Essays in Mathematical Finance
Modeling the Futures Price

Magnus Blix

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To Elisabeth ♥

Acknowledgments

Back at the end of 1992, I was at a friend's house enjoying the last Christmas break of high school. Between the many rounds of scrabble and table hockey, the main topic of discussion was our future profession, what kind of career we should pursue. On that particular occasion, my friend's older brother was home for Christmas. He had just finished his first semester of the PhD program in finance at the Stockholm School of Economics, and his opinions on the matter were strong: First study lots and lots of mathematics and perhaps mathematical statistics, and then apply to the program he was studying. With a solid technical background, being accepted to the program would not be a problem, and after that, a stimulating and rewarding career would be waiting. He was very convincing, and I was young and naive – it sounded like a great idea. I nodded along enthusiastically, completely oblivious to the amount of work and anxiety I had just agreed to take on.

Today, almost twelve years later, and with my PhD in clear sight, I am glad I did not understand what I was getting myself into, and that once I did, I was in too deep to pull out.

The person I am most indebted to for this thesis ever coming into existence is, without question, my supervisor, Tomas Björk. He gave me the opportunity to share some of his vast knowledge of mathematical finance, and advised me on how to approach mathematics and research in general. I especially appreciate the many tailor-made lectures he gave me when my knowledge of a particular aspect of mathematics was considered too poor (and that happened a lot), and his generosity in sharing his research ideas. I am also grateful to the rest of the faculty, and in particular to Joel Reneby for co-supervision on the first paper (and for good times after working hours), and to Inaki Rodrigues for providing me with the data for the empirical paper. I am indebted to my fellow PhD students: Per Östberg for countless hours of moral support in the coffee room at SSE, and for always helping out whenever my computer or any other technical devices gave me a hard time; Anders and Mia (who were also my roommates); Sven, whose help was invaluable, particularly in the final stages of the thesis; and Irina, Raquel and Roland. The others who came and went at Holländargatan 40 during my five years there are too many to mention, but you know who you are; thanks – I really enjoyed your company.

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Amsterdam, November 2004

Magnus Blix

Introduction

In terms of trading volume, the futures contract is by far the most important financial derivative. In 2003 alone, almost three billion futures contracts were executed. The futures contract is popular for several reasons, all of which acknowledge one of its most attractive features – it facilitates trading. For commodity and energy markets, this is especially noticeable, since futures contracts allow market participants to trade without the obligation of physical delivery. Instead, positions are *marked-to-market* on a daily basis, and hence the party for which the futures price moves in an unfavourable direction pays the amount required to make the position neutral again. This procedure makes entering into a futures contract costless, and the only obligation it brings is the cash flow described above. Thus the futures contract is an excellent tool for hedging purposes. Further, high volumes of futures trading mean liquidity. For this reason, the futures price is often used as the underlying asset in derivatives transactions. In fact, for some assets, the futures market is more liquid than the spot market.

This thesis consists of four papers dealing with the futures price process.

In the first paper, we propose a two-factor futures volatility model designed for the US natural gas market, but applicable to any futures market where volatility decreases with maturity and varies with the seasons. A closed form analytical expression for European call options is derived within the model and used to calibrate the model to implied market volatilities. The result is used to price swaptions and calendar spread options on the futures curve.

In the second paper, a financial market is specified where the underlying asset is driven by a d-dimensional Wiener process and an M dimensional Markov process. On this market, we provide necessary and, in the time homogenous case, sufficient conditions for the futures price to possess a semi-affine term structure. Next, the case when the Markov process is unobservable is considered. We show that the pricing problem in this setting can be viewed as a filtering problem, and we present explicit solutions for futures. Finally, we present explicit solutions for options on futures both in the observable and unobservable case.

The third paper is an empirical study of the SABR model, one of the latest contributions to the field of stochastic volatility models. By Monte Carlo simulation we test the accuracy of the approximation the model relies on, and we investigate the stability of the parameters involved. Further, the model is calibrated to market implied volatility, and its dynamic performance is tested.

