Essays in Real Estate Finance

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Essays in Real Estate Finance

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Stockholm, August 2005
Eric Clapham

Contents

Introduction and Summary

This dissertation contains five papers in real estate finance and two asset pricing papers. In the following, the papers are introduced and main results are summarized.

The leasing literature has traditionally focused perhaps primarily on tax aspects (Grenadier 1995, 2005). However, from a pricing perspective, leasing may be viewed as purchasing the right to use an asset over a specific period, as noted by Miller and Upton (1976). This insight was applied using discrete time models by McConnell and Schallheim (1983) and Schallheim and McConnell (1985). In this approach, an equilibrium lease pricing condition is that the present value of expected lease payments must be equal to the present value (PV) of the expected asset service flow over the period as well as the value of any embedded options, such as renewal or purchase options. Heuristically

$$PV(\text{lease payments}) = PV(\text{asset serviceflow}) + PV(\text{embedded options}). \quad (1)$$

If liquid markets exist, the asset service flow over a period may be locked in for example by establishing a portfolio consisting of the asset and a short call option with zero strike price (Grenadier, 1995).

An important paper by Grenadier (1995) develops a continuous time model involving stochastic demand and supply of real estate. Due to restrictions on the liquidity of the underlying asset in typical real estate markets, risk neutral pricing is justified not through arbitrage arguments, but by means of an equilibrium pricing approach. The spot rent is the current rate for an infinitesimally short lease and gives the value of the asset service flow, while the asset price is defined as the present value of the spot rent to infinity. In Grenadier (1995, 2005) the spot rent is determined endogenously. McConnell and Schallheim (1983) and Schallheim and McConnell (1985), as well as the study of leasing and credit risk in Grenadier (2005), however use exogenously specified rent dynamics and that is typically the approach taken in the applied literature.

Grenadier (1995) popularized the concept of a term structure of lease rates, or the lease rate as a function of contract length. An important qualitative result in Grenadier's model is that the slope of the term structure reflects expectations of the future development of market lease rates. In Grenadier (2005) this is qualified by introducing the concept of an infinitesimal forward rate, or the lease rate contracted today for a future infinitesimally short period. In interest rate theory, the expectations hypothesis is well known and dates back at least to Fisher (1896). The pure expectations hypothesis states that forward rates are unbiased estimators of future spot rates; a classic treatment can be found in Cox *et al.* (1981). The pure expectations hypothesis holds in general only when interest rates are deterministic, while an assumption of risk neutrality is not sufficient (Björk, 1998).

The first paper in the thesis, coauthored with Åke Gunnelin, is an attempt to characterize general properties of the term structure of lease rates assuming only equilibrium pricing restrictions as in (1) above. The risk free interest rate is allowed to be stochastic. A main result is that a sufficient condition for the expectations hypothesis to hold for lease rates is risk neutrality and a deterministic short interest rate. The general framework is parameterized using specific assumptions about the dynamics of the short rent and short interest rate. It is demonstrated that for realistic parameter values, varying risk aversion and correlation between the interest rate and short rent, can significantly alter the term structure of lease rates, holding objective expectations constant.

The second paper considers lease contracts that have upward only characteristics in the sense that they are indexed to some process, but cannot be adjusted below a floor rate. Several empirically relevant lease contracts fall into this category. Following a general analysis, three specific contract types are considered: a modified upward only contract, the Swedish standard commercial contract and the percentage lease.

In an upward only lease, common in the United Kingdom and some Commonwealth countries, the lease rate is periodically adjusted if market rates have risen and otherwise left unchanged. The upward only lease is priced by Ambrose *et al.* (2002); the historical background is discussed by Crosby *et al.* (2005). The lease type has become controversial in the UK and one alternative is a modified upward only contract that has up or down rent reviews, with the initial rate as a floor (see Commercial Leases Working Group, 2002, as well as ODPM, 2004 and 2005). The second paper compares the properties of the standard and modified upward only leases.

The third paper considers a lease with an embedded renewal option, where the strike price is inflation indexed. It is demonstrated that a result by Fischer (1978) can be used to value the option analytically, which is an alternative to the numerical approach suggested by Buetow and Albert (1998). The properties of the indexed renewal option are further explored.

The fourth paper, coauthored with Tomas Björk, analyzes a commercial real estate index swap, which involves the exchange of the cumulative returns on a real estate index and the risk free rate. Using an analytic approach, it is found that the price of the contract is exactly zero under general conditions. The intuitive explanation is that exchanging equal nominal amounts of two assets should be a zero net present value transaction. The result sharpens previous work by Buttimer *et al.* (1998), who using numerical methods found a value close to zero under specific assumptions.

The fifth paper, coauthored with Peter Englund, Christian L. Redfearn and John M. Quigley, considers index revision, or the updating of earlier estimates in an index series as new information becomes available, in the context of home equity insurance.

Home equity represents a large fraction of the net worth of many households (e.g., Englund *et al.*, 2002), leaving them potentially vulnerable to price fluctuations in real estate markets. This is an example of a macroeconomic risk that is often difficult to insure against. Shiller (1993) has argued in favor of establishing "macro markets" that would create opportunities to hedge against such risks. A way of minimizing moral hazard risk in a home equity insurance scheme is to let payouts be based on the development of a local house price index and not on the selling price of the individual dwelling. It is desirable that the index is subject to limited revision as that may cause a contract settlement to appear unreasonable or require delays.

The paper uses a data set that includes all arm's-length housing transactions in Sweden during the period 1981-1999. For tax reasons, Swedish authorities collect extensive information on the characteristics of dwellings; the data set is documented in Englund *et al.* (1998, 1999). It is therefore possible to compute accurate indexes and analyze their revision properties.

The two main types of transaction based constant quality indexes are the hedonic and repeat sales indexes (for surveys see e.g. Cho, 1996 and Palmquist, 2003). Hedonic indexes use information about the characteristics and attributes of the dwellings to control for quality; the standard theoretical interpretation of hedonic pricing is due to Rosen (1974). The repeat sales index, introduced by Bailey *et al.* (1963), provides a method for constructing constant quality indexes in the absence of hedonic information by using dwellings that have been on the market at least twice. If both the level and the price of the hedonic characteristics stay constant, the change in the house price will reflect the course of a constant quality index. The repeat sales index is currently widely used, at least in the U.S. In particular, the Office of Federal Housing Enterprise Oversight uses the repeat sales method to construct its House Price Index (for technical details see Calhoun, 1996).

For a repeat sales index, the sample size may increase significantly over time as new pairs are formed and enter into the data set. Clapp and Giacotto (1999) find that the repeat sales index exhibits large and systematic downward revision. They emphasize the role of "flip sales" as a source of revision, or repeats with a short time span between sales. Hoesli *et al.* (1997) find significant but unsystematic index revision in a sample from Geneva, Switzerland. Abraham and Schaumann (1991) find significant revision that does not settle down rapidly even in large samples.

The fifth paper considers repeat sales and hedonic indexes, which are potentially subject to revision. Consistent with Clapp and Giacotto (1999), revision in repeat indexes tends to be systematic and mostly downward. It appears that if possible, pure repeat sales indexes should be avoided for purposes of claims settlement. If they are used, the systematic nature of revision implies that basing settlement on index numbers that have been updated an equal number of times since the first estimate will tend to reduce exposure to revision.

The two final papers consider asset pricing, in particular the consumption based asset pricing model and measures of misspecification. The model, derived in the seminal work by Lucas (1978) and Breeden (1979), postulates that an asset is priced according to its ability to provide the owner with a high and stable level of consumption. However, since Hansen and Singleton (1982, 1983) the model is known to have a poor empirical fit. In particular, the significant volatility of many asset returns is not well explained by the typically smoothly developing aggregate consumption. The link between consumption and asset prices is however so fundamental that the theory has an almost axiomatic standing (Cochrane, 2000). As a result, disappointing empirical results have lead to attempts to find alternative specifications and approaches that will give the theory greater explanatory power. For surveys of studies on consumption based asset pricing see e.g. Kocherlakota (1996) and Campbell (2000).

Two approaches to improving the performance of the consumption based asset pricing model is to use alternative utility specifications or to consider market frictions such as bid-ask spreads and short-sell constraints. He and Modest (1995) as well as Luttmer (1996) have shown how the model can be adjusted to take market frictions into account. Hansen and Jagannathan (1997) introduced the concept of a measure of misspecification. In this approach, rather than accepting or rejecting a model, the extent of mispricing is instead quantified.

The sixth paper, coauthored with Roland Nilsson, is an empirical test of the consumption based asset pricing model using Swedish data. Both varying utility specifications and market frictions are considered. Market frictions are calibrated to available historical data and to account for small sample statistical properties, a bootstrapping approach is used. Model performance is assessed using several statistics, including the Hansen and Jagannathan (1997) measure adjusted for market frictions. In summary, especially market frictions seem to go some way towards explaining the empirical failure of the traditional test of the consumption based capital asset pricing model on Swedish data. The seventh paper, coauthored with Iñaki R. Longarela and Roland Nilsson, uses Monte Carlo simulations to evaluate the performance of several different measures of misspecification, using varying distributional assumptions for asset returns. In addition, the parameters of a linear factor model are chosen so as to minimize the measures of misspecification.

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Chapter 1

Rental Expectations and the Term Structure of Lease Rates

Abstract

We consider the term structure of lease rates in a general setting where both rents and interest rates are stochastic. The framework is applicable to any leasing market, but we focus on real estate. We find that the "expectations hypothesis", i.e. forward rates are unbiased estimators of future rents, requires similar assumptions as in interest rate theory to hold. To study bias magnitude, simulations are performed using a parameterization of the general framework. Different realistic values for risk aversion and interest rate stochastics can generate widely different shapes of the term structure, holding objective expectations constant. Thus an expected increase in rent is consistent with a downward-sloping term structure and vice versa.

1 Introduction

There has recently been an increasing interest in lease valuation. Particular attention has been given to the term structure of lease rates, that is, the determination of equilibrium lease rates for different contractual terms. One important issue is the extent to which the shape of the term structure is related to expectations of future market rents.

This paper uses a continuous time lease valuation framework in which both the short lease rate and the short interest rate are stochastic. Previous work has used a deterministic interest rate. The framework is applicable to any valid rent and interest rate process, as well as any leasing object, although we mainly discuss real estate. By using a non-parametric setting it is possible to distinguish among relationships that hold in general and those that arise from specific models. Several general expressions relating to the equilibrium term structure of lease rates are derived. Among other things, this allows us to analyze the effect on the term structure of interest rate uncertainty, risk aversion and objective market expectations.

[0] This chapter is coauthored with Åke Gunnelin and appeared in *Real Estate Economics* 31(4), 647-670 (2003); the appendix is extended. We are grateful for valuable comments from Tomas Björk, Peter Englund, David Geltner (the editor), John M. Quigley, two anonymous referees as well as seminar participants at the University of Southern California and the Stockholm Institute for Financial Research.

Our work builds on the earlier literature on lease and term structure analysis. Although this strand of the finance literature is far smaller than that dealing with the term structure of interest rates, several papers do exist. Miller and Upton (1976) provide an early analysis based on the equilibrium condition that the present value of lease payments should equal the present value of the service flow from the asset. McConnell and Schallenheim (1983) as well as Schallenheim and McConnell (1985) extend the analysis and value several common types of lease contracts in a parameterized discrete time model.

An important contribution is made by Grenadier (1995). He explicitly considers the term structure in a continuous time setting. Grenadier derives the term structure in a competitive industry equilibrium where the short rent process is endogenously determined from expectations of future demand and supply. Grenadier (2002) extends the perfect competition equilibrium by analyzing the term structure of lease rates in an oligopolistic property market. Grenadier (1995, 2002) also provide valuation formulas for many common leasing arrangements, such as forward leases, leases with cancellation or renewal options and indexed leases.

The shape of the term structure in Grenadier's models is determined by expectations of future short term lease rates. In a market in which the short rate is expected to increase, Grenadier argues that the term structure should be upward-sloping since lessors otherwise would prefer to roll over short term leases to take advantage of increasing short lease rates. The opposite will be the case when the short rate is expected to decrease. In an intermediate case, in which the short rate is expected to increase in the short run but thereafter come down again, the term structure can be expected to be single humped (that is, at first upward- and then downward-sloping).

The equilibrium term structure in Grenadier (1995, 2002) thus suggests an "expectations hypothesis" similar to that of interest rates. It is well known that the expectations hypothesis for interest rates does not hold in general, i.e. the forward rate is not an unbiased estimator of the future short rent (for a thorough discussion, see Cox *et al.*, 1981; a more recent contribution is Frachot, 1996). We show that the same factors that bias the expectations hypothesis for interest rates, namely stochastic interest rates and risk aversion, also bias the expectations hypothesis for lease rates. Furthermore, by parameterizing the framework and undertaking simulations we show that the magnitude of the bias can be large. Different sets of realistic parameter values for the interest rate and risk aversion can generate widely different term structures even though the rent expectation under the objective probability measure is kept constant. For example, the slope of the term structure is very sensitive to risk aversion towards rent. A small increase in risk aversion can make an upward-sloping term structure turn downward-sloping and vice versa. Similarly, changing the correlation between interest rates and rents or changing the interest rate volatility can invert the term structure. Hence, our simulations show that careful analysis of the risk aversion prevailing in the market and of interest rate- and rent stochastics is required before trying to infer objective rental expectations from the term structure.

The paper is organized as follows: In the following section, we present the continuous time framework used and review forward rental agreements. Thereafter we derive and analyze the term structure of lease rates. In particular we consider the expectations hypothesis for lease rates. Further, the framework is then applied and a simple parameterized model is presented. The final section presents a brief conclusion.

2 The model

We consider a frictionless market containing bonds and contracts on the service flow from a standardized asset. We will think of the asset mainly as property, but it could be any asset. The basic objects of study are two stochastic processes:

- The price of the service flow per unit of time, or the spot rent, denoted X_t. The rental income over an infinitesimally short time period dt is thus $X_t dt$.
- The short interest rate, denoted r_t.

For ease of presentation we assume that the asset corresponds to only one leasing unit and we also abstract from operating costs. Hence, the value of the asset H_t equals the present value of all future lease payments from the single leasing unit.

The spot rent may be modelled as an exogenous process, or be endogenously determined through some form of equilibrium condition. For instance, several papers model X_t as an exogenous geometric Brownian motion with constant drift rate (e.g. Stanton and Wallace, 2002), while in the model of Grenadier (1995), the spot rent is a function of current demand and supply. We do not at this stage parameterize X_t, as we wish to study properties that hold for any valid spot rent process. For the rent process to be valid it must be non-negative and such that the asset H_t is always finite.

The technical setup is the general continuous time framework, as presented in e.g. Björk (1998). We thus use the "risk-neutral pricing" method where drift rates are adjusted to reflect risk aversion in the market and discounting is done with the risk free short interest rate. Originally the use of risk-neutral pricing was motivated using arbitrage arguments for a traded asset. However, as argued by Rubinstein (1976) and commonly applied in e.g. the real options literature, risk-neutral drift rates can also be inferred from general equilibrium arguments. This makes it possible to use risk-neutral pricing even in the presence of market imperfections. Risk-neutral expectations are denoted using notation $E_t^Q[\cdot]$.

The asset value is defined as the present value of future spot rents:

$$H_t = E_t^Q \left[\int_t^{\infty} e^{-\int_t^s r_u du} X_s ds \right]. \tag{1}$$

We also introduce notation $\delta_t = X_t/H_t$. The quantity δ_t is thus the continuous dividend yield, or payout ratio.

It is well-known that the local drift rate of a traded asset with no dividends is equal to the short interest rate under the risk-neutral measure. However, the short rent is not an asset but an income flow and therefore its drift rate is not directly defined by the interest rate. At a given time, the drift of the short rent can be either higher or lower than the interest rate. However, since the asset value H_t is to be well defined, the short rent is indirectly restricted by the interest rate to have an average drift rate over time that leads to a finite present value. Further, H_t will by construction have a local drift rate equal to the risk free rate less the payout ratio, or $r_t - \delta_t$, under the risk-neutral measure. To see this, note that by computing the time differential of (1) we obtain

$$E_t^Q [dH_t] = r_t H_t dt - X_t dt = (r_t - \delta_t) H_t dt. \tag{2}$$

Recently the technique known as change of numeraire has seen increased use (for a textbook treatment see Björk, 1998). From a computational viewpoint, this technique implies that the expectation of products may be split, which simplifies calculations since we avoid computing the joint distribution of two stochastic variables and then integrate with respect to that distribution. Instead, covariance is accounted for by changing drift rates. The economic interpretation is that the unit of account is changed; usually we think of the bank account as numeraire, but any traded asset may be used, for instance a bond. For the arbitrage free price process π_t on some stochastic claim $\mathcal{X}$ paid out at T we therefore obtain

$$\pi_t = E_t^Q \left[e^{-\int_t^T r_s ds} \cdot \mathcal{X} \right] = p(t,T) E_t^T [\mathcal{X}], \tag{3}$$

where $E_t^T[\cdot]$ denotes expectation with respect to the forward neutral measure Q^T. Simply put, we account for the correlation between the discount factor and the payoff by switching to a new measure[1]. The term $p(t,T)$ is the price at t of a standardized zero coupon bond giving one certain unit of account at time T.

It is as usual defined as the risk-neutral expectation over the short interest rate r_t:

$$p(t,T) = E_t^Q\left[e^{-\int_t^T r_s ds}\right]. \tag{4}$$

2.1 Forward contracts

When deriving an expression for the term structure of lease rates, it is convenient to have an expression for forward contracts on the spot rent stream or the asset. This is also necessary for a discussion of the expectations hypothesis. We therefore begin by deriving expressions for these objects.

A forward contract is such that the holder of the contract pays a fixed amount determined today and receives a stochastic amount at a future date. The contract is set up such that its initial value is zero. In our case, forward contracts could be made on both the underlying rent stream and the asset.

We define $f(t,T)$ as the forward rate for renting the asset over an infinitesimally short interval at T. In equilibrium at time t the present value of the expected difference at time T between the forward rate and the instantaneous spot rent must equal zero. Thus

$$E_t^Q\left[e^{-\int_t^T r_s ds}\left(f(t,T) - X_T\right)\right] = 0. \tag{5}$$

Since $f(t,T)$ is a known at time t it can be moved out of the expectation:

$$E_t^Q\left[e^{-\int_t^T r_s ds}\right] f(t,T) = E_t^Q\left[e^{-\int_t^T r_s ds} X_T\right]. \tag{6}$$

Applying the definition of the bond price and rearranging gives

$$f(t,T) = \frac{E_t^Q\left[e^{-\int_t^T r_s ds} X_T\right]}{p(t,T)}. \tag{7}$$

It follows directly from (3) and (7) that $f(t,T)$ can be written as

$$f(t,T) = E_t^T\left[X_T\right]. \tag{8}$$

The asset forward price $F(t,T)$ must similarly be

$$F(t,T) = E_t^T\left[H_T\right]. \tag{9}$$

By inserting the definition of H_T into (9), applying iterated expectations we obtain

$$\begin{aligned} F(t,T) &= \frac{E_t^Q\left[\int_T^\infty e^{-\int_t^s r_u du} X_s ds\right]}{p(t,T)} = \frac{\int_T^\infty E_t^Q\left[e^{-\int_t^s r_u du} X_s\right] ds}{p(t,T)} \\ &= \frac{\int_T^\infty p(t,s) E_t^s\left[X_s\right] ds}{p\left(t,T\right)} = \frac{\int_T^\infty p(t,s) f(t,s) ds}{p(t,T)}. \end{aligned} \quad (10)$$

In particular as time to maturity goes to zero, the forward price converges to the asset price, H_t.

3 Term structure of lease rates

The underlying approach for determining the equilibrium term structure follows the basic principle used in Miller and Upton (1976), McConnell and Schallenheim (1983), Schallenheim and McConnell (1985), Grenadier (1995, 2002) and Stanton and Wallace (2002). The starting point is that leasing is equivalent to purchasing the service flow from the underlying asset for a specified period of time. In equilibrium the rent on leases of all maturities must adjust in such a way that the present value of the rental payments equal the present value of the acquired service flow.[2] From this equilibrium relationship it is straightforward to derive the term structure. The most natural analogy to the term structure of rents in the bond market is the term structure of swap rates. In both cases a fixed rate is exchanged for a fluctuating rate, which is either the short rent or the short interest rate depending on the contract.[3]

We denote by $R(t,T)$ the fixed rate at time t for using the asset from time t to time T. The present value of the lease payments must be equal to the present value of the service flow the asset provides during the same period. Equally put, the present value of the difference between the two flows must be equal to zero:

$$E_t^Q\left[\int_t^T e^{-\int_t^s r_u du}\left(R(t,T) - X_s\right) ds\right] = 0. \quad (11)$$

Since $R(t,T)$ is a known at time t it can be moved out of the expectation:

$$E_t^Q\left[\int_t^T e^{-\int_t^s r_u du} ds\right] R(t,T) = E_t^Q\left[\int_t^T e^{-\int_t^s r_u du} X_s ds\right] \quad (12)$$

Applying the definition of the bond price and rearranging, the term structure of lease rates is given by the following expression:

$$R(t,T) = \frac{E_t^Q\left[\int_t^T e^{-\int_t^s r_u du} X_s ds\right]}{\int_t^T p(t,s)ds}. \tag{13}$$

Expression (13) can be re-expressed in terms of forward rates, either on the short rent or the asset:

$$R(t,T) = \frac{\int_t^T p(t,s)f(t,s)ds}{\int_t^T p(t,s)ds} \tag{14}$$

$$R(t,T) = \frac{H_t - p(t,T)F(t,T)}{\int_t^T p(t,s)ds}. \tag{15}$$

We now show the results in (14) and (15). The nominator in (13) can be re-expressed in terms of the instantaneous forward rates or in terms of the forward asset price. In the former case we have

$$\begin{aligned} E_t^Q\left[\int_t^T e^{-\int_t^s r_u du} X_s ds\right] &= \int_t^T E_t^Q\left[e^{-\int_t^s r_u du} X_s\right] ds = \\ \int_t^T p(t,s) E_t^s\left[X_s\right] ds &= \int_t^T p(t,s)f(t,s)ds. \end{aligned} \tag{16}$$

From this formula (14) follows immediately. In order to prove (15) we have the following calculations:

$$\begin{aligned} & E_t^Q\left[\int_t^T e^{-\int_t^s r_u du} X_s ds\right] = \int_t^T p(t,s)f(t,s)ds \\ = \; & \int_t^\infty p(t,s)f(t,s)ds - \int_T^\infty p(t,s)f(t,s)ds \\ = \; & H_t - p(t,T)F(t,T). \end{aligned} \tag{17}$$

Expression (17) has the interpretation of the cost of the following two transactions: buying the asset at t while simultaneously entering into a forward contract to sell the asset at T. That is, the present value of leasing the asset over time $T - t$ equals the value of owning the asset over the same time period. This representation is similar to the one used by Grenadier (1995, 2002).[4]

3.1 Properties of the term structure

By taking the limit of (13) we obtain

$$\lim_{T\to\infty} R(t,T) = \frac{H_t}{\int_t^\infty p(t,s)ds} = \widetilde{r}_t H_t. \tag{18}$$

In the above, $\widetilde{r}_t$ denotes the yield-to-maturity of a consol bond.[5] Further, when the length of the lease goes to zero:

$$R(t,t) = f(t,t) = X_t = \delta_t H_t. \tag{19}$$

By differentiating (14) with respect to T we obtain

$$\frac{\partial R(t,T)}{\partial T} = \frac{p(t,T)}{\int_t^T p(t,s)ds}\left[f(t,T) - R(t,T)\right]. \tag{20}$$

As seen by expression (20) the term structure of lease rates is locally increasing in T whenever $f(t,T) > R(t,T)$ and vice versa. When $f(t,T) = R(t,T)$ it follows that $R(t,T)$ has a stationary point. The relationship between $f(t,T)$ and $R(t,T)$ is very similar to the relationship between marginal and average costs in basic microeconomics. It can be seen from expression (14) that $R(t,T)$ is a weighted average of instantaneous forward rates. Therefore if $f(t,T)$ is a consistently increasing or decreasing function of T, then so is $R(t,T)$. Also, if $f(t,T)$ is strictly concave or convex, then so is $R(t,T)$. Note also that even if $f(t,T)$ tends to infinity or zero as T increases, $R(t,T)$ still converges to the annuity of the asset price. Very informally, $R(t,T)$ is a smoothed version of $f(t,T)$.

3.2 The expectations hypothesis

Grenadier (1995, 2002) argues that the shape of the term structure of lease rates should reveal expectations of future short lease rates. This hypothesis has an obvious parallel to the expectations hypothesis of interest rates, which loosely says that the slope of the yield curve is related to expectations of future short interest rates. A more rigorous form of the expectations hypothesis states that forward interest rates are unbiased estimates of expected future interest rates. As is familiar, this form of the expectations hypothesis only holds if the interest rate is deterministic. With uncertain interest rates, the bias of the hypothesis will depend on the stochastic properties of the short interest rate and the degree of risk aversion against interest rate uncertainty that is prevailing in the market.

In this section we examine the factors that might bias a similar expectations hypothesis of lease rates. Continuing the parallel to interest rates and denoting objective expectations by $E_t^P[\cdot]$ we formulate the expectations hypothesis as in Grenadier (2002):

$$f(t,T) = E_t^P[X_T], \tag{21}$$

that is, the forward lease rate is an unbiased estimate of the future short lease rate.

We now study the expectations hypothesis more carefully and we give the instantaneous forward rate again for convenience:

$$f(t,T) = E_t^T[X_T] = \frac{E_t^Q\left[e^{-\int_t^T r_s ds} X_T\right]}{p(t,T)}. \tag{22}$$

The expectations hypothesis is therefore always true under the forward neutral measure. This measure is different for each maturity however, and has no simple link with objective expectations.

The forward rate in (22) can be rewritten using the definition of covariance[6]:

$$f(t,T) = E_t^Q[X_T] + \frac{Cov_t^Q\left(e^{-\int_t^T r_s ds}, X_T\right)}{p(t,T)}. \tag{23}$$

The forward rate is therefore an increasing function of the covariance between the discount factor and the short rent (or equivalently a decreasing function of the covariance between the short interest rate and the short rent). A high covariance with the discount factor implies that low short rents tend to be discounted less severely, and vice versa. This increases the present value of the rent flow and thus the forward rate. Note that covariance is in levels rather than stochastic increments. Even if the increments of the stochastic processes for the short rent and short interest rate are independent under the objective probability measure, it may still be that the processes are dependent in levels under the subjective probability measure Q. This is because the drift adjustments when moving to the subjective probability measure may in general induce correlation; in particular the drift adjustments may both be functions of the same stochastic market price of risk.

We further see that if at least one the two processes r_t and X_t is deterministic or they are uncorrelated under the risk-neutral measure Q, then the result reduces to

$$f(t,T) = E_t^Q[X_T], \tag{24}$$

that is, the expectations hypothesis holds true under the risk-neutral measure. Note that risk aversion against the rent decreases its risk-neutral drift rate, making the forward rate lower than the objective expectation. This is in contrast to the bond market, where risk aversion drives down the prices of bonds, or conversely pushes up the yield, making forward rates higher than objective interest rate expectation.

Assume the expectations hypothesis holds true under the risk-neutral measure. If in addition the short rent is deterministic or the economy is risk-neutral with respect to at least that process, then the distinction between the objective and the risk-neutral measure disappears. We then obtain

$$f(t,T) = E_t^P\left[X_T\right]. \tag{25}$$

To summarize, we find that if the short rent is deterministic, then the instantaneous forward lease rate is an unbiased estimator of future rents. This is somewhat similar to the expectations hypothesis of interest rates, which holds if and only if the interest rate is deterministic. However, the expectations hypothesis for rents may hold even if the short rent is stochastic. This is the case if the market is risk-neutral towards the rent process and in addition either the short interest rate is deterministic or the short rent and the short interest rate are uncorrelated in levels under Q. Since these requisites are unlikely to hold empirically, we would not expect to find a property market in which the expectations hypothesis of rents holds fully.

The fact that the expectations hypothesis does not hold in the general case implies that the shape of the term structure is not directly related to objective rent expectations. The shape instead depends on (*i*) the risk-neutral drift rate of the rent, (*ii*) the Q-covariance between the short rent and the short interest rate and (*iii*) the term structure of interest rates. Some intuition for this can be given by first examining expression (23). First, all else equal, the forward rent is an increasing function of the risk-neutral drift rate of the short rent. Second, the forward rate also depends on the Q-covariance between rent changes and interest rate changes. The more positive (negative) the covariance, the lower (higher) the forward rate. Third, the covariance term is weighted by the inverse of the corresponding zero coupon bond price. Thus, the forward rate also depends on the term structure of interest rates.

The definition of the fixed lease rate $R(t,T)$ implies that it is a weighted average of instantaneous forward rates with declining weights. Thus the level of forward rates feeds into the level of the fixed rate. Further, from expression (20) we see that the term structure curve, given by $R(t,T)$, is locally upward-sloping (downward-sloping) when the instantaneous forward lease rate is higher (lower) than the fixed lease rate of the same maturity. In conclusion, the shape of the term structure is a function of the forward rates and therefore depends on the three factors given above.

3.3 Interpretation of the term structure

Since the analysis above holds for any valid parameterization of the short lease rate and interest rate processes, we can interpret the term structure results in the previous literature within our framework. In Grenadier (1995) and Stanton and Wallace (2002) investors are risk averse but the interest rate is deterministic. This yields the version of the expectations hypothesis given by expression (24), i.e. the expectations hypothesis holds under the risk-neutral measure. Hence the shape of the term structure is determined by risk-neutral rent expectations. The monotonically upward-sloping (downward-sloping) term structure in the simulations in Grenadier (1995) is obtained when the risk-neutral drift rate is strictly positive (negative). The single humped term structure is obtained when the risk-neutral drift rate of the short rent is decreasing in the time argument and goes from positive to negative. Similarly, the empirically estimated term structures in Stanton and Wallace (2002) should be related to risk-neutral rent expectations. In Grenadier (2002) investors are risk-neutral and the interest rate is deterministic. With this setup the expectations hypothesis, according to expression (25), holds under the objective probability measure. Hence in this model the shape of the term structure has a one-to-one correspondence to rental expectations under the objective probability measure.

Our results underline that one needs to be careful when interpreting empirical term structures in the property market. Only with the tight restrictions that either the rent is deterministic or alternatively that market participants are risk-neutral and that interest rates are deterministic or uncorrelated with the rent, is it possible to directly infer objective market expectations from the shape of the term structure. As a result, the short rent may easily be expected to decrease under the risk-neutral measure but increase under the objective measure. Hence, an expected increase in the rent level may very well be consistent with a downward-sloping term structure. The opposite scenario, i.e. that the term structure is upward-sloping when the short rent is expected to decrease is, however, arguably more unlikely. Ignoring the effect of interest rate uncertainty, this would require the risk-neutral drift rate to be higher than the objective drift rate, which is a less likely scenario in the real world. Furthermore, different scenarios of interest rate uncertainty can also lead to different term structures for the same objective rent expectations. In the next section we parameterize our model to develop an understanding for the degree to which the expectations hypothesis is distorted in different scenarios for risk aversion and interest rate uncertainty.

4 A parameterized model

As demonstrated in the previous section, the ability to infer market expectations about the level of future lease rates from the term structure depends on how seriously the expectations hypothesis is distorted by risk aversion and interest rate uncertainty. In this section, we parameterize the framework derived earlier and perform simulations to study these questions. For tractability we first assume a simple geometric Brownian motion rent process and a Vasicek (1977) short interest rate process. Finally, we also consider a mean reverting rent process.

Thus assume the following under Q:

$$dr_t = \eta(\alpha - r_t)dt + \sigma_r dW_t \tag{26}$$

$$dX_t = \mu X_t dt + \sigma_x X_t dV_t \tag{27}$$

$$dW_t dV_t = \rho dt. \tag{28}$$

The well-known Vasciek interest process is computationally relatively tractable and incorporates the empirically relevant property of mean reversion. The parameter η quantifies the speed of mean reversion towards the long term value of the short interest rate, α. The term σ_r gives the rental volatility. Further, the parameter μ is the risk-neutral drift rate and σ_x denotes the volatility of the short rent. The term ρ is the correlation between the driving Wiener processes.

The risk-neutral drift rate of the short rent μ is further defined as

$$\mu = \mu^P - \lambda\sigma_x, \tag{29}$$

where μ^P is the objective drift rate and λ is the market price of risk. In the general case, the market price of risk can be stochastic and time dependent; in this parameterization we however assume a constant market price of risk.

Further, by using the definition of $f(t,T)$ it can be shown that (see the first section of the appendix for a derivation, where Vasicek bond prices are also given)

$$f(t,T) = X_t e^{\mu(T-t) - \frac{\sigma_x\sigma_r\rho}{\eta}\left(T-t-\frac{1-e^{-\eta(T-t)}}{\eta}\right)}. \tag{30}$$

The term structure of lease rates is easily derived using earlier results. That is

$$R(t,T) = X_t \frac{\int_t^T p(t,s) e^{\mu(s-t) - \frac{\sigma_x\sigma_r\rho}{\eta}\left((s-t)-\frac{1-e^{-\eta(s-t)}}{\eta}\right)} ds}{\int_t^T p(t,s)ds}. \tag{31}$$

Note that the expression for the instantaneous forward rate is analytical, while expression (31) is easily solved by means of a numerical integration.

4.1 Effect of risk aversion

In the following we analyze how risk aversion affects the term structure of lease rates, holding other factors constant. To suppress the effect of interest rate uncertainty, we assume a constant interest rate, that is $\eta = 0$ and $\sigma_r = 0$. From an economic viewpoint, the varying price of risk can be interpreted as the market reacting to new economic information that makes it more or less willing to take on risk. Alternatively, we could also think of separate markets that differ only with respect to their risk aversion.

For the numerical analysis we will use the following base case parameters:

$$\begin{array}{ll} \mu^P = 0.04 & \lambda = 0.15 \\ \sigma_x = 0.20 & r = 0.06 \\ X_t = 1. & \end{array}$$

In the base case the risk-neutral drift rate of the rent, $\mu = \mu^P - \lambda\sigma_x = 0.04 - 0.15 \cdot 0.20 = 0.01$, or 1%.

Figure 1a displays the base case. A constant positive risk-neutral drift rate implies, as discussed in the previous section, a monotonic upward-sloping term structure of the rent when the interest rate is deterministic.

Figure 1b shows the term structure when the risk aversion parameter is increased to 0.25, i.e. the risk-neutral drift rate is reduced to -1%. As discussed earlier, a negative risk-neutral drift rate implies a downward-sloping term structure.

Note that the objective drift rate of the rent is the same in both scenarios, i.e. the dramatic difference between the shape of the term structure in the two figures is only attributed to a slight change in assumptions regarding risk aversion. It is worth stressing that both choices of risk aversion parameter are compatible with risk-return relationships that typically can be found in the property market. This statement can be motivated in the following way.

Assume for tractability that the interest rate is constant and that the short rent follows a geometric Brownian motion, i.e. the base case scenario. With these assumptions the definition of the property value given by expression (1) simplifies to $H_t = X_t/(\mu - r)$. Hence, since r and μ are constants, the rent process and the property value processes are identical and consequently they have the same risk-neutral drift rate. Since we know from expression (2) that the risk-neutral drift rate of the asset price is equal to the risk free rate less the payout ratio (δ), we have that $\mu = r - \delta$. A risk-neutral drift rate of the rent and the property value of $\mu = 1\%$, as in the base case, implies a property market in which the payout ratio equals $\delta = r - \mu = 6\% - 1\% = 5\%$ and the required total rate of return equals $\mu^P + \delta = 4\% + 5\% = 9\%$. In the second scenario, the payout ratio and the required total rate of return equals 7% and

11% respectively. Both these scenarios can realistically be found in the property market.

Figure 1c shows the term structure when the base case is changed by substituting the positive objective drift rate for a negative drift rate of 1%, that is $\mu^P = -0.01$. As expected the term structure is downward-sloping since the risk-neutral drift rate is even more negative, namely -4%. Theoretically it would be possible to obtain a positive term structure when the objective drift rate of the short rent is negative, but this would require an assumption of a negative risk aversion parameter (since we assume a constant interest rate), which we find less likely.

The analysis here is consistent with that of Geltner and Miller (2001, chapter 30). In a discrete time setting, they argue that if the rent is expected to stay constant and the rent is discounted at a rate that is higher than the risk free rate, then the term structure will be downward-sloping. To say that the rent is to be discounted at a higher rate than the risk free rate is equivalent to saying that the risk-neutral drift rate is lower than the objective drift rate. So if the objective rent is flat, then risk aversion implies that the risk-neutral drift rate is negative, producing a downward-sloping term structure.

4.2 Effect of interest rate uncertainty

We now go on to study the impact of interest rate uncertainty and the correlation between the short interest rate and the short rent. The following parameters are used:

$$\begin{array}{ll} \mu^P = 0.04 & \lambda = 0.15 \\ \sigma_x = 0.20 & \eta = 0.12 \\ \alpha = 0.06 & \sigma_r = 0.03 \\ X_t = 1. & r_t = 0.06. \end{array}$$

The above parameters again gives a risk-neutral drift rate of the short rent equal to 1%, i.e. $\mu = 0.01$. Figure 2a-d show the simulation results. In addition to the case with $\sigma_r = 0.03$ there is also a dotted line corresponding to $\sigma_r = 0.02$. As seen in the figures, different assumptions regarding correlation changes the term structure significantly. The higher the correlation, the less upward-sloping the term structure becomes. Also the volatility of the short interest rate matters. If the correlation between the short rent and the short interest rate is positive, then a higher interest rate volatility will make the term structure more downward sloping (or as an intermediate effect single humped). In case of negative correlation, the reverse relationship holds true.

In Grenadier (1995, 2002), a single humped term structure is associated with expectations of new property supply in the medium term. However, as the above shows, the humped shape can also be the result of interest rate uncertainty.

4.3 Effect of trend reverting rent

In the previous sections rents are assumed to follow a lognormal distribution with constant drift rate. The literature on rental adjustment processes suggest, however, that the rent process is better described as mean-reverting or trend-reverting. Recent examples are Hendershott, Lizieri and Matysiak (1999) and Hendershott, MacGregor and Tse (2002). In these adjustment models, rents are constrained to return to their long run average. The gap between actual and trend level rent is found to have explanatory power for rent changes and suggests that rents revert towards the average rent level. Hendershott, MacGregor and Tse (2002) also estimate error correction models, in which they find significant error correction coefficients, which also indicates reverting rents.

In this section we present a parameterization of our basic model that allow for trend reversion in the rent process. As in Lo and Wang (1995), we implement trend reversion in a continuous time setting by modelling the logarithm of the stochastic process as a trending Ornstein-Uhlenbeck process. Thus assume under Q a Vasicek short interest rate and a log spot rent process, $Z_t = \ln X_t$, that follows a trending Ornstein-Uhlenbeck process:

$$dr_t = \eta(\alpha - r_t)dt + \sigma_r dW_t \tag{32}$$

$$dZ_t = [\gamma(\mu t - Z_t) + \mu]\, dt + \sigma_Z dV_t \tag{33}$$

$$dW_t dV_t = \rho dt. \tag{34}$$

The term structure becomes (for a derivation of $f(t,s)$ see the second section of the appendix)

$$R(t,T) = \frac{\int_t^T p(t,s) \left(\frac{X_t}{e^{\mu s}}\right)^{e^{-\gamma(s-t)}} e^{\mu s - \frac{\sigma_Z \sigma_r \rho}{\eta} G(s,t) + \frac{\sigma_Z^2}{4\gamma} H(s,t)} ds}{\int_t^T p(t,s)\, ds}, \tag{35}$$

where the following definitions apply:

$$\begin{aligned} G(s,t) &= \frac{1-e^{-\gamma(s-t)}}{\gamma} - \frac{1-e^{-(\eta+\gamma)(s-t)}}{\eta+\gamma} && (36) \\ H(s,t) &= 1 - e^{-2\gamma(s-t)}. && (37) \end{aligned}$$

Equation (33) is a simple way of modelling rents that follow a long term trend but due to occasional shocks or business cycles, are pushed away from the trend level. When this happens, rents tend to be pulled back to the trend level. When the economy is off the trend, the best forecast is that it will eventually converge back to trend, but nothing beyond that, i.e. when new major shocks might occur, can be predicted. In the second section of the appendix it is further shown that the specification of Z_t is also consistent with a trend-reverting rent under the objective probability measure.

To study the effect of trend-reversion on the term structure of fixed rates we use a constant interest rate and further

$$\begin{array}{ll} \mu^P = 0.04 & \lambda = 0.15 \\ \sigma_Z = 0.20 & \gamma = 0.2 \\ X_t = 1. & r = 0.06. \end{array}$$

This again gives a long term risk-neutral drift of $\mu = \mu^P - \lambda\sigma_Z = 0.01$, or 1%.

Figure 3a-c displays the term structure in the case when the initial rent is at its trend value ($X_t = 1$) as well as 50% below and 50% above. Since the rent in Figure 3b initially is below its trend value, the risk-neutral drift rate is initially higher than when rent is at its equilibrium. Hence, for shorter terms, the slope of the term structure curve is steeper compared to Figure 3a. When the rent follows a geometric Brownian motion the term structure can only be single humped as an effect of interest rate uncertainty. When rents are trend reverting, a single humped term structure can also occur due to the trend reversion. The humped term structure in Figure 3c, in which the rent is initially above the trend, occurs because the risk-neutral drift rate of the rent goes from negative (the short term reversion effect) to positive.

Finally, Figure 3d plots the case when the risk-neutral drift rate is negative ($\mu^P = 0.02$ which gives $\mu = \mu^P - \lambda\sigma_Z = -0.01$), but the rent is currently 50% below trend ($X_t = 0.5$). This leads to a term structure with humped shape that is initially upward-sloping. For shorter lease, the positive mean reversion effect is larger than the effect of the long term drift rate.

4.4 Discussion of simulation results

The simulations show that risk-aversion, interest rate uncertainty and trend reversion significantly affect the shape of the term structure in our model. These results once again underline that great care is needed when attempting to infer expectations of future rent from the term structure. A specific shape can be attributed to a number of scenarios for the economy and the specific property market. Hence, in order not to draw erroneous conclusions regarding objective rent expectations from the term structure, careful analysis is required.

Gunnelin and Söderberg (2002) and Englund *et al.* (2002) provide empirical evidence that can be interpreted in the light of the above simulations. Gunnelin and Söderberg found mainly positive term structures during the pronounced upswing in the office market of the Stockholm CBD during the 1980's. Since risk-neutral rent expectations, as discussed earlier, are pivotal for the slope of the term structure, this indicates that the expectations were mainly positive during this time period. Although it is difficult to measure risk-neutral expectations correctly, the fact that the risk free interest rate was higher than the payout ratio in the Stockholm CBD during the whole time period supports that this was the case.

Since the risk-neutral drift rate typically is lower than the objective, we would, however, expect the term structure to be less steeply upward-sloping than would be the case if objective rental expectations was the main determinant of the term structure. This is also consistent with the findings in Gunnelin and Söderberg. During the second half of the 1980's, rental expectations were extremely high in the Stockholm CBD and rents doubled during this time period. Although positive term structures were found, they were not as steeply upward-sloping as one would expect if objective rental expectations were the main factor determining the shape of the term structure.

