113	0.3060	0.2976	0.2907
20	0.2230	0.2147	0.2148	0.2077	0.2020	0.1938	0.1882

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	0.6872	0.6706	0.6700	0.6670	0.6644	0.6452	0.6400
6	0.6372	0.6185	0.6176	0.6138	0.6098	0.5888	0.5811
7	0.5911	0.5707	0.5696	0.5650	0.5597	0.5373	0.5274
8	0.5484	0.5264	0.5252	0.5199	0.5132	0.4900	0.4781
10	0.4720	0.4478	0.4463	0.4397	0.4305	0.4067	0.3918
12	0.4063	0.3808	0.3792	0.3713	0.3604	0.3368	0.3207
15	0.3244	0.2983	0.2967	0.2872	0.2748	0.2529	0.2369
20	0.2230	0.1982	0.1969	0.1862	0.1741	0.1561	0.1438

Table 4.5: (a) Credit Spreads and (b)Price of defaultable bond with recovery for several higher maturities $T = 5, 6, 7, 8, 10, 12, 15, 20$.

(a) ZERO-COUPON DEFAULTABLE BOND PRICES W/O RECOVERY

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
5	0.6064	0.6203	0.6209	5	0.6064	0.6010	0.6003	5	0.6064	0.5714	0.5674
6	0.5483	0.5624	0.5633	6	0.5483	0.5411	0.5400	6	0.5483	0.5090	0.5030
7	0.4962	0.5099	0.5109	7	0.4961	0.4874	0.4856	7	0.4961	0.4537	0.4461
8	0.4489	0.4621	0.4630	8	0.4489	0.4387	0.4363	8	0.4489	0.4041	0.3950
10	0.3676	0.3788	0.3795	10	0.3675	0.3549	0.3514	10	0.3675	0.3199	0.3090
12	0.3009	0.3100	0.3105	12	0.3009	0.2865	0.2823	12	0.3009	0.2528	0.2414
15	0.2229	0.2287	0.2287	15	0.2229	0.2071	0.2024	15	0.2229	0.1770	0.1663
20	0.1352	0.1370	0.1373	20	0.1352	0.1199	0.1165	20	0.1352	0.0973	0.0897

(b) IMPLIED SURVIVAL PROBABILITIES

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
5	77.865%	79.641%	79.729%	5	77.865%	77.164%	77.080%	5	77.865%	73.372%	72.850%
6	74.008%	75.912%	76.037%	6	74.008%	73.043%	72.885%	6	74.008%	68.712%	67.903%
7	70.398%	72.357%	72.493%	7	70.398%	69.161%	68.912%	7	70.398%	64.379%	63.300%
8	66.965%	68.931%	69.070%	8	66.965%	65.447%	65.094%	8	66.965%	60.280%	58.932%
10	60.592%	62.458%	62.572%	10	60.592%	58.505%	57.931%	10	60.592%	52.750%	50.953%
12	54.826%	56.491%	56.577%	12	54.826%	52.201%	51.430%	12	54.826%	46.071%	43.988%
15	47.189%	48.424%	48.412%	15	47.189%	43.839%	42.847%	15	47.189%	37.473%	35.204%
20	36.751%	37.242%	37.313%	20	36.751%	32.598%	31.660%	20	36.751%	26.447%	24.388%

(c) DEFAULT DIGITAL PAYOFFS

Case A : Bull Market				Case B : Normal Market				Case C : Bear Market			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
5	0.19572	0.17983	0.17908	5	0.19572	0.20179	0.20250	5	0.19572	0.23542	0.23993
6	0.21978	0.20407	0.20299	6	0.21978	0.22820	0.22960	6	0.21978	0.26456	0.27158
7	0.24522	0.22920	0.22805	7	0.24522	0.25560	0.25765	7	0.24522	0.29506	0.30397
8	0.26824	0.25224	0.25108	8	0.26824	0.28054	0.28331	8	0.26824	0.32250	0.33321
10	0.30792	0.29263	0.29166	10	0.30792	0.32378	0.32791	10	0.30792	0.36927	0.38268
12	0.34040	0.32631	0.32551	12	0.34040	0.35930	0.36456	12	0.34040	0.40680	0.42174
15	0.37843	0.36654	0.36624	15	0.37843	0.40093	0.40729	15	0.37843	0.44949	0.46528
20	0.42119	0.41237	0.41169	20	0.42119	0.44694	0.45306	20	0.42119	0.49454	0.50944

Table 4.6: (a) Prices of zero-recovery, zero-coupon bond prices for several maturities (top table). (b)Implied survival probabilities in the interval $[t, T]$ for several maturities. (c) Price of a digital payoff at default if it occurs in the interval $[t, T]$.

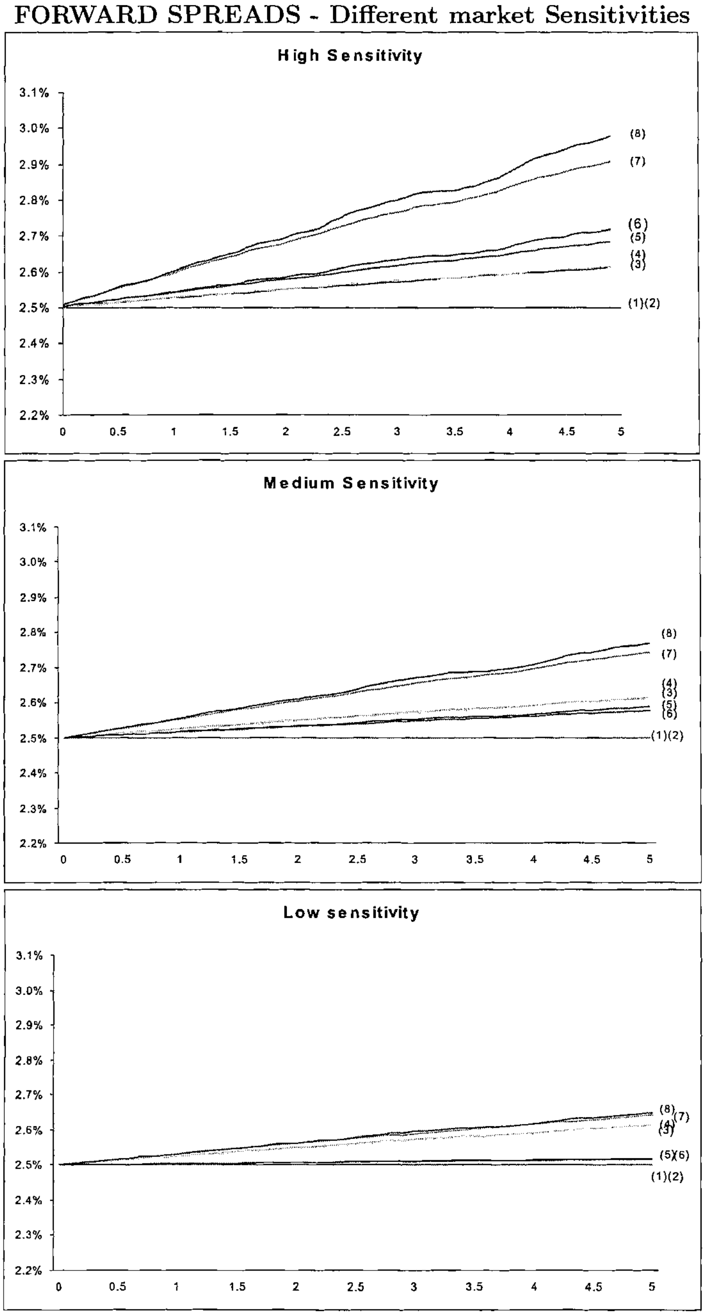


Figure 4.9: Forward spreads for all scenarios, under normal market conditions. For three different values of ϵ: high $\epsilon = \frac{1}{2}$, medium $\epsilon = \frac{1}{4}$ and low $\epsilon = \frac{1}{16}$.

(a) SPREADS

High

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5038%	2.5038%	2.5088%	2.5088%
0.5	2.5000%	2.5124%	2.5126%	2.5189%	2.5193%	2.5440%	2.5450%
1	2.5000%	2.5246%	2.5247%	2.5376%	2.5386%	2.5869%	2.5894%
1.5	2.5000%	2.5362%	2.5378%	2.5573%	2.5612%	2.6327%	2.6416%
2	2.5000%	2.5479%	2.5492%	2.5755%	2.5816%	2.6736%	2.6875%
3	2.5000%	2.5700%	2.5736%	2.6142%	2.6300%	2.7606%	2.7953%
5	2.5000%	2.6121%	2.6145%	2.6836%	2.7178%	2.9082%	2.9769%

Medium

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5016%	2.5016%	2.5054%	2.5053%
0.5	2.5000%	2.5124%	2.5126%	2.5079%	2.5080%	2.5266%	2.5271%
1	2.5000%	2.5246%	2.5247%	2.5157%	2.5160%	2.5525%	2.5535%
1.5	2.5000%	2.5362%	2.5378%	2.5236%	2.5252%	2.5792%	2.5837%
2	2.5000%	2.5479%	2.5492%	2.5313%	2.5335%	2.6038%	2.6099%
3	2.5000%	2.5700%	2.5736%	2.5471%	2.5527%	2.6544%	2.6697%
5	2.5000%	2.6121%	2.6145%	2.5768%	2.5879%	2.7433%	2.7701%

Low

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5003%	2.5003%	2.5032%	2.5032%
0.5	2.5000%	2.5124%	2.5126%	2.5017%	2.5017%	2.5156%	2.5159%
1	2.5000%	2.5246%	2.5247%	2.5033%	2.5034%	2.5310%	2.5313%
1.5	2.5000%	2.5362%	2.5378%	2.5050%	2.5053%	2.5460%	2.5482%
2	2.5000%	2.5479%	2.5492%	2.5067%	2.5070%	2.5606%	2.5629%
3	2.5000%	2.5700%	2.5736%	2.5100%	2.5110%	2.5891%	2.5951%
5	2.5000%	2.6121%	2.6145%	2.5165%	2.5184%	2.6419%	2.6491%

(b) ZERO-COUPON DEFAULTABLE BONDS W/ RECOVERY

High

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9630	0.9630
1	0.9277	0.9275	0.9275	0.9275	0.9275	0.9272	0.9272
1.5	0.8935	0.8933	0.8933	0.8931	0.8931	0.8926	0.8926
2	0.8606	0.8602	0.8602	0.8600	0.8599	0.8591	0.8590
3	0.7984	0.7976	0.7975	0.7971	0.7970	0.7953	0.7950
5	0.6872	0.6852	0.6852	0.6840	0.6836	0.6800	0.6792

Medium

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9630	0.9630
1	0.9277	0.9275	0.9275	0.9276	0.9276	0.9274	0.9274
1.5	0.8935	0.8933	0.8933	0.8934	0.8933	0.8930	0.8930
2	0.8606	0.8602	0.8602	0.8604	0.8603	0.8597	0.8597
3	0.7984	0.7976	0.7975	0.7979	0.7978	0.7966	0.7964
5	0.6872	0.6852	0.6852	0.6859	0.6858	0.6829	0.6826

Low

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9631	0.9631
1	0.9277	0.9275	0.9275	0.9276	0.9276	0.9275	0.9275
1.5	0.8935	0.8933	0.8933	0.8935	0.8935	0.8932	0.8932
2	0.8606	0.8602	0.8602	0.8606	0.8606	0.8601	0.8601
3	0.7984	0.7976	0.7975	0.7983	0.7983	0.7974	0.7973
5	0.6872	0.6852	0.6852	0.6869	0.6869	0.6847	0.6846

Table 4.7: (a) Credit Spreads and (b)Price of defaultable bond with recovery for several maturities and three different values of ϵ: high $\epsilon = \frac{1}{2}$, medium $\epsilon = \frac{1}{4}$ and low $\epsilon = \frac{1}{16}$.

(a) ZERO-COUPON DEFAULTABLE BOND PRICES W/O RECOVERY

High				Medium				Low			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	0.9899	0.9899	0.9899	0.1	0.9899	0.9899	0.9899	0.1	0.9899	0.9899	0.9899
0.5	0.9510	0.9510	0.9510	0.5	0.9510	0.9510	0.9510	0.5	0.9510	0.9510	0.9510
1	0.9047	0.9043	0.9043	1	0.9047	0.9045	0.9045	1	0.9047	0.9046	0.9046
1.5	0.8605	0.8598	0.8598	1.5	0.8605	0.8602	0.8602	1.5	0.8605	0.8605	0.8605
2	0.8186	0.8173	0.8173	2	0.8186	0.8181	0.8180	2	0.8186	0.8185	0.8185
3	0.7407	0.7382	0.7380	3	0.7407	0.7396	0.7396	3	0.7407	0.7405	0.7404
5	0.6064	0.6010	0.6003	5	0.6064	0.6041	0.6039	5	0.6064	0.6059	0.6059

(b) IMPLIED SURVIVAL PROBABILITIES

High				Medium				Low			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	99.481%	99.481%	99.481%	0.1	99.481%	99.481%	99.481%	0.1	99.481%	99.481%	99.481%
0.5	97.511%	97.502%	97.502%	0.5	97.511%	97.508%	97.508%	0.5	97.511%	97.511%	97.511%
1	95.104%	95.068%	95.067%	1	95.104%	95.089%	95.089%	1	95.104%	95.101%	95.101%
1.5	92.756%	92.677%	92.673%	1.5	92.756%	92.723%	92.722%	1.5	92.756%	92.749%	92.749%
2	90.466%	90.329%	90.321%	2	90.466%	90.409%	90.406%	2	90.466%	90.454%	90.453%
3	86.054%	85.765%	85.740%	3	86.054%	85.933%	85.924%	3	86.054%	86.028%	86.026%
5	77.865%	77.164%	77.080%	5	77.865%	77.568%	77.539%	5	77.865%	77.800%	77.795%

(c) DEFAULT DIGITAL PAYOFFS

High				Medium				Low			
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)	T	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	0.00495	0.00496	0.00496	0.1	0.00495	0.00495	0.00495	0.1	0.00495	0.00495	0.00495
0.5	0.02426	0.02437	0.02437	0.5	0.02426	0.02430	0.02430	0.5	0.02426	0.02427	0.02427
1	0.04733	0.04771	0.04772	1	0.04733	0.04749	0.04750	1	0.04733	0.04737	0.04737
1.5	0.06928	0.07008	0.07012	1.5	0.06928	0.06962	0.06963	1.5	0.06928	0.06935	0.06936
2	0.09016	0.09150	0.09157	2	0.09016	0.09072	0.09075	2	0.09016	0.09028	0.09029
3	0.12892	0.13161	0.13184	3	0.12892	0.13004	0.13013	3	0.12892	0.12916	0.12918
5	0.19572	0.20179	0.20250	5	0.19572	0.19828	0.19854	5	0.19572	0.19627	0.19632

Table 4.8: (a) Prices of zero-recovery, zero-coupon bond prices for several maturities (top table). (b)Implied survival probabilities in the interval $[t, T]$ for several maturities. (c) Price of a digital payoff at default if it occurs in the interval $[t, T]$. Three different values of ϵ: high $\epsilon = \frac{1}{2}$, medium $\epsilon = \frac{1}{4}$ and low $\epsilon = \frac{1}{16}$. Higher maturities and three different market conditions: bull, normal and bear.

Figure 4.9 and Tables 4.7 and 4.8 show the simulation results for the different ϵ values under normal market conditions.

The key feature that results from considering different sensitivities is that the TS of less sensitive firms has a smaller TS slope. This is particularly obvious for scenarios (7) and (8) when the index influences both PD and LGD, and less obvious when it affects only one of them.

Thus, from a practical point of view, it is more important to take into account the correlation with the market index when considering a portfolio of securities with low credit ratings. We believe that the value of the portfolio will fluctuate significantly with the business cycle.

The effect will be even more pronounced for firms with low credit ratings and especially in the case when we have stochastic index volatility dependent of the index level.

Using implied ATM volatilities as credit spread trackers

An interesting side effect of our model, when we take the index volatility to be stochastic and negatively related to the index level, is that short spread dynamics can be tracked quite well by observing the index volatility. See Figure 4.10 with three possible volatility paths and compare with the short spread evolution for the same paths in Figure 4.6.

If the (spot) volatility seems to be a good tracker of the short spread, then implied volatilities of options with longer maturities may be good trackers of the forward spread TS. This is all due to the negative correlation between the index level and its volatility.

Furthermore, it provides a fundamental reason for using implied volatilities of options on indices as predictors of the forward spread term structure and is in line with what seems to be common practice among traders who typically use at-the-money (ATM) volatility term structures in predicting forward credit spreads.

For example, Collin-Dufresne, Goldstein, and Martin (2001) investigated the determinants of credit spread changes. They showed that credit spreads are mostly driven by a single common factor and that implied volatilities of index options contain important information for credit spreads[8].

[8]Recent papers (see e.g. Cremers, Dreissen, and Weinbaum (2004)) start using measures of volatility and skewness based also on individual stock options to explain credit spreads on corporate bonds.

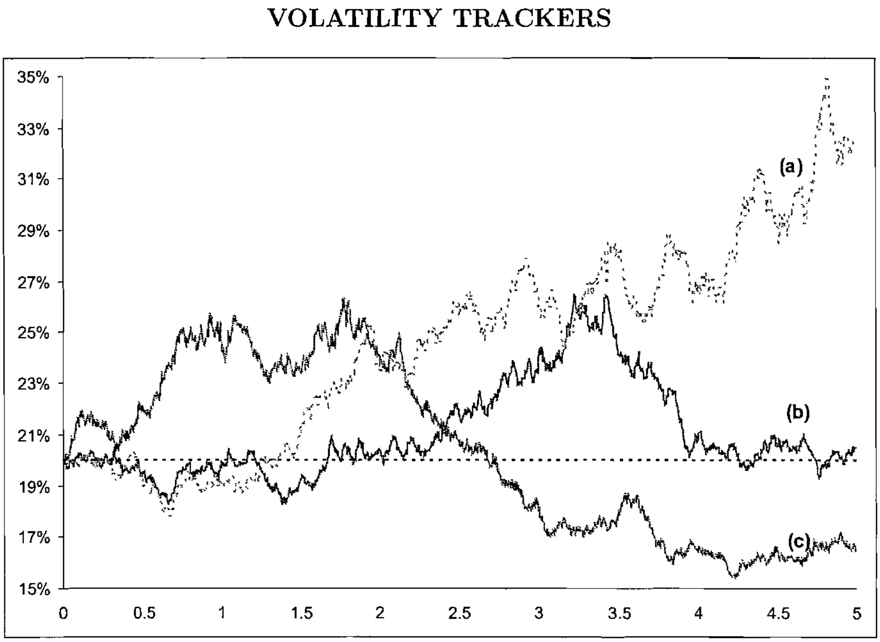


Figure 4.10: Volatility paths corresponding to the spot spread paths in Figure 4.6.

4.5 Conclusions and Future Research

We propose a concrete reduced-form model where both the PD and LGD are dependent on a macroeconomic index. Furthermore, during recessions both the PD and the LGD increase (the reverse happens during economic booms). Finally, recessions are periods of higher market volatility, while booms are characterized by low volatility.

Through simulation we show that many of the realistic features of our model depend on:

(i) the fact that *both* PD and LGD are driven by the same factors, and

(ii) the negative relation of the index level and its volatility.

Unfortunately, these two aspects together make the model untractable from an analytical point of view. However, this may be the right price to pay given all the behaviors it captures.

The main realistic features captured by our model can be summarized as follows:

- The difference between short spreads in bull versus bear markets can be up to three times more than the difference produced by models that consider the market influence in the PD or LGD only [mainly property (i)].
- Convergence to long-run levels is faster, originating flat TS for maturities higher than 15 years [properties (i) and (ii)].
- Market volatility tracks the short credit spread dynamics quite well, suggesting that the TS of ATM implied volatilities of index options may do the same for forward credit spreads [mainly property (ii)].

It seems clear that a model with both characteristic (i) and (ii) will not allow for closed-form solutions. We would, nonetheless, like to conclude the chapter on a positive note related to future research.

Given the tractability drawback, and since one must rely on numerical simulations, the ideal situation would be to model the intensity and the distribution of the loss quota as realistically as possible (this may involve different functional forms and a different market price of jump risk assumption). A study of the credit TS shapes observed in the market can help to define such functional forms. Calibration of our "toy model", or other more complex models, to market data seems to be another obvious next step. In addition to the single firm, this framework could also be extended to several firms and help deal with portfolio credit risk issues. For portfolio credit risk the interdependence between PD and LGD is likely to have a much more relevant impact than at the individual firm level. Indeed, portfolio losses depend upon both quantities, and the fact that periods when default is more likely may also be periods when recovery is lower suggests caution in using naive models to establish bank reserves and related precautionary measures. Finally, our last comment on volatility trackers may help to construct a bridge between equity and credit markets, and deserves further investigation.

Part III

Forward Price Models

Chapter 5

GQTS for Futures and Forward Prices

"A conventional forward contract is distinct from a futures contract in that forward positions are not regularly marked-to-market. Depending on the behavior of interest rates, this distinction may not be negligible. "

Darrell Duffie, *Futures markets*, 1989.

For finite dimensional factor models, the chapter studies general quadratic term structures of forward prices. Forward prices are the most well-known financial martingales under the forward measures. This martingale property under measures where the bond prices are the *numeraire*, make their term structure equation dependent on properties of the term structure of bond prices. We exploit this connection and show that even in quadratic short rate settings we can have affine term structures for forward prices.

Finally, we show how the study of futures prices is naturally embedded in a study of forward prices, that the difference between the two term structures may be deterministic in some (non-trivial) stochastic interest rate settings.

5.1 Introduction

While the literature on term structures is rich, it has to date mainly focused on the study of the term structure of *bond prices*, much less frequently on the term structure of *futures prices*, and rarely on the term structure of *forward prices*.

The literature on the term structure of *bond prices* is extremely vast on both the theoretical and empirical levels. The two most studied classes of term structures are the so-called affine term structures (ATS) and the Gaussian-quadratic term structures (Gaussian-QTS). Since the well-known papers of Vasiček (1977) and of Cox, Ingersoll, and Ross (1985) on ATS of bond prices, and in particular after some more recent empirical work, there seems to be the belief that ATS are attractive from a computational point of view, but lack the flexibility needed to explain market data.[1] Gaussian-QTS introduced by Longstaff (1989) and Beaglehole and Tenney (1991) try to introduce the necessary flexibility, but unfortunately at the cost of imposing Gaussian dynamics on the stochastic state variables that enter the term structures. Despite this fact, the empirical evidence seems to show that these term structures have better fitting properties (see, for instance Ahn, Dittmar, and Gallant (2002)). Due to their Gaussian properties these early studies on QTS could not be seen as extensions of the ATS. Chapter 2, however, has characterized what is defined to be *general quadratic term structures*, which include as special cases both the ATS and the Gaussian-QTS but also allow for non-Gaussian QTS.

On the *futures prices* side, the theoretic literature has focused mainly on ATS (extended sometimes to also include jump processes) and on option valuation (see, for example, Björk and Landén (2001) and Jamshidian (1996)). The empirical literature, on the other hand, has pointed out that some term structures of futures prices are not affine, and especially when dealing with commodity futures, ATS are not flexible enough (see, for instance Escobar, Hernandés, and Seco (2003) and Miltersen and Schwartz (1998)).

The term structure of *forward prices*, however, has been almost entirely ignored by the literature. The main exception is Björk and Landén (2001) who devote a small section to the study of ATS of forward prices in an affine interest rate setting. The main focus of their study, however, is futures prices and no attempt is made to exploit the connection between the two prices. While this chapter is particularly inspired by that study on forward prices, it generalizes it in three different ways: first, we consider a more general stochastic setting of interest rates; second, we study general quadratic term structures (that include ATS as a special case); and finally we exploit the relationship between the term structures of forward prices and futures prices. Forward prices are martingales under forward measures and their term structure equation has the particularity that it depends on the term structure of bond prices. This feature makes their term structure considerably harder to handle.

[1]The literature on ATS of bond prices is so vast that we refer to recent surveys (for instance, Ahn, Dittmar, and Gallant (2002), Dai and Singleton (2003), Leippold and Wu (2002), Rebonato (2003)) for an updated list of references.

The main contributions of the chapter are as follows:

In Section 5.2, we present the general framework for the term structure dynamics. We model the entire term structure of forward and futures prices, *a la* Heath-Jarrow-Morton, and recover from Chapter 2 the definition of *general quadratic term structures* (GQTS). We introduce our main assumption and restate useful results.

In Section 5.3, we study GQTS of *futures prices.* We show that the adequate classification of factors for futures prices is similar in spirit to the one used in Chapter 2 when studying bond prices, and we derive sufficient conditions as well as a way to explicitly compute GQTS of futures prices. Finally, we compare these GQTS with the ones previously studied in the futures prices literature.

Section 5.4 focuses on the study of *forward prices* themselves and their relationship with bond prices. We show which term structures of bond prices are compatible with GQTS of forward prices. We prove that GQTS of forward prices are consistent with GQTS of bond prices and, in particular, that ATS of forward prices are consistent with volatility restricted GQTS of bond prices.

We exploit the connection with futures prices in Section 5.5. We show how the study of the term structure of futures prices is (in some sense) included in the study of the term structure of forward prices and analyze the differences between the two term structures. We are able to characterize qualitatively the difference between the two term structures for any *general quadratic short rate* (GQSR) setting and to quantify it in one (still non-trivial) special case where we show that futures and forward prices differ only by a multiplicative deterministic term (an adjustment factor) for which we give a close form solution.

In Section 5.6 we exemplify our GQTS theory for forward and futures prices. Our examples include a number of factor models that have been proposed in the literature, as well as some new models. Section 5.7 concludes this study and identifies directions for future research.

5.2 Basic Concepts

Our main goal is the study of *general quadratic term structures* (GQTS) of *forward prices* when those term structures can be expressed as functions of a finite dimensional state process Z. As mentioned before, it turns out that in order to study forward prices term structures we will need also *bond prices* term structures and we will obtain *futures* *prices* term structures as a side result. Therefore, in this setup, we need to consider the three types of term structures.

We take an m-dimensional (column) vector of factors Z and assume that, at time t, the bond, futures or forward price with maturity T is given by some deterministic functions, H_p, H_F and H_f, respectively, so that we have:

$$\begin{aligned}
p(t,T) &= H_p(t,Z(t),T) \\
F(t,T) &= H_F(t,Z(t),T) \\
f(t,T) &= H_f(t,Z(t),T).
\end{aligned}$$

In the above functions it is natural to view t and $Z(t)$ as variables and T (the maturity of the prices) as a parameter. For a fixed t and z, the mappings $T \to H_p(t,z,T)$, $T \to H_F(t,z,T)$ and $T \to H_f(t,z,T)$ are then the term structure of bond, futures and forward prices, respectively.

We recall the definitions of a general quadratic term structure (GQTS) and of factor impacts from Chapter 2.

Definition 5.1. The term structure H is said to be *generally quadratic* if it has the form[2]

$$\ln H(t,z,T) = A(t,T) + B^\top(t,T)z + z^\top C(t,T)z \tag{5.1}$$

where $\top$ stands for transpose, $C_{(m\times m)}$ is symmetric[3] and not necessarily different from $\mathbf{0}$, $B_{(m\times 1)}$ and $A_{(1\times 1)}$ are matrices of deterministic and smooth functions.

Definition 5.2. A factor Z_i is called *quadratic* and is said to have a *quadratic impact* on a GQTS

$$\ln H(t,z,T) = A(t,T) + B(t,T)z + z^\top C(t,T)z$$

if

$$\exists\; t,T \text{ s.t. } C_i(t,T) \neq \mathbf{0}. \tag{5.2}$$

A factor Z_j is called *linear* and is said to have at most a *linear impact* on a GQTS if it does not satisfy (5.2), i.e. if for all t,T $C_i(t,T) = \mathbf{0}$.

[2]Whenever we will refer to a property of term structures that does not depend on the particularities of bond, forward of futures prices we will not use any sub index. Instead we will denote the term structure by a generic function $H(t,Z(t),T)$.

[3]As noted in Chapter 2 requiring C to be symmetric is not restrictive, as any non-symmetric quadratic form can always be rewritten in an equivalent symmetric way with the advantage that the symmetric representation is unique. Since later on we will be interested in determining C the uniqueness property will be useful.

In terms of the specific notation of each price we have a GQTS if [4]:

- Bond prices

$$\ln H_p(t,z,T) = A_p(t,T) + B_p^\top(t,T)z + z^\top C_p(t,T)z \tag{5.3}$$

- Futures prices

$$\ln H_F(t,z,T) = A_F(t,T) + B_F^\top(t,T)z + z^\top C_F(t,T)z \tag{5.4}$$

- Forward prices

$$\ln H_f(t,z,T) = A_f(t,T) + B_f^\top(t,T)z + z^\top C_f(t,T)z \tag{5.5}$$

Finally, we note that considering the same state process Z for all prices represents no limitation to the analysis. Any irrelevant factor Z_i for, say, the futures prices will not show up in the term structure and we would have $B_{Fi}(t,T) = 0 \quad C_{Fi}(t,T) = \mathbf{0} \quad \forall\, t,T$. Here, and from now on, "$\cdot_i$" stands for the i-th component/row of a vector/matrix.

From the specification of any term contract, the market provides us also with an *a priori* given boundary function h that gives us the maturity values, possibly as a function of our factors Z. So we have

$$H(T,Z(T),T) = h(T,Z(T)).$$

The most well know boundary function is that of the bond prices, since at maturity we always have $p(T,T) = 1$. This means that we have $h_p(T,z) = 1$ for all T, thus $A_p(T,T) = 0$, $B_p(T,T) = \mathbf{0}$ and $C_p(T,T) = \mathbf{0}$.

In the case of futures and forward prices the boundary functions h_F and h_f will equal the value process of the underlying (to the futures or forward contract) at maturity[5]. As before this will give us some boundary conditions for $A_F(T,T)$, $B_F(T,T)$,$C_F(T,T)$ or $A_f(T,T)$, $B_f(T,T)$,$C_f(T,T)$, but also will allow us to identify the *natural factors*.

[4]Recall from Definition 2.2 that pure quadratic term structures (PQTS) and affine term structures (ATS) are special cases of GQTS.

[5]In particular, when we are dealing with futures and forward contracts on the same underlying we, obviously have, $h_F(T,z) = h_f(T,z)$.

Definition 5.3. A term structure is said to have a *generally quadratic boundary condition* (GQBC) if the boundary function h is of the form

$$\ln h(T,z) = a(T) + \mathbf{b}^\top(T)z + z^\top \mathbf{c}(T)z \tag{5.6}$$

where $\mathbf{c}(T)_{(m\times m)}$ is symmetric, and not necessarily different from $\mathbf{0}$, $\mathbf{b}(T)_{(m\times 1)}$ and $\mathbf{a}(T)_{1\times 1}$ are matrices of smooth and deterministic functions.[6] Moreover, for all i such that $\mathbf{b}_i(T) \neq 0$ or $\mathbf{c}_i(T) \neq \mathbf{0}$, Z_i is said to be a *natural factor.*

We note that, if the boundary function h is not of the form (5.6), then we cannot have a GQTS because the definition of a GQTS would fail at maturity.

Lemma 5.4. *If we denote by S the spot price of the underlying to a futures or forward contract, then $\ln S$ is a linear combination of linear factors.*

Proof. It follows from the definition of a *natural factor* and the fact that, at maturity, we must have both $h(T, Z(T)) = S(T)$, and $\ln h(T, Z(T)) = a(T) + \mathbf{b}^\top(T)Z(T) + Z^\top(T)\mathbf{c}(T)Z(T)$. ■

Before we go on with the analysis, let us set the scene.

Assumption 5.5. *We assume that zero-coupon bond prices are of the form*

$$p(t,T) = H_p(t, Z(t), T) \tag{5.7}$$

where H_p is a smooth function with the boundary condition

$$H_p(T, z, T) = h_p(T, z) = 1. \tag{5.8}$$

Furthermore, we assume that futures prices $F(t,T)$, can be written as

$$F(t,T) = H_F(t, Z(t), T) \tag{5.9}$$

where H_f is also a smooth function with the boundary condition

$$H_F(T, z, T) = h_F(T, z). \tag{5.10}$$

for an a priori given function h_F.

---

[6] Definition 5.3 can also be applied to bond prices. In this case the "underlying asset" is the non-risky asset that pays 1 at maturity and $a(T) = 1$, $\mathbf{b}(T) = \mathbf{0}$ and $\mathbf{c}(T) = \mathbf{0}$ for all T, satisfy (5.6).

Likewise, we assume that forward prices $f(t,T)$, can be written as

$$f(t,T) = H_f(t, Z(t), T) \tag{5.11}$$

where H_f is also a smooth function with the boundary condition

$$H_f(T, z, T) = h_f(T, z). \tag{5.12}$$

for an a priori given function h_f.

As in Chapter2 we will also consider that our m-dimensional factor model that, under the martingale measure $\mathbb{Q}$, general quadratic dynamics and is driven by an n-dimensional Wiener process W.

Assumption 5.6. *The dynamics of Z, under the $\mathbb{Q}$-measure are given by*

$$dZ(t) = \alpha(t, Z(t))dt + \sigma(t, Z(t))dW(t) \tag{5.13}$$

where $\alpha(t,z)$ is a $m \times 1$ vector and $\sigma(t,z)$ is a $m \times n$ matrix, and W is a n-dimensional Wiener process.

Definition 5.7. The vector of factors Z is said to have *general quadratic $\mathbb{Q}$-dynamics* if $\alpha(t,z)$ and $\sigma(t,z)$ in (5.13) are such that

$$\alpha(t,z) = \mathbf{d}(t) + \mathbf{E}(t)z \tag{5.14}$$

$$\sigma(t,z)\sigma^{\top}(t,z) = \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t) z_u + \sum_{u,k=1}^{m} z_u \mathbf{g_{uk}}(t) z_k \tag{5.15}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$, $\mathbf{k_u}$ and $\mathbf{g_{uk}}$ for $\mathbf{u}, \mathbf{k} = \mathbf{1}, \cdots, \mathbf{m}$ are matrices of deterministic smooth functions.