In the fourth paper, co-authored with Tomas Björk and Camilla Landén, we consider HJM type models for the term structure of futures prices, where the volatility is allowed to be an arbitrary smooth functional of the present futures price curve. Using a Lie algebraic approach we investigate when the infinite dimensional futures price process can be realized by a finite dimensional Markovian state space model, and we give general necessary and sufficient conditions, in terms of the volatility structure, for the existence of a finite dimensional realization. We study a number of concrete applications including the model developed in the first paper of this thesis. In particular, we provide necessary and sufficient conditions for when the induced spot price is a Markov process. We prove that the only HJM type futures price models with spot price dependent volatility structures, generically possessing a spot price realization, are the affine ones. These models are thus the only generic spot price models from a futures price term structure point of view.

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Nov 2004

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A Gas Futures Volatility Model

1.1 Introduction

The classical approach to derivatives pricing in the commodity literature is to model the spot price of the commodity along with other key variables such as the convenience yield or interest rates, and from this derive the prices of various contingent claims [17], [11], [12], [14], [16]. There are several problems with this approach. First of all, the convenience yield is unobservable, which makes the models hard to estimate. Secondly, the futures curve is endogenous, and may not be consistent with the one observed in the market. This is particularly unfortunate, since most derivatives are written on the futures price, rather than the spot price itself. These were some of the considerations that led to the idea of using the futures curve as the underlying object, similar to the HJM framework from interest rate theory (see e.g. [15], [13], [10], [9]). Direct modelling of the futures price and using the current observed futures curve as the initial condition eliminates the problem of fitting the model to the current futures prices.

What still needs to be done, however, is to ensure that the model fits the implied volatility curve. If the primary object is to price derivatives, it is convenient to model the futures price directly under the risk neutral martingale measure. Under this measure, the futures price is a martingale, so only the volatility structure needs to be chosen. This structure varies from commodity to commodity, but there are some common features. For instance, if climate dependence affect either the demand side (e.g., energy in the winter time) or production (e.g. crops being harvested in the summer), volatility usually has a seasonality component. Further, volatility often decreases with maturity of the futures contract, corresponding to a mean reversion in the spot price. Finally, volatility tends to converge to a "minimum volatility level", which is separate from zero.

To date, no model in the literature has taken all of these features into account. Richter & Sørensen [16] incorporate seasonality in their model, but they work within a stochastic volatility framework, and so do not aim for closed form analytical solutions for futures prices. Reisman [15], and Cortazar and Schwartz [10] model futures prices directly, but they do not explicitly specify the functional form of the volatility. The model proposed by Clewlow & Strickland [9] is the model that comes closest to ours, and we will therefore use that model as a benchmark. They incorporate the fact that volatility is a decreasing function of the maturity of the underlying futures contract, but they do not model the seasonality effect. The model presented in this paper takes all of the above considerations into account, producing a significant effect when the model is calibrated to market implied volatilities - the mean squared error is reduced by between 50% and 75% compared to the benchmark model. What effect does this have on derivative prices? Ideally, we would like to price existing derivatives using both our model and the benchmark model to see which model comes closest to the traded prices. Unfortunately, the only derivative traded on NYMEX is the standard European call option, which we already use for calibration; everything else is OTC. It is still interesting, however, to know how our model's prices differ from those of the benchmark model. If the difference is substantial, we could argue that since our model uses volatilities closer to the true ones, the prices it produces should also be closer to the true prices. We would expect the benchmark model to underestimate the price of derivatives maturing in the winter, when volatility peaks, and overestimate the price on derivatives maturing in the summer. This is confirmed when we implement the models and use them to price swaptions. The price differences vary between 6.55% and -8.1%, depending on when the swaption in question expires.