Englund *et al.* (2002) study the term structure in the same office market during the time period 1998-2002. The rent increase during this period was of similar magnitude as that during the peak of the boom in the late 1980's. As was the case in the study of Gunnelin and Söderberg, the very high rental increase does not translate into steeply upward-sloping term structures. Instead, for most of the years the estimated term structures are trendless or slightly positive. If we once again look at risk-neutral rental expectations, this result seems plausible. The risk-free interest rate was lower than the payout ratio during the whole period under study, indicating that risk-neutral expectations were low or even negative. It should, however, be pointed out that the use of the difference between the interest rate and the payout ratio as an indicator of risk-neutral expectations is based on the assumption that the rent is at its trend value. The fact that rents nearly doubled from 1998 to 2002 indicates that this was not the case. Considering the severe down-turn of the Stockholm property market in the first half of the 1990's, it is more likely that the rental market exhibited a positive mean- or trend-reversion. As shown in Figure 3b and 3d, when rents are below trend, the term structure is more upward-sloping compared to when rents are at the trend level. Hence, taking trend reversion into consideration, the negative difference between interest rates and payout ratio may very well be consistent with a slightly positive term structure. Another possible explanation could be that correlation between interest rates and rents in the Stockholm office market was negative, which as shown in Figure 2, tends to increase the slope of the term structure. However, since no study has attempted to model the covariance under the risk-neutral measure between interest rate levels and rent levels in the Stockholm CBD, we cannot confirm or reject this hypothesis.

5 Conclusion

In this paper we have extended previous work on the term structure of lease rates by deriving equilibrium relationships in a general continuous time setting where both the short rent and the short interest rate is uncertain. Since the framework is non-parametric our results hold under very general conditions.

We show that risk aversion and interest rate uncertainty can significantly bias an expectations hypothesis of lease rates similar to that of interest rates. It is the risk-neutral expectation of future rents, not the objective, that in combination with the characteristics of the interest rate process determine the relationship between expected future lease rates and forward lease rates. As a result, objective expectations about future rent levels can not be directly inferred from an inspection of the lease term structure. The effect of risk-aversion and interest rate uncertainty on equilibrium rents in the local property market must first be taken into consideration when interpreting the term structure. To directly infer market expectations from an inspection of the term structure, without considering these aspects, can lead to erroneous conclusions.

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Notes

[1]For example, assuming that the rent and the short interest rate only have one source of uncertainty each, the T-forward neutral drift rate of the rent can be expressed as

$$\mu\left(X_t, t\right) + \rho_{X,r}\sigma_X\left(X_t, t\right) v\left(t, T\right),$$

where μ is the risk-neutral drift rate, σ_X is the volatility of the rent, v is the bond price volatility term and $\rho_{X,r}$ is the correlation between the rent and the interest rate. That is, the covariance $\rho_{X,r}\sigma_X v$ between the increments of the short rent and the bond price is used to adjust the risk-neutral drift rate. Note that the volatility term v will be a negative number to reflect that the short interest rate and bond prices move in opposite directions.

[2]This approach abstracts from transaction costs. For an in-depth discussion of transaction costs see Miceli and Sirmans (1999). In their static two-period model transaction costs are pivotal and induce landlords to offer lower rent on longer leases in order to minimize the turnover. Thus, their model implies that the term structure of real estate lease rates should generally be downward-sloping. However, this result is partly an effect of keeping the rent constant over the two periods. Higher rent in the second period would allow for an upward-sloping term structure. Nevertheless the result suggests that incorporating transaction costs in our model would result in less upward-sloping (more downward sloping) term structure for any given set of model parameter values.

[3]The term structure of swap rates is derived by Duffie and Singleton (1993). Denoting the swap rate by $R^r\left(t, T\right)$, a continuous payment version of their result is

$$R^r\left(t, T\right) = \frac{1 - p\left(t, T\right)}{\int_t^T p\left(t, s\right) ds}.$$

Observe the similarity between this result and equation (15).

[4]Grenadier (1995, 2002) prefers to work in terms of a call option with zero exercise price, denoted $C(H_t, 0, T)$. This contract differs from a forward on the asset only in that payment is made at origin rather than at maturity. Thus, to avoid arbitrage:

$$C(H_t, 0, T) = p(t, T)F(t, T) = \int_T^\infty p(t, s)f(t, s)ds.$$

This also gives an alternative characterization of the forward rate:

$$f(t, T) = -\frac{\partial C(H_t, 0, T)/\partial T}{p(t, T)}.$$

[5]A consol bond (denoted Co_t) gives an infinite continuous payment stream of

one unit of account. Its yield-to-maturity ($\widetilde{r}_t$) can be obtained as follows:

$$Co_t = \int_t^{\infty} p(t,s)ds = \int_t^{\infty} e^{-\widetilde{r}_t(s-t)}ds = \frac{1}{\widetilde{r}_t}.$$

[6]The definition of covariance is $Cov\,(u,v) = E\,[uv] - E\,[u]\,E\,[v]$. Thus,

$$\begin{aligned} Cov_t^Q\left(e^{-\int_t^T r_s ds}, X_T\right) &= E_t^Q\left[e^{-\int_t^T r_s ds} X_T\right] - E_t^Q\left[e^{-\int_t^T r_s ds}\right] E_t^Q\left[X_T\right] \\ &= E_t^Q\left[e^{-\int_t^T r_s ds} X_T\right] - p\,(t,T)\,E_t^Q\left[X_T\right]. \end{aligned}$$

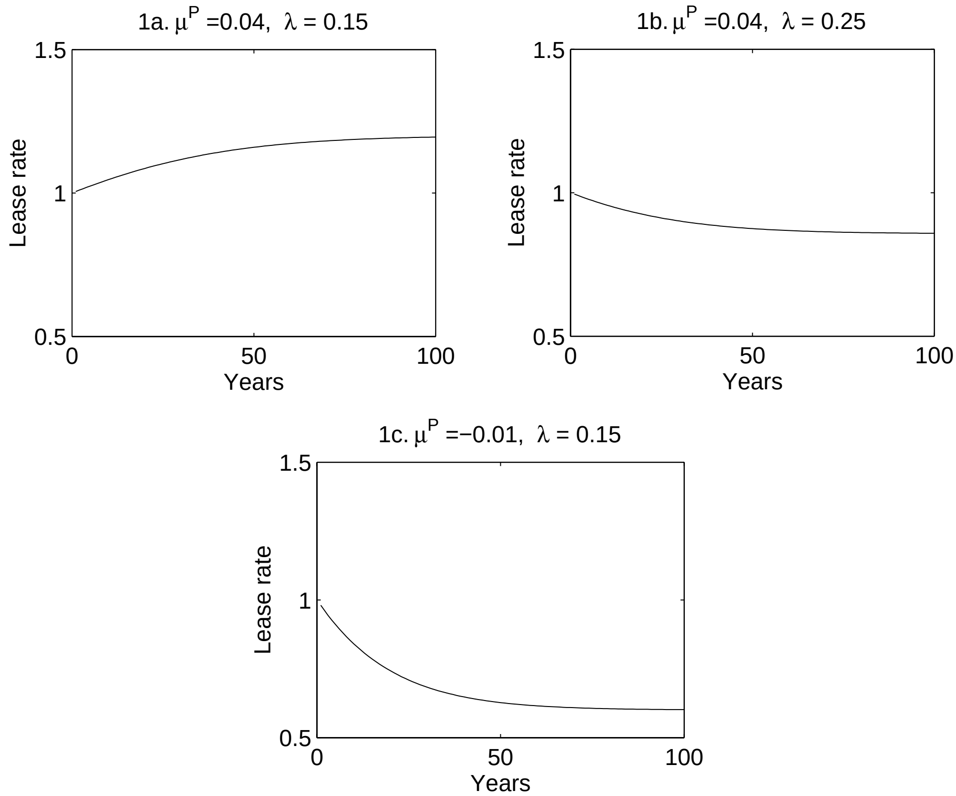

Figure 1a-1c. Effect of risk aversion (λ) on the term structure of lease rates.

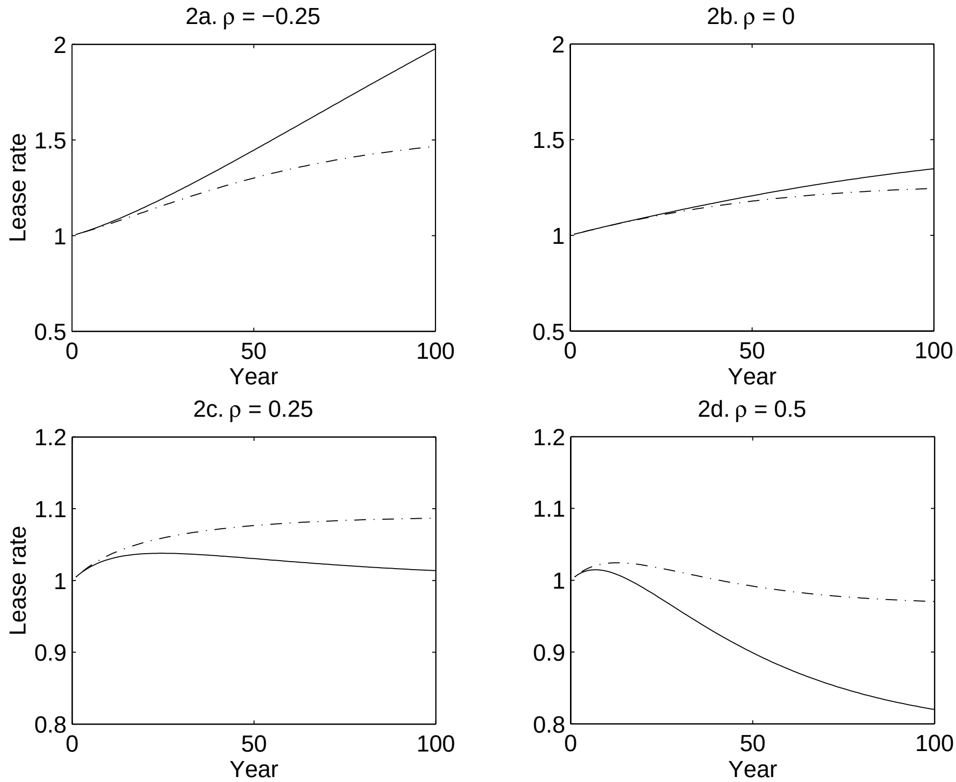

Figure 2a-2d. Effect of interest rate uncertainty on the term structure of lease rates. (Solid line is for rental volatility $\sigma_r = 0.02$, dotted line is for $\sigma_r = 0.03$.)

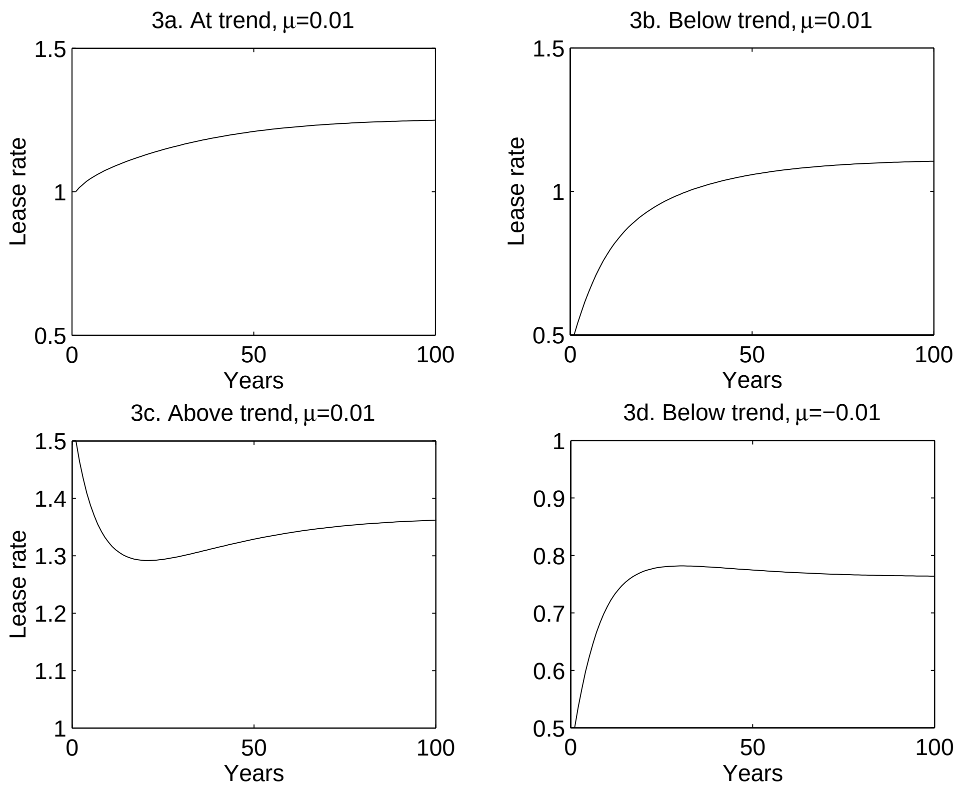

Figure 3a-3d. Effect of trend reversion on the term structure of lease rates. The trend level at origin is 1.

A Appendix

A.1 The parameterized model

The short interest rate and the short rent are defined by (26)-(28). Thus analytic bond prices follow Vasicek (1977):

$$p(t,T) = E_t^Q \left[e^{-\int_t^T r_s ds} \right] = e^{A(t,T) - B(t,T) r_t}. \tag{A.1}$$

Here the following definitions apply:

$$B(t,T) = \frac{1}{\eta} \left[1 - e^{-\eta(T-t)} \right] \tag{A.2}$$

$$A(t,T) = \frac{[B(t,T) - (T-t)] \left[\eta\alpha^2 - \sigma_r^2/2 \right]}{\eta^2} - \frac{\sigma_r^2 B^2(t,T)}{4\eta}. \tag{A.3}$$

Risk-neutral bond dynamics may be expressed as

$$dp(t,T) = r_t p(t,T) dt + v(t,T) p(t,T) dW_t \tag{A.4}$$

$$v(t,T) = -\frac{\sigma_r}{\eta} \left[1 - e^{-\eta(T-t)} \right]. \tag{A.5}$$

The rent process thus has the following Q^s dynamics:

$$dX_t = (\mu + \rho\sigma_x v(t,s)) X_t dt + \sigma_x X_t dV_t^s. \tag{A.6}$$

Further

$$\begin{aligned} & \int_t^s v(u,s) du = \\ = & -\frac{\sigma_r}{\eta} \int_t^s \left[1 - e^{-\eta(s-u)} \right] du \\ = & -\frac{\sigma_r}{\eta} \left| u - \frac{e^{-\eta(s-u)}}{\eta} \right|_t^s \\ = & -\frac{\sigma_r}{\eta} \left((s-t) - \frac{1 - e^{-\eta(s-t)}}{\eta} \right). \end{aligned} \tag{A.7}$$

Hence

$$f(t,s) = E_t^s [X_s] = X_t e^{\mu(s-t) + \rho\sigma_x \int_t^s v(u,s) du} \tag{A.8}$$

$$= e^{\mu(s-t) - \frac{\rho\sigma_x\sigma_r}{\eta} \left((s-t) - \frac{1-e^{-\eta(s-t)}}{\eta} \right)}. \tag{A.9}$$

A.2 The parameterized model with trend reversion

The short interest rate and the log short rent are defined by (32)-(34). Thus bond prices again follow Vasicek (1977) while the log rent process has the following Q^s dynamics:

$$dZ_t = \left[\gamma\left(\mu t - Z_t\right) + \mu + \rho\sigma_Z v(t,s)\right]dt + \sigma_Z dV_t^s. \tag{A.10}$$

The value of Z_s under Q^s at time t is given by the following expression:

$$\begin{aligned} Z_s &= e^{-\gamma(s-t)}Z_s + \gamma\mu\int_t^s e^{-\gamma(s-u)}udu + \mu\int_t^s e^{-\gamma(s-u)}du + \\ &\quad +\rho\sigma_Z\int_t^s e^{-\gamma(s-u)}v(t,s)du + \sigma_Z\int_t^s e^{-\gamma(s-u)}dV_u^s \qquad \text{(A.11)} \\ &= \left(Z_t - \mu t\right)e^{-\gamma(s-t)} + \mu s - \frac{\rho\sigma_Z\sigma_r}{\eta}\left(\frac{1-e^{-\gamma(s-t)}}{\gamma} - \frac{1-e^{-(\eta+\gamma)(s-t)}}{\eta+\gamma}\right) \\ &\quad +\sigma_Z\int_t^s e^{-\gamma(s-u)}dV_u^s. \qquad \text{(A.12)} \end{aligned}$$

Since Z_s has a driving Wiener process it is normally distributed with variance at time t equal to

$$V_t\left[Z_s\right] = \sigma_Z^2\int_t^s e^{-2\gamma(s-u)}du = \frac{\sigma_Z^2}{2\gamma}\left(1-e^{-2\gamma(s-t)}\right). \tag{A.13}$$

Hence

$$\begin{aligned} f(t,s) &= E_t^s\left[X_s\right] = E_t^s\left[e^{Z_s}\right] = e^{E_t^s[Z_s]+V_t[Z_s]/2} \\ &= \left(\frac{X_t}{e^{\mu t}}\right)^{e^{-\gamma(s-t)}} e^{\mu s - \frac{\rho\sigma_Z\sigma}{\eta}\left(\frac{1-e^{-\gamma(s-t)}}{\gamma} - \frac{1-e^{-(\eta+\gamma)(s-t)}}{\eta+\gamma}\right) + \frac{\sigma_Z^2}{4\gamma}\left(1-e^{-2\gamma(s-t)}\right)}. \end{aligned}$$

Note that

$$\lim_{\gamma\to 0}\left[\frac{1-e^{-\gamma(s-t)}}{\gamma}\right] = s-t. \tag{A.14}$$

This gives the special case when $\gamma = 0$.

The rent process may follow a trending Urnstein-Ohlenbeck process both under P and Q, given specific assumptions of risk aversion. We assume that risk aversion against the log rent process is $q = \lambda(\gamma t + 1)$. If we define $\mu^P = \mu + \lambda\sigma_Z$ then the objective dynamics are

$$\begin{aligned} dZ_t &= \left[\gamma(\mu t - Z_t) + \mu + q\sigma_Z\right]dt + \sigma_Z dV_t \\ &= \left[\gamma(\mu^P t - Z_t) + \mu^P\right]dt + \sigma_Z dV_t. \qquad \text{(A.15)} \end{aligned}$$

That is, the rent is trend reverting under the objective measure.

A.3 Term structure relationships

The fixed forward rate at t for renting over the future period $[T_1, T_2]$, to be paid continuously during the contract period, is denoted $R^F(t, T_1, T_2)$. Based on the principle that equivalent contracts have the same price, the following obtains:

$$R(t,T_2)\int_t^{T_2} p(t,s)ds = R(t,T_1)\int_t^{T_1} p(t,s)ds + R^F(t,T_1,T_2)\int_{T_1}^{T_2} p(t,s)ds.$$

By solving for the forward rate and using the definitions for the term structure of lease rates in (14) and (15) in section 2, we can express the forward contract as follows:

$$R^F(t,T_1,T_2) = \frac{\int_{T_1}^{T_2} p(t,s)f(t,s)ds}{\int_{T_1}^{T_2} p(t,s)ds}. \tag{A.16}$$

$$R^F(t,T_1,T_2) = \frac{F(t,T_1)p(t,T_1) - F(t,T_2)p(t,T_2)}{\int_{T_1}^{T_2} p(t,s)ds}. \tag{A.17}$$

Grenadier (1995, 2002) prefers to work in terms of call options with zero exercise price rather than forward contracts. This contract differs from a forward on the house only in that payment is made at origin rather than at maturity. Thus, to avoid arbitrage:

$$C(H_t,0,T) = p(t,T)F(t,T) = \int_T^{\infty} p(t,s)f(t,s)ds. \tag{A.18}$$

Section 2 expresses several results in terms of the instantaneous forward rate, but this can also also be reversed:

$$f(t,T) = -\frac{\partial C(H_t,0,T)/\partial T}{p(t,T)} \tag{A.19}$$

$$f(t,T) = R(t,T) + \frac{\partial R(t,T)}{\partial T}\cdot\frac{\int_t^T p(t,s)ds}{p(t,T)} \tag{A.20}$$

$$f(t,T) = \lim_{\delta\to 0} R^F(t,T,T+\delta). \tag{A.21}$$

The result (A.19) is obtained by differentiating (A.18) with respect to T. Result (A.20) follows by differentiating the expression for the term structure of lease rates in (14) with respect to T. Finally, (A.21) follows from (A.16). The first and last relationships above are given by Grenadier (2002) for the case with constant interest rate. In principle the above relationships could be used to compute an empirically observed term structure, and then calibrate it to some model.

Although we have considered continuous rent payments, one could also think of payments in discrete installments. If payments are made at n time points $t_1, ..., t_n$ then the fixed rate becomes, $R^d(t,T) = R(t,T) \int_t^T p(t,s)ds / \sum_{k=1}^n p(t,t_k)$.

A.4 Term structure with indexing

A real contract denoted $R^R(t,T)$ is adjusted according to the change in some stochastic index, denoted I_t, which can be thought of for instance as the consumer price index. Grenadier (1995) considers this case with a constant interest rate. With a stochastic interest rate, it is useful to introduce notation $p^R(t,T)$ for a real bond that gives one inflation adjusted unit of account:

$$p^R(t,T) = E_t^Q \left[e^{-\int_t^T r_s ds} \cdot \frac{I_T}{I_t} \right] = p(t,T) E_t^T \left[I_T \right] / I_t. \tag{A.22}$$

The payment streams must have the same present value:

$$R^R(t,T) = \frac{\int_t^T p(t,s)ds}{\int_t^T p^R(t,s)ds} R(t,T) = \frac{\int_t^T f(t,s)p(t,s)ds}{\int_t^T p^R(t,s)ds}. \tag{A.23}$$

From (A.22) we can see that the difference in expected nominal yield between the nominal and real bond depends on three factors: expected increase in the price index (inflation), risk aversion against inflation, and correlation between nominal interest rates and the price index. A negative risk-neutral drift rate in the price index, or correlation between the index and the short rent may lead to a higher real rent than nominal rent. The correlation between the rent process and the index has no affect on the term structure of real lease rates.

Rather than paying a constant rate, it could be specified to adjust according to some deterministic scheme. We denote that by φ_s. That is, the initial rent becomes,

$$R^{adj}(t,T) = \frac{\int_t^T p(t,s)f(t,s)ds}{\int_t^T \varphi_s p(t,s)ds} = \frac{\int_t^T p(t,s)ds}{\int_t^T \varphi_s p(t,s)ds} R(t,T). \tag{A.24}$$

Arguably the simplest implementation is the case when the rent is adjusted at a constant rate ϕ,

$$\varphi_s = e^{\phi(s-t)}. \tag{A.25}$$

Chapter 2

Leases with Upward Only Characteristics

Abstract

This paper considers a class of leases with indexed rents subject to a floor. Such leases have an upward only flavour to them, although not in exactly the same sense as in the traditional upward only institutional lease. After deriving a general result, three empirically important cases are considered: a modified upward only lease, the Swedish standard contract for commercial leases and the percentage lease. Analytical results and numerical examples are used to illustrate how the leases relate to other contract types.

1 Introduction

The real options approach to lease valuation has seen increased use in recent years, not least due to the seminal contributions by Grenadier (1995, 1996). This paper uses the options methodology to analytically consider a type of lease agreement, which has a rate indexed to some stochastic process as well as a binding floor. Such leases thus have an upward only flavour to them, although not in exactly the same sense as in the traditional upward only institutional lease. In the following an analytical pricing result is first derived for the general case. Thereafter the result is applied to three specific and empirically relevant cases, namely a modification of the upward only contract, the standard Swedish commercial lease agreement and finally the percentage lease.

The commercial leasing market in the UK has been undergoing significant change since the early 1990s (see for instance McAllister, 2001; Crosby *et al.*, 2003). The traditional institutional lease, typically a 25 year lease with upward only rent reviews every fifth year, is less common and average lease length has been reduced. The UK government has also attempted to promote greater flexibility in leasing conditions. The *Commercial Leases Working Group* has developed the voluntary *Code of Practice for Commercial Leases,* which was originally launched in December 1995. A second version of the code was presented in 2002, (Commercial Leases Working Group, 2002; Hewitt and Innocent, 2002). The new code consists of 22 recommendations and the sixth recommendation deals with rent reviews. The recommendation stresses that landlords

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should offer alternatives to upward only rent reviews and specifically mentions "up/down reviews to open market rent with a minimum of the initial rent". This paper will therefore evaluate such a modified upward only contract and in particular consider how it relates to the traditional upward only lease under varying assumptions.

The upward only lease has been analysed by several authors (notably Adams *et al.,* 2001; Ambrose *et al.* 2002; Baum *et al.*, 1996; Booth and Walsh 2001a, 2001b; Ward and French, 1997; Ward *et al.*, 1998). In terms of the pricing formula derived, best practice is arguably the paper by Ambrose *et al.* (2002), which is also the most recent paper. They present a closed form pricing equation that builds on the framework of Grenadier (1996) and this paper therefore uses that result for the upward only lease. Boot and Walsh (2001b) derive a similar pricing result and consider different assumptions for the rental process. Since they use the rate of a given upward only contract as a starting point, it becomes less convenient to make systematic comparisons across lease types. However, they consider the implication of different approaches to valuation (which is discussed in section 2 and 4.3 below) and provide empirical estimates of UK rental volatility (which are referred to in section 4.1).

The second contract considered is the standard Swedish commercial agreement. This contract is indexed to the consumer price index with the initial rate as a binding floor. As a result it is a close parallel to the modified upward only lease, except the indexation is to the consumer price index rather than the general rent level. This lease type is used in the majority of commercial lease agreements in Sweden (Gunnelin and Söderberg, 2003), but until now it appears never to have been explicitly priced. Historically, the floor has had little impact since Sweden, like other countries, has typically experienced inflation. There have however been suggestions that globalisation and increasing competition could lead to an environment of falling prices (e.g. Farrell, 2004). A decline in the Swedish consumer price index over the course of a twelve month period last happened in March 2004. While the likelihood of a sustained fall in prices is no doubt quite small, even the fact that the probability is positive has a potential pricing implication that may be interesting to explore.

Thirdly, the paper revisits the standard percentage contract. In such a lease, the rent may be raised above a floor level depending on the productivity of the leased space, usually measured by a business variable such as profit or turnover. The most common variant has payments tied to turnover, which is often referred to as a turnover or overage lease. McAllister (1996) surveys the use of turnover rents in the UK and the US, noting that they are much more common in the US. He also compares the traditional UK valuation approach, with a base rent and a yield incorporating both risk and expected growth, and traditional US techniques based on a discounted cash flow approach. Hendershott and

Ward (1999) study option-like features in the valuation of shopping centres. In particular, they conclude that ignoring the option implicit in a turnover lease, as in conventional discounted cash flow analysis, can lead to significant undervaluation of the lease. Further, Hendershott (2002) compares the turnover and upward only leases and finds that the turnover lease is better at aligning the interests of landlord and tenant. Hendershott and Ward (2002) employ the framework of Grenadier (1995, 1996) and consider leases with embedded renewal options and turnover clauses with different initial ratios of sales to threshold. Using Monte Carlo simulations, they find that by calibrating the renewal option and initial ratio it is possible to achieve an *ex post* distribution of internal rate of returns that is very similar to a simple lease with no options attached. Further, they extend their simulation to include a stochastic risk free interest rate, but find that it has very limited effect on the results. In this paper the standard percentage lease is revisited and explored using the derived analytical results. Finally, a combined inflation indexed and percentage lease with a binding floor is analysed.

The remainder of the paper is organised as follows. In section 2 the framework and some basic pricing results are presented. The indexed contract with a floor rate is analysed for the general case in section 3. Further, in section 4 the result is applied to the three specific leases mentioned above (modified upward only lease, Swedish standard contract and the percentage lease). Finally, a contract that combines both inflation indexation and a percentage clause is also analysed. Section 5 concludes.

2 Framework

Leasing can be interpreted as purchasing the use of the asset, and hence the service flow it provides, over a certain period (Miller and Upton, 1976). Equilibrium in a leasing contract is then such that the present value of lease payments equals the present value of the expected service flow of the leased asset. This approach assigns a price to every well defined contract, but does not in itself single out some contracts as optimal. In real estate, the seminal contribution is the continuous time model of Grenadier (1995). A risk neutral pricing methodology is used and the starting point in the analysis is the short rent process, which gives the cost of renting an infinitesimally short period. The short rent process can be derived within the context of a formal model as in Grenadier (1995, 2004) or, as is typically done in the applied literature, taken as exogenously given.

Due to transaction costs and lack of liquidity, prices can in general not be viewed as arbitrage free and should instead be interpreted as equilibrium prices. This is discussed in Grenadier (1995), and a similar reasoning is also found in Booth and Walsh (2001a, 2001b). Finding the risk neutral drift of a process is conceptually equivalent to finding its discount rate (discussed for instance in the textbook by Cochrane, 2000). Much like discount rates, risk neutral drifts can be based on intuitive reasoning, derived by calibrating pricing results to market data or by using a formal model such as the CAPM (Capital Asset Pricing Model).[1]

The framework used here is closely related that of Grenadier (1996). Thus assume there exists a short rent process S_u and a constant short interest rate r. All quantities and processes are expressed in nominal terms. Denoting the risk neutral expectation by $E_0^Q[\cdot]$, the present value of the service flow over a T period, $Y(T)$, is given by the following expression:

$$Y(T) = \int_0^T e^{-ru} E_0^Q [S_u] \, du. \tag{1}$$

The expression is of course valid for any short rent process. In particular, if the short rent follows geometric Brownian motion with a drift α under the risk neutral measure it holds that

$$Y(T) = S_0 \frac{1 - e^{-(r-\alpha)T}}{r - \alpha}. \tag{2}$$

We now review some basic contracts, which will later be used as benchmarks when analysing leases with a fluctuating rate subject to a floor. The simple fixed rate lease is a contract over a T time period with a fixed rate R paid in advance at a t interval. Using that that the present value of the lease payments must equal the present value of the service flow over the period, the pricing equation is as follows:

$$R\left[1 + e^{-rt} + ... + e^{-r(T-t)}\right] = Y(T). \tag{3}$$

By applying the formula for the sum of a geometric series it follows that

$$R = Y(T) \frac{1 - e^{-rt}}{1 - e^{-rT}}. \tag{4}$$

A slight modification of the fixed rate lease is an indexed contract $R^I(P_0)$ where rental payments are tied to an index P_0 that follows geometric Brownian motion with risk adjusted drift and volatility given by (μ_P, σ_P). Rent payment and review are assumed to occur at t and τ intervals respectively (τ must be a whole multiple of t); for a T period lease, the number of rent reviews is thus $n = T/\tau - 1$. Introducing notation $\widetilde{r} = r - \mu_P$ gives

$$R^I(P_0) = Y(T) \left(\frac{1 - e^{-rt}}{1 - e^{-r\tau}}\right) \left(\frac{1 - e^{-\widetilde{r}\tau}}{1 - e^{-\widetilde{r}T}}\right). \tag{5}$$

A special case of the indexed contract is when the indexation is to the short rent itself. Such a contract is usually referred to as an up- or downward adjusting agreement. It is in effect a series of discrete floating rate agreements. The initial up- and down lease rate is thus equal to

$$R^{UD} = R^{I}\left(S_0\right) = Y\left(\tau\right)\frac{1-e^{-rt}}{1-e^{-r\tau}}, \tag{6}$$

which is simply the annuity of the present value of the service flow until the next rent review.

The pricing result for the traditional UK upward only lease by Ambrose *et al.* (2002) is as follows in the case of discrete rental payments:

$$\Re = \frac{Y\left(T\right)\frac{1-e^{-rt}}{1-e^{-r\tau}}}{\sum_{k=0}^{n} e^{-r\tau k} E_0^Q\left[\max\limits_{0\leq j\leq k}\left\{S_{j\tau}\right\}\right]/S_0}. \tag{7}$$

This formula is based on the upward only lease adjusting to the evolution of the short rent. For this to hold for the upward only lease, as well as for the modified upward only lease below, it is necessary that the sport rent is time homogenous (i.e. at any time, expectations depend only on the value of the short rent today, and not on absolute time; this is the case for geometric Brownian motion). The upward only lease type can be computed for multiple reviews using the algorithm presented in Clapham (2003), based on a result by Öhgren (2001); see also Appendix A.4.

3 The general case

This section considers a contract type with a lease rate that is indexed to some stochastic process, but in addition has a binding floor. The pricing result is then applied to the specific cases of the modified upward only lease, the Swedish standard lease and the percentage contract.

In the general case, the indexation process may be denoted ϕ_τ and the floor rate F. Again rent payment and rent review occur at t and τ intervals respectively; the number of rent reviews in a T year lease is thus as earlier $n = T/\tau - 1$. From the definition of the contract it follows that rental payments are a function of $F\max\left\{1, \phi_{k\tau}\right\}$. The pricing equation therefore becomes:

$$\sum_{k=0}^{n} e^{-r\tau k} E_0^Q\left[\frac{1-e^{-r\tau}}{1-e^{-rt}} F\max\left\{1, \phi_{k\tau}\right\}\right] = Y\left(T\right). \tag{8}$$

That is

$$F = \frac{Y(T)\,\frac{1-e^{-rt}}{1-e^{-r\tau}}}{\sum_{k=0}^{n} e^{-r\tau k} E_0^Q\left[\max\{1, \phi_{k\tau}\}\right]}. \tag{9}$$

The analysis can easily be related to specific cases: The modified upward lease is indexed to the prevailing rent level, giving $\phi_\tau = S_\tau/S_0$. For the Swedish standard contract, the fluctuating rate is the floor increased by the consumer price index, which implies $\phi_\tau = P_\tau/P_0$. Finally, in a percentage lease, the fixed rate is adjusted by the ratio between a business variable, Z_τ, and a threshold, B. That is, in the percentage lease it holds that $\phi_\tau = Z_\tau/B$.[2]

Before considering specific cases in detail, some general properties will be discussed. For a contract that begins out of the money or at the money (i.e. $\phi_0 \leq 1$), the floor rate (F) and initial rate ($F \max\{1, \phi_0\}$) will coincide. However, for an in the money lease ($\phi_0 > 1$), the initial rate will be higher than the floor rate. Among the three lease types introduced above, both the modified upward only lease and the Swedish standard contract are by definition originally at the money, i.e. $\phi_0 = 1$. However, this obviously need not hold for the percentage lease.

The option feature of the floor implies that, all else equal, the higher the volatility of the indexation function ϕ_τ, the lower the floor and initial rate. Further, again all else equal, the higher the risk neutral drift of the indexation process, the lower the floor and initial rate. It is clear that the ratio between the floor and the fixed rate, F/R, is always less than one, implying that F is in the interval $[0, R]$. The ratio F/R does not depend directly on the stochastics of the short rent; computing it therefore does not require knowledge about the dynamics of the short rent (unless ϕ_τ itself is a function of the short rent).

If the lease starts at the money or out of the money and it is certain that it will never go in the money, then it is effectively a fixed rate lease. In this case it must hold that $F \max\{1, \phi_0\} = F = R$. This might result if the lease begins strongly out of the money or if drift is negative and large in relation to volatility. Further, if the lease starts at the money or in the money and it is certain that the lease will never go out of the money, then it is effectively a lease indexed to the ϕ_τ-process. It must therefore hold that $F \max\{1, \phi_0\} = \phi_0 F = R^I(\phi_0)$. This might result if the lease begins strongly in the money or if drift is positive and large in relation to volatility.

If the stochastic process ϕ_τ follows geometric Brownian motion with risk neutral drift and volatility given by (μ_ϕ, σ_ϕ), the following pricing formula can be derived:

$$F = \frac{Y(T)\,\frac{1-e^{-rt}}{1-e^{-r\tau}}}{\sum_{k=0}^{n} e^{-rk\tau}\left[\phi_0 e^{\mu_\phi k\tau} N(d_{1k}) + N(-d_{2k})\right]}. \tag{10}$$

Here $N(\cdot)$ denotes the cumulative normal distribution and further for $k > 0$

$$\begin{aligned} d_{1k} &= \frac{\ln \phi_0 + \left(\mu_\phi + \sigma_\phi^2/2\right) k\tau}{\sigma_\phi \sqrt{k\tau}} \\ d_{2k} &= d_{1k} - \sigma_\phi \sqrt{k\tau}. \end{aligned}$$

For $k = 0$, the parameter d_{1k} is defined as either plus or minus infinity depending on the sign of $\ln \phi_0$. In appendix A.1, there is a *Matlab* routine that implements the result.

By differentiating the initial rate with respect to ϕ_0 we can see how the initial rate reacts as the lease goes from being out of the money to being in the money at origin. Derivation gives the following result for $\phi_0 \leq 1$ (see appendix A.2 for a full discussion):

$$\frac{\partial \left[F \max\left\{1, \phi_0\right\}\right]}{\partial \phi_0} = \frac{\partial F}{\partial \phi_0} \leq 0. \tag{11}$$

Further, for $\phi_0 > 1$

$$\frac{\partial \left[F \max\left\{1, \phi_0\right\}\right]}{\partial \phi_0} = \frac{\partial \left[\phi_0 F\right]}{\partial \phi_0} \geq 0. \tag{12}$$

That is, the floor rate is decreasing in ϕ_0, which means the initial rate also decreases in ϕ_0 for an out of the money lease. However, the initial rate is increasing in ϕ_0 for an in the money lease. Taken together this implies that the initial rate has a minimum when the contract begins at the money ($\phi_0 = 1$). This is because the option value for the landlord is then maximized - every increase in the stochastic process raises the rent but a fall cannot cut the rent, because it is already at the floor level. As mentioned, a lease that starts heavily in the money is like an indexed lease and a contract that starts heavily out of the money is like a fixed rate lease. Hence for an at the money lease it must thus hold that:

$$F \leq \min\left\{R, R^I\left(\phi_0\right)\right\}. \tag{13}$$

Note that if the risk neutral drift of the indexation process is positive (negative) then it is the rate R (rate $R^I(\phi_0)$) that is higher among R and $R^I(\phi_0)$. The inequality is replaced by an equality when the volatility of the indexation process goes to zero.

4 Applying the general result

In the following the general result will be used to analyse three specific and empirically relevant cases in greater detail. Those contracts are the modified upward only lease, the Swedish standard contract and the percentage lease. In addition a contract that includes both inflation indexation subject to a floor as well as a percentage clause is considered.

4.1 Modified upward only lease

The traditional UK upward only lease may, as discussed in the introduction, become replaced by a modified lease that has the initial rate as a binding floor. Both types of upward only leases have the interpretation that they adjust based on the new initial level of a similar lease. It is also clear that they coincide in the case of only one review date. However, with more than one review date, the initial rate of the traditional upward only lease must be weakly lower than that of the modified (i.e. $\mathfrak{R} \leq F$). This is because the traditional contract depends on the maximum rent level at contract origin and at all review dates, while the modified contract only takes the original and present rent levels into account. Further, the traditional upward only feature becomes more important as the number of rent reviews and lease length increase - it boosts the likelihood that rents get stuck at a high level.

Using that the initial rate is lower in the traditional than the modified upward only lease, as well as the result in equation (**13**), we obtain

$$\mathfrak{R} \leq F \leq \min\left(R, R^{UD}\right). \tag{14}$$

The fixed rate R will be higher than the up and down rate R^{UD} when rents tend to increase over time under the risk neutral measure, and vice versa. The inequalities are replaced by equalities if the volatility of the short rent goes to zero.

Boot and Walsh (2001b) as well as Ambrose *et al.* (2002) present numerical results. Booth and Walsh (2001b) consider the present value of a traditional upward only contract for one or four upward only reviews and different assumptions for the rental process. They use both a geometric Brownian motion and a mean reverting process, but conclude that it is difficult to make comparisons across the two types of processes. Further, they show how, all else equal, the value of a given upward only contract increases with the number of rent reviews as well as the drift and volatility of the lease rate but decreases in the discount rate used (an equilibrium interpretation is that the initial rate instead varies so as to keep total value constant). Ambrose *et al.* (2002) compare the upward only and up and down rates for the case of one rent review for varying assumptions about short rent drift and volatility as well as real interest rate. Their numerical results reflect that the initial upward only rate is always lower than the initial up and down rate and, in particular, the difference increases with higher volatility of the short rent and decreases with higher real interest rates. In the case of one review date, the modified and traditional upward only contracts obviously coincide.

The differences between the four lease rates (R, R^{UD}, F, $\mathfrak{R}$) will be illustrated in this section using a figure and a table. It will be assumed that rents are paid quarterly in advance ($t = 0.25$). Regarding the volatility, Booth and Walsh (2001b) present an empirical estimate, based on IPD annual rental data over the period 1976-97, of just over 11% annually; Ambrose *et al.* (2002) use 10% and 20% for their numerical examples. Given that highly aggregated data might well smooth away some market volatility, 20% seems plausible.

Figure 1 illustrates the relationship between the four lease rates for a range of risk neutral drifts in the rental process. The short rent is assumed to follow a geometric Brownian motion with a volatility of 20% annually, the contracts are for 25 years and rent review, when applicable, is once every year (all parameter values given below the figure). Consistent with equation (14), the figure shows that the initial rate of the modified upward only contract is bounded from above by the minimum of the fixed rate (R) and the up and down rate (R^{UD}). The modified upward only lease converges to the fixed rate when rental growth is sharply negative and to the up and down rate when the short rent has a large rate of increase. The modified upward only contract always has a higher initial rate than the traditional upward only contract.

Table 1 further explores the initial rates of the contracts and the empirically important issue whether the modified upward only lease is likely to behave more like a traditional upward only lease or an up down contract in typical cases. The fixed rate R is always standardised to 1 in the table, which considers 10 and 25 year leases for different frequencies of rent review and rental growth rates. As expected, the longer the lease and the more frequent the rent reviews are, the greater the discrepancy between the traditional and modified upward only leases. The initial modified upward only rate is at the most 15% higher than that of the traditional upward only rate for the 25 year lease. As was mentioned in the introduction, average lease lengths have decreased since the early 1990s in the UK. This will tend to reduce the difference between the traditional and the modified upward only contract. Therefore, the impact of using the initial rate as a lower barrier, rather than having traditional upward only reviews, becomes less significant in a typical contract.

4.2 The Swedish standard contract

Most commercial rental agreements in Sweden follow the standard contract of the *Swedish Federation of Rental Property Owners* (in Swedish *Fastighetsägarna Sverige*). According to the "Index Clause for Non-Residential Premises", a certain fraction of the rent is linked to the monthly consumer price index published by Statistics Sweden[3]. The contract stipulates an annual rent review in October, but the indexed part of the rent cannot fall below the originally contracted rent. As a result, the rent may be left unchanged in the case of a fall in the consumer price index. That happened between October 1997 and October 1998 when the monthly consumer price index dropped by just below 1%. Gunnelin and Söderberg (2003) report that an average office lease length is about three to four years. The single most common indexation level is 100%, but lower levels of indexation also occur. For simplicity only the fully indexed variety will be considered here; a partially indexed contract will obviously be a linear combination of the fixed rate and fully indexed contract.

Denoting the floor rate of the fully indexed Swedish standard contract by F, and the initial rate of the contract indexed to the price level by R^I, it follows from (**13**) that

$$F \leq \min \left\{ R, R^I \right\}. \tag{15}$$

That is, the floor rate will be lower than both the fixed rate and the initial indexed rate.

Figure 2 shows the difference between the fixed rate, the indexed rate and the Swedish standard contract (R, R^I, F) as functions of the risk neutral drift in the price index. This is obviously a partial equilibrium analysis: the risk neutral drift of the price index is changed, holding everything else constant. In a full equilibrium analysis, presumably all other variables would change in response to a change in inflationary prospects. Both the short rent and the price index are assumed to follow geometric Brownian motion (parameters are given below the figure). Note that the fixed rate R is independent of the price index and is therefore simply a straight line in the figure. Further, the initial indexed rate R^I is larger than the fixed rate when the drift in the price index is negative and vice versa. The figure shows that when the drift in the price index is highly negative, the floor rate of the standard Swedish contract converges to the fixed rate. The reason is obviously that in this case the Swedish standard contract is unlikely ever to be increased. Further, when the drift in the risk neutral price is high, the Swedish standard contract converges to the indexed contract. With a zero growth rate, the fixed and indexed rates coincide. At the same time, the option value of the Swedish standard contract is maximised.