We also recall the notation

$$\mathbf{K}(t) = \begin{pmatrix} \mathbf{k_1}(t) \\ \mathbf{k_2}(t) \\ \vdots \\ \mathbf{k_m}(t) \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{g_{11}}(t) & \mathbf{g_{12}}(t) & \cdots & \mathbf{g_{1m}}(t) \\ \mathbf{g_{21}}(t) & \mathbf{g_{22}}(t) & \cdots & \mathbf{g_{2m}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g_{m1}}(t) & \mathbf{g_{m2}}(t) & \cdots & \mathbf{g_{mm}}(t) \end{pmatrix}. \tag{5.16}$$

Having defined the exact setup, we can go on with the analysis. Applying the Itô formula to equation (5.9) and (5.11) and using the dynamics for the factors in (5.13) we can find the dynamics of the futures prices and forward prices under the martingale measure $\mathbb{Q}$. Lemma 5.8 give these dynamics

Lemma 5.8. *Assume that the dynamics of Z are as in (5.13), then,*
if the futures prices are given by (5.9), their $\mathbb{Q}$-dynamics are (likewise) described by

$$dF(t,T) = \left\{ \frac{\partial H_F}{\partial t} + \sum_{i=1}^{m} \frac{\partial H_F}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H_F}{\partial z_i \partial z_j}\sigma_i\sigma_j^\top \right\} dt + \sum_{i=1}^{m} \frac{\partial H_F}{\partial z_i}\sigma_i dW(t)$$

if the forward prices are given by (5.11), their $\mathbb{Q}$-dynamics are (likewise) described by

$$df(t,T) = \left\{ \frac{\partial H_f}{\partial t} + \sum_{i=1}^{m} \frac{\partial H_f}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H_f}{\partial z_i \partial z_j}\sigma_i\sigma_j^\top \right\} dt + \sum_{i=1}^{m} \frac{\partial H_f}{\partial z_i}\sigma_i dW(t)$$

All partial derivatives should be evaluated at $(t, Z(t), T)$, and α_i and σ_i at $(t, Z(t))$.

In order to derive the term structure equation of futures and forward prices, we now recall a well-known fact (see for instance, Björk (2004a)).

Result 5.9. *For a T-claim $\mathcal{X}$, futures prices are martingales under the risk neutral measure $\mathbb{Q}$ and forward prices are martingales under the forward measures $\mathbb{Q}^T$*

$$F(t,T) = \mathbb{E}_t^{\mathbb{Q}}[\mathcal{X}] \qquad\qquad f(t,T) = \mathbb{E}_t^T[\mathcal{X}],$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ and $\mathbb{E}_t^T[\cdot]$ stand for expectations conditional on $\mathcal{F}_t$ under $\mathbb{Q}$ and $\mathbb{Q}^T$, respectively.

Looking first at futures prices, and using $F(t,T) = H_F(t, Z(t), T) = \mathbb{E}_t^{\mathbb{Q}}[h_F(T, Z(T))]$, we can then recover the term structure equation for futures prices in Björk and Landén (2001).

Result 5.10. **(Björk and Landén)** *Suppose the future prices are given by (5.9) and Assumption 5.6 holds. Then H_F satisfies the following differential equation*

$$\left\{ \begin{array}{rcl} \dfrac{\partial H_F}{\partial t} + \displaystyle\sum_{i=1}^{m} \frac{\partial H_F}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H_F}{\partial z_i \partial z_j}\sigma_i\sigma_j^\top & = & 0 \\ H_F(T,z,T) & = & h_F(T,z) \end{array} \right. . \tag{5.17}$$

All partial derivatives should be evaluated at (t, z, T), and α_i and σ_i at (t, z).

For forward prices, we proceed similarly, and use

$$f(t,T) = H_f(t, Z(t), T) = \mathbb{E}_t^T[h_f(T, Z(T))]$$

to get a differential equation for the forward prices H_f. Proposition 5.11 gives us that result.

Proposition 5.11. *Suppose the zero-coupon bond prices are given by (5.7), the forward prices are given by (5.11) and Assumption 5.6 holds. Then H_f satisfies the following differential equation*

$$\begin{cases} \dfrac{\partial H_f}{\partial t} + \displaystyle\sum_{i=1}^{m} \frac{\partial H_f}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H_f}{\partial z_i \partial z_j}\sigma_i\sigma_j^\top + \sum_{i,j=1}^{m} \frac{\partial H_f}{\partial z_i}\sigma_i \left(\frac{\partial H_p}{\partial z_j}\frac{1}{H_p}\sigma_j^\top\right) & = \quad 0 \\ H_f(T,z,T) & = \quad h_f(T,z) \end{cases} \tag{5.18}$$

All partial derivatives should be evaluated at (t,z,T), and α_i and σ_i at (t,z).

Proof. It follows (applying the Itô formula) from (5.7) and (5.13) that the bond prices volatility, σ_p, is given by $\sigma_p^\top = \sum_{j=1}^{m} \frac{\partial H_p}{\partial z_j}\frac{1}{H_p}\sigma_j^\top$.
Recall that the numeraires[7] under $\mathbb{Q}$ and $\mathbb{Q}^T$ are, respectively, the money account B and the bond price with maturity T, $p(\cdot,T)$. The likelihood process $L^T(t) = \dfrac{d\mathbb{Q}^T}{d\mathbb{Q}}$ for $0 \le t \le T$ is, thus, given by $L^T(t) = \dfrac{p(t,T)}{B(t)p(0,T)}$, and its $\mathbb{Q}$-dynamics by $dL^T(t) = L^T(t)\{\sigma_p(t,T)\}\,dW_t$. That is, the Girsanov Kernel for the transition from $\mathbb{Q}$ to $\mathbb{Q}^T$ is given by the bond price volatility and we can write $dW = \sum_{j=1}^{m} \frac{\partial H_p}{\partial z_j}\frac{1}{H_p}\sigma_j^\top dt + dW^T$. Using this, the dynamics under the $\mathbb{Q}^T$ forward measure of $f(t,T)$ become

$$\begin{aligned} df(t,T) = &\left\{ \frac{\partial H_f}{\partial t} + \sum_{i=1}^{m} \frac{\partial H_f}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H_f}{\partial z_i \partial z_j}\sigma_i\sigma_j^\top + \sum_{i,j=1}^{m} \frac{\partial H_f}{\partial z_i}\sigma_i \left(\frac{\partial H_p}{\partial z_j}\frac{1}{H_p}\sigma_j^\top\right)\right\} dt \\ &+ \sum_{i=1}^{m} \frac{\partial H_f}{\partial z_i}\sigma_i dW^T(t), \end{aligned}$$

and the result follows from the fact that the forward prices with maturity T are martingales under the $\mathbb{Q}^T$ forward measure, so its drift term (which shows up in the l.h.s. of equation (5.18)) must be zero. ∎

Comparing equations (5.17) and (5.18) one soon realizes that *forward prices* have a more complex term structure equation. Indeed, the term structure equation for forward prices does not depend only on the properties of its own term structure H_f and of the factors dynamics (trough its drift α and volatility σ), but also on properties of the *bond prices* term structures. This dependence shows up in the last term of (5.18): $\displaystyle\sum_{i,j=1}^{m} \frac{\partial H_f}{\partial z_i}\sigma_i \left(\frac{\partial H_p}{\partial z_j}\frac{1}{H_p}\sigma_j^\top\right)$.

[7]For further details on change of numeraire see Geman, El Karoui, and Rochet (1995).

This is however not surprising, in fact it goes in line with our intuition: since bond prices are the *numeraire* under forward measures and forward prices are martingales under those measures, the term structure of bond prices should, of course, play some role in the forward prices term structure. Indeed, any study of forward prices term structures need to be done under some interest rate setting and the difference between futures and forward prices results from the role that bond prices have on the forward prices term structure.

The main goal of this chapter is to study GQTS of forward prices in an *as general as possible* interest rate setting. For reasons that will become clear later on, that setting will be what has been called, in Chapter 2, a *general quadratic short rate* (GQSR) setting. We repeat here the basic definitions and a few results to avoid references across chapters and notation misunderstandings.

Definition 5.12. A *general quadratic short rate* setting is defined by a short rate, r, with the following functional quadratic form

$$r(t, Z(t)) = Z^{\top}(t)Q(t)Z(t) + g^{\top}(t)Z(t) + f(t) \tag{5.19}$$

where $Q(t)_{(m\times m)}$, symmetric, and not necessarily different from $\mathbf{0}$, $g(t)_{(m\times 1)}$ and $f(t)_{1\times 1}$ are matrices of smooth and deterministic functions.

In a setting with GQSR and Z with general quadratic $\mathbb{Q}$-dynamics we can classify the factors as follows.

Definition 5.13. (Classification for Bond prices) Given a GQSR as in (5.19) and the general quadratic $\mathbb{Q}$-dynamics for Z (so that (5.14) and (5.15) hold for $\alpha(t, z)$ and $\sigma(t, z)$ in (5.13)).

- Z_i is a $Z^{(q)}$*-factor* if it satisfies *at least one* of the following requirements:
  - (i) it has a quadratic impact on the short rate of interest $r(t)$, i.e., there exists t such that $Q_i(t) \neq \mathbf{0}$;
  - (ii) it has a quadratic impact on the functional form of the matrix $\sigma(t, z)\sigma^{\top}(t, z)$, i.e., there exist k and t such that $\mathbf{g_{ik}}(t) \neq \mathbf{0}$;
  - (iii) it affects the drift term of the factors satisfying (i) or (ii), i.e., for Z_j satisfying (i) or (ii) we have $\mathbf{E}_{ji}(t) \neq 0$, at least for some t.
- Z_i is a $Z^{(l)}$*-factor* if it does not satisfy (i)-(iii).

Theorem 5.14. *Suppose that Assumption 5.5 and 5.6 are in force. Furthermore suppose that we are in a GQSR, so that (5.19) hold and that Z has a general quadratic $\mathbb{Q}$-dynamics, (i.e., that α and σ from the factor dynamics (5.13), satisfy (5.14)-(5.15)). Finally assume that the factors are reordered as $Z = \begin{pmatrix} Z^{(q)} \\ Z^{(l)} \end{pmatrix}$ (according to Definition 5.13), and that the following restrictions apply to $\mathbf{K}$ and $\mathbf{G}$ in (5.16):*

$$\mathbf{k}_{\mathbf{u}}(t) = \begin{pmatrix} 0 & 0 \\ & \\ 0 & \mathbf{k}_{\mathbf{u}}^{(\mathbf{ll})}(t) \end{pmatrix} \quad \forall\, t, \forall\, \mathbf{u} \tag{5.20}$$

$$\mathbf{g}_{\mathbf{uk}}(t) = \begin{pmatrix} 0 & 0 \\ & \\ 0 & \mathbf{g}_{\mathbf{uk}}^{(\mathbf{ll})}(t) \end{pmatrix} \quad \forall\, t \text{ and } \forall\, \mathbf{u}, \mathbf{k} \text{ s.t. } z_u, z_k \in \mathbf{z}^{(\mathbf{q})}. \tag{5.21}$$

Then the term structure of bond prices is generally quadratic, i.e. H_p from (5.7) can be written in the form (5.3) where A_p, B_p and C_p solve the following system of ordinary differential equations.

$$\begin{cases} \dfrac{\partial A_p}{\partial t} + \mathbf{d}^\top(t)B_p + \dfrac{1}{2}B_p^\top \mathbf{k}_0(t)B_p + \operatorname{tr}\{C_p\mathbf{k}_0(t)\} = f(t) \\ A_p(T,T) = 0 \end{cases} \tag{5.22}$$

$$\begin{cases} \dfrac{\partial B_p}{\partial t} + \mathbf{E}^\top(t)B_p + 2C_p\mathbf{d}(t) + \dfrac{1}{2}\bar{B}_p^{\ \top}\mathbf{K}(t)B_p + 2C_p\mathbf{k}_0(t)B_p = g(t) \\ B_p(T,T) = 0 \end{cases} \tag{5.23}$$

$$\begin{cases} \dfrac{\partial C_p}{\partial t} + C_p\mathbf{E}(t) + \mathbf{E}^\top(t)C_p + 2C_p\mathbf{k}_0(t)C_p + \dfrac{1}{2}\bar{B}_p^{\ \top}\mathbf{G}(t)\bar{B}_p = Q(t) \\ C_p(T,T) = 0 \end{cases} \tag{5.24}$$

where C_p has the special form $C_p = \begin{pmatrix} C_p^{(qq)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ and A_p, B_p and $C_p^{(qq)}$ should be evaluated at (t,T). $\mathbf{E}$, $\mathbf{d}$, $\mathbf{k}_0$, $\mathbf{K}$, $\mathbf{G}$ are the same as in (5.14)-(5.16), and

$$\bar{B}_p = \begin{pmatrix} B_p & 0 & \cdots & 0 \\ 0 & B_p & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B_p \end{pmatrix}. \tag{5.25}$$

Moreover, if there exists a $Z_i \in Z^{(q)}$, we have a QTS for bond prices (i.e., $C_p(t,T) \neq \mathbf{0}$ for some t and t), and if all $Z_i \in Z^{(l)}$ we have an ATS of bond prices (i.e.$C_p(t,T) = \mathbf{0}$ for all t and T).

Since the intuition underlying the results on *futures prices* is easier, we start by briefly analyzing futures prices.

5.3 GQTS for Futures Price Models

Recall from Definition 5.1 that a GQTS for futures prices H_F has the form

$$\ln H_F(t, z, T) = A_F(t, T) + B_F^\top(t, T)z + z^\top C_F(t, T)z \tag{5.26}$$

where $C_{F(m\times m)}$ symmetric and not necessarily different from $\mathbf{0}$, $B_{F(m\times 1)}$ and $A_{F(1\times 1)}$ are matrices of deterministic and smooth functions.

And that a *necessary condition* for a GQTS of futures prices is that GQBC holds

$$\ln h_F(T, z) = a_F(T) + \mathbf{b}_F^\top(T)z + z^\top \mathbf{c}_F(T)z. \tag{5.27}$$

Now, we give the relevant classification of factors for futures prices. As in Chapter 2, this will have to do with the different impacts factors may have on the term structure. After presenting the main result on GQTS for futures prices, we will be able to give that motivation and to show that, indeed, this classification has to do with the impact that the factors may end up having.

5.3.1 Futures prices' classification of factors

Definition 5.15. (Classification for Futures Prices) Given a general quadratic $\mathbb{Q}$-dynamics for Z (so that (5.14) and (5.15) hold for the α and σ in (5.13)), and a GQBC as in (5.27), we say

- Z_i is a $\bar{Z}^{(q)}$*-factor* if it satisfies *at least one* of the following requirements:
  (i) it has a quadratic impact on the boundary condition h_F, i.e., there exists T such that $\mathbf{c}_F(T)_i \neq \mathbf{0}$;
  (ii) it has a quadratic impact on the functional form of the matrix $\sigma(t, z)\sigma(t, z)^\top$, i.e., there exist $\mathbf{k}$ and t such that $\mathbf{g_{ik}}(t) \neq \mathbf{0}$;
  (iii) it affects the drift terms of the factors satisfying in (i) or (ii), i.e., for Z_j satisfying (i) or (ii) we have $\mathbf{E}(t)_{ji} \neq \mathbf{0}$ at least for some t.
- Z_i is a $\bar{Z}^{(l)}$*-factor* if it does not satisfy (i)-(iii).

We note that from the classification of factors it results, by definition, special forms for the matrices involved in this classification.

Remark 5.16. *We note that given Classification 5.15,*

- *it is always possible to reorder the vector of factors Z and its correspondent value vector, so that we have*

$$Z = \begin{pmatrix} \bar{Z}^{(q)} \\ \bar{Z}^{(l)} \end{pmatrix} \qquad z = \begin{pmatrix} \bar{z}^{(q)} \\ \bar{z}^{(l)} \end{pmatrix}.$$

- *with this reordering of factors we have,* by definition, *the following forms for $\mathbf{c}_F$ in (5.19) and for $\mathbf{E}$ and $\mathbf{G}$ in (5.14) and (5.16), respectively*

$$\mathbf{c}_F(t) = \begin{pmatrix} \mathbf{c}^{(\mathbf{qq})}{}_F(t) & 0 \\ & \\ 0 & 0 \end{pmatrix} \ \mathbf{E}(t) = \begin{pmatrix} \mathbf{E}^{(\mathbf{qq})}(t) & 0 \\ & \\ \mathbf{E}^{(\mathbf{lq})}(t) & \mathbf{E}^{(\mathbf{ll})}(t) \end{pmatrix} \ \mathbf{G}(t) = \begin{pmatrix} \mathbf{G}^{(\mathbf{qq})}(t) & 0 \\ & \\ 0 & 0 \end{pmatrix}. \tag{5.28}$$

5.3.2 Main result on Futures Prices

Theorem 5.17. *Suppose that Assumption 5.5 and 5.6 are in force. Furthermore suppose that Z follows a general quadratic $\mathbb{Q}$-dynamics, (i.e., that α and σ from the factor dynamics (5.13), satisfy (5.14)-(5.15)) and that we have a GQBC, so that the boundary condition h_F from (5.10), has the quadratic form in (5.27).*

Finally assume that the factors are reordered as $Z = \begin{pmatrix} \bar{Z}^{(q)} \\ \bar{Z}^{(l)} \end{pmatrix}$ (using Classification 5.15), and that the following restrictions apply to k_u and g_{uk} in (5.15):

$$\mathbf{k}_\mathbf{u}(t) \ = \ \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{k}_\mathbf{u}^{(\mathbf{ll})}(t) \end{pmatrix} \quad \forall\ \mathbf{u}, \forall\ t \tag{5.29}$$

$$\mathbf{g}_{\mathbf{uk}}(t) \ = \ \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{g}_{\mathbf{uk}}^{(\mathbf{ll})}(t) \end{pmatrix} \quad \forall\ \mathbf{u}, \mathbf{k} : (z_u \wedge z_k) \in \bar{\mathbf{z}}^{(\mathbf{q})}, \forall\ t \tag{5.30}$$

Then the term structure of futures prices is generally quadratic, i.e. H_F from (5.9) can be written on the form (5.26)and A_F, B_F and C_F can be obtained by solving the following system of ordinary differential equations.

$$\begin{cases} \dfrac{\partial A_F}{\partial t} + \mathbf{d}^\top(t) B_F + \dfrac{1}{2} B_F^\top \mathbf{k}_0(t) B_F + \operatorname{tr}\{C_F \mathbf{k}_0(t)\} & = \ 0 \\ A_F(T,T) & = \ a_F(T) \end{cases} \tag{5.31}$$

$$\begin{cases} \dfrac{\partial B_F}{\partial t} + \mathbf{E}^\top(t) B_F + 2C_F\mathbf{d}(t) + \dfrac{1}{2}\bar{B_F}^\top \mathbf{K}(t) B_F + 2C_F\mathbf{k}_0(t) B_F & = \mathbf{0} \\ B_F(T,T) & = \mathbf{b}_F(T) \end{cases} \tag{5.32}$$

$$\begin{cases} \dfrac{\partial C_F}{\partial t} + C_F\mathbf{E}(t) + \mathbf{E}^\top(t) C_F + 2C_F\mathbf{k}_0(t) C_F + \bar{B_F}^\top \mathbf{G}(t)\bar{B_F} & = \mathbf{0} \\ C_F(T,T) & = \mathbf{c}_F(T) \end{cases} \tag{5.33}$$

where C_F has the special form $C_F = \begin{pmatrix} C_F^{(qq)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ and A_F, B_F, $C_F^{(qq)}$ should be evaluated at (t,T). $\mathbf{E}$ and $\mathbf{d}$ are the same as in (5.14), $\mathbf{k}_0$, $\mathbf{K}$ and $\mathbf{G}$ the same as in (5.15) (with the same notation as in (5.16)), and where $\bar{B_F}$ follows the same idea as the definition in (5.25) and have dimension $(m^2 \times m)$.

Proof. First of all, the restrictions imposed on the matrices $\mathbf{c}_F$, $\mathbf{E}$ and $\mathbf{G}$ in (5.28) guarantee that $C_F^{(ll)}(t,T) = \mathbf{0}$ and $C_F^{(ql)}(t,T) = C_F^{(lq)\top}(t,T) = \mathbf{0}$ for all t,T always solves (5.33). This proves the last statement of the Theorem.

The main result follows from the fact that $\ln H_F(t,z,T) = A_F(t,T) + B_F^\top(t,T)z + z^\top C_F(t,T)z$ where A_F, B_F and C_F satisfy (5.31)-(5.33), solves the PDE (5.17) which uniquely characterizes the futures prices in this setting.

(i) We start by showing that the term structure equation (5.17), is equivalent to the PDE

$$\begin{cases} \dfrac{\partial A_F}{\partial t} + \dfrac{\partial (B_F)}{\partial t} z + z^\top \dfrac{\partial C_F}{\partial t} z + \displaystyle\sum_{i=1}^m \left[(B_F)_i + (C_F)_i z + (C_F^\top)_i z\right] \alpha_i & \\ + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^m \left[(B_F)_i + (C_F)_i z + (C_F^\top)_i z\right] \sigma_i\sigma_j^\top \left[(B_F)_j + (C_F)_j z + (C_F^\top)_j z\right] & \\ \qquad + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^m \left((C_F)_{ij} + (C_F)_{ji}\right)\sigma_i\sigma_j^\top & = 0 \\ A_F(T,T) + B_F(T,T)^\top z + z^\top C_F(t,T) z & = h_F(t,z) \end{cases} \tag{5.34}$$

for $\ln H_F(t,z,T) = A_F(t,T) + B_F^\top(t,T)z + z^\top C_F(t,T)z$. This follows from the partial derivatives,

$$\begin{aligned} \frac{\partial H_F}{\partial t} &= \left[\frac{\partial A_F}{\partial t} + \frac{\partial B_F}{\partial t}^\top z + z^\top \frac{\partial C_F}{\partial t} z\right] H_F \\ \frac{\partial H_F}{\partial z_i} &= \left[B_{Fi} + C_{Fi} z + C_{F\,i}^\top z\right] H_F \\ \frac{\partial^2 H_F}{\partial z_i \partial z_j} &= \left[C_{Fij} + C_{Fji}\right] H_F + \left[B_{Fi} + C_{Fi} z + C_{F\,i}^\top z\right]\left[B_{Fj} + C_{Fj} z + C_{F\,j}^\top z\right] H_F \end{aligned}$$

(ii) It remains to show that the system of ODE (5.31)-(5.33) solves the PDE (5.34). To see this, we substitute α, $\sigma\sigma^\top$ and h_F from (5.14), (5.15) and (5.27) respectively, and apply the restrictions (5.29)-(5.30). The PDE (5.34) becomes, then, a separable equation. In addition, by making use of the fact that $C_F^{(ll)}(t,T) = \mathbf{0}$ and $C_F^{(ql)}(t,T) = C_F^{(lq)\top}(t,T) = \mathbf{0}$ for all t, T are always a solution to (5.33), we can guarantee that all terms of order higher than two disappear. Hence, equation (5.34) becomes separable up to quadratic terms of z. Identification of terms, and the use of the assumption on the symmetry of C_F (which helps to simplify the expressions), leads to the system of ODEs (5.31)-(5.33). ■

The system of ODEs for futures prices, (5.31)-(5.33), is of the same level of difficulty as the system (5.22)-(5.24) for bond prices. And here, as there, it is possible to split the interrelated Riccati equations (5.32)-(5.33) into simpler ODEs for $B_F^{(q)}$, $B_F^{(l)}$ and $C_F^{(qq)}$ that can be solved in iterative order.

5.3.3 Understanding the $\bar{Z}^{(q)}$ and $\bar{Z}^{(l)}$ factors

In terms of the *impact*[8] of various factors on the futures prices term structure, it follows from $C_F^{(ll)}(t,T) = \mathbf{0}$ and $C_F^{(ql)}(t,T) = C_F^{(lq)\top}(t,T) = \mathbf{0}$ for all t, T that the $\bar{Z}^{(l)}$ factors have a linear impact.

Corollary 5.18. **(Linear factors)** *The $\bar{Z}^{(l)}$-factors are linear factors in a GQTS of futures prices.*

Moreover, it follows from this corollary that the matrices $\mathbf{c}_F$ and $\mathbf{G}$ play quite and important role in determining the type of terms structure. If $\mathbf{c}_F(T) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t, T, there will be no $\bar{Z}^{(q)}$-factors and the terms structure will be affine. In other words, *necessary conditions* for existence of a quadratic term structure are that either $\mathbf{c}_F(T) \neq \mathbf{0}$ at least for some T or $\mathbf{G}(t) \neq \mathbf{0}$ at least for some t.

Besides this, *some* $\bar{Z}^{(q)}$-factors have obviously a quadratic impact.

Lemma 5.19. *A factor $Z_i \in \bar{Z}^{(q)}$ for which $\mathbf{c}_{Fi}(T) \neq \mathbf{0}$ at least for some T, has a quadratic impact in the futures prices term structure.*

Proof. It follows from $C_{Fi}(T,T) = \mathbf{c}_{Fi}(T)$ and the Definition 5.2. ■

[8]Recall Definition 5.2

The remaining $\bar{Z}^{(q)}$-factors will also have a quadratic impact as long as we exclude redundant factors from the analysis and consider that some, not at all restrictive, regularity condition is satisfied.

Assumption 5.20. *For any i, k, t, T such that $E_{ji}(t)C_{Fjk}(t,T) \neq 0$ for some j* or $B_{Fu}(t,T)\mathbf{g}_{\mathbf{ik}_{uu}}(t)B_{Fu}(t,T) \neq 0$ *for some u the following* regularity condition *hold*

$$\sum_j E_{ji}(t)C_{Fjk}(t,T) + \sum_{u,v} B_{Fu}(t,T)\mathbf{g}_{\mathbf{ik}_{uv}}(t)B_{Fv}(t,T) \neq 0.$$

Proposition 5.21. **(Quadratic factors)** *As long as we exclude from the analysis any redundant factors and under the regularity condition of Assumption 5.20, the $\bar{Z}^{(q)}$-factors are quadratic factors in a GQTS of futures prices.*

Proof. Lemma 5.19 proves that any $Z_i \in \bar{Z}^{(q)}$ such that $\mathbf{c}_{Fi}(T) \neq \mathbf{0}$ at least for some T has a quadratic impact. It remains to show that for all $Z_i \in \bar{Z}^{(q)}$ for which $\mathbf{c}_{Fi}(T) = \mathbf{0}$ for all T (all remaining $\bar{Z}^{(q)}$-factors), we also have, at least for some t, T, $C_{Fi}(t,T) \neq \mathbf{0}$ as a solution to (5.33). It follows from Classification 5.15 that, as long as we do not consider the redundant factors, for such Z_i there will always be a k, t, T such that $E_{ji}(t)C_{Fjk}(t,T) \neq 0$ and/or $B_{Fu}(t,T)\mathbf{g}_{\mathbf{ik}_{uu}}(t)B_{Fu}(t,T) \neq 0$ for some u. The regularity condition of Assumption 5.20 then guarantees that for that k, t, T, $C_{Fik}(t,T)$ solves a Riccati equation with a non-zero independent term and that, thus $C_{Fik}(t,T) \neq 0$. This guarantees the quadratic impact of Z_i according to Definition 5.2. ∎

Many of the results proved for futures prices will be extremely useful in providing the right intuition for the more complex situation we face when dealing with *forward prices*. Moreover, since forward prices are martingales under the forward measures (and bond prices are numeraires under those measures), the term structure of bond prices will play a special role. As we will show, it is the role of bond prices that makes the term structure of futures and forward prices on a same underlying differ.

5.4 GQTS for Forward price models

In this section we will be looking at GQTS of forward prices,

$$\ln H_f(t,z,T) = A_f(t,T) + B_f^\top(t,T)z + z^\top C_f(t,T)z \tag{5.35}$$

where $C_{f(m\times m)}$ symmetric and not necessarily different from $\mathbf{0}$, $B_{f(m\times 1)}$ and $A_{f(1\times 1)}$ are matrices of deterministic and smooth functions.

According to Definition 5.3, we have a GQBC for the term structure of forward prices if

$$\ln h_f(T,z) = a_f(T) + \mathbf{b}_f(T)z + z^\top \mathbf{c}_f z. \tag{5.36}$$

Already taking into account the specific form of the forward prices term structures in (5.35), the term structure equation for forward prices previously derived (recall equation (5.18)), can be rewritten in terms of A_f, B_f, C_f, the elements characterizing the factor dynamics (α and σ), and the properties of the term structure of bond prices H_p in (5.7).

Lemma 5.22. *Suppose that the Z dynamics, as before, are given by (5.13). Suppose, furthermore, that the term structure of forward prices is generally quadratic so that equation (5.35) holds. Then the differential equation (5.18) can be written in the following terms*

$$\begin{cases} \begin{aligned} \frac{\partial A_f}{\partial t} + \frac{\partial B_f}{\partial t}z + z^\top \frac{\partial C_f}{\partial t}z + \sum_{i=1}^m \left[B_{f_i} + 2C_{f_i}z\right]\alpha_i + \frac{1}{2}\sum_{i,j=1}^m 2C_{f_{ij}}\sigma_i\sigma_j^\top & \\ +\frac{1}{2}\sum_{i,j=1}^m \left[B_{f_i} + 2C_{f_i}z\right]\sigma_i\sigma_j^\top \left[B_{f_j} + 2C_{f_j}z\right] & \\ + \sum_{i,j=1}^m \left[B_{f_i} + 2C_{f_i}z\right]\sigma_i\sigma_j^\top \frac{\partial H_p}{\partial z_j}\frac{1}{H_p} &= 0 \\ H_f(T,z,T) &= h_f(T,z) \end{aligned} \end{cases} . \tag{5.37}$$

All partial derivatives should be evaluated at (t,z,T), and α_i and σ_i at (t,z_i).

Proof. If we have $H_f(t,z,T) = \exp\left\{A_f(t,T) + B_f(t,T)^\top z + z^\top C_f(t,T)z\right\}$, for symmetric C_f then we have $\frac{\partial H_f}{\partial t} = \left[\frac{\partial A_f}{\partial t} + \frac{\partial B_f}{\partial dt}^\top z + z^\top \frac{\partial C_f}{\partial t}z\right]H_f$, $\frac{\partial H_f}{\partial z_i} = \left[B_{f_i} + 2C_{f_i}z\right]H_f$, $\frac{\partial^2 H_f}{\partial z_i \partial z_j} = \left[2C_{f_{ij}}\right]H_f + \left[B_{f_i} + 2C_{f_i}z\right]\left[B_{f_j} + 2C_{f_j}z\right]H_f$.

Substituting this partial derivatives into (5.18) and canceling the H_f present in all terms of the l.h.s. give us the result. ∎

In equation (5.37) we see clearly how the term structure of forward prices is linked to the term structure H_p of bond prices, through the terms

$$\left[B_{f_i} + 2C_{f_i}z\right]\sigma_i\sigma_j^\top \frac{\partial H_p}{\partial z_j}\frac{1}{H_p} \qquad \forall i,j. \tag{5.38}$$

Thus, any model for forward prices must necessarily also include a model for interest rates.[9] In this study we are interested in finding a *general* interest rate setting to study GQTS of forward prices. The exact reason why a *general quadratic short rate* (GQSR) turns out to be the interesting setting, can now be fundamented.

Given the form of equation (5.37), it is natural to think in terms of separation of variables, i.e., to think of conditions that would guarantee that each of the terms in (5.37) are polynomials of z.

This will lead us to the usual conditions on α, $\sigma\sigma^\top$ and the boundary condition h_f but this time also to conditions on the functional form of H_p, indirectly via the terms (5.38). Concretely we need the terms (5.38) to be also polynomials of z and we see that this can happen only if *either* they are all (i.e. for all i, j) zero and we get a trivial polynomial, *or* if $\frac{\partial H_p}{\partial z_j}\frac{1}{H_p}$ is itself a polynomial of z and the term structure of bond prices is an exponential of polynomials of z.

We look at each of these hypotheses.

A *sufficient condition* for all terms (5.38) to be zero is that the term structure of bond prices is deterministic. Then, for all j such that $\frac{\partial H_p}{\partial z_j}\frac{1}{H_p} \neq 0$, (i.e. for all Z_j in the bond prices term structure), we have $\sigma_j^\top(t) = \mathbf{0}$. This gives us the classical result that in a deterministic interest rate setting the measures $\mathbb{Q}$ and $\mathbb{Q}^T$ are the same and, thus, futures and forward prices are also the same. For the purposes of this section this is however an uninteresting case[10] and we exclude it by introducing the next assumption.

Assumption 5.23. *The term structure of bond prices is* stochastic.

Another trivial circumstance, when we would also have all terms (5.38) equal to zero, is when all factors related to the bond prices term structure are not *connected* in any sense to the factors related to underlying of the forward contract. That is, when there is full separability between the two sets of factors.[11] In full generality this is, of course, hard to formalize. Nonetheless, making use of the fact that bond prices and futures prices term structures can be independently determined we can also exclude this case from the analysis (Assumption 5.25).

[9]For instance, in Björk and Landén (2001), the study of ATS of forward prices is done in an affine interest rate setting.

[10]In deterministic interest rate settings the futures' results of the previous section apply trivially to forward prices.

[11]Also in this case there would be equivalence between futures and forward prices and the results of the previous sections would apply.

Definition 5.24. Given a forward contract on some underlying $\mathcal{X}$ and an interest rate setting. We say that

- Z_i is a $Z^{(u)}$*-factor* if it shows up in the futures price term structure.[12]
- Z_i is a $Z^{(p)}$*-factor* if it shows up in the bond price term structure.

Assumption 5.25. *We assume that at least one of the following conditions hold:*

- $Z^{(u)} \cap Z^{(p)} \neq \emptyset$.
- $\sigma_i(t)\sigma_j^\top(t) \neq 0$ *form some* $Z_i \in Z^{(u)}$ *and* $Z_j \in Z^{(p)}$ *and for some* t.

Given Assumptions 5.23 and 5.25, we exclude the possibility that all terms (5.38) are zero, and hence the condition for separability of the forward prices term structure equation (5.37) is now that

$$\frac{\partial H_p}{\partial z_j}\frac{1}{H_p} \qquad \forall j$$

have to be polynomial in z, and thus the term structure of bond prices to be exponential of a polynomials of z.