The outline of the paper is as follows: in section 2 we begin by describing the features of the most commonly traded contracts on the gas market. In section 3 we present the data, and in section 4 the model is presented. In section 4.1 we derive the price of a European call option within the model, and in section 4.2 we use Monte Carlo simulation to price a swaption. In section 5 the calibration procedure is explained, and in section 6 the empirical results are summarized. In section 7 we present an alternative way of modelling swaption prices, very similar to the market models of interest rate theory. Finally we note that the futures model can be realized by means of a finite (six) dimensional state space process and we provide the explicit dynamics for that process. In section 8 we conclude.

1.2 The Market for Gas Derivatives

A *forward contract* is an agreement to buy or sell an asset at a certain future date and at a certain price, the *forward price.* The forward price is decided when the contract is initiated, and is set to give the contract zero value at that time. No money changes hands until maturity of the contract, when the party with the long position in the contract has to buy the asset at the predetermined forward price. The cash flow she faces is thus the difference between the current asset spot price and the forward price, an amount that could be either positive or negative. In reality, however, most contracts are closed out prior to expiry, and the asset is never delivered. A more formal definition of the forward contract is as follows:

Definition 2.1 *A forward contract on the asset S, contracted at time t, with time of maturity T and with forward price $f(t,T)$, is defined by the following properties:*

- *At time t, the forward price $f(t,T)$ is decided.*
- *At time T, the holder of the contract receives the stochastic (positive or negative) amount $S(T) - f(t,T)$.*
- *The value of the contract at time t is zero.*

By general theory, the forward price is given by

$$f(t,T) = E_t^T[S(T)], \tag{1.1}$$

where E^T denotes the forward neutral measure, i.e., the measure under which all assets scaled by the bond price $p(t,T)$ are martingales.

Since it is free to enter a forward contract, but the future financial loss can be huge, there is an obvious risk that the party with a negative cash flow at maturity may not honor her obligation. For this reason, it would be convenient to have a contract that was settled on a daily basis rather than on the expiry date. This is exactly the idea behind the *futures contract.* The futures contract is similar to the forward contract, but with the difference that each day, profits and losses are cleared, and the traders automatically receive a position in a new futures contract with a new futures price. Another difference between the futures contract and the forward contract lies in liquidity. Future contracts are standardized instruments widely traded on all exchanges, whereas forwards typically are OTC products. Formally:

Definition 2.2 *A futures contract on the asset the S, contracted at time t, with time of maturity T and with futures price $F(t,T)$ is defined by the following properties:*

- *At time t, the futures price $F(t,T)$ is decided.*
- *During an arbitrary time interval, $(s,u]$ the holder of the contract receives the amount*

$$F(u,T) - F(s,T).$$

- *At time T, the holder of the contract receives $S(T)$ and pays $F(T,T)$*
- *The value of the contract at each point in time is zero.*

By general theory, the futures price is given by

$$F(t,T) = E_t^Q[S(T)], \tag{1.2}$$

where E^Q denotes the risk neutral measure, i.e., the measure under which all assets scaled by the bank account process $B(t)$ are martingales. At Nymex, futures contracts on natural gas exist with maturity ranging from 1 to 72 months. One futures contract is for delivery of 10,000 million British thermal units (mmBtu) of gas, evenly spread throughout the delivery month. In reality, the marking to market procedure does not take place continuously, but rather once a day. However, for tractability, futures prices are usually modelled within a continuous framework, and thus also in the model presented later.

Remark 2.1 *Under deterministic interest rates, E^T and E^Q coincide exactly. For that reason, the model presented later can be regarded either as a model for the futures price or as a model for the forward price.*

There is also a liquid market for call options on the futures. The holder of a call option with strike price K and exercise date T_0, on an underlying futures contract with maturity $T_1 > T_0$ will, if the option is exercised, at time T_0 obtain

- the amount $(F(T_0, T_1) - K)^+$ in cash, and
- a long position in the underlying futures contract.