The figure illustrates how the different contracts are connected in principle. Table 2 casts further light on the discrepancy between the three contracts for representative parameter values. We consider the Swedish standard contract and the indexed contract in relation to the fixed rate, and that ratio is independent of the short rent. As a result, short rent dynamics need not be considered. It is however desirable to obtain some benchmark values for the risk neutral drift and volatility of the index process. Maintaining the assumption that the Swedish monthly consumer price index follows geometric Brownian motion, the volatility has been 1.8% annually during 1953-2002 (i.e. since the Korean War) while since 1995, volatility has averaged 1.3% per year. However, in the 1970s and 1980s volatility occasionally reached peaks of almost 4% annually. Regarding the drift rate, the Swedish central bank (*Riksbanken*) has an inflation target of 2% since 1992. Assuming that the target is credible and the market does not have a preference for inflation uncertainty, the risk adjusted drift of the consumer price index is less than 2%. Even if objective inflationary expectations are positive, risk neutral expectations could be zero or even negative.

Table 2 shows the fractions in R^I/R and F/R for different risk adjusted drift rates and volatilities in the price index. In line with Swedish practice rents are assumed to be paid monthly in advance and reviewed yearly ($t = 1/12$, $\tau = 1$). Results show that as long as the price index is expected to increase under the risk neutral measure and is not very volatile, the floor has virtually no pricing implications compared to the normal indexed contract (i.e. $F \approx R^I$). With negative drift and low volatility, the Swedish standard contract is indistinguishable from a fixed rate contract (i.e. $F \approx R$). With some volatility added, the initial rate of the standard contract may drop measurably below both of the two benchmark indexes. In particular, for a zero risk neutral rate of inflation and 3% annual volatility, a five year contract will have an initial rate that is 1.4% below both of the two benchmark contracts, which in this case will have the same rates. That reduction is thus wholly attributable to index volatility. On the other hand, if the risk neutral drift in the index is -2% then the standard contract will have an initial rate that is 4% lower than the indexed contract. However, this difference is mainly attributable to the fact that the floor makes the standard contract behave like a fixed contract when the drift is negative, and is not related to index volatility.

If the objective is to calibrate the model to current market conditions, it should be possible to extract information from inflation linked bonds. The Swedish central government has issued inflation linked bonds since 1994 and there are currently five series (called 3001-3005; Swedish National Debt Office, 2004). The two most recently issued series, namely series 3004 and 3005, have deflation protection for the capital amount. That means that the principal is at least redeemed at its nominal value, while coupon payments have no such restriction (Björkmo, 2002). Series 3004 and 3005 were issued 1 December 1998 and will mature 1 December of 2015 and 2028, respectively. As a result these bonds contain an implicit valuation of the risk that there will be a cumulative fall in the consumer price index from 1998 to 2015 and 2028, respectively.

Boot and Walsh (2001a, 2001b) use the price of an existing upward only lease as a starting point for their analysis of that lease type. This relates to the pricing equation for the Swedish standard contract, which is in fact a sequence of inflation linked zero coupon bonds with deflation protection. In a theoretically ideal world, there would be a liquid market in such zero coupon bonds of all maturities and their prices could be inserted directly into the pricing equation of the standard Swedish contract. In practice it would be necessary to use some form of interpolation from the limited number of inflation linked bonds traded in the market place.

4.3 Percentage lease

The percentage lease will now be considered using our analytic results. As mentioned, the lease type can be characterised as follows: If the ratio of a business variable such as turnover to a threshold value ($\phi_\tau = Z_\tau/B$) is larger than one, lease payments fluctuate with that ratio, but otherwise lease payments are set to a floor rate F; that is, payments vary according to $F \max(1, \phi_\tau)$. The percentage lease differs from the modified upward only lease and Swedish standard contract in the sense that it need not start "at the money" ($\phi_0 = Z_0/B = 1$). If the percentage lease begins out of the money, the initial level of the turnover is less than the threshold value ($Z_0 < B$) and the lease rate will be adjusted if and when the turnover rises above the threshold at a future review date. On the other hand it is also possible to start with a percentage lease that is "in the money" ($Z_0 > B$). Then the floor rate will only become relevant if the business variable decreases below the threshold level at a future review date.

As mentioned in the introduction, Hendershott and Ward (2002) employ Monte Carlo simulations to consider the case of a percentage contract combined with a lease renewal option using the Grenadier approach. This section will consider a much more limited exercise, namely to explore the impact of varying initial values of $\phi_0 = Z_0/B$ on the initial and floor rates. In section two it was concluded that the lease converges to a fixed contract for low ϕ_0, to an indexed contract for high ϕ_0 and finally that for $\phi_0 = 1$ the initial rate has a minimum. This will now be illustrated further.

Figure 3 plots the equilibrium floor, fluctuating and initial rate (that is $F, \phi_0 F$ and $F \max(\phi_t, 1)$) as functions of the initial ratio of turnover to threshold ($\phi_0 = Z_0/B$). For comparability across the numerical examples, the rates have been standardized by dividing by the fixed rate (hence F/R rather than F and so forth is plotted). As can be seen the result is that for a low initial ratio of Z_0/B, the percentage lease will have the same initial rate as the fixed lease. That is because it is highly improbable that the variable rate in the percentage lease will ever kick in. However, as the initial ratio Z_0/B increases, the floor and initial rates of the percentage lease gradually decrease, which reflects the growing option value in the percentage lease. When the option is at the money, the option value is maximised and a global minimum is attained for the initial rate. At this point, any increase in the turnover will boost the rent, but can never decrease it. As the lease becomes more strongly in the money, the initial rate increases. If the risk neutral drift in the business variable is negative, then the initial rate converges to a rate higher than the fixed rate, otherwise to a rate lower than the fixed rate. The floor rate goes to zero as the percentage lease becomes more strongly in the money at origin.

4.4 Combining contracts

The previous analysis can be extended to consider a contract that includes both inflation indexation subject to a floor as well as a percentage clause. The idea is that in an inflation indexed contract, it makes sense for a percentage clause to kick in only if growth in turnover has exceeded inflation, i.e. there has been a real increase in sales. The pricing equation for the contract is

$$F^I \frac{1-e^{-r\tau}}{1-e^{-rt}} \sum_{k=0}^{n} e^{-r\tau k} E_0^Q \left[\max\left\{1, \frac{I_{k\tau}}{I_0}, \frac{Z_{k\tau}}{B}\right\}\right] = Y(T)\,. \tag{16}$$

That is

$$F^I = \frac{Y(T)\,\frac{1-e^{-rt}}{1-e^{-r\tau}}}{\sum_{k=0}^{n} e^{-r\tau k} E_0^Q \left[\max\left\{1, \frac{I_{k\tau}}{I_0}, \frac{Z_{k\tau}}{B}\right\}\right]}. \tag{17}$$

The expectation is evaluated analytically in appendix A.3. A simple application of this result is to compare the initial rates for the following three contracts: a) Inflation indexing with floor, b) at the money percentage lease and c) the above combination of inflation indexation with floor and percentage lease. Note that the ratios of these contracts and the fixed rate (R) are independent of the short rent. Table 3 displays the initial rate of contracts a)-c) just mentioned. All else equal, the combined contract will always have a lower initial rate than any of the two individual contracts it is based on. However, depending on parameter values, the difference may be smaller or larger. For instance, if turnover has strong growth, the percentage clause is likely to dominate the indexation clause in the combined contract; the combined contract will then be very similar to the percentage contract. On the other hand, it could also be that inflation is significant while growth in turnover is low but volatile. The combined contract will then have the growth of the indexed contract and in addition upside potential due to the high volatility in turnover. In this case, the combined contract has a noticeably lower initial rate than either of its two constituent contracts.

5 Conclusion

This paper considers indexed lease agreements that are subject to a floor. Specifically, a modified upward only lease, the Swedish standard commercial contract and the percentage lease are discussed. The contracts all reflect that it is empirically common for lease agreements with variable rates to have a lower barrier.

The three specific variants considered in the paper are very similar from a pricing point of view, but differ somewhat in their economic implications. The floor matters significantly more for the modified upward only lease than for the Swedish standard contract that is tied to the consumer price index. The reason is of course the much lower volatility of the price index compared to rents. For a percentage contract the initial rate is also related to whether the contract begins out of the money, at the money or in the money. The paper further considers a contract combining a percentage and an inflation clause with an added floor. Such a contract would provide the tenant with both a guarantee in real terms as well as the incentive provided by a percentage arrangement. In return, the tenant obtains a further reduced initial rate.

There are many possible extensions. The explicit pricing results of this paper are based on the geometric Brownian motion and a constant interest rate. For the Swedish standard contract, the relationship between a stochastic interest rate and the price index could be important. For a percentage lease, a turnover process with mean reversion would potentially make it important to consider which part of the business cycle the contract is signed. Empirically, it should be possible to use data on rental contracts to value review clauses using an explicit options methodology.

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Notes

[1]If a stochastic process follows geometric Brownian motion and has a constant objective drift μ and volatility σ, its risk neutral drift can be expressed as $\alpha = \mu - \lambda\sigma$, where λ is the market price of risk. Given the continuous time CAPM, referred to in Grenadier (1995), it holds that $\lambda\sigma = \beta\left(r^M - r\right)$ where β is the beta of the process in relation to the market portfolio, whose expected excess return is denoted $r^M - r$. By solving for the market price of risk and using the definition of beta it follows that $\lambda = \left(r^M - r\right)\rho/\sigma_M$, where ρ is the correlation between the stochastic process and the market portfolio, which has standard deviation σ_M. Boot and Walsh (2001a, 2001b) suggest using not the market portfolio, but a traded asset that is highly correlated with the stochastic process.

[2]An alternative but equivalent characterisation of the percentage lease is obtained by noting that above the threshold B, the lease rate can be expressed a fraction, denoted $\eta = F/B$, of the business variable Z_τ (e.g. Hendershott and Ward, 1999). That is, following this characterisation the percentage lease rate is equal to the floor rate plus a fraction η of the business variable above the threshold. Algebraically

$$F \max\left\{\frac{Z_\tau}{B}, 1\right\} = \frac{F}{B}\max\left\{Z_\tau, B\right\} = F + \eta\max\left\{Z_\tau - B, 0\right\}.$$

[3]The monthly consumer price index is available at www.scb.se/default____2154.asp (the English version of Statistics Sweden's site) and the index clause (with a translation in English) is available at www.dokumentconcept.se/senaste/senastek1.htm.

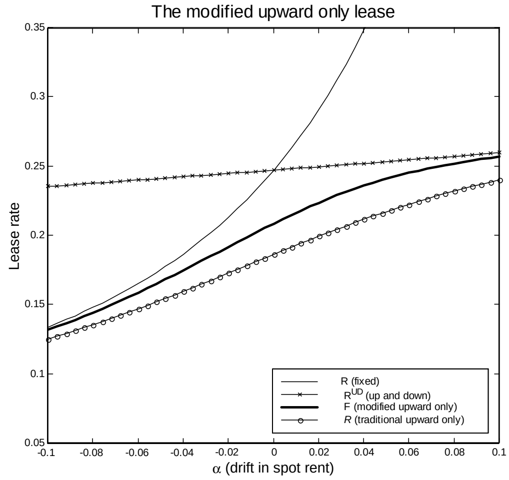

Figure 1. The graph gives the simple fixed rate (R) and the initial rates of the up and down, traditional upward only and modified upward only leases (R^{UD},$\mathfrak{R}$,F) as functions of the risk neutral growth rate of the spot rent (α). Parameters: $T = 25$, $\tau = 1$, $t = 0.25$, $S_0 = 1$, $\sigma_S = 0.20$, $r = 0.10$.

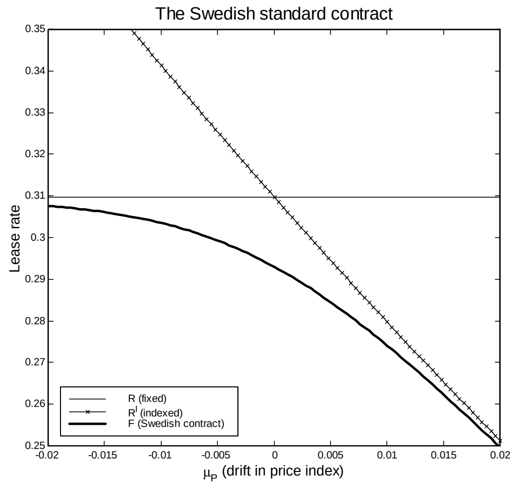

Figure 2. The graph gives the simple fixed rate (R) and the initial rates of the indexed and standard Swedish contract (R^I, F) as functions of the risk neutral growth rate of the price index (μ_P). Parameters: $T = 25$, $\tau = 1$, $t = 1/12$, $\alpha = 0.02$, $S_0 = 1$, $\sigma_P = 0.05$, $r = 0.04$.

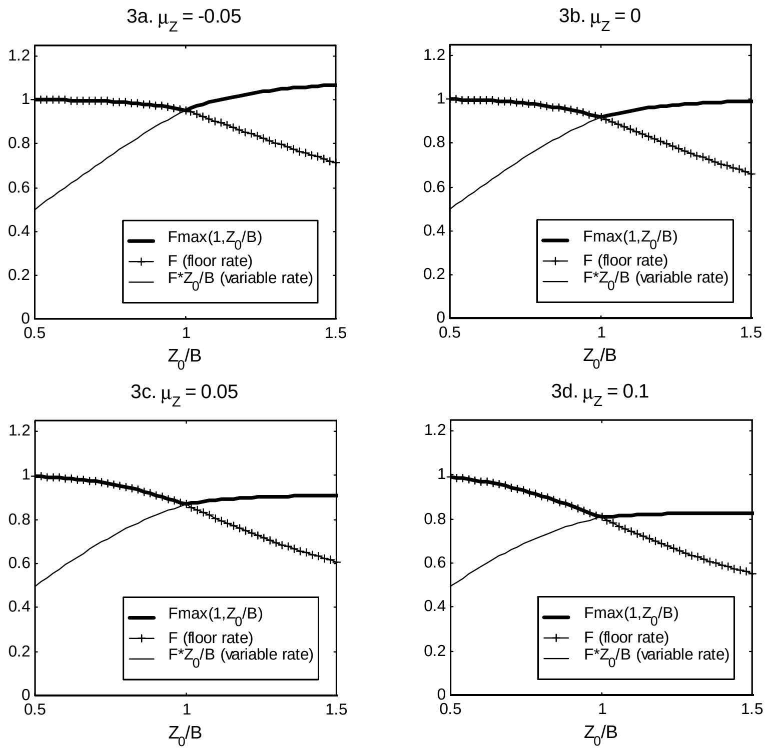

Figure 3. Graphs give percentage lease to fixed rate ratio ($F \max\{1, \phi_0\}/R$), as a function of the initial business variable to threshold ratio ($\phi_0 = Z_0/B$). Figure a)-d) give graphs for different risk neutral growth rates μ_Z in the business variable. Parameters: $T = 5$, $\tau = 1$, $t = 0.25$, $\sigma_Z = 0.11$, $r = 0.11$. (Spot rent parameters are irrelevant as they do not affect the ratio between the percentage lease and the fixed rate.)

Table 1. Modified upward only lease

Initial rates of up or down (R^{UD}), modified upward only (F)
and traditional upward only ($\mathfrak{R}$) leases in relation to fixed rate (R).

Rent are paid quarterly in advance. Interest rate is r=0.06.
Spot rent follows geometric Brownian motion with drift α and volatility $\sigma_S = 0.20$.
.

α	τ	R^{UD}/R	F/R	$\mathfrak{R}/R$	$F/\mathfrak{R}$
Lease length is T=10 years.					
-0.02	5	104%	95%	95%	100%
0.00	5	100%	93%	93%	100%
0.02	5	96%	91%	91%	100%
0.05	5	89%	86%	86%	100%
-0.02	1	108%	91%	85%	106%
0.00	1	100%	88%	82%	107%
0.02	1	92%	84%	78%	107%
0.05	1	81%	76%	72%	106%
Lease length is T=25 years.					
-0.02	5	107%	89%	85%	104%
0.00	5	100%	86%	82%	104%
0.02	5	93%	82%	79%	105%
0.05	5	66%	63%	61%	104%
-0.02	1	109%	86%	76%	114%
0.00	1	100%	83%	72%	115%
0.02	1	91%	78%	68%	115%
0.05	1	60%	57%	51%	113%

Table 2. Swedish standard contract

Initial rates of inflation indexed contract (R^I) and Swedish standard contract (F) in relation to fixed rate (R).

Rent is paid monthly in advance with yearly rent review. Interest rate is r=0.04.
The variables μ_P and σ_P give drift and volatility in price index; T gives length of contract.

μ_P	T	R^I/R	F/R (Swedish standard contract)		
			σ_P=0.01	σ_P=0.03	σ_P=0.05
-0.02	2	101.0%	100.0%	99.8%	99.4%
-0.01	2	100.5%	100.0%	99.6%	99.3%
0.00	2	100.0%	99.8%	99.4%	99.0%
0.01	2	99.5%	99.5%	99.1%	98.8%
0.02	2	99.0%	99.0%	98.8%	98.5%
0.03	2	98.5%	98.5%	98.4%	98.1%
-0.02	5	103.9%	100.0%	99.7%	99.1%
-0.01	5	101.9%	100.0%	99.3%	98.5%
0.00	5	100.0%	99.5%	98.6%	97.7%
0.01	5	98.1%	98.1%	97.4%	96.6%
0.02	5	96.2%	96.2%	95.9%	95.3%
0.03	5	94.3%	94.3%	94.2%	93.8%
-0.02	10	108.5%	100.0%	99.8%	99.1%
-0.01	10	104.2%	100.0%	99.3%	98.1%
0.00	10	100.0%	99.3%	97.9%	96.5%
0.01	10	95.9%	95.9%	95.2%	94.1%
0.02	10	91.9%	91.8%	91.7%	91.0%
0.03	10	87.9%	87.9%	87.9%	87.5%

Table 3. Combined contract

Initial rates of percentage lease, Swedish standard contract
(i.e. inflation indexed contract with floor) and combined lease in relation to fixed rate.

The contract is five years, with yearly rent review and rent paid quarterly in advance.
Volatilities in turnover and price index are 0.1 and 0.01 annually respectively and uncorrelated.
Percentage lease begins at the money. Interest rate is r=0.04.

Drifts		**Percentage**	**Inflation**	**Combined**
Turnover	Price index		(with floor)	
-0.03	0.02	98%	96%	95%
0.00	0.02	96%	96%	93%
0.03	0.02	92%	96%	91%
0.05	0.02	90%	96%	89%

A Appendix

A.1 *Matlab* implementation

The following *Matlab* routine computes fixed, up down and indexed rates as well as the floor and initial rates of the indexed contract subject to a floor (i.e. R, R^{UD}, R^I, F and $F \max\{1, \phi_0\}$).

Both the short rent and the index process are assumed to follow geometric Brownian motion. The risk neutral drift in the short rent is given by the parameter `a` (the initial short rent has been normalised to 1; the volatility of the short rent process does not affect any of the leases). Frequency of lease payments, frequency of rent reviews and lease length are parameterised by the time parameters `t`,`tau` and `T`. The risk free interest rate is given by `r`. Further, there are two parameters for the risk neutral drift and volatility of the index process (`mu`,`s`) and one for its initial value (`phi`).

```
function Rates = General(a,t,tau,T,r,mu,s,phi);
n=T/tau-1;
time=(1:n)*tau;
tfrac=(1-exp(-r*t))/(1-exp(-r*tau));
Y=1/(r-a)*(1-exp(-(r-a)*T));
d1=(log(phi)+(mu+s^2/2)*time)./(s*sqrt(time))/2^.5;
d2=d1-s*sqrt(time)/2^.5;
DR=exp(-r*time).*(phi*[1+erf(d1)].*exp(time*mu)+[1+erf(-d2)]);
RFixed=Y*(1-exp(-r*t))/(1-exp(-r*T));
RUpDown=1/(r-a)*(1-exp(-(r-a)*tau))*tfrac;
RIndexed=Y*tfrac*(1-exp(-(r-mu)*tau))/(1-exp(-(r-mu)*T));
F=Y/(max(phi,1)+0.5*sum(DR))*tfrac;
Initial=F*max(1,phi);
Rates=[RFixed RUpDown RIndexed F Initial];
```

A.2 Initial rate

In the following we show the following result given in equations (11) and (12) in the main text:

$$\frac{\partial\left[F \max\{1, \phi_0\}\right]}{\partial \phi_0} = \begin{cases} \frac{\partial F}{\partial \phi_0} \leq 0 & \phi_0 \leq 1 \\ \frac{\partial[\phi_0 F]}{\partial \phi_0} \geq 0 & \phi_0 > 1. \end{cases}$$

From equation (9) the definition of F is as follows:

$$F = \frac{Y(T)\,\frac{1-e^{-rt}}{1-e^{-r\tau}}}{\sum_{k=0}^{n} e^{-rk\tau}\left[\phi_0 e^{\mu_\phi k\tau} N(d_{1k}) + N(-d_{2k})\right]} = \frac{A}{B}. \tag{A.1}$$

Differentiation with respect to ϕ_0 is denoted $(\cdot)'$ .Thus:

$$B' = \sum\nolimits_{k=0}^{n} e^{(\mu_\phi - r)k\tau} N(d_{1k}) \geq 0. \tag{A.2}$$

By direct insertion:

$$\frac{\phi_0 B'}{B} \leq 1. \tag{A.3}$$

Finally:

$$\frac{\partial F}{\partial \phi_0} = F' = -\frac{A}{B^2}B' = -F\frac{B'}{B} \leq 0 \tag{A.4}$$

$$\begin{aligned} \frac{\partial[\phi_0 F]}{\partial \phi_0} &= [\phi_0 F]' = F + \phi_0 F' = F - \phi_0 F \frac{B'}{B} \\ &= F\left(1 - \phi_0 \frac{B'}{B}\right) \geq 0. \end{aligned} \tag{A.5}$$

A.3 Combining contracts

To complete the pricing exercise, the expectation must be evaluated. We assume that the price index and turnover follow the same geometric Brownian motion as before, i.e. they have risk neutral drifts and volatilities of (μ_I, σ_I) and (μ_Z, σ_Z). In addition, the correlation between the two processes is denoted ρ. The pricing result in Stulz (1982) is for a call option on the maximum of two assets, rather than the expected maximum of two stochastic processes and a constant. The following rewriting shows how the two expressions relate:

$$\begin{aligned} &\max\left\{1, \frac{I_{k\tau}}{I_0}, \frac{Z_{k\tau}}{B}\right\} = \max\left\{\max\left\{\frac{I_{k\tau}}{I_0}, \frac{Z_{k\tau}}{B}\right\}, 1\right\} \\ = \; &\max\left\{\max\left\{\frac{I_{k\tau}}{I_0}, \frac{Z_{k\tau}}{B}\right\} - 1, 0\right\} + 1. \end{aligned}$$

Based on the above, it is possible to use the result of Stulz (1982) and then add one. Using notation $\phi_0 = Z_0/B$:

$$\begin{aligned}
& E_0^Q\left[\max\left\{1, \frac{I_{k\tau}}{I_0}, \frac{Z_{k\tau}}{B}\right\}\right] \\
= \; & e^{\mu_I k\tau} N_2\left\{d_1\left(1, 1, \sigma_I^2, \mu_I\right), d_1\left(1, \phi_0, \sigma_{IZ}^2, \mu_I - \mu_Z\right), \widehat{\rho}_I\right\} \\
& + \phi_0 e^{\mu_Z k\tau} N_2\left\{d_1\left(\phi_0, 1, \sigma_Z^2, \mu_Z\right), d_1\left(\phi_0, 1, \sigma_{IZ}^2, \mu_Z - \mu_I\right), \widehat{\rho}_Z\right\} \\
& + N_2\left\{-d_2\left(1, 1, \sigma_I^2, \mu_I\right), -d_2\left(\phi_0, 1, \sigma_Z^2, \mu_Z\right), \rho_{IZ}\right\}. \qquad \text{(A.6)}
\end{aligned}$$

Here $N_2(\cdot)$ denotes the cumulative bivariate standard normal distribution and further

$$\begin{aligned}
d_1\left(X_1, X_2, \sigma^2, \mu\right) &= \frac{\ln\left(X_1/X_2\right) + \left(\mu + \sigma^2/2\right) k\tau}{\sigma\sqrt{k\tau}} \\
d_2 &= d_1 - \sigma\sqrt{k\tau} \\
\sigma_{IZ}^2 &= \sigma_I^2 - 2\rho_{IZ}\sigma_I\sigma_Z + \sigma_Z^2 \\
\widehat{\rho}_I &= \frac{\sigma_I - \rho_{IZ}\sigma_Z}{\sigma_{IZ}} \\
\widehat{\rho}_Z &= \frac{\sigma_Z - \rho_{IZ}\sigma_I}{\sigma_{IZ}}.
\end{aligned}$$

A.4 The Upward Only Lease

For the upward only lease given in (7) in the main text, it is necessary to evaluate the expectation of the maximum of the short rent observed at rent review dates with an interval of τ. This can be done analytically using a method presented by Öhgren (2001). More precisely, if the short rent follows a geometric Brownian motion with drift α and volatility σ, the following holds for positive k:

$$\begin{aligned}
x_k &= E_0^Q\left[\max_{0 \leq j \leq k}\left\{S_{j\tau}\right\}\right] = \frac{1}{k}\sum_{j=0}^{k-1} a_{k-j} x_j \qquad \text{(A.7)} \\
a_k &= N\left(d_{1k}\right) + e^{\alpha k\tau} N\left(d_{2k}\right) \\
d_{1k} &= -\left(\alpha - \sigma^2/2\right)\sqrt{k\tau/\sigma^2} \\
d_{2k} &= d_{1k} + 2\alpha\sqrt{k\tau/\sigma^2}.
\end{aligned}$$

Here $x_0 = S_0$ and $N(\cdot)$ denotes the standard normal cumulative density function.

Chapter 3

A Note on Embedded Lease Options

Abstract

Buetow and Albert (1998) discuss options embedded in lease contracts. They present a pricing framework, calibrate it using data from the National Real Estate Index and apply it using a numerical method known as the finite difference method with absorbing boundaries. This note extends the analysis. Analytic solutions are presented and some of the findings are discussed. The framework developed by Grenadier is used to compare indexed renewal options for different lease lengths.

1 Introduction

In a previous article, Buetow and Albert (1998) discuss the pricing of options embedded in real estate lease contracts. They focus on renewal options where the strike price is either tied to a price index or is a fraction of the prevailing lease price. The authors argue that correctly valuing such options is of importance to practitioners, but conclude that the complexity of their numerical method precludes its use to individual properties. The message is repeated in Albert and Buetow (2000) and the methodology is also referred to in a real estate case study (Albert *et al.*, 2000).

This study extends the analysis by avoiding the use of numerical methods, such as the finite difference approach with absorbing boundaries in Buetow and Albert (1998). The analytical derivatives of the indexed option are compared to the results earlier inferred numerically. There is a discussion of how the various parameters enter into the pricing formulas of the options. In addition, the results are related to the framework of Grenadier (1995, 1996), which enables a systematic comparison of the indexed option for different lease lengths.

1.1 Framework

Buetow and Albert (1998) value options using continuous time arbitrage pricing theory. The value of a contingent claim can often be expressed in terms of a so-called risk neutral expectation (i.e. where drifts have been adjusted to reflect the risk aversion prevailing in the market). The risk neutral pricing approach is

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traditionally motivated by arbitrage arguments in a frictionless market (Björk, 1998), but this is not always a realistic assumption for real estate. As argued by Grenadier (1995), risk neutral dynamics can also be inferred from general equilibrium arguments. In this approach, the price is seen as that which would tend to prevail in general equilibrium. This makes it possible to motivate the use risk neutral pricing even in the presence of market imperfections.

In the following, notation $E_t^Q[\cdot]$ denotes the risk neutral expectation at time t. The standard Black and Scholes (1973) call option formula, $C(\cdot) = C(A_t, K, \sigma, T-t, r)$ will also sometimes be used. This denotes the call option price as a function of the price of the underlying, strike price, volatility, time to maturity and the interest rate.

2 Valuing the options

Buetow and Albert (1998) consider the value at a European call option to enter into a τ period lease at a future time period T. Denoting the lease price and strike price by $R(T,\tau)$ and K respectively, the payoff of the option becomes $\max(R(T,\tau) - K, 0)$. In the following the notation of Buetow and Albert (1998) is used, where the lease length is assumed to be $\tau = 5$:

$$\max(R_T - K, 0). \tag{1}$$

Two different types of strike prices are considered:

- The lease price prevailing at origin at time 0, adjusted by the change in a price index. The strike price becomes $K = R_0 X_T$, if the index is normalized to 1 at origin.

- A fraction p of the prevailing market lease price at time T, or $K = pR_T$. This type of option is always in the money by construction and has the simple payoff $(1-p)R_T$.

In the following the more interesting indexed option is first discussed, and thereafter the second option is analyzed.

2.1 Indexed strike price

The value of the indexed option can be expressed as a risk neutral expectation:

$$O_t = e^{-r(T-t)} E_t^Q \left[\max(R_T - R_0 X_T, 0)\right]. \tag{2}$$

This general expression is valid for any stochastic processes. The case when the risk neutral processes are correlated geometric Brownian motions is:

$$\begin{aligned} dR_t &= \mu R_t dt + \sigma_R R_t dW_t^R \quad (3) \\ dX_t &= \pi X_t dt + \sigma_X X_t dW_t^X. \quad (4) \end{aligned}$$

The driving Wiener processes have correlation coefficient ρ.

A similar pricing problem was studied by Fischer (1978) and the value of the option is as follows at any time up to maturity

$$\begin{aligned} O_t &= R_t e^{(\mu-r)(T-t)} N(d_1) - R_0 X_t e^{(\pi-r)(T-t)} N(d_2) \quad (5) \\ d_1 &= \frac{\ln\left(R_t/\left[R_0 X_t\right]\right) + \left(\mu - \pi + \widehat{\sigma}^2/2\right)(T-t)}{\widehat{\sigma}\sqrt{T-t}} \quad (6) \\ d_2 &= d_1 - \widehat{\sigma}\sqrt{T-t} \quad (7) \\ \widehat{\sigma} &= \sqrt{\sigma_R^2 + \sigma_X^2 - 2\rho\sigma_R\sigma_X}. \quad (8) \end{aligned}$$

Here $N(\cdot)$ denotes the cumulative standard normal distribution.

2.2 A different viewpoint

By rewriting the payoff of the option, it becomes clearer how the pricing formula can be derived and some intuition can be provided. With notation $Z_T = R_T/X_T$, the option price in (2) can be expressed as

$$O_t = e^{-r(T-t)} E_t^Q \left[X_T \max\left(Z_T - R_0, 0\right)\right]. \quad (9)$$

This representation relates to methods known as reduction of the state space in differential equation theory or change of numeraire in probabilistic theory (Björk, 1998). The intuition is that while the strike price is stochastic in nominal terms, it is fixed in relation to the index X_t. As a result, it is possible to price the option in terms of the index ("real terms") using the Black-Scholes formula, and then convert back to the current price level:

$$O_t = X_t C(Z_t e^{(\mu-r)(T-t)}, R_0, \widehat{\sigma}, T-t, \widetilde{r}). \quad (10)$$

The following definitions apply,

$$\begin{aligned} Z_t &= R_t/X_t \quad (11) \\ \widehat{\sigma} &= \sqrt{\sigma_R^2 + \sigma_X^2 - 2\rho\sigma_R\sigma_X} \quad (12) \\ \widetilde{r} &= r - \pi. \quad (13) \end{aligned}$$

This is the same result as in (5), but expressed using the Black-Scholes formula. Here, Z_t may be interpreted as the real lease price and $\widetilde{r}$ as the real interest rate. Further, at origin the option is at the money and has value

$$O_0 = R_0 C(e^{(\mu-r)T}, 1, \widehat{\sigma}, T, \widetilde{r}).$$

By indexing the option, its value changes through two effects (Fischer, 1978):

- Adjusting the interest rate.
- Adjusting volatility.

First, a positive (negative) risk neutral drift of the price index lowers the adjusted interest rate, reducing (increasing) the value of a call option. Second, higher (lower) volatility increases (decreases) the value of the call option. Volatility is changed by the index itself as well as through the covariance between the two processes, as can be seen from (12) above. The first effect increases the value, while the second can go either way depending on the sign of the correlation coefficient. The more positive the correlation coefficient is, the lower the value of the option.

One would probably believe that indexing tends to reduce the value of the option: a generally increasing price level means on average a higher strike price at maturity. However, the indexed option could be more valuable as both the interest and volatility effects may in general go either way.

2.3 Greeks

By differentiating the value of the option with respect to its parameters, the so-called "greeks" are obtained. Buetow and Albert (1998) inferred the signs using numerical simulations. The analytical expressions are given in Table 1, and the signs conform to their results with one exception. They report that the price of the option is always an increasing function of the volatility in the lease price process. However the sign is ambiguous for the same reason as discussed above for the volatility of the price index. That is, there is a direct volatility effect and a covariance effect, where the latter may go either way. More specifically, the derivative of the option with respect to the volatility of the lease price can be written as $\kappa_R = M(\sigma_R - \rho\sigma_X)$ where M is a complicated expression that is always positive. Since the correlation coefficient can be at most equal to 1, a necessary condition for the derivative to be negative is that volatility of the lease price is smaller than that of the price index. Further, the option's reaction to changes in the volatility of the two stochastic processes is completely symmetric, i.e. the derivative with respect to volatility in the price index can be written as $\kappa_X = M(\sigma_X - \rho\sigma_R)$, with the same definition of M as before. As a result, the option can be decreasing in the volatility of at most one of the two processes at the same time.

2.4 Method of Buetow and Albert

Buetow and Albert (1998) price the payoff $\max(R_T - X_T R_0, 0)$ using the pricing PDE (partial differential equation). Risk neutral pricing and the PDE approach are equivalent, and the pricing formula presented above may be viewed alternatively as the risk neutral expectation or as the analytic solution to the pricing PDE. Buetow and Albert (1998) use a numerical method known as finite differences with absorbing boundaries. Looking at the PDE one sees that the authors have used the drift r, which is thus the postulated risk neutral drift of the index and lease price processes.

Regarding the lease price, Buetow and Albert (1998) make the following statement: "Since the value of income-producing real estate is a direct function of the expected rental stream, then it is easily assumed that both rent and price follow the same stochastic process". It is well known that the risk neutral drift of a traded asset is equal to the short interest rate less any dividend yield (e.g. Björk, 1998). Since the asset pays out a dividend (the value of the lease service flow), its risk neutral drift must be less than the risk free rate. When considering a stochastic process that is not a traded asset (e.g. a price index) then the risk neutral drift is not defined by the interest rate in general. In order to find the risk neutral drift in this case, one must infer its value from traded contracts that depend on the price index (such as real bonds), or resort to some theoretical equilibrium model (such as the capital asset pricing model).

Buetow and Albert (1998) price indexed renewal options by calibrating volatility using data from the National Real Estate Index. Their results would also be affected by the choice of risk neutral drifts for the lease price and index.

2.5 Numerical algorithms and an example

For more complicated derivatives or stochastic processes it may be difficult to obtain simple analytic results. Although one could then consider the numerical method of Buetow and Albert (1998), a Monte Carlo valuation is also feasible. This proceeds as follows: (1) Find the joint risk neutral distribution of the lease price and the index at time T. (2) Generate an outcome from the joint distribution. (3) Compute the option payoff for this outcome and discount back to obtain the present value. (4) Repeat steps 2 and 3 a large number of times, for instance 10000 times. (5) The average in step 4 is the estimated price of the call option.

The above can be used to evaluate the case when the initial value of the index is be specified as a lower bound, which is suggested as a task for possible future work by Buetow and Albert (1998). The option's price then becomes

$$O_t = e^{-r(T-t)} E_t^Q \left[\max\left(R_T - R_0 \max(X_T, 1), 0\right)\right]. \quad (14)$$

This is the same expression as in (2), except that X_T has been replaced by $\max(X_T, 1)$. In Table 2 the ratio between the indexed option with and without lower bound is given (for parameter values see below the table). As might be expected, the impact of the lower bound is less significant, the higher the risk neutral drift of the price index. Higher volatility of the price index will tend to be offset by the fact that the lower bound cuts off the positive potential of the volatility.

2.6 Strike price as a fraction of market price

We now consider the option where strike price is equal to a fraction p of the market price prevailing at the time of maturity. The drift of the lease price is the same as previously, i.e. it follows (3). Thus the value of the option at origin is given by

$$O_0 = e^{-rT}(1-p) E_0^Q [R_T] = (1-p) R_0 e^{(\mu - r)T}. \quad (15)$$

This contradicts the result reported by Buetow and Albert (1998):

$$O_0 = e^{-rT}(1-p) E [R_T] = (1-p) R_0 e^{(\mu - r)T + \sigma^2 T/2}. \quad (16)$$

Firstly, there is no information regarding under which measure the expectation is taken. For pricing purposes, it is the risk neutral measure that is appropriate. Secondly, a variance term appears in the result. The authors therefore conclude that the option would be "less valuable when attached to a lease in stable market, such as the northern New Jersey office market, than it would be in a volatile market, such as the Boston office market". However, since the value of the option is a fraction of the risk neutral expectation of the lease price, volatility does not matter all else equal. In a standard option on the other hand, the holder only has upside potential and therefore volatility increases the expected payoff.

3 Renewal options for different lease lengths

The aim of this section is to relate the previous results to the lease literature in finance. This will give the same type of valuation formula as before for the indexed renewal option, but in addition makes it possible to systematically compare the effect of the renewal option across lease lengths. It is of interest to analyze how the value of the indexed renewal option depends on the length of the lease. Also, since the work by Buetow and Albert (1998) is one of few continuous time real estate papers with an empirical section on lease option pricing, this helps to establish a link between the theoretical and empirical work.

Several earlier papers consider lease options. Grenadier (1995) presents a general discussion based on the premise that the price of a lease with an option should be equal to the value of its two parts, namely the pure lease and the option. Among other things, he specifically compares the value of a lease with and without a nominal renewal option for different lease lengths (illustrated in his Figure 3). The renewal option becomes more valuable for longer lease lengths with an increasing rent level. Grenadier (1995, 2002) also discusses tying lease payments to a price index. This arrangement does not include an option, but highlights the importance of indexing. Beardsley *et al.* (2000) use a Monte Carlo approach to value multiple indexed renewal options. They find that the value to repeatedly renew a lease at an indexed price can become significant, especially if real rents are increasing. Ambrose *et al.* (2002) further analyze an upward only adjusted lease common in the UK and many Commonwealth countries. In this case it is the lessor, rather than the lessee, that has an option to increase the lease rate at certain points in time to the currently prevailing lease rate. This section uses an analytic expression for the indexed renewal option to analyze its impact on the lease price for varying contract lengths.

The approach used follows the seminal work of Grenadier (1995, 1996), which is set in continuous time and abstracts from transaction costs. Further, he assumes that in equilibrium, the present value of lease payments should equal the present value of the service flow from the leased asset, which is also the case in Miller and Upton (1976), McConnell and Schallenheim (1983) and Schallenheim and McConnell (1985). Grenadier (1995, 2002) proceeds to derive models with endogenously determined supply sides and uses those models to consider the term structure of lease rates and many different leasing arrangements. Grenadier (1996) considers the effect of credit risk, i.e. a risky lessee. In that paper, exogenously specified real estate dynamics are used, and that is also the approach followed here.

3.1 Parameterization

In the following an exogenous short rent (similar to a dividend yield for a stock) is assumed to follow a geometric Brownian motion, as in e.g. Grenadier (1996), Beardsley *et al.* (2000), Ambrose *et al.* (2002) as well as Stanton and Wallace (2002). This gives a well known and simple expression for the lease price for different contract lengths. It is then straightforward to derive results for renewal options with indexed strike price.

The short rent (V_t) is thus defined by the following dynamics:

$$dV_t = \mu V_t dt + \sigma V_t dW_t. \tag{17}$$

Following Grenadier (1995) the equilibrium price of the real estate asset (H_t) is assumed equal to the present value of the short rent stream

$$H_t = E_t^Q \left[\int_t^\infty e^{-r(s-t)} V_s ds \right] = V_t \int_t^\infty e^{(\mu - r)(s-t)} ds = \frac{V_t}{r - \mu}. \tag{18}$$

It is thus necessary that the interest rate is lager than the drift of the short rent for the real estate asset to have a finite price. Further, introduce notation for the difference between the interest rate and the drift in the short rent, $\delta = r - \mu$. Since $V_t = \delta H_t$, it is possible to interpret δ as a yield or payout ratio. The risk neutral dynamics of the real estate asset price, which in our case is the same as for the short rent, can now be written

$$dH_t = (r - \delta) H_t dt + \sigma H_t dW_t. \tag{19}$$

This rewriting is convenient because it expresses risk neutral dynamics in terms of observable quantities (real estate price, payout ratio and volatility). It has been frequently used in the real options literature (e.g. Dixit and Pindyck, 1994). A short rent that follows geometric Brownian motion thus implies constant payout ratio and volatility. This may be empirically less plausible for lease lengths where business cycle dynamics are likely to be important.

The price $R(t, \tau)$ for leasing over a τ period is equal to the present value of the corresponding short rent stream (Grenadier 1995, 1996)

$$R(t, \tau) = E_t^Q \left[\int_t^{t+\tau} e^{-r(s-t)} V_s ds \right] = \frac{1 - e^{-\delta\tau}}{\delta} V_t = \left(1 - e^{-\delta\tau}\right) H_t. \tag{20}$$

Typically, payments are made in periodical installments throughout the lease and not as a lump sum at the beginning of the lease. The continuous lease *rate* paid throughout the lease is just the annuitized value of the lease price

$$\overline{R}(t, \tau) = \left[\frac{r}{1 - e^{-r\tau}} \right] R(t, \tau) = \left[\frac{1 - e^{-\delta\tau}}{1 - e^{-r\tau}} r \right] H_t. \tag{21}$$

The lease price $R(t, \tau)$ observed a different times t is not a traded asset, but in fact a different asset for each t. However with a constant payout ratio, the lease price is just a constant times the asset price and therefore the two have the same dynamics. Now consider a call option that expires at T to rent over a τ time period at indexed price K. Since the lease price follows a geometric Brownian motion, the same type of Black-Scholes analysis as before is valid. If the index continues to follow the geometric Brownian motion of (4) and be normalized to 1 at origin, we obtain from earlier results the price at any time prior to maturity

$$O_t = X_t C\left(\frac{R(t,\tau)}{X_t}e^{-\delta(T-t)}, K, \widehat{\sigma}, T-t, \widetilde{r}\right). \tag{22}$$

This is a repetition of the result given in (10), where $\widehat{\sigma} = \sqrt{\sigma^2 + \sigma_X^2 - 2\rho\sigma\sigma_X}$, $\widetilde{r} = r - \pi$ and $\delta = r - \mu$. The volatility of the lease price earlier referred to as σ_R is now given as σ, since all lease prices and the short rent have the same volatility. The special case of a constant strike price results when drift and volatility of the index is equal to zero.

3.2 Indexed renewal option

In order to employ the above, we consider a τ year lease with an indexed option to renew for a further τ years. The option will be valued at origin, i.e. at time 0. As also noted by Beardsley *et al.* (2000), there are two possible approaches to handle this:

- The lessee makes a separate payment to cover the cost of the option and leases at the standard price $R(0,\tau)$. The strike price in the renewal option is $R(0,\tau)$ adjusted by the index.

- The cost of the renewal option is embedded into the lease price. The strike price in the renewal option is equal to this new lease price adjusted by the increase in the price index.