This is very good news, not only the GQTS for bond prices studied in Chapter 2 are a natural candidate, but given that exponential term structures of order higher than two suffer from consistency and computational problems, they are (except of pathological cases) the most general setting that can be considered. In order to exclude from the analysis the (pathological) cases where higher-order term structures of bond prices may exist, we state the needed short rate setting in the form of an assumption.

Assumption 5.26. *We are in a* GQSR *setting (recall Definition 5.12), that is, the short rate of interest is of the following form*

$$r(t, Z(t)) = Z(t)^\top Q(t) Z(t) + g(t)^\top Z(t) + f(t). \tag{5.39}$$

Moreover, since for a GQTS of forward prices, we need to require general quadratic $\mathbb{Q}$-dynamics anyway (from the standard requirements on α and $\sigma\sigma^\top$ for separability of (5.37)). The only additional condition[13] that has to be included to guarantee as well a GQTS for bond prices is on the functional form of the short-rate r.

As we will see, general quadratic short rates mix very well with general quadratic forward prices, and GQTS of bond prices and forward prices may be coupled in great

[12]Considering the same underlying $\mathcal{X}$ and hence that $h_F(t,z) = H_f(T,z)$ for all T, z.

[13]Besides a careful classification of factors, see Definition 5.27.

variety. In particular we will see below that, perhaps surprisingly, a GQTS of bond prices may be coupled with an ATS of forward prices.

5.4.1 Forward prices' classification of factors

We can now give the adequate classification of factors needed for dealing with GQTS of *forward prices*. Once again, this classification can be justified using the intuition from both bond prices and futures prices term structures. One can guess the classification will have to do with the impact that the various factors may have on the term structure of forward prices, and this time also on their impact on bond prices term structures (since whenever bond prices affect forward prices they need then to have also a GQTS).

Definition 5.27. (**Classification for Forward Prices**) Given a general quadratic $\mathbb{Q}$-dynamics for Z (so that (5.14) and (5.15) hold for the α and σ in (5.13)), a GQBC as in (5.36), and a GQSR[14] as in (5.39).

- Z_i is a $\tilde{Z}^{(q)}$*-factor* if it satisfies *at least one* of the following requirements:
  - (i) it has a quadratic impact on the boundary condition h_f, i.e., there exists T such that $\mathbf{c_f}_i(T) \neq \mathbf{0}$;
  - (ii) it has a quadratic impact on the short rate of interest $r(t)$, i.e., there exists t such that $Q_i(t) \neq \mathbf{0}$;
  - (iii) it has a quadratic impact on the functional form of the matrix $\sigma(t,z)\sigma^\top(t,z)$, i.e., there exists k and t such that $\mathbf{g_{ik}}(t) \neq \mathbf{0}$;
  - (iv) it affects the drift term of factors satisfying (i), (ii) or (iii) i.e., for Z_j satisfying (i), (ii) or (iii) we have $\mathbf{E}_{ji}(t) \neq 0$, at least for some t.
- Z_i is a $\tilde{Z}^{(l)}$*-factor* if it *does not* satisfy (i)-(iv).

Note that considering a futures contract on a same underlying as our forward contract, (i.e. $h_F = h_f$) and the classifications of factors, in Definitions 5.13 and 5.15, the following hold

$$\tilde{Z}^{(q)} = Z^{(q)} \cup \bar{Z}^{(q)} \qquad \text{and} \qquad \tilde{Z}^{(l)} = Z^{(l)} \cap \bar{Z}^{(l)}.$$

The classification of the factors in Definition 5.27 have, thus, some implications for many matrices in our standard setup.

[14] If Assumptions 5.23 and 5.25 do not hold, we do not have to have a GQSR, and this classification of factors reduces to that of Definition 5.15.

Remark 5.28. *We note that given Definition 5.27:*

- *it is always possible to reorder the vector of factors Z, so that we have* $Z = \begin{pmatrix} \tilde{Z}^{(q)} \\ \tilde{Z}^{(l)} \end{pmatrix}$
- *with this reordering of factors we have,*by definition, *the following forms for* $\mathbf{E}$ *and* $\mathbf{G}$ *in (5.14) and (5.16),*

$$\mathbf{E}(t) = \begin{pmatrix} \mathbf{E}^{(\mathrm{qq})}(t) & 0 \\ \mathbf{E}^{(\mathrm{lq})}(t) & \mathbf{E}^{(\mathrm{ll})}(t) \end{pmatrix} \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{G}^{(\mathrm{qq})}(t) & 0 \\ 0 & 0 \end{pmatrix} \tag{5.40}$$

and for $\mathbf{c}_f$ *and* $\mathbf{Q}$ *in (5.36) and (5.39)*

$$\mathbf{c}_f(T) = \begin{pmatrix} \mathbf{c}^{(\mathrm{qq})}{}_f(T) & 0 \\ 0 & 0 \end{pmatrix} \quad Q(t) = \begin{pmatrix} Q^{(qq)}(t) & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.41}$$

5.4.2 Main result on Forward prices

Theorem 5.29. *Suppose that Assumptions 5.5 and 5.6 are in force. Furthermore suppose that Z has general quadratic $\mathbb{Q}$-dynamics, (i.e., that α and σ from the factor dynamics (5.13), satisfy (5.14)-(5.15)), that we have a GQBC, so that the boundary condition h_f from (5.12), has the quadratic form in (5.36) and that we are in a GQSR setting (so that (5.39) hold).*

Finally assume that the factors are reordered as $Z = \begin{pmatrix} \tilde{Z}^{(q)} \\ \tilde{Z}^{(l)} \end{pmatrix}$ *(according to Definition 5.27), and that the following restrictions apply to $\mathbf{k}_{\mathrm{u}}$ and $\mathbf{g}_{\mathrm{uk}}$ in (5.15):*

$$\mathbf{k}_{\mathrm{u}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{k}_{\mathrm{u}}^{(\mathrm{ll})}(t) \end{pmatrix} \qquad \forall\ \mathbf{u} \textit{ and } \forall\ t \tag{5.42}$$

$$\mathbf{g}_{\mathrm{uk}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{g}_{\mathrm{uk}}^{(\mathrm{ll})}(t) \end{pmatrix} \qquad \forall\ t \textit{ and } \forall\ \mathbf{u},\mathbf{k} \textit{ s.t. } z_u, z_k \in \tilde{\mathbf{z}}^{(\mathrm{q})}. \tag{5.43}$$

Then the term structure of forward prices is generally quadratic, i.e. H_f from (5.11) can be written on the form (5.35) and A_f, B_f and C_f can be obtained by solving the following system of ordinary differential equations.

$$\begin{cases} \dfrac{\partial A_f}{\partial t} + \mathbf{d}(t)^\top B_f + \dfrac{1}{2}B_f^\top \mathbf{k}_0(t) B_f + B_f^\top \mathbf{k}_0(t) B_p + \mathrm{tr}\left\{C_f \mathbf{k}_0(t)\right\} &= 0 \\ A_f(T,T) &= a_f(T) \end{cases} \tag{5.44}$$

$$\left\{\begin{array}{rcl} \frac{\partial B_f}{\partial t} + \mathbf{E}(t)^\top B_f + 2C_f\mathbf{d}(t) + \frac{1}{2}\bar{B}_f{}^\top \mathbf{K}(t) B_f + 2C_f\mathbf{k_0}(t)B_f & & \\ +\bar{B}_f{}^\top \mathbf{K}(t)B_p + 2C_f\mathbf{k_0}(t)B_p + 2C_p\mathbf{k_0}(t)B_f & = & \mathbf{0} \\ B_f(T,T) & = & \mathbf{b}_f(T) \end{array}\right. \tag{5.45}$$

$$\left\{\begin{array}{rcl} \frac{\partial C_f}{\partial t} + C_f\mathbf{E}(t) + \mathbf{E}(t)^\top C_f + 2C_f\mathbf{k_0}(t)C_f + \frac{1}{2}\bar{B}_f{}^\top \mathbf{G}(t)\bar{B}_f & & \\ +4C_f\mathbf{k_0}(t)C_p + \bar{B}_f{}^\top \mathbf{G}(t)\bar{B}_p & = & \mathbf{0} \\ C_f(T,T) & = & \mathbf{c}_f(T) \end{array}\right. \tag{5.46}$$

where C_f has the special form $C_f = \begin{pmatrix} C_f^{(qq)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, A_f, B_f, $C_f^{(qq)}$ should be evaluated at (t,T) and B_p, C_p solve (5.23)-(5.24). $\mathbf{E}$ and $\mathbf{d}$ are the same as in (5.14), $\mathbf{k_0}$, $\mathbf{K}$ and $\mathbf{G}$ the same as in (5.15)-(5.16), and where $\bar{B}_f$ and $\bar{B}_p$ follows the same idea as in (5.25) and have dimension $(m^2 \times m)$.

Proof. We start by noting that, from the functional form of α, $\sigma\sigma^\top$ and r, in (5.14),(5.15), and (5.39), respectively, and the fact that $Z^{(q)} \subset \tilde{Z}^{(q)}$, Theorem 5.14 guarantee a GQTS for bond prices with $C_p^{(ll)}(t,T) = \mathbf{0}$ and $C_p^{(lq)}(t,T) = C_p^{(ql)}(t,T) = \mathbf{0}$ for all t,T. This together with restrictions on $\mathbf{E}$, $\mathbf{G}$, $\mathbf{c}_f$ and Q (check (5.40) and (5.41)) guarantee that $C_f^{(ll)}(t,T) = \mathbf{0}$ and $C_f^{(ql)}(t,T) = C_f^{(lq)\top}(t,T) = \mathbf{0}$ for all t,T always solve (5.46), which proves the last statement of the Theorem.

It remains to show that the system of ODEs (5.44)-(5.46) solves the PDE (5.37) (recall that in Lemma 5.22 we proved the equivalence between (5.18) and (5.37) when $H_f(t,z,T)$ is of the form (5.35)). To see this, we start by using the fact that we have a GQTS for bond prices to get $\frac{\partial H_p}{\partial z_j}\frac{1}{H_p} = (B_p)_j + 2(C_p)_j z$. Substitute in (5.37) this partial derivative, as well as α, $\sigma\sigma^\top$ and r, from (5.14), (5.15), and (5.39), respectively, and applying the restrictions (5.42)-(5.43). The PDE (5.37) becomes, then, a separable equation. In addition, using the fact that $C_p^{(ll)}(t,T) = \mathbf{0}$, $C_p^{(lq)}(t,T) = C_p^{(ql)}(t,T) = \mathbf{0}$, $C_f^{(ll)}(t,T) = \mathbf{0}$ and $C_f^{(ql)}(t,T) = C_f^{(lq)\top}(t,T) = \mathbf{0}$ for all t,T, we can guarantee that all terms of order higher than two disappear. Hence, equation (5.34) becomes separable up to quadratic terms of z. Identification of terms, and the use of the assumption on the symmetry of C_p and C_f (which helps to simplify the expressions), leads to the system of ODEs (5.44)-(5.46). ∎

Solving explicitly the TS of Forward prices

The system of ODEs (5.44)-(5.46) seems quite complex, but once again, in most practical situations, it is possible to decompose it in (much) easier smaller systems. It is also important to note that B_p and C_p are independently obtained. So the natural steps to compute a *forward prices* term structure are:

1. obtain B_p and C_p solving (5.23)-(5.24) and substitute into (5.44)-(5.46),
2. split the ODEs (5.45)-(5.46) into simpler ODEs for $B_f^{(l)}$, $B_f^{(q)}$ and $C_f^{(qq)}$ and iteratively solve them.
3. substitute the solutions into (5.44) and integrate to obtain A_f.

In Section we exemplify this method.

The special case of Björk and Landén

If we assume an affine term structure for bond prices, and we are only interested in affine forward price models, as in Björk and Landén (2001), we have $C_p(t,T) = \mathbf{0}$, and $C_f(t,T) = \mathbf{0}$, for all t and T.

Moreover, since the need to distinguish between factors has to do with the necessity of imposing a deterministic volatility structure for *quadratic factors*, non-existent in this setup, we do not need to classify factors. All factors are *linear factors.* Also all matrices $\mathbf{g_{uk}}(t)$ in (5.43) will be zero since they can only be associated with quadratic factors, thus $\mathbf{G}(t) = \mathbf{0}$ for all t. Using $C_p(t,T) = \mathbf{0}$,$C_f(t,T) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ we recover their results from (5.44)-(5.45).

It is, however, important to stress that some quadratic short rate settings are consistent with an ATS for forward prices. This will become obvious once we get a better understanding of the $\tilde{Z}^{(q)}$ and $\tilde{Z}^{(l)}$ factors.

5.4.3 Understanding the $\tilde{Z}^{(q)}$ and $\tilde{Z}^{(l)}$ factors

Important consequences of Theorem 5.29 are the following.

Corollary 5.30. **(Linear Factors)** *The $\tilde{Z}^{(l)}$-factors are linear factors in a GQTS of forward prices.* [15]

[15] Recall Definition 5.2.

Corollary 5.31. *Necessary conditions of a QTS of forward prices are* $\mathbf{c}_f(T) \neq \mathbf{0}$ *for some* T *or* $\mathbf{G}(t) \neq \mathbf{0}$ *for some* t.

Corollary 5.30 follows immediately from $C_f^{(ll)}(t,T) = \mathbf{0}$ and $C_f^{(ql)}(t,T) = C_f^{(ql)\top}(t,T) = \mathbf{0}$ for all t, T. Corollary 5.31 can be checked by taking $\mathbf{c}_f(T) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ in (5.46) and noting that $C_f(t,T) = \mathbf{0}$ solves (5.46), for all t, T.

It follows that factors satisfying requirement (ii) of Definition 5.27, i.e., factors having a quadratic impact on the short rate r, do not necessarily have a quadratic impact on the *forward prices* term structure. The next Remark is a direct consequence of Corollary 5.31.

Remark 5.32. *For some models with* $\mathbf{c}_f(T) = \mathbf{0}$ *for all* T, *a volatility restricted (by* $\mathbf{G}(t) = \mathbf{0}$ *for all* t*) GQTS for bond prices is consistent with and ATS of forward prices.*

A concrete example of a model with an ATS for forward prices but a QTS for bond prices in now presented. In Section 6 we explicitly compute both term structures.

Example 5.33. *Consider the model*

$$\begin{aligned} dZ_1(t) &= [\beta_1(t) - \alpha_1 Z_1(t)]\, dt + \sigma_1 dW_1(t) \\ dZ_2(t) &= [\beta_2(t) - \alpha_2 Z_2(t)]\, dt + \sigma_2 dW_2(t) \end{aligned}$$

where α_1, α_2, σ_1 *and* σ_2 *are deterministic constants and* $\beta_1(\cdot)$, $\beta_2(\cdot)$, $a_1(\cdot)$, $a_2(\cdot)$, $a_3(\cdot)$, $k_0(\cdot)$ *and* $k_3(\cdot)$ *deterministic functions of time and* W_1, W_2 *are independent Wiener processes.*

The following relations between the factors above, the underlying of the forward contract S, *and the short rate* r, *hold*

$$\ln S(t) = Z_1(t) \quad , \qquad r(t) = \frac{1}{2}\left[Z_1^2(t) + Z_2^2(t)\right] .$$

First, note that in this model we have general quadratic $\mathbb{Q}$*-dynamics with* $\mathbf{K}(t) = \mathbf{0}$ *and* $\mathbf{G}(t) = \mathbf{0}$ *for all* t. *Secondly,* Z_1 *and* Z_2 *are* $Z^{(q)}$*-factors according to Definition 5.13, so the term structure of bond prices will actually be quadratic in both these factors. Finally, since* Z_1 *shows only linearly in the boundary condition* $\ln S = Z_1$ *(and* Z_2 *does not show up), we have* $\mathbf{c}_f(T) = \mathbf{0}$ *for all* T *and (Corollary 5.31) and ATS for forward prices.*

By now, one easily realizes that *not all* $\tilde{Z}^{(q)}$ factors will have a *quadratic impact*[16] on the *forward prices* term structure. In fact the $\tilde{Z}^{(q)}$-factors should be interpreted has

[16] Recall Definition 5.2.

the factors that, under the some regularity conditions, will have *either* a quadratic impact only on the term structure of bond prices, *or* a quadratic impact only on the term structure of forward prices, *or* on both term structures.

The only easy answer is given in the next Lemma.

Lemma 5.34. *A factor $Z_i \in \tilde{Z}^{(q)}$ for which $\mathbf{c}_{f_i}(T) \neq \mathbf{0}$ at least for some T, has a quadratic impact in the term structure of forward prices.*

Proof. It follows from $C_{f_i}(T,T) = \mathbf{c}_{f_i}(T)$ and Definition 5.2. ∎

To understand the exact role of bond prices is the same as to understand the difference between futures and forward prices term structures. The task is not an easy one and it is complicated by the various relations that may exist among various types of factors and because the same factors may be of different types depending on the specific term structure we are looking at.

In the next section, we study the difference between futures and forward prices term structures in the (stochastic) GQSR setting. Most results will be, however, of a qualitative nature. Only in a very concrete situation it is possible to quantify this difference.

5.5 Futures versus Forward prices

As is well known, forward prices (with maturity T) are martingales under the T-forward measure, while futures prices are martingales under the risk-neutral measure Q. Loosely speaking, this means that except for deterministic interest rates or full separability between the factors related to the underlying of a forward contract ($Z^{(u)}$ in Definition 5.24) and to the short rate ($Z^{(p)}$ in Definition 5.24), we have no reason to believe that they are equal.

Despite this fact, and with the exception of the obvious equivalence between the two prices in these extreme settings, few studies in the literature have analyzed the relation between the two prices. We now show how the study of GQTS of futures prices is somehow included in the study of GQTS of forward prices and analyze their relation in a *general quadratic short rate*(GQSR) setting.

Generally speaking, the reason for this is that the system of ODEs in the second part of Theorem 5.29 is equivalent to the one in Theorem 5.17 if we "delete" the terms dependent on the bond price term structure (i.e. terms with B_p or C_p). Thus the comparison of the systems of ODEs (5.31)-(5.33) and (5.44)-(5.46) should be useful in understanding the difference between futures and forward prices.

The only point of caution is that the classification of factors in these two theorems is not the same. We note that we typically have to deal with more factors in the term structure of forward prices than in futures prices, since we also have to model the term structure of bond prices.

Remark 5.35. *In some situations there may be a GQTS for* futures prices *but not for* forward prices.

The following example may help to clarify this point.

Example 5.36. *Consider the following two-factor model*

$$dZ(t) = (\cdots)dt + \begin{pmatrix} \sqrt{k_{01}(t) + k_1(t)Z_1(t)} & 0 \\ 0 & \sqrt{k_{02}(t) + k_2(t)Z_2(t)} \end{pmatrix} dW(t)$$

where we have $r(t) = \frac{1}{2}\left[Z_1(t)^2 + Z_1(t)^2\right]$ *and the spot price, of the underlying to the futures and forward contract, is given by* $S(t) = e^{b_1 Z_1(t) + b_2 Z_2(t)}$.

In this model we have a linear boundary function h_F *and* $\sigma(t,z)\sigma(t,z)^\top$*, is also obviously linear in* z*, so this model is compatible with an ATS for futures (there are only* $\bar{Z}^{(l)}$ *factors according to Definition 5.15).*

However, since the term structure of bond prices will be quadratic in both Z_1 *and* Z_2 *(both* Z_1 *and* Z_2 *are* $Z^{(q)}$ *hence* $\bar{Z}^{(q)}$ *factors, Definitions 5.13 and 5.27), and the volatility structure is not deterministic for both factors, we are* not *under the conditions for a GQTS for bond prices and hence also* not *under the conditions for an or GQTS for forward prices.*

In the situations when futures prices have a GQTS but forward prices do not, the comparison the ODEs systems (5.31)-(5.33) and (5.44)-(5.46) cannot help us, our goal is this section is to compare the term structure of futures and forward prices when *both* are GQTS.

Assumption 5.37. *We are under the conditions for a GQTS for forward prices.*[17]

As we will see the following classification of factors will be useful.

[17]A consequence of Assumption 5.37 is that we also have GQTS for bond prices and for futures prices.

Definition 5.38. Consider that we are under the conditions for a GQTS of bond prices and for a GQTS of futures prices. Then we have the following classification of factors.

- Z_i is a $\mathbf{Z^{(qp)}}$*-factor* if it has a quadratic impact on the term structure of bond prices.
- Z_i is a $\mathbf{Z^{(lp)}}$*-factor* if it a linear impact on the term structure of bond prices.
- Z_i is a $\mathbf{Z^{(p)}}$*-factor* if $Z_i \in Z^{(qp)} \cup Z^{(lp)}$.
- Z_i is a $\mathbf{Z^{(qu)}}$*-factor* if it has a quadratic impact on the term structure of futures prices.
- Z_i is a $\mathbf{Z^{(lu)}}$*-factor* if it has a linear impact on the term structure of futures prices.
- Z_i is a $\mathbf{Z^{(u)}}$*-factor* if $Z_i \in Z^{(qu)} \cup Z^{(lu)}$.

Before we go on we note that given the *a priori* classification of factors for bond and futures prices (Definitions 5.13 and 5.15, respectively), and under some regularity conditions, we know which factors will have a linear or a quadratic impact. So, we do not need to solve any system of ODEs to identify all the factors $Z^{(qp)}$, $Z^{(lp)}$, $Z^{(qu)}$ and $Z^{(lu)}$. We also note that these classifications are not mutually exclusive. The following example may help to illustrate this point.

Example 5.39. *Consider the following naive 5-factor model*

$$\begin{aligned}
dZ_1(t) &= [\beta_1 + \alpha_1 Z_1(t)]\, dt + \sigma_1 dW_1(t) \\
dZ_2(t) &= [\beta_2 + \alpha_2 Z_3(t)]\, dt + \sigma_2 dW_2(t) \\
dZ_3(t) &= [\beta_3 + \alpha_3 Z_3(t)]\, dt + \sigma_3 dW_3(t) \\
dZ_4(t) &= [\beta_4 + \alpha_4 Z_4(t)]\, dt + \sigma_4 \sqrt{{Z_2}^2(t) + Z_4(t)} dW_4(t) \\
dZ_5(t) &= [\beta_5 + \alpha_5 Z_5(t)]\, dt + \sigma_5 dW_5(t) \\
r(t) &= {Z_1}^2(t) + {Z_2}^2(t) \\
h_f(T, Z(t)) &= Z_4(t) + {Z_5}^2(t)
\end{aligned}$$

where all W_i are independent Wiener processes.

According to Definition 5.38 we have the following classification of factors:

$$Z^{(qu)} = \{Z_5\} \quad Z^{(lu)} = \{Z_4, Z_2\} \quad Z^{(qp)} = \{Z_1, Z_2, Z_3\} \quad Z^{(lp)} = \emptyset$$

We now define various concepts of stochastic correlation.

Definition 5.40. Any two stochastic Z_i and Z_j are said to be *stochastic correlated* in a general quadratic $\mathbb{Q}$-dynamics if $\sigma\sigma^\top$ have the form in (5.15) and we have at least one of the following conditions satisfied:

$$\begin{array}{rcl} \text{Deterministic correlation:} & \text{there exist a } t \text{ such that} & \mathbf{k}_{0ij}(t) \neq 0. \\ \text{Linear correlation:} & \text{there exists a } t \text{ and an } \mathbf{u} \text{ such that} & \mathbf{k}_{\mathbf{u}ij}(t) \neq 0. \\ \text{Quadratic correlation:} & \text{there exists a } t \text{ and } \mathbf{u}, \mathbf{k} \text{ such that} & \mathbf{g}_{\mathbf{uk}ij}(t) \neq 0. \end{array}$$

The use of the name "correlated" in Definition 5.40 is justified by noting that off-diagonal terms in a volatility matrix will imply nonzero correlation between factors. In the way we define correlation, any stochastic factor is correlated to itself but that a deterministic process is not. This turns out to be a crucial point since only stochastic factors may play a role in the difference between futures and forward prices. The next example may help to clarify these concepts.

Example 5.41. *Consider the following four-factor model*

$$d\begin{pmatrix} Z_1(t) \\ Z_2(t) \\ Z_3(t) \\ Z_4(t) \end{pmatrix} = \left[\mathbf{d}(t) + \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix}\begin{pmatrix} Z_1(t) \\ Z_2(t) \\ Z_3(t) \\ Z_4(t) \end{pmatrix}\right] dt + \sigma(t, Z(t))dW_t$$

with

$$\sigma(t,Z(t))\sigma^\top(t,Z(t)) = \overbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & 0 \\ 0 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}^{\mathbf{k}_0} + \overbrace{\begin{pmatrix} \sigma_1^2 & 0 & 0 & \delta\sigma_1\sigma_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta\sigma_1\sigma_4 & 0 & 0 & \sigma_4^2 \end{pmatrix}}^{\mathbf{k}_3} Z_3(t)$$

$$+ \overbrace{\begin{pmatrix} \sigma_1^2 & 0 & 0 & \delta\sigma_1\sigma_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta\sigma_1\sigma_4 & 0 & 0 & \sigma_4^2 \end{pmatrix}}^{\mathbf{g}_{22}} Z_2^2(t).$$

In this case Z_2 and Z_3 are deterministicly correlated, while Z_1 and Z_4 are both linearly (via $\mathbf{k}_3$) and quadraticly correlated (via $\mathbf{g}_{22}$).

We are now in conditions to our main result on the comparison between futures and forward prices GQTS.

Proposition 5.42. *Consider a given underlying T-claim $\mathcal{X}$. Given Definition 5.38, there will be differences between the term structures of forward and futures prices for $\mathcal{X}$ only if $Z^{(p)}$-factors are correlated with $Z^{(u)}$-factors.*

Moreover, if $Z^{(p)}$-factors are correlated with $Z^{(u)}$-factors, the following hold

- *the term structure of futures and forward prices will differ in the* quadratic term, *i.e. $C_F(t,T) \neq C_f(t,T)$ for some t,T, if at least one of the following conditions hold:*

  (i) $Z^{(qp)}$-factors are deterministicly correlated with $Z^{(qu)}$-factors

  (ii) $Z^{(lp)}$-factors are quadraticly correlated with $Z^{(lu)}$-factors.

- *the term structure of futures and forward prices will differ in the* linear term, *i.e. $B_F(t,T) \neq B_f(t,T)$ for some t,T, if at least one of the following conditions hold:*

  (iii) $C_F(t,T) \neq C_f(t,T)$ at least for some t,T

  (iv) $Z^{(qp)}$-factors are deterministicly correlated with $Z^{(lu)}$-factors

  (v) $Z^{(qu)}$-factors are deterministicly correlated with $Z^{(lp)}$-factors

  (vi) $Z^{(lp)}$-factors are linearly correlated with $Z^{(lf)}$-factors.

- *the term structure of futures and forward prices will differ in the* deterministic *term, i.e. $A_F(t,T) \neq A_f(t,T)$ for some t,T, if at least one of the following conditions hold:*

  (vii) $C_F(t,T) \neq C_f(t,T)$ at least for some t,T

  (viii) $B_F(t,T) \neq B_f(t,T)$ at least for some t,T

  (ix) $\bar{Z}^{(lp)}$ are deterministicly correlated with $\bar{Z}^{(lu)}$-factors.

Proof. Consider that B_p, C_p, B_F and C_F solve (5.23), (5.24),(5.32) and (5.33), respectively. If any of the conditions (i)-(ii) hold, C_F is not a solution for (5.46), since $C_f\mathbf{k_0}(t)C_p \neq \mathbf{0}$, or $\bar{B}_f{}^\top \mathbf{G}(t)\bar{B}_p \neq \mathbf{0}$ at least for some t,T and, under those conditions, the equations (5.33) and (5.46) are not the same. Otherwise C_f and C_F have the same ODE so $C_f = C_F$ solves (5.46). If any of the conditions (iii)-(vi) hold, B_F is not a solution for (5.45), since $C_F \neq C_f$, or $\bar{B}_f{}^\top \mathbf{K}(t)B_p \neq \mathbf{0}$, or $C_f\mathbf{k_0}(t)B_p \neq \mathbf{0}$ or $C_p\mathbf{k_0}(t)B_f \neq \mathbf{0}$ at least for some t,T and under those conditions the equations (5.32) and (5.45) are not the same. Otherwise, B_f and B_F have the same ODE so $B_f = B_F$ solves (5.45). If any of the conditions (vii)-(ix) hold, A_F is not a solution for (5.44), since $C_F \neq C_f$ or $B_F \neq B_f$ or $B_f^\top \mathbf{k}_0(t) B_p \neq \mathbf{0}$ at least for some t, T and under those conditions the equations (5.31) and (5.44) are not the same. Otherwise, A_f and A_F have the same ODE so $A_f = A_F$ solves (5.44).

■

Unfortunately, Proposition 5.42 gives us only *qualitative* differences. Given the complexity of the matrix system of ODEs, it is impossible to quantify the differences when they occur in the quadratic or the linear terms. However, if the difference occurs *only* in the deterministic term, we can explicitly compute it. In that case, we say that the forward and futures prices differ by a deterministic *adjustment term.*

Proposition 5.43. *Consider the classification of factors in Definition 5.38. If the only correlations (as defined by Definition 5.40) between $Z^{(p)}$ and $Z^{(u)}$ factors are via deterministic correlations of $Z^{(lp)}$ and $Z^{(lu)}$ factors, then the term structures of futures and forward prices differ by a deterministic adjustment term $D(t,T)$, and we have*

$$H_f(t,z,T) = D(t,T) H_F(t,z,T) \quad \textit{with} \quad D(t,T) = e^{\left\{\int_t^T B_f(s,T)^\top \mathbf{k}_0(s) B_p(s,T) ds\right\}}.$$

Proof. Proposition 5.42 show that the impact is only deterministic. The exact form of the adjustment term $D(t,T)$ follows form the fact that both (5.44) and (5.31) case be solved by simple integration and we have

$$A_f(t,T) = A_F(t,T) + \int_t^T B_f(s,T)^\top \mathbf{k}_0(s) B_p(s,T) ds.$$

■

To give an example of the use of Proposition 5.43 we can simply say that, as long as its conditions are satisfied, one can extend results from futures prices to forward prices by simply considering the adjustment term $D(t,T)$.

As far as the conditions themselves are concerned, even though they may seem quite restrictive given the generality of the previous sections, in practice they are actually not that restrictive: most multi-factor models previously studied in the literature assume constant volatility structure for the factors (so only deterministic correlations) and most studies are preformed in an affine interest rate setting. Note that these two conditions are even more restrictive than those of Proposition 5.43.

We now apply the results on GQTS of *futures* and *forward* prices and their connections to some models previously studied in the literature as well as to some original models.

5.6 Applications

5.6.1 Schwartz spot price with Vasiček short rate

We consider a spot price S driven by a Schwartz (1997) type of model and a short rate process following the Hull and White (1990) extension of the Vasiček model. Furthermore, we consider both processes to be deterministicly correlated.

Hence, we have

$$\begin{aligned} dS(t) &= [\beta(t) + \alpha(t)\ln S(t)]\, S(t)dt + \sigma_s(t)S(t)d\bar{W}_s(t) \\ dr(t) &= [b(t) - ar(t)]\, dt + \sigma_r d\bar{W}_r(t) \\ d\bar{W}_s(t)d\bar{W}_r(t) &= \rho(t)dt \end{aligned}$$

The parameters a and σ_r are considered to be deterministic constants, all others are allowed to be deterministic functions of time.