Since the value of a futures contract is by definition zero, the last part can be disregarded. The call option is said to be *at the money* at t if

$$K = F(t, T_1).$$

Another contract of interest on the natural gas market is the *swap contract.* Generally, a swap contract is any arrangement between two parties to exchange cash flows in the future according to a prearranged formula. In the gas market, entering a swap is similar to entering several forward contracts at the same time. The difference is that instead of one forward price for each maturity, with a swap there is one common price for all maturities, *the swap price.*

Definition 2.3 *A swap contract on the asset S, contracted at time t with reset dates T_i, $i = n, n+1.., N$, and swap price K is defined by the following properties:*

:• *At time t, the swap price K is decided.*

• *At time* T_i, $i = n, n+1, ..., N$, *the holder of the long position in the swap receives the stochastic (positive or negative) amount*

$$S(T_i) - K.$$

In analogy with the forward price, the swap price does not change during the life of the swap. The value of the swap itself, however, is likely to change over time. The *par swap price* $K(t)$ is the swap price that gives the swap zero value. Let

$$\mathcal{SW}_n^N(t, K(s))$$

denote the value at time t of a swap that was initiated at time s with par swap price $K(s)$. By definition of the par swap price

$$\mathcal{SW}_n^N(t, K(t)) = 0 \ \forall t.$$

Further, by noting that

$$\mathcal{SW}_n^N(t; K(t)) = E_t^Q[\sum_{i=n}^{N}(S(T_i) - K(t))e^{-r(T_i - t)}] = \sum_{i=n}^{N}(F(t, T_i) - K(t))e^{-r(T_i - t)} \tag{1.3}$$

we can solve for the par swap price $K(t)$:

$$K(t) = \frac{\sum_{i=1}^{n} F(t, T_i)e^{-r(T_i - t)}}{\sum_{i=1}^{n} e^{-r(T_i - t)}}. \tag{1.4}$$

A *swaption* is a call option on a swap. It gives the holder the right to enter into a swap at a predetermined swap price. Formally:

Definition 2.4 *A swaption with expiration date* T_0, *written on a swap with reset dates* $T_n, T_{n+1}, ...T_N$ *and swap price* M, *is defined by the following payoff:*

• *At time* T_0 *the holder of the swaption has the right, but not the obligation, to enter into the above swap at swap price* M.

It is clear that the holder of the swaption will only exercise his right to enter the swap if it gives him a positive payoff. Hence we can write the payoff of the swaption at time T_0 as

$$\mathcal{SWN}_n^N(T_0; M, T_0) = \max\{\mathcal{SW}_n^N(T_0; M), 0\}$$

which by (1.3) is equal to

$$\max\{\sum_{i=n}^{N}(F(T_0,T_i)-M)e^{-r(T_i-T_0)},0\} \tag{1.5}$$

We can view the value of a swaption in a different way by noting that the swaption will only be exercised at T_0 if the strike price is less than the prevailing par swap price, $K(T_0)$. This difference will then be portioned out to the holder of the swaption at times $T_n, T_{n+1}, ..., T_N$. The value of the swaption is given by the present value of these payments, i.e.,

$$\mathcal{SWN}_n^N(T_0;M,T_0)=\sum_{i=n}^{N}e^{-r(T_i-T_0)}*\max\{K(T_0)-M,0\}.$$

Written in this form, it is clear that the payoff from a swaption is the payoff from a call option on the swap price, with an appropriate discount factor attached to it. It is also intuitive to say that the swaption is *at the money* at t if M is given by the par swap price at t, $K(t)$. Further, if we assume that the futures price is lognormally distributed (as we will later), it is clear from (1.4) that the payoff from a swaption is distributed as the sum of lognormal variables. This issue is dealt with in section 4.2.

1.3 The Data

The data consist of daily futures prices from the New York Mercantile Exchange, with maturities from 1 to 18 months observed between 5 January 2000 and 31 July 2002, and daily volatilities from the same period, implied from futures options with maturities up to 72 months at most. One futures contract is for delivery of 10,000 million British thermal units (mmBtu) of gas, evenly spread throughout the delivery month.