In the first case the indexed option need only be evaluated at origin and its value is a fraction of the standard lease price, $R(0,\tau)C\left(e^{-\delta\tau}, 1, \widehat{\sigma}, \tau, \widetilde{r}\right)$. In the second case, the premium for the renewal option must be incorporated in the lease price. Since the price for the contract including the option must be equal to its two parts, the following equation is obtained

$$R^{renew}(0,\tau) = R(0,\tau) + C\left(e^{-\delta\tau}R(0,\tau), R^{renew}(0,\tau), \widehat{\sigma}, \tau, \widetilde{r}\right). \tag{23}$$

Once the lease price has been solved for, the corresponding rate can be computed

$$\overline{R}^{renew}(0,\tau) = \frac{r}{1-e^{-r\tau}}R^{renew}(0,\tau). \tag{24}$$

If it is the rate including the option that is known, the above reasoning can be used in reverse to solve for the pure lease rate without an option.

Figure 1 plots the rate as a function of lease length (parameter values are given below the graph). The graph shows the term structure with (upper line) and without (lower line) an embedded renewal option. With no indexed renewal option, the term structure is downward sloping in this case, i.e. the risk neutral drift of the lease price is negative. However, with the option the term structure becomes hump shaped. That is a result of two counteracting forces: As the lease period increases, effective volatility becomes larger, which drives up the price of the option. On the other hand, the expected future lease price decreases, which reduces the value of the option. The first effect dominates the second one for short horizons, widening the gap between the lease with and without the renewal option. For longer maturities however, the lease with a renewal option slowly converges to the one without.

The above analysis for specific contract types and dynamics is a special case of the general discussion in Grenadier (1995). In particular, the lease price with an option must always be at least as high as the price without an option.

4 Conclusion

Renewal options of many types are common in real world leasing arrangements. They are therefore an important phenomenon and as a result, tractable methods could potentially be very useful. Buetow and Albert (1998) present work involving just that, and here an attempt to extend the analysis has been made. The pricing was implemented in a manner that involved little more than the Black-Scholes formula. Further work could try to apply more flexible lease option models, perhaps by drawing on the theoretical lease valuation literature that has recently been developed.

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Table 1. The greeks.

$$
\begin{aligned}
\Delta_R &= \frac{\partial O}{\partial R} = N(d_1)e^{(\mu-r)(T-t)} \\
\Gamma_R &= \frac{\partial^2 O}{\partial R^2} = \frac{\varphi(d_1)e^{(\mu-r)(T-t)}}{R_t\sqrt{\sigma_R^2+\sigma_X^2-2\rho\sigma_R\sigma_X}} \\
\Delta_X &= \frac{\partial O}{\partial X} = N(d_2)R_0e^{-(r-\pi)(T-t)} \\
\Gamma_X &= \frac{\partial^2 O}{\partial X^2} = \frac{\varphi(d_2)R_0e^{-(r-\pi)(T-t)}}{X_t\sqrt{\sigma_R^2+\sigma_X^2-2\rho\sigma_R\sigma_X}} \\
\Theta &= \frac{\partial O}{\partial t} = -R_te^{(\mu-r)(T-t)}\left[\frac{\varphi(d_1)\sqrt{\sigma_R^2+\sigma_X^2-2\rho\sigma_R\sigma_X}}{2\sqrt{T-t}} - (\mu-r)N(d_1)\right] \\
&\quad -(r-\pi)R_0X_te^{-(r-\pi)(T-t)}N(d_2) \\
\kappa &= \frac{\partial O}{\partial \widehat{\sigma}} = R_0e^{(\mu-r)T}\varphi(d_1)\sqrt{T-t} \\
\kappa_\rho &= \frac{\partial O}{\partial \rho} = -\frac{R_0e^{(\mu-r)(T-t)}\varphi(d_1)\sigma_X\sigma_R\sqrt{T-t}}{\sqrt{\sigma_R^2+\sigma_X^2-2\rho\sigma_R\sigma_X}} \\
\kappa_R &= \frac{\partial O}{\partial \sigma_R} = \frac{R_0e^{(\mu-r)(T-t)}\varphi(d_1)\sqrt{T-t}\,(\sigma_R-\rho\sigma_X)}{\sqrt{\sigma_R^2+\sigma_X^2-2\rho\sigma_R\sigma_X}} \\
\kappa_X &= \frac{\partial O}{\partial \sigma_R} = \frac{R_0e^{(\mu-r)(T-t)}\varphi(d_1)\sqrt{T-t}\,(\sigma_X-\rho\sigma_R)}{\sqrt{\sigma_R^2+\sigma_X^2-2\rho\sigma_R\sigma_X}}
\end{aligned}
$$

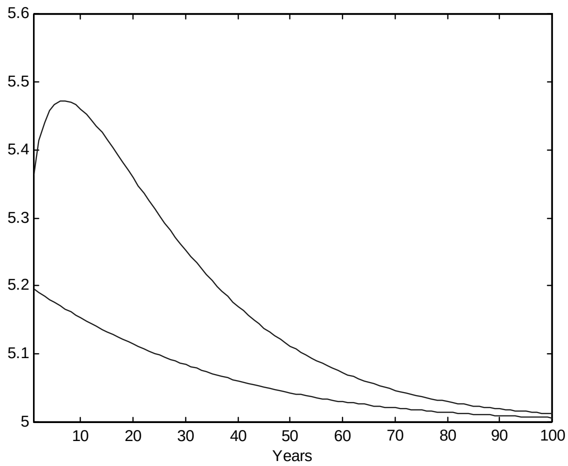
Here, as before $N(\cdot)$ denotes the cumulative standard normal distribution and $\varphi(\cdot)$ the standard normal density function. Further, d_1 and d_2 are defined as in (6) and (7).

Table 2.
Renewal option. Impact of lower bound on indexing.

		π				
		-0.02	**-0.01**	**0.00**	**0.01**	**0.02**
σ_X	**0.02**	0.68	0.81	0.96	1.00	1.00
	0.04	0.68	0.80	0.92	0.99	1.00
	0.06	0.68	0.78	0.89	0.96	0.99
	0.08	0.67	0.76	0.85	0.92	0.97
	0.10	0.66	0.74	0.82	0.88	0.93

The table shows value of the indexed renewal option with lower bound as fraction of that without lower bound (see also the subsection about numerical algorithms). Other parameter values: $\mu = 0.04$, $r = 0.05$, $\sigma_R = 0.10$, $\rho = 0$ and $T = 5$.

Figure 1.
Term structure of lease rates. Impact of embedded renewal option.



The lower downward sloping curve shows the standard term structure of lease rates. The hump-shaped curve shows the term structure of lease rates where a τ year lease includes a renewal option on a τ year lease, with strike price equal to the original lease rate adjusted by the increase in an index (see also the subsection about indexed renewal options for different lease lengths). Parameter values are, $\mu = r - \delta = -0.002$, $r = 0.05$, $\pi = 0.01$, $\sigma = 0.10$, $\sigma_X = 0.05$, $\rho = 0.5$, $H_t = 100$.

Chapter 4

A Note on Real Estate Index Linked Swaps

Abstract

In this paper we discuss the pricing of commercial real estate index linked swaps (CREILS). This particular pricing problem has been studied by Buttimer *et al.* (1997) in a previous paper in this journal. We show that their results are only approximately correct and that the true theoretical price of the swap is in fact equal to zero. This result is shown to hold regardless of the specific model chosen for the index process, the dividend process, and the interest rate term structure. We provide an intuitive economic argument as well as a full mathematical proof of our result. In particular we show that the nonzero result in the previous paper is due to two specific numerical approximations introduced in that paper, and we discuss these approximation errors from a theoretical as well as from a numerical point of view.

1 Introduction

The object under study in the present paper is a commercial real estate index linked swap (CREILS). The basic construction of such a swap is that the appreciation and yield of a given real estate index is swapped, quarterly, against the three months spot LIBOR. In an interesting paper previously published in this journal, Buttimer *et al.* (1997) presented a two-state model for pricing securities dependent upon a real estate index as well as upon an interest rate, and the model was then used to calculate the arbitrage free value of a CREILS. For this concrete application, the authors in Buttimer *et al.* (1997) used a numerical method based upon replacing their original continuous time model by a bivariate binomial tree, and it was found that, for a notional amount of $10,000,000, the value (to the receiver of the swap) was around $50. Buttimer *et al.* (1997) then proceed to discuss the sensitivity of their numerical results to changes in volatilities, correlations and the initial term structure.

The object of the present paper is to show that the results in Buttimer *et al.* (1997) are only approximatively true, in the sense that the arbitrage free theoretical value of the CREILS is in fact exactly equal to zero. More precisely we carry out the following program.

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- In Section 2 we present the institutional setup of the swap.
- We begin the theoretical analysis in Section 3 where we give a simple verbal arbitrage argument showing that the theoretical value of the swap in fact equals zero.
- In Section 4 we add to the verbal discussion in the previous section by presenting a very general mathematical framework for the swap along the following lines.
  - The real estate index is allowed to be a general (semimartingale) process with the only requirement that it should be possible to view it as the price of a traded asset.
  - The income (dividend) process associated to the index is allowed to be completely general.
  - The interest rate model is allowed to be completely general.
  - We assume absence of arbitrage.

  This framework is considerably more general than that of Buttimer *et al.* (1997) where the index is assumed to be lognormal with a constant dividend yield, and where the interest rate structure is given by a CIR short rate model.
- Within the above framework, and using the standard (martingale) machinery of arbitrage theory, we prove formally that the arbitrage free value of the swap is exactly equal to zero.
- In Section 5 we discuss why the pricing results in Buttimer *et al.* (1997) differ (although not much) from the correct value zero. We show that the reasons for the nonzero computational results in Buttimer *et al.* (1997) are due to two specific approximation errors introduced in the numerical calculations. We discuss these errors from a theoretical as well as a numerical perspective.

2 Institutional setup

In this section we give a description of the institutional setup of a commercial real estate index linked swap (CREILS). We follow Buttimer *et al.* (1997).

- The swap is assumed to be active over a prespecified time period. This period is subdivided by equidistant time points $t_0 < t_1 < \ldots < t_n$, and we denote by Δ the length of an elementary time interval, i.e. $\Delta = t_{k+1} - t_k$.

In a typical example the length $t_n - t_0$ of the total time period could be five years, whereas the length Δ of the elementary time interval would be three months.

- One leg of the CREILS is based upon a real estate index, henceforth denoted by I_t. This index varies stochastically over time and it also carries with it a (possibly stochastic) income (dividend).
- The other leg of the CREILS is based upon a market rate, such as the spot LIBOR rate, over the elementary time intervals.
- The CREILS is a sum of individual "swaplets", where the individual swaplet is active over an elementary interval $[t_{k-1}, t_k]$.
- At the end of each elementary interval $[t_{k-1}, t_k]$ the CREILS **receiver** will have the following cash flows from the swaplet active over the interval:
  - A cash inflow consisting of appreciation of the index plus all income generated by the index over the interval $[t_{k-1}, t_k]$.
  - A cash outflow equal to the spot LIBOR, plus a given spread δ, for the period $[t_{k-1}, t_k]$, acting on the ingoing index $I_{t_{k-1}}$.
- The CREILS **payer** will have the same cash flows with opposite signs.
- The CREILS would in real life be operating on a *notional amount*, rather than directly on the index value. This however is only a scaling factor, and without loss of generality we disregard this (or rather set it equal to one).

We assume that we are standing at time t, and that $t \leq t_0$. Our problem is to find the arbitrage free value, at time t, of the CREILS.

A typical (see Buttimer *et al.*, 1997) value of the spread δ could be $\delta = 0.00125\%$. For the rest of the paper we will follow Buttimer *et al.* (1997) in assuming that there is no spread, i.e. we assume that $\delta = 0$.

The main object of the present paper is to show that, regardless of any specific assumptions about the dynamics of the index, the income process, or the interest rate model, the arbitrage free value of the CREILS is in fact equal to zero.

For this strong result to hold, we will however need the following important assumption.

Assumption 2.1. We assume that the real estate index process I_t can be treated as the price process of a traded asset with a certain associated dividend (income) process.

The practical relevance of this assumption can of course be questioned, and in any concrete case its applicability depends on whether it is possible or not to replicate the index through continuous trading in a frictionless market. In particular, the assumption can be seen as somewhat unrealistic in the case of an index where the underlying is an illiquid asset such as commercial real estate. The assumption is however in complete agreement with Buttimer *et al.* (1997), who in fact assume Geometric Brownian Motion for the index, and model the income process as a constant dividend yield.

3 Verbal discussion

We begin our analysis by giving an simple verbal arbitrage argument, which shows that the value at an arbitrary time $t \leq t_0$ of the CREILS equals zero. Let us thus consider a trading strategy starting at time t and ending at t_n. The strategy consists of the following simple scheme which is repeated at each elementary time period $[t_{k-1}, t_k]$, for $k = 1, \ldots, n$.

- At time t_{k-1}, borrow the sum $I_{t_{k-1}}$ over the period $[t_{k-1}, t_k]$ at the spot LIBOR $L = L(t_{k-1}, t_k)$. Use all the borrowed money to buy one unit of the index.
- All the income generated by the index holdings during the interval $[t_{k-1}, t_k]$ is invested in the bank.
- At t_k sell the index to obtain I_{t_k}. Repay the loan, i.e. the principal $I_{t_{k-1}}$ plus the accrued interest $\Delta \cdot L \cdot I_{t_{k-1}}$ where $L = L(t_{k-1}, t_k)$ is the spot LIBOR for $[t_{k-1}, t_k]$. Collect the income that was generated and invested during the elementary period.

The net result of this strategy is that we obtain the following cash flow at each t_k for $k = 1, \ldots, n$.

- Plus: I_{t_k} (selling the index).
- Minus: $I_{t_{k-1}}$ (repayment of the principal of the loan).
- Plus: the value, at t_k, of the invested income during the period $[t_{k-1}, t_k]$.
- Minus: LIBOR on the borrowed capital during the period, i.e. $\Delta \cdot L \cdot I_{t_{k-1}}$.

We have thus exactly replicated the cash flow of the receiver of a CREILS. Since the strategy is self financing and the initial cost of setting up the strategy is zero, the arbitrage free value of the strategy, and hence that of the CREILS, has to equal zero.

4 Formal analysis

In this section we present a formal mathematical proof of our claim that the arbitrage free price of the CREILS is zero. The reasons for including this "extra" proof are as follows.

- It highlights the logical structure of the argument and shows more precisely where the various assumptions are needed.
- By presenting a formalized argument we can more easily compare our calculations to the computations made in Buttimer *et al.* (1997). In particular we will see that certain approximation errors are in fact introduced into the computations in Buttimer *et al.* (1997) and we will be able to study the relative importance of these numerical errors.

4.1 The mathematical model

Our chosen framework is a very general one (see Björk, 1999; Harrison *et al.*, 1981; Musiela *et al.*, 1997). We consider a financial market living on a stochastic basis (filtered probability space) $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$. Here the measure P is interpreted as the objective probability measure, whereas the σ-algebra $\mathcal{F}_t$ formalizes the idea of the information available to the agents in the economy at time t. We assume that the basis carries the following basic financial objects:

- An index process I_t. As a notational convention we consider the index process *ex dividend.*
- A cumulative dividend (income) process D_t. The interpretation is that if you hold the index over an infinitesimal interval $(t, t + dt]$ then you will receive the amount $dD_t = D_{t+dt} - D_t$ in dividend payments. Put in other words; over the interval $(s, t]$ the holder of the index will receive the (undiscounted) amount $D_t - D_s$.
- A short rate process r_t.
- A liquid bond market (at any time) for bonds of all possible maturities. The market price at time t for a zero coupon bond maturing at T is denoted by $p(t, T)$.
- A money market account process denoted by B_t, where by definition

$$dB_t = r_t B_t dt.$$

Note that we make no assumptions whatsoever about any specific dynamical structure of the index, the short rate, or the dividend process. For example; we do not assume that the processes above are driven by Wiener processes or that they are Markov processes (they are allowed to be arbitrary semimartingales). Our setup is thus extremely general and in particular it includes the model considered in Buttimer *et al.* (1997). In that paper, I_t is assumed to be Geometric Brownian Motion, the dividend process is assumed to be of the form

$$dD_t = \eta I_t dt,$$

(i.e. a constant dividend yield η), and the short rate process is assumed to be of Cox-Ingersoll-Ross type. The one assumption we make is that the market is free of arbitrage possibilities in the sense that there exists an equivalent martingale measure $Q \sim P$. We recall (see Björk, 1999) the following standard properties of Q.

- The normalized gains process G_t^B, defined by

$$G_t^B = \frac{I_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s. \tag{1}$$

  is a Q-martingale.

- Bond prices are given by the expression

$$p(t,T) = E^Q \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right]. \tag{2}$$

- For any contingent claim X, payed out at time T, the corresponding arbitrage free price process $\Pi(t;X)$ is given by the "risk neutral valuation formula"

$$\Pi(t;X) = E^Q \left[e^{-\int_t^T r_s ds} \cdot X \middle| \mathcal{F}_t \right]. \tag{3}$$

4.2 Pricing

We now proceed to price the CREILS within the above framework, and by convention this is done from the point of view of the receiver. Denoting the arbitrage free value (always for the receiver) at time t of the CREILS by $\Pi(t;CREILS)$ we have

$$\Pi(t;CREILS) = \sum_{k=1}^{n} \Pi(t;X_k)$$

where X_k denotes the net payments, at time t_k, to the CREILS receiver. We now go on to compute $\Pi(t;X_k)$ and by the "risk neutral valuation formula" (3) we have

$$\Pi(t; X_k) = E^Q \left[e^{-\int_t^{t_k} r_s ds} \cdot X_k \middle| \mathcal{F}_t \right].$$

Now, from the definition of the CREILS, it follows that X_k is given by

$$X_k = I_{t_k} - I_{t_{k-1}} + \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} r_u du} dD_s - \Delta L(t_{k-1}, t_k) I_{t_{k-1}}. \tag{4}$$

In this expression, the first term $I_{t_k} - I_{t_{k-1}}$ equals the appreciation of the index.

The integral term represents the total value, at time t_k, of all dividends generated by the index during the interval $(t_{k-1}, t_k]$. By convention all dividends are being invested in the bank account until time t_k, so the dividend dD_s generated during the small time interval $(s, s + ds]$ will, at time t_k have grown to

$$e^{\int_s^{t_k} r_u du} dD_s.$$

The integral term is thus the total value at time t_k of the entire income stream generated during the interval.

The third term obviously represents the cash outflow, which by definition is the spot LIBOR operating on $I_{t_{k-1}}$.

Remark 4.1 *Note the reinvestment of the dividends into the bank account. This is of course an institutional assumption, and to a certain extent it is crucial to our results below. The exact logical situation is as follows.*

- *For our results below to hold it is essential that dividends either are paid out directly at the time they are generated by the index, (i.e. the CREILS receiver obtains the amount dD_s at time s) or are being reinvested in a traded asset and paid out at time t_k.*
- *Exactly which asset that is used for reinvesting the dividends is, from the point of view of our calculations, irrelevant. We have by convention chosen the bank account, but the dividends could in fact be invested in any traded asset (or reinvested in the index) without affecting our results.*

We now go on to compute $\Pi(t; X_k)$ and to this end we recall that the spot LIBOR $L(t_{k-1}, t_k)$ for the period $[t_{k-1}, t_k]$ is given by the relation

$$p(t_{k-1}, t_k) = \frac{1}{1 + \Delta L(t_{k-1}, t_k)},$$

so in particular we have

$$\Delta L(t_{k-1}, t_k) = \frac{1}{p(t_{k-1}, t_k)} - 1.$$

We can thus write X_k as

$$\begin{aligned} X_k &= I_{t_k} - I_{t_{k-1}} + \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} r_u du} dD_s - \left(\frac{1}{p(t_{k-1}, t_k)} - 1 \right) I_{t_{k-1}} \\ &= I_{t_k} - \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} + \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} r_u du} dD_s, \end{aligned}$$

and we obtain

$$\begin{aligned} \Pi\left(t; X_k\right) &= E^Q \left[e^{-\int_t^{t_k} r_s ds} I_{t_k} \middle| \mathcal{F}_t \right] \\ &- E^Q \left[e^{-\int_t^{t_k} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} \middle| \mathcal{F}_t \right] \\ &+ E^Q \left[e^{-\int_t^{t_k} r_s ds} \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} r_u du} dD_s \middle| \mathcal{F}_t \right]. \end{aligned}$$

In this expression we can, by iterated conditional expectation, write the second term as

$$E^Q \left[e^{-\int_t^{t_k} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} \middle| \mathcal{F}_t \right] = E^Q \left[E^Q \left[e^{-\int_t^{t_k} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} \middle| \mathcal{F}_{t_{k-1}} \right] \middle| \mathcal{F}_t \right].$$

The inner expectation can now be simplified as

$$\begin{aligned} & E^Q \left[e^{-\int_t^{t_k} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} \middle| \mathcal{F}_{t_{k-1}} \right] \\ &= e^{-\int_t^{t_{k-1}} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} E^Q \left[e^{-\int_{t_{k-1}}^{t_k} r_s ds} \middle| \mathcal{F}_{t_{k-1}} \right] \\ &= e^{-\int_t^{t_{k-1}} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} \cdot p(t_{k-1}, t_k) = e^{-\int_t^{t_{k-1}} r_s ds} I_{t_{k-1}}, \end{aligned}$$

where we have used (2) together with the fact that the objects

$$I_{t_{k-1}}, \quad p(t_{k-1}, t_k), \quad e^{-\int_t^{t_{k-1}}}$$

are known at t_{k-1} and can thus be brought outside the conditional expectation.

The third term in the expression for $\Pi\left(t; X_k\right)$ can be written

$$E^Q \left[e^{-\int_t^{t_k} r_s ds} \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} r_u du} dD_s \middle| \mathcal{F}_t \right] = E^Q \left[\int_{t_{k-1}}^{t_k} e^{-\int_t^{s} r_u du} dD_s \middle| \mathcal{F}_t \right].$$

Collecting our results we finally obtain

$$\begin{aligned}
\Pi(t; X_k) &= E^Q\left[e^{-\int_t^{t_k} r_s ds} I_{t_k}\middle|\mathcal{F}_t\right] - E^Q\left[e^{-\int_t^{t_{k-1}} r_s ds} I_{t_{k-1}}\middle|\mathcal{F}_t\right] \\
&+ E^Q\left[\int_{t_{k-1}}^{t_k} e^{-\int_t^s r_u du} dD_s\middle|\mathcal{F}_t\right] \\
&= B_t \cdot E^Q\left[\frac{I_{t_k}}{B_{t_k}} - \frac{I_{t_{k-1}}}{B_{t_{k-1}}} + \int_{t_{k-1}}^{t_k} \frac{1}{B_s} dD_s\middle|\mathcal{F}_t\right] \\
&= B_t \cdot E^Q\left[G_{t_k}^B - G_{t_{k-1}}^B\middle|\mathcal{F}_t\right] = 0.
\end{aligned}$$

In the last equality we have used the fact that the normalized gains process G_t^B defined in (1) is a martingale under Q.

Since this holds for every k and the receiver value of the CREILS is given by

$$\Pi(t; CREILS) = \sum_{k=1}^{n} \Pi(t; X_k),$$

we have thus formally proved the following proposition, which is the main result of this paper.

Proposition 4.1 *Under assumption 2.1, the arbitrage free price of the CREILS equals zero at any time $t \leq t_0$, i.e.*

$$\Pi(t; CREILS) = 0.$$

Remark 4.2 In this section we have formally worked within a continuous time model. We note, however, that the results above are true also in a discrete time framework and that, in particular, Proposition 4.1 remains unchanged. The proofs remain the same, the difference simply being that all integrals are interpreted as sums.

5 The effects of approximation errors

As we have seen above, the theoretical value of the swap equals zero regardless of the model under consideration. In Buttimer *et al.* (1997), however, the authors computed a numerical (receiver) value of \$50 on a notional amount of \$10,000,000 and even if, in relative terms, this is very close to the true value it may still be interesting to see exactly where the numerical errors were introduced in Buttimer *et al.* (1997).

A closer look at Buttimer *et al.* (1997) reveals that, compared to their original model, the following three approximations were made by the authors:

- The original continuous time model was approximated by a discrete time tree model.
- The dividend process was approximated.
- There was an approximation made in the computation of the bond prices and hence of the LIBOR rates.

It is clear that numerical computations will in general produce approximation errors that cause estimated and theoretical values to diverge. However, as explained below the discrete time method employed by Buttimer *et al.* (1997) should still produce the zero result. The reason is that it is a pure arbitrage argument that holds also in a correctly specified discrete time model.

Even so, Buttimer *et al.* (1997) find a value that differs from zero. The basic reason for this appears to be that simple interest and dividend rates are approximated by continuous rates. We estimate that the errors introduced by these approximations are in the order of 10^{-5} to 10^{-6}, which is in line with the reported results. The side that is long in the real estate swap gains from an increase in the absolute difference between the short rate and the dividend yield.

Once simple rates are replaced by continuous rates, other parameters may also come into play though it is not obvious what the net effect will be. In Buttimer *et al.* (1997) it is for instance reported that the variance of the index and the short rate both have an impact on the value of the real estate swap. Understanding these results would require precise knowledge of the numerical implementation and we therefore do not pursue the issue further.

5.1 The discrete time approximation

Within a continuous time model, a discrete time approximation can be introduced in essentially two ways:

- The continuous time **valuation equation**, such as a pricing partial differential equation, may be replaced by a discrete time approximation, such as a finite difference scheme. This can be done as a purely numerical approximation, and it is not necessarily the case that the numerical approximation scheme has an economic interpretation as an arbitrage free discrete time financial model. Thus; in such a case the numerical result is typically not an exact arbitrage free price, neither in the original continuous time model, nor in any discrete time financial model.
- The continuous time **model** for the evolution of asset prices may be replaced by a discrete time model. The binomial model of Cox *et al.* (1979),

and its extension to the bivariate binomial model used by Buttimer *et al.* (1997), is an example of this procedure. In this approach one requires that the discrete time model should be arbitrage free, and also that it should converge to the continuous time model as the length of the time step goes to zero. As a consequence, in this approach the numerical prices produced are arbitrage free within the discrete time model. They are, however, not necessarily arbitrage free w.r.t. the continuous time model, but they are (hopefully good) approximations to the continuous time arbitrage free prices.

In Buttimer *et al.* (1997) the approach taken is clearly the latter one, i.e. the intention is to approximate the original continuous time model by an arbitrage free discrete time model. In doing so, an approximation error will typically be introduced, but for the CREILS the situation is in fact different:

Since, by the results of Sections 3 and 4, the arbitrage free value of a CREILS equals zero **regardless of the model**, this should also hold for any arbitrage free discrete time approximation of the original model. In other words, the result that the real estate swap has value zero should remain intact.

As noted in Remark 4.2, the result of Proposition 4.1, that the arbitrage free value of the CREILS equals zero, remains true also in any arbitrage free discrete time model. However, for the sake of completeness we now also give an independent proof of this discrete time result within the framework of Buttimer *et al.* (1997).

Let us thus consider the pricing of a swaplet in the discrete model from the viewpoint of the side that is long in the real estate index. We stand at time t_{k-1} and consider the payoff at time t_k with time step $\Delta = t_k - t_{k-1}$. We denote by R_k the dividends paid out at time t_k, and by L the LIBOR rate for the period. The value of the swaplet X_k at time t_k is then given by the expression

$$X_k = I_{t_k} - I_{t_{k-1}} + R_k - \Delta \cdot L \cdot I_{t_{k-1}}.$$

From equation (6) in Buttimer *et al.* (1997) it is clear that the short rate is quoted as a continuously compounded rate. We denote the level of this short rate at t_{k-1} by r, and hence the price at t_{k-1} of a zero coupon bond maturing at t_k is given by

$$p(t_{k-1}, t_k) = e^{-r\Delta}.$$

To ensure that the discrete time model is arbitrage free it is necessary and sufficient that the discounted gain process of the index is a Q-martingale, i.e.

that

$$e^{-r\Delta}E^{Q}[I_{t_k}+R_k|\mathcal{F}_{t_{k-1}}]=I_{t_{k-1}}.$$

Furthermore, the (simple) LIBOR rates are given by the standard definition

$$\begin{aligned} L(t_{k-1},t_k) &= \frac{1/p(t_{k-1},t_k)-1}{\Delta} \\ &= \frac{1/e^{-r\Delta}-1}{\Delta}=\frac{e^{r\Delta}-1}{\Delta}. \end{aligned} \tag{5}$$

The arbitrage free value of the swaplet at t_{k-1} is given by

$$\Pi\left(t_{k-1};X_k\right)=e^{-r\Delta}E^{Q}\left[X_k|\mathcal{F}_{t_{k-1}}\right],$$

and we have the simple calculation

$$\begin{aligned} E^{Q}\left[X_k|\mathcal{F}_{t_{k-1}}\right] &= E^{Q}\left[I_{t_k}+R_k|\mathcal{F}_{t_{k-1}}\right]-I_{t_{k-1}}-\Delta\cdot L\cdot I_{t_{k-1}} \\ &= I_{t_{k-1}}\left[e^{r\Delta}-1-(e^{r\Delta}-1)\right] \\ &= 0. \end{aligned}$$

From this it follows that the value of the swap is indeed equal to zero in a correctly specified discrete model. Thus the discrete time approximation in Buttimer *et al.* (1997) should not *per se* introduce an approximation error for the swap. However, in Buttimer *et al.* (1997) two further approximations are introduced, which cause the calculated price to differ from zero, and we now go on to discuss these.

5.2 The dividend approximation

From the discussion in Buttimer *et al.* (1997) and especially equation (2) it follows that

$$E^{Q}\left[I_{t_k}|\mathcal{F}_{t_{k-1}}\right]=I_{t_{k-1}}e^{(r-\eta)\Delta}, \tag{6}$$

which indicates that the constant dividend yield η is (implicitly) quoted as a *continuously compounded* rate. However, according to equation (9) in Buttimer *et al.* (1997), the dividend in the discrete model paid out at time t_k is set to a fraction of the index at time t_{k-1}:

$$R_k^s=\Delta\eta I_{t_{k-1}}. \tag{7}$$

In this expression the dividend yield η is thus quoted on a *simple* basis, (hence the superscript) which is not consistent with the continuously compounded interpretation of η in (6) above. In fact it follows directly from (6)-(7) that the expected payoff at time t_k discounted back to t_{k-1} is:

$$e^{-r\Delta}E^Q\left[I_{t_k} + R_k^s \middle| \mathcal{F}_{t_{k-1}}\right] = e^{-r\Delta}I_{t_{k-1}}\left[e^{(r-\eta)\Delta} + \eta\Delta\right].$$

Thus the discounted gain process is not a martingale under Q, which means that the model is not arbitrage free. In more concrete terms: if one insists on quoting η as continuously compounded as in equation (6), then one has to use this convention consistently and replace (7) by the continuously compounded counterpart

$$R_k = e^{r\Delta}I_{t_{k-1}}\left[1 - e^{-\eta\Delta}\right]. \tag{8}$$

With this expression for R_k, the no arbitrage condition

$$e^{-r\Delta}E^Q\left[I_{t_k} + R_k \middle| \mathcal{F}_{t_{k-1}}\right] = I_{t_{k-1}},$$

is indeed satisfied, and the precise interpretation of (8) is that the amount $\left(1 - e^{-\eta\Delta}\right)I_{t_{k-1}}$ is put into the bank account at t_{k-1}, and paid out at t_k.

What the authors do in Buttimer *et al.* (1997) is thus to approximate R in (8) with R^s in (7). The approximation error can easily be calculated, and is as follows for the benchmark values in Buttimer *et al.* (1997):

$$\begin{aligned} R_k^s - R_k &= \eta\Delta I_{t_{k-1}} + e^{(r-\eta)\Delta}I_{t_{k-1}} - e^{r\Delta}I_{t_{k-1}} \\ &= I_{t_{k-1}}\left[0.04\cdot 0.25 + e^{(0.05-0.04)\cdot 0.25} - e^{0.05\cdot 0.25}\right] \\ &= -I_{t_{k-1}}\cdot 7.53\times 10^{-5}. \end{aligned}$$

Thus Buttimer *et al.* (1997) somewhat underestimate the dividends.

We end this section by computing the discounted expected dividends over the interval $[t_{k-1}, t_k]$ at time t_{k-1} in the continuous model. We thus want to compute

$$E^Q\left[\int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{u} r_s ds}\eta I_u du \middle| \mathcal{F}_{t_{k-1}}\right],$$

and in order to do this we note that, under Assumption 2, standard arbitrage theory implies that the Q dynamics of I are of the form

$$dI_t = I_t(r_t - \eta)dt + I_t\delta_t dM_t,$$

where M is a Q martingale, and where δ is the volatility process for I. (In the setting of Buttimer *et al*, δ would be constant and M would be Wiener). Defining Z by

$$Z_t = e^{-\int_{t_{k-1}}^{t} r_s ds}I_t,$$

it then follows directly from the Itô formula that the Z dynamics are given by

$$dZ_t = -\eta Z_t dt + Z_t \delta_t dM_t.$$

From this it is easily seen that

$$E^Q\left[Z_u \middle| \mathcal{F}_{t_{k-1}} \right] = Z_{t_{k-1}} e^{-\eta(u-t_{k-1})} = I_{t_{k-1}} e^{-\eta(u-t_{k-1})},$$

and we obtain

$$\begin{aligned} E^Q\left[\int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{u} r_s ds} \eta I_u du \middle| \mathcal{F}_{t_{k-1}} \right] &= I_{t_{k-1}} \int_{t_{k-1}}^{t_k} \eta e^{-\eta(u-t_{k-1})} du \\ &= I_{t_{k-1}} \left[-e^{-\eta(u-t_{k-1})} \right]_{t_{k-1}}^{t_k} \\ &= I_{t_{k-1}} \left[1 - e^{-\eta\Delta} \right]. \end{aligned}$$

If paid out at t_k this amount will in the discrete model grow with interest to

$$R_k = e^{r\Delta} I_{t_{k-1}} \left[1 - e^{-\eta\Delta} \right],$$

which is exactly equal to the expression in (8). With this method the discounted gain process is of course a Q-martingale:

$$e^{-r\Delta} E[I_{t_k} + R_k] = I_{t_{k-1}} e^{-\eta\Delta} + I_{t_{k-1}} \left[1 - e^{-\eta\Delta} \right] = I_{t_{k-1}}.$$

It is interesting to note that

$$e^{-\eta\Delta} = 1 - \eta\Delta + \frac{(\eta\Delta)^2}{2} - ...$$

Therefore a first order Taylor expansion gives

$$I_{t_{k-1}} \left[1 - e^{-\eta\Delta} \right] \approx I_{t_{k-1}} \eta\Delta.$$

This is the term used by Buttimer *et al.* (1997) and given in (7), apart from the interest factor.

5.3 The interest rate approximation

In Buttimer *et al.* (1997), the LIBOR rate is set equal to the spot rate, i.e.

$$\widehat{L}(t_{k-1}, t_k) = r.$$

Now, as noted earlier, r is quoted as continuously compounded, while the LIBOR rate should be a simple rate, so this is again an inconsistency introduced in Buttimer *et al.* (1997). The correct expression for the simple LIBOR rate is given in (5) and it is easily seen that

$$\widehat{L}(t_{k-1}, t_k) = r < \frac{e^{r\Delta} - 1}{\Delta} = L(t_{k-1}, t_k),$$

so Buttimer *et al.* (1997) underestimate the LIBOR rate and thus overestimate the value of the swap for the side that is long in the real estate index. For the benchmark values used by Buttimer *et al.* (1997) we have that

$$\begin{aligned} I_{t_{k-1}} \left[(e^{r\Delta} - 1) - r\Delta \right] &= \left(e^{0.05 \cdot 0.25} - 1 \right) - 0.05 \cdot 0.25 \\ &= I_{t_{k-1}} \cdot 7.85 \times 10^{-5}. \end{aligned}$$

This is the magnitude of the overestimate of the value of the swap due to the interest rate approximation.

5.4 Combined effect

With the approximations introduced in Buttimer *et al.* (1997) we have

$$\begin{aligned} E\left[X_k | \mathcal{F}_{t_{k-1}} \right] &= I_{t_{k-1}} e^{(r-\eta)\Delta} + I_{t_{k-1}} \eta\Delta - I_{t_{k-1}} + \Delta \cdot r \cdot I_{t_{k-1}} \\ &= I_{t_{k-1}} \left[e^{(r-\eta)\Delta} - (1 + (r - \eta)\Delta) \right]. \end{aligned}$$

The net effect is roughly that continuous and simple interest rates are confused.

With the benchmark values in Buttimer *et al.* (1997) the following is obtained

$$\begin{aligned} E\left[X_k | \mathcal{F}_{t_{k-1}} \right] &= I_{t_{k-1}} \left[e^{(0.05-0.04) \cdot 0.25} - (1 + (0.05 - 0.04) \cdot 0.25) \right] \\ &= I_{t_{k-1}} \cdot 3.13 \times 10^{-6}. \end{aligned}$$

This is of the same order of magnitude as the numbers reported in Buttimer *et al.* (1997). Those numbers are however also affected by the accuracy of the numerical method used. Due to the systematic behavior of the Buttimer *et al.* (1997) results, it is likely that the approximations we have analyzed are important factors.

6 Conclusion

We studied the pricing of a real estate index linked swap and found that in a very general setting, its arbitrage free price is exactly equal to zero. This sharpens the result of Buttimer *et al.* (1997) who, under specific assumptions, found a result close to zero using numerical methods.

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Chapter 5

Revisiting the Past and Settling the Score: Revision in Repeat Sales and Hedonic Indexes of House Prices

Abstract

This paper examines index revision in measuring the prices for owner-occupied housing. We consider the context of equity insurance and the settlement of futures contracts. In addition to other desirable characteristics for aggregate price indexes, their usefulness in these contexts requires stability as they are revised. Methods that are subject to substantial or complex revision raise questions about the viability of derivatives markets. Of course, all indexes are subject to revision as the result of new information. Nevertheless, we find that the most-widely used house price indexes are not equally exposed to volatility in revision. Hedonic indexes appear to be substantially more stable than repeat-sales indexes and are less prone to substantial revision in the light of new information. Moreover, we find that the repeat-sales indexes are subject to systematic downward revision. We analyze alternative settlement procedures and contracts to mitigate the impact of revision associated with repeat sale indexes.

1 Introduction

Most of the statistical series used to describe the workings of the economy are subject to periodic reexamination and reestimation. Indexes are revised when either the method used to incorporate data or the data themselves are updated. Index revision of the former type is illustrated by the changes in the CPI which arose from implementing the findings of the Advisory Commission to Study the CPI (the so-called "Boskin Commission"). In this case, the CPI was rebenchmarked through the adoption of new practices based on economic theory.[1] Index revision of the latter type results from the reestimation of the

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index after the arrival of new information.[2] We focus on index revision of this latter type. Revision from either source has direct implications for public policy and private investment: these indexes are relied upon in policy formulation, investment decisions, and economic modeling, and they may form the basis for the development of markets for products such as housing price futures and home equity insurance.

Our interest is the dynamic performance of commonly-used methods of housing price index construction: those based on repeat sale and hedonic models. While there is an extensive literature on the asymptotic characteristics of these indexes, little is known about their stability as they are reestimated with the arrival of new information about housing prices. At issue is not the influence of observations that are added to the data due to informational lags. Rather it is the extent of index revision as additional sales are included in the set of observations used in their construction.

Index stability is often overlooked as a desirable characteristic of price indexes. This is especially relevant for house price indexes, given their wide use. Economic models of mortgage prepayment and default, measures of household wealth and cost-of-living calculations, among many other applications, are all informed by indexes based on the "latest" data. If, in fact, initial estimates of aggregate prices are subject to substantial revision, results from these models and measures may be misleading. Furthermore, the perception of instability in measured prices may preclude the development of markets for financial assets based on housing price indexes.

In the United States, the only widely available set of quality-controlled housing price indexes are based on so-called repeat sale models. We present extensive evidence on the extent to which revision for this class of indexes is "large." Our benchmark for making this assessment is a chained Fisher Ideal index derived from a series of cross-sectional hedonic regressions. Repeat sale indexes rely on strong assumptions regarding the time-invariance of both dwelling characteristics and their implicit prices in order to recover aggregate housing prices from a sample of dwellings that sell two or more times. In contrast, the chained Fisher Ideal index is based on a series of cross-sectional regressions that employ all available information about housing sales, while allowing both housing characteristics and their implicit prices to change over time. We compare this benchmark index to two indexes based on repeat sale models and one index based on a longitudinal hedonic model, where implicit prices of housing characteristics are assumed to remain constant over time.

We find that there are significant differences among the price indexes; revisions to estimated series are larger for indexes based upon repeat sales models than those based upon the longitudinal hedonic model. Our benchmark index, the chained Fisher Ideal index, is not subject to revision from the addition of new sales as the index horizon is extended. While larger in magnitude, we also find that revision to the repeat-sale indexes is systematically downward. Furthermore, we find that these indexes are subject to revision that can persist over several years. In some part, this arises because of the inherent nature of the information embodied in the arrival of an additional paired sale: while one observation reveals information about current market conditions, the accompanying paired-sale reveals information about past prices. Moreover, the linked nature of the repeat sale indexes implies that all previous index-level estimates are subject to revision whenever new paired sales are added to the data set.

We also examine the impact of index revision in the context of home equity insurance. Home equity insurance schemes seek to reduce the exposure of homeowners to fluctuations in the values of their homes by developing a market for derivatives based on an index of local house prices. By trading in such an index, households may hedge their long positions in housing by taking short positions in contracts derived from local housing prices (Shiller, 1993). The popular press has prominently featured discussions of this possibility,[3] and a pilot project is underway offering home equity insurance to homeowners in a large U.S. metropolitan area.[4] For home equity insurance to be attractive to homeowners, the reliability of the index is crucial. Preliminary price estimates must be credible, accurately reflecting systematic movements in local housing prices, period by period. Indeed, the integrity of the index may be the most important factor in developing a successful market for index swaps, futures, or other derivative contracts that would reduce the risks of homeownership.

By construction, the chained Fisher Ideal index is not revised as it is extended. In contrast, revision is an implicit feature of the other indexes we examine; all exhibit some revision. We seek to establish whether the revision for any index is large enough to limit its usefulness. In particular, we ask whether the change over time in the estimated index – and the length of time required to achieve a stable estimate of price levels – is likely to inhibit the development of a market for contracts based on the indexes. More specifically, we address the question of whether the level of revision found in the repeat sales indexes – which currently form the basis for regional house price measurement and provide the only feasible basis for index derivatives in the United States – will cause complications in practice. It would seem natural to settle insurance compensation on the basis of the index value at the date of the transaction. But if the index were subject to large revisions subsequently, any settlement may seem unfair or arbitrary in hindsight. We explore several approaches to contract settlement that may mitigate the impact of index revision.

This comparative analysis is only made possible with large samples of house sales and hedonic characteristics unavailable in the U.S.[5] Thus, we rely upon a unique body of data that reports sales prices and hedonic characteristics for each of the 600,000 single-family dwellings sold in Sweden during the period 1980-1999. This body of data – detailed government records on dwelling characteristics and market transactions (maintained for national property tax administration in Sweden) – form the basis for our empirical comparison.

Section 2 below describes the problem of index revision in the specific context of hedonic and repeat sales models. Section 3 provides a description of data analyzed and an overview of Swedish house prices. Sections 4-6 present a comparative analysis of the problem of index revision based upon these rich samples. Section 7 is a brief conclusion.

2 Price Indexes for Housing

Housing values are reported in the units of price times quantity, so it is natural to consider a model

$$\log V_{it} = \log P_{it} + \log Q_{it} + \varepsilon_{it}, \tag{1}$$

where V_{it} is the value of house i at time t, P_{it} is an index of house prices at t, Q_{it} is the quantity of housing (e.g. the quality of the house), and ε_{it} is an error term. Of the variables in (1) only V is directly observable and only at the time of sale. We must use indirect statistical methods to disentangle the price index from the measure of quality.