The factors that we consider are $Z = \begin{pmatrix} Y \\ r \end{pmatrix}$, for $Y = \ln S$, and we have $h_f(T, Z(T)) = e^{Y(T)} = S(T)$. Thus, our state variable dynamics in a multi-dimensional framework are:

$$dZ(t) = \left[\begin{pmatrix} \beta(t) - \frac{1}{2}\sigma_s^2(t) \\ b(t) \end{pmatrix} + \begin{pmatrix} \alpha(t) & 0 \\ 0 & -a \end{pmatrix} Z(t)\right] dt + \begin{pmatrix} \sigma_s(t) & 0 \\ \rho(t)\sigma_r & \sqrt{1-\rho(t)}\sigma_r \end{pmatrix} dW(t).$$

Given the dynamics of Z, we start by recalling the Hull and White extension of the Vasiček model of the short rate allows for an ATS of bond prices with

$$B_p(t,T) = \begin{pmatrix} 0 \\ \frac{1}{a}\left\{e^{-a(T-t)} - 1\right\} \end{pmatrix}.$$

We can also check, using Definition 5.15, that we have only $\bar{Z}^{(l)}$ factors and thus an ATS for futures prices. For ATS, A_F and B_F satisfy the system (5.31)-(5.32), which in our case becomes

$$\begin{cases} \dfrac{\partial A_F}{\partial t} + \begin{pmatrix} \beta(t) - \frac{1}{2}\sigma_s^2(t) & b(t) \end{pmatrix} B_F + \dfrac{1}{2} B_F^\top \begin{pmatrix} \sigma_s^2(t) & \rho(t)\sigma_s(t)\sigma_r \\ \rho(t)\sigma_s(t)\sigma_r & \sigma_r^2 \end{pmatrix} B_f = 0 \\ A_F(T,T) = 0 \end{cases}$$

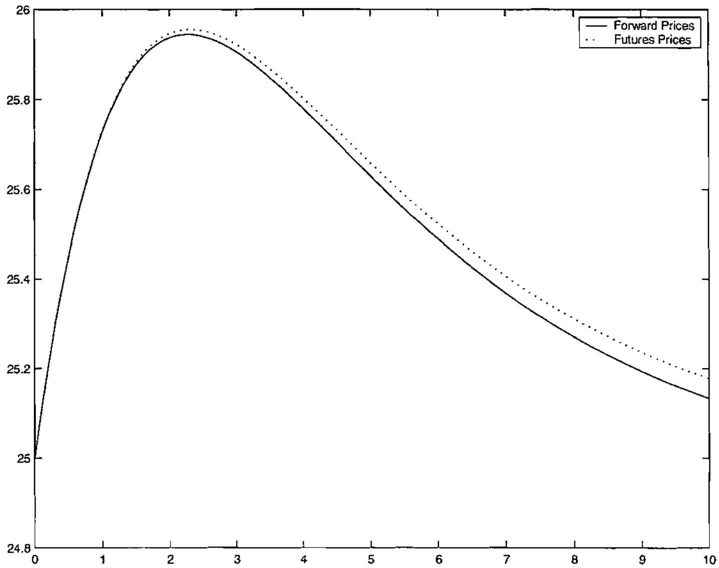


Figure 5.1: Illustration of forward vs. futures prices term structure. Parameter values: $S_t = 25$,$\alpha = -0.3$,$\beta = -\alpha \ln(S_t)$,$\sigma_s = 0.3$, $r_t = 0.05$,$\sigma_r = 0.01$, $a = 0.2$, $b = ar_t$, $\rho = 0.1$.

$$\begin{cases} \dfrac{\partial B_F}{\partial t} + \begin{pmatrix} \alpha(t) & 0 \\ 0 & -a \end{pmatrix} B_F &= 0 \\ B_F(T,T) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}.$$

The solutions are given by

$$B_F(t,T) = \begin{pmatrix} B_F^{(y)}(t,T) \\ B_F^{(r)}(t,T) \end{pmatrix} \quad \text{where} \quad B_F^{(y)}(t,T) = \exp\left\{ \int_t^T \alpha(s) ds \right\} \qquad B_F^{(r)}(t,T) = 0$$

and

$$A_F(t,T) = \int_t^T \left[\beta(s) - \frac{1}{2}\sigma_s(s) \right] B_F^{(y)}(s,T) ds + \frac{1}{2} \int_t^T \sigma^2(s) \left[B_F^{(y)}(s,T) \right]^2 ds.$$

The term structure of *forward prices* could likewise be obtained by noting that both Y and r are $\tilde{Z}^{(l)}$ factors (according to Definition 5.27) and solving the system of ODEs (5.44)-(5.45). Alternatively, since there are only *deterministic* correlations and $Y \in Z^{lu}$ while $r \in Z^{lp}$, we could use Proposition 5.43 and compute forward prices term structures using the adjustment term. In our case,

$$D(t,T) = e^{\frac{\sigma_r}{a}\int_t^T \rho(s)\sigma_s(s)B_f^{(y)}(s,T)\left\{e^{-a(T-s)}-1\right\}ds}.$$

For specified $\beta(\cdot)$, $\alpha(\cdot)$, $\rho(\cdot)$ and $\sigma_s(\cdot)$ exact solutions can be obtained. In the particular case where β, α, ρ and σ_s are deterministic constants we have

$$\begin{aligned} F(t,T) &= S(t)e^{e^{\alpha(T-t)}}e^{\frac{\beta-\frac{1}{2}\sigma_s^2}{\alpha}\left\{e^{\alpha(T-t)}-1\right\}+\frac{\sigma_s^2}{2\alpha}\left\{e^{2\alpha(T-t)}-1\right\}} \\ f(t,T) &= S(t)e^{e^{\alpha(T-t)}}e^{\frac{\beta-\frac{1}{2}\sigma_s^2}{\alpha}\left\{e^{\alpha(T-t)}-1+\frac{\rho\sigma_s\sigma_r}{a(\alpha-a)\alpha}\left\{\alpha\left[e^{(\alpha-a)(T-t)}-1\right]-(\alpha-a)\left[e^{\alpha(T-t)}-1\right]\right\}\right\}} \end{aligned}$$

Figure 5.1 illustrates the difference between forward and futures prices for concrete parameter values.

5.6.2 Two-factor model with QTS for bond Prices

Consider the following model

$$\begin{aligned} dZ_1(t) &= [\beta_1(t) - \alpha_1 Z_1(t)]\,dt + \sigma_1 dW_1(t) \\ dZ_2(t) &= [\beta_2(t) - \alpha_2 Z_2(t)]\,dt + \sigma_2 dW_2(t) \\ dW_1(t)dW_2(t) &= 0dt \\ S(t) &= e^{Z_1(t)} \\ r(t) &= \frac{1}{2}\left[Z_1^2(t) + Z_2^2(t)\right] \end{aligned}$$

where α_1, α_2, σ_1 and σ_2 are deterministic constants and $\beta_1(.)$ and $\beta_2(.)$ deterministic functions of time.

Taking $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ we have

$$dZ(t) = \left[\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} + \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} Z(t)\right] dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} dW(t)\ .$$

Concerning the interest rate setting this model was also considered in Chapter 2, hence, we can use the solutions for B_p and C_p previously obtained:

$$
\begin{aligned}
B_p^{(i)}(t,T) &= 2\int_t^T e^{\int_t^s\left(\alpha_i - 2\sigma_i^2 C^{(ii)}(u,T)\right)du}\beta_i(s)C^{(ii)}(s,T)ds \qquad i=1,2 \\
C_p(t,T) &= \begin{pmatrix} C^{(11)}(t,T) & 0 \\ 0 & C^{(22)}(t,T) \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{1-e^{2\gamma_1(T-t)}}{2(\alpha_1+\gamma_1)\left(e^{2\gamma_1(T-t)}-1\right)+4\gamma_1} & 0 \\ 0 & \dfrac{1-e^{2\gamma_2(T-t)}}{2(\alpha_2+\gamma_1)\left(e^{2\gamma_2(T-t)}-1\right)+4\gamma_1} \end{pmatrix}
\end{aligned}
$$

where $\gamma_i = \sqrt{\alpha_i^2+\sigma_i^2}$ for $i=1,2$.

In this case, despite the fact that we have a QTS for bond prices, we have a linear boundary function and a deterministic volatility structure so will have ATS for both futures and forward prices. Moreover, since $Z_1 \in \left[Z^{(qp)} \cap Z^{lu}\right]$ and is (obviously) deterministicly correlated to itself (recall Definitions 5.38 and 5.40), we know that the term structures of futures and forward prices will differ in both the deterministic and the linear terms. Taking the (more complex) case of forward prices, A_f and B_f should solve system of ODEs (5.44)-(5.45), which becomes

$$
\begin{cases}
\dfrac{\partial A_f}{\partial t} + \begin{pmatrix}\beta_1(t)\\ \beta_2(t)\end{pmatrix} B_f + \dfrac{1}{2}B_f^\top \begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix} B_f \\
\qquad + B_f^\top \begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix}\begin{pmatrix}B_p^{(1)}(t,T)\\ B_p^{(1)}(t,T)\end{pmatrix} = 0 \\
A_f(T,T) = 0
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial B_f}{\partial t} + 2\begin{pmatrix}C_p^{(11)}(t,T) & 0\\ 0 & C_p^{(22)}(t,T)\end{pmatrix}\begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix} B_f \\
\qquad + \begin{pmatrix}-\alpha_1 & 0\\ 0 & -\alpha_2\end{pmatrix} B_f = 0 \\
B_f(T,T) = \begin{pmatrix}1\\0\end{pmatrix}
\end{cases}
$$

and the solutions are given by $B_f(t,T) = \begin{pmatrix} B_f^{(1)}(t,T) \\ 0 \end{pmatrix}$ where

$$\begin{aligned} B_f^{(1)}(t,T) &= \exp\left\{\alpha_1(T-t) - 2\sigma_1^2 \int_t^T C_p^{(11)}(s,T)ds\right\} \\ A_f(t,T) &= \int_t^T \beta_1(s) B_f^{(1)}(s,T)ds + \frac{1}{2}\sigma_1^2 \int_t^T \left[B_f^{(1)}(s,T)\right]^2 ds \\ &\quad + \sigma_1^2 \int_t^T B_f^{(1)}(s,T) B_p^{(1)}(s,T)ds. \end{aligned}$$

Given the dependence B_f on C_p, we must rely on numerical integration to obtain both B_f and A_f.

The futures price term structure is in this case is given by (note that both terms are different)

$$B_F(t,T) = \begin{pmatrix} e^{\alpha_1(T-t)} \\ 0 \end{pmatrix} \quad A_F(t,T) = \int_t^T \beta_1(s) e^{\alpha_1(T-s)} ds + \frac{1}{2}\sigma_1^2 \int_t^T \left[e^{\alpha_1(T-s)}\right]^2 ds.$$

5.6.3 A QTS model for forward and futures prices

Consider a model specified by the following equations

$$\begin{aligned} dZ_1(t) &= \beta_1(t)dt + \sigma_1(t)dW_1(t) \\ dZ_2(t) &= \beta_2(t)dt + \sigma_2(t)dW_2(t) \\ dr(t) &= \left[b(t) - ar(t)\right]dt + \sigma_r dW_3(t) \\ W_1, W_2 &\text{ and } W_3 \text{ are independent Wiener processes.} \end{aligned}$$

where $\ln S(t) = q_1 [Z_1(t)]^2 + q_2 [Z_2(t)]^2 + g_1 Z_1(t) + g_2 Z_2(t) + f(t)$.

The parameters a,σ_r, q_1, q_2, g_1, g_2 are considered to be deterministic constants but all others are allowed to be deterministic functions of time.

The factors that we consider are $Z = \begin{pmatrix} Z_1 \\ Z_2 \\ r \end{pmatrix}$ and, as boundary condition, we have that

$$h_f(T, Z(T)) = \exp\left\{q_1 [Z_1(T)]^2 + q_2 [Z_2(T)]^2 + g_1 Z_1(T) + g_2 Z_2(T) + f(T)\right\} = S(T) .$$

So,

$$dZ(t) = \left[\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ b(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} Z(t)\right] dt + \begin{pmatrix} \sigma_1(t) & 0 & 0 \\ 0 & \sigma_2(t) & 0 \\ 0 & 0 & \sigma_r \end{pmatrix} dW(t)$$

Given the dynamics of Z and recalling that the Hull and White extension of the Vasiček model of the short rate allows an ATS of bond prices with

$$B_p(t,T) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{a}\left\{e^{-a(T-t)} - 1\right\} \end{pmatrix} .$$

Since $Z^{(p)} = \{r\}$ and $Z^{(u)} = \{Z_1, Z_2\}$ (check Definition 5.38), and the factors $Z^{(p)}$ are not correlated to the factors $Z^{(u)}$, the term structures of futures and forward prices will be the same, i.e., we have $A_F(t,T) = A_f(t,T)$, $B_F(t,T) = B_f(t,T)$ and $C_F(t,T) = C_f(t,T)$. The forward prices system (equal do the futures price system) is:

$$\left\{\begin{array}{rcl} \dfrac{\partial A_f}{\partial t} + \begin{pmatrix}\beta_1(t) & \beta_2(t) & b(t)\end{pmatrix} B_f + \dfrac{1}{2} B_f^\top \begin{pmatrix} \sigma_1^2(t) & 0 & 0 \\ 0 & \sigma_2^2(t) & 0 \\ 0 & 0 & \sigma_r^2 \end{pmatrix} B_f & & \\ + B_f^\top \begin{pmatrix} \sigma_1^2(t) & 0 & 0 \\ 0 & \sigma_2^2(t) & 0 \\ 0 & 0 & \sigma_r^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{a}\left\{e^{-a(T-t)} - 1\right\} \end{pmatrix} & & \\ + \operatorname{tr}\left\{ C_f \begin{pmatrix} \sigma_1^2(t) & 0 & 0 \\ 0 & \sigma_2^2(t) & 0 \\ 0 & 0 & \sigma_r^2 \end{pmatrix} \right\} & = & 0 \\ A_f(T,T) & = & f(T) \end{array}\right.$$

$$\left\{\begin{array}{rcl} \dfrac{\partial B_f}{\partial t} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} B_f + 2C_f \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ b(t) \end{pmatrix} + 2C_f \begin{pmatrix} \sigma_1^2(t) & 0 & 0 \\ 0 & \sigma_2^2(t) & 0 \\ 0 & 0 & \sigma_r^2 \end{pmatrix} B_f & & \\ + 2C_f \begin{pmatrix} \sigma_1^2(t) & 0 & 0 \\ 0 & \sigma_2^2(t) & 0 \\ 0 & 0 & \sigma_r^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{a}\left\{e^{-a(T-t)} - 1\right\} \end{pmatrix} & = & 0 \\ B_f(T,T) & = & \begin{pmatrix} g_1 \\ g_2 \\ 0 \end{pmatrix} \end{array}\right.$$

$$\begin{cases} \dfrac{\partial C_f}{\partial t} + 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} C_f + 2C_f \begin{pmatrix} \sigma_1^2(t) & 0 & 0 \\ 0 & \sigma_2^2(t) & 0 \\ 0 & 0 & \sigma_r^2 \end{pmatrix} C_f & = 0 \\ C_f(T,T) & = \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{cases}$$

The solutions are given by

$$C_f(t,T) = \begin{pmatrix} C_p^{(11)}(t,T) & 0 & 0 \\ 0 & C_p^{(22)}(t,T) & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B_f(t,T) = \begin{pmatrix} B_f^{(1)}(t,T) \\ B_f^{(2)}(t,T) \\ 0 \end{pmatrix}$$

where for $i = 1, 2$,

$$C_f^{(ii)}(t,T) = \frac{q_i}{1 - 2q_i \int_t^T \sigma_i^2(s)ds}, \quad B_f^{(i)}(t,T) = 2\int_t^T e^{2\int_t^s \sigma_i^2(u) C_f^{(ii)}(u,T)du} C_f^{(ii)}(s,T)\beta_i(s)ds,$$

and

$$A_f(t,T) = \sum_{i=1}^{2} \left\{ \int_t^T \beta_i(s) B_f^{(i)}(s,T)ds + \frac{1}{2}\sigma_i^2 \int_t^T \left[B_f^{(i)}(s,T)\right]^2 ds + \sigma_i^2 \int_t^T C_f^{(ii)}(s,T)ds \right\}$$

For specified $\beta_1(\cdot)$,$\beta_2(\cdot)$, $\sigma_1(\cdot)$, $\sigma_2(\cdot)$ and $f(\cdot)$ exact solutions can be obtained.

In the particular case where β_1,β_2, σ_1, σ_2 and f are deterministic constants we have the exact expressions

$$\begin{aligned} f(t,T) = F(t,T) \quad = \quad & \exp\left\{ \sum_{i=1}^{2} \frac{q_i}{1 - \delta_i(T-t)} (Z_i(t))^2 \right\} \times \exp\left\{ \sum_{i=1}^{2} M_i(t,T) \right\} \\ & \times \exp\left\{ \sum_{i=1}^{2} \left(2\beta_i \left[q_1(T-t) + \frac{1}{\sigma_1^2}(1-\delta_i)\ln(1-\delta_i) \right] Z_i(t) \right) \right\} \end{aligned}$$

where

$$
\begin{aligned}
&M_i(t,T) = \\
&= \frac{\beta_i^2}{\sigma_i^2}\left[\left(2(T-t) - \delta_i(T^2-t^2)\right) L_i(t,T) - \delta_i(T-t) - \delta_i t^2 L_i(t,T) - \delta_i(T^2-t^2)\right] \\
&\quad + 2\beta_i\sigma_i^2\left[\frac{4T^3}{3} - \frac{t^3}{3} - T^2t - T^2 - t^2\right] + 2\beta_i q_i T\left[\left(2T - \delta_i T\right) L_i(t,T) - (T^2-t^2)\delta_i\right] \\
&\quad - 2\beta_i q_i\left[\left(t^2 - \delta_i T t^2 + \frac{2t^3}{3}\delta_i\right) L_i(t,T) + \left(T^2 - \delta_i\frac{T^3}{3}\right) L_i(t,T) + \left(\frac{T^3}{3} - \frac{t^3}{3}\right)\delta_i\right] \\
&\quad - 2\beta_i q_i\left[\left(\frac{t^3}{2} - \frac{t^2}{2} + \frac{t}{4} - \frac{1}{4}\right) L_i(t,T)\right] \\
&\quad - 2\frac{\beta_i}{\sigma_i^2}\left[\frac{\left(1-\delta_i(T_t)\right)^3}{3\delta_i} L_i(t,T)^2 + \frac{2\left(1-\delta_i(t,T)\right)^3}{27\delta_i}\right] \\
&\quad + 2\frac{\beta_1}{\sigma_1^2}\left[\frac{2\left(1-\delta_i\right)^3}{9\delta_i} L_i(t,T) + \frac{2}{27\delta_i}\right] - \frac{1}{2} L_i(t,T)
\end{aligned}
$$

and $\qquad L_i(t,T) = \ln\left(1 - \delta_i(t,T)\right) \qquad$ with $\qquad \delta_i = 2q_i\sigma_i^2.$

5.7 Concluding remarks

This chapter investigates the term structures of *futures* and *forward prices* when we assume that these prices are functions of a finite dimensional state process. The GQTS is a considerable generalization of the literature on the term structure of both futures and forward prices.

Motivated by the fact that forward prices are martingales under measures where bond prices are *numeraires*, we exploit the relationship between bond prices and forward prices. We show, on the one hand, that bond prices will only play a role in the term structure of forward prices through stochastic correlations. On the other hand, we show that if they do play a role, volatility-restricted GQTS of bond prices are consistent with ATS of forwards while the non-restricted version is only consistent with QTS of forward prices.

Finally we show that, in some sense, the study of the GQTS of futures prices is included in the study of the GQTS of forward prices. Their difference is related to the impact that bond prices may have on the term structure of forward prices. We *qualify* this difference in a quadratic short rate setting and provide a *quantification* for situations where the two term structures differ only by a deterministic adjustment term.

The examples section highlights the applicability of the theoretical results derived.

We conclude by pointing out that the GQTS results presented here can be (almost directly) applied to study term structures of *any* martingale either under the risk neutral measure or the forward measure, besides futures and forward prices.

Chapter 6

Finite Dimensional Realizations of FPM

"In the end, a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that it is correct and relevant."

Fisher Black, 1986.

In this chapter we study a fairly general Wiener driven model for the term structure of forward prices. The model, under a fixed martingale measure, $\mathbb{Q}$, is described by using two infinite dimensional stochastic differential equations (SDEs). The first system is a standard HJM model for (forward) interest rates, driven by a multidimensional Wiener process W. The second system is an infinite SDE for the term structure of forward prices on some specified underlying asset driven by the same W.

We are primarily interested in the *forward prices.* However, since for any fixed maturity T, the forward price process is a martingale under the T-forward neutral measure, the zero coupon bond volatilities will enter into the drift part of the SDE for these forward prices. The interest rate system is, thus, needed as input into the forward price system.

Given this setup we use the Lie algebra methodology of Björk *et al.* to investigate under what conditions on the volatility structure of the forward prices and/or interest rates, the inherently (doubly) infinite dimensional SDE for forward prices can be realized by a finite dimensional Markovian state space model.

6.1 Introduction

In this chapter we study forward price models and, in particular, we want to understand when the inherently infinite forward price process can be realized by means of a (Markovian) finite state space model.

As we saw in the previous chapter, term structures of forward prices are more complex than term structures of interest rates or term structures of futures prices. The extra complexity is due to the fact that forward prices, with maturity T, are martingales under the T-forward measure, $\mathbb{Q}^T$. Under such *T-forward measures*, zero-coupon bonds (with maturity T) are numeraires, which implies that zero-coupon bond price volatilities will enter into the dynamics of forward prices. Consequently, in general, the term structures of forward prices cannot be studied in isolation; they must be studied under some interest rate setting and a needed input to the term structures of forward price is the term structure of interest rates.

We model the dynamics directly under the risk-neutral measure $\mathbb{Q}$ and our forward price model is described by using two infinite dimensional stochastic differential equations (SDEs), one defining the interest rate setting, and the other defining the forward contract setting. For the interest rate setting we consider a standard HJM model for (forward) interest rates, driven by a multidimensional Wiener process W. For the forward contract setting we use the $\mathbb{Q}^T$-martingale property of forward prices and the bond prices dynamics induced by the interest rate setting to get a second infinite SDE for the term structure of forward prices on a specified underlying asset. Without loss of generality, we consider that the Wiener process W is the same for both SDEs.

The theoretical literature on the term structure of forward prices is not substantial and has mainly focused on understanding under what conditions - on the dynamics of the state space variables (which are *assumed* to be finite) - the term structure is of an *a priori given* specific functional form. Included in this traditional approach are the studies on affine and quadratic term structures of forward prices of Chapter 5.

In this chapter we choose a fundamentally different approach. We do *not* assume that the state space model is finite, nor that the term structure of forward prices is of a given specific functional form. Instead, we try to understand under what conditions, in terms of the volatility of forward prices and interest rates, we can have a finite dimensional realization (FDR) for the term structure models of forward prices.

This more systematic way of thinking about term structures was proposed by Björk and Christensen (1999) and Björk and Svensson (2001), and a more geometric way of thinking about FDR of term structures was then introduced. The main technical tool of these studies is the Frobenius Theorem, and the main result is that there exists an FDR if, and only if, the Lie algebra generated by the drift and diffusion terms of the underlying infinite dimensional (Stratonovich) SDE is finite dimensional. Filipović and Teichmann (2003) and Filipović and Teichmann (2002) increased the applicability of the geometrical approach by showing how the theory can be extended to even more general settings than initially considered. Finally, Björk and Landén (2002) addressed the question of the actual construction of finite dimensional realizations, making this geometrical analysis interesting also from an application point of view.

The main area of application of these ideas has been (forward) interest rate term structures, which was the object of study in all the above-mentioned papers (for a review study on the geometry of interest rate models, see also Björk (2004b)). More recently this geometric machinery has also been applied to study the term structure of futures prices (see Björk, Blix, and Landén (2004)). To the best of our knowledge, these techniques have not yet been applied to study forward prices (or any other $\mathbb{Q}^T$-martingales). In the present chapter we, thus, take this next natural step.

The main contributions of this paper are as follows.

- We adapt the geometrical analysis of term structures to the case of doubly infinite systems.
- We obtain necessary and sufficient conditions for the existence of an FDR of forward rate term structure models.
- Given that such conditions are satisfied, we derive the dynamics of the underlying finite state space variable.

The paper is organized as follows.

In Section 4.2 we present the basic setup, derive the doubly infinite SDE that will be the object of study and present the main questions to be answered. Section 6.3 explains the method of analysis and adapts important results from the previous literature.

Sections 6.4, 6.5 and 6.6 are devoted to the actual study of forward price models answering the questions proposed.

Section 6.7 summarizes our main conclusions and discusses the applicability of the results.

6.2 Setup

The main goal of this study is the study of *forward prices* in a general stochastic interest rate setting.

We consider a financial market living on a filtered probability space $\left(\Omega,\ \mathcal{F},\ \mathbb{Q},\ (\mathcal{F}_t)_{t\geq 0}\right)$ carrying an m-dimensional Wiener process W. For reasons that will soon become clear, the main assets we consider are forward contracts (written on some given underlying asset under consideration) and zero-coupon bonds[1].

Let $f_0(t,T)$ denote the forward prices at time t of a forward contract maturing at time T, and $p_0(t,T)$ denote the price at time t of a zero-coupon bond maturing at time T.

Besides the trivial boundary conditions

$$\begin{aligned} f_0(T,T) &= S(T) \\ p_0(t,T) &= 1 \end{aligned}$$

where S is the price process of the underlying asset to the forward contract, arbitrage arguments yields

$$p_0(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T R(s)ds}\right] \tag{6.1}$$

$$f_0(t,T) = \mathbb{E}_t^T\left[f_0(T,T)\right] \tag{6.2}$$

where R is the short rate of interest and $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$, $\mathbb{E}_t^T[\cdot]$ denote, respectively, expectations, conditional on $\mathcal{F}_t$, under the martingale measure $\mathbb{Q}$ and under the *forward* martingale measure $\mathbb{Q}^T$.

It is also well-known that under *deterministic* interest rate settings, or complete orthogonality between the underlying and the interest rate random sources, forward prices are the same as futures prices (a similar but conceptually different contract).

In this study we focus on *forward prices* and, in particular, we are interested in analyzing the settings where they are not equivalent to futures prices. Our analysis will, therefore, assume *stochastic interest rates* and that there are at least some *common radom sources* driving *both* the interest rates *and* the underlying to the forward contract.

In our context, a *forward price model* is, thus, only fully defined once we have specified both forward prices dynamics and interest rates dynamics under a same measure (which

[1]Recall the definitions of bond and forward contracts from Section 1.1.1

we choose to be $\mathbb{Q}$) and assumed that these two dynamics are, at least partially, driven by common elements of our multidimensional Wiener process.

Before we present in detail our setting, we start by reparameterizing our variables. A more suitable parameterization for our purposes is the so called Musiela parameterization (Brace and Musiela (1994) and Musiela (1993)). Under the Musiela parameterization, forward prices and bond prices are given in terms of t and x, where x denotes time *to* maturity, in contrast to T which defined time *of* maturity. Therefore, we will use

$$f(t,x) = f_0(t,t+x) \qquad\qquad p(t,x) = p_0(t,t+x) \tag{6.3}$$

6.2.1 The interest rate curve

We consider a standard HJM model for the (forward) interest rates, driven by a multidimensional Wiener process W. Using the Musiela parameterization the dynamics for the interest rates, under $\mathbb{Q}$, are given by[2]

$$dr(t,x) = \left\{ \frac{\partial}{\partial x} r(t,x) + \sigma(t,x) \int_0^x \sigma^\top(t,s)ds \right\} dt + \sigma(t,x)dW_t \tag{6.4}$$

where $\sigma(t,x)$ is a given adapted process in $\mathbb{R}^m$ and $\top$ denotes transpose.

From the relation between (forward) interest rates and bond prices, we can derive the bond price $\mathbb{Q}$-dynamics.

Lemma 6.1. *Assume the (forward) interest rates dynamics in (6.4). Then the dynamics of the zero-coupon bond prices, using the Musiela parameterization, is given by*

$$dp(t,x) = \{R(t) - r(t,x)\} p(t,x)dt + p(t,x)v(t,x)dW_t$$

where R is the short interest rate[3] *and the bond prices' volatility, v, is obtained from the (forward) interest rate volatilities as*

$$v(t,x) = -\int_0^x \sigma(t,s)ds \tag{6.5}$$

and hence also adapted.

[2] For a textbook treatment of HJM models and the Musiela parameterization for such models, see Björk (2004a).

[3] Recall that in the Musiela parametrization the short rate of interest is $R(t) = r(t,0)$.

Proof. Recall the standard relation between (forward) interest rates and bond prices

$$p(t,x) = e^{-\int_0^x r(t,s)ds}.$$

Let us set $y(t,x) = -\int_0^x r(t,s)ds$. Applying the Itô lemma we get

$$\begin{aligned}
dy(t,x) &= -\int_0^x dr(t,s)ds \\
&= -\int_0^x \left[\left(\frac{\partial}{\partial s}r(t,s) + \sigma(t,s)\int_0^s \sigma^\top(t,u)du\right)dt + \sigma(t,s)dW_t\right]ds \\
&= -\int_0^x \frac{\partial}{\partial s}r(t,s)dsdt - \int_0 \sigma(t,s)\int_0^s \sigma^\top(t,u)dudsdt \underbrace{-\int_0^x \sigma(t,s)ds\, dW_t}_{v(t,x)} \\
&= \left[\underbrace{r(t,0)}_{R(t)} - r(t,x) - \int_0 \sigma(t,s)\int_0^s \sigma^\top(t,u)duds\right]dt + v(t,x)dW_t
\end{aligned}$$

The result follows from $dp(t,x) = p(t,x)dy(t,x) + \frac{1}{2}p(t,x)\,[dy(t,x)]^2$ and by notting that

$$\begin{aligned}
\sigma(t,x)\int_0^x \sigma^\top(t,u)du &= \frac{1}{2}2\sigma(t,x)\int_0^x \sigma^\top(t,u)du \\
\int_0^x \sigma(t,s)\int_0^x \sigma^\top(t,u)duds &= \frac{1}{2}\left(\int_0^x \sigma^\top(t,u)du\right)^2
\end{aligned}$$

■

6.2.2 The forward price curve

Since the forward prices are $\mathbb{Q}^T$-martingales (recall (6.2)), we assume $\mathbb{Q}^T$–dynamics of the form

$$df_0(t,T) = f_0(t,T)\gamma_0(t,T)dW_t^T \tag{6.6}$$

where we also take γ_0 to be a given adapted process.

Here we use the fact that martingales have zero drift. Note however that, by choosing to model the forward price dynamics as in (6.6), forward prices with different maturities T are modeled under a different martingale measures $\mathbb{Q}^T$.

Reparameterizing using $f(t,x) = f_0(t,t+x)$ give us

$$df(t,x) = \left\{\frac{\partial}{\partial x}f(t,x)\right\}dt + f(t,x)\gamma(t,x)dW_t^T.$$

where $T = t + x$ and $\gamma(t,x) = \gamma_0(t,T)$.

It will also simplify matters if we work with the logarithm of forward prices instead of the forward prices themselves. Thus setting

$$q(t,x) = \ln f(t,x) \tag{6.7}$$

we have

$$dq(t,x) = \left\{ \frac{\partial}{\partial x} q(t,x) - \frac{1}{2} \|\gamma(t,x)\|^2 \right\} dt + \gamma(t,x) dW_t^T. \tag{6.8}$$

Note that analyzing the logarithm of forward prices is equivalent to analyzing the forward prices themselves, as we can always use (6.7) to transfer any results on the logarithm of forward prices into results on forward prices.

Finally, to obtain the dynamics of (the logarithm) of forward prices, under the risk-neutral martingale measure, $\mathbb{Q}$, we use the change of numeraire technique (introduced in Geman, El Karoui, and Rochet (1995)). Denoting by L the Radon-Nikodym derivative,

$$L(t) = \frac{d\mathbb{Q}}{d\mathbb{Q}^T} \qquad \text{on } \mathcal{F}_t, \quad 0 \le t \le T,$$

and recalling the money account B is the numeraire under $\mathbb{Q}$, we have in Musiela parameterization

$$L(t) = p(0,T) \frac{B(t)}{p(t,x)}$$

where $x = T - t$.

Thus, the dynamics of our likelihood process L are given by

$$dL(t) = L(t) \left\{ -v(t,x) \right\} dW_t^T$$

i.e., the Girsanov kernel, for the transition from $\mathbb{Q}^T$ to $\mathbb{Q}$, is the symmetric of the volatility of zero-coupon bond price with maturity T or, equivalently, with time to maturity $x = T - t$.

Using the above Girsanov kernel, we can easily obtain the (logarithm of) forward prices $\mathbb{Q}$-dynamics from (6.8), that is

$$dq(t,x) = \left\{ \frac{\partial}{\partial x} q(t,x) - \frac{1}{2} \|\gamma(t,x)\|^2 - \gamma(t,x) v^\top(t,x) \right\} dt + \gamma(t,x) dW_t \tag{6.9}$$

where, w.l.o.g., we can take W to be multidimensional and the same as in (6.4).

Taking a geometrically oriented interpretation of equations (6.4) and (6.9), we can see each of these equations as infinite dimensional objects. The main infinite dimensional object under study in this chapter is (the logarithm) of the *forward price curve*, i.e., the curve $x \to q(t,x)$. This object, however, for general adapted processed σ and γ may depend on, the *interest rate curve*, i.e., the curve $x \to r(t,x)$, another infinite dimensional object.

In principle, both adapted processes σ and γ could depend on q and r. It seems, however, unrealistic to assume that a forward price on a specific underlying (be it the price of a stock, or any other asset) should influence the interest rate volatility.

The opposite is true for forward prices. As mentioned before, these prices are only interesting to study in *stochastic* interest rate settings. This tell us that, maybe, it is realistic that the forward price volatility depends on the interest rates' curve.

With this basic intuition in mind we set some more structure on the volatility processes σ and γ.

Assumption 6.2. *The adapted processes $\gamma(t,x)$, and $\sigma(t,x)$ have the following functional form in terms of r and q*

$$\gamma(t,x) = \gamma(q_t, r_t, x) \tag{6.10}$$

$$\sigma(t,x) = \sigma(r_t, x) \tag{6.11}$$

where, with a slight abuse of notation, the r.h.s. occurrence of γ and σ denotes deterministic mappings

$$\begin{aligned} \gamma &: \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \to \mathbb{R}^m \\ \sigma &: \mathcal{H}_r \times \mathbb{R}_+ \to \mathbb{R}^m. \end{aligned}$$

where $\mathcal{H}_q$ and $\mathcal{H}_r$ are special Hilbert spaces of functions where forward price curves and interest rate curves live, respectively. [4]

Note that by imposing Assumption 6.2, σ does not depend on q_t (the logarithm of the forward price process) and that we restrict ourselves to the study of time homogenous models. Extensions to non-homogeneous models have been considered in Björk, Blix, and Landén (2004) and turn out to be straightforward generalization of the homogeneous results.

From now on we will use the short-hand notation $q_t = q(t,x)$, $r_t = r(t,x)$ where we suppress the x-dependence. This shorter notation will be helpful when the expressions get messy, and it is also more intuitive from a geometrical point of view.

[4]For details on the construction of the Hilbert spaces $\mathcal{H}_q$ and $\mathcal{H}_r$ we refer to Björk and Svensson (2001), Filipović and Teichmann (2003) and Filipović and Teichmann (2002).

Using the short-hand notation and Assumption 6.2, we can rewrite equations (6.4)-(6.9) as

$$dq_t = \left\{ \mathbf{F}q_t - \frac{1}{2}\|\gamma(q_t, r_t)\|^2 - \gamma(q_t, r_t)v^\top(r_t) \right\} dt + \gamma(q_t, r_t)dW_t \tag{6.12}$$

$$dr_t = \left\{ \mathbf{F}r_t - \sigma(r_t)v^\top(r_t) \right\} dt + \sigma(r_t)dW_t \tag{6.13}$$

where $\mathbf{F} = \frac{\partial}{\partial x}$ and we can interpret the entire system as an object $\hat{q} = (q_t\ ,\ r_t)^\top \in \mathcal{H}_q \times \mathcal{H}_r$.

As we can see from equations (6.12)-(6.13), the interest rates equation (6.13) does not depend on the forward prices equation (6.12), so the interest rates curve r exist and can be studied in isolation. For a survey study on the geometry of interest rate models see Björk (2004b). In contrast to this, the (logarithm of the) forward price equation (6.12), is linked to the interest rate equation (6.13) through $\gamma(q, r)$ and/or $v(r)$ [5]. This means, that in general, to study forward prices we will have to study the entire system (6.12)-(6.13).

In the following analysis we will refer to *forward price equation* when referring only to (6.12), to *interest rate equation* when referring only to (6.13), and to *forward price model* when referring to the entire system (6.12)-(6.13).

We can now formulate our main problems.

6.2.3 Main Problems

Problem 1 : Under what conditions we have Markovian forward prices?

Problem 2 : Is it possible to have a finite realization for the forward prices equation (6.12) but not for interest rates equation (6.13)?

Problem 3 : When can the inherently infinite forward price system (6.12)-(6.13) be realized by means of a finite dimensional state space model?

Problem 4 : In the cases when a finite dimensional realization (FDR) exists, can we determine the finite dimensional state space model?

The next section introduces the method of analysis.

[5] Given the definition of the bond price volatility, v, in (6.5), if $\sigma(t, x) = \sigma(r_t, x)$ then also $v(t, x) = v(r_t, x)$.

6.3 Method and Existing results

In this section we describe the general method we will use to attack the presented problems.

The method relies on geometric results from differential geometry and was firstly applied to finance in Björk and Christensen (1999) and Björk and Svensson (2001). In this section, we adapt the framework of Björk and Christensen (1999) and Björk and Svensson (2001) to our doubly-infinite system case.

To be able to apply the concepts and intuitions of ordinary differential geometry to (stochastic) Itô calculus, we need to rewrite the analysis in terms of Stratonovich integrals instead of Itô integrals.[6]

Given the volatility mappings $\gamma : \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \to \mathbb{R}^m$ and $\sigma : \mathcal{H}_r \times \mathbb{R}_+ \to \mathbb{R}^m$ from Assumption 6.2, the forward prices equation will, in general, depend on the interest rate curve. Thus, our main object of study will be a Stratonovich forward price *system* of the following form:

$$d\hat{q}_t = \begin{bmatrix} \mu^q(q_t, r_t) \\ \mu^r(r_t) \end{bmatrix} dt + \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix} \circ dW_t, \tag{6.14}$$

where $\hat{q} = \begin{bmatrix} q \\ r \end{bmatrix} \in \mathcal{H}_q \times \mathcal{H}_r$.

In special cases, the forward price dynamics may be independent of the interest rate curve, then our object of study is the Stratonovich forward price *equation*,

$$dq_t = \mu^q(q_t)dt + \gamma(q_t) \circ dW_t. \tag{6.15}$$

where $q \in \mathcal{H}_q$, and we say that the (logarithm of) forward prices is *Markovian*.

When referring to the *forward price model* we refer to either (6.14) or (6.15), depending on the circumstances.

In this study, we will consider these two possibilities.

Let us begin by specifying exactly what we mean with a finite dimensional realization for the forward prices generated by volatilities.

[6] Recall discussion on Section 1.2.2.

Definition 6.3. We say that the doubly-infinite SDE (6.14) has a (local) d-dimensional realization at $\hat{q}^0 = (q^0\ ,\ r^0)^\top$, if there exists a point $z_0 \in \mathbb{R}^d$, smooth vector fields $\hat{a}, \hat{b}_1 \cdots, \hat{b}_m$ on some open subset $\hat{\mathcal{Z}}$ of $\mathbb{R}^d$ and a smooth (sub manifold) map $\hat{G} : \hat{\mathcal{Z}} \to \mathcal{H}_q \times \mathcal{H}_r$, such that $\hat{q} = (q\ ,\ r)^\top$ has a local representation

$$\hat{q}_t = \hat{G}(Z_t) \qquad \text{i.e.,} \quad \begin{bmatrix} q_t \\ r_t \end{bmatrix} = \begin{bmatrix} \hat{G}^q(Z_t) \\ \hat{G}^r(Z_t) \end{bmatrix} \quad a.s.$$

where Z is the strong solution of the d-dimensional Stratonovich SDE

$$\begin{cases} dZ_t &= \hat{a}(Z_t) + \hat{b}(Z_t) \circ dW_t \\ Z_0 &= z_0 \end{cases} \tag{6.16}$$

and where W is the same as in (6.14).

Likewise, we say that the SDE (6.15) has a (local) n-dimensional realization at q^0, if there exist $z_o \in \mathbb{R}^n$, smooth vector fields $a, b_1 \cdots, b_m$ on some open subset $\mathcal{Z}$ of $\mathbb{R}^n$ and a smooth (sub manifold) map $G : \mathcal{Z} \to \mathcal{H}_q$, such that q has a local representation

$$q_t = G(Z_t) \quad a.s.$$

where Z is the strong solution of the d-dimensional Stratonovich SDE

$$\begin{cases} dZ_t &= a(Z_t) + b(Z_t) \circ dW_t \\ Z_0 &= z_0 \end{cases} . \tag{6.17}$$

where W is the same as in (6.15).

If the SDE under analysis, (6.14) or (6.15), has a finite dimensional realization (FDR), we say that our forward rate model admits a FDR.

The method of studying existence and construction of FDR for forward price models, relies on some basic concepts from infinite dimensional differential geometry, previously introduced, in Section 1.2.2 of the preliminaries chapter.

The Hilbert space under analysis will be either $\mathcal{H}_q$ when studying Markovian forward prices or $\mathcal{H}_q \times \mathcal{H}_r$ when dealing with the entire forward price system.

We can now adapt two important theorems from Björk and Svensson (2001) to our forward price problem. The first theorem gives us the general necessary and sufficient conditions for existence of a FDR.

Theorem 6.4 (Björk and Svensson). *Consider the SDE in (6.14) and denote $\gamma_1, \cdots, \gamma_m$ and $\sigma_1, \cdots, \sigma_m$ the elements of γ and σ, respectively. Assume that the dimension of the Lie algebra*

$$\left\{ \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix}, \begin{bmatrix} \gamma_1 \\ \sigma_1 \end{bmatrix}, \ldots, \begin{bmatrix} \gamma_m \\ \sigma_m \end{bmatrix} \right\}_{LA}$$

is constant near the initial point $\hat{q}^0 = (q^0\ ,\ r^0)^\top \in \mathcal{H}_q \times \mathcal{H}_r$.

Then (6.14) possesses an FDR if and only if

$$\dim \left\{ \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix}, \begin{bmatrix} \gamma_1 \\ \sigma_1 \end{bmatrix}, \ldots, \begin{bmatrix} \gamma_m \\ \sigma_m \end{bmatrix} \right\}_{LA} < \infty$$

in a neighborhood of $\hat{q}^0$.

Likewise, for Markovian forward prices we consider the SDE (6.15), and assume that the dimension of the lie algebra of $\{\mu, \gamma_1, \ldots, \gamma_m\}_{LA}$ is constant near $q^0 \in \mathcal{H}_q$. Then (6.15) possesses and FDR if and only if

$$\dim \{\mu, \gamma_1, \ldots, \gamma_m\}_{LA} < \infty$$

Remark 6.5. *To shorten notation we will sometimes use $\left\{ \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix}, \begin{bmatrix} \gamma \\ \sigma \end{bmatrix} \right\}_{LA}$ instead of $\left\{ \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix}, \begin{bmatrix} \gamma_1 \\ \sigma_1 \end{bmatrix}, \ldots, \begin{bmatrix} \gamma_m \\ \sigma_m \end{bmatrix} \right\}_{LA}$ and $\{\mu, \gamma\}_{LA}$ instead of $\{\mu, \gamma_1, \ldots, \gamma_m\}_{LA}$.*

The second theorem gives us a parameterization of the curves produced by the forward price model and is a crucial step to the understanding of the construction algorithm.

Theorem 6.6 (Björk and Svensson). *Assume that the Lie algebra $\left\{ \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix}, \begin{bmatrix} \gamma \\ \sigma \end{bmatrix} \right\}_{LA}$ is spanned by the smooth vector fields $\hat{f}_1, \ldots, \hat{f}_d$ in $\mathcal{H}_q \times \mathcal{H}_r$.*
Then, for the initial point $\hat{q}^0 = \left(q^0\ ,\ r^0\right)^\top$, all forward price and interest rate curves produced by the model will belong to the manifold $\hat{\mathcal{G}} \in \mathcal{H}_q \times \mathcal{H}_r$, which can be parameterized as $\hat{\mathcal{G}} = \mathrm{Im}[\hat{G}]$, where

$$\hat{G}(z_1, \ldots, z_d) = e^{\hat{f}_d z_d} \ldots e^{\hat{f}_1 z_1} \begin{bmatrix} q^0 \\ r^0 \end{bmatrix}, \tag{6.18}$$

and where the operator $e^{\hat{f}_i z_i}$ is given in Definition 1.45.

Likewise, in the case of Markovian forward prices, and assuming that the Lie algebra $\{\mu, \gamma\}_{LA}$ is spanned by the smooth vector fields $f_1, \ldots, f_d$ in $\mathcal{H}_q$. Then, for the initial *point q^0, all forward price curves produced by the model will belong to the manifold $\mathcal{G} \in \mathcal{H}$, which can be parameterized as $\mathcal{G} = \mathrm{Im}[G]$, where*

$$G(z_1, \ldots, z_d) = e^{f_d z_d} \ldots e^{f_1 z_1} q^0 \tag{6.19}$$

and where the operator $e^{f_i z_i}$ is given in Definition 1.45.

The manifolds $\hat{\mathcal{G}}$ and $\mathcal{G}$ in the above theorem are obviously invariant under the forward price model dynamics. Therefore, they will be referred to as the *invariant manifolds* in the sequel. $\hat{G}$ and G are, thus, local parameterizations of the invariant manifolds $\hat{\mathcal{G}}$ and $\mathcal{G}$, respectively.

The construction algorithm introduced in Björk and Landén (2002) is based on idea that, if we are in the case when the forward price system generated by the volatilities $\gamma : \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \to \mathbb{R}^m$ and $\sigma : \mathcal{H}_r \times \mathbb{R}_+ \to \mathbb{R}^m$ admits a FDR, we have

$$\begin{bmatrix} q \\ r \end{bmatrix} = \hat{G}(Z)$$

and for

$$dZ = \hat{a}(Z)dt + \hat{b}(Z) \circ dW_t,$$

it must hold that

$$\hat{G}_\star \hat{a} = \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix} \qquad \hat{G}_\star \hat{b} = \begin{bmatrix} \gamma \\ \sigma \end{bmatrix}. \tag{6.20}$$

Equivalently for Markovian forward prices, if forward price model generated by $\gamma : \mathcal{H}_q \times R_+ \to R^m$ admits a FDR, then we have

$$q = G(Z)$$

and or

$$dZ = a(Z)dt + b(Z) \circ dW_t,$$

it must hold that,

$$G_\star a = \mu \qquad G_\star b = \gamma. \tag{6.21}$$

Since we can compute $\hat{G}$ and G from (6.18) and (6.19), we can solve the system (6.20) for $\hat{a}$ and $\hat{b}$, or the system (6.21) for a and b. We note that the equations in (6.20) and (6.21) do not necessarily have unique solutions, but it is enough to find *one* solution.

Note also that by solving (6.20) or (6.21), we obtain the Stratonovich dynamics of our FDR. The Itô dynamics, (which, in general, looks nicer) can be easily obtained using (1.8).

6.3.1 Strategy of analysis

In the next few sections we will address our four main problems.

In Section 6.4, we answer our *problem one* characterizing the settings when forward prices are Markovian. As it turns out, during this analysis, we will also be able to give a partial answer to *problem two.*

In Section 6.5, we study existence and construction of FDR for Markovian forward prices.

In Section 6.6, we deal with the cases when forward prices are *not* Markovian, studying existence and construction of FDR for the entire forward price system. Studying the entire system we are able to give a complete answer to *question two.*

In sections 6.5 and 6.6, *problems three and four* we will be answered following the scheme.

- Choose a number of vector fields $f_0, f_1, \cdots$ that spans the Lie-algebra we are interested on. For that purpose Lemma 1.44 is useful to help simplifying the vector fields.
- Conclude under what conditions our Lie-algebra is of finite dimension in view of Theorem 6.4.
- Assuming that those conditions hold, compute a local parameterization of the invariant manifold using Theorem 6.6.
- Given that parameterization, solve a system of equations of the type (6.20) or (6.21) to obtain the finite state variables dynamics.

6.4 On the Existence of Markovian Forward Prices

Having described the setup and the general method, we now start our analysis.

Recall that our main object of study is the forward price system

$$\begin{cases} dq_t &= \left\{\mathbf{F}q_t - \frac{1}{2}\|\gamma(q_t, r_t)\|^2 - \gamma(q_t, r_t)v^\top(r_t)\right\} dt + \gamma(q_t, r_t)dW_t \\ dr_t &= \left\{\mathbf{F}r_t - \sigma(r_t)v^\top(r_t)\right\} dt + \sigma(r_t)dW_t \end{cases} \tag{6.22}$$

where $\mathbf{F} = \dfrac{\partial}{\partial x}$ and $v(x, r) = -\displaystyle\int_0^x \sigma(s, r)ds.$

Before we go on, and to exclude patholigical cases from the analysis, we need to impose a regularity condition on forward price models.

Assumption 6.7. *If $\gamma_i(q_t, r_t) \neq 0$ for some $i \in \{1, \cdots, m\}$, then the following regularity condition holds:*

$$\frac{1}{2}\|\gamma(q_t, r_t)\|^2 + \gamma(q_t, r_t)v^\top(r_t) \neq 0.$$

Given Assumption 6.7 and by mere inspection of (6.22), we see that the answer to our *first problem* – on whether forward prices can be studied without considering the interest rate equation – is *yes* if and only if the terms $\gamma(q_t, r_t)$ and $\gamma(q_t, r_t)v^\top(r_t)$ do not depend on r_t.

Remark 6.8. *The (logarithm of the) forward price equation is Markovian if and only if the mappings $\gamma : \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \longrightarrow \mathbb{R}^m$ and $\gamma v^\top : \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \longrightarrow \mathbb{R}^m$ are constant w.r.t. r.*

The first condition – that γ cannot depend on r – is quite straightforward, but let us take a moment to understand what "$\gamma v^\top$ constant w.r.t. to r" really means.

Given that γ is not dependent on r, does this mean that v must also be independent of r? The answer to this question is trivially *no*, when we take into consideration the fact that both γ and v are multidimensional. The exercise of explaining this answer, however, helps to establish crucial notation.

Recall that our m- dimensional Wiener process W drives both forward prices and interest rates, and that a multidimensional Wiener process can be seen as a vector of scalar independent Wiener processes. With this in mind, it is possible to understand that, depending on the applications, we may face all the following situations.

- The scalar Wiener processes driving interest rates are orthogonal to the scalar Wiener processes driving the forward prices;
- The scalar Wiener processes driving interest rates and forward prices are the same;
- A part of the scalar Wiener processes driving the interest rates also drives the forward prices (or vice versa);
- Interest rates and forward prices are partially driven by orthogonal scalar Wiener processes and partial driven by the same Wiener processes.

Without loss of generality, we can reorder the scalar Wiener processes inside a multidimensional Wiener process. Assumption 6.9 bellow, give us the reordering we will assume for our multidimensional Wiener process W.

Assumption 6.9. *The $\mathbb{Q}$-Wiener process, W, driving both the forward prices and the interest rates is m-dimensional, and the same for both processes. Furthermore, we suppose that W has been reordered as*

$$W = \begin{bmatrix} W^A \\ W^B \\ W^C \end{bmatrix}$$

where W^A, W^B and W^C are, possibly multidimensional, Wiener process such that

- *W^A drives only the forward prices q,*
- *W^B drives only the interest rates r,*
- *W^C drives both forward prices q and interest rates r.*

Finally we establish that $i \in A$ means " W_i is a element of W^A ", and similarly for $i \in B$ and $i \in C$.

Assumption 6.9 has obvious implications for the matrices γ and σ which become then of the following form.

$$\begin{aligned} \gamma &= \begin{bmatrix} \gamma_A & 0 & \gamma_C \end{bmatrix} \\ \sigma &= \begin{bmatrix} 0 & \sigma_B & \sigma_C \end{bmatrix} \end{aligned}$$

thus, using $v(x,r) = -\int_0^x \sigma(s,r)ds$, we have

$$v = \begin{bmatrix} 0 & v_B & v_C \end{bmatrix},$$

and

$$\gamma v^\top = \begin{bmatrix} \gamma_A & 0 & \gamma_C \end{bmatrix} \begin{bmatrix} 0 \\ v_B^\top \\ v_C^\top \end{bmatrix} = \gamma_C v_C^\top.$$

From this we see that requiring γ and $\gamma v^\top$ independent of r, is nothing but requiring that, γ_A, γ_C and v_C do not depend on r.

The important point here is that *no condition* is imposed on σ_B.

We can now restate Remark 6.8, using the notation introduced by Assumption 6.9.

Lemma 6.10. *Suppose that Assumptions 6.7 and 6.9 holds. The (logarithm of) forward prices will be Markovian if and only if the volatility mappings γ_A, γ_C and σ_C are constant w.r.t. r. No condition is imposed on σ_B.*

Proof. If γ_A, γ_C and σ_C are constant w.r.t. r, so are $\|\gamma_A\|$, $\|\gamma_C\|$, v_C and $\gamma_C v_C$. The dynamics in of q in (6.22) does not depend on r and forward prices are, thus, Markovian.

To prove the "only if" part we show that dependence of r by γ_A, γ_C or σ_C suffices, under the regularity conditions of Assumption 6.7, to guarantee non-Markovian forward prices. Suppose, first, that γ_A depends on r. Then $\|\gamma_A\|^2$ also depends on r making the forward prices non-Markovian. Suppose now that γ_C depends on r, then $\|\gamma_C\|^2$ and $\gamma_C v_c^\top$ also depend on r. Assumption 6.7 guarantees that there is no full cancelation and the forward prices are non-Markovian. Finally suppose that σ_C depends on r, then v_c depends on r (since the integral is w.r.t. the variable s and we know $\sigma_C \neq 0$). Since v_c depends on r so does $\gamma_C v_c^\top$ and the forward prices are non-Markovian. ∎

Having established conditions for the forward prices being Markovian, we can go on and try to answer our *second problem* – on whether there exist models which admit a FDR for forward prices but not for interest rates. It turns out that, our unrestricted σ_B for Markovian forward prices, together with general results from the previous literature on interest rates FDR, allows us to give a partial answer already now.

From the previous literature on FDR of *interest rates* we know that only some particular functions $\sigma : \mathcal{H}_r \times \mathbb{R}_+ \longrightarrow \mathbb{R}^m$ will generate interest rate models that admit a FDR. Concretely, it is shown in Filipović and Teichmann (2003) that *every* component σ_i must be a weighted sum of quasi-exponential deterministic functions of x, weighted by scalar fields in $\mathcal{H}_r$.

Hence, the fact that σ_B is not driving the forward price equation and can be of *any* form for Markovian forward price models, tell us that, existence of FDR for Markovian forward prices is, in some sense, independent from existence of FDR for interest rates. This leads us to an early answer to our *second question.*

Remark 6.11. *As long as there are FDR for Markovian forward prices, there exist forward price models that allow for a FDR for forward prices but* not *for interest rates.*

6.5 Markovian Forward Prices

We now focus on the task of studying FDR for the forward price equation, in the special case where we have *Markovian* forward prices. Thus, in this section the following assumption holds (recall Lemma 6.10).

Assumption 6.12. *Consider Assumption 6.2 and 6.9 We, assume that the mappings γ_A, γ_C and σ_C are of the following special forms,*

$$\gamma_A(q,r,x) = \gamma_A(q,x) \qquad \gamma_C(q,r,x) = \gamma_C(q,x) \qquad \sigma_C(r,x) = \sigma_C(x).$$

Note that the specific functional form of σ_C implies we have deterministic σ_C-volatilities and we can interpret σ_i for $i \in C$ as *constant* vector fields in $\mathcal{H}_q$.

Given Assumption 6.12, the $\mathbb{Q}$-dynamics of (the logarithm) of forward prices can be written as

$$dq_t = \left\{ \mathbf{F}q_t - \frac{1}{2}\left[\|\gamma_A(q_t)\|^2 + \|\gamma_C(q_t)\|^2\right] - \gamma_C(q_t)v_C{}^\top \right\} dt + \left[\gamma_A(q_t) \quad 0 \quad \gamma_C(q_t)\right] dW_t. \tag{6.23}$$

Now, we rewrite equation (6.23) in Stratonovich form and obtain

$$\begin{aligned} dq_t \;&=\; \left\{ \mathbf{F}q_t - \frac{1}{2}\left[\|\gamma_A(q_t)\|^2 + \|\gamma_C(q_t)\|^2\right] - \gamma_C(q_t)v_C{}^\top \right\} dt - \frac{1}{2}\sum_{i\in A,C} d\langle \gamma_i(q_t), W_t^i\rangle \\ &\quad + \left[\gamma_A(q_t), \quad 0, \quad \gamma_C(q_t)\right] \circ dW_t. \end{aligned}$$

To compute the Stratonovich correction we use the infinite dimensional Itô formula (see Da Prato and Zabzcyk (1992)) to obtain

$$d\gamma_i(q_t) = (\cdots)\, dt + \gamma_i'(q_t)\gamma_i(q_t)dW_t^i \qquad i \in A, C$$

and, thus

$$d\langle \gamma_i(q_t), W_t^i\rangle = \gamma_i'(q_t)\gamma_i(q_t)dt \qquad i \in A, C.$$

Given the above computations we can write the Stratonovich dynamics of q as

$$\begin{aligned} dq_t =& \left\{ \mathbf{F}q_t - \frac{1}{2}\left[\|\gamma_A(q_t)\|^2 + \|\gamma_C(q_t)\|^2\right] - \gamma_C v_C{}^\top - \frac{1}{2}\left[\gamma_A'(q_t)\gamma_A(q_t) + \gamma_C'(q_t)\gamma_C(q_t)\right] \right\} dt \\ &+ \left[\gamma_A(q_t) \quad 0 \quad \gamma_C(q_t)\right] \circ dW_t \end{aligned}$$

where γ'_A and γ'_B denotes the Frechet derivative. The terms $\gamma'_A(q_t)\gamma_A(q_t)$ and $\gamma'_C(q_t)\gamma_C(q_t)$ should be interpreted as follows,

$$\gamma'_A(q_t)\gamma_A(q_t) = \sum_{i\in A} \gamma'_i(q_t)\gamma_i(q_t) \qquad \gamma'_C(q_t)\gamma_C(q_t) = \sum_{i\in C} \gamma'_i(q_t)\gamma_i(q_t) \; .$$

We start by studying the two easier cases:

(i) the case when γ (i.e, γ_A and γ_C) is also deterministic (σ_C is deterministic by Assumption 6.12), and

(ii) the case when γ is not deterministic, but has deterministic direction.

6.5.1 Deterministic Volatility

We first consider the case when the functions γ_A and γ_C do not depend on q, so they have the special form

$$\gamma_i(q,x) = \gamma_i(x) \qquad i \in A, C. \tag{6.24}$$

γ_i for $i \in A, C$ are, thus, *constant* vector fields in $\mathcal{H}_q$.

Recall from Assumption 6.12 that $\sigma_C(r,x) = \sigma_C(x)$.

In this case, the Stratonovich correction term is zero, and equation (??) becomes

$$dq_t(x) = \mu(q_t,x)dt + \gamma(x) \circ dW_t \tag{6.25}$$

where

$$\mu(q,x) \;=\; \mathbf{F}q - \frac{1}{2}\left[\|\gamma_A(x)\|^2 + \|\gamma_C(x)\|^2\right] - \gamma_C(x) v_C{}^{\top}(x) \tag{6.26}$$

$$\gamma(x) \;=\; \begin{bmatrix}\gamma_A(x) & 0 & \gamma_C(x)\end{bmatrix}. \tag{6.27}$$

Since this is a simple case, we choose to include all computations behind the results in the main text to exemplify the technique. In the next sections, when dealing with more complex cases, most of the computations will instead be presented in the appendix, leaving to the main text the intuition behind the results and their discussion.

Existence of a FDR

From Theorem 6.4 we know that a FDR exists if and only if

$$\dim\{\mu, \gamma_i; \quad i \in A, C\}_{LA} < \infty.$$

We, thus, need to compute the Lie-algebra, $\{\mu,\gamma\}_{LA}$. Computing the Lie brackets we have, for each i

$$\begin{aligned}
[\mu,\gamma_i] &= \mathbf{F}\gamma_i =: f_{1i} \\
[\gamma_i, f_{1i}] &= 0 \\
[\mu, f_{1i}] &= \mathbf{F}f_{1i} = \mathbf{F}^2\gamma_i =: f_{2i} \\
&\vdots
\end{aligned}$$

It follows that

$$\{\mu,\gamma\}_{LA} = \operatorname{span}\left\{\mu,\ \mathbf{F}^k\gamma_i\ ; \qquad k=0,1\cdots \quad i\in A, C\right\}. \tag{6.28}$$

Obviously, if a FDR exists, there must exist an n_i for each i, such that

$$\mathbf{F}^{n_i+1}\gamma_i = \sum_{k=0}^{n_i} c_{i,k}\mathbf{F}^k\gamma_i \tag{6.29}$$

where the $c_{i,k}$ are real numbers.

Proposition 6.13 tell us under what conditions we will have $\dim\{\mu,\gamma\}_{LA} < \infty$.

Proposition 6.13. *The (logarithm of the) forward price equation (6.25) admits a finite dimensional realization (FDR) if and only if each component of γ is quasi-exponential (QE). No functional restriction is imposed on the deterministic function σ_C, so in particular, σ_C* does not *have to be a QE function, it can be any deterministic function.*

Proof. Recall from Lemma 1.47 that γ_i solves the ODE (6.29) if and only if it is a QE function. ∎

Note that, for Markovian forward prices, the interest rate volatility plays no role in determining existence of FDR. The only restriction on interest rate volatility is that σ_C is deterministic, but that is a result of the Markovian property, not an added requirement imposed to guarantee existence. One other way to see this is to note that only γ shows up in (6.28). As we will soon see, this is specific to the totally deterministic setting.

Remark 6.14. *In the simple deterministic setting, where $\gamma : \mathbb{R}_+ \to \mathbb{R}^m$, Markovian realizations of forward prices are generated only by the volatility of forward prices γ.*

In the next example, and to stress this point, we check existence of FDR in a simple model without even specifying the deterministic function σ_C.

Example 6.15. *Assume that forward prices are driven by a one-dimensional Wiener-process W which also drives the interest rates*[7]. *Furthermore, assume that the interest rate volatility associated to W is deterministic and that we have for the forward price volatility,*

$$\gamma(x) = \gamma_C(x) = \alpha e^{-ax}$$

where $\alpha, a \in \mathbb{R}$.

In this case, we have $A \cup B = \emptyset$, $C = \{1\}$ and

$$\mathbf{F}\gamma(x) = -a\alpha e^{-ax} \qquad \Rightarrow \qquad n_1 = 0 \quad c_{1,0} = -a.$$

Hence,

$$\{\mu, \gamma\}_{LA} = \operatorname{span}\{\mu, \gamma\},$$

and the Lie-algebra $\{\mu, \gamma\}_{LA}$ has dimension two. Consequently, there exist a FDR for forward prices in this case. We will get back to this example in the construction part.

Finally, we want to make a remark on the exact dimension of the lie-algebra.

Remark 6.16. *It follows from (6.29) that*

$$\dim\{\mu, \gamma\}_{LA} = \dim\left\{\mu, \mathbf{F}^k\gamma_i\ ; \qquad k = 0, \cdots, n_i \quad i \in A, C\right\} \leq 1 + \sum_{i=1}^{m} n_i. \tag{6.30}$$

The "$\leq$" in (6.30) just reminds us that, given the possibility of Gaussian elimination, there may exist some cancelation effects.

To a better understanding of the above remark, we take the following example.

Example 6.17. *Suppose that*

$$\gamma_1(x) = e^{-bx} \qquad \gamma_2(x) = xe^{-bx}.$$

Thus, $n_1 = 0$, $n_2 = 1$, and $\dim\{\mu, \gamma_1, \gamma_2, \mathbf{F}\gamma_2\} \leq 4$.

However since $\mathbf{F}\gamma_2 = \gamma_1 - b\gamma_2$,

$$\operatorname{span}\{\mu, \gamma_1, \gamma_2, \mathbf{F}\gamma_2\} = \operatorname{span}\{\mu,\ \gamma_1,\ \gamma_2\}.$$

Hence, in this case, we actually have $\dim\{\mu,\ \gamma\}_{LA} = 3$.

[7]This does not exclude the possibility of more Wiener-processes driving only the interest rates.

Construction of FDR

We now go on to the construction of FDR, in the totally deterministic volatility setup.

First, we obtain a parameterization G of the invariant manifold $\mathcal{G}$. In this case we have that

$$\{\mu, \gamma\}_{LA} = \text{span}\left\{\mu, \mathbf{F}^k\gamma_i\,; \quad k = 0, 1\cdots, n_i \quad i \in A, C\right\},$$

and we recall γ_i solves the ODE (6.29).

Using Theorem 6.6 we obtain G by computing the operators $e^{(\mu z_0)}$ and $e^{\left(\mathbf{F}^k\gamma_i z_{i,k}\right)}$. In order to get $\exp\{\mu z_0\}\, q_0$ we solve

$$\begin{cases} \dfrac{dy_t}{dt} &= \mu(y_t, x) \\ &= \mathbf{F}y_t + \mathcal{D}(x) \\ y_0 &= q_0 \end{cases}$$

for

$$\mathcal{D}(x) = -\frac{1}{2}\sum_{i\in A,C}\gamma_i^2(x) - \sum_{i\in C}\gamma_i(x)v_i(x). \tag{6.31}$$

Hence by Definition 1.45, we have [8]

$$\begin{aligned} e^{\mu t}q_0(x) &= e^{\mathbf{F}t}q_0(x) + \int_0^t e^{\mathbf{F}(t-s)}\mathcal{D}(x)ds \\ &= q_0(x+t) + \int_0^t \mathcal{D}(x+t-s)ds. \end{aligned}$$

To obtain the remaining operators we solve

$$\begin{cases} \dfrac{dy_t}{dt} &= \mathbf{F}^k\gamma_i \\ y_0 &= y \end{cases}.$$

Because γ does not depend on t, the solution is

$$e^{\mathbf{F}^k\gamma_i(x)t}y = y + \mathbf{F}^k\gamma_i(x)t.$$

[8]From the context, it is clear that $e^{\mathbf{F}t} : \mathcal{H}_q \to \mathcal{H}_q$. From the usual series expansion of the exponential function we have, $e^{\mathbf{F}t}f = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathbf{F}^n f$. In our case, $\mathbf{F}^n = \frac{\partial^n}{\partial x^n}$, so we have $\left[e^{\mathbf{F}t}f\right](x) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\partial^n f}{\partial x^n}(x)$, which is a Taylor expansion of f around the point x, so for analytic f we have $\left[e^{\mathbf{F}t}f\right](x) = f(x+t)$.

It follows that

$$G(z_0, z_{i,k}; i \in A, C \quad k = 0, 1, \cdots, n_i) =$$

$$\begin{aligned} &= \prod_{i \in A, C; k=0,\dots,n_i} \left(e^{\mathbf{F}^k \gamma_i(x) z_{i,k}} \right) e^{\mu(q,x) z_0} q_0 \\ &= q_0(x + z_0) + \int_0^{z_0} \mathcal{D}(x + z_0 - s) ds + \sum_{i \in A, C} \sum_{k=0}^{n_i} \mathbf{F}^k \gamma_i(x) z_{i,k} \, . \end{aligned} \tag{6.32}$$

Note that the volatility of interest rates show up (through the terms v_i for $i \in C$) only in the deterministic term $\mathcal{D}$ defined in (6.31).