Observable information about the hedonic attributes of dwellings and transactions dates yields an empirical relationship:

$$\log V_{it} = X_{it}\beta_t + D_{it}\delta_t + \varepsilon_{it}. \tag{2}$$

In this formulation, X_{it} is a vector of hedonic characteristics of dwelling i, D_{it} is a vector of dummy variables with a value of one in the time period of sale, and zero otherwise. Further, β_t represents the implicit prices of hedonic characteristics at time t, and δ_t is the intercept at t. β and δ are estimated statistically, by making suitable assumptions about the error term, ε.[6]

In the simplest application of the hedonic model represented by (2), it is assumed that the vector of implicit prices of characteristics is time-invariant, i.e. $\beta_t = \beta$ for all t. In this case, the time varying intercepts (δ_t) have direct interpretations as index levels.[7] Since implicit prices are stable, by assumption, all dwellings appreciate at the same rate irrespective of hedonic characteristics. The arrival of data in later time periods will increase the precision of the estimates of price indexes and implicit prices. Further, if the implicit prices are time-invariant, then estimates of β and of δ_t will only change because the arrival of new information increases the sample size. As data accumulate, revisions will be smaller, and sampling variances will be reduced. Estimates will converge toward the true parameter values.

However, the assumption that the implicit prices of housing attributes are stable has neither theoretical foundation nor empirical support. Nevertheless, even in that case an index based on stable hedonic prices need not yield biased estimates of aggregate housing prices. Only if the implicit prices change in a systematic fashion will the covariance between time and attributes have a systematic impact on the price index estimates.

With large samples it is possible to estimate (3) for each temporal cross-section:

$$\log V_{it} = \delta_t + X_{it}\beta_t + \varepsilon_{it}.$$

An index may then be constructed by pricing a constant-quality dwelling in each period, valuing it based on the implicit attribute prices estimated in (3). In this approach, the choice of the representative dwelling can have an important impact on the estimated index. The choice of a "standard" house is typically either the average in the initial or final period, yielding Laspeyres or Paasche indexes, respectively. A Fisher Ideal index is the geometric average of the Laspeyres and Paasche indexes. Equation (3) and the Fisher Ideal index form the basis for the national index of new single-family home prices currently produced by the U. S. Census (Moulton, 2001).

The benchmark index used in this paper is a chained version of the Fisher Ideal index proposed by Thibodeau (1995). In this application, we estimate period-by-period changes in aggregate housing prices – "chaining" each periodic price change to the existing index. Here, the Laspeyres and Paasche indexes are constructed for adjacent periods – the last period covered by the existing Fisher price index and the new period to be appended to it. The Fisher index is the geometric average of the one-period change in house price as measured by the Laspeyres and Paasche indexes. This change is then applied to the current level of the chained Fisher index to arrive at the updated index. Clearly, this procedure precludes revision of past index values.

In the absence of measures of hedonic characteristics, price indexes may be constructed based on transactions of the same dwelling at two points in time, t and τ. Taking the difference of (2) yields

$$\log V_{it} - \log V_{i\tau} = X_{it}\beta_t - X_{i\tau}\beta_\tau + D_{it}\delta_t - D_{i\tau}\delta_\tau + \varepsilon_{it} - \varepsilon_{i\tau}. \qquad (3)$$

Two further assumptions make the estimation tractable without hedonic characteristics: unchanged quality of dwellings, $X_{it} = X_{i\tau}$, and unchanged implicit prices, $\beta_{it} = \beta_{i\tau}$.

Under these circumstances, the log difference in sales prices is related only to the dummy variables identifying the timing of the first and second sale, and equation (4) simplifies to

$$\log V_{it} - \log V_{i\tau} = D_{it\tau}\delta + \varepsilon_{it\tau}. \tag{4}$$

Here D is a matrix of dummy variables taking a value of -1 in the period of the first sale, $+1$ in the period of the second sale, and 0 otherwise. This method of producing house price indexes was first proposed by Bailey *et al.* (1963).

It is likely that some dwellings have different characteristics at the two sale dates, violating one of the assumptions required for the consistency of the repeat-sales approach. If data on hedonic characteristics are available, repeat sales indexes can include dwellings that have been modified between sales by including the changes explicitly in the regression:

$$\log V_{it} - \log V_{i\tau} = (X_{it} - X_{i\tau})\,\beta + D_{it\tau}\delta + \varepsilon_{it\tau}. \tag{5}$$

Note again that hedonic prices are assumed to be constant over time.

We estimate equations (2) through (6) using sales over a 19-year period. From these statistical models, we compute four indexes of housing prices based upon repeat sales and hedonic methods.

Index 1 is a repeat sales index based on equation (5). It assumes that neither the physical characteristics of the dwelling nor the implicit characteristic prices are changed between sales. This "naïve" index is analogous to the most widely-used regional housing price indexes in the United States.[8]

Index 2 is based upon equation (6), a repeat sales model in which all changes in the hedonic characteristics of dwellings are explicitly controlled for, but in which hedonic prices are assumed to be constant.

Index 3 is based upon the hedonic method reported in equation (2), but it imposes the restriction that the implicit prices of the hedonic characteristics are time-invariant. We refer to this model of housing price as the "longitudinal hedonic" model.[9]

Index 4 is based upon equation (3), estimated separately for each time period. Index 4 is the chained Fisher index, constructed from a series of geometric averages of Paasche and Laspeyres indexes estimated over adjacent time periods.

Note that the statistical models are clearly nested. A comparison of indexes 3 and 4 permits a test of the importance of changes in the prices of hedonic characteristics over time in explaining the course of prices.[10] A comparison of indexes 1 and 2 permits a test of the importance of changes in the hedonic characteristic of dwellings between sales and dates in explaining the course of prices.

Whatever the index, revision arises from the arrival of new information in the form of dwelling sales. The indexes derived from the repeat sales model are particularly exposed to revision because they utilize paired sales – two sales of the same dwelling at different points in time. Thus, when observations for a new period are added to a repeat sales database, they will augment not only sales prices in the new period, but also purchase prices in earlier periods. Thus, all previous index estimates will be revised in the light of this new information.[11]

3 The Data and an Overview of Swedish Housing Prices

We rely upon data describing all sales of owner-occupied single-family dwellings in Sweden during the 19-year period, 1981-1999. They are compiled by Statistics Sweden from two sources: tax assessment records, which contain physical characteristics of the dwellings; and sales records, which contain dates of sale and transactions prices. Dwellings are assigned a unique identification number, making it possible to identify multiple sales of the same dwelling. The detailed physical description of each dwelling enables verification of the assumption that quality remains constant between sales. Transactions between family members have been eliminated, and the data set is confined to arm's-length transactions, as far as can be ascertained. The data sources are described more fully in Englund *et al.* (1998).

In the full data set there are nearly 1,000,000 transactions involving more than 600,000 dwellings. These data are reported separately for eight administrative regions. Only the Stockholm metropolitan region covers a single housing metropolitan market, and we restrict the analysis in this paper to this region. The distribution of dwellings by the number of times they are sold during the sample period is reported in Table 1. It shows that almost a third of all dwellings that transacted were sold more than once during the 19-year period, and multiple sales constitute more than half of all transactions. The owner-occupied housing stock in Stockholm during this period averaged over 200,000 units. We observe 93,584 sales of distinct units, or almost half of the entire owner-occupied housing stock.

Englund *et al.* (1998) present a hedonic model of housing prices which relates measures of size, age, amenities, quality and location to the selling prices of Swedish dwellings. All specifications of the hedonic models presented below are based upon regressions including these measures. Column 1 of Appendix Table A1 presents summary information on these hedonic attributes. In the empirical analysis, time is expressed in quarter years, and estimates of quarterly price indexes for dwellings are presented.

Figure 1 reports the estimated price levels for five indexes – the longitudinal hedonic model, the chained Fisher index based on cross-sectional hedonic regressions, the naïve and hedonic-adjusted weighted repeat sale indexes, as well as median sales prices. The indexes reflect the broad evolution of prices in

Stockholm over the 19-year time period, and the patterns are similar. Figure 2 shows each index relative to the chained Fisher Ideal index; it makes clear that the indexes are far from identical.[12]

The repeat-sales indexes differ significantly from the chained Fisher Ideal index and from the other hedonic-based indexes. By the end of the sample period, the repeat sales index with hedonic adjustment exceeds the chained Fisher Ideal index by over ten percent. The "naïve" repeat sales model lags the Fisher Ideal index during the middle of the sample period by approximately 3 percent, but by the end of the time series it exceeds the Fisher index by six percent. The remarkable similarity of the longitudinal and chained Fisher indexes may be surprising given the clear rejection of the implicit assumption of fixed hedonic prices over time (see endnote 10). The similarity suggests that temporal variation in attribute prices is not systematically correlated with time.[13]

It is worth comparing the quality-adjusted indexes to the simple median sales price index, which is also included in Figures 1 and 2. The median is commonly used when more precise hedonic information is unavailable. The most well known example in the U.S. is probably the median price index produced by the National Association of Realtors. The main disadvantage of the median index is that it is not quality-adjusted, implying that for an increase in the median index may reflect either a higher price for a constant quality unit or quality variation in the sample of sold dwellings from period to period.

On theoretical grounds the median index is therefore undesirable for contract settlement. Nevertheless, it is easy to compute and simple to understand. Moreover, the median index is not subject to revision. Figure 2 also compares the median index relative to the Fisher Ideal index. As can be seen, the median and Fisher indexes at times diverge, especially so during the peak of the house price cycle around 1990. Despite the lack of quality control in the median price index, it is worth noting that it appears no worse than the index based on repeat sales with hedonic adjustment.

These indexes are based on all data that are available between the first quarter of 1981 and the fourth quarter of 1999. Neither Figure 1 nor Figure 2 indicates the extent to which the estimates of prices change as new information arrives with additional sales. We now consider the evolution of the indexes over time when new information is incorporated into existing indexes.

4 Aggregate Price Indexes and the Sources of Revision

All four of the quality-controlled housing price indexes examined in this paper are based on hedonic models. But because each makes selective use of the available data on sales and imposes different assumptions about the bundle of attributes and their implicit prices, estimated aggregate housing prices can vary across indexes. These distinct approaches to index construction also result in differences in the exposure to revision. While revision ultimately arises from new dwelling sales, the differences in measured revision arise from variations in the way that this new information is incorporated into index estimates.[14] This section examines the mechanics of revision for the four quality-controlled indexes.

In the case of the longitudinal hedonic index, new sales observations are pooled with existing data and the model coefficients are reestimated. Both the implicit prices of the dwelling characteristics and time indicators (from which price indexes are constructed) are revised. If the assumption of time-invariant coefficients (made when pooling sales from across different time periods) is appropriate, revision represents more accurate estimation as standard errors decrease. However, if this assumption is inappropriate, changes in the implicit prices over time are reflected in the estimates of the time dummies, and index revision represents an evolving bias resulting from model misspecification.

The chained Fisher Ideal index is the geometric average of the Paasche and Laspeyres indexes estimated over adjacent periods, so revision in the chained Fisher cannot arise due to the extension of the index with the passage of time. Both the Paasche and Laspeyres indexes are based on a series of cross-sectional hedonic regressions that once estimated are not recalculated – revision cannot arise from reestimation.

For the repeat sales indexes, additional observations on paired sales over time include sales in the current period as well as their associated sales in the past. An updated repeat sales index then incorporates new information about current selling prices as well as prices in the period of the earlier paired sale; the nature of the repeat sales approach implies that these indexes are subject to revision in every period.

As the sample period is extended, some dwellings sell a second time during the sample period – this results in an additional observed sale in the current period as well as an additional sale in the period of the pair's first sale – yielding a larger sample from which to estimate past prices. Figures 3 and 4 illustrate how the sample of repeat sales in any quarter changes as the sample period is extended.

Figure 3 plots the fraction of the dwelling stock available for use in index estimation at each quarter from 1986:I (quarter 21) through 1999:IV (quarter 76). The lines running roughly horizontally represent quarterly sub-samples of sales at various sampling periods. For example, the line at the bottom of the figure shows quarterly sales – as a percentage of the stock of dwellings – observed in each quarter when data are sampled in quarter 21; the top-most line reports the percentage of the housing stock represented by the full quarterly samples using all of the available data (through 76 quarters).

In addition to the sample evolutions, Figure 3 also shows substantial variation in quarterly sales activity over time. In particular, the data exhibit a strong seasonality in dwelling sales, with troughs typically in the first quarter and peaks in the second quarter. There is also some longer-term variation, with a gradual increase in number of sales during the boom of the late 1980s and a general decrease accompanying the downturn in the early 1990s.[15]

It is clear from the figure that only a small fraction of the dwelling stock is used in the construction of a repeat sales price index. It shows, for example, that the fraction of the stock available to estimate prices using repeat-sales indexes in quarter 21 is less than 0.2 percent; 50 quarters later the share of the dwelling stock included in the repeat-sales sample for this quarter doubled to more than 0.6 percent. Further, the initial fraction tends to increase over time as the sample grows, being less than 0.2 percent for the first 20 quarters but generally above 0.4 percent beyond quarter 60. Despite the ever-increasing representation of the stock within the repeat-sales sample, only about one-sixth of the housing stock is represented over this 19-year sample period. During the same period, roughly half of the stock is traded at least once, but only dwellings sold two or more times are used in the constructing the repeat sales indexes.

Figure 4 rescales the data in Figure 3 to underscore the exposure the repeat sale indexes face from the arrival of new information about past prices. Figure 4 uses the full repeat sales sample – those paired sales observed through 76 quarters – as a benchmark to illustrate the evolution of quarterly sub-samples. That is, the horizontal line at 100 percent indicates that all of the appropriate paired-sales available in the 19-year sample are used to estimate prices for the entire 76-quarter sample period. The lines below this benchmark are thus the percentages of the full sample that are available to estimate prices in earlier quarters. Figure 4 shows, for example, that the initial estimate of prices in quarter 21 is based on 40 percent of the number of sales in that quarter available by quarter 76. If the first 40 percent of the sales in the quarter differ in average appreciation from the remaining 60 percent, revision will occur. This can be contrasted with the data sets used in the hedonic indexes. Where the repeat sale indexes are significantly exposed to revision from the arrival of new information about the past, the data used to construct the hedonic indexes are unchanging over time.

5 Quantifying Revision

The general issue of index revision for repeat sales indexes has received limited previous attention. Abraham and Schaumann (1991) analyzed repeat-sales data from Freddie Mac and Fannie Mae, finding that index revisions are significant (on the order of ten percent in certain cases) even with large samples. Hoesli *et al.* (1997) also reported sizeable revisions based upon a smaller body of multiple sales for Geneva. Clapp and Giacotto (1999), in the most comprehensive study on index revision, highlighted the role of "flip sales," i.e., transactions with very short holding periods. They showed that the extent of price revision in housing is reduced considerably if samples are confined to dwellings sold after intervals of two or more years.[16] This suggests that there is a link between the index revision and the sample selectivity of repeat-sales data. Several authors, including Clapp and Giacotto (1992), Case, Pollakowski and Wachter (1997), and Gatzlaff and Haurin (1997) have found that smaller properties – "starter homes" – trade more frequently than houses in general. This indicates that index revision may be related to differences in price development between starter homes and other properties.

The empirical issue we consider is whether the index revisions are "large." In this section, we compare the magnitudes of revision among our four indexes. In the following section, we make inferences about the relative usefulness of indexes based on hedonic methods and those based on repeat sales for contract settlement.

We define "revision" as a change in estimated price levels that results from the remeasurement that occurs when new information is revealed in the form of additional sales. For the indexes based on repeat sales, revision results from the addition of new paired sales to the sample – paired sales that contain information about both current and past prices. "Revision" is then the change in measured housing prices as they are recalculated using the new, appended data sets. We define an estimate of the price level in period t using information extending from an "initial" period 1 to the "current" period t as $P(t, 1, \tau)$, where $\tau \geq t$. Revision is the process as the estimated index evolves from the initial estimate $P(t, 1, \tau)$ to a current estimate $P(t, 1, T)$, as data are extended beyond the initial period t. Tracking the time series evolution of the index estimates from "initial" to "current", i.e. for $t = t, t+1, \ldots, T$, yields a revision path. Examples of revision paths using the "naïve" repeat sales index are shown in Figures 5a through 5d.

These figures demonstrate the systematic remeasurement of parameter estimates from their "initial" to "current" values. For example, Figure 5a reports that the "initial" estimate of the aggregate price level in 1990:I was just over 2.43. Subsequently, as sales occurred in the following period, these data and their associated paired sales were added to the data set, and prices were remeasured. As of 1990:II, the "current" estimate of the aggregate price level in 1990:I was just over 2.42. Index revision is this change in estimated prices due to remeasurement.

Figures 5a through 5d show clear trends in revision resulting from the arrival of new information – the revision paths do not appear to be random. The figures suggest that revision is generally downward; "initial" estimates are higher than "current" estimates. This indicates systematic differences in the rates of price appreciation between the first sets of paired sales relative to those that enter the data set later. Downward revision is not universal, however, as is evident in Figure 5a. All four figures show economically and statistically significant revision in the level of prices and revision paths that last many years. Figures 5b and 5d suggest that the current index estimates have reached apparent stability, with estimates over the past several years relatively close to the current estimate. This is less clearly apparent in the revision paths shown in Figures 5a and 5c, even after years of data.

These figures also underscore the need for care in the interpretation of summary measures of the extent of index revision. Figure 5a, for example illustrates a revision dynamic that would be greatly understated by comparing initial and current index estimates. We present several measures of index revision in Tables 2 and 3 that summarize these dynamics.

These measures are derived from the estimated housing price indexes using our data on sales spanning the period of 76 quarters, 1981-1999. We estimate indexes using a rolling sample interval, beginning with $T = 31$ (1987:III) and ending with $T = 76$ (1999:IV). That is, we start with a subsample restricted to all sales up to and including 1987:III, and we estimate price indexes using the appropriate observations from this sample (all sales for the hedonic-based indexes, paired sales for those based on repeat sales). We then extend the sample interval quarter by quarter, amend the samples accordingly, and reestimate the price indexes. This is repeated until the last period, 1999:IV. We then extract comparable revision paths – the first 25 estimates of aggregate price levels for quarters 31 though 51, i.e. $P(t, 1, t)$, $P(t, 1, t + 1)$,..., $P(t, 1, t + 24)$ for $t = 31, 32, \ldots, 51$. By doing so, we are able to compare the behavior of the price level revision paths across index models.

The statistics in Table 2 focus on the panel of period-by-period index revisions (25 revisions to each of the price level estimates for quarters 31 through 51). The average quarterly revision is generally small, but it is larger for the repeat sales indexes than for the hedonic indexes. The standard deviations indicate wide variation along the revision paths and substantially wider revision – by a factor of two – for the repeat sale indexes when compared to the longitudinal hedonic index. These figures indicate that initial estimates provided by repeat sales indexes are less stable and are far more likely to be revised significantly relative to the indexes based on hedonic approaches.

Table 2 also shows the extent to which the revision of price indexes occurs in periods immediately following initial estimation. It reports the mean and standard deviation of the "early" and "late" revisions – changes in the first 10 and remaining 15 price level estimates, respectively. For the repeat sales indexes, more significant revision occurs in the early quarters than in the later: periodic revision is larger on average and more varied over the first 10 estimates. Conversely, there is essentially no change in the typical revision for the longitudinal hedonic index from early to late estimates and, of course, no revision in the chained Fisher index.

In order to assess index quality, it is important to distinguish short-run noise from long-run tendencies. Long-run trends in revision to the four indexes are illustrated in Table 3. This table reports summary statistics for the current estimates of the price levels at $T = t + 24$ relative to the initial estimates at $T = t$. Table 3 confirms that there is a clear tendency for the repeat-sales indexes to be revised downwards, on average by 2.4 and 1.7 percent, respectively. There is a wide range around these averages, extending to a maximum cumulative downward revision of 4.8 percent for the repeat-sales index with hedonics. The amount of cumulative revision for the longitudinal hedonic index, by comparison, is rather small, minus 1.0 and plus 0.6 percent, respectively. The range around these averages is only about a third to half the range for the repeat-sales indexes.[17] The observation that downward revisions of repeat-sales indexes are more common than upward revisions confirms earlier results by Clapp and Giacotto (1999).

In order to understand better the impact of quarterly revision on overall index stability, it is useful to recast the statistics in Tables 2 and 3, which report the extent of revision, to demonstrate the incidence of revision of particular magnitudes. If a futures market requires index stability, it would be useful to know how often revision – either period-by-period or cumulative – exceeds some level. Say, for example, that futures markets could tolerate 0.5 percent revision in any one quarter and 2 percent cumulative revision to the initial estimate – how often do the four indexes violate these criteria? We address this question in Tables 4 and 5.

In Table 4, periodic revision limits are set and the frequencies that quarterly changes in price level estimates exceed these limits are tabulated. There is a far greater frequency of significant quarterly revision for the repeat sales indexes. While over no two quarters is the longitudinal hedonic index revised by more that 1 percent, revision occurs in 25 percent of the quarters for the repeat sale with hedonic adjustment and 10 percent of the quarters for the standard repeat sale index. These changes are compared with the benchmark index – the chained Fisher index – which is time invariant.

Table 5 reports an analogous set of frequencies for cumulative revision. These figures are perhaps more relevant for contract settlement. If, for example, the standard for an index requires cumulative revision of no more than two percent at any point after initial estimation, the table indicates that neither of the repeat sale indexes is adequate. 62 percent of all revision paths estimated using repeat sales index with hedonic controls vary by at least two percent at some point over the 25 quarters examined; this frequency is 33 percent for the standard repeat sales index. For the tighter standard of plus or minus one percent relative to the initial price level estimate, the repeat sales indexes fails consistently whereas the longitudinal hedonic produces a revision path that exceeds this level in just less than one-half of the cases.

We can compare the magnitudes of revision in the housing price indexes with other widely accepted indexes. In particular, these revisions may be compared to revisions in the chained CPI in the U.S.[18] Currently, the Bureau of Labor Statistics publishes initial, interim and final index figures. Generally, the update from initial to interim and from interim to final is available at the beginning of the following calendar year. The relative changes in the index levels are approximately 0.2 percent for both revisions (Bureau of Labor Statistics, 2003). These are changes for the seasonally unadjusted series; the revisions in the seasonally adjusted series are larger and continue for five years. The revisions in our longitudinal hedonic index are slightly larger than for the unadjusted U.S. CPI, but still roughly comparable. The repeat sales indexes display revisions of an entirely different magnitude. The 2-4 percent revision that can occur in a repeat sales index are a good bit larger.[19]

6 Contract Settlement

Index revision may pose a significant impediment to the development of equity insurance products or futures markets based on aggregate housing prices. For example, consider a house that is purchased today for \$400,000, corresponding to the median price in San Diego County or in affluent Stockholm suburbs like Djursholm or Lidingö. Suppose the owner purchased equity insurance to protect herself from metropolitan-level shocks to housing prices. Over the next year, the price index – based on initial estimates – remains unchanged. At the end of the year, the owner sells her house at a loss.[20] Because the index initially reports no change in aggregate prices, this loss is attributed to idiosyncratic movements in the value of this particular home. Later, the arrival of new information that yields a downward revision to the index of one percent would imply that \$4000 of the loss was due to aggregate price changes (for which the owner had purchased the insurance). This number would double to \$8,000 or more if the downward revision exceeded the two percent level that the repeat sale indexes frequently do. By comparison, consider the impact of CPI-level revision on an insurance contract written in real terms: a change by 0.2 percent, as is typical of CPI revisions would imply a loss of only \$800 dollars. This would be far more tolerable than the error associated with the repeat sales index.[21]

In this example, the change in housing prices is mismeasured initially; subsequent information reveals that the estimate was too high. Consider the specifics of contract settlement. Contractual payments typically arise from the difference between two price indexes, $P_T - P_t$. In practice, the contract could be implemented as the difference between two initial estimates at the time of each transaction, $P(T,1,T) - P(t,1,t)$, as the difference between the two estimates at the time of the latter transaction, $P(T,1,T) - P(t,1,T)$, or some other variation on these themes.

We examine three types of contracts: The first employs the difference in initial estimates, $P(T,1,T) - P(t,1,t)$. In this case, the index estimate at the time of purchase and sale is set based on information available at t and at T – with no subsequent revision based on new information allowed. The second approach measures price change based on two current indexes, $P(T,1,T) - P(t,1,T)$, making use of all available information at contract settlement to evaluate the change in aggregate prices. In our third approach to contract settlement, we recognize that revision is most significant in the quarters immediately following an initial estimate. The third approach delays index measurement in order to mitigate subsequent revision. We examine delaying measurement of aggregate prices in both the difference in the initial indexes as well as differences in the current indexes. That is, we examine both $P(T,1,T+d) - P(t,1,t+d)$ and $P(T,1,T+d) - P(t,1,T+d)$. For our example, we delay measurement for eight quarters ($d = 8$).

The difference in initial indexes is surely the most straightforward to administer – it would involve no revision to the base during the course of the contract. It would also give a better measure of the true rate of index change if index revision were mainly systematic. Conversely, systematic revision would present a problem for contract settlement based on the difference between two current indexes. In such a case, aggregate prices at time of settlement would already account for (much of) the ultimate revision of the earlier purchase date index but none of the revision ultimately attributed to the sales date. Where revision is typically downward, this approach provides too little compensation to the holder of the insurance contract, as in the insurance example above. On the other hand if revision were predominantly random, then basing the contract on the difference between the initial estimates would waste useful information – it would be preferable to base settlement on the current estimates.

To illustrate the implications of each of these settlement methods in the light of revision in repeat sales indexes, we simulate the set of possible futures contracts possible during quarters 21 through 66.[22] For each contract – $[P_T - P_t]$ where $T = 22, \ldots, 66$; $t = 21, \ldots, T-1$ – we calculate the estimated change in prices based on the four settlement methods described above using the standard repeat sales approach. We then calculate the deviations between the settlements using the repeat sales index and the benchmark Fisher Ideal index. The "best" settlement approach is that which minimizes the deviation in contract settlement amounts.

Table 6 reports the results. The average settlement using current indexes is 1.26 percent higher than the average settlement using the benchmark index. Use of initial indexes substantially mitigates the average bias, but the settlement bias exhibits substantially higher variation. The "best" method for contract settlement appears to be the use of the difference in delayed initial indexes. On average, this approach yields settlements only 0.20 percent different from settlements based on the benchmark index. Both delayed contracts exhibit lower variance in settlement bias, but come at the cost of a delay of two years.

7 Conclusion

In contrast to the extensive literature that focuses on the static properties of housing price indexes, we focus on the dynamic characteristics of several commonly-used indexes: those based on repeat sale and hedonic models. In general, relatively little is known about the stability of these housing price indexes. We address several questions: What are the mechanics of revision? Are different indexes equally susceptible to revision? Is substantial revision common? And, is revision "large enough" to complicate the development of financial instruments based on aggregate housing prices? We focus centrally on the performance of the repeat sales indexes as the sample period lengthens and as new paired sales are added to the data. We do so because this methodology forms the basis for the only quality-controlled indexes systematically available for metropolitan areas and regions in many countries, including the United States. These price indexes are themselves inputs to a great deal of research, and they inform many of the discussions about price trends in owner-occupied housing in the United States.

We assess the importance of index revision in repeat-sales indexes against the benchmark of a chained Fisher index, which by definition is free of revision. The Fisher index employs all sales information – including characteristics of the transacted dwellings – in a flexible form that obviates the need to make the strong assumptions associated with the repeat sale indexes. This is possible because the basis for the Fisher indexes is a series of cross-sectional regressions; housing characteristics and their implicit prices are allowed to vary over time. We also compare revision in the repeat-sales indexes against revision in an index constructed from a commonly-used longitudinal hedonic model.

We find revision to be an inherent feature of repeat sale indexes. The range of revision in the price level estimates is two to six times greater for the repeat-sales indexes relative to the longitudinal hedonic index. We find revision of the repeat sale indexes to be asymmetric, with downward revision more prevalent than upward. We also find that revision in the repeat-sales indexes is primarily an "early" arrival problem. That is, most of the revision occurs in the first ten quarterly estimates, and price estimates become more stable thereafter. This suggests systematic differences in the relative appreciation of those early entrants to the sample compared to those that arrive later. The hedonic-based index is not subject to this type of revision as it uses all sales data as they become available – no "new" information is added to data used to calculate prior hedonic regressions.

The second approach we take to assess the importance of index revision is examining its impact on a hypothetical market for home equity insurance and aggregate housing price futures. Given the difficulty in hedging the wealth represented by owner-occupied housing, these types of contracts may offer great benefits to home owners and investors. However, both contracts require a settlement procedure that is based on an aggregate housing price index. Accuracy and stability are essential features for the functioning of these markets.

We find that levels of revision in the repeat sales indexes are not inconsequential for the settlement process. We examine several modes of contract settlement in an effort to mitigate the impact of index revision with mixed results. The intuitive approach of settlement based on current indexes is problematic in the case of the repeat sales indexes as revision of the initial price levels is both common and substantial. The "best" solution – one designed to reduce the instability of revision – requires lengthy delays in the final settlement of contracts. Neither approach appears conducive to the development of markets for housing futures or equity insurance. This suggests that the development of futures markets in housing prices would be better served by hedonic-based indexes, and that care must be taken when using a repeat sales index as the only basis for the settlement of financial contracts.

We leave several issues for further research. First, we have not addressed the "cause" of revision in the repeat-sales indexes. The empirical regularity we document – systematic downward revision – may have its roots in nominal-loss aversion, positive price-volume correlation, or some other reasons. Second, we have made no attempt to guide the design of specific financial instruments based on aggregate housing price indexes. We proceed in this paper on the assumption that any metropolitan-level index would represent a new opportunity for diversification and would therefore be of some value. Of course, there are many issues, for example, the geographic definition of markets, needs close examination. However, independent of these and other challenges, the level of revision found in the repeat-sales indexes suggests that the OFHEO indexes may be inadequate for contract settlement.

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Notes

[1]The Boskin Commission identified four sources of potential bias based on observed behaviors of consumers: the substitution effect, the sample rotation effect, the outlet substitution effect, and the quality-improvement effect. For more on these and the suggestions for revisions to the construction of the CPI, visit www.bls.gov/cpip/home.htm.

[2]Revision of this type can be found in the CPI as well. Each year with the release of the January CPI, seasonal adjustment factors are recalculated to reflect price movements from the just-completed calendar year. This routine annual recalculation may result in revisions to seasonally adjusted indexes for the previous 5 years.

[3]See, for example, *Forbes Magazine*, August 28, 2002. (www.forbes.com/2002/08/28/0829whynot.html.)

[4]See Caplin *et al.* (2003) for an extensive description.

[5]Since the U.S. does not have a national real property tax, there is no administrative reason for assembling consistent national or regional data on hedonic attributes of houses and their sale prices.

[6]Of course, neither the set of X variables nor the log linear relationship represented by (2) is deduced from theory. For hedonic models of housing prices, there is some evidence that semi-log and Box-Cox specifications "do best." (Cropper *et al.*, 1988). However, these complications – specification and choice of variables – are no different in this context than in other applications in economics.

[7]Because the specification includes an intercept, the price index, I_t, is merely $\exp(\delta_t)$. The price index for period $t = 1$ is 1.00.

[8]The Office of Federal Housing Enterprise Oversight (OFHEO) house price indexes for states and metropolitan areas are based on equation (6), the repeat-sale index technique first introduced by Bailey *et al.* (1963). The so-called "weighted repeat-sale" (WRS) approach, developed by Case and Shiller (1987), assumes a random-walk rather than mean reversion in the error structure. For more details on the WRS approach and its application by OFHEO, see Calhoun (1996; www.ofheo.gov/house/hpi_tech.pdf). For a commercial application of the WRS, see www.cswv.com/products/redex/case.

[9]Englund *et al.* (1999) explore in more detail the relationship between the samples of paired sales with and without verification of the constant quality assumption. They find aggregate quality improvement over time and a significant overstatement of housing prices in the naïve (unverified assumption of constant quality) sample relative to the unchanged (verified) sample. This bias is attributable to unmeasured quality change. They do not investigate the influence on index revision.

[10]If hedonic prices are stable, then it is more efficient to pool the data and estimate hedonic prices over the whole sample. Alternatively, if implicit prices are time varying,

it is necessary to estimate cross-sectional regressions. Our results clearly indicate that implicit prices are unstable, which thus justifies the cross-sectional approach. The formal test of stable implicit prices has previously been employed for the same purpose by Crone and Voith (1992), Meese and Wallace (1997) and others. Like them, we reject the temporal stability of the hedonic prices on housing attributes.

[11] An example may illustrate the importance of this new information in revising the index. Assume there are three samples of repeat sales from two periods. Assume dwellings in sample A were sold at time 0 for an average price of 100 and at time 1 for an average of 110. This yields a price index estimate of 110 in period 1. Houses in sample B were sold at time 1 for 110 and at time 2 for 125. With no other observations, the augmentation of sample A with sample B will provide a price estimate of 125 for period 2 and with no need to revise the index for period 1. Now assume that we also observe a sample C of houses that sold at time 0 for 100 and at time 2 for 150. The new price information from the long-interval repeat sale sample C suggests that the price increase between time 0 and time 2 is greater than is indicated by samples A and B. Hence, further augmentation of sample A and B with sample C will cause the index for period 1 to be revised upwards. Estimates of the new index values will be a weighted average of the original estimate 110, and the period 1 estimate implied by the combined sample of B and C. Note that revisions will be systematic if dwellings with long holding periods appreciate at a rate that is different from houses with short holding periods: if long-interval sales typically exhibit higher (lower) average rates of price increase, then upward (downward) revisions will be the norm.

[12] The null hypothesis that the repeat sale and hedonic indexes are indistinguishable from one another can be rejected at standard levels of significance.

[13] As reported in Table A1 in the appendix, the averages of the implicit prices from the cross-sectional hedonic regressions are similar to those from the longitudinal hedonic regression. However, the temporal variation in the hedonic prices is apparent in the relatively large standard deviations of the cross-sectional coefficients.

[14] Indeed, we are interested in the relative stability as new sales data become available. Others have examined the revision due to the lag between sale date and the date the data become available for use in index construction. Butler et al, 2005 find a significant upward revision that effectively is complete over one month. Their focus was on the OFHEO indexes and the lag between the origination of a mortgage and the date it is sold to the GSEs. Their finding of downward revision may be an indication of “creaming” on the part of mortgage sellers. There is no such lag in our data set; a dwelling enters the data at the date of sale.

[15] The peak in the first two quarters of 1991 is explained by the tax reform effective in January 1991 and reduced the tax rate on capital income (including capital gains on house sales) from 50 percent to 30 percent.

[16] We also find that the removal of short-hold paired sales reduces the magnitude of revisions. However, we find that the pattern of revision is still significant. Moreover, we fail to observe a natural cutoff between “short” and “typical” holding periods. Removal

of the short-holds seems ad hoc for this particular sample.

[17] The repeat sales index appears to perform worse with hedonic information included. This may reflect the maintained assumption of constant implicit prices.

[18] The revisions in that index occur for reasons that are technically different from the Fisher Index discussed above. In particular, the cost-of-living approach used to compute the CPI utilizes expenditure data in adjacent time periods in order to reflect the effect of any substitution that consumers make across item categories in response to changes in relative prices (Bureau of Labor Statistics, 2003). Nevertheless, since the CPI is widely used for contract settlement and escalation clauses, it gives an indication of the revision sizes that may be tolerated.

[19] While we employ the Swedish data to learn about the potential magnitude of index revision associated with the OFHEO indexes, it should be noted that the data differ in two important ways. First, the Swedish data are arm's-length transactions only; there are no assessments from refinancing. Second, there is no limit on the price of dwellings we use in the analysis that might act like the conforming loan limitations used by the GSEs. To the extent that "starter homes" transact more frequently, our repeat sales data are likely to be more heavily represented by those dwellings analogous to those that would be included in the OFHEO data. Moreover, if the arrival of dwelling sales into the data is explained by Stein's (1995) liquidity constraints, then our results may in fact understate the magnitude of revision. In this case, "winners" trade first, making the systematic downward revision found here an artifact of the late arrival of those dwellings that did not appreciate as much as the early arrivals.

[20] Note that this type of idiosyncratic price variation is likely to be far larger than aggregate index revision. Though true, it is not idiosyncratic risk that the homeowner insures against. While the homeowner can exercise some control over maintenance and – to a lesser extent – the neighborhood, he or she is not able to protect against systematic movements in house prices at the metropolitan level. Revision at this level is the motivation for this example.

[21] It is important to recognize that while a median index offers a more stable index, a quality-controlled index is required for these types of instruments. Consider an example of a metropolitan area in decline, precisely the application for which equity insurance was conceived. Here the composition of sold houses may shift to toward higher quality dwellings as higher skilled, more mobile workers migrate to better employment opportunities. In the extreme, aggregate house prices may be declining at the same time a median index would report rising house prices.

[22] Recall, that we define 21 quarters as the minimum sample period needed to ensure an adequate number of paired sales. We stop this exercise at quarter 66 in order to be able to impose the settlement delay, where required.

Table 1. Dwellings by Frequency of Sale

Number of Sales	Dwellings	Transactions
1	61,614	61,614
2	22,815	45,630
3	7,037	21,111
4	1,728	6,912
5	346	1,730
6	39	234
7	4	28
8	1	8
Total	93,584	137,267

Table 2. Index Revision: Quarter-by-Quarter

Percent Change in Price Level Estimate Relative to Previous Estimate
(Percent Change at Quarter s = 100*[P(t,1,s)/P(t,1,s-1) - 1]; t = 31,…,51,s=t+1,…,t+24)

	All Revisions		Early Revisions[1]		Late Revisions[1]	
	Average	Std. Dev.	Average	Std. Dev	Average	Std. Dev
Index 1: WRS (naive)	-0.07%	0.51%	-0.11%	0.24%	-0.05%	0.15%
Index 2: WRS (with hedonics)	-0.10%	0.65%	-0.17%	0.41%	-0.06%	0.18%
Index 3: Hedonic (Longitudinal)	-0.04%	0.31%	-0.04%	0.02%	-0.04%	0.02%
Index 4: Hedonic (Chained Fisher)	-	-	-	-	-	-

[1] 'Early' is defined as the first 10 revised index values, 'late' as the remaining 15.

Table 3. Index Revision: Cumulative Change from Initial to Current Estimates

Percent Change in Final Price Level Estimates Relative to Initial Estimate
(Percent Change = 100*[P(t,1,t)/P(t,1,t+24) - 1]: t = 31,…,51)

	Average	Min	Max	Standard Deviaion
Index 1: WRS (naive)	-1.72%	-2.87%	-0.21%	0.69%
Index 2: WRS (with hedonics)	-2.38%	-4.84%	-0.56%	0.91%
Index 3: Hedonic (Longitudinal)	-1.03%	-1.84%	-0.35%	0.51%
Index 4: Hedonic (Chained Fisher)	-	-	-	-

Table 4. Quarterly Index Revision Limits

Frequency that Quarterly Revision Exceeds Some Limit

(Limit Exceeded at Quarter s if (100*[P(t,1,s)/P(t,1,s-1) - 1]) > L; t = 31,…,51)

	Limit & Frequency Exceeded			
	0.10%	*0.25%*	*0.50%*	*1.00%*
Index 1: WRS (naive)	100%	100%	57%	10%
Index 2: WRS (with hedonics)	100%	95%	76%	24%
Index 3: Hedonic (Longitudinal)	33%	0%	0%	0%
Index 4: Hedonic (Chained Fisher)	0%	0%	0%	0%

Table 5. Cumulative Index Revision Limits

Frequency that Cumulative Revision Exceeds Some Limit

(Limit Exceeded at any Quarter s if abs(100*[P(t,1,s)/P(t,1,t) - 1]) > L: t = 31,…,51; s=t+1,…,t+24)

	Limit & Frequency Exceeded			
	0.50%	*1.0%*	*2.0%*	*3.0%*
Index 1: WRS (naive)	90%	90%	33%	0%
Index 2: WRS (with hedonics)	86%	48%	0%	0%
Index 3: Hedonic (Longitudinal)	62%	24%	0%	0%
Index 4: Hedonic (Chained Fisher)	0%	0%	0%	0%

Table 6. Contract Settlement & Settlement Bias

Contracting Approach	**Settlement Bias**			
Difference in:	*Average*	*Min*	*Max*	*Standard Deviation*
Initial Indexes	0.75%	-2.74%	5.02%	1.44%
Current Indexes	1.26%	-2.75%	4.03%	1.06%
Initial Indexes (delayed)[1]	0.20%	-2.05%	2.91%	0.84%
Current Indexes (delayed)[1]	0.62%	-2.30%	3.12%	0.79%

[1] Delay is 8 quarters.

Figure 1: Price Indexes for the Stockholm Region 1981-1999

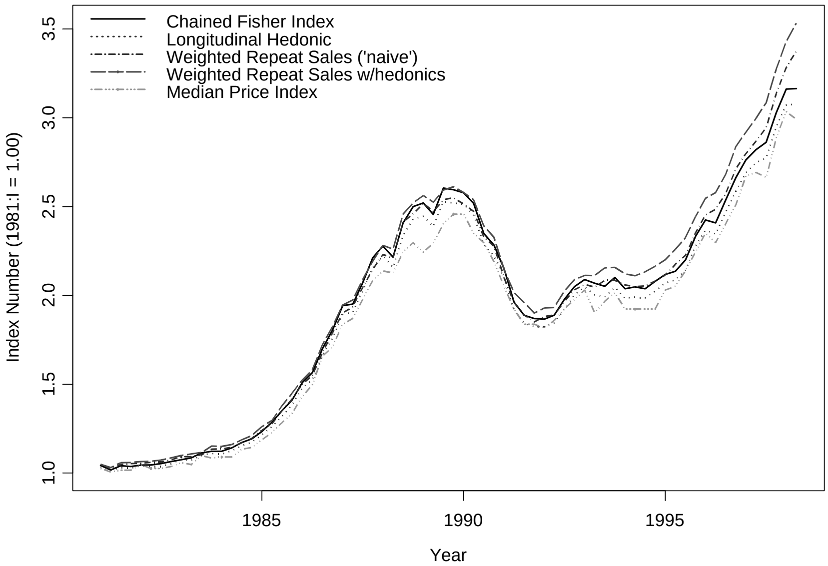

Figure 2: Price Indexes Relative to the Chained Fisher Ideal Index

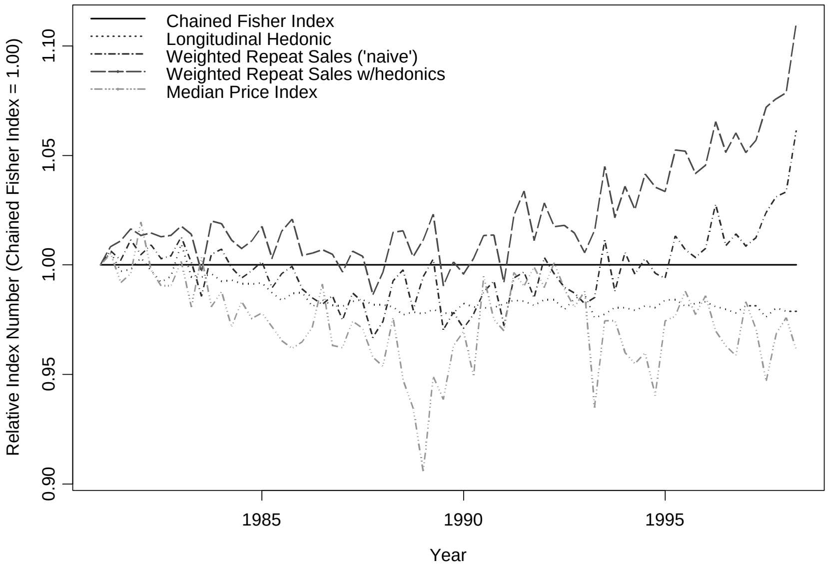

Figure 3: Evolution of Repeat Sale Sample
Sample Size for Various Time Intervals as a Percent of Dwelling Stock

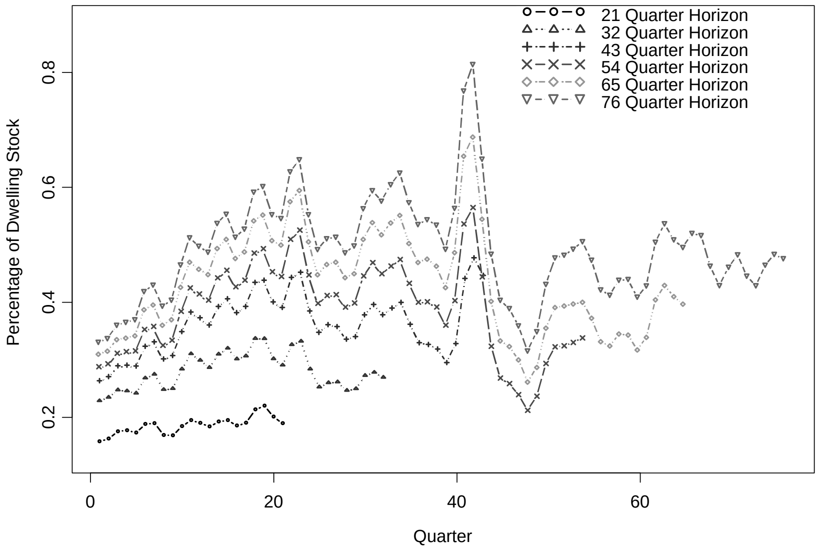

Figure 4: Evolution of Repeat Sale Sample
Sample Size for Various Time Intervals as a Percent of Full Sample

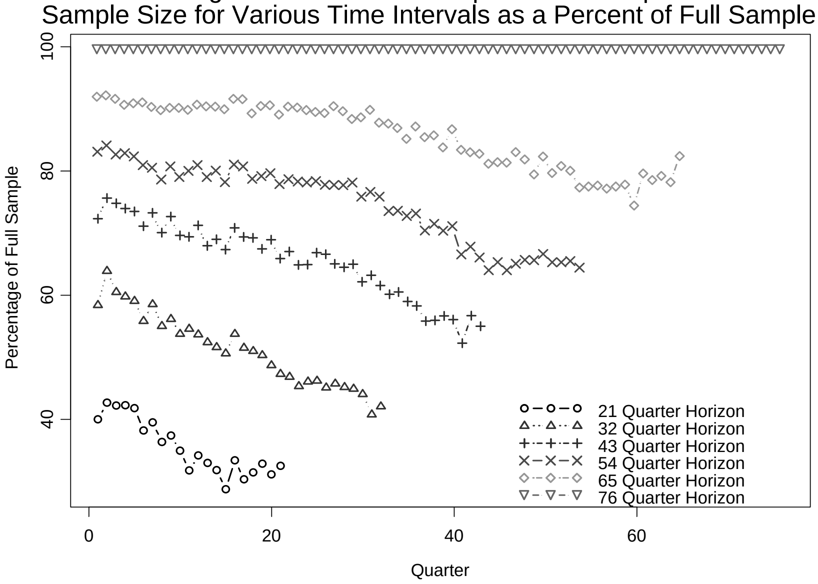

Figure 5: Estimated Paths of Revision.