We are now interested in finding a set of factors Z such that

$$q_t = G(Z_t)$$

while Z is given by a strong solution to the SDE

$$\begin{cases} dZ_t &= a(Z_t)dt + b(Z_t) \circ dW_t \\ Z_0 &= z_0 \end{cases} .$$

For that we need to find a solution to

$$G_\star a = \mu \qquad\qquad G_\star b^i = \gamma_i \qquad i \in A, C.$$

Simple computations yields

$$G'(z_0, z_{j,k}; j \in A, C \quad k = 0, 1, \cdots, n_j) \begin{pmatrix} h_0 \\ h_{1,0} \\ h_{1,1} \\ \vdots \\ h_{m,n_m} \end{pmatrix} (x)$$

$$= \left[\frac{\partial q_0}{\partial z_0}(x + z_0) + \mathcal{D}(x) + \int_0^{z_0} \frac{\partial \mathcal{D}}{\partial z_0}(x + z_0 - s) ds \right] h_0 + \sum_{i \in A, C} \sum_{k=0}^{n_j} \mathbf{F}^k \gamma_i(x) h_{i,k}.$$

We can now use the fact that $q = G(Z)$ and that γ satisfies the ODE (6.29) to get

$$\begin{aligned}
\mu(q_t, x) &= \mathbf{F}q_t + \mathcal{D}(x) \\
&= \frac{\partial}{\partial x} q_0(x+z_0) + \int_0^{z_0} \frac{\partial}{\partial x}\mathcal{D}(x+z_0-s)ds + \sum_{j\in A,C}\sum_{k=0}^{n_j} \mathbf{F}^{k+1}\gamma_j(x) z_{j,k} + \mathcal{D}(x) \\
&= \frac{\partial}{\partial x} q_0(x+z_0) + \int_0^{z_0} \frac{\partial}{\partial x}\mathcal{D}(x+z_0-s)ds + \mathcal{D}(x) + \sum_{j\in A,C}\sum_{k=1}^{n_j} \mathbf{F}^{k}\gamma_j(x) z_{j,k-1} \\
&\quad + \sum_{j\in A,C} \mathbf{F}^{n_j+1}\gamma_j(x) z_{j,n_j} \\
&= \frac{\partial}{\partial x} q_0(x+z_0) + \int_0^{z_0} \frac{\partial}{\partial x}\mathcal{D}(x+z_0-s)ds + \mathcal{D}(x) + \sum_{j\in A,C}\sum_{k=1}^{n_j} \mathbf{F}^{k}\gamma_j(x) z_{j,k-1} \\
&\quad + \sum_{j\in A,C}\sum_{k=0}^{n_j} c_{j,k}\mathbf{F}^{k}\gamma_j(x) z_{j,n_j}.
\end{aligned}$$

Thus, from $G_\star a = \mu$, we get

$$\begin{aligned}
&\left[\frac{\partial q_0}{\partial z_0}(x+z_0) + \mathcal{D}(x) + \int_0^{z_0} \frac{\partial \mathcal{D}}{\partial z_0}(x+z_0-s)ds\right] a_0 + \sum_{j\in A,C}\sum_{k=0}^{n_j} \mathbf{F}^{k}\gamma_j(x) a_{j,k} = \\
&= \frac{\partial}{\partial x} q_0(x+z_0) + \int_0^{z_0} \frac{\partial}{\partial x}\mathcal{D}(x+z_0-s)ds + \mathcal{D}(x) + \sum_{j\in A,C}\sum_{k=1}^{n_j} \mathbf{F}^{k}\gamma_j(x) z_{j,k-1} \\
&+ \sum_{j\in A,C}\sum_{k=0}^{n_j} c_{j,k}\mathbf{F}^{k}\gamma_j(x) z_{j,n_j}
\end{aligned}$$

and, since the above expression should hold for all x, while a is not allowed to depend on x, it is possible to identify the following expressions for a

$$\begin{aligned}
a_0 &= 1 \\
a_{j,0} &= c_{j,0} z_{j,n_j}, && j \in A, C \\
a_{j,k} &= c_{j,k} z_{j,n_j} + z_{j,k-1} && j \in A, C \quad k = 1, \cdots, n_j.
\end{aligned}$$

Likewise, from $G_\star b^i = \gamma_i$ for each $i \in A, C$ we get

$$\left[\frac{\partial q_0}{\partial z_0}(x+z_0) + \mathcal{D}(x) + \int_0^{z_0} \frac{\partial \mathcal{D}}{\partial z_0}(x+z_0-s)ds\right] b_0 + \sum_{i\in A,C}\sum_{k=0}^{n_j} \mathbf{F}^{k}\gamma_j(x) b^i_{j,k} = \gamma_i(x)$$

and by simple identification of terms

$$\begin{aligned}
b^i_0 &= 0 && i \in A, C \\
b^i_{j,0} &= 1 && i = j \quad i, j \in A, C \\
b^i_{j,k} &= 0 && \text{for all other } j \text{ and } k.
\end{aligned}$$

In this totally deterministic setting, the Stratonovich and the Itô dynamics are equivalent, so we have proved the following result.

Proposition 6.18. *Given the initial forward price curve q_0, the forward prices system generated by γ_A, γ_C and σ_C has a finite dimensional realization given by*

$$q_t = G(Z_t)$$

where G is defined as in (6.32) and the dynamics of the state space variables Z are given by

$$\begin{cases} dZ_0 &= dt & \\ dZ_{j,0} &= c_{j,0} Z_{j,n_j} dt + dW_t^j & j \in A, C \\ dZ_{j,k} &= \left(c_{j,k} Z_{j,n_j} + Z_{j,k-1}\right) dt & j \in A, C \ k = 1, \cdots, n_j \end{cases}$$

We first take the easiest example: the one-dimensional deterministic constant volatility.

Example 6.19. *Assume that forward prices q are driven by a one-dimensional Wiener process that also drive the interest rates r ($C = \{1\}$). Furthermore, assume that the forward price volatility γ is of the following form*

$$\gamma(x) = \gamma_C(x) = \alpha$$

where $\alpha \in \mathbb{R}$.

We leave the (scalar) function $\sigma_C(x)$ (thus $v_C(x)$) unspecified to stress the point that it plays no role determining the dimension of the Lie-algebra or constructing the realization.

Then we know $\mathbf{F}\gamma = 0$, thus $n = 1$, $c_{1,0} = 0$ and the dimension of the $\{\mu, \gamma\}_{LA}$ is two. The invariant manifold is given by

$$G(z_0, z_{1,0}) = q_0(x + z_0) - \frac{1}{2}\alpha^2 z_0 - \alpha \int_0^{z_0} v_C(x + z_0 - s) ds + \alpha z_{1,0}$$

for some deterministic function v_C.

Using Proposition 6.18, we have $q_t = G(Z_t)$ for the state variable $Z = \begin{bmatrix} Z_0 \\ Z_{1,0} \end{bmatrix}$ with dynamics given by

$$\begin{cases} dZ_0 &= dt \\ dZ_{1,0} &= dW_t \end{cases}$$

We now recover again Example 6.15.

Example 6.15 (Cont.) *Recall that we assumed*

$$\gamma(x) = \gamma_C(x) = \alpha e^{-ax}$$

which implies $n_1 = 0$, $c_{1,0} = -a$.

Once again we leave σ_C *(thus* v_C*) as an unspecified deterministic function.*

In the previous comments it was explained that the Lie-algebra is of dimension 2, *so the invariant manifold can be obtained from (6.32),*

$$G(z_0, z_{1,0}) = q_0(x + z_0) + \int_0^{z_0} \mathcal{D}(x + z_0 - s)ds + \gamma(x)z_{1,0}.$$

In this case

$$\mathcal{D}(x) = -\frac{1}{2}\gamma_C^2(x) - \gamma_C(x)v_C(x) = -\frac{1}{2}\alpha^2 e^{-2ax} - \alpha e^{-ax}v_C(x).$$

Thus,

$$\begin{aligned} G(z_0, z_{1,0}) =& q_0(x + z_0) - \int_0^{z_0} \left(\frac{1}{2}\alpha^2 e^{-2a(x+z_0-s)} + \alpha e^{-a(x+z_0-s)}v_C(x + z_0 - s)\right) ds \\ &+ \alpha e^{-ax}z_{1,0} \\ =& q_0(x + z_0) + \frac{1}{2}\frac{\alpha^2}{2a}\left[e^{-2ax} - e^{-2a(x+z_0)}\right] \\ &- \alpha\int_0^{z_0} \left(e^{-a(x+z_0-s)}v_C(x + z_0 - s)\right) ds + \alpha e^{-ax}z_{1,0} \end{aligned}$$

and from Proposition 6.18 it follows that the FDR is given by

$$\begin{cases} dZ_0 &= dt \\ dZ_{1,0} &= -aZ_{1,0}dt + dW_t \end{cases}.$$

6.5.2 Deterministic Direction Volatility

We now deal with the second simplest case, that of having *deterministic direction* forward prices volatilities.

Then, we have the following special functional forms for γ_A and γ_C in Assumption 6.12.

$$\gamma_i(q, x) = \lambda_i(x)\varphi_i(q) \qquad i \in A, C$$

where λ_i is a deterministic function of x (constant vector field in $\mathcal{H}_q$) and φ_i is a scalar vector field in $\mathcal{H}_q$ (i.e., it does not depend on x and depends only on the current forward price curve).

Omitting the x-dependence,

$$\gamma_i(q) = \lambda_i \varphi_i(q) \qquad i \in A, C \tag{6.33}$$

and for future reference we also define

$$\lambda = \begin{bmatrix} \lambda_A, & 0, & \lambda_C \end{bmatrix}. \tag{6.34}$$

On what concerns interest rate volatilities, we maintain the requirement that σ_C is deterministic (since we are still dealing with Markovian forward prices).

In this particular case, the forward price equation can be rewritten as

$$dq_t = \mu(q_t)dt + \gamma(q_t) \circ dW_t \tag{6.35}$$

for

$$\mu(q_t) = \mathbf{F}q_t - \frac{1}{2}\left[\|\gamma_A(q_t)\|^2 + \|\gamma_C(q_t)\|^2\right] - \gamma_C(q_t)v_C{}^\top - \frac{1}{2}\left[\gamma_A'(q_t)\gamma_A(q_t) + \gamma_C'(q_t)\gamma_C(q_t)\right] \tag{6.36}$$

$$\gamma(q_t) = \begin{bmatrix} \gamma_A(q_t) & 0 & \gamma_C(q_t) \end{bmatrix},$$

and given the functional form for γ_A and γ_C in (6.33) we have the following Frechet derivatives

$$\gamma_A'(q_t)\gamma_A(q_t) = \sum_{i \in A} \lambda_i \varphi_i'(q_t)\left[\lambda_i \varphi_i(q_t)\right] \qquad \gamma_C'(q_t, x)\gamma_C(q_t, x) = \sum_{i \in C} \lambda_i(x)\varphi_i'(q_t)\left[\lambda_i \varphi_i(q_t)\right].$$

We see that μ in (6.36) is much more complex than the one previously studied (compare to (6.26)), so, the task of actually computing the Lie algebra $\mathcal{L} = \{\mu, \gamma\}_{LA}$ will not be as straightforward as before.

Using the specific functional forms of γ_A and γ_C in (6.33) we have

$$\begin{aligned}
\mu(q) &= \mathbf{F}q - \frac{1}{2}\sum_{i \in A,C} \underbrace{\lambda_i^2}_{D_i} \underbrace{\varphi_i^2(q)}_{\phi_i(q)} - \sum_{i \in C} \overbrace{\varphi_i(q)}^{\text{scalar field}} \underbrace{\lambda_i v_i}_{V_i} - \frac{1}{2}\sum_{i \in A,C} \overbrace{\varphi_i'(q)\left[\lambda_i\right]\varphi_i(q)}^{\text{scalar field}} \lambda_i \\
\gamma_i(q) &= \lambda_i \overbrace{\varphi_i(q)}^{\text{scalar field}}
\end{aligned}$$

and, given the possibility of Gaussian elimination (Lemma 1.44), we see that the Lie algebra is in fact generated by the simpler system of vector fields,

$$\begin{aligned} f_0(q) &= \mathbf{F}q - \frac{1}{2}\sum_{k\in A,C} D_k\phi_k(q) - \sum_{k\in C}\varphi_k(q)V_k \\ f_{1i}(q) &= \lambda_i \qquad\qquad\qquad\qquad\qquad\qquad i\in A,C. \end{aligned}$$

where

$$D_k = \lambda_k^2 \qquad\qquad V_k = \lambda_k v_k.$$

We now start computing Lie brackets and simplifying. For all $i \in A, C$ we have

$$\begin{aligned} [f_0, f_{1i}] &= f_0' f_{1i} - f_{1i}' f_0 \\ &= \mathbf{F}\lambda_i - \frac{1}{2}\sum_{k\in A,C} D_k \underbrace{\phi_k'[\lambda_i]}_{\text{scalar field}} - \sum_{k\in C} V_k \underbrace{\varphi_k'[\lambda_i]}_{\text{scalar field}} \\ &= f_{2i} \end{aligned}$$

and using this new field we have for all $i, j \in A, C$

$$\begin{aligned} [f_{1i}, f_{2j}] &= f_{1i}' f_{2j} - f_{2j}' f_{1i} \\ &= +\frac{1}{2}\sum_{k\in A,C} D_k \underbrace{\phi_k''[\lambda_j;\lambda_i]}_{\text{scalar field}} + \sum_{k\in C} V_k \underbrace{\varphi_k''[\lambda_j;\lambda_i]}_{\text{scalar field}} \\ &= f_{3ij}. \end{aligned}$$

We know realize that the our lie algebra is hard to handle (even in the one-dimensional Wiener process case[9]). At this point it seems a good idea to note the following.

Remark 6.20. *The Lie-algebra*

$$\mathcal{L} = \{\mu, \gamma\}_{LA} \tag{6.37}$$

is included in the larger Lie-algebra

$$\bar{\mathcal{L}} = \{\mathbf{F},\ \lambda_i,\ D_i,\ V_j\ ;\quad i\in A,C \quad j\in C\}_{LA}. \tag{6.38}$$

That is

$$\{\mu,\gamma\}_{LA} = \{f_0,\ f_{1i}\ ;\quad i\in A,C\}_{LA} \subseteq \{\mathbf{F},\ \lambda_i,\ D_i,\ V_j\ ;\quad i\in A,C \quad j\in C\}_{LA}.$$

[9]We note once again, that if we consider the Wiener process to be one-dimensional, the only interesting case to consider is when that Wiener process drives both forward prices and interest rates, i.e. it belongs to the W^C set. Otherwise, we fall into the futures case already studied in Björk, Blix, and Landén (2004). So, even in that case, we cannot avoid having two parcels not easy to simplify.

There are three important points to make here.

- The fields in the larger Lie-algebra, $\bar{\mathcal{L}}$, are simpler than those in $\mathcal{L}$. That is, none of the field contains sums.

- From the inclusion $\mathcal{L} \subseteq \bar{\mathcal{L}}$ it is obvious that if $\bar{\mathcal{L}}$ has finite dimension also $\mathcal{L}$ does. So, studying the conditions that guarantee $\bar{\mathcal{L}}$ to have finite dimension, give us, at least, *sufficient* conditions for $\mathcal{L}$ to have also finite dimension.

- We conjecture that conditions that guarantee $\bar{\mathcal{L}}$ to have finite dimension are also *necessary* conditions for $\mathcal{L}$ to have also finite dimension. The intuition is that since the fields in $\bar{\mathcal{L}}$ are all contained in the fields of $\mathcal{L}$ (as parcels of various sums), if they are "nasty" enough to make the dimension of $\bar{\mathcal{L}}$ infinite, they should make the fields that contain them in $\mathcal{L}$ even "nastier". We will formalize this intuition below.

- Finally, even if the analysis of $\bar{\mathcal{L}}$ is, in the above sense, equivalent to the analysis of $\mathcal{L}$, in the construction sense, studying $\bar{\mathcal{L}}$ will, in principle, generate finite realizations with state variables of higher dimension. This is obviously the price one has to pay for dealing with easier fields. We call these realizations *non-minimal realizations.* An advantage of *non-minimal* realizations is that they are always possible to obtain (as long as the dimension of $\bar{\mathcal{L}}$ is finite).

The following conjecture formally states the idea behind our third point above (and the sketch of the proof, for the one-dimensional case, can be found in the appendix).

Conjecture 6.21. *Consider $\mathcal{L}$ in (6.37) and $\bar{\mathcal{L}}$ in (6.38). Then the following holds*

$$\dim(\mathcal{L}) < \infty \qquad \Leftrightarrow \qquad \dim(\bar{\mathcal{L}}) < \infty \ .$$

These ideas can be applied in a more complex setting. They will be used extensively in Section 6.6, when dealing with the entire forward price system.

We now continue our analysis studying the larger Lie-algebra $\bar{\mathcal{L}}$.

Existence of FDR

As mentioned before, in the current deterministic direction setting, the larger Lie-algebra, $\bar{\mathcal{L}}$, is given by

$$\bar{\mathcal{L}} = \{\mathbf{F},\ \lambda_i,\ D_i,\ V_j\ ; \quad i \in A, C \quad j \in C\}_{LA} \cdot$$

thus, the basic fields of the enlarged Lie-algebra are

$$
\begin{aligned}
g_0(q) &:= \mathbf{F}q \\
g_{1i}(q) &:= \lambda_i \qquad\qquad i \in A, C \\
g_{2i}(q) &:= D_i \qquad\qquad i \in A, C = 1, \cdots, m \\
g_{3j}(q) &:= V_j \qquad\qquad j \in C
\end{aligned}
$$

Computing the Lie-brackets we have, in the first step,

$$
\begin{aligned}
[g_0, g_{1i}] &= \mathbf{F}\lambda_i =: g_{4i} \\
[g_0, g_{2i}] &= \mathbf{F}D_i =: g_{5i} \\
[g_0, g_{3j}] &= \mathbf{F}V_j =: g_{6j}
\end{aligned}
$$

all remaining combinations of lie-brackets from the fields in (6.38) are zero. Using the new vector fields, we easily see that

$$
\begin{aligned}
[g_0, g_{4i}] &= \mathbf{F}^2\lambda_i \\
[g_0, g_{2i}] &= \mathbf{F}^2 D_i \\
[g_0, g_{3j}] &= \mathbf{F}^2 V_j
\end{aligned}
$$

and again all others lie-brackets are zero.

Continuing with similar iterations, it is easy to check that

$$
\bar{\mathcal{L}} = \text{span}\left\{\mathbf{F},\ \mathbf{F}^k\lambda_i,\ \mathbf{F}^k D_i,\ \mathbf{F}^k V_j\ ;\quad i \in A, C \quad j \in C \quad k = 0, 1, \cdots\right\}.
$$

Thus, for $\dim(\bar{\mathcal{L}}) < \infty$ there must exist orders n_j^1, n_j^2 and n_j^3 for which

$$
\begin{aligned}
\mathbf{F}^{n_j^1+1}\lambda_j &= \sum_{k=0}^{n_j^1} c_{j,k}^1 \mathbf{F}^k \lambda_j \qquad\qquad \mathbf{F}^{n_j^2+1} D_j = \sum_{k=0}^{n_j^2} c_{j,k}^2 \mathbf{F}^k D_j \qquad\qquad j \in A, C \\
\mathbf{F}^{n_j^3+1} V_j &= \sum_{k=0}^{n_j^3} c_{j,k}^3 \mathbf{F}^k V_j \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \in C
\end{aligned}
$$

where $c_{j,k}^I$ are real constants, $k = 0, \cdots, n_j^I$, $j \in A, C$, and $I = 1, 2, 3$.

Indeed, if (6.39) holds,

$$
\dim(\bar{\mathcal{L}}) \leq 1 + \sum_{j \in A, C} (n_j^1 + 1) + (n_j^2 + 1) + \sum_{j \in C} (n_j^3 + 1) < \infty. \tag{6.39}
$$

We now start a brief side discussion on the exact dimension of $\bar{\mathcal{L}}$. As before, we note that the sign "$\leq$" above results from the possibility of Gaussian elimination across the various derivatives of various different fields. This is particularly relevant for the concrete fields of $\bar{\mathcal{L}}$ since $V_j = \lambda_j v_j$ and thus derivatives of λ_j may help to Gaussian simplify the derivatives of V_j. This point will be made clear in Example 6.24 bellow.

Unfortunately, these simplifications are instance dependent and, thus, in an abstract way it is impossible to be more exact about the dimension of $\bar{\mathcal{L}}$ than in (6.39). The consequence is that when doing the abstract construction of realizations we cannot take into account case-specific Gaussian eliminations. Therefore, when applying the abstract results to concrete models, we may get unnecessarily large realizations. However, since we are considering non-minimal realizations anyway (because we analyze $\bar{\mathcal{L}}$ instead of $\mathcal{L}$), this does not seem a major disadvantage[10]. With this we conclude the side discussion and go on with the abstract analysis.

Proposition 6.22 give us the necessary and sufficient conditions that guarantee $\dim(\bar{\mathcal{L}}) < \infty$.

Proposition 6.22. *The dimension of the Lie-algebra $\bar{\mathcal{L}}$ in (6.38) is finite if and only if each component of λ and σ_C is QE.*

Proof. It follows from Lemma 1.47 that (6.39) holds if and only if λ_i, D_i for $i \in A, C$ and V_j for $j \in C$ are QE functions. It also follows from the properties of QE functions that, λ_i for $i \in A, C$ and σ_j for $j \in C$ QE, suffices to guarantee this requirement. To check the last statement note that λ_i QE $\Rightarrow D_i = \lambda_i^2$ QE; σ_j QE, $\Rightarrow v_j(x) = -\int_0^x \sigma_j(s)ds$ QE; and finally λ_j and v_j QE $\Rightarrow V_j = \lambda_j v_j$ QE. ■

Taking together Propositions 6.22 and Conjecture 6.21 we have the following general result for forward prices with deterministic direction volatilities.

Proposition 6.23. *Assume that Conjecture 6.21 holds. The (logarithm of) forward price equation (6.35) admits a finite dimensional realization if and only if each component of λ and σ_C are quasi-exponential.*

We note that, in contrast to the deterministic forward price volatilities case, in our present setting of deterministic direction forward price volatilities, existence of FDR imposes requirements on the concrete functional form of the deterministic function σ_C: it must be a QE function. This was to be expected, since this time, σ_C actually drives the forward price equation indirectly trough the fields $V_j = \lambda_j v_j$ for $j \in C$.

[10]The unsatisfied reader can always, when faced with a concrete situation, use the techniques presented and, whenever possible, derive a smaller realization instead of using the abstract results. In Example 6.24 below, we use both approaches to exemplify the kind of difference one can expect.

The following example gives us one, very simple instance where we have a finite dimensional realization for forward prices, considering deterministic direction volatilities.

Example 6.24. *Suppose that forward prices are driven by a one-dimensional Wiener process ($i = 1$) that also drives interest rates ($i \in C$) and that*

$$\gamma(q,x) = \gamma_C(q,x) = \underbrace{\alpha e^{-bx}}_{\lambda(x)} \underbrace{q}_{\varphi(q)} \qquad \sigma_C(x) = \delta e^{-ax}.$$

for $\alpha, \beta, a, b \in \mathbb{R}$.

Then we have

$$\begin{aligned}
v(x) &= -\delta \int_0^x e^{-as} ds = \frac{\delta}{a}\left(e^{-ax} - 1\right) \\
\lambda(x) &= \alpha e^{-bx} \\
D(x) &= \lambda^2(x) = \alpha^2 e^{-2bx} \\
V(x) &= \lambda(x) v(x) = \alpha e^{-bx} \frac{\delta}{a}\left(e^{-ax} - 1\right) = \frac{\alpha\delta}{a}\left[e^{-(a+b)x} - e^{-bx}\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}\lambda(x) &= -b\alpha e^{-bx} = -b\lambda(x) && \Rightarrow\ n_1^1 = 0,\ \ c_{1,0}^1 = -b \\
\mathbf{F}D(x) &= -2b\alpha^2 e^{-2bx} = -2bD(x) && \Rightarrow\ n_1^2 = 0,\ \ c_{1,0}^2 = -2b
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}V(x) &= \tfrac{\alpha\delta}{a}\left[-(a+b)e^{-(a+b)x} - (-b)^k e^{-bx}\right] \\
&= -bV(x) - \alpha\delta e^{-(a+b)x} \\
\mathbf{F}^2 V(x) &= -b\mathbf{F}V(x) + (a+b)\alpha\delta e^{-(a+b)x} \\
&= -b\mathbf{F}V(x) + (a+b)\alpha\delta\left[-bV(x) - \mathbf{F}V(x)\right] \\
&= -(a+2b)\mathbf{F}V(x) - (a+b)bV(x) && \Rightarrow\ n_1^3 = 1,\ \ c_{1,0}^3 = -(a+b)b \\
&&& \qquad\qquad\quad c_{1,1}^3 = -(a+2b)
\end{aligned}$$

so we have

$$\{\mu, \gamma\}_{LA} \subseteq \{\mathbf{F}, \lambda, D, V, \mathbf{F}V\}_{LA}.$$

Thus, $\dim\{\mu,\gamma\}_{LA} \leq \dim\{\mathbf{F}, \lambda, D, V\}_{LA} \leq 5$.

Alternatively, we may note that

$$V(x) = \frac{\alpha\delta}{a}\left[e^{-(a+b)x} - e^{-bx}\right] = \frac{\alpha\delta}{a} e^{-(a+b)x} - \frac{\delta}{a} \underbrace{\alpha e^{-bx}}_{\lambda(x)}$$

thus, using Gaussian elimination we can substitute V by $\tilde{V}(x) = e^{-(a+b)x}$. And since

$$\mathbf{F}\tilde{V}(x) = -(a+b)e^{-(a+b)x} = -(a+b)\tilde{V}(x) \quad \Rightarrow \quad \tilde{n}_1^3 = 0 \quad \tilde{c}_{1,0}^3 = -(a+b)$$

and we can compute the exact dimension of $\{\mathbf{F}, \lambda, D, V, \mathbf{F}V\}_{LA}$,

$$\dim \{\mathbf{F}, \lambda, D, V, \mathbf{F}V\}_{LA} = \dim \left\{\mathbf{F}, \lambda, D, \tilde{V}\right\}_{LA} = 4.$$

Obviously, in either way we are able to conclude that the forward prices admit a FDR.

Construction of FDR

We would now like to construct a FDR for forward prices, whenever we know that it exists. Since we are using the larger Lie-algebra $\bar{\mathcal{L}}$ and we cannot use case-specific Gaussian elimination in the general case, we are aiming to get *non-minimal* finite realizations.

As before we would like the derive a parameterization $\bar{G}$ of the invariant manifold $\bar{\mathcal{G}}$ and infer, from the functional form of that parameterization, the dynamics of the state variables.

Given the simplicity of the fields spanning $\bar{\mathcal{L}}$, it is straightforward to compute the operators:

$$e^{\mathbf{F}z_0}q_0 = q_0(x + z_0),$$

$$e^{\mathbf{F}^k\lambda_j z^1_{j,k}}q = q + \mathbf{F}^k\lambda_j z^1_{j,k}, \quad e^{\mathbf{F}^k D_j z^2_{j,k}}q = q + \mathbf{F}^k D_j z^2_{j,k} \qquad j \in A, C; \quad k = 0, 1, \cdots, n_j$$

$$e^{\mathbf{F}^k V_j z^3_{j,k}}q = q + \mathbf{F}^k V_j z^3_{j,k} \qquad j \in C; \qquad k = 0, 1, \cdots, n_j.$$

Thus,

$$\begin{aligned} \bar{G}(z_0, z^1_{j,k}, z^2_{j,k}, z^3_{j,k}) &= \prod_{j\in A,C} e^{\mathbf{F}^k\lambda_j(x)z^1_{j,k}} \cdot \prod_{j\in A,C} e^{\mathbf{F}^k D_j(x)z^2_{j,k}} \cdot \prod_{j\in C} e^{\mathbf{F}^k V_j(x)z^3_{j,k}} \\ &= q_0(x + z_0) + \sum_{j\in A,C}\sum_{k=0}^{n^1_j} \mathbf{F}^k\lambda_j z^1_{j,k} + \sum_{j\in A,C}\sum_{k=0}^{n^2_j} \mathbf{F}^k D_j z^2_{j,k} \\ &\quad + \sum_{j\in C}\sum_{k=0}^{n^3_j} \mathbf{F}^k V_j z^3_{j,k}. \end{aligned} \tag{6.40}$$

Hence, in order to find the dynamics of the state variable Z such that $q = \bar{G}(Z)$, and as in the previous section, we take the Stratonovich $\mathbb{Q}$-dynamics to be given by

$$\left\{ \begin{array}{rcl} dZ_t &=& \bar{a}(Z_t)dt + \bar{b}(Z_t) \circ dW_t \\ Z_0 &=& z_0 \end{array} \right. , \tag{6.41}$$

and we solve

$$\bar{G}_\star \bar{a} = \mu \qquad \bar{G}_\star \bar{b} = \gamma \tag{6.42}$$

with μ and γ given in (6.33)-(6.36), and $\bar{G}$ from (6.40), to get a strong solution to the SDE (6.41).

The steps are then the usual ones, but with much more messy computations. Using the functional form of $\bar{G}$ in (6.40), it is possible to compute the Frechet derivatives. Then, from the expression for $q = \bar{G}(Z)$ and equations (6.39), we can find an concrete expression to the term $\mathbf{F}q$ in our μ (equation (6.36)). Identification of term in equations (6.42) allows us to determine the Stratonovich drift and diffusion terms. And finally, to obtain the Itô dynamics we calculate the Stratonovich correction term. Following these steps gives us the result in Proposition 6.25 (the actual computations of can be found in the appendix).

Proposition 6.25. *Given the initial forward price curve q_0, the forward prices system generated by γ_A, γ_C as in (6.33) and σ_C deterministic, has a finite dimensional realization given by*

$$q_t = \bar{G}(Z_t)$$

where $\bar{G}$ is defined as in (6.40) and the dynamics of the state space variables Z are given by

$$\left\{\begin{array}{rcllr}
dZ_0 & = & dt & & \\
dZ^1_{j,0} & = & c^1_{j,0} Z^1_{j,n^1_j} dt + \varphi(\bar{G}(Z)) dW^j_t & j \in A, C & \\
dZ^1_{j,k} & = & \left(Z^1_{j,k-1} + c^1_{j,k} Z^1_{j,n^1_j}\right) dt & j \in A, C & k = 1, \cdots, n^1_j \\
dZ^2_{j,0} & = & \left(c^2_{j,0} Z^2_{j,n^2_j} - \frac{1}{2}\varphi^2_j(\bar{G}(Z))\right) dt & j \in A, C & \\
dZ^2_{j,k} & = & \left(Z^2_{j,k-1} + c^2_{j,k} Z^2_{j,n^2_j}\right) dt & j \in A, C & k = 1, \cdots, n^2_j \\
dZ^3_{j,0} & = & \left(c^3_{j,0} Z^3_{j,n^3_j} + \varphi(\bar{G}(zZ))\right) dt & j \in C & \\
dZ^3_{j,k} & = & \left(Z^3_{j,k-1} + c^3_{j,k} Z^3_{j,n^3_j}\right) dt & j \in C & k = 1, \cdots, n^3_j
\end{array}\right.$$

Example 6.24 (cont.) *Recall that we studied an one-dimensional model where $A \cap B = \emptyset$, $C = \{1\}$ and*

$$\gamma(q,x) = \gamma_C(q,x) = \underbrace{\alpha e^{-bx}}_{\lambda(x)} \underbrace{q}_{\varphi(q)} \qquad \sigma_C(x) = \delta e^{-ax}.$$

As in the first part of the example, we will first directly apply the abstract results (in the construction part, that is Proposition 6.25). Then we derive a smaller realization that can be obtained from the case-specific simpler fields.

We start by directly applying Proposition 6.25. Recall from previous computations that we had $n_1^1 = 0$, $c_{1,0}^1 = -b$, $n_1^2 = 0$, $c_{1,0}^2 = -2b$, $n_1^3 = 1$, $c_{1,0}^3 = -(a+b)b$ *and* $c_{1,1}^3 = -(a+2b)$ *and*

$$V(x) = \frac{\alpha\delta}{a}\left[e^{-(a+b)x} - e^{-bx}\right] \quad \mathbf{F}V(x) = \frac{\alpha\delta}{a}\left[-(a+b)e^{-(a+b)x} - (-b)^k e^{-bx}\right].$$

Using this we get from (6.40) the parameterization of the realization to be

$$\begin{aligned}\bar{G}(z_0, z_{1,0}^1, z_{1,0}^2, z_{1,0}^3, z_{1,1}^3) =& q_0(x+z_0) + \lambda(x)z_{1,0}^1 + D(x)z_{1,0}^2 + V(x)z_{1,0}^3 + \mathbf{F}V(x)z_{1,1}^3 \\ =& q_0(x+z_0) + \alpha e^{-bx}z_{1,0}^1 + \alpha^2 e^{-2bx}z_{1,0}^2 + \frac{\alpha\delta}{a}\left[e^{-(a+b)x} - e^{-bx}\right]z_{1,0}^3 \\ &+ \frac{\alpha\delta}{a}\left[-(a+b)e^{-(a+b)x} - (-b)^k e^{-bx}\right]z_{1,1}^3.\end{aligned}$$

We now note that in our case $\varphi(q) = q$ *and by Proposition 6.25 it follows that the realization is*

$$\begin{cases} dZ_0 &= dt \\ dZ_{1,0}^1 &= -bZ_{1,0}^1 dt + \bar{G}(Z)dW_t \\ dZ_{1,0}^2 &= \left(-2bZ_{1,0}^2 - \frac{1}{2}(\bar{G}(Z))^2\right)dt \\ dZ_{1,0}^3 &= \left(-b(a+b)Z_{1,1}^3 + \bar{G}(Z)\right)dt \\ dZ_{1,1}^3 &= \left(Z_{1,0}^3 - (a+2b)Z_{1,1}^3\right)dt \end{cases}$$

Now recall that for this particular model we have $\tilde{\mathcal{L}} = \left\{\mathbf{F}, \lambda, D, \tilde{V}\right\}_{LA}$ *with* $\tilde{V}(x) = e^{-(a+b)x}$. *Thus another (smaller) parameterization,* $\tilde{G}$ *is given by*

$$\begin{aligned}\tilde{G}(\tilde{z}_0, \tilde{z}_{1,0}^1, \tilde{z}_{1,0}^2, \tilde{z}_{1,0}^3) &= q_0(x+\tilde{z}_0) + \lambda(x)\tilde{z}_{1,0}^1 + D(x)\tilde{z}_{1,0}^2 + \tilde{V}(x)\tilde{z}_{1,0}^3 \\ &= q_0(x+\tilde{z}_0) + \alpha e^{-bx}\tilde{z}_{1,0}^1 + \alpha^2 e^{-2bx}\tilde{z}_{1,0}^2 + e^{-(a+b)x}\tilde{z}_{1,0}^3.\end{aligned}$$

Proposition 6.25 cannot be used directly, but we can compare the two parameterizations $\bar{G}$ *and* $\tilde{G}$ *(of the same invariant manifold) above, to get*

$$\tilde{z}_0 = z_0 \quad \tilde{z}_{1,0}^1 = z_{1,0}^1 - \frac{\delta}{a}z_{1,0}^3 + b\frac{\delta}{a}z_{1,1}^3 \quad \tilde{z}_{1,0}^2 = z_{1,0}^2 \quad \tilde{z}_{1,0}^3 = \frac{\alpha\delta}{a}z_{1,0}^3 - \frac{\alpha\delta}{a}(a+b)z_{1,1}^3.$$

Finally using Itô and simplifying we have

$$\begin{cases} d\tilde{Z}_0 &= dt \\ d\tilde{Z}^1_{1,0} &= \left(-b\tilde{Z}^1_{1,0} - \frac{\delta}{a}\tilde{G}(Z)\right) dt + \tilde{G}(Z)dW_t \\ d\tilde{Z}^2_{1,0} &= \left(-2b\tilde{Z}^2_{1,0} - \frac{1}{2}(\tilde{G}(Z))^2\right) dt \\ d\tilde{Z}^3_{1,0} &= \left(-(a+b)\tilde{Z}^3_{1,0} + \frac{\alpha\delta}{a}\tilde{G}(Z)\right) dt \end{cases}$$

The two realizations are equivalent.

6.5.3 The General Case

In the sections 6.5.1 and 6.5.2, we analyzed existence and construction of FDR of Markovian forward prices, under the specific setting of *deterministic* and *deterministic direction* forward price volatility.

A natural question at this point is: *what is the most general functional form, for the volatilities γ and σ, consistent with FDR of Markovian forward prices?* The answer follows from previous results in Filipović and Teichmann (2003). In the following proposition we adapt it to the Markovian forward prices case. We state it in the form of a proposition.

Proposition 6.26. *Suppose Assumption 6.2 holds. There exist a FDR for Markovian forward prices if and only if*

$$\gamma_j(q,r,x) = \sum_{k=0}^{N^j_\gamma} \varphi^j_k(q)\lambda^j_k(x) \qquad j \in A, C$$

$$\sigma_j(r,x) = \begin{cases} \beta_j(x) & \text{if } \gamma_j(q,r,x) = \gamma_j(x) \\ \\ \delta_j(x) & \text{if } \gamma_j(q,r,x) = \gamma_j(q,x) \end{cases} \qquad j \in C$$

where β_j are unrestricted deterministic functions, δ_j,λ^j_k are QE deterministic functions, and φ^j_k are scalar vector fields in $\mathcal{H}_q$.

From Proposition 6.26, we see the most general situation can be attained by extending *deterministic direction* forward price volatilities to finite sums of deterministic direction parcels. This represents, of course, a relevant extension in terms of model flexibility, but not in terms of complexity of analysis.

The results from section 6.5.2 extend naturally to this most general case, the computations are exactly the same and, in concrete applications, easy to derive. In abstract terms, however, computation get much messier given the additional indices one must keep track of[11].

We now will consider the case when forward prices are *not* Markovian.

6.6 Non-Markovian Forward Prices

Recall that under Assumption 6.2 – our basic assumption on the volatility processes for forward prices and interest rates γ and σ, respectively – the (logarithm of the) forward price curve q cannot, in general, be studied without incorporating in the analysis the interest rate curve r (recall (6.12)-(6.13)).

In this section, we want to study the circumstances which were not covered by Section 6.5. In that case our forward price model is a doubly infinite system and we set

$$\hat{q} = \begin{bmatrix} q \\ r \end{bmatrix}.$$

and $\hat{q}$ belongs to $\mathcal{H}_q \times \mathcal{H}_r$.

The Itô dynamics of $\hat{q}$ can, thus, also be written in block matrix notation as

$$d\begin{bmatrix} q_t \\ r_t \end{bmatrix} = \left\{ \mathbf{F}\begin{bmatrix} q_t \\ r_t \end{bmatrix} - \frac{1}{2}\begin{bmatrix} \|\gamma(q_t, r_t)\|^2 \\ 0 \end{bmatrix} - \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix} v^\top(r_t) \right\} dt + \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix} dW_t.$$

where, as before, we take W to be an m-dimensional Wiener process, and

$$\mathbf{F} = \frac{\partial}{\partial x} \qquad\qquad v(r, x) = -\int_0^x \sigma(r, s)ds.$$

In this case the Stratonovich correction term is given by

$$-\frac{1}{2}d\langle \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix}, W_t\rangle = -\frac{1}{2}\sum_{i=1}^{m} d\langle \begin{bmatrix} \gamma_i(q_t, r_t) \\ \sigma_i(r_t) \end{bmatrix}, W_t^i\rangle$$

Since we have (from the infinite Itô formula)

$$d\begin{bmatrix} \gamma_i(q_t, r_t) \\ \sigma_i(r_t) \end{bmatrix} = (\cdots)dt + \begin{bmatrix} \gamma'_{iq}(q, r) & \gamma'_{ir}(q, r) \\ 0 & \sigma'_{ir}(r) \end{bmatrix} \begin{bmatrix} \gamma_i(q_t, r_t) \\ \sigma_i(r_t) \end{bmatrix} dW_t$$

[11]We do not present the abstract results and derivations, as we believe the reader would spend more time understanding the notation, than extending the results of section 6.5.2 to concrete, slightly more general, applications.

with γ'_q, γ'_r and σ'_r the partial Frechet derivatives.

Then for $i = 1, \cdots, m$ we have

$$d\langle \begin{bmatrix} \gamma_i(q_t, r_t) \\ \sigma_i(r_t) \end{bmatrix}, W_t^i \rangle = \begin{bmatrix} \gamma'_{iq}(q_t, r_t) & \gamma'_{ir}(q_t, r_t) \\ 0 & \sigma'_{ir}(r_t) \end{bmatrix} \begin{bmatrix} \gamma_i(q_t, r_t) \\ \sigma_i(r_t) \end{bmatrix} dt$$

and the Stratonovich dynamics for $\hat{q}$,

$$d\hat{q}_t = \mu(q_t, r_t)dt + \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix} \circ dW_t$$

$$\mu(q,r) = \mathbf{F} \begin{bmatrix} q \\ r \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \|\gamma(q,r)\|^2 \\ 0 \end{bmatrix} - \begin{bmatrix} \gamma(q,r) \\ \sigma(r) \end{bmatrix} v^\top(r) - \frac{1}{2} \begin{bmatrix} \gamma'_q(q,r) & \gamma'_r(q,r) \\ 0 & \sigma'_r(r) \end{bmatrix} \begin{bmatrix} \gamma(q,r) \\ \sigma(r) \end{bmatrix}.$$

Given that

$$\gamma(q,r) = \begin{bmatrix} \gamma_A(q,r) & 0 & \gamma_C(q,r) \end{bmatrix} \qquad \sigma(r) = \begin{bmatrix} 0 & \sigma_B(r) & \sigma_C(r) \end{bmatrix}$$

we note that

$$\begin{aligned} \begin{bmatrix} \gamma(q,r) \\ \sigma(r) \end{bmatrix} v(r)^\top &= \begin{bmatrix} \gamma_A(q,r) & 0 & \gamma_C(q,r) \\ 0 & \sigma_B(r) & \sigma_C(r) \end{bmatrix} \begin{bmatrix} 0 \\ v_B(r)^\top \\ v_C(r)^\top \end{bmatrix} \\ &= \begin{bmatrix} \gamma_C(q,r) v_c^\top(r) \\ \sigma_B(r) v_c^\top(r) + \sigma_C(r) v_c^\top(r) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \gamma'_q(q,r) & \gamma'_r(q,r) \\ 0 & \sigma'_r(r) \end{bmatrix} \begin{bmatrix} \gamma(q,r) \\ \sigma(r) \end{bmatrix} &= \begin{bmatrix} \gamma'_{A_q}(q,r) & \gamma'_{A_r}(q,r) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_A(q,r) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma'_{B_r} \end{bmatrix} \begin{bmatrix} 0 \\ \sigma_B(r) \end{bmatrix} \\ &\quad + \begin{bmatrix} \gamma_{C'_q}(q,r) & \gamma_{C'_r}(q,r) \\ 0 & \sigma_{C'_r}(r) \end{bmatrix} \begin{bmatrix} \gamma_C(q,r) \\ \sigma_C(r) \end{bmatrix} \\ &= \begin{bmatrix} \gamma_{A'_q}(q,r)\gamma_A(q,r) + \gamma_{C'_q}(q,r)\gamma_C(q,r) + \gamma_{C'_r}(q,r)\sigma_C(r) \\ \sigma_{B'_r}(r)\sigma_B(r) + \sigma_{C'_r}(r)\sigma_C(r) \end{bmatrix} \end{aligned}$$

where

$$\gamma_{A'_q}(q,r)\gamma_A(q,r) = \sum_{i\in A}\gamma_{A'_{iq}}(q,r)\gamma_{Ai}(q,r),$$

$$\gamma'_{C_r}(q,r)\sigma_C(r) = \sum_{i\in C}\gamma_{C'_{ir}}(q,r)\sigma_{Ci}(r), \qquad \gamma'_{C_q}(q,r)\gamma_C(q,r) = \sum_{i\in C}\gamma_{C'_{iq}}(q,r)\gamma_{Ci}(q,r),$$

$$\sigma'_{B_r}(r)\sigma_B(r) = \sum_{i\in B}\sigma_{B'_{ir}}(r)\sigma_{Bi}(r), \qquad \sigma'_{C_r}(r)\sigma_C(r) = \sum_{i\in C}\sigma_{C'_{ir}}(r)\sigma_{Ci}(r).$$

We can, finally, identify our main object of study as the following doubly infinite Stratonovich SDE

$$d\hat{q}_t = \mu(q_t,r_t)dt + \begin{bmatrix}\gamma(q_t,r_t)\\ \sigma(r_t)\end{bmatrix} \circ dW_t$$

where

$$\begin{aligned}
\mu(q,r) &= \mathbf{F}\begin{bmatrix}q\\ r\end{bmatrix} - \frac{1}{2}\begin{bmatrix}\|\gamma_A(q,r)\|^2 + \|\gamma_C(q,r)\|^2\\ 0\end{bmatrix} - \begin{bmatrix}\gamma_C(q,r)v_C^\top(r)\\ \sigma_B(r)v_B^\top(r) + \sigma_B(r)v_B^\top(r)\end{bmatrix}\\
&\quad -\frac{1}{2}\begin{bmatrix}\gamma_{A'_q}(q,r)\gamma_A(q,r) + \gamma_{C'_q}(q,r)\gamma_C(q,r) + \gamma_{C'_r}(q,r)\sigma_C(r)\\ \\ \sigma_{B'_r}(r)\sigma_B(r) + \sigma_{C'_r}(r)\sigma_C(r)\end{bmatrix}\\
\begin{bmatrix}\gamma(q,r)\\ \sigma(r)\end{bmatrix} &= \begin{bmatrix}\gamma_A(q,r) & 0 & \gamma_C(q,r)\\ 0 & \sigma_B(r) & \sigma_C(r)\end{bmatrix}
\end{aligned} \tag{6.43}$$

Given the general functional forms of γ_A, γ_C, σ_B and σ_C the study of all possible special cases[12] would be exhausting.

In this section, we take, therefore a more agressive strategy: we consider immediately the scenario where each element in γ and σ have *deterministic direction volatility.*

As before, the situation of deterministic direction volatilities can be extended to the case where each element of γ and σ is a finite sum of deterministic direction parcels, and that is the most general possible scenario consistent with existence of FDR[13]. We omit the analysis of this most general scenario because the results can be easily derived from the ones on deterministic direction volatilities, and, in abstract terms, notation becomes almost untractable.

[12]One particular special case would be to take, say, γ_A an σ_B to have deterministic direction and γ_C, σ_C to be deterministic.

[13]For details on why this is the most general scenario we refer to Filipović and Teichmann (2003).

The deterministic direction setting we will work with is formally stated by the next assumption.

Assumption 6.27. *The mappings $\gamma_i : \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \to \mathbb{R}^m$ and $\sigma_i : \times\mathcal{H}_r \times \mathbb{R}_+ \to \mathbb{R}^m$ are of the following functional form.*

$$\begin{aligned} \gamma_i(x,q,r) &= \lambda_i(x)\varphi_i(q,r) \qquad & i \in A, C \\ \sigma_i(x,r) &= \beta_i(x)\phi_i(r) \qquad & i \in B, C \end{aligned} \tag{6.44}$$

where λ_i, β_i are deterministic functions of x and φ_i, ϕ_i are scalar vector fields in $\mathcal{H}_q \times \mathcal{H}_r$ (i.e., they do not depend on x and depend only on the current forward price and interest rate curves).

We note that under Assumption 6.27

$$v_i(x,r) = -\int_0^x \sigma_i(r,s)ds = -\int_0^x \beta_i(s)\phi_i(r)ds = -\phi_i(r)\int_0^x \beta_i(s)ds \qquad i \in B, C.$$

Defining

$$B_i(x) = \int_0^x \beta_i(s)ds \tag{6.45}$$

we, thus, have

$$v_i(x) = -\phi_i(r)B_i(x) \qquad i \in B, C.$$

To check if our forward prices model admits a finite dimensional realization, we need to see if

$$\dim\left\{\mu, \begin{bmatrix}\gamma(q,r)\\ \sigma(r)\end{bmatrix}\right\}_{LA} < \infty.$$

Considering (6.43), under Assumption 6.27, our basic vector fields can be written as

$$\begin{aligned} \mu(q,r) \;=\; & \mathbf{F}\begin{bmatrix}q\\ r\end{bmatrix} - \frac{1}{2}\sum_{i\in A,C}\varphi_i^2(q,r)\begin{bmatrix}\lambda_i^2\\ 0\end{bmatrix} + \sum_{i\in C}\varphi_i(q,r)\phi_i(r)\begin{bmatrix}\lambda_i B_i\\ 0\end{bmatrix} + \sum_{i\in B,C}\phi_i^2(r)\begin{bmatrix}0\\ \beta_i B_i\end{bmatrix} \\ & -\frac{1}{2}\left\{\sum_{i\in A,C}\varphi_{iq}'(q,r)\,[\lambda_i]\,\varphi_i(q,r)\begin{bmatrix}\lambda_i\\ 0\end{bmatrix}\right. \\ & \left. + \sum_{i\in C}\varphi_{ir}'(q,r)\,[\beta_i]\,\phi_i(r)\begin{bmatrix}\lambda_i\\ 0\end{bmatrix} + \sum_{i\in B,C}\phi_{ir}'(r)\,[\beta_i]\,\phi_i(r)\begin{bmatrix}0\\ \beta_i\end{bmatrix}\right\}, \end{aligned}$$

$$\begin{bmatrix} \gamma_i(q,r) \\ \sigma_i(r) \end{bmatrix} = \begin{cases} \varphi_i(q,r) \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} & i \in A \\ \phi_i(r) \begin{bmatrix} 0 \\ \beta_i \end{bmatrix} & i \in B \\ \varphi_i(q,r) \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} + \phi_i(r) \begin{bmatrix} 0 \\ \beta_i \end{bmatrix} & i \in C. \end{cases}$$

Following the strategy described in Section 6.3.1, we would now start computing Lie brackets of all possible combinations of these fields and, through Gaussian elimination, hopefully, get to a simple set of generators of our Lie-algebra $\mathcal{L} = \{\mu, \delta\}_{LA}$. Based on properties of these generators we would, also hopefully, be able to understand which γ and σ would guarantee a FDR for forward prices.

The particular complex expression for μ above, and the almost impossibility of Gaussian elimination that results from having to handle two infinite SDE at the same time[14], leads to the conclusion that our best hope is again to study a larger Lie-algebra, $\bar{\mathcal{L}}$, and to choose such a Lie-algebra so that the basic fields would be simple.

The following Lemma give us the desired (simple enough) Lie-algebra $\bar{\mathcal{L}}$.

Lemma 6.28. *Consider the following set of fields in $\mathcal{H}$.*

$$\begin{aligned} f^0 &= \begin{bmatrix} \mathbf{F}q \\ \mathbf{F}r \end{bmatrix}, \\ f_i^1 &= \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix}, \quad f_i^2 = \begin{bmatrix} D_i \\ 0 \end{bmatrix}, && i \in A, C \\ f_i^3 &= \begin{bmatrix} V_i \\ 0 \end{bmatrix}, && i \in C \\ f_i^4 &= \begin{bmatrix} 0 \\ \beta_i \end{bmatrix}, \quad f_i^5 = \begin{bmatrix} 0 \\ H_i \end{bmatrix} && i \in B, C \end{aligned} \tag{6.46}$$

[14]To the usual complexity of dealing with multidimensional cases, there is an additional complexity specific of forward price models that results from the fact that $\varphi_i(q,r) \neq \phi_i(r)$. However even under the unrealistic assumption (since the forward price volatility could not depend on the forward prices) where we would assume $\varphi_i(r) = \phi_i(r)$, the complexity of the Stratonovich correction term would not allow us to obtain simple generators for $\mathcal{L}$.

where λ_i, β_i and B_i are deterministic functions of x and defined as in (6.44) and (6.45) and we further define

$$D_i(x) = \lambda_i^2(x) \qquad V_i(x) = \lambda_i(x)B_i(x) \qquad H_i(x) = \beta_i(x)B_i(x). \tag{6.47}$$

Then the following holds

$$\mathcal{L} = \{\mu, \delta\}_{LA} \quad \subseteq \quad \bar{\mathcal{L}} = \left\{ \mathbf{F}, \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix}, \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \begin{bmatrix} V_j \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \beta_n \end{bmatrix}, \begin{bmatrix} 0 \\ H_n \end{bmatrix}; \begin{matrix} i \in A, C, j \in C, \\ n \in B, C \end{matrix} \right\}_{LA}$$

Proof. First, note that

$$\begin{aligned} \varphi_i(q,r), \quad \varphi_i^2(q,r), \quad \varphi'_{iq}(q,r)\,[\lambda_i]\,\varphi_i(q,r) \qquad & i \in A, B \\ \phi_i(r), \quad \phi_i^2(r), \quad \phi'_{ir}(r)\,[\beta_i]\,\phi_i(r) \qquad & i \in B, C \\ \varphi_i(q,r)\phi_i(r) \qquad & i \in C \end{aligned}$$

are scalar fields in $\mathcal{H}_q \times \mathcal{H}_r$. The conclusion now follows using Gaussian Elimination (Lemma 1.44). ■

6.6.1 Existence of FDR

Computing Lie-brackets on the basic fields of $\bar{\mathcal{L}}$ is not hard, and the conclusion on the existence of a FDR becomes a straightforward generalization of the easier setups studied in previous sections. Proposition 6.29 give us the needed conditions.

Proposition 6.29. *The lie-algebra $\bar{\mathcal{L}}$ is spanned by*

$$\operatorname{span}\left\{ \mathbf{F}, \mathbf{F}^k \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix}, \mathbf{F}^k \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \mathbf{F}^k \begin{bmatrix} V_j \\ 0 \end{bmatrix}, \mathbf{F}^k \begin{bmatrix} 0 \\ \beta_n \end{bmatrix}, \mathbf{F}^k \begin{bmatrix} 0 \\ H_n \end{bmatrix}; \begin{matrix} i \in A, C;\ j \in C \\ n \in B, C \end{matrix}; k = 0, 1, \cdots \right\}$$

and will have a finite dimension if and only if each component of λ and β is QE.

Moreover, under those conditions also each component of D, V, W in (6.47) are QE and

$$\dim\{\bar{\mathcal{L}}\} \leq 1 + \sum_{j \in A,C} (n_j^1 + n_j^2) + \sum_{j \in C} n_j^3 + \sum_{j \in B,C} (n_j^4 + n_j^5) \tag{6.48}$$

for $n_j^i \in \mathbb{N}$, such that,

$$\begin{aligned}
&\mathbf{F}^{n_j^1+1}\lambda_j = \sum_{k=1}^{n_j^1} c_{j,k}^1 \mathbf{F}^k \lambda_j && \mathbf{F}^{n_j^2+1}D_j = \sum_{k=1}^{n_j^2} c_{j,k}^2 \mathbf{F}^k D_j && j \in A, C \\
&\mathbf{F}^{n_j^3+1}V_j = \sum_{k=1}^{n_j^3} c_{j,k}^3 \mathbf{F}^k V_j && && j \in C \qquad (6.49)\\
&\mathbf{F}^{n_j^4+1}\beta_j = \sum_{k=1}^{n_j^4} c_{j,k}^4 \mathbf{F}^k \beta_j && \mathbf{F}^{n_j^5+1}H_j = \sum_{k=1}^{n_j^5} c_{j,k}^5 \mathbf{F}^k H_j && j \in B, C
\end{aligned}$$

hold for some $c_{j,k}^I \in \mathbb{R}$ with $k = 0, 1, \cdots, n_j^I$, $j = 1, \cdots, m$, and $I = 1, \cdots, 5$.

Proof. Given the fields of $\bar{\mathcal{L}}$, we have a FDR if and only if (6.49) hold and it follows that (6.48) holds (the "$\leq$" in (6.48) accounts for possible case-specific Gaussian elimination across terms). Finally, (6.49) can be interpreted as ODEs whose solution are QE functions, thus λ_i, for $i \in A, C$; D_j, V_j, for $j \in C$ and H_k for $k \in B, C$ solve (6.49) if and only if they are QE. It remains to show that requiring λ_i and β_j for $i \in A, C$, $j \in B, C$ is sufficient to guarantee that. Given that

$$D_i(x) = \lambda_i^2(x) \qquad V_j(x) = \lambda_j(x) \int_0^b \beta_j(s)ds \qquad H_j(x) = \beta_j(x) \int_0^b \beta_j(s)ds,$$

the result follows from Lemma 1.47. ■

6.6.2 Construction of FDR

Knowing the conditions for existence of a FDR for forward prices, we can now construct the finite dimensional realization. Proposition 6.30 gives us a *non-minimal* (since it is based on $\bar{\mathcal{L}}$ and cannot take into account case-specific Gaussian elimination) parameterization $\hat{G}$ of the invariant manifold $\hat{\mathcal{G}}$.

Note that our parameterization, $\hat{q} = \hat{G}(Z)$, will be of the following block matrix form

$$\begin{bmatrix} q \\ r \end{bmatrix} = \begin{bmatrix} \hat{G}^q(Z) \\ \hat{G}^r(Z) \end{bmatrix}.$$

Furthermore, by close inspection of (6.46) we realize that the operator generated by f^0,

$$e^{\mathbf{F}z_0} \begin{bmatrix} q_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} \tilde{q}_0 \\ \tilde{r}_0 \end{bmatrix},$$

is the only that will affect both $\hat{G}^q$ and $\hat{G}^r$. The remaining operators will only affect one component at the time.

The operators generated by f_i^1,f_i^2 for $i \in A, C$ and f_j^3 for $j \in C$, will only affect $\hat{G}^q$. So,

$$\begin{aligned} e^{\mathbf{F}^k \lambda_j z^1_{j,k}} q &= q + \lambda_j z^1_{j,k} \qquad e^{\mathbf{F}^k D_j z^2_{j,k}} q = q + D^1_j z^2_{j,k} \qquad && j \in A, C \\ e^{\mathbf{F}^k V_j z^3_{j,k}} q &= q + V_j z^3_{j,k} && j \in C\ . \end{aligned}$$

On the other hand, $\hat{G}^r$ will be affected by the operators generated by f_j^4,f_j^5 for $j \in B, C$ and we have

$$e^{\mathbf{F}^k \beta_j z^4_{j,k}} r = r + \beta_j z^4_{j,k} \qquad e^{\mathbf{F}^k H_j z^5_{j,k}} r = r + H_j z^5_{j,k} \qquad j \in B, C$$

Once the parameterization has been derived, we can infer the dynamics of the finite dimensional realization, exactly as before. The actual construction of the realization, though cumbersome (and thus presented in the appendix), follow the same ideas of the constructions in previous sections.

Proposition 6.30. *Suppose Assumption 6.27 holds. Given the initial forward price curve q_0 and the initial interest rate curve r_0, the system generated by forward price and interest rate volatilities defined as in (6.44) has a finite dimensional realization, given by*

$$\begin{bmatrix} q_t \\ r_t \end{bmatrix} = \hat{G}(Z)$$

where $\hat{G}$ is defined by

$$\hat{G}(z_0, z^1_{j,k}, z^2_{j,k}, z^3_{j,k}, z^4_{j,k}, z^5_{j,k}) =$$

$$= \begin{bmatrix} \tilde{q}_0 + \sum_{j\in A,C} \sum_{k=0}^{n^1_j} \mathbf{F}^k \lambda_j z^1_{j,k} + \sum_{j\in A,C} \sum_{k=0}^{n^2_j} \mathbf{F}^k D_j z^2_{j,k} + \sum_{j\in C} \sum_{k=0}^{n^3_j} \mathbf{F}^k V_j z^3_{j,k} \\ \tilde{r}_0 + \sum_{j\in B,C} \sum_{k=0}^{n^4_j} \mathbf{F}^k \beta_j z^4_{j,k} + \sum_{j\in B,C} \sum_{k=0}^{n^5_j} \mathbf{F}^k H_j z^5_{j,k} \end{bmatrix} \tag{6.50}$$

for $\tilde{q}_0(x) = q_0(x + z_0)$ and $\tilde{q}_0(x) = q_0(x + z_0)$.

Moreover, the dynamics of the state space variables Z are given by

$$\begin{cases} dZ_0 &= dt & & \\ dZ^1_{j,0} &= c^1_{j,0} Z^1_{j,n^1_j} dt + \varphi_j(\hat{G}^q(Z), \hat{G}^r(Z)) dW^j_t & j \in A, C & \\ dZ^1_{j,k} &= \left\{ Z^1_{j,k-1} + c^1_{j,k} Z^1_{j,n^1_j} \right\} dt & j \in A, C & k = 1, \cdots, n^1_j \\ dZ^2_{j,0} &= \left\{ c^2_{j,0} Z^2 j, n^2_j - \frac{1}{2}\varphi^2_j(\hat{G}^q(Z), \hat{G}^r(Z)) \right\} dt & j \in A, C & \\ dZ^2_{j,k} &= \left\{ Z^2_{j,k-1} + c^2_{j,k} Z^2_{jn^2_j} \right\} dt & j \in A, C & k = 1, \cdots, n^2_j \\ dZ^3_{j,0} &= \left\{ c^3_{j,0} z^3_{j,n^3_j} + \varphi(\hat{G}^q(Z), \hat{G}^r(Z)) \phi(\hat{G}^r(Z)) \right\} dt & j \in C & \\ dZ^3_{j,k} &= \left\{ Z^3_{j,k-1} + c^3_{j,k} Z^3_{j,n^3_j} \right\} dt & j \in C & k = 1, \cdots, n^3_j \\ dZ^4_{j,0} &= c^4_{j,0} Z^4_{j,n^4_j} dt + \phi_j((\hat{G}^r(Z)) dW^j & j \in B, C & \\ dZ^4_{j,k} &= \left\{ c^4_{j,0} Z^4_{j,n^4_j} \right\} dt & j \in B, C & k = 1, \cdots, n^4_j \\ dZ^5_{j,0} &= \left\{ c^5_{j,0} Z^5_{j,n^5_j} + \phi^2_j(\hat{G}^r(Z)) \right\} dt & j \in B, C & \\ dZ^5_{j,k} &= \left\{ Z^5_{j,k-1} + c^5_{j,k} Z^5_{j,n^5_j} \right\} dt & j \in B, C & k = 1, \cdots, n^5_j \end{cases}$$

The next example may help to understand Proposition 6.30.

6.6.3 Example

Consider a model with the following volatility matrix

$$\begin{bmatrix} \gamma(q, r) \\ \sigma(r) \end{bmatrix} = \begin{bmatrix} \alpha e^{-bx} q & \rho \\ 0 & \delta e^{-ax} \sqrt{r} \end{bmatrix}.$$

Using the usual notation we have two Wiener processes, one of them of type W^A and another of type W^C. So $A = \{1\}$ and $C = \{2\}$. In this example, since the is only one element of each type, we write for the reader's convinience "$\cdot_A$" instead of "$\cdot_1$" and "$\cdot_C$" instead of "$\cdot_2$" all over.

We have

$$\begin{array}{llll} \gamma_A(q, r, x) = \lambda_A(x) \varphi_A(q, r) & \Rightarrow & \lambda_A(x) = \alpha e^{-bx} & \varphi_A(q, r) = q \\ \gamma_C(q, r, x) = \lambda_C(x) \varphi_C(q, r) & \Rightarrow & \lambda_C(x) = \rho & \varphi_C(q, r) = 0 \\ \sigma_C(r, x) = \beta_C(x) \phi_C(x) & \Rightarrow & \beta_C(x) = \delta e^{-ax} & \phi_C(q) = \sqrt{r} \\ & \Rightarrow & v_C(r, x) = -B_C(x) \phi(r) & B_C(x) = -\frac{1}{a}\left[e^{-ax} - 1\right]. \end{array}$$

Moreover, we have

$$\begin{aligned}
D_A(x) &= \lambda_A^2(x) = \alpha^2 e^{-2bx} \\
V_C(x) &= \lambda_C(x) B_C(x) = -\frac{\rho}{a}\left[e^{-ax} - 1\right] \\
H_C(x) &= \beta_C(x) B_C(x) = -\frac{\delta}{a}\left[e^{-2ax} - e^{-ax}\right].
\end{aligned}$$

Taking all this into account we easily get

$$\begin{array}{lllll}
\mathbf{F}\lambda_A = -b\lambda_A & & \Rightarrow n_A^1 = 0, & c_{A,0}^1 = -b \\
\mathbf{F}\lambda_C = 0 & & \Rightarrow n_C^1 = 0, & c_{C,0}^1 = 0 \\
\mathbf{F}D_A = -2bD_A & & \Rightarrow n_A^2 = 0, & c_{A,0}^2 = -2b \\
\mathbf{F}D_C = 0 & & \Rightarrow n_C^2 = 0, & c_{C,0}^2 = 0 \\
\mathbf{F}V_C = \rho e^{-ax} & \mathbf{F}^2 V_C = -a\mathbf{F}V_C & \Rightarrow n_C^3 = 1, & c_{C,0}^3 = 0 \\
 & & & c_{C,1}^3 = -a \\
\mathbf{F}\beta_C = -a\beta_C & & \Rightarrow n_C^4 = 0, & c_{C,0}^4 = -a \\
\mathbf{F}H_C = -aH_C + \delta e^{-2ax} & \mathbf{F}^2 H_C = -3a\mathbf{F}H_C - 2a^2 H_C & \Rightarrow n_C^5 = 2, & c_{C,0}^5 = -2a^2 \\
 & & & c_{C,1}^5 = -3a.
\end{array}$$

Given this computations, we see the following fields span $\bar{\mathcal{L}}$

$$\left\{\mathbf{F}, \begin{bmatrix}\lambda_A \\ 0\end{bmatrix}, \begin{bmatrix}\lambda_C \\ 0\end{bmatrix}, \begin{bmatrix}D_A \\ 0\end{bmatrix}, \begin{bmatrix}D_C \\ 0\end{bmatrix}, \begin{bmatrix}V_C \\ 0\end{bmatrix}, \begin{bmatrix}\mathbf{F}V_C \\ 0\end{bmatrix}, \begin{bmatrix}0 \\ \beta_C\end{bmatrix}, \begin{bmatrix}0 \\ H_C\end{bmatrix}, \begin{bmatrix}0 \\ \mathbf{F}H_C\end{bmatrix}\right\} \tag{6.51}$$

thus we know that our forward price model admits a FDR since

$$\dim(\bar{\mathcal{L}}) \leq 10.$$

From (6.50) we get the parameterization

$$\begin{bmatrix}\hat{G}^q(x, z_0, z_{A,0}^1, z_{C,0}^1, z_{A,0}^2, z_{C,0}^2, z_{C,0}^3, z_{C,1}^3) \\ \\ \hat{G}^r(x, z_0, z_{C,0}^4, z_{C,0}^5, z_{C,1}^5)\end{bmatrix} =$$

$$= \begin{bmatrix} q_0(x+z_0) + \alpha e^{-bx} z_{A,0}^1 + \rho z_{C,0}^1 + \alpha^2 e^{-2bx} z_{A,0}^2 + \rho^2 z_{C,0}^2 - \frac{\rho}{a}\left[e^{-ax} - 1\right] z_{C,0}^3 + \rho e^{-ax} z_{C,1}^3 \\ r_0(x+z_0) + \delta e^{-ax} z_{C,0}^4 - \frac{\delta}{a}\left[e^{-2ax} - e^{-ax}\right] z_{C,0}^5 - \frac{\delta}{a}\left[-2ae^{-2ax} + ae^{-ax}\right] z_{C,1}^5 \end{bmatrix} \tag{6.52}$$

and from Proposition 6.30 we get the Z dynamics

$$\begin{cases} dZ_0 = dt & \\ dZ^1_{A,0} = -bZ^1_{A,0}dt + \hat{G}^q(Z)dW^j_t & dZ^1_{C,0} = \hat{G}^q(Z)dW^j_t \\ dZ^2_{A,0} = \left(-2bZ^2_{A,1} - \frac{1}{2}\left(\hat{G}^q(Z)\right)^2\right)dt & dZ^2_{C,0} = -\frac{1}{2}\left(\hat{G}^q(Z)\right)^2 dt \\ dZ^3_{C,0} = \hat{G}^q(z)\sqrt{\hat{G}^r(Z))}dt & dZ^3_{C,1} = \left(Z^3_{C,0} - aZ^3_{C,1}\right)dt \\ dZ^4_{C,0} = -aZ^4_{C,0}dt + \sqrt{\hat{G}^r(Z)}dW^j & \\ dZ^5_{C,0} = \left(-2a^2Z^5_{C,1} + \hat{G}^r(Z)\right)dt & dZ^5_{C,1} = \left(Z^5_{C,0} - 3aZ^5_{C,1}\right)dt \end{cases} \tag{6.53}$$

where $\hat{G}^q$ and $\hat{G}^r$ are as in (6.52).

It is obvious however, from both (6.52) and (6.53), that this realization is unnecessarily larger. Using the following change of variables we can find a realization of dimension 7 (which is the minimal possible with the fields in (6.51)),

$$\begin{aligned} \tilde{Z}_0 &= Z_0 \\ \tilde{Z}^1_{A,0} &= Z^1_{A,0} \\ \tilde{Z}^2_{A,0} &= Z^2_{A,0} \\ \tilde{Z}^3_{A,0} &= -\frac{\rho}{a}Z^3_{C,0} + \rho Z^3_{C,1} \\ \tilde{Z}^1_{C,0} &= \rho Z^1_{C,0} + \rho^2 Z^2_{C,0} + \frac{\rho}{a}Z^3_{C,0} \\ \tilde{Z}^4_{C,0} &= \delta Z^4_{C,0} + \frac{\delta}{a}Z^5_{C,0} - \delta Z^5_{C,1} \\ \tilde{Z}^5_{C,0} &= -\frac{\delta}{a}Z^5_{C,0} + 2\delta Z^5_{C,1}. \end{aligned}$$

We can then use Itô and (6.53) to derive the dynamics of the new state variables.

6.6.4 FDR of forward prices versus FDR of interest rates

We finish this study giving a complete answer to our *problem four* – on whether there it is possible to have forward price model which allows for a FDR for forward prices but not for interest rates[15].

We recall a forward price term structure model consists of the following two infinite

[15] Recall the partial answer given in Remark 6.11.