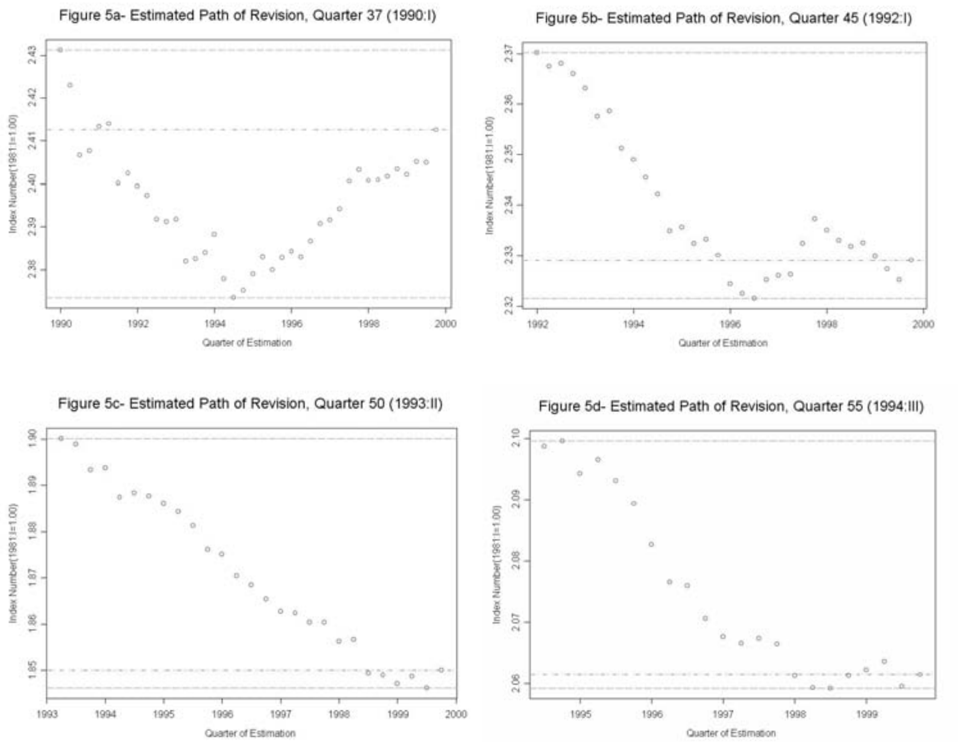

Table A1: Summary data

Summary data for arm's-length transactions in the Stocholm region 1981-99. Average hedonic characteristics and OLS coefficients for quarterly regressions.

	Average hedonic characteristics[1]	Average cross-sectional regression[2]	Final longitudinal regression[2]
Continuous variables			
Living area[3]	123	0.592	0.594
Square meters	(40)	(0.021)	(0.002)
Lot size[3]	999	0.090	0.088
Square meters	(2299)	(0.014)	(0.002)
Distance to regional center[3]	19	-0.307	-0.311
Kilometers	(15)	(0.011)	(0.001)
Vintage	62	-0.002	-0.002
19xx	(18)	(0.000)	(0.000)
Dichotomous variables			
Tile bath	0.170	0.036	0.054
		(0.017)	(0.002)
Sauna	0.221	0.044	0.043
		(0.015)	(0.002)
Stone/brick	0.220	0.020	0.020
		(0.015)	(0.002)
Single detached	0.617	0.079	0.082
		(0.022)	(0.003)
Recroom	0.156	0.049	0.049
		(0.017)	(0.002)
Fireplace	0.410	0.066	0.064
		(0.014)	(0.002)
Laundry	0.752	0.029	0.040
		(0.016)	(0.002)
One car garage	0.686	-0.004	-0.010
		(0.014)	(0.002)
Two car garage	0.046	0.053	0.045
		(0.031)	(0.004)
Waterfront	0.011	0.372	0.411
		(0.061)	(0.007)
Insulation			
Walls only	0.166	0.048	0.056
		(0.090)	(0.010)
Walls and windows	0.829	0.117	0.131
		(0.091)	(0.010)
Kitchen			
Good	0.166	0.163	0.146
		(0.007)	(0.007)
Excellent	0.825	0.168	0.172
		(0.069)	(0.007)
Roof			
Good	0.707	0.040	0.039
		(0.014)	(0.002)
Excellent	0.012	0.082	0.064
		(0.057)	(0.006)
Price	976		
Thousand Swedish kronor	(648)		
$\mathbf{R^2}$		0.675	0.806

[1] Standard deviations in parantheses.
[2] Standard errors in parantheses.
[3] Variables expressed in logarightms.

Chapter 6

The CCAPM: Interaction Between Market Frictions and Habit Formation

Abstract

This paper tests the consumption capital asset pricing model on American quarterly data. To address the empirical failure of the CCAPM, considerable attention has been devoted to market frictions as well as alternative utility specifications. Both of these strands can claim some success, but they have until now been considered separately. The focus of this study is therefore to consider both in the same study. Furthermore, market frictions have in the previous literature been found to have a potentially large impact. However, a weak test based on the Hansen and Jagannathan bound, which only gives a necessary condition for model correctness, has been used. To shed further light on the issue, we therefore consider the same authors' measure test, which is based on a condition that is both necessary and sufficient. Finite sample versions of the tests will also be used since the asymptotic distribution has been shown to be a poor approximation for the sample sizes considered in this study. Different utility specifications are found to have considerably less impact than the introduction of market frictions, which in general has a significant impact, and the model is often accepted.

1 Introduction

The Consumption Capital Asset Pricing Model (CCAPM) postulates a relationship between household consumption decisions and asset returns. The seminal work by Lucas (1978) and Breeden (1979) established the theory in its basic version and since then considerable effort has been devoted to empirically evaluate the model. Hansen and Singleton (1982, 1983) were the first to comprehensively test the model and establish its weak performance. Subsequent work has characterized this failure in terms of several puzzles, the most well known being the

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equity premium puzzle of Mehra and Prescott (1985) and the risk free puzzle of Weil (1989).

The literature can be divided into a few broad approaches of how to solve these puzzles, two of which will be the focus of this paper. The first strand starts from the observation that CCAPM is not based on a specific utility assumption. As a result of the failure of the standard utility function with constant relative risk aversion (CRRA), attempts have been made to consider alternatives. We will consider the approaches of Epstein and Zin (1989, 1991) and Abel's (1990) habit formation. The literature indicates that they are promising alternatives, and habit formation has a nice behavioral interpretation (see also the discussion on utility functions below). The second strand of the literature has addressed the fact that capital markets are not frictionless. Frictions include bid/ask spreads and other transaction costs, commissions, margin requirements, taxes, restrictions on trade such as short sale restrictions and solvency constraints. This will affect behavior and should be taken into account when testing the model, which we will do in the spirit of Luttmer (1996) and He and Modest (1995)[1].

Both of these strands have had some success. However, although results improve, the approach can either not adequately explain the observed puzzles (utility considerations) or the results can be questioned on the grounds of the test procedure (market frictions). In much of the market frictions literature the focus has been on the so called Hansen and Jagannathan (1991) bounds and extensions thereof as means of testing the model. However, this is a weak test that only provides a necessary, but not a sufficient criterion, for a model to hold. It is therefore difficult to use this tool to assess the overall fit of the model.

The aim of this article is therefore three-fold. Firstly, a more stringent test will be applied to more carefully assess the impact of market frictions. To this end the Hansen and Jagannathan (1997) measure will be used which is both necessary and sufficient. Extensions of this test that can cope with the case of market frictions will be used, which to our knowledge has not been done before in published research. Secondly, since the two strands have been considered separately so far, we also explore if the joint effect of market frictions and some recent utility specifications could be sufficient to explain the observed puzzles, given that a more stringent test procedure is applied. Finally, several studies have shown that the asymptotic properties of the tests used in this paper can be a poor approximation of the distribution in finite samples. The asymptotic distribution is also simply not available in several cases. For these reasons we rely on bootstrapping to obtain small sample properties.

The rest of the paper is structured as follows. In section two we review some of the remedies proposed to improve the basic CCAPM. In particular we discuss utility specification, market frictions and statistical considerations. In section three we present the Hansen and Jagannathan framework, which is our main tool to evaluate the performance of the models. We discuss the Hansen and Jagannathan bounds, measure and how they can be adapted to take market frictions into account. The details of the empirical study and our results are presented in section four. Section five concludes.

2 Remedies

Given that the law of one price holds, any asset pricing model must satisfy the Euler equations (e.g. Hansen and Jagannathan, 1991),

$$E\left(m\mathbf{R}\right) = \mathbf{i}, \tag{1}$$

where m is the pricing kernel or the stochastic discount factor (SDF) proposed by the model, $\mathbf{R}$ is a vector of asset returns containing all assets in the economy and $\mathbf{i}$ is a vector of ones. In a complete market setting the SDF will be unique. In an incomplete market, however, a whole set of SDF:s complies with the Euler equations and are referred to as the set of admissible stochastic discount factors (ASDF), denoted by M. Among these only the subset that are strictly positive are compatible with absence of arbitrage.

In its basic form the CCAPM considers a representative agent, maximizing time separable utility function over consumption (c_t), subject to a frictionless dynamic budget restriction (as in Lucas, 1978)

$$\max E_0 \sum_{t=1}^{\infty} \beta^t u(c_t). \tag{2}$$

This leads to an SDF given by the intertemporal marginal rate of substitution (IMRS) times the subjective discount factor (β)

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}. \tag{3}$$

The standard approach in the literature has been to use CRRA utility, $u = \frac{c^{1-\gamma}}{1-\gamma}$ (where γ represents relative risk aversion), and to test whether (3) satisfies (1). This has typically has lead to rejection of the model (e.g. Hansen and Singleton, 1982, 1983). This benchmark utility function will be used to compare our results to previous findings. In this paper m^*, called the SDF proxy, will denote the SDF when a specific utility function such as the CRRA utility has been imposed. In the following we review how different utility functions, market frictions and non-parametric inference can improve the performance of the CCAPM.

2.1 Utility considerations

We consider two alternatives to the standard CRRA utility function: firstly the approach of Epstein and Zin (1989, 1991) towards separating intertemporal substitution and risk aversion and secondly habit formation.

Hall (1988) pointed out that CRRA utility implies that the intertemporal elasticity of substitution, (ψ), which describes the agents willingness to substitute consumption across time periods, is the reciprocal of the agent's coefficient of relative risk aversion (γ), describing the willingness to substitute consumption over states of the world. He argued that this link is inappropriate since the elasticity of substitution is well defined even if there is no uncertainty, while the risk aversion is well defined even in models with no time dimension.

In two papers by Epstein and Zin (1989, 1991) this close relationship was broken, implying that the agent is no longer risk neutral with respect to future consumption. The utility function they proposed is as follows

$$u_t = \left\{ (1-\beta)\, c_t^{\frac{1-\gamma}{\theta}} + \beta \left(E_t \left[u_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}, \tag{4}$$

where $\theta = (1-\gamma)\,/\,(1-1/\psi)$. That (4) is a more general version of CRRA utility can be seen by setting $\theta = 1$. Epstein and Zin also showed that it is possible to derive a non-recursive stochastic discount factor from their utility function.

The separation between γ and ψ might, as suggested in Kocherlakota (1996), make it possible to resolve the risk free rate puzzle since it enables higher risk aversion without reducing the IMRS.

Another approach is that of habit formation which implies that utility is not provided by absolute consumption, but rather consumption in relation to some function of previous levels of consumption. The motivation behind this model is that well-being is defined by earlier experiences and the expectations they have formed. A problem in consumption based asset pricing is that measured aggregate consumption is too smooth to fit the volatile asset returns, but a relative measure of consumption might be more volatile than absolute consumption (Kocherlakota, 1996). A habit ratio model was introduced by Abel (1990):

$$u = \frac{(c_t/x_t)^{1-\gamma} - 1}{1-\gamma}. \tag{5}$$

The habit level will be of the form $x_t = c_{t-1}^{\kappa}$ in this study. Campbell and Cochrane (1999) report some success using a habit difference model (where c_t/x_t above is replaced $c_t - x_t$). In this model relative risk aversion is time varying, since it is equal to $\gamma c_t/(c_t - x_t)$. As we will consider the CCAPM for fixed levels of relative risk aversion, we focus on the ratio model to facilitate the comparison of results between competing utility functions. Also, we use the external habit specification.

2.2 Market frictions

Instead of considering different utility specifications, another strand of the literature has tried to incorporate the various market frictions that exist in real world capital markets. The introduction of transaction costs such as bid/ask spreads and commission will imply that the price of the asset can fluctuate within a range bounded by the transaction cost. The intuition is of course that it is only possible to profit from mispricing if it exceeds transaction costs in the market. This also implies a relaxation of the conditions a correct asset pricing model must fulfill.

In the case of a bid/ask spread, the Euler equation in (1) for an individual asset i changes to (He and Modest, 1995):

$$\frac{1}{1+c} \leq E\left[mR^i\right] \leq 1+c. \tag{6}$$

Here the transaction cost associated with each trade equals roughly $100c\%$. Note that (6) can be rewritten as in terms of two joint conditions:

$$\begin{aligned} E\left[m\left(\frac{R^i}{1+c}\right)\right] &\leq 1 && (7)\\ E\left[m\left(-R^i\left[1+c\right]\right)\right] &\leq -1. && (8) \end{aligned}$$

Hence, an alternative but equivalent way of modelling transaction costs, dating from Foley (1970), is to replace the original asset with two new assets where only positive portfolio weights are allowed. This excludes the possibility of buying and selling at the ask and bid price respectively. One of the assets will have a negative price and return and the other a positive price and return. In He and Modest (1995) the transaction cost is assumed to be constant and the prices of the assets are adjusted for the transaction cost. Luttmer (1996) adopts Foley's setting in which the returns rather than the prices are adjusted, and this will be the approach followed in this paper.

He and Modest (1995) also show how the Euler condition changes when a utility maximizing agent faces short sale constraints. In this case traders can not take advantage of arbitrage opportunities that occur when an asset is overpriced. This results in an equilibrium where the Euler equation becomes an inequality

$$E\left[mR^i\right] \leq 1. \tag{9}$$

Several other types of market frictions can be incorporated to deal with solvency constraints, borrowing constraints and margin requirements. Their implementation is very similar in spirit, as they all are based on exchanging the Euler equality for an inequality. We will focus on the short sale constraint, as it is arguably the most important type of friction.

In this study a bond and a stock market index will be the assets under consideration and both short sale constraint on the bond and the stock will be considered. A short sale restriction on the bond implies that agents in the market are not allowed to borrow at all. No short sale constraint implies no restriction to borrowing at all. It is relevant to consider both these cases.

In the U.S., stock shorting is allowed. Nevertheless, there are still many implicit short sale constraints such as the "up tick rule", the short squeeze to name a few examples[2]. In practice they could be so severe that agents might choose not to go short. Furthermore, neither He and Modest (1995) nor Luttmer (1996) directly consider the impact of a short sale restriction on the stock.

In the case of bonds, the bid/ask spread is negligible, whereas the difference between lending and borrowing rates are not. We therefore propose to use the above framework to differentiate between the two. Since we are considering a representative agent model, all agents would like to borrow when the bond is overvalued, i.e. whenever (9) holds. In this case the borrowing rate will be employed, while in the reverse case, when the bond is undervalued, the lending rate is used.

2.3 Statistical issues

A rejection of the CCAPM could be the product of purely statistical considerations. Parametric statistical tests depend on distributional assumptions that may or may not hold. For instance, Kocherlakota (1997) argues that the assumption of a finite variance of the Euler residuals could be wrong. In this case residuals are not even asymptotically normally distributed and Kocherlakota proposes an alternative test procedure based on the jackknife method which is able to cope with this situation.

Another and less extreme possibility is that the test under consideration converges to its asymptotic distribution very slowly, making that distribution a poor approximation to the actual small sample distribution usually studied. Such slow convergence has been noted for the Hansen and Jagannathan (1997) measure in for example Ahn and Gadarowski (1999). Similar observations have been made for the widely used GMM in Hansen, Heaton and Yaron (1996), among others. Also, as argued by Engsted *et al.* (2000), variance may be finite but residuals sufficiently heavy tailed to make the asymptotic normal approximation inappropriate.

Because of these considerations and the fact that for some of the tests used no asymptotic distribution has been derived, we will only consider the finite sample distributions obtained by bootstrapping. When choosing a bootstrap method, it is desirable that it takes the observed serial correlation in asset returns into account. We will use the adjusted block bootstrap method by Politis and Romano (1994), known as the stationary bootstrap, for primarily two reasons. Firstly, it generates a stationary data series. Secondly, it is a non-parametric approach with few assumptions about the structure of the bootstrapped time series. It is essentially an extension of the basic block bootstrap method: When sampling from the original data set, it either picks observations sequentially or with some probability starts over again at a random point in the time series. The probability that the sampling will begin at a new random point is determined by the so called smoothing parameter. It follows that the average sample length will be equal to the reciprocal of the smoothing parameter. This parameter determines how much of the serial correlation prevailing in the original data series is carried over to the bootstrapped data. It should be selected in such a way that the properties of the bootstrapped series resemble that of the original series. Two opposing forces must be taken into account when setting the smoothing parameter. On the one hand, a too low value (i.e. the probability of picking observations sequentially from the original data is high) will result in a bootstrapped distribution with virtually no dispersion. On the other hand, when it is too high virtually no observations are picked sequentially and thus the serial correlation in the original data will not be captured in the bootstrapped data. The actual parameter choice is described further in section 4.

Bootstrapped p-values should be computed under the null hypothesis that the model is true. If the model under consideration is true, the standard Euler equation (1) would hold for each individual asset, i.e. $E(mR^i) = 1$ (here we have abstracted from market frictions, in which case the Euler equation would be an inequality). However, in the bootstrapped sample this is usually not the case, which is why either the bootstrap procedure or the final tests must be adjusted. We will do this following the method suggested in Engsted *et al.* (2000) who adjust the final test rather than the bootstrap procedure. The basic principle behind the adjustments stems from the observation that the bootstrap will be generated under the null hypothesis of a correct model if the price of the asset return is set equal to the average price actually observed in the bootstrapped samples. So by exchanging $E(mR^i) = 1$ for

$$E(mR^i) = p_*^i, \tag{10}$$

the assets will on average be correctly priced, and thus generated under the null. Here p_*^i represents the bootstrapped average; that is if $E^*(\cdot)$ denotes expectation over bootstrapped samples then $p_*^i \equiv E^*(mR^i)$. The details of the adjustments are addressed for each test separately in their respective sections.

3 The Hansen and Jagannathan framework

The Hansen and Jagannathan (1991, 1997) framework is the main method of model evaluation. We discuss the bounds and the measure and how they can be incorporated to include market frictions.

3.1 Bound

A commonly used test to assess asset pricing models stems from Hansen and Jagannathan (1991) where a non-parametric bound on the standard deviation of the set of ASDF:s is presented, henceforth the HJ bound. The HJ bound is based on the observation that given a mean of the pricing kernel, its standard deviation must exceed some minimum level in order to satisfy the Euler equations and hence price the assets correctly. This diagnostic is relevant because due to the smoothness in aggregate consumption, many consumption based models will fall short of this minimum level.

For a given mean of the SDF, denoted by v, the HJ bound will give the minimum corresponding standard deviation, denoted $B(v)$, that a SDF with a mean of v must have in order to price the assets correctly. Thus the computation of the bound can be reduced to finding the admissible stochastic discount factor that has the smallest second moment

$$B\left(v\right)=\min_{m\in M}\sigma(m)=\left[\min_{m\in M}E\left(m^{2}\right)-v^{2}\right]^{\frac{1}{2}}. \tag{11}$$

The risk free rate is defined by the reciprocal of v if one is available. If there is no asset that is truly risk free, i.e. with a constant payoff over time, then the data does not specify v and it has to be prespecified to calculate $B(v)$. Furthermore, if arbitrage is ruled out, manifested in restricting an ASDF to be positive in all states of nature (also referred to as positivity constraint), the bound can not be expressed in closed form but must be solved for numerically; see the appendix for details. However, if that restriction is dropped and there are no market frictions, a simple analytic expression results. Therefore, this version has seen widespread use in empirical research. Given the mean SDF and the observed vector of asset returns $\mathbf{R}$, the solution to (11) can be written as

$$B\left(v\right)=\left[(vE(\mathbf{R})-\mathbf{i})'\mathbf{\Omega}^{-1}(vE(\mathbf{R})-\mathbf{i})\right]^{\frac{1}{2}}. \tag{12}$$

In the above equation $\mathbf{\Omega}$ is the covariance matrix of the asset returns and $\mathbf{i}$

is a vector of ones. To generate the bootstrapped distribution under the null, $vE(\mathbf{R}) - \mathbf{i}$ is exchanged for $vE(\mathbf{R}) - \mathbf{p}_*$ in (12).

If a proxy SDF (m^*) given from some asset pricing model falls below the HJ bound for some significance level one can reject the hypothesis that it prices the assets correctly at that level. Using this so-called *vertical distance* as a way of evaluating the performance of a model was originally proposed by Burnside (1994) and Cecchetti *et al.* (1994). It is thus based on the difference between the standard deviation of the SDF proxy, $\sigma(m^*)$, and the HJ bound, $B\left[E(m^*)\right]$:

$$D = \sigma(m^*) - B\left[E(m^*)\right]. \tag{13}$$

Note that here v is prespecified and set equal to $E(m^*)$. If D is positive, the proxy is consistent with observed data in terms of the HJ bounds. However, the SDF that achieves the lowest variance is perfectly correlated with the returns, as it is a projection onto the space of returns (Hansen and Jagannathan, 1991). Cochrane (1999) therefore proposes an adjusted vertical distance that takes into account the degree of correlation between the SDF and the returns:

$$\widetilde{D} = \sigma(m^*) - \frac{B\left[E(m^*)\right]}{r}. \tag{14}$$

Here r is the multiple correlation coefficient from the regression of the demeaned SDF proxy on the demeaned returns.

The above discussion illustrates the important point that even if the proxy SDF is within the bounds, this merely means that the first two moments of the SDF comply with data observed in the market. For the model to truly price the assets correctly the entire distribution, or put otherwise, also higher moments, of the pricing kernel must comply with the data (Snow, 1991, gives asset pricing bounds on higher moments). This illustrates that the HJ bound gives a necessary but not a sufficient condition for testing asset pricing models.

3.2 Bounds with frictions

He and Modest (1995) and Luttmer (1996) show how the HJ bounds can be extended to include market frictions. The definition of the bounds is still given by (11) but the set of admissible stochastic discount factors changes. This is so because an admissible discount factor satisfies the pricing restrictions, and they change with market frictions. With market frictions, typically Euler equalities are replaced by Euler inequalities, which makes the set of admissible stochastic discount factors larger (section 2.2 discussed this in greater detail). This means that the bounds are weakly pushed down. As a result, a lower standard deviation of the stochastic discount factor will be consistent with the data.

For bounds with frictions, it is not possible to find an analytic expression even if the restriction of a positive SDF is dropped. In this study we therefore use only the specification that excludes the possibility of arbitrage opportunities. The details of the computation of the bounds with frictions is given in the appendix.

3.3 Measure

Hansen and Jagannathan (1997) provide a further diagnostic, henceforth the HJ measure. It is closely linked to their HJ bound, but with the difference that it provides both a necessary and sufficient condition for a model to hold.

The HJ measure is defined as the smallest least squares distance between the set of ASDF:s (M) and the proxy SDF (m^*)

$$\delta = \min_{m \in M} \left\{ E\left[(m^* - m)^2 \right] \right\}^{1/2}. \tag{15}$$

The HJ measure thus gives a quantification of the models capacity to price the assets. The HJ measure can also be interpreted as a model's maximum percentage pricing error, and as such should be considered a conservative test. Thus, a HJ measure of 0.2 implies that the model misprices the assets by at the most 20% on average.

If arbitrage is ruled out the measure must be solved for numerically; see the appendix for details. However, as with the bound, an analytic solution exists if the restriction of a positive SDF is dropped. The solution to the problem is then given by the following quadratic expression

$$\delta = \left[(E(m^*\mathbf{R}) - \mathbf{i})' E(\mathbf{R}\mathbf{R}')^{-1} (E(m^*\mathbf{R}) - \mathbf{i}) \right]^{\frac{1}{2}}. \tag{16}$$

Thus, loosely speaking the HJ measure squares the pricing errors, $E(m^*\mathbf{R}) - \mathbf{i}$, and weights them together using the inverse of the returns' second moment matrix which is model independent. Put differently, the HJ measure can be rephrased as the solution to a GMM problem with a *constant* weight matrix across different models (Jagannathan and Wang, 1996). This facilitates the comparison of models, unlike the commonly used optimal GMM procedure where the weight matrix is different across models. In principle this makes it possible to rank the performance of models, something which will be important in this paper since we try to compare the degree of mispricing across models. Another issue with optimal GMM is that it either accepts or rejects the null hypothesis of a correct model, rather than quantifies the extent of the pricing error of a model that in practice is often viewed as an approximation.

To generate the model under the null hypothesis, $E(m^*\mathbf{R}) - \mathbf{i}$ is exchanged for $E(m^*\mathbf{R}) - \mathbf{p}_*$ in (16) as discussed in section 2.3. This will make it possible to test the hypothesis that $\delta = 0$ which is not possible using the asymptotic results in Hansen and Jagannathan (1997) since it assumes that $\delta > 0$. In the GMM procedure the same method is used; since the Euler equation is used as the moment condition the $\mathbf{i}$ is exchanged for the average price over all bootstrapped samples.

3.4 Measure with frictions

Just as with the bound, also the measure can be adjusted to take market frictions into account. Again, the definition of the measure in (15) remains the same, and only the definition of the set of admissible stochastic discount factors changes. The larger set of admissible stochastic discount factors will make the HJ distance to the proxy SDF weakly smaller, and thus the model is more easily accepted. The intuition for this is that the larger set may but need not include a point that is closer, in the sense of the HJ measure, to the proxy SDF.

With frictions no simple analytic expression exists for the measure; the details of the computation of the measure with frictions is given in the appendix. We focus on the arbitrage free version with positivity constraint imposed on the stochastic discount factor.

4 Empirical study

In the following the empirical study is presented. We begin by presenting and discussing the data used as well as the setup of the bootstrap study. Finally results for the various computed statistics are evaluated.

4.1 Data

We use a data set with American quarterly data for the period 1959 to 2000. The length of the holding period is crucial, since expected asset returns increase with the holding period, while transaction costs are constant. Therefore, the impact of transaction costs are much smaller in, for instance, yearly data sets. The time horizon used in prior studies varies, mostly because it is difficult to get an empirical measure of the representative agent's time horizon. For instance, He and Modest (1995) consider a monthly time horizon while Luttmer (1996) uses both monthly and quarterly data. However, using a data set also considered in Bodnaruk (2002), that covers the transactions of all Swedish investors owning more than 500 shares, we found that approximately 30% of all investors divest a share within the first 6 months. Another 25% divest in the following 6 month period. This only gives an indication of the holding period of a share and not the whole asset portfolio of the representative investor. Nevertheless, a quarterly time horizon seems in better agreement with the data than a monthly horizon. Further, one can argue that since corporations file reports on a quarterly basis, major new public information is available at this frequency. It is therefore reasonable to assume that investors reallocate their portfolios quarterly.

The consumption measure used is real per-capita consumption of nondurables and services. We include two assets; the risk free rate is the real return on quarterly. Treasury bills and the market return as proxied by the total return (including dividends) on the Standard and Poor 500 index[3].

Real-world market frictions are proxied in different ways for the market return and the risk free rate. The market return is firstly corrected to take dividend tax into account by adjusting the dividend part of the return downwards. McGrattan and Prescott (2001) conduct a survey of dividend taxation, and following their evidence we set the rate to 30%, as an average of post-war tax rates. Secondly, we implement bid/ask spreads. Jones (2001) conducts an extensive survey of bid/ask spreads and other market frictions and fees. In line with his results, we set the bid/ask spread plus commission to 0.8%. By using these fees we are likely to underestimate the transaction cost since the cost used is the one that applies to big institutions and not the representative agent. However, as will be shown in the empirical section, these moderate fees are still sufficient to get statistically significant results. We implement a constant bid/ask spread because the statistics used depend primarily on the average and not on the time profile or frictions. Also, an average can be estimated more robustly than for individual years. For the risk free rate, we differentiate between borrowing and lending rates. The lending rate is set equal to the return on quarterly treasury bills as above, while the borrowing rate is equal to the prime commercial rate.

4.2 Setup

We include three utility functions in the simulations: standard CRRA utility function as a reference, Epstein-Zin utility with intertemporal elasticity of substitution (IES) equal to 0.5 and the Abel ratio habit utility with habit set to lagged consumption with habit power as $\kappa = 0.5$. The parameter values are admittedly somewhat ad hoc but are consistent with evidence presented in Dynan (2000), Fuhrer (2000) and Patterson and Pesaran (1992). The subjective discount factor is set to $\beta = \sqrt[4]{0.99}$,on a quarterly basis which corresponds to a discount factor of 0.99 on an annual level. The SDF of these utility functions are as follows on a quarterly basis: Each of these three utility functions is considered in six different settings: with no market frictions, with transaction costs, with a short sale restriction on the bond and the stock separately, with both transaction costs and short sell restriction on the bond, and finally with both transaction costs and short sell restriction on the stock. In each such case a relative risk aversion coefficient (γ) of 0, 1, 2, 4, 10 and 15 is evaluated.

The three utility functions, six market friction settings and six risk aversion levels gives in total 108 different cases. For each such case we compute several diagnostics. Significance levels are computed using the stationary bootstrap approach of Politis and Romano (1994) which was discussed in section 2.3. We use 1000 bootstrap replications and set the smoothing parameter to 0.1. This was based on the method suggested by Politis and Romano (1994), which involves evaluating the autocovariance structure of the included series. According to Politis and Romano (1994) the method is robust towards the choice of the smoothing parameter. For each utility function, friction and risk aversion setting, the following diagnostics are computed: (1) the vertical distance to the HJ bound, (2) the vertical distance to the HJ bound with adjustment for less than perfect correlation between the SDF and asset returns, (3) the HJ measure, (4) GMM using a weight matrix with lagged consumption as instrument. For each diagnostic (1)-(4) the sample value as well as the bootstrapped significance level, standard deviation and 5% cutoff value is presented.

4.3 Results

Table 1 gives the vertical distance to the Hansen and Jagannathan bound with positivity restriction, while table 2 gives the same statistic with adjustment for less than perfect correlation. Obviously, the statistic is adjusted strictly upwards by taking the less than perfect correlation into account. That adjustment is of significant size, and reflects the relatively low correlation between quarterly consumption growth and asset returns. Differences in bootstrapped significance levels are however much smaller, which is likely to reflect the fact that the estimated adjusted vertical distances also have much higher standard deviations.

The distance to the bound becomes very large for the upper range of the risk aversion parameter, that is the gamma values. This is partly the result of the positivity restriction, where minimum variance goes to infinity for finite values of the mean of the stochastic discount factor. The reason for this is that to exclude arbitrage, the risk free rate, which is defined by the mean of the stochastic discount factor, may not dominate or be dominated by any combination of the risky assets. This aspect is not included in the bounds without positivity restriction.

Figure 1 illustrates the effect of introducing a short sale constraint or transaction cost. As can be seen the effect is significant in both cases, but more pronounced for the short sale constraint.

Without frictions, the stochastic discount factor is almost always outside the valid bounds region, although it falls within in some cases, especially with the Epstein-Zin utility specification. Overall the ranking seems to be that Epstein-Zin performs better than the habit formation specification which in turn does better than the constant relative risk aversion utility function. However, the differences are comparatively small in relation to the impact of market frictions. It is clear that if our estimated transaction costs are a reasonable approximation of real world frictions, and a representative agent rebalances the entire portfolio at least quarterly, then there is little evidence against the consumption based asset pricing model for low values of gamma. For higher values the CCAPM is rejected.

When transaction costs are exchanged for short sale constraint on the bond, the CCAPM can not be rejected for any value of risk aversion (governed by the gamma parameter). A short sale constraint on the stock on the other hand has a more limited effect. Examining He and Modest's (1995) expression of the bounds with frictions but without positivity restriction provides some intuition. They express the bound as the minimization over $\boldsymbol{\lambda}$ of $w'\boldsymbol{\Omega}^{-1}w$ where $\boldsymbol{\Omega}$ is the return covariance matrix and $w = \boldsymbol{\lambda} - vE(\mathbf{R})$. Here $\boldsymbol{\lambda}$ is a vector where the elements are equal to 1 for assets with no frictions but lies in the region [0,1] with short sale constraint. If the optimal lambda is above or close to one, then imposing the short sale constraint will have little effect, which is the case for the stock but not for the bond.

The HJ measure with positivity restriction is presented in table 3. Here the model is just as in the bounds case strongly rejected without market frictions. However, when frictions are introduced the measure behaves differently compared to the vertical distance. In the case of transaction costs the distance to the bound increases sharply for larger gammas and the model is rejected. While the measure remains roughly constant and the model is accepted for virtually all gammas. Since the bound is based on a necessary, rather than sufficient, condition for model correctness it need not give much information on overall model performance. Obviously by only considering the bound one would falsely conclude that the CCAPM is rejected in this specific setting. As for the bound imposing a short sale constraint on the bond results in an acceptance of the model, while a short sale constraint on the stock has very little impact.

The intuition behind this result can, just as for the bound, be explained by the fact that the average price of the stock return is above or slightly below one with the used SDF proxy. The average price of the bond price is on the other hand always below one for the risk aversion parameters considered in this study.

By considering the p-values, transaction costs seems to be significant even when compared to the short sale constraint. Even though one should be careful in this comparison, since the transaction cost data and short sale data are generated by two different bootstrap samples, this tendency is too strong to be attributed to small sample effects. In previous studies transaction costs have been found to have less impact (Luttmer, 1996, and He and Modest, 1995), which should be compared to the more significant effect that is found in this study. The difference can not be explained by the holding period, since both used quarterly data as well. However, they did only consider the bound whereas this effect only can be observed for the measure. Another contributing factor is that we include the difference between the borrowing and the lending rate in our transaction cost measure, which is much higher than the bid/ask spread for bonds used in their studies. Further, we take dividend taxation into account.

Both for the bound as well as the measure the Epstein-Zin utility behaves somewhat differently from the other two, even though Epstein-Zin is generalization of the standard CRRA utility. The overall result seems to be that increasing gamma results in a worse fit, contrary to Abel and CRRA where for increased risk aversion no dramatic change in the result can be observed (for the measure as well as bound in case of short sale constraint and when transaction costs are added). This is consistent with the bounds and the SDF:s as depicted in figure 1.

Following Luttmer (1996), a chi square test can be implemented with moment equalities replaced by moment inequalities where appropriate. This implies optimizing over an additional so called nuisance parameter for every inequality. In effect this is the same as GMM with moment inequalities. When computing the weight we used the Newey-West (1987) method with one lag. In table 4, the results are given. As was noted above, the use of an optimal weight matrix in GMM precludes the direct comparison between models. Having said that, the behavior of the GMM is more similar to the bounds than the measure. However, Epstein-Zin utility consistently performs better here.

In the simulations the case when gamma is equal to zero is also included. This setting is equal to a constant SDF case for the CRRA utility, i.e. a case where assets are not adjusted for risk (all agents are risk neutral) and therefore equivalent to the absence of any risk adjusting model.

Without market frictions, there is a tendency for the model to perform comparatively well, regardless of the test, when gamma is set to zero. Intuition for this result can be obtained by looking at the position of the SDF for gamma equal to zero in figure 1. Furthermore, when some market friction is imposed not only is the CCAPM accepted, as noted above, but this is also true for the zero gamma case. These should be interpreted as, first of all, that the risk neutral model performs as well as the consumption based model in the standard frictionless setting. Secondly, since some market frictions also imply that the zero gamma case is accepted, one should not say that the inclusion of market frictions result in an acceptance of the CCAPM. Rather, the interpretation should be that market frictions, as defined in the literature, is such a severe restriction that virtually all models can be accepted, including the case of no risk adjusting model at all.

It appears that the method of implementing the short sell constraint is more restrictive than the method used for the bid/ask spread. Consider the Euler inequality (9), which according to He and Modest (1995) and Luttmer (1996) is the result of introducing a short sale constraint. The interpretation is that an asset can remain overpriced since agents are not allowed to go short. That is, an underpriced asset will be bid up, but an overpriced asset cannot be pressed down as shortselling is disallowed. For this argument to work, agents cannot have existing stocks of the asset that can be sold off. If that were the case, an overpriced asset could be bid down even without short sell constraints when the owners of the asset decided to sell off parts of their holdings. Thus the short sell constraint implies that agents automatically sell off their assets in each period (or they perish as in Lucas, 1978). Agents then use their endowments to buy a new portfolio. As a result, the simulation results involving short sell constraints should be interpreted with caution.

5 Conclusion

In this paper we evaluated the effect of both different utility specifications and market frictions on the consumption based capital asset pricing model using a bootstrap approach. To test whether previous results, mainly in the market frictions literature, were not due to the weak test used a more stringent procedure is applied relying on the Hansen and Jagannathan measure.

The standard constant relative risk aversion model without market frictions is strongly rejected by our quarterly data, regardless of risk aversion setting. Furthermore, the other popular utility specifications considered here, Epstein-Zin and Abel's habit formation utility, only perform somewhat better in this setting. However, the impact of market frictions is highly significant and often makes it possible to accept the model. By considering the more stringent HJ measure instead of the HJ bound the results did not change significantly, lending support to the findings in previous studies such as in Luttmer (1996) and He and Modest (1995). The two types of market frictions considered, transaction costs and short sale constraints, was shown both to have a potentially important impact. As opposed to previous studies transaction costs are found to have a much bigger impact than a short restriction. This can partly be explained by the fact that our transaction costs are larger since they in addition to bid/ask spread on stocks and commission also include tax on dividends as well as the differentiation between lending and borrowing rates. The results with bid/ask spreads have a greater intuitive appeal as the short sell constraint is modelled in a way which excludes the possibility of bidding down prices using sale of existing stocks of the asset. An extension of this work could involve the empirical testing of models where market frictions are endogenously determined, rather than exogenously given.

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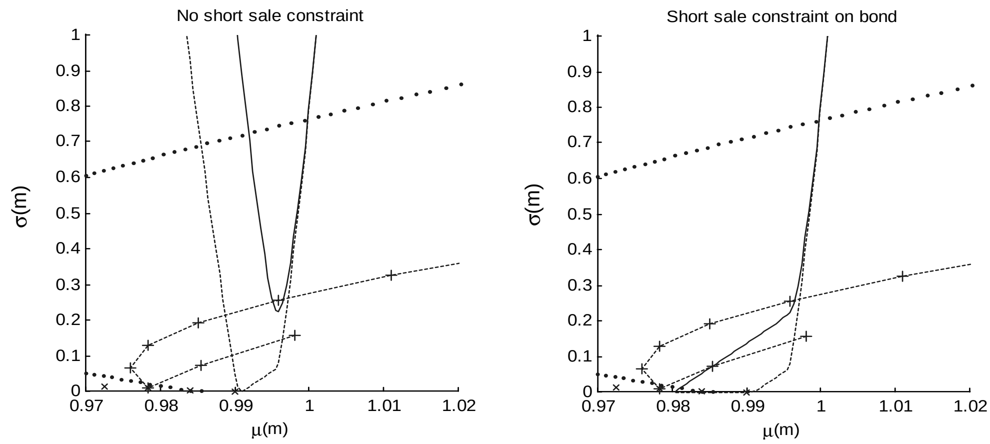
Notes

[1]Other ways to come to terms with the failure of the CCAPM includes consumption data considerations (see for instance Ferson and Harvey, 1992), investor heterogeneity (Heaton and Lucas, 1996) and irrational decision making (see Benartzi and Thaler, 1995).

[2]The up tick rule implies that investors only are allowed to short when the stock has risen. A short squeeze refers means the owner of the stock wants the stock back prematurely and the borrower is forced to buy the stock in the market since no one is willing to lend to her.

[3]The stock data was obtained from Robert Shiller's homepage (aida.econ.yale.edu/~shiller), while all other data was obtained through the EcoWin database (the data used is originally from the American Department of Commerce).

Figure 1. Frictions and different utility functions



In Figure 1 solid and dotted lines refer to Hansen Jaganathan bounds with and without bid/ask spread. Crosses (x) refer to CRRA utility, dots (.) to Abel utility and plus (+) to Epstein-Zin for increasing levels of risk aversion, gamma. Note that Abel utility disappears and then returns.