SDEs.

$$dq_t = \left\{ \mathbf{F}q_t - \frac{1}{2}\|\gamma(q_t,r_t)\|^2 - \gamma(q_t,r_t)v^\top(r_t) \right\} dt + \gamma(q_t,r_t)dW_t \tag{6.54}$$

$$dr_t = \left\{ \mathbf{F}r_t - \sigma(r_t)v^\top(r_t) \right\} dt + \sigma(r_t)dW_t \tag{6.55}$$

Proposition 6.31. *In forward price term structure models, inexistence of a FDR for the interest rate equation (6.54) and existence of a FDR for the forward price equation (6.55), is possible only if forward prices are Markovian and the conditions of Proposition 6.26 hold i.e.,*

$$\begin{aligned}
\gamma_j(q,r,x) &= \sum_{k=0}^{N_\gamma^j} \varphi_k^j(q)\lambda_k^j(x) && j \in A, C \\
\sigma_j(r,x) &= \beta_j(x) && j \in B \\
\sigma_j(r,x) &= \begin{cases} \beta_j(x) & \text{if } \gamma_j(q,r,x) = \gamma_j(x) \\ \sum_{k=0}^{N_\sigma^j} \omega_k^j \delta_k^j(x) & \text{if } \gamma_j(q,r,x) = \gamma_j(q,x) \end{cases} && j \in C
\end{aligned} \tag{6.56}$$

where ω_k^j are deterministic constants, β_j are unrestricted deterministic functions, δ_j^k,λ_k^j are QE deterministic functions, and φ_k^j are scalar vector fields in $\mathcal{H}_q$.

Proof. If forward prices are *not* Markovian, than either γ or $\gamma v^\top$ depend on r. If that is the case, we know from Proposition 6.30 that the dynamics of the state variables of type Z^1, Z^2, Z^3 (the ones showing up directly on the parameterization $\hat{G}^q$) depend on $\hat{G}^r$. That is, the forward price realization depend on the interest rate realization, indirectly, through the dynamics of the factors showing up in $\hat{G}^q$. Thus, if $r = \hat{G}^r(Z)$ only holds for infinite Z, the forward prices will also be a function on an infinite state variable and, by definition, do not admit a FDR.

If forward prices are Markovian, we know, from Proposition 6.26, that forward prices admit a FDR if and only if (6.56) hold. On the other hand, we see (6.56) imposes no restriction on σ_B, so we can choose σ_B *not to be* a weighted finite sum of quasi-exponential functions, weighted by scalar fields in $\mathcal{H}_r$, making existence of FDR for interest rates impossible (for further details on this result from the previous literature see Filipović and Teichmann (2003)). ∎

6.7 Conclusions

Forward prices are only interesting objects of study in settings where the forward measures $\mathbb{Q}^T$ differ from the risk-neutral measure $\mathbb{Q}$. In these settings, the study of forward prices depends on zero-coupon bond price volatilities. Using the Lie algebraic approach of Björk *et al.*, we have shown that term structure models for forward prices consist of a system of two infinite dimensional SDEs, one describing the dynamics of the forward prices themselves and the other characterizing the interest rate setting and where the interest rate equation is an input to the forward price equation.

Despite the apparent non-Markovian nature of forward prices, we were able to identify models for which forward prices are in fact Markovian along with the necessary and sufficient conditions for this Markovian property to hold. Studying Markovian forward prices, we concluded that the existence of a finite dimensional realization (FDR) for Markovian forward prices is, in some sense, independent of the existence of an FDR for interest rates.

We studied existence and construction of FDR for Markovian forward prices and derived general conditions for existence of FDR. We considered with special detail the cases of purely deterministic and deterministic direction volatility. From this analysis, we concluded that some results from previous literature can be extended to the forward price term structure case, but also that forward price term structure models are particularly complex.

The dynamics of forward prices have a particularly complex drift under the risk-neutral measure $\mathbb{Q}$. A direct consequence of this complexity is that as soon as we leave the purely deterministic volatility setting, the best we can hope for is to study *non-minimal* Lie algebras and to find a *non-minimal* FDR. The existence of non-minimal realizations is, of course, sufficient to prove existence of FDR, but is in general not necessary. We showed, however, that given the specificity of the forward price equation drift, and for a specific enlargement of the Lie algebra, existence of an FDR for non-minimal realizations is also necessary for existence of FDR, at least for the one-dimensional case. We then conjectured that this holds for the higher dimension case.

Even if *non-minimal* Lie algebras are, in the above sense, satisfactory for existence results, they are not as satisfactory for construction results, since we are bound to find realizations with too many variables. In spite of this, we exemplified how, given a concrete application, we can use the abstract results to obtain a smaller realization (sometimes the minimal one) simply by using a smart change of variables and Itô's lemma.

For non-Markovian forward prices, we showed that whenever there exists an FDR for the forward price equation, the dynamics of the state variable depend on the interest rates. Consequently, term structures of forward prices will always (indirectly) depend on interest rates, and the existence of an FDR for the interest rate equation is *necessary* for the existence of an FDR for the forward price equation. In order to study non-Markovian forward price term structure models, we handle a system of two infinite dimensional SDEs, and computations hence become quite cumbersome. Nevertheless, most results are the expected ones, given the previous literature on FDR of interest rates and the study of the forward price equation in the easier Markovian setting.

In terms of the applicability of the results presented here, it is important, first of all, to stress that the characterization of the conditions that guarantee the existence of an FDR for forward price term structure models is crucial to distinguish the "good" forward price models from the "bad" forward price models. After all, a term structure model that does not allow for an FDR cannot be useful for any practical application. After all, it is impossible to estimate the dynamics of an infinite state variable.

In addition to this *selection applicability*, perhaps the most important application results from the actual local parameterization of the term structure. This parameterization can help in understanding what are the conditions (on the driving volatility vector fields) needed to produce term structures consistent with case-specific realities, helping to design good models. In the present study this *design applicability* of the Lie algebraic approach was left untouched because it is case dependent, and our focus has been on general results.

Finally, let alone forward prices, the results derived here can be used to study term structures of *any* $\mathbb{Q}^T$-martingale. Swap rates and lease rates, for example, are financial instruments with strong connections to $\mathbb{Q}^T$-martingales.

A Appendix : Technical details and Proofs

Conjecture 6.21

Proof. (sketch) The implication $\Leftarrow$ follows immediately from $\mathcal{L} \subseteq \bar{\mathcal{L}}$.

The implication $\Rightarrow$ is much harder to prove. Here, as an illustration, we consider the one-dimensional Wiener process case. We will show the equivalent result that if $\dim(\bar{\mathcal{L}}) = \infty$ then we must also have $\dim(\mathcal{L}) = \infty$.

In this case

$$\begin{aligned}
\mathcal{L} &= \{\mu, \gamma\}_{LA} = \{f_0, f_1\}_{LA} \\
\bar{\mathcal{L}} &= \{\mathbf{F}, \lambda, D, V\}.
\end{aligned}$$

for

$$\begin{aligned}
f_0(q) &= \mathbf{F}q - \frac{1}{2}D\phi(q) + \varphi(q)V \\
f_1(q) &= \lambda \\
D &= \lambda^2 \\
V &= \lambda v
\end{aligned}$$

and as usual we take $v(x) = -\int_0^x \sigma(s)ds$.

From Lemma 6.22 we know that $\dim(\bar{\mathcal{L}}) = \infty$ if and only if at least one of the functions γ and σ is not QE. So, assume that γ and σ are *not* QE functions. Then, also D and V are not QE functions.

Let us now, have a look at the original (smaller) Lie-algebra, $\mathcal{L}$. We will try to see if

$$\mathcal{L} = \{\mu, \gamma\}_{LA} = \{f_0, f_1\}_{LA} < \infty.$$

Computing lie-brackets and simplifying

$$\begin{aligned}
[f_0, f_1](q) &= f_0'(q)f_1(q) - f_1{}'(q)f_0(q) \\
&= \mathbf{F}\lambda - \frac{1}{2}D \underbrace{\phi'(q)[\lambda]}_{\text{scalar field}} - V \underbrace{\varphi'(q)[\lambda]}_{\text{scalar field}} \\
&= f_2(q).
\end{aligned}$$

Continuing this way we need to compute $[f_0, f_2] = f_0'(q)f_2 - f_2{}'f_0(q)$. The first parcel, however give us

$$\begin{aligned}
f_0'(q)f_2(q) &= \mathbf{F}^2\lambda - \frac{1}{2}(\mathbf{F}D)\,\phi'(q)[\lambda] + (\mathbf{F}V)\,\varphi'(q)[\lambda] \\
&= \mathbf{F}^2\lambda + \mathbf{F}D.(\text{scalar field}) + \mathbf{F}V.(\text{scalar field})
\end{aligned}$$

and the second parcel (though involving more messy computations) is of the form

$$f_2'(q)f_0(q) = D.(\text{scalar filed}) + V.(\text{scalar field}).$$

So,

$$\begin{aligned}
[f_0, f_2] &= \mathbf{F}^2\lambda + \mathbf{F}D(\text{scalar field}) + \mathbf{F}V.(\text{scalar field}) + D.(\text{scalar field}) \\
&\quad + V.(\text{scalar field}) =: f_3 \\
[f_0, f_3] &= \mathbf{F}^3\lambda + \mathbf{F}^2D(\text{scalar field}) + \mathbf{F}^2V.(\text{scalar field}) + D.(\text{scalar field}) \\
&\quad + V.(\text{scalar field}) \\
&\vdots
\end{aligned}$$

It is now easy to see why the dimension of our $\mathcal{L}$ will also be infinite, in this case. It would only be finite if λ, D and V were QE functions, but this hypothesis is excluded by assumption. ∎

Proposition 6.25

Proof. Using $\bar{G}$ from (6.40) we have

$$\bar{G}'(z_0, z_{j,k}^1, z_{j,k}^2, z_{j,k}^3)\begin{bmatrix} h_0 \\ h_{1,0}^1 \\ \vdots \\ h_{1,0}^2 \\ \vdots \\ h_{1,0}^3 \\ \vdots \\ h_{m,n_j}^3 \end{bmatrix} = \frac{\partial}{\partial z_0} q(x+z_0)h_0 + \sum_{j\in A,C}\sum_{k=0}^{n_j^1} \mathbf{F}^k\lambda_j h_{j,k}^1$$

$$+ \sum_{j\in A,C}\sum_{k=0}^{n_j^2} \mathbf{F}^k D_j h_{j,k}^2 + \sum_{j\in C}\sum_{k=0}^{n_j^3} \mathbf{F}^k V_j h_{j,k}^3.$$

From $q = \bar{G}(Z)$ and once again the functional form of $\bar{G}$ in (6.40) we get

$$\begin{aligned}
\mathbf{F}q =& \mathbf{F}\tilde{q}_0 + \sum_{j\in A,C}\sum_{k=0}^{n_j^1}\mathbf{F}^{k+1}\lambda_j z_{j,k}^1 + \sum_{j\in A,C}\sum_{k=0}^{n_j^2}\mathbf{F}^{k+1}D_j z_{j,k}^2 + \sum_{j\in C}\sum_{k=0}^{n_j^3}\mathbf{F}^{k+1}V_j z_{j,k}^3 \\
=& \frac{\partial}{\partial x}\tilde{q}_0 + \sum_{j\in A,C}\left[\sum_{k=1}^{n_j^1}\mathbf{F}^k\lambda_j z_{j,k-1}^1 + \mathbf{F}^{n_j^1+1}\lambda_{n_j} z_{j,n_j^1}^1\right] \\
&+ \sum_{j\in A,C}\left[\sum_{k=1}^{n_j^2}\mathbf{F}^k D_j z_{j,k-1}^2 \mathbf{F}^{n_j^2+1}D_{n_j} z_{j,n_j}^2\right] + \sum_{j\in C}\left[\sum_{k=1}^{n_j^3}\mathbf{F}^k V_j z_{j,k-1}^3 + \mathbf{F}^{n_j^3+1}V_{n_j} z_{j,n_j^3}^3\right] \\
=& \frac{\partial}{\partial x}\tilde{q}_0 + \sum_{j\in A,C}\left[\sum_{k=1}^{n_j^1}\mathbf{F}^k\lambda_j z_{j,k-1}^1 + \sum_{k=0}^{n_j^1} c_{j,k}^1\lambda_k z_{j,n_j}^1\right] \\
&+ \sum_{j\in A,C}\left[\sum_{k=1}^{n_j^2}\mathbf{F}^k D_j z_{j,k-1}^2 + \sum_{k=0}^{n_j^2} c_{j,k}^2 D_k z_{j,n_j}^2\right] + \sum_{j\in C}\left[\sum_{k=1}^{n_j^3}\mathbf{F}^k V_j z_{j,k-1}^3 + \sum_{k=0}^{n_j^3} c_{j,k}^3 V_k z_{j,n_j}^3\right]
\end{aligned}$$

where we omitted the x-dependence and used $\tilde{q}_0(x) = q_0(x + z_0)$.

We can now use the above expression to substitute into our μ (recall (6.36)). Thus, $\bar{G}_\star\bar{a} = \mu$ becomes

$$\begin{aligned}
&\frac{\partial}{\partial z_0}\tilde{q}_0\bar{a}_0 + \sum_{j\in A,C}\sum_{k=0}^{n_j^1}\mathbf{F}^k\lambda_j\bar{a}_{j,k}^1 + \sum_{j\in A,C}\sum_{k=0}^{n_j^2}\mathbf{F}^k D_j\bar{a}_{j,k}^2 + \sum_{j\in C}\sum_{k=0}^{n_j^3}\mathbf{F}^k V_j\bar{a}_{j,k}^3 = \\
&= \frac{\partial}{\partial x}\tilde{q}_0 + \sum_{j\in A,C}\left[\sum_{k=1}^{n_j^1}\mathbf{F}^k\lambda_j z_{j,k-1}^1 + \sum_{k=0}^{n_j^1} c_{j,k}^1\lambda_k z_{j,n_j}^1\right] \\
&+ \sum_{j\in A,C}\left[\sum_{k=1}^{n_j^2}\mathbf{F}^k D_j z_{j,k-1}^2 + \sum_{k=0}^{n_j^2} c_{j,k}^2 D_k z_{j,n_j}^2\right] + \sum_{j\in C}\left[\sum_{k=1}^{n_j^3}\mathbf{F}^k V_j z_{j,k-1}^3 + \sum_{k=0}^{n_j^3} c_{j,k}^3 V_k z_{j,n_j}^3\right] \\
&- \frac{1}{2}\sum_{j\in A,C} D_j\varphi_j^2(q) - \frac{1}{2}\sum_{j\in A,C}\varphi_j'(q)\left[\lambda_j\right]\varphi_j(q)\lambda_j
\end{aligned}$$

and $\bar{G}_*\bar{b}^i = \gamma_i$ for all $i \in A, C$

$$\frac{\partial}{\partial z_0}\tilde{q}_0\bar{b}_0^i + \sum_{j\in A,C}\sum_{k=0}^{n_j^1}\mathbf{F}^k\lambda_j\bar{b}_{j,k}^{1i} + \sum_{j\in A,C}\sum_{k=0}^{n_j^2}\mathbf{F}^k D_j\bar{b}_{j,k}^{2i} + \sum_{j\in C}\sum_{k=0}^{n_j^3}\mathbf{F}^k V_j\bar{b}_{j,k}^{3i} = \varphi_i(q)\lambda_i$$

Identification of terms, and again use of $q = \bar{G}(Z)$ yields

$$\begin{array}{rcllr}
\bar{a}_0 &=& 1 & & \\
\bar{a}_{j,0}^1 &=& c_{j,0}^1 z_{j,n_j^1}^1 - \frac{1}{2}\varphi_j'(\bar{G}(z))\,[\lambda_j]\,\varphi_j(\bar{G}(z)) & j\in A,C & \\
\bar{a}_{j,k}^1 &=& z_{j,k-1}^1 + c_{j,k}^1 z_{j,n_j^1}^1 & j\in A,C & k=1,2,\cdots n_j^1 \\
\bar{a}_{j,0}^2 &=& c_{j,0}^2 z^2 j, n_j^2 - \frac{1}{2}\varphi_j^2(\bar{G}(z)) & j\in A,C & \\
\bar{a}_{j,k}^2 &=& z_{j,k-1}^2 + c_{j,k}^2 z_{jn_j^2}^2 & j\in A,C & k=1,2,\cdots n_j^2 \\
\bar{a}_{j,0}^3 &=& c_{j,0}^3 z^3 j, n_j^3 + \varphi(\bar{G}(z)) & j\in C & \\
\bar{a}_{j,k}^3 &=& z_{j,k-1}^3 + c_{j,k}^3 z_{j,n_j^3}^3 & j\in C & k=1,2,\cdots n_j^3
\end{array}$$

$$\begin{array}{rcllr}
\bar{b}_0^i &=& 0 & & \\
\bar{b}_{j,k}^{1i} &=& \begin{cases}\varphi_j(\bar{G}(z)) & \text{if } i=j \text{ and } k=0\\ 0 & \text{if } i\neq j \text{ or } k=1,2,\cdots n_j^1\end{cases} & j\in A,C & \\
\bar{b}_{j,k}^{2i} &=& 0 & j\in A,C & k=1,2,\cdots n_j^2 \\
\bar{b}_{j,k}^{3i} &=& 0 & j\in A,C & k=1,2,\cdots n_j^3.
\end{array}$$

Note that the factors $z_{j,0}^1$ are driven by the scalar wiener process W^j and that all remaining factors have diffusion terms equal to zero. Thus only for $z_{j,0}^1$ the Itô dynamics differ from the Stratonovich dynamics. The above $\bar{a}_{j,0}^1$ is the Stratonovich drift. Given the form of $\bar{a}_{j,0}^1$, the diffusion for $z_{j,0}^1$. We easily get the Itô drift to be simply $\bar{a}_{j,0}^{1\ \text{Itô}} = c_{j,0}^1 z_{j,n_j^1}^1$. ■

Proposition 6.30

Proof. We first compute the parameterization $\hat{G}$. From Theorem 6.6 and the special shape of the basic fields in (6.46), we have

$$
\begin{bmatrix} \hat{G}^q(Z) \\ \hat{G}^r(Z) \end{bmatrix} = \begin{bmatrix} \prod_{j\in A,C}\left(e^{\mathbf{F}^k\lambda_j z_{j,k}^1}e^{\mathbf{F}^k D_j z_{j,k}^2}\right)\prod_{j\in C}\left(e^{\mathbf{F}^k V_j z_{j,k}^3}\right)e^{\mathbf{F}z_0}q_0 \\ \prod_{j\in B,C}\left(e^{\mathbf{F}^k\beta_j z_{j,k}^4}e^{\mathbf{F}^k H_j z_{j,k}^5}\right)e^{\mathbf{F}z_0}r_0 \end{bmatrix} \tag{6.57}
$$

$$
= \begin{bmatrix} \tilde{q}_0 + \sum_{j\in A,C}\sum_{k=0}^{n_j^1}\mathbf{F}^k\lambda_j z_{j,k}^1 + \sum_{j\in A,C}\sum_{k=0}^{n_j^2}\mathbf{F}^k D_j z_{j,k}^2 + \sum_{j\in C}\sum_{k=0}^{n_j^3}\mathbf{F}^k V_j z_{j,k}^3 \\ \tilde{r}_0 + \sum_{j\in B,C}\sum_{k=0}^{n_j^4}\mathbf{F}^k\beta_j z_{j,k}^4 + \sum_{j\in B,C}\sum_{k=0}^{n_j^5}\mathbf{F}^k H_j z_{j,k}^5 \end{bmatrix}
$$

Using $\hat{G}$ from (6.57) we have

$$
\hat{G}^{q'}(z_0, z_{j,k}^1, z_{j,k}^2, z_{j,k}^3)\begin{bmatrix} h_0 \\ h_{1,0}^1 \\ \vdots \\ h_{1,0}^2 \\ \vdots \\ h_{m,n_j^3}^3 \end{bmatrix} = \frac{\partial}{\partial z_0}\tilde{q}_0 h_0 + \sum_{j\in A,C}\sum_{k=0}^{n_j^1}\mathbf{F}^k\lambda_j h_{j,k}^1 + \sum_{j\in A,C}\sum_{k=0}^{n_j^2}\mathbf{F}^k D_j h_{j,k}^2
$$

$$
+\sum_{j\in C}\sum_{k=0}^{n_j^3}\mathbf{F}^k V_j h_{j,k}^3.
$$

and

$$
\hat{G}^{r'}(z_0, z_{j,k}^4, z_{j,k}^5)\begin{bmatrix} h_0 \\ h_{1,0}^4 \\ \vdots \\ h_{m,n_j^3}^5 \end{bmatrix} = \frac{\partial}{\partial z_0}\tilde{r}_0 h_0 + \sum_{j\in B,C}\sum_{k=0}^{n_j^4}\mathbf{F}^k\beta_j h_{j,k}^4 + \sum_{j\in A,C}\sum_{k=0}^{n_j^5}\mathbf{F}^k H_j h_{j,k}^2
$$

where we omitted the x-dependence and used $\tilde{q}_0(x) = q_0(x+z_0)$, $\tilde{r}_0(x) = r_0(x+z_0)$.

From $q = \hat{G}^q(Z)$, $r = \hat{G}^r(Z)$ and once again the functional form of $\hat{G}$ in (6.57) we get

$$\begin{aligned}
\mathbf{F}q &= \mathbf{F}\tilde{q}_0 + \sum_{j\in A,C}\sum_{k=0}^{n_j^1}\mathbf{F}^{k+1}\lambda_j z_{j,k}^1 + \sum_{j\in A,C}\sum_{k=0}^{n_j^2}\mathbf{F}^{k+1}D_j z_{j,k}^2 + \sum_{j\in C}\sum_{k=0}^{n_j^3}\mathbf{F}^{k+1}V_j z_{j,k}^3 \\
&= \frac{\partial}{\partial x}\tilde{q}_0 + \sum_{j\in A,C}\left[\sum_{k=1}^{n_j^1}\mathbf{F}^k\lambda_j z_{j,k-1}^1 + \mathbf{F}^{n_j^1+1}\lambda_{n_j^1} z_{j,n_j^1}^1\right] \\
&\quad + \sum_{j\in A,C}\left[\sum_{k=1}^{n_j^2}\mathbf{F}^k D_j z_{j,k-1}^2 \mathbf{F}^{n_j^2+1} D_{n_j^2} z_{j,n_j^2}^2\right] + \sum_{j\in C}\left[\sum_{k=1}^{n_j^3}\mathbf{F}^k V_j z_{j,k-1}^3 + \mathbf{F}^{n_j^3+1}V_{n_j^3} z_{j,n_j}^3\right] \\
&= \frac{\partial}{\partial x}\tilde{q}_0 + \sum_{j\in A,C}\left[\sum_{k=1}^{n_j^1}\mathbf{F}^k\lambda_j z_{j,k-1}^1 + \sum_{k=0}^{n_j^1} c_{j,k}^1\lambda_k z_{j,n_j}^1\right] \\
&\quad + \sum_{j\in A,C}\left[\sum_{k=1}^{n_j^2}\mathbf{F}^k D_j z_{j,k-1}^2 \sum_{k=0}^{n_j^2} c_{j,k}^2 D_k z_{j,n_j^2}^2\right] + \sum_{j\in C}\left[\sum_{k=1}^{n_j^3}\mathbf{F}^k V_j z_{j,k-1}^3 + \sum_{k=0}^{n_j^3} c_{j,k}^3 V_k z_{j,n_j^3}^3\right]
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}r &= \mathbf{F}\tilde{r}_0 + \sum_{j\in b,C}\sum_{k=0}^{n_j^4}\mathbf{F}^{k+1}\beta_j z_{j,k}^4 + \sum_{j\in B,C}\sum_{k=0}^{n_j^5}\mathbf{F}^{k+1}H_j z_{j,k}^5 \\
&= \frac{\partial}{\partial x}\tilde{r}_0 + \sum_{j\in B,C}\left[\sum_{k=1}^{n_j^4}\mathbf{F}^k\beta_j z_{j,k-1}^4 + \mathbf{F}^{n_j^4+1}\beta_{n_j^4} z_{j,n_j^4}^4\right] \\
&\quad + \sum_{j\in B,C}\left[\sum_{k=1}^{n_j^5}\mathbf{F}^k H_j z_{j,k-1}^5 \mathbf{F}^{n_j^5+1} H_{n_j^5} z_{j,n_j^5}^5\right] \\
&= \frac{\partial}{\partial x}\tilde{r}_0 + \sum_{j\in B,C}\left[\sum_{k=1}^{n_j^4}\mathbf{F}^k\beta_j z_{j,k-1}^4 + \sum_{k=0}^{n_j^4} c_{j,k}^4\beta_k z_{j,n_j^4}^4\right] \\
&\quad + \sum_{j\in B,C}\left[\sum_{k=1}^{n_j^5}\mathbf{F}^k H_j z_{j,k-1}^5 + \sum_{k=0}^{n_j^5} c_{j,k}^5 H_k z_{j,n_j^5}^5\right]
\end{aligned}$$

We can now use the above expressions to substitute into our μ, recall we have

$$
\begin{aligned}
\mu(q,r) &= \mathbf{F}\begin{bmatrix} q \\ r \end{bmatrix} - \frac{1}{2}\sum_{i\in A,C}\varphi_i^2(q,r)\begin{bmatrix} \lambda_i^2 \\ 0 \end{bmatrix} + \sum_{i\in C}\varphi_i(q,r)\phi_i(r)\begin{bmatrix} \lambda_i B_i \\ 0 \end{bmatrix} + \sum_{i\in B,C}\phi_i^2(r)\begin{bmatrix} 0 \\ \beta_i B_i \end{bmatrix} \\
&\quad -\frac{1}{2}\Bigg\{\sum_{i\in A,C}\varphi'_{iq}(q,r)\,[\lambda_i]\,\varphi_i(q,r)\begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} \\
&\quad +\sum_{i\in C}\varphi'_{ir}(q,r)\,[\beta_i]\,\phi_i(r)\begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} + \sum_{i\in B,C}\phi'_{ir}(r)\,[\beta_i]\,\phi_i(r)\begin{bmatrix} 0 \\ \beta_i \end{bmatrix}\Bigg\},
\end{aligned}
$$

$$
\begin{bmatrix} \gamma_i(q,r) \\ \sigma_i(r) \end{bmatrix} = \begin{cases} \varphi_i(q,r)\begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} & i\in A \\ \phi_i(r)\begin{bmatrix} 0 \\ \beta_i \end{bmatrix} & i\in B \\ \varphi_i(q,r)\begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} + \phi_i(r)\begin{bmatrix} 0 \\ \beta_i \end{bmatrix} & i\in C. \end{cases}
$$

Defining

$$
\mu(q,r) = \begin{bmatrix} \mu^q(q,r) \\ \mu^r(r) \end{bmatrix}
$$

identification of terms in $\hat{G}^q_\star \hat{a}^q = \mu^q$, $\hat{G}^r_\star \hat{a}^r = \mu^r$, gives us

$$
\hat{a}_0 = 1
$$

$$
\hat{a}^1_{j,0} = \begin{cases} c^1_{j,0} z^1_{j,n^1_j} - \frac{1}{2}\varphi'_j(\hat{G}^q(z),\hat{G}^r(z))\,[\lambda_j]\,\varphi_j(\hat{G}^q(z),\hat{G}^r(z)) & j\in A \\ c^1_{j,0} z^1_{j,n^1_j} - \frac{1}{2}\varphi'_j(\hat{G}^q(z),\hat{G}^r(z))\,[\lambda_j]\,\varphi_j(\hat{G}^q(z),\hat{G}^r(z)) & \\ \qquad -\frac{1}{2}\varphi'_j(\hat{G}^q(z),\hat{G}^r(z))\,[\beta_j]\,\phi_j(\hat{G}^r(z)) & j\in C \end{cases}
$$

$$
\begin{array}{rcllll}
\hat{a}^1_{j,k} & = & z^1_{j,k-1} + c^1_{j,k} z^1_{j,n^1_j} & j \in A, C & k = 1, 2, \cdots n^1_j \\
\hat{a}^2_{j,0} & = & c^2_{j,0} z^2_{j,n^2_j} - \frac{1}{2}\varphi^2_j(\hat{G}^q(z), \hat{G}^r(z)) & j \in A, C & \\
\hat{a}^2_{j,k} & = & z^2_{j,k-1} + c^2_{j,k} z^2_{jn^2_j} & j \in A, C & k = 1, 2, \cdots n^2_j \\
\hat{a}^3_{j,0} & = & c^3_{j,0} z^3_{j,n^3_j} + \varphi(\hat{G}^q(z), \hat{G}^r(z))\phi(\hat{G}^r(z)) & j \in C & \\
\hat{a}^3_{j,k} & = & z^3_{j,k-1} + c^3_{j,k} z^3_{j,n^3_j} & j \in C & k = 1, 2, \cdots n^3_j \\
\hat{a}^4_{j,0} & = & c^4_{j,0} z^4_{j,n^4_j} - \frac{1}{2}\phi'_j(\hat{G}^r(z))\,[\beta_j]\,\phi_j(\hat{G}^r(z)) & j \in B, C & \\
\hat{a}^4_{j,k} & = & z^4_{j,k-1} + c^4_{j,k} z^4_{j,n^4_j} & j \in B, C & k = 1, 2, \cdots n^4_j \\
\hat{a}^5_{j,0} & = & c^5_{j,0} z^5_{j,n^5_j} + \phi^2(\hat{G}^r(z)) & j \in B, C & \\
\hat{a}^5_{j,k} & = & z^5_{j,k-1} + c^5_{j,k} z^5_{j,n^5_j} & j \in B, C & k = 1, 2, \cdots n^5_j
\end{array}
$$

To get the diffusion terms we identify terms in $\hat{G}^q_\star \hat{b}^{qi} = \gamma_i$ for $i \in A, C$ and $\hat{G}^r_\star \hat{b}^{ri} = \sigma_i$ for $i \in B, C$.

$$
\begin{array}{rcll}
\hat{b}^i_0 & = & 0 & \\
\hat{b}^{1i}_{j,k} & = & \begin{cases} \varphi_j(\hat{G}^q(z), \hat{G}^r(z)) & \text{if } i = j, \text{ and } k = 0 \\ 0 & \text{otherwise} \end{cases} & j \in A, C \\
\hat{b}^{2i}_{j,k} & = & 0 & j \in A, C \quad k = 1, 2, \cdots n^2_j \\
\hat{b}^{3i}_{j,k} & = & 0 & j \in A, C \quad k = 1, 2, \cdots n^3_j \\
\hat{b}^{4i}_{j,k} & = & \begin{cases} \phi_j(\hat{G}^r(z)) & \text{if } i = j \text{ and } k = 0 \\ 0 & \text{if } i \neq j \text{ or } k = 1, 2, \cdots n^4_j \end{cases} & j \in B, C \\
\hat{b}^{5i}_{j,k} & = & 0 & j \in B, C \quad k = 1, 2, \cdots n^5_j
\end{array}
$$

Note that this implies that the factors $Z^1_{j,0}$, $Z^4_{j,0}$ are driven by the scalar wiener process W^j so, in particular for $j \in C$ the same Wiener process drives the two variables. All remaining factors have diffusion terms equal to zero. Thus only for $z^1_{j,0}$ and $z^4_{j,0}$, the Itô dynamics differ from the Stratonovich dynamics. We easily get the Itô drifts to be simply

$$
\begin{array}{rcl}
\hat{a}^{1\ \text{Itô}}_{j,0} & = & c^1_{j,0} z^1_{j,n^1_j} \\
\hat{a}^{4\ \text{Itô}}_{j,0} & = & c^4_{j,0} z^4_{j,n^4_j}
\end{array}
$$

■

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Charpentier, Claes. *IT inom omsorgen. Förväntade effekter av införande av IT-system för utförarna inom äldre- och handikappomsorgen.*
Lind, Johnny och Göran Nilsson (redaktörer). *Ekonomistyrningens metoder, sammanhang och utveckling – En vänbok till Lars A Samuelson.*
Samuelson, Lars A. *Organizational governance and control – a summary of research in the Swedish society.*

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Andersson, Martin. *Making a Difference – Project Result Improvement in Organizations.*
Arvidsson, Per. *Styrning med belöningssystem – Två fallstudier om effekter av belöningssystem som styrmedel.*
Bems, Rudolfs. *Essays in International Macroeconomics.*
Berg-Suurwee, Ulrika. *Nya trender, nya nämnder – effekter av en stadsdelsnämndsreform inom kultur och fritid.*
Björkman, Hans. *Learning from members – Tools for strategic positioning and service innovation in trade unions.*
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Tolis, Christofer. *Framing the Business – Business Modelling for Business Development.*
Östberg, Per. *Corporate Disclosure and Investor Recognition.*

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Lind, Johnny. *Strategi och ekonomistyrning. En studie av sambanden mellan koncernstrategi, affärsstrategi och ekonomistyrning.*
Lind, Johnny och Walter Schuster (red). *Redovisningens teori, praktik och pedagogik. En vänbok till Lars Östman.*
Sevón, Guje och Lennart Sjöberg (red). *Emotioner och värderingar i näringslivet.* EFIs Årsbok 2004.
Wijkström, Filip and Stefan Einarsson. *Foundations in Sweden – Their scope, roles and visions.*

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Bengtsson, Lars, Johnny Lind och Lars A. Samuelson (red). *Styrning av team och processer – Teoretiska perspektiv och fallstudier.*
Berg-Suurwee, Ulrika. *Styrning före och efter stadsdelsnämndsreform inom kultur och fritid – Resultat från intervjuer och enkät.*
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Ericson, Mona. *Strategi, kalkyl, känsla.*
Gunnarsson, Jonas, Richard Wahlund och Helena Flink. *Finansiella strategier i förändring: segment och beteenden bland svenska hushåll.*

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Schwarz, Brita och Susanne Weinberg. *Serviceproduktion och kostnader – Att söka orsaker till kommunala skillnader.*
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Bergkvist, Lars. *Advertising Effectiveness Measurement: Intermediate Constructs and Measures.*
Bornefalk, Anders. *Essays on Social Conflict and Reform.*
Edman, Jan. *Information Use and Decision Making in Groups – A Study of an Experimental Oligopoly Market with the Use of a Business Game.*
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