Table 1. Vertical distance (with positivity restriction)

No market frictions

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	-0.32	0.15	0.13	-0.45	-0.90	0.34	0.61	-1.89	-0.36	0.08	0.12	-0.41
1	-0.78	0.00	0.13	-0.43	-2.26	0.00	0.52	-1.69	-0.78	0.00	0.13	-0.43
2	-1.38	0.00	0.15	-0.50	-2.20	0.00	0.17	-0.55	-2.20	0.00	0.17	-0.54
4	-3.17	0.00	0.20	-0.65	-0.90	0.33	0.60	-1.94	-1.4E+22	0.00	0.25	-0.80
10	-3.0E+32	0.00	0.33	-1.01	-2.2E+34	0.00	0.53	-1.64	-7.9E+31	0.00	0.50	-1.67
15	-2.4E+27	0.00	0.45	-1.43	-6.6E+36	0.00	0.51	-1.36	-3.7E+32	0.00	0.55	-1.79

Transaction costs

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	-0.05	0.93	0.18	-0.58	0.14	1.00	0.64	-2.05	-0.31	0.31	0.19	-0.64
1	0.00	1.00	0.19	-0.63	-0.66	0.35	0.50	-1.66	-0.01	0.99	0.18	-0.62
2	-0.23	0.47	0.19	-0.65	-0.66	0.04	0.19	-0.65	-0.66	0.05	0.21	-0.67
4	-1.09	0.00	0.21	-0.72	-0.90	0.41	0.67	-2.07	-3.4E+00	0.00	0.26	-0.83
10	-711.1	0.00	0.36	-1.12	-11997.2	0.00	0.61	-1.85	-2.1E+03	0.00	0.48	-1.52
15	-8689.7	0.00	0.44	-1.42	-6141.8	0.00	0.62	-1.77	-1.8E+03	0.00	0.55	-1.74

Short sell restriction on bond

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	-0.20	0.19	0.11	-0.36	0.01	0.42	0.54	-1.60	-0.36	0.04	0.11	-0.35
1	-0.16	0.29	0.11	-0.35	0.00	0.83	0.49	-1.47	-0.16	0.29	0.12	-0.37
2	-0.11	0.38	0.12	-0.37	-0.07	0.57	0.13	-0.41	-0.07	0.59	0.13	-0.43
4	-0.02	0.80	0.16	-0.46	-0.90	0.13	0.44	-1.45	-0.07	0.57	0.24	-0.66
10	-0.16	0.35	0.29	-0.85	-1314.2	0.00	0.91	-2.73	-0.55	0.22	0.46	-1.36
15	-0.33	0.26	0.41	-1.15	-13074.0	0.00	1.83	-5.58	-1.06	0.10	0.47	-1.37

Transaction costs and short sell restriction on bond

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	-0.05	0.39	0.12	-0.33	0.16	1.00	0.51	-1.55	-0.31	0.25	0.18	-0.56
1	0.00	0.42	0.13	-0.37	0.07	1.00	0.46	-1.35	-0.01	0.96	0.17	-0.55
2	0.01	0.76	0.13	-0.36	0.01	1.00	0.17	-0.55	0.01	1.00	0.18	-0.55
4	0.02	0.72	0.16	-0.45	-0.90	0.13	0.44	-1.44	0.02	0.99	0.22	-0.65
10	-0.01	0.42	0.29	-0.76	-744.1	0.00	0.78	-2.40	-0.38	0.34	0.42	-1.28
15	-0.17	0.33	0.41	-1.09	-55069.5	0.00	1.61	-4.94	-0.83	0.14	0.47	-1.42

Short sell restriction on stock

	Abel				Epstein-Zin				CRRA			
Gamma		P-value	Std	CutOff		P-value	Std	CutOff		P-value	Std	CutOff
0	-0.32	0.15	0.13	-0.43	0.00	0.00	0.00	0.00	-0.36	0.11	0.14	-0.45
1	-0.78	0.00	0.14	-0.45	-2.26	0.01	0.48	-1.37	-0.78	0.00	0.14	-0.44
2	-1.38	0.00	0.15	-0.48	-2.20	0.00	0.18	-0.55	-2.20	0.00	0.17	-0.56
4	-3.16	0.00	0.19	-0.60	-0.90	0.41	0.69	-2.13	-52.2	0.00	0.27	-0.84
10	-3579.3	0.00	0.35	-1.10	-20285.0	0.00	0.78	-1.92	-367.0	0.00	0.52	-1.70
15	-1962.8	0.00	0.46	-1.42	-8.1E+06	0.00	0.87	-1.47	-654.3	0.00	0.58	-1.87

Transaction costs and short sell restriction on stock

	Abel				Epstein-Zin				CRRA			
Gamma		P-value	Std	CutOff		P-value	Std	CutOff		P-value	Std	CutOff
0	-0.05	0.88	0.16	-0.52	0.14	1.00	0.61	-1.94	-0.31	0.28	0.17	-0.55
1	0.00	0.97	0.17	-0.53	-0.66	0.38	0.49	-1.57	-0.01	0.95	0.17	-0.52
2	-0.23	0.44	0.18	-0.58	-0.66	0.02	0.18	-0.57	-0.66	0.04	0.19	-0.60
4	-1.09	0.00	0.20	-0.63	-0.90	0.38	0.71	-2.32	-3.41	0.00	0.25	-0.80
10	-100.3	0.00	0.31	-1.01	-1554.9	0.00	0.80	-2.19	-279.6	0.00	0.46	-1.53
15	-222.8	0.00	0.41	-1.26	-33390.2	0.00	0.88	-1.71	-2960.2	0.00	0.39	-1.15

See section 4.2 in main text for description of the setup. For the Abel utility function a kappa of 0.5 has been used, for the Epstein-Zin utility function an intertemporal elasticity of substitution is set to 0.5 and the sujective discount rate on an annual basis is 0.99. P-value gives bootstrapped p-value, Std is the standard deviation of the bootstrapped sample and CutOff is the 5% one-sided cut-off value.

Table 2. Vertical distance (with positivity restriction and correlation adjusted)

No market frictions

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
1	-2.86	0.01	0.74	-1.99	-2.28	0.00	0.53	-1.70	-2.87	0.02	0.66	-1.90
2	-5.49	0.01	0.94	-2.48	-8.03	0.01	1.18	-2.71	-8.03	0.00	0.91	-2.43
4	-13.32	0.00	1.48	-3.44	-0.92	0.33	0.61	-1.99	-5.0E+22	0.00	1.89	-3.94
10	-1.3E+33	0.00	1.91	-5.60	-2.4E+34	0.00	0.58	-1.83	-2.8E+32	0.00	3.19	-7.95
15	-1.1E+28	0.00	3.41	-7.71	-8.1E+36	0.00	0.58	-1.98	-1.3E+33	0.00	5.56	-8.36

Transaction costs

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
1	-0.02	1.00	0.50	-1.62	-0.66	0.35	0.51	-1.67	-0.02	0.99	0.48	-1.54
2	-0.71	0.45	0.62	-2.04	-1.64	0.07	0.61	-1.93	-1.64	0.06	0.56	-1.77
4	-3.91	0.01	0.80	-2.55	-0.91	0.41	0.69	-2.11	-8.44	0.00	0.68	-2.15
10	-2746.0	0.00	1.53	-4.53	-12996.6	0.00	0.65	-2.03	-5050.8	0.00	1.29	-4.01
15	-34054.5	0.00	1.86	-5.72	-7478.7	0.00	0.72	-2.31	-4421.9	0.00	1.48	-4.65

Short sell restriction on bond

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
1	-0.49	0.36	2.6E+11	-1.88	0.00	0.90	NaN	-1.48	-0.48	0.37	NaN	-1.80
2	-0.39	0.43	NaN	-2.12	-0.23	0.62	1.9E+12	-1.94	-0.23	0.65	8.2E+11	-2.15
4	-0.12	0.82	2.3E+11	-2.64	-0.94	0.13	0.45	-1.50	-0.24	0.60	NaN	-3.12
10	-0.68	0.39	12.95	-4.16	-1481.7	0.00	0.99	-3.02	-1.75	0.25	1.3E+12	-4.99
15	-1.38	0.28	NaN	-5.33	-16804.0	0.00	2.26	-6.98	-3.25	0.13	NaN	-5.64

Transaction costs and short sell restriction on bond

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
1	-0.02	0.96	0.85	-2.40	0.07	1.00	0.47	-1.37	-0.03	0.96	0.73	-2.17
2	0.01	0.99	0.92	-2.55	0.01	1.00	0.79	-2.35	0.01	1.00	0.88	-2.42
4	0.02	0.99	0.97	-2.62	-0.92	0.13	0.44	-1.47	0.02	0.99	0.93	-2.63
10	-0.16	0.80	1.54	-4.01	-809.7	0.00	0.85	-2.67	-1.37	0.34	1.42	-4.11
15	-0.95	0.44	1.60	-4.98	-67489.0	0.00	1.97	-6.13	-2.92	0.14	1.69	-4.81

Short sell restriction on stock

	Abel				Epstein-Zin				CRRA			
Gamma		P-value	Std	CutOff		P-value	Std	CutOff		P-value	Std	CutOff
1	-2.70	0.02	0.71	-2.07	-2.28	0.01	0.48	-1.40	-2.71	0.02	0.77	-2.12
2	-5.43	0.01	1.01	-2.36	-7.23	0.01	0.65	-2.47	-7.23	0.00	1.20	-2.60
4	-11.75	0.00	1.22	-3.20	-0.92	0.41	0.71	-2.18	-171.3	0.00	1.90	-4.59
10	-12578.0	0.00	15.92	-6.18	-22207.9	0.00	0.84	-2.13	-1305.8	0.00	2.88	-7.94
15	-7543.0	0.00	11.71	-9.13	-1.0.E+07	0.00	1.05	-2.06	-2254.3	0.00	3.16	-8.75

Transaction costs and short sell restriction on stock

	Abel				Epstein-Zin				CRRA			
Gamma		P-value	Std	CutOff		P-value	Std	CutOff		P-value	Std	CutOff
1	-0.02	0.96	0.46	-1.50	-0.66	0.38	0.50	-1.58	-0.02	0.95	0.49	-1.48
2	-0.72	0.42	0.61	-1.91	-1.69	0.04	0.54	-1.64	-1.69	0.06	0.55	-1.74
4	-3.95	0.01	0.82	-2.43	-0.92	0.38	0.72	-2.35	-8.7	0.00	0.75	-2.37
10	-390.6	0.00	1.40	-4.16	-1689.6	0.00	0.87	-2.39	-710.3	0.00	1.34	-4.18
15	-880.6	0.00	1.83	-5.61	-40975.7	0.00	1.05	-2.30	-7512.9	0.00	1.10	-2.87

As table 1, except that vertical Luttmer measure with adjustment for imperfect correlation is used.

Table 3. HJ measure (with positivity restriction)

No market frictions

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	0.22	0.00	0.04	0.14	0.07	0.05	0.02	0.07	0.22	0.00	0.04	0.14
1	0.22	0.00	0.04	0.13	0.15	0.00	0.03	0.11	0.22	0.00	0.04	0.14
2	0.22	0.00	0.04	0.14	0.22	0.00	0.04	0.13	0.22	0.00	0.04	0.13
4	0.22	0.00	0.04	0.13	0.35	0.00	0.02	0.08	0.22	0.00	0.04	0.13
10	0.22	0.01	0.05	0.14	0.79	0.00	0.04	0.15	0.22	0.00	0.04	0.13
15	0.21	0.00	0.04	0.14	1.38	0.00	0.11	0.38	0.22	0.00	0.04	0.13

Transaction costs

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	0.00	0.93	0.16	0.47	0.00	0.95	0.14	0.43	0.01	0.91	0.17	0.50
1	0.00	0.96	0.17	0.50	0.01	0.92	0.18	0.54	0.00	0.96	0.17	0.51
2	0.00	0.95	0.17	0.52	0.00	0.90	0.15	0.45	0.00	0.94	0.16	0.50
4	0.01	0.93	0.17	0.53	0.12	0.41	0.17	0.52	0.02	0.89	0.17	0.49
10	0.02	0.83	0.17	0.51	0.56	0.03	0.14	0.42	0.05	0.70	0.16	0.49
15	0.04	0.72	0.16	0.49	1.16	0.00	0.19	0.54	0.08	0.59	0.15	0.46

Short sell restriction on bond

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	0.01	0.44	0.04	0.11	0.00	0.96	0.02	0.06	0.02	0.39	0.04	0.10
1	0.01	0.45	0.04	0.11	0.00	1.00	0.03	0.10	0.01	0.43	0.04	0.12
2	0.01	0.52	0.04	0.11	0.01	0.62	0.04	0.11	0.01	0.61	0.04	0.11
4	0.00	0.84	0.04	0.11	0.03	0.44	0.04	0.12	0.01	0.61	0.04	0.11
10	0.01	0.42	0.04	0.11	0.22	0.08	0.07	0.24	0.04	0.29	0.04	0.11
15	0.03	0.35	0.03	0.10	0.54	0.00	0.11	0.33	0.07	0.17	0.04	0.11

Transaction costs and short sell restriction on bond

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	0.00	0.83	0.16	0.50	0.00	1.00	0.17	0.50	0.01	0.78	0.17	0.49
1	0.00	0.90	0.17	0.50	0.00	0.81	0.17	0.51	0.00	0.89	0.16	0.50
2	0.00	0.91	0.16	0.47	0.00	1.00	0.15	0.45	0.00	1.00	0.15	0.47
4	0.00	1.00	0.16	0.51	0.02	0.69	0.14	0.43	0.00	0.91	0.15	0.46
10	0.00	0.81	0.15	0.47	0.21	0.27	0.16	0.50	0.03	0.59	0.15	0.46
15	0.02	0.69	0.15	0.46	0.53	0.06	0.18	0.57	0.06	0.47	0.16	0.47

Short sell restriction on stock

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	0.22	0.00	0.05	0.12	0.00	0.00	0.00	0.00	0.22	0.00	0.05	0.14
1	0.22	0.00	0.04	0.13	0.15	0.01	0.02	0.06	0.22	0.00	0.04	0.12
2	0.22	0.00	0.05	0.14	0.22	0.00	0.04	0.12	0.22	0.00	0.04	0.12
4	0.22	0.00	0.05	0.13	0.35	0.00	0.02	0.06	0.22	0.00	0.04	0.13
10	0.22	0.00	0.04	0.12	0.79	0.00	0.04	0.12	0.22	0.00	0.04	0.11
15	0.21	0.00	0.04	0.12	1.38	0.00	0.09	0.26	0.22	0.00	0.04	0.12

Transaction costs and short sell restriction on stock

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	0.00	0.81	0.15	0.43	0.00	0.87	0.12	0.34	0.01	0.77	0.16	0.46
1	0.00	0.88	0.15	0.44	0.01	0.79	0.12	0.36	0.00	0.85	0.14	0.42
2	0.00	0.84	0.15	0.45	0.00	0.77	0.14	0.41	0.00	0.78	0.15	0.42
4	0.01	0.80	0.15	0.44	0.12	0.30	0.13	0.36	0.02	0.71	0.14	0.42
10	0.02	0.66	0.15	0.41	0.56	0.01	0.12	0.33	0.05	0.55	0.13	0.38
15	0.04	0.60	0.14	0.41	1.16	0.00	0.16	0.46	0.08	0.12	0.08	0.21

As table 1, except that Hansen Jagannathan measure with positivity restriction is used.

Table 4. Chi-squared test with GMM weight

No market frictions

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	30.7	0.26	9.2	42.8	27.2	0.08	7.9	30.2	32.2	0.06	9.1	26.6
1	49.6	0.01	8.5	28.9	12.8	0.11	4.7	15.7	50.5	0.00	7.8	30.2
2	78.6	0.00	7.4	24.5	118.0	0.00	8.1	32.0	118.0	0.00	8.6	33.7
4	114.7	0.00	7.2	26.6	173.7	0.00	5.4	22.3	201.6	0.00	8.2	34.1
10	130.3	0.00	6.7	27.9	67.9	0.00	8.0	29.2	251.4	0.00	8.3	33.5
15	129.2	0.00	6.5	29.2	87.7	0.00	14.6	42.7	256.8	0.00	8.6	34.5

Transaction costs

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	22.6	0.91	24.6	84.7	17.1	0.81	15.1	52.3	29.9	0.55	22.3	82.3
1	18.2	0.89	23.8	82.2	20.2	0.62	51.4	81.0	18.4	0.86	22.2	71.3
2	19.5	0.82	22.8	72.9	52.7	0.11	19.5	72.4	52.7	0.14	23.8	75.5
4	52.5	0.14	20.9	77.0	25.7	0.56	18.8	66.4	212.4	0.00	21.0	73.0
10	102.1	0.02	22.8	79.7	30.1	0.26	14.1	52.2	206.6	0.00	21.6	74.8
15	110.2	0.01	19.3	73.2	28.5	0.37	22.8	54.6	243.2	0.00	18.1	71.9

Short sell restriction on bond

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	27.7	0.31	8.5	39.5	13.8	0.43	6.3	24.4	25.7	0.02	5.7	20.5
1	19.3	0.18	6.1	25.2	6.6	0.33	4.1	13.0	19.5	0.17	6.8	25.7
2	12.2	0.26	5.9	19.5	19.8	0.29	7.1	29.1	19.8	0.28	6.8	29.1
4	8.1	0.63	5.9	22.0	13.5	0.37	8.2	23.0	17.9	0.47	8.1	32.4
10	18.4	0.27	6.2	26.2	17.6	0.12	8.4	24.7	66.0	0.00	7.5	29.7
15	37.2	0.01	14.8	28.3	21.9	0.13	13.3	36.4	138.4	0.00	7.6	30.3

Transaction costs and short sell restriction on bond

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	20.6	0.69	7.2	37.5	12.7	0.82	13.7	49.7	18.1	0.38	14.1	47.2
1	14.7	0.50	6.3	27.1	10.8	0.67	13.4	45.7	14.8	0.72	13.2	46.9
2	10.7	0.53	7.4	25.4	17.0	0.73	12.2	50.0	17.0	0.72	12.4	46.8
4	11.5	0.59	7.5	29.2	13.3	0.66	9.6	35.4	18.3	0.72	11.8	47.5
10	12.7	0.74	7.9	32.0	16.2	0.22	9.6	31.1	43.0	0.07	11.7	46.9
15	22.7	0.25	7.7	33.9	21.1	0.18	15.7	43.0	113.1	0.00	12.1	47.8

Short sell restriction on stock

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	32.3	0.25	20.5	44.6	0.0	0.00	0.0	0.0	24.2	0.10	8.8	30.6
1	55.0	0.00	8.2	31.0	13.5	0.04	4.6	12.0	55.9	0.00	8008.4	30.9
2	80.0	0.00	41.4	29.3	120.3	0.00	8.9	32.7	120.3	0.00	14948.7	35.5
4	111.4	0.00	492671.4	25.1	175.8	0.00	159.0	20.9	179.4	0.00	10.0	35.7
10	121.7	0.00	43.0	28.2	65.0	0.00	208399.9	13.1	203.6	0.00	15.3	35.3
15	116.5	0.00	6.9	29.2	24.4	0.01	4.9	14.2	214.9	0.00	8.3	33.6

Transaction costs and short sell restriction on stock

	Abel				Epstein-Zin				CRRA			
Gamma	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff	Measure	P-value	Std	CutOff
0	22.3	0.75	15.0	58.1	17.1	0.50	11.5	40.6	25.5	0.36	14.9	51.8
1	18.2	0.64	15.9	48.4	16.5	0.70	18.5	60.4	18.4	0.61	14.4	46.3
2	17.3	0.67	13.0	44.7	45.8	0.06	13.5	47.9	45.8	0.08	15.5	50.9
4	41.4	0.07	11.5	45.1	11.7	0.75	9.5	34.3	203.7	0.00	13.9	48.7
10	97.8	0.00	12.2	46.8	22.8	0.07	8.0	25.9	406.5	0.00	13.5	47.5
15	110.1	0.00	12.0	46.8	21.1	0.11	9.4	28.4	457.3	0.00	12.8	32.4

As table 1, except that GMM is used.

A Appendix

In the following we discuss in greater detail the computation of the measure and bounds of Hansen and Jagannathan (1991, 1997) in settings which may include market frictions.

A.1 Measure

The definition of the HJ measure can be expressed as (Hansen and Jagannathan, 1997)

$$\delta^2 = \min_{\substack{m \in M \\ (m>0)}} E\left[(m^* - m)^2\right]. \tag{A.1}$$

Here M is the set of admissible stochastic discount factors, and it may or may not include a positivity constraint. For generality, we consider any set of asset returns $\mathbf{x}$ with prices $\mathbf{p}$, rather than only returns with price 1. That is, the set of admissible stochastic discount factors is the set of all m such that (Hansen, Heaton and Luttmer, 1995)

$$E\,(m\mathbf{x}) - \mathbf{p} \in C^*. \tag{A.2}$$

The elements of C^* will depend on the type of market friction imposed. For an asset i with no market friction, the corresponding part of C^* is equal to 0 and the constraint reduces to

$$E\left(mx^i\right) = p^i. \tag{A.3}$$

However, for an asset i with short sale constraint, the corresponding part of C^* is the interval $(-\infty, 0]$, implying

$$E\left(mx^i\right) \leq p^i. \tag{A.4}$$

This also includes bid/ask spreads as they can be implemented, as described in section 2.2, by introducing two assets, both with short sale constraints. The HJ measure may thus be expressed as follows:

$$\delta^2 = \min_{(m>0)} E\left[(m^* - m)^2\right] \text{ subject to } E\,(m\mathbf{x}) - \mathbf{p} \in C^*. \tag{A.5}$$

Hansen, Heaton and Luttmer (1995) show how this can be re-expressed in terms of its conjugate problem. The result differs slightly depending on whether the positivity constraint on m is not imposed

$$\delta^2 = \max_{\boldsymbol{\lambda} \in C} E\{(m^*)^2 - \left(m^* - \boldsymbol{\lambda}'\mathbf{x}\right)^2 - 2\boldsymbol{\lambda}'\mathbf{p}\}, \tag{A.6}$$

or is imposed

$$\delta^{+2} = \max_{\boldsymbol{\lambda}\in C} E\{(m^*)^2 - [(m^* - \boldsymbol{\lambda}'\mathbf{x})^+]^2 - 2\boldsymbol{\lambda}'\mathbf{p}\}. \tag{A.7}$$

The expression x^+ denotes $max(x, 0)$. The elements of the vector of lagrange multipliers $\boldsymbol{\lambda}$ are restricted to the set C which corresponds to the complement set of C^* including $\{0\}$. That is, for an asset with no short sale constraint the corresponding part of C is $[-\infty, \infty]$, and $[0, \infty]$ with a short sale constraint. Note that the above is a generalization that encompasses the HJ measure without frictions as well as the HJ bounds with and without frictions as special cases (see below). In this paper we have focused on the measure with positivity constraint, as it avoids arbitrage opportunities. When actually computing the measure for a discrete data set with observations on returns with price 1, the numerical problem can be expressed as follows

$$\begin{aligned}
\delta^{+2} &= \max_{\boldsymbol{\lambda}\in C} \frac{1}{T}\sum_{t=1}^{T}\left\{(m_t^*)^2 - [(m_t^* - \boldsymbol{\lambda}'\mathbf{R}_t)^+]^2 - 2\boldsymbol{\lambda}'\mathbf{i}\right\} \\
&= -\min_{\boldsymbol{\lambda}\in C} \frac{1}{T}\sum_{t=1}^{T}\left\{(m_t^*)^2 - [(m_t^* - \boldsymbol{\lambda}'\mathbf{R}_t)^+]^2 - 2\boldsymbol{\lambda}'\mathbf{i}\right\}.
\end{aligned} \tag{A.8}$$

Finally consider the simplest case of the HJ measure with no positivity constraint and no market frictions

$$\delta^2 = \max_{\boldsymbol{\lambda}\in\Re^n} E\{(m^*)^2 - \left(m^* - \boldsymbol{\lambda}'\mathbf{R}\right)^2 - 2\boldsymbol{\lambda}'\mathbf{i}\}. \tag{A.9}$$

First order condition with respect to the vector $\boldsymbol{\lambda}$ is

$$E\{\mathbf{R}\left(m^* - \boldsymbol{\lambda}'\mathbf{R}\right) - \mathbf{i}\} = \mathbf{0}. \tag{A.10}$$

Solving for $\boldsymbol{\lambda}$ and reinserting this into (A.9) yields

$$\delta = \left[(E(m^*\mathbf{R}) - \mathbf{i})' E(\mathbf{R}\mathbf{R}')^{-1}(E(m^*\mathbf{R}) - \mathbf{i})\right]^{\frac{1}{2}}, \tag{A.11}$$

This is the same expression as (16) in the main text. When bootstrapping under the null, $\mathbf{i}$ will of course be replaced by $\mathbf{p}_*$, i.e. the average of the Euler equation over all bootstrapped samples. The reason for this is discussed in greater detail in section 2.3.

A.2 HJ bounds

As noted in Hansen, Heaton and Luttmer (1995) there is a neat relationship between the bound and the measure. The HJ measure corresponds to the bound when, firstly, the SDF proxy m^* is set to zero. Secondly, if there is no truly risk free asset, the asset vector is augmented with such an asset with unit payoff and price v. Starting with (A.6), we perform these modifications and the risk free asset is given a Lagrange multiplier denoted γ. Thus we arrive at the following result, with positivity constraint imposed:

$$\delta^{+2} = \max_{\gamma \in \Re, \boldsymbol{\lambda} \in C} \left\{ E\left[(\gamma - \boldsymbol{\lambda}'\mathbf{R})^{+2} \right] - 2\gamma v + 2\boldsymbol{\lambda}'\mathbf{i}) \right\}. \tag{A.12}$$

Which corresponds to the result obtained in Luttmer (1996). For the simplest case with no positivity constraint and no market frictions, Hansen and Jagannathan (1991) derive an analytic expression:

$$\sigma(m) \geq \left[(vE(\mathbf{R}) - \mathbf{i})'\boldsymbol{\Omega}^{-1}(vE(\mathbf{R}) - \mathbf{i}) \right]^{\frac{1}{2}}. \tag{A.13}$$

Here $\boldsymbol{\Omega}$ denotes the return covariance matrix. This is the same equation as (12) in the main text. To generate the bound under the null replace $\mathbf{i}$ by $\mathbf{p}_*$ as previously mentioned.

Chapter 7

A Monte Carlo Study of L1-Norm Based Measures of Model Misspecification

Abstract

Hansen and Jagannathan (1997) introduced a measure of model misspecification based on the L2-norm. It is however well-known that L1-norm methods may show good properties in the presence of non-normal distributions. In this paper we therefore introduce some L1-norm based measures of misspecification. We also provide an easy algorithm which simplifies the computation of the gain-loss ratio presented by Bernardo and Ledoit (2000). Two Monte Carlo simulations are undertaken to assess the performance of the measures under varying distributional assumptions. We provide evidence that L1-norm based measures tend to perform better in small and non-normally distributed samples.

1 Introduction

Many things have changed in the asset pricing literature since the arrival of the stochastic discount factor (SDF) methodology. Its major virtue is to identify an asset pricing model with a random variable which adjusts for risk by discounting future payoffs state by state. This random variable is usually set as a function of real data and a set of parameters. With these constructs at hand it becomes much easier to develop simple tools to evaluate asset pricing models.

One of the contributions that initiated the transition towards this new approach is Hansen's generalized method of moments (GMM). This method allows the researcher to estimate the parameters of the SDF by minimizing a quadratic form of the sample mean of the pricing errors. This form is asymptotically χ^2 distributed which provides a simple test of the overall fit of the model. However, this test can only be used to determine whether a given model should be

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rejected or not and it is not valid to establish a comparison between competing asset pricing models nor to characterize the performance of a false model.

To get around these drawbacks, Hansen and Jagannathan (1997) introduce a measure of model misspecification which equals the least square distance between the postulated SDF and the set of SDF's that correctly price a given group of asset payoffs. This measure (hereafter HJ) also has an appealing interpretation since it can be proved to minimize among all admissible SDF's the maximum mispricing the benchmark model implies for those payoffs with unit second norm in the span of all contingent claims.

Other proposals include Hansen and Jagannathan's (1991) volatility bounds which can be used to test whether a given model satisfies the necessary conditions of volatility that the pricing equations imply and Kocherlakota's (1996) t-test which only considers two basis assets one at a time. In any case, all the above tests share a common feature: they are L2-norm based.

It is well-known that L1-norm methods may show good properties in the presence of non-normal distributions, for instance heavy-tailed and/or asymmetric distributions. These methods provide more robust estimators, since they are less easily influenced by outliers or other extreme observations. The basic intuition for this is that L2-norm methods involve squaring errors, which magnifies large deviations, while L1-norm methods are based on absolute deviations (Green, 1997; an extensive survey of L1-norm estimators is given by Amemiya, 1985). Since financial data are known to frequently display non-normal properties, L1-norm methods have seen considerable use in financial economics. Asset pricing models can for instance be estimated using regression techniques based on L1-norm methods such as minimum absolute error criterion or quantile regression. To obtain more robust estimates, stock betas have been computed using such regressions methods (e.g. Chan and Lakonishok, 1992; Mills and Coutts, 1996).

Recently, the Hansen and Jagannathan (1997) measure has seen increasing use. Although its economic interpretation is appealing, it is subject to the same possible drawbacks as any L2-norm based method. For instance, a small numbers of outliers can potentially distort model evaluation and comparison. Given the general interest in L1-norm methods, it is therefore relevant to develop also measures of model misspecification that are L1-norm based. Such measures will also be presented as well as implemented in the following. This should help researchers to choose a measure of misspecification that is suitable to the object of study.

Bernardo and Ledoit (2000) use the gain-loss ratio to derive asset price bounds. They also suggest that its use as a measure of model misspecification could be well justified in the presence of heavily skewed returns. Longarela (2001) presents a new method to derive asset price bounds which is also based on an L1-norm measure of mispricing. Furthermore, its computation is very simple since it only involves solving a linear program. However, these two methods present in principle one major disadvantage since their asymptotic theory is very hard to obtain.

1.1 Structure of paper

The goal of this paper is twofold. On one hand, we introduce alternative ways of measuring misspecification in the confines of the L1-norm and we discuss the existing ones. We also provide an easy algorithm which simplifies the computation of the gain-loss ratio. On the other hand, we carry out a study of their behavior in different scenarios (normal, heavy-tailed and asymmetric distributions) and compare their relative performance to the L2 based HJ measure. We perform two different types of Monte Carlo studies. Firstly, we examine the properties of the measures as tests of misspecification, more specifically size and power. Secondly, we use the measures of misspecification to estimate the parameters of a linear factor model, and compare the results with the HJ measure.

The remainder of the paper is organized as follows. In Section 2 we introduce notation and preliminaries. We then review some existing measures of model misspecification and present new L1-norm based alternatives. Section 3 describes the design and results of our study of the statistical properties of measures of misspecification. Section 4 describes the study of parameter estimation and reports our results. Section 5 concludes.

2 Measures of Model Misspecification

2.1 Preliminaries

We will only examine the i.i.d. case and therefore, we will concentrate on a two-period economy endowed with a probability space $\{\Omega, \mathcal{F}, \mu\}$. In this economy N basis assets are traded today at a known price given by a vector $\mathbf{p}$ and they deliver a continuous random payoff denoted by a vector $\mathbf{x}$. No frictions are allowed and hence any linear combination of the basis payoffs constitutes an attainable payoff x whose set will be denoted by X. Let $Z \subset X$ be the subset of zero-price payoffs and $L \supset X$ the space of payoffs in the span of all contingent claims.[1] For any payoff $x \in L$ we denote its positive and negative part $x^+ \equiv \max(x, 0)$ and $x^- \equiv \max(-x, 0)$, respectively. Clearly, $x = x^+ - x^-$.

The law of one price (LOP) is assumed to hold which implies the existence of random variables m (admissible SDF's) which satisfy

$$\mathbf{p} = E(m\mathbf{x}). \tag{1}$$

If we are considering the special case of a set of returns ($\mathbf{R}$) whose prices are a vector ones ($\mathbf{i}$), then the pricing restriction is obviously instead

$$\mathbf{i} = E\left(m\mathbf{R}\right). \tag{2}$$

Denote by M the set of all random variables m satisfying (1). If we rule out arbitrage opportunities, it is well-known there will be at least one strictly positive $m \in M$. These random variables give the price of any payoff $x \in L$ through their *pricing extension* defined as

$$\pi_m\left(x\right) \equiv E\left(mx\right),$$

which obviously assigns the same price for any $m \in M$ if $x \in X$. Since our objective is to evaluate asset pricing models, denoted by m^*, a postulated SDF and for any given x consider also its implied pricing function $\pi^*\left(x\right) = E\left(m^*x\right)$.

For any $m \in M$ denote by $d\left(m, m^*\right)$ a distance between m and the benchmark model m^*. A measure of model misspecification δ can be defined as the solution to the following optimization:

$$\delta_{m^*} \equiv \min_{m \in M} d\left(m, m^*\right),$$

and we will denote by $\delta^+_{m^*}$ the above value when the restriction $m > 0$ is incorporated. All the measures of model specification we discuss in the following section are particular cases of the above formulation. The difference between them only depends on the functional form of the distance $d\left(m, m^*\right)$ which delivers a distinct economic meaning. In each case we will present a duality result in the form of a proposition which provides the corresponding economic interpretation.

In the following we discuss the gain-loss ratio in Section 2.2 and the HJ measure as well as its L1 alternatives in Section 2.3. We also discuss estimation of model parameters using measures of misspecification in Section 2.4.

2.2 About the gain-loss ratio

Bernardo and Ledoit (2000) introduce the gain-loss ratio as the underlying measure behind their derivation of asset price bounds. They also suggest its use as a measure of model misspecification especially in the presence of heavily skewed returns. In our context, this measure results from choosing

$$d\left(m, m^*\right) = \frac{\sup \frac{m}{m^*}}{\inf \frac{m}{m^*}}, \tag{3}$$

where m^* must be a strictly positive benchmark SDF.

Proposition 2.1 *For the distance in (3), the following equality holds*

$$\delta^{+}_{m^*} = \max_{x \in Z, x \neq 0} \frac{\pi^* (x^+)}{\pi^* (x^-)}.$$

Proof. See Bernardo and Ledoit (2000). ∎

To understand the gain-loss ratio, recall from Section 2.1 that Z denotes the set of zero price payoffs. This implies that if $x \in Z$, then an admissible stochastic discount factor $m \in M$ gives $\pi_m(x) = E(mx) = 0$. Further, recall that x^+ and x^- are the positive and negative part of the asset payoff with $x = x^+ - x^-$. Thus if $\pi_m(x) = 0$ then $\pi_m(x^+) = \pi_m(x^-)$ and hence the gain-loss ratio is $\pi_m(x^+)/\pi_m(x^-) = 1$ for any admissible stochastic discount factor. This means if the gain-loss ratio is higher than one for the benchmark model, we have a misspecified model and the amount the gain-loss ratio exceeds one is a quantification of the misspecification.

Hence, for the gain-loss ratio, model misspecification follows from the price discrepancy for zero-price payoffs between M and the benchmark model m^*, summarized by the ratio of the price of the positive part and the negative part under m^*. Again, the further this quotient is from one, the worse specified the benchmark model is.

Even though the computation of the gain-loss ratio may seem troublesome, it is simplified with the following tractable algorithm. First, note that for a given value δ, the set of constraints given by (1) and

$$\begin{aligned} \delta_1 \leq \frac{m}{m^*} &\leq \delta_2 \\ \delta \delta_1 &= \delta_2 \\ m &> 0 \\ \delta_1, \delta_2 &> 0, \end{aligned} \tag{4}$$

are linear on m and the constants δ_1 and δ_2. It is straightforward to see that $\delta^{+}_{m^*}$ is equal to the smallest value δ for which it is possible to find constants δ_1 and δ_2 and a random variable m which satisfy (1) and (4). In practice the problem must be approximately solved by restricting ourselves to a finite number of states of nature. Any linear programming solver allows to check for feasibility of a finite number of linear constraints. Thus, one trick could be to choose a large enough starting value for δ so that feasibility is guaranteed. From that point on, we try smaller values and check each time for feasibility until we get to the smallest δ for which the constraints hold. Obviously, if the starting value is not large enough and it does not pass the feasibility check, we try larger values instead.

There is an alternative procedure which may be easier to implement and less costly in terms of computing time. It is again straightforward to see that

$$\delta_{m^*}^{+} = \left\{ \max_{m \in M, m>0} \frac{\inf \frac{m}{m^*}}{\sup \frac{m}{m^*}} \right\}^{-1}.$$

Thus, we can concentrate on solving the maximization in braces instead. Replace the second constraint in (4) by $\delta\delta_2 = \delta_1$. Reasoning as above, the optimal value in the above maximization is the largest value of δ for which it is possible to find constants δ_1 and δ_2 and a random variable m which satisfy (1) and (4). This value is obviously between zero and one. A simple rule will be to set in each iteration the value of δ according to the following scheme:

$$\delta_{i+i} = \begin{cases} (\delta_i + \gamma_i)/2 & \text{in case of feasibility for } \delta_i \\ (\alpha_i + \delta_i)/2 & \text{otherwise,} \end{cases}$$

where γ_i (α_i) is the value that δ took in the latest iteration without (with) feasibility and $\delta_0 = 0.5$, $\gamma_0 = 1$ and $\alpha_0 = 0$.

2.3 About the HJ measure and L1 analogues

Hansen and Jagannathan (1997) introduce the most popular measure of misspecification of asset pricing models. Their contribution constitutes the first real answer to the question of how a given model performs when it is understood that it is inherently misspecified. Up to then, the wildly spread GMM methodology had imposed an approach to the evaluation of asset pricing models which only involved checking whether a model was exactly true or not except for sample variation. Furthermore, the HJ measure makes it possible to compare the degree of misspecification across models. This is not possible with standard GMM, since the optimal weight matrix used is model specific. The HJ measure thus implied an enrichment of the methodology addressing model evaluation.

In our framework, the HJ measure obtains by setting

$$d(m, m^*) = \left\{ E\left[(m - m^*)^2 \right] \right\}^{1/2}. \tag{5}$$

Proposition 2.2 *For the distance given in (5), the following equalities hold*

$$\delta_{m^*} = \min_{m \in M} \max_{\substack{x \in X \\ E(x^2)=1}} |\pi_m(x) - \pi^*(x)|$$

and

$$\delta^{+}_{m^*} = \min_{\substack{m\in M\\ m>0}} \max_{\substack{x\in L\\ E\left(x^2\right)=1}} \left|\pi_m\left(x\right) - \pi^*\left(x\right)\right|.$$

Proof. See Hansen and Jagannathan (1997). ■

Thus, $\delta^{+}_{m^*}$ gives the maximum pricing discrepancy between π_m and π^* for payoffs in the span of all contingent claims whose second moments are equal to one. An alternative and more intuitive interpretation goes as follows. Suppose there are two different (complete) financial markets where the whole set of contingent claims are traded. In one market, prices are set according to m and in the other one, prices are set according to the benchmark. Arbitrage opportunities do not exist within each market since both m and m^* are strictly positive. However, there are cross-market strategies that give infinite riskless benefits as long as there exist pricing discrepancies between π_m and π^* ($\delta^{+}_{m} > 0$). The role of the normalization $E\left(x^2\right) = 1$ is simply to guarantee the boundedness of $\delta^{+}_{m^*}$ and therefore, to give a measure of the size of the above benefits in relative terms; it has no economic meaning beyond that. In other words, $\delta^{+}_{m^*}$ gives the optimal value of cross-market arbitrage strategies for those payoffs whose second moment is equal to one. In this way, $\delta^{+}_{m^*}$ can be seen as a measure of market integration.[2] The interpretation of δ_{m^*} is identical with the exception that only payoffs in X are considered. Hansen, Heaton and Luttmer (1995) derived the corresponding asymptotic theory.

We turn now to present an L1-norm based alternative which partially retain this economic interpretation. If we look at (5) the obvious L1-norm equivalent of the HJ measure can be obtained by setting,

$$d\left(m, m^*\right) = E\left|m - m^*\right|. \tag{6}$$

Proposition 2.3 *Assume there is no duality gap. For the distance in (6), the following equality holds*

$$\delta^{+}_{m^*} = \min_{\substack{m\in M\\ m>0}} \max_{\substack{x\in L\\ |x|=1}} \left|\pi_m\left(x\right) - \pi^*\left(x\right)\right|.$$

Proof. This is the dual of the linear problem implied by (6). ■

Thus, the above measure (which will be referred to as the L1 equivalent) has an interpretation that can also be read in terms of a maximum pricing discrepancy. The difference lies on the target set of payoffs in the space of all contingent claims implied in the normalization. In this case, we look at those payoffs whose absolute value equals one, while the HJ measure considers payoff with second moment equal to one.

Obviously, any measure will need some kind of normalization, or target set, within which the pricing discrepancy is computed. Without such a normalization, the discrepancy can be arbitrarily large or small. The advantage of setting the normalization to payoffs with absolute value equal to one is that it may be easier to correctly estimate than the second norm, especially in a small sample of asset payoffs or when payoffs are non-normal. Such pragmatic considerations are valid, since the normalization does not necessarily have a meaningful economic interpretation in itself.

One important feature of this measure is further the simplicity of its computation. A simple linear program gives the solution. To see this note that the problem to be solved to obtain δ_m^+ can be rewritten as

$$\min_{y,m} \int y\left(s\right) ds,$$

subject to the pricing constraint, the positivity constraint and

$$-y\left(s\right) \leq \mu\left(s\right)\left[m\left(s\right) - m^*\left(s\right)\right] \leq y\left(s\right) \quad \forall s,$$

where $y\left(s\right), m\left(s\right), m^*\left(s\right)$ and $\mu\left(s\right)$ is the value of the corresponding variables at the state of nature s and μ is the real probability measure.

Finally, the following distance (referred to as L1sup) also has an L1-norm interpretation:

$$d\left(m, m^*\right) = \sup\left|\mu\left(m - m^*\right)\right|. \tag{7}$$

Proposition 2.4 *For the distance in (7), the following equality holds*

$$\delta_{m^*}^+ = \min_{\substack{m \in M \\ m > 0}} \max_{\substack{x \in L \\ E(|x|)=1}} \left|\pi_m\left(x\right) - \pi^*\left(x\right)\right|.$$

Proof. This is the dual of the linear problem implied by (7). ■

This measure normalizes by restricting the maximization to payoffs whose absolute expectation equals one. For this measure, it is straightforward to see that $\delta_{m^*}^+$ is equal to the smallest value δ for which it is possible to find a strictly positive random variable m that satisfies (1) and the linear constraint

$$-\delta \leq \mu\left(m - m^*\right) \leq \delta.$$

2.4 Parameter estimation

Measures of misspecification are primarily used to evaluate a given benchmark SDF; this is also the main focus in this paper. However, a measure of misspecification can also be used to estimate model parameters. That is, the parameters of an asset pricing model are chosen to minimize a measure. This is mentioned in a theoretical context for the HJ measure by Hansen, Heaton and Luttmer (1995) and is carried out empirically for instance by Jagannathan and Wang (1996). The latter use the HJ measure without positivity constraint on the SDF, which is equal to GMM with the inverse of the second moment of returns as weight matrix.

If the benchmark SDF is a function of a parameter vector $b \in B$, i.e. $m^* = m^*(b)$, then we obtain:

$$b = \arg\min_{b \in B} \left[\min_{m \in M} d(m, m^*(b)) \right]. \tag{8}$$

Clearly, any of the discussed linear measures could be used to choose model parameters. The economic interpretation of the methodology is that the parameters are chosen to minimize the pricing discrepancy associated with the measure. It is arguably appealing to be able to motivate the choice of parameters using an economic argument, although it is obviously highly desirable that also the statistical properties of the estimates are favorable. This will be considered in the Monte Carlo study of parameter estimation in Section 4.

An advantage of the L1-norm measures is that they can be computed using linear programming techniques. If we wish to keep the optimization program linear even when the parameters of the SDF are unknown, we must restrict ourselves to linear factor models. Due to the widespread use of linear factor models in empirical work, this is a relevant case. The benchmark SDF must thus be of the of the following type:

$$m^* = b'f,$$

where one of the factor may be a constant. For the L1-norm equivalent of the HJ measure we have that

$$b = \arg\min_{b \in B} \left[\min_{m \in M} E \left| m - b'f \right| \right]. \tag{9}$$

It can be computed as a linear program. That is, the optimization program can be expressed as

$$\min_{y,m,b} \int y(s) ds,$$

subject to the pricing constraint, the positivity constraint and

$$-y(s) \leq \mu(s) \left[m(s) - b'f(s)\right] \leq y(s) \quad \forall s.$$

A similar argument applies to the L1sup measure. The performance of these measures will be compared with the well-known HJ measure without positivity restriction in Section 4. The gain-loss ratio, however, is not considered for parameter estimation as it cannot be carried out as a linear program once the benchmark SDF is a function of unknown parameters.

3 Statistical properties of measures

In this section we carry out a Monte Carlo simulation where asset returns and a benchmark SDF are generated. We compute several measure of misspecification and consider their ability to accept a correct model (size) and reject an incorrect model (power) under different distributional assumptions.

In the following, we firstly discuss the Monte Carlo setup and then give details about the implementation. Finally, we present our results.

3.1 Monte Carlo setup

We will consider different specifications of the benchmark model in scenarios which will vary in two dimensions: the distribution of the pricing errors and the validity of the model. We will start by describing the case of a standard consumption-based asset pricing model with constant relative risk aversion (CRRA model). Thus, m^* is specified as

$$m^* = \beta G^{-\gamma}, \tag{10}$$

where G denotes consumption growth and β and γ are the subjective discount factor and the risk-aversion parameter, respectively. To study the properties of the above measures with the benchmark model, we need to fix the expectation of the pricing errors and hence it follows that

$$E(m^* R_i) = \eta, \qquad i = 1, \ldots, N, \tag{11}$$

where R_i is a return under consideration. If assets are correctly priced, then $\eta = 1$, which will also be our base case in the following. By equation (11) we may write if assets are correctly priced we may write:

$$m^* R_i = 1 + \varepsilon_i, \qquad i = 1, \ldots, N,$$

where $E(\varepsilon_i) = 0$. Inserting (10) in the above equation and taking logs we get

$$r_i - \gamma g + \log \beta = \lambda_i, \qquad i = 1, \ldots, N,$$

where $g = \log G$, $r_i = \log R_i$ and $\lambda_i = \log(1 + \varepsilon_i)$. So the pricing equations can be written as

$$E(\exp \lambda_i) = 1, \qquad i = 1, \ldots, N. \tag{12}$$

We will take the second moments, i.e. the variance-covariance matrix involving g and the λ_i's, as parameters which will be calibrated with real data. A nontrivial exercise is to implement the generation of the λ_i's with different distributional assumptions so that equations (12) hold.

Three types of distributions will be used, namely normal, asymmetric and heavy-tailed. In dealing with asymmetry of market returns it is common to use Tukey's asymmetric distribution[3]. A discussion of its theoretical properties can be found in Tukey (1977) and Hoaglin (1983) and some empirical applications are Chatterjee and Badrinath (1988) and Mills (1995). This is the approach we follow here. For the heavy-tailed case, we use a mixture of normals; again, this is commonly found in the literature (e.g. Mills, 1995). The distributions are thus as follows:

- In the normal case the λ_i's are given by

  $$\lambda_i = \mu_i + \sigma_i Z_i, \qquad i = 1, \ldots, N, \tag{13}$$

  where the Z_i's are standard Gaussian random variables which are possibly correlated with each other.

- In the asymmetric case, the λ_i's follow Tukey's distribution, that is, they are given by,

  $$\lambda_i = a_i + \frac{b_i}{c_i}\left[\exp(c_i Z_i) - 1\right], \qquad i = 1, \ldots, N, \tag{14}$$

  where a_i, b_i and c_i are parameters and the Z_i's are standard Gaussian random variables which are possibly correlated with each other.

- In the heavy-tailed case, the λ_i's are implemented using a mixture of normals, that is, they are given by,

  $$\lambda_i = \mu_i + I_p^a Z_i^a + I_{1-p}^b Z_i^b, \qquad i = 1, \ldots, N. \tag{15}$$

  Here Z_i^a's and Z_i^b's are Gaussian random variables with zero expectation. Further, I_p^a is an indicator function that takes value 1 with probability p

and zero otherwise with probability $1-p$. Also, I^b_{1-p} takes the opposite value of the first indicator function.

In the appendix we show how to calibrate the parameters of the distributions such that (12) holds and that pricing residuals and consumption growth have a prespecified covariance matrix. Also, in the case of the asymmetric distribution, the asymmetry given by the c_i parameter is prespecified. The extent of heavy tails can also be prespecified in the case of mixed normals.

The Monte Carlo model will thus be calibrated to a data set, and we may use the estimated parameters to generate asset returns that are correctly priced in the population, or have a prespecified average error. The former can be used to test the size of a test and the latter to test power.

3.2 Implementation

We wish to study the performance of a selection of measures of misspecification on returns with the distributions discussed above. To calibrate the model we use American quarterly data on growth of aggregate per-capita consumption of durables and services as well returns on ten size sorted portfolios.[4] All data are in quarterly real terms for the period 1960-1999.

When computing the SDF for CRRA utility we use risk aversion parameter $\gamma = 3$ and the subjective discount factor is set to $\beta = 0.99$ on an annual basis. As before a measure of misspecification is defined according to

$$\delta^+_{m^*} = \min_{\substack{m \in M \\ m>0}} d(m, m^*).$$

The following three measures are included in the study:

- $d(m, m^*) = \left\{ E\left[(m - m^*)^2 \right] \right\}^{1/2}$. (The HJ measure.)
- $d(m, m^*) = E\left| m - m^* \right|$ (L1 equivalent.)
- $d(m, m^*) = \frac{\sup \frac{m}{m^*}}{\inf \frac{m}{m^*}}$. (The gain-loss ratio.)

These measures have previously been discussed in the context of continuous payoffs. However, practical implementation implies a discrete state space. This is exemplified using the L1-norm based measure. Its continuous version for returns is as follows:

$$\min_{y,m} \int y(s)\, ds$$
$$s.t. \begin{cases} E(m\mathbf{R}) = \mathbf{i} \\ -y(s) \le \mu(s)\left[m(s) - m^*(s)\right] \le y(s) \quad \forall s. \end{cases}$$

When actually implementing this on return data with T time periods and N assets we use the following form:

$$\min_{y,m} \sum_{t=1}^{T} y_t$$
$$s.t. \begin{cases} \frac{1}{T}\sum_{t=1}^{T} m_t R_t^i = 1, & i = 1, ..., N \\ -y_t \leq \frac{1}{T}\left(m_t - m_t^*\right) \leq y_t, & t = 1, ..., T. \end{cases}$$

A Monte Carlo simulation with 1000 repetitions is carried out for the normal, asymmetric and heavy-tailed cases for different sample lengths and pricing errors. For each case, we repeatedly generate a Monte Carlo sample and compute the measure of misspecification. Further, for each Monte Carlo sample and measure we also obtain a bootstrapped p-value for the test

$$\begin{aligned} H_0 &: m^* \in M \\ H_1 &: m^* \notin M. \end{aligned}$$

Since by construction our generated series are i.i.d., the bootstrapping method need not take serial correlation into account. The bootstrapping method used therefore simply consists of drawing randomly with replacement, using a uniform distribution, from the original series. The Monte Carlo simulation in this way produces a distribution of p-values for each measure.

P-values should be computed under the null hypothesis that the model is true, i.e. that the benchmark SDF under consideration is admissible. To achieve this we follow the methodology suggested by Engsted *et al.* (2000). This implies that when the measures are computed from bootstrapped samples we do not use the regular pricing constraint, which is $E(mR_i) = 1$. Instead the correct price is taken as the average over all bootstrapped samples, i.e. the adjusted constraint is as follows:

$$E(mR^i) = p_*^i.$$

Here p_i represents the bootstrapped average; that is if $E^*(\cdot)$ denotes expectation over bootstrapped samples then $p_*^i \equiv E^*(mR^i)$.

3.3 Results

The Monte Carlo simulations were performed for $T = 60$ and $T = 120$ time periods, and also with no pricing error, 20% pricing error and 50% pricing error.

In Figure 1 and 2 histograms for the Monte Carlo sample values of the measures, when the model is true, are given. In this case, it is clear that any deviations from of the measure from its minimum (0 except for the gain-loss ratio where it is 1) is due to random fluctuations. A larger sample means that the random noise becomes relatively less important. Consistent with that, the histograms do also show that for the larger sample size, the measures take smaller values and are concentrated over a shorter interval. The distributions are also considerable smoother and more regular with the larger sample size.

Table 1 gives information about the size of the test, or how often a correct null hypothesis of a true model is rejected at different significance levels. Obviously, when the model is true, we want it to be rejected in a manner that is consistent with absence of size distortions.

Again, the results indicate that the test for the most part works better for the larger sample size - the null tends to be rejected more seldom in this case compared to the same situation with a smaller sample size. This is likely to be true as the random noise becomes less important for the larger sample size.

There is some reason to believe that the L1-norm works relatively better compared to the HJ measure when the distribution is heavy-tailed. This is at least the case for the smaller sample size where L1-norm measure does slightly worse in the normal case, but better in the heavy-tailed case.

Looking at the gain-loss ratio, it performs fairly well in all scenarios. For the asymmetric case, it appears to perform similary to the HJ measure for the large sample, while worse in the small sample. Thus, there is some evidence that the gain-loss ratio works relatively better with asymmetric asset returns. Nevertheless, even here, it does not appear to do much better than the alternative measures, and thus the suggestion by Bernardo and Ledoit (2000) that the gain-loss ratio is suitable as a measure of misspecification with asymmetric returns is only partially born out. An explanation for this may bee the more robust nature of the gain-loss ratio, which is insensitive to near arbitrage opportunities.

Overall a significant impression is that all measures appear not to behave drastically different from each other. Nevertheless, there appears to be some systematic variation depending on the type of distribution.

Table 2 gives information about the power of the test, or how often an incorrect null hypothesis of a true model is rejected at different significance levels. When the model is false, we naturally want it to be rejected as often as possible.

With the very large error of 50% the model is essentially always rejected for the larger sample size. For the shorter sample size, rejection rates are significantly less than unity for the lowest significance levels, but rise rapidly. When the error is at the more moderate 20%, rejection rates are lower. This indicates that the power of the bootstrap test is not ideal, but still acceptable.

For the normal case, the HJ measure is better than the L1, which in turn is better than the GL measure. The results for the asymmetric case are mixed, and overall, the measures perform very similarly. In the heavy-tailed case, the L1 measure performs by far the best, followed by the HJ and then by the GL measure. In summary, the power of the L1 measure is better than the size of the measure.

4 Parameter estimation

In this section we consider the ability of measures of misspecification to choose model parameters in a linear factor model. Here it is thus the parameter estimates produced by the measure of misspecification that is of interest, rather than the value of the measure. In the following we describe the Monte Carlo setup, then give some information about the implementation and finally discuss our results.

4.1 Monte Carlo Setup

The Monte Carlo setup for the estimation of parameters in a linear factor model, follows Ahn and Gadarowski (1999). As in Fama-French (1993), a linear three-factor model is used, although here factors are artificial. Returns are i.i.d. and generated by a linear three-factor model, well-known model with N assets:

$$R_{it} = \alpha + f_{1t}\beta_{1i} + f_{2t}\beta_{2i} + f_{3t}\beta_{3i} + e_{it}, \quad i = 1, ..., N. \tag{16}$$

From standard arguments this implies a linear SDF of the following type:

$$m = a + b_1 f_1 + b_2 f_2 + b_3 f_3.$$

The relationship between the two representations is given by the following when, as in our study, the stochastic factors are i.i.d.:

$$\begin{aligned} a &= \frac{1}{\alpha}\left[1 + \sum\nolimits_{j=1}^{3} \frac{E^2(f_j)}{\sigma^2(f_j)}\right] \qquad (17) \\ b_j &= -\frac{1}{\alpha}\frac{E(f_j)}{\sigma^2(f_j)}, \quad j = 1, 2, 3. \end{aligned}$$

4.2 Implementation

We will follow Ahn and Gadarowski (1999), who choose parameter values to match historical U.S. data. Betas are chosen from a uniform distribution with mean one. More precisely the following applies with N assets and T time periods:

$$\begin{array}{ll} \alpha = 1.033 & \\ f_{jt} \sim N(0.0022, 6.944 \cdot 10^{-5}) & j = 1,2,3 \;\; ; \;\; t = 1, ..., T \\ \beta_{ij} \sim U(0,2) & i = 1, ..., N \; ; \;\; j = 1,2,3. \end{array} \tag{18}$$

This implies the following population values for the SDF parameters:

$$
\begin{aligned}
a &= 1.21 \\
b_j &= -31.89, \quad j = 1, 2, 3.
\end{aligned}
$$

The objective of this section is to estimate a and b_j by using the framework outlined in Section 2.4. That is, the parameters are chosen so as to minimize one of several alternative measures of misspecification. Jagannathan and Wang (1996) show that the HJ measure is minimized by

$$\mathbf{b}_{HJ} = (\mathbf{D}_T'\mathbf{G}_T^{-1}\mathbf{D}_T)^{-1}\mathbf{D}_T'\mathbf{G}_T^{-1}\mathbf{i}_N.$$

Here $\mathbf{b}_{HJ}$ is the vector of estimated constant and factor coefficients, $\mathbf{i}_N$ is a $N \times 1$ vector of ones and

$$
\begin{aligned}
\mathbf{D}_T &= T^{-1}\mathbf{R}'\mathbf{Y} \\
\mathbf{G}_T &= T^{-1}\mathbf{R}'\mathbf{R}.
\end{aligned}
$$

The matrix $\mathbf{R}$ is a $T \times N$ matrix of returns and $\mathbf{Y}$ is the $T \times 4$ matrix of ones and factors.

Again we carry out the Monte Carlo study for three different types of distributions: normal, asymmetric and heavy-tailed. This simply implies using different distributions for the error term in equation (16).

- In the normal case we have used an error term with the half the variance of the factors given in table (18) above.
- In the asymmetric case, we have again used Tukey asymmetric distribution, as discussed in the previous simulation, with asymmetry parameter $c = -1$. The other two parameters a and b in the distribution are determined by the facts that the variance is to be the same as in the normal case, and expected value equal to zero.
- In the heavy-tailed case we have again used mixed normals. The probability to draw the variable with higher variance is 0.1 and it has 5 times higher variance than the normally distributed variable. This also implicitly determines the variance of the other variable, as the mixed normal error is to have the same variance as in the normal case.

4.3 Results

Simulations were carried out for sample lengths of 60, 250, 500 and 1000 periods. For each time length, the SDF parameters are estimated using the L1, the L1sup and the HJ measures. The results are given in Table 3 and in Figures 3 to 4. Note that the three stochastic factors in the model are identical and all have the same population parameter value, so we only provide results for the average over these factors as well as the constant.

In Figures 3-4 we plot histograms over the parameter estimates. Note that even though the errors have characteristics such as being normally or asymmetrically distributed, this need not imply that the parameter estimates are. As can be seen, the estimates obtained using the HJ measure tends to be more spread out over a larger interval than for the other two measures. It can also be seen that the constant is more precisely estimated and that the distributions converge with an increasing sample length.

In Table 3, the performance of the measures is evaluated using average bias and root mean squared error. Generally, the L1 measures perform relatively better in relation to the HJ measure in terms of root mean squared error, except for the largest sample size. The HJ measure clearly outperforms the other measures in terms of small sample bias. For the L1sup measure it is doubtful whether parameter estimates are consistent as they retain a bias even in the longest samples. The L1-norm measure has a consistently declining small sample bias, although the rate of decline is slower than for the HJ measure. There is no reason to assume that is asymptotically biased.

Regarding the different distributions, the relative performance of the L1 measures tends to be better for the non-normal error distributions. This is quite apparent for the L1-norm measure, but slightly less clear for the L1sup measure. For the HJ measure, the constant is less well estimated for non-normal error distribution, while the factor parameter is often more precisely estimated in those cases.

Overall, the intuition that L1 measures can be more robust in small and non-normally distributed samples is largely born out. The L1sup measure focuses entirely on minimizing the maximum discrepancy between the set of admissible stochastic discount factors and the benchmark. It is not surprising that this reduces squared error in small samples but leads to highly inefficient estimation in large samples.

The histograms show that parameter estimates have a larger range for the HJ measure. If one is concerned with even a fairly small possibility of getting estimates that are widely off the mark, the L1 measures seem to be a possible alternative. Especially for larger samples, there is evidence that the L2-norm based HJ measure is superior.

Ahn and Gadarowski (1999) use the same kind of linear factor model framework to compare the HJ measure with standard GMM methods. In summary, they find that the HJ measure performed somewhat better than the two-step GMM and considerably better than iterated GMM in small samples. The reason is the difficulty of correctly estimating the weight matrix with little data, which can introduce significant noise with iterated GMM in small samples. Even the HJ measure however requires the second moment matrix of returns, which may be difficult to estimate precisely. This may be a reason for preferring L1-norm based measures in small and highly non-normally distributed samples.

5 Conclusion

Recently, the Hansen and Jagannathan measure of model misspecification has seen increased use. As it is an L2-norm based method, it is subject to the same strengths and drawbacks as any L2 method. It is likely to be efficient in normally distributed samples, but sensitive to outliers and therefore possibly not very robust, especially in smaller samples. An alternative route could be to consider L1-norm based measures of misspecification. This paper has tried to shed some light on the issue by presenting some L1-norm measures of misspecification and discussing a method for computing the gain-loss ratio as a measure. All of these measures can be solved using linear programming techniques. A drawback, however, is the difficulty of obtaining analytic or asymptotic distributions: for this reason we used bootstrapping in our Monte Carlo simulations. The simulations provided evidence that L1-norm based measures perform relatively better, mainly in terms of power with non-normal distributions. We also attempted to estimate a linear factor model using measures of misspecification. The results indicated that with larger and normally distributed samples, the L2 based HJ measure was superior. However, for small and non-normally distributed samples, L1-norm based measures may give superior precision, at least in a mean squared error sense.

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Notes

[1] The space L we consider in each case depends on the particular formulation of the problem. For example, for the HJ measure we have that $L = L^2$.

[2] Chen and Knez (1995) apply these ideas to develop a measure of market integration in real markets.

[3] Or simply Tukey's g distribution.

[4] Data was downloaded from Kenneth French's homepage, and includes stocks from NYSE, AMEX and NASDAQ.

(See http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/.)

Table 1. Size of test

Panel A: T=60

Normal				Asymmetric				Heavy-Tailed			
α	HJ	L1	GL ratio	α	HJ	L1	GL ratio	α	HJ	L1	GL ratio
0.01	0.4%	0.2%	0.7%	0.01	1.1%	1.6%	1.0%	0.01	1.6%	1.2%	0.4%
0.05	2.6%	2.6%	3.8%	0.05	4.0%	4.3%	3.7%	0.05	6.0%	6.4%	6.4%
0.10	5.9%	5.5%	8.0%	0.10	9.1%	8.0%	8.4%	0.10	10.8%	10.0%	10.8%
0.25	19.1%	16.1%	24.6%	0.25	22.6%	20.9%	22.7%	0.25	24.0%	21.6%	20.4%
0.50	46.2%	40.0%	50.6%	0.50	50.9%	46.4%	47.9%	0.50	48.8%	46.0%	45.2%

Panel B: T=120

Normal				Asymmetric				Heavy-Tailed			
α	HJ	L1	GL ratio	α	HJ	L1	GL ratio	α	HJ	L1	GL ratio
0.01	0.1%	0.1%	0.1%	0.01	0.9%	0.9%	0.8%	0.01	1.0%	0.4%	0.8%
0.05	1.4%	1.9%	1.6%	0.05	4.5%	3.9%	4.5%	0.05	4.8%	3.8%	4.4%
0.10	4.1%	4.8%	5.7%	0.10	8.6%	6.6%	8.9%	0.10	10.5%	7.9%	8.7%
0.25	15.5%	15.6%	20.2%	0.25	23.9%	19.1%	24.0%	0.25	25.3%	20.2%	23.9%
0.50	41.6%	38.3%	47.1%	0.50	49.1%	42.2%	48.5%	0.50	49.5%	42.6%	46.8%

Table 1 gives the fraction of times the null hypothesis is rejected at various significance levels, alpha, given that the model is correct. This is done using a sample of 60 observations (panel A) and 120 observations (panel B). Three distributions are used to generate residuals from the Euler equations, normal, asymmetric and heavy-tailed. Three measures of misspecifiacation are then compared, the Hansen Jagannathan meassure (HJ), the L1-norm measure and the Gain-Loss (GL).

Table 2. Power of test

Panel A: T=60, Error=0.2

				Asymmetric				Heavy-Tailed			
α	HJ	L1	GL ratio	α	HJ	L1	GL ratio	α	HJ	L1	GL ratio
0.01	6.4%	5.6%	5.2%	0.01	6.8%	6.4%	7.2%	0.01	6.2%	7.2%	4.8%
0.05	24.6%	22.6%	20.4%	0.05	25.4%	23.8%	24.2%	0.05	23.4%	25.6%	19.8%
0.10	42.4%	41.4%	35.6%	0.10	43.6%	42.4%	45.2%	0.10	41.8%	43.8%	31.4%
0.25	68.4%	66.8%	52.1%	0.25	71.2%	70.2%	70.8%	0.25	66.4%	68.8%	51.6%
0.50	83.2%	79.4%	67.6%	0.50	85.6%	82.4%	85.4%	0.50	82.6%	85.6%	67.8%

Panel B: T=60, Error=0.5

Normal				Asymmetric				Heavy-Tailed			
α	HJ	L1	GL ratio	α	HJ	L1	GL ratio	α	HJ	L1	GL ratio
0.01	29.0%	68.8%	23.6%	0.01	31.0%	99.0%	25.6%	0.01	32.0%	87.4%	31.6%
0.05	75.4%	99.8%	66.5%	0.05	69.3%	99.8%	84.6%	0.05	71.6%	98.8%	85.2%
0.10	99.0%	99.8%	98.0%	0.10	99.8%	99.8%	99.8%	0.10	99.0%	99.6%	39.6%
0.25	99.8%	99.8%	99.8%	0.25	99.8%	99.8%	99.8%	0.25	98.8%	99.6%	98.8%
0.50	99.8%	99.8%	99.8%	0.50	99.8%	99.8%	99.8%	0.50	99.6%	99.6%	99.6%

Panel C: T=120, Error=0.2

Normal				Asymmetric				Heavy-Tailed			
α	HJ	L1	GL ratio	α	HJ	L1	GL ratio	α	HJ	L1	GL ratio
0.01	18.0%	17.2%	15.4%	0.01	21.2%	20.2%	22.2%	0.01	17.8%	21.2%	15.2%
0.05	51.2%	48.4%	41.4%	0.05	53.4%	51.6%	55.6%	0.05	51.0%	55.4%	41.6%
0.10	71.2%	68.6%	65.2%	0.10	73.4%	71.6%	75.4%	0.10	70.8%	72.1%	65.6%
0.25	85.8%	81.4%	78.4%	0.25	86.2%	84.2%	88.4%	0.25	84.2%	86.4%	77.8%
0.50	95.2%	90.0%	89.6%	0.50	96.2%	91.4%	97.2%	0.50	95.0%	97.2%	88.6%

Panel D: T=120, Error=0.5

Normal				Asymmetric				Heavy-Tailed			
α	HJ	L1	GL ratio	α	HJ	L1	GL ratio	α	HJ	L1	GL ratio
0.01	99.0%	99.0%	92.0%	0.01	99.0%	99.0%	99.0%	0.01	99.0%	99.0%	99.0%
0.05	99.0%	99.0%	99.0%	0.05	99.0%	99.0%	99.0%	0.05	99.0%	99.0%	99.0%
0.10	99.0%	99.0%	99.0%	0.10	99.0%	99.0%	99.0%	0.10	99.0%	99.0%	99.0%
0.25	99.0%	99.0%	99.0%	0.25	99.0%	99.2%	99.0%	0.25	99.0%	99.6%	99.8%
0.50	99.6%	99.8%	99.0%	0.50	99.2%	99.8%	99.0%	0.50	99.2%	99.8%	99.8%

As Table 1 except that Table 2 gives the fraction of times the null hypothesis is rejected at various significance levels given that the model is false, where a 20% or a 50% pricing error is applied.

Table 3. Results from linear factor model estimation

Panel A: Relative bias, **a**

	T=60			T=250			T=500			T=1000		
Distribution	L1	L1sup	HJ	L1	L1sup	HJ	L1	L1sup	HJ	L1	L1sup	HJ
Normal	-9.5%	-1.3%	1.5%	-4.7%	-1.8%	0.1%	-3.4%	-2.3%	0.0%	-2.4%	-2.8%	0.1%
Heavy-Tailed	-7.9%	-2.0%	2.0%	-3.2%	-1.7%	0.6%	-2.5%	-2.1%	0.3%	-1.7%	-2.5%	0.4%
Asymmetric	-5.9%	-3.0%	2.5%	-1.7%	-2.9%	0.3%	-1.3%	-2.5%	0.4%	-1.1%	-2.7%	0.1%

Panel B: Relative bias, $\mathbf{b_j}$

	T=60			T=250			T=500			T=1000		
Distribution	L1	L1sup	HJ	L1	L1sup	HJ	L1	L1sup	HJ	L1	L1sup	HJ
Normal	-29.7%	-30.1%	-12.3%	-12.3%	-16.0%	-4.2%	-8.0%	-17.9%	-2.3%	-4.8%	-21.2%	-0.8%
Heavy tails	-26.8%	-30.5%	-10.2%	-9.4%	-14.9%	-2.1%	-7.0%	-17.0%	-1.6%	-3.6%	-20.5%	0.6%
Asymmetric	-21.8%	-29.8%	-6.7%	-7.8%	-15.8%	-2.9%	-5.4%	-16.5%	-0.9%	-4.5%	-20.2%	-0.7%

Panel C: Relative RMSE, **a**

	T=60			T=250			T=500			T=1000		
Distribution	L1	L1sup	HJ	L1	L1sup	HJ	L1	L1sup	HJ	L1	L1sup	HJ
Normal	14.4%	11.3%	13.8%	7.0%	5.0%	6.4%	4.9%	3.6%	4.5%	3.7%	3.1%	3.4%
Heavy tails	14.1%	11.8%	14.6%	6.0%	5.0%	6.4%	4.6%	3.6%	4.6%	3.1%	2.9%	3.2%
Asymmetric	13.0%	13.0%	13.7%	5.7%	5.8%	6.3%	4.2%	4.2%	4.7%	3.0%	3.2%	3.3%

Panel D: Relative RMSE, $\mathbf{b_j}$

	T=60			T=250			T=500			T=1000		
Distribution	L1	L1sup	HJ	L1	L1sup	HJ	L1	L1sup	HJ	L1	L1sup	HJ
Normal	81.8%	76.6%	98.5%	43.8%	37.3%	47.0%	32.1%	26.4%	34.5%	23.0%	18.6%	24.1%
Heavy tails	85.9%	77.5%	102.4%	43.8%	37.5%	48.2%	32.6%	25.8%	33.5%	23.3%	19.1%	23.8%
Asymmetric	82.7%	74.0%	95.1%	43.5%	36.8%	45.0%	31.8%	26.0%	33.1%	23.4%	18.4%	23.9%

In Table 3 the parameters of the following linear SDF have been estimated, using the same distributions and measures as in Table 1, with the exception that the L1sup is used instead of the gain-loss measure:

$$m = a+b_1f_1+b_2f_2+b_3f_3.$$

For definition of the parameters see section 4, main text. The relative bias and the root mean square error of the parameter estimates in relation to the true values are given in the table for sample lengths 60, 250, 500 and 1000. 10 assets have been used with a signal to noise ratio of 0.5. The variance of the asymmetric and heavy-tailed distributions are set equal to that of the normal. For the asymmetric case c is set to -1. For the heavy-tailed case p is set equal to 0.1 and k to five.

Figure 1. Distribution of measures of misspecification for T=60

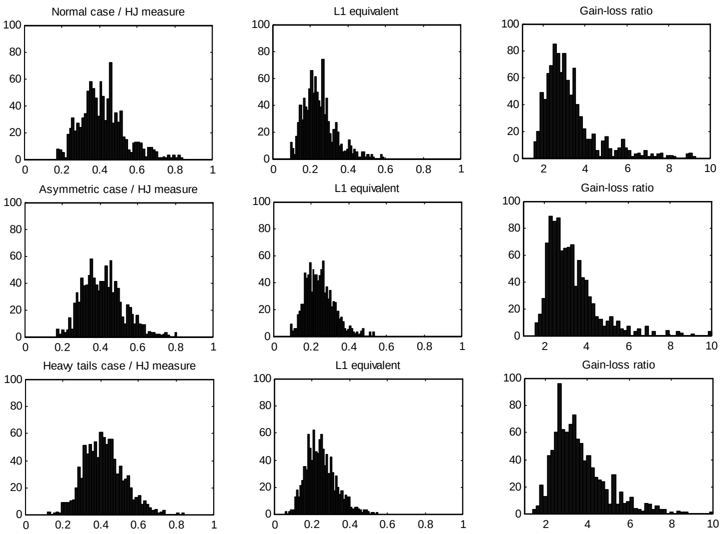
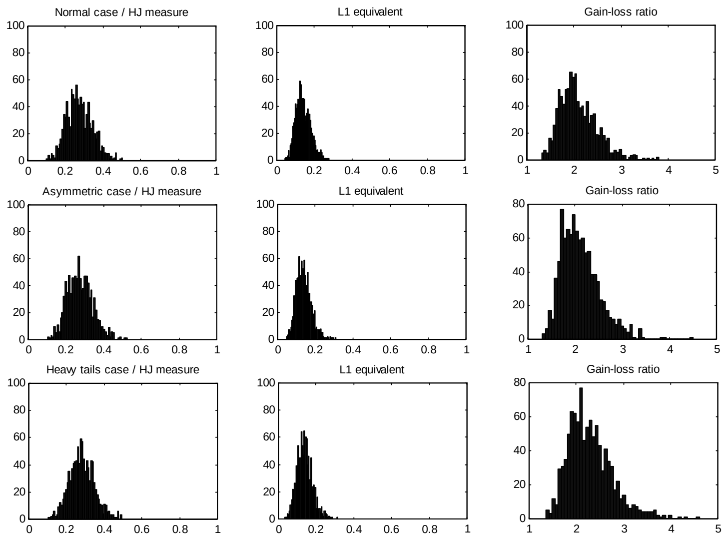


Figure 1 gives the distribution of the Hansen Jagannathan (HJ) measure, the L1 equivalent to HJ, and the Gain-Loss measure given that the Euler residuals have been generated using normal, asymmetric and heav-tailed distributions. 60 observations have been used.

Figure 2. Distribution of measures of misspecification for T=120



As Figure 1, except that 120 observations have been used.

Figure 3. Parameter estimates for T=250

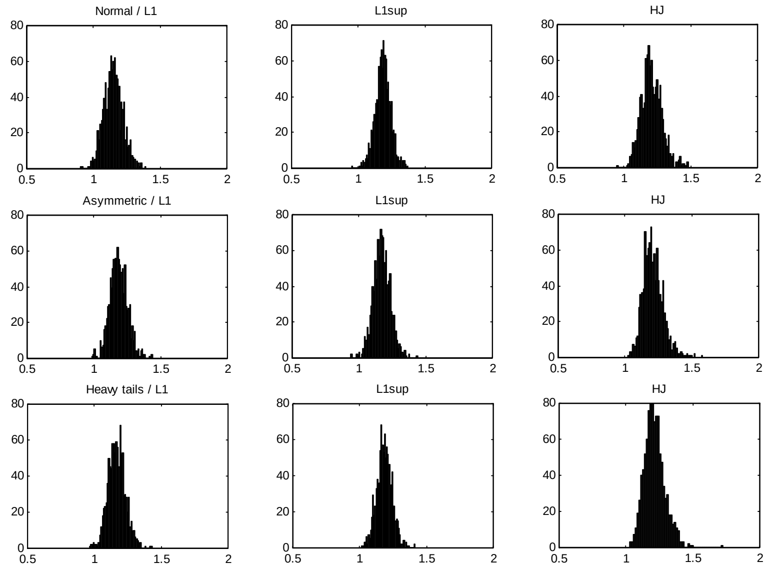
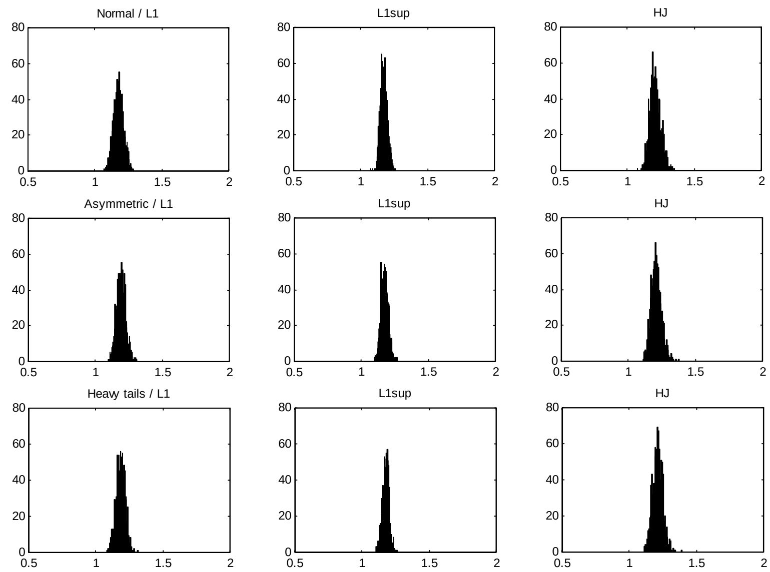


Figure 3 illustrates the result given in Table 3. The parameter shown is the intercept, where the true value is equal to 1.21 and the number of observations used is 250.

Figure 4. Parameter estimates for T=1000



As Figure 3, except that 250 observations have been used.

A Appendix

In the following we show in greater detail how to calibrate the distributions for the Monte Carlo simulation in Section 3. As already mentioned in the main text, given that the assets are correctly priced we may by equation (11) write

$$m^* R_i = 1 + \varepsilon_i, \qquad i = 1, \ldots, N,$$

where $E(\varepsilon_i) = 0$ and $m^* = \beta G^{-\gamma}$. Now, if we take logs in the above equation we get

$$r_i - \gamma g + \log \beta = \lambda_i, \qquad i = 1, \ldots, N, \tag{A.1}$$

where $g = \log G$, $r_i = \log R_i$ and $\lambda_i = \log(1 + \varepsilon_i)$ So the pricing equations (11) can be written as

$$E(\exp \lambda_i) = 1, \qquad i = 1, \ldots, N. \tag{A.2}$$

Now, define the first and second moments of the above variables as $E(g) = \mu_g$, $var(g) = \sigma_g^2$, $E(\lambda_i) = \mu_i$, $var(\lambda_i) = \sigma_i^2$, $cov(\lambda_i, g) = \sigma_{ig}$ and $cov(\lambda_i, \lambda_j) = \sigma_{ij}$, $i = 1, \ldots, N$. We will take the second moments as parameters which will be calibrated with real data. In the following we show how to implement the generation of the λ_i's, given that the the second moments are known, with different distributional assumptions so that equations (A.2) hold.

A.1 Normal pricing errors

This is the simplest case. Suppose that λ_i's are given by

$$\lambda_i = \mu_i + \sigma_i Z_i, \qquad i = 1, \ldots, N, \tag{A.3}$$

where Z_i 's are standard Gaussian random variables that are possibly correlated with each other. We assume that g is normally distributed (or, equivalently G lognormally distributed). Thus, the generation of the deviates which delivers a given variance-covariance matrix of the λ_i's and g is straightforward. The only thing that remains is to guarantee that the pricing errors are zero, that is, they must satisfy

$$E(\exp[\mu_i + \sigma_i Z_i]) = 1, \qquad i = 1, \ldots, N,$$

which by using the properties of the lognormal distribution gives

$$\exp\left(\mu_i + \frac{\sigma_i^2}{2}\right) = 1, \qquad i = 1, \ldots, N,$$

or equivalently

$$\mu_i = -\frac{\sigma_i^2}{2}, \qquad i = 1, \ldots, N.$$

A.2 Asymmetric pricing errors

In dealing with asymmetry of market returns it is common to use Tukey's asymmetric distribution. A discussion of its theoretical properties can be found in Tukey (1977) and Hoaglin (1983) and some empirical applications are Chatterjee and Badrinath (1988) and Mills (1995). This is the approach we follow here. Suppose the λ_i's follow Tukey's distribution, that is, they are given by

$$\lambda_i = a_i + \frac{b_i}{c_i}\left[\exp\left(c_i Z_i\right) - 1\right], \qquad i = 1, \ldots, N, \tag{A.4}$$

where a_i, b_i and c_i are parameters and the Z_i's are standard Gaussian random variables which are possibly correlated with each other. Suppose we know σ_g^2, σ_i^2, σ_{ig}, σ_{ij} and c_i, $i = 1, \ldots, N$. The c_i's can be estimated from real data by the method discussed in Badrinath and Chatterjee (1988; see Appendix A.4 below). From (A.4) we have that

$$b_i = \pm \frac{\sigma_i c_i}{\exp\left(2c_i^2\right) - \exp\left(c_i^2\right)},$$

which gives the value of the b_i's. Now by using Stein's Lemma and (A.4) we have that

$$\sigma_{ig} = \frac{b_i}{c_i} \exp\left(c_i^2/2\right) cov\left(Z_i, g\right),$$

which gives

$$cov\left(Z_i, g\right) = \frac{\sigma_{ig} c_i}{b_i \exp\left(c_i^2/2\right)}.$$

Obviously, the values of the variance-covariance matrix only depend on b_i, c_i and $cov\left(Z_i, g\right)$, $i = 1, \ldots, N$. For the covariances between the λ_i's we have by (A.4) that

$$\begin{aligned}
\sigma_{ij} &= \frac{b_i b_j}{c_i c_j} cov\left(\exp\left[c_i Z_i\right], \exp\left[c_j Z_j\right]\right) = \\
&= \frac{b_i b_j}{c_i c_j}\left\{E\left(\exp\left[c_i Z_i + c_j Z_j\right]\right) - E\left(\exp\left[c_i Z_i\right]\right) E\left(\exp\left[c_j Z_j\right]\right)\right\} = \\
&= \frac{b_i b_j}{c_i c_j}\left\{\exp\left(\frac{c_i^2 + c_j^2 + 2c_i c_j cov\left(Z_i, Z_j\right)}{2}\right) - \exp\left(\frac{c_i^2 + c_j^2}{2}\right)\right\},
\end{aligned}$$

where the last equality can be obtained by applying the properties of the log-normal distribution. This gives us

$$cov\left(Z_i, Z_j\right) = \frac{\log\left[\frac{\sigma_{ij}c_i c_j}{b_i b_j} + \exp\left(\frac{c_i^2+c_j^2}{2}\right)\right]}{c_i c_j} - \frac{c_i^2 + c_j^2}{2c_i c_j}.$$

Finally, the a_i's must be chosen so that the pricing errors are equal to zero, that is, they must satisfy

$$E\left\{\exp\left(a_i + \frac{b_i}{c_i}\left[\exp\left(c_i Z_i\right) - 1\right]\right)\right\} = 1, \qquad i = 1, \ldots, N, \tag{A.5}$$

or equivalently

$$a_i = \frac{b_i}{c_i} - \log E\left\{\exp\left[\frac{b_i}{c_i}\exp\left(c_i Z_i\right)\right]\right\}, \qquad i = 1, \ldots, N.$$

A.3 Heavy tails: a mixture of normals

Suppose the λ_i's are given by,

$$\begin{aligned} \lambda_i &= I_p^a\left(\mu_i + Z_i^a\right) + I_p^b\left(\mu_i + Z_i^b\right) \\ &= \mu_i + I_p^a Z_i^a + I_{1-p}^b Z_i^b, \qquad i = 1, \ldots, N. \end{aligned} \tag{A.6}$$

Here Z^a and Z^b are independent Gaussian random vectors with zero expectation and so far unknown covariance matrix. Further, I_p^a is an indicator function that takes value 1 if a draw from a uniform random variable U satisfies $U \leq p$ and zero otherwise. I_{1-p}^b takes the opposite value of the first indicator function.

We have that

$$\begin{aligned} \sigma_{ij} &= p\sigma_{ij}^a + (1-p)\sigma_{ij}^b \\ \sigma_{ig} &= p\sigma_{ig}^a + (1-p)\sigma_{ig}^b. \end{aligned}$$

Again we assume knowledge of σ_g^2, σ_i^2, σ_{ig} and σ_{ij}. It is easy to see that the number of parameters to be determined in (A.6) is much larger than the number of available constraints. When imposing restrictions, it is necessary to keep in mind that the correlation between assets is high, and therefore the covariance structure needs to be spread out on both random vectors. Otherwise, it may be that individual covariance terms exceed variance terms. One approach is to assume that

$$\begin{aligned}\sigma_{ij}^{a} &= k\sigma_{ij}\\ \sigma_{ij}^{b} &= m\sigma_{ij}.\end{aligned}$$

Also, the equivalent holds for covariance with log consumption growth. We prespecify $p \in [0, 1]$ and $k \in [0, 1/p]$. This gives

$$\begin{aligned}\sigma_{ij} &= pk\sigma_{ij} + (1-p)m\sigma_{ij}\\ m &= \frac{1-pk}{1-p}.\end{aligned}$$

For the zero pricing error condition note that

$$\begin{aligned}&E\left(\exp\left[\mu_i + I_p^a Z_i^a + I_p^b Z_i^b\right]\right) =\\ &= E\left(\exp\left[\mu_i + I_p^a Z_i^a + I_p^b Z_i^b\right]\right)\\ &= pE\left[\exp\left(\mu_i + Z_i^a\right)\right] + (1-p)\,E\left[\exp\left(\mu_i + Z_i^b\right)\right] =\\ &= \exp\left(\mu_i\right)\left[p\exp\left(\frac{k\sigma_i^2}{2}\right) + (1-p)\exp\left(\frac{m\sigma_i^2}{2}\right)\right]. \qquad \text{(A.7)}\end{aligned}$$

which gives the equations

$$\mu_i = -\log\left[p\exp\left(\frac{k\sigma_i^2}{2}\right) + (1-p)\exp\left(\frac{m\sigma_i^2}{2}\right)\right], \qquad i = 1, \ldots, N. \qquad \text{(A.8)}$$

A.4 Implementation

As described text in Section 3.2 of the main text, the simulation in Section 3 is calibrated to American quarterly data for the period 1960-1999. The average logarithmic consumption growth (g) as well as the covariance matrix of asset pricing residuals (λ_i) and g is estimated.

For the normal case, this directly defines all parameters. By generating simulated consumption growth and asset pricing residuals, asset returns can be computed from (A.1). For the mixed normal case, we used the same covariance as in the normal case, and split this on two random variables (using the notation above we set $p = 0.1$ and $k = 5$).

Regarding the c_i parameter for Tukey's asymmetric distribution we follow Badrinath and Chatterjee (1988). If z_p denotes the p:th percentile of the standard normal distribution and x_p denotes the p:th percentile of the distribution, then the following relationship holds:

$$c_i = -\frac{1}{z_p} \ln \left[\frac{x_{1-p} - x_{0.5}}{x_{0.5} - x_p} \right]. \tag{A.9}$$

If returns follow the Tukey's asymmetric distribution exactly, then the estimate of c_i will be the same regardless at which percentile it is computed. In practice it will vary somewhat when actual returns are used and therefore we estimate c_i as the average over the 5:th to 40:th percentiles (the same would be obtained over the 60:th to 95:th percentiles due to symmetry in expression (A.9) above). Typically firms with larger market capitalization have returns with lower variance and distributions that are closer to normal. For the Monte Carlo study, the c vector was multiplied by a factor of two to get slightly more pronounced asymmetry.

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