A figure showing gas futures volatility as a function of maturity is shown below. The behavior can be characterized by the following properties:

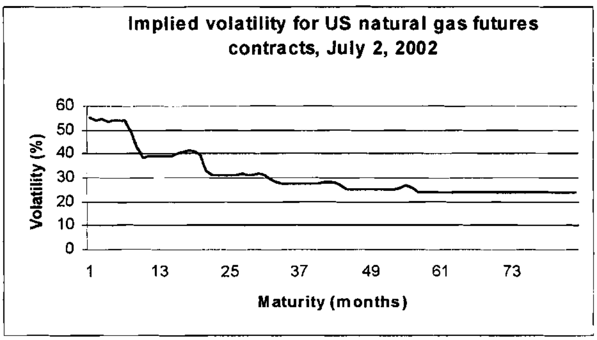
1) Overall, the function declines exponentially.

2) There is a "min level volatility", i.e., the volatility converges to a constant for long contracts (in the picture 24%).

3) We observe clear seasonality effects in maturity, i.e. contracts maturing during the colder months are more volatile than those maturing during the summer.

The final important feature that cannot be observed in the figure, is the seasonality in running time. In the figure, time to maturity varies, but running time is fixed. Hence systematic changes in volatility related to when the contract was initiated are lost. It is a well-known fact amongst practitioners in the natural gas industry that the increase in volatility experienced during the colder months spills over to contracts maturing much later in the year. In other words, there is a seasonality effect not only in maturity, but also in running time.

These are the features we will try to capture in the model we present in the next section.



1.4 The Model

In this section, we present the model and explain how it incorporates all the features of the previous section. Then we derive an expression for the value of a European call option, which we will use to calibrate the model to market data. Finally we show how Monte Carlo simulation can be used to price more complicated derivatives within this framework.

Let $F(t,T)$ denote the futures price at time t of a futures contract maturing at time T, and consider the following model, specified under an equivalent martingale measure Q:

$$\frac{dF(t,T)}{F(t,T)} = \alpha_1 e^{-\beta_1(T-t)} dW_1(t) + \tag{1.6}$$
$$(\alpha_2 e^{-\beta_2(T-t)} \sin(\phi + 2\pi T) + \alpha_3 e^{-\beta_3(T-t)} \sin(\phi + 2\pi t) + \gamma) dW_2(t).$$

Here $W_1(t)$ and $W_2(t)$ are independent Wiener processes, and α_i, β_j, γ and ϕ are real numbers $\forall i,j$. The first term

$$\alpha_1 e^{-\beta_1(T-t)}$$

measures the general exponential decrease of the volatility. If we consider only this term we obtain the model proposed by Clewlow and Strickland

(1999). An appealing feature of their model is that it gives rise to Markovian spot prices. However, the graph on the previous page, shows quite clearly that it will never produce a good fit to implied market volatilities. To manage that, we need to introduce a seasonal component,

$$\sin(\phi + 2\pi T).$$

Since the seasonality effect dies out for long maturity contract, the component is multiplied by another negative exponential,

$$\alpha_2 e^{-\beta_2 (T-t)}.$$

With these components, and with γ, the level to which the volatility converges, it is possible to achieve a close to perfect fit to the volatility curve observed in the market. However, one more important aspect needs to be addressed. As mentioned in the description of the data, there is a seasonality effect in running time as well. That is the motivation behind the term

$$\alpha_3 e^{-\beta_3 (T-t)} \sin(\phi + 2\pi t)).$$

Note that we need three different negative exponentials, since there is no reason to believe that the speed at which the curve as a whole decays is the same as the speed at which either of the seasonality effects dies out. This is confirmed from the parameter estimates, where we find significantly different values for β_1, β_2 and β_3.

To be able to calibrate the model to market data, we need an expression for the price of a European call option within the model. This is derived in the next subsection. We then show how Monte Carlo simulation can be used to price more complex contracts, letting a swaption serve as an example.

1.4.1 Futures Options

If we expand $\ln F(t,T)$ by the *Itô* formula, we get

$$\ln F(t,T) = \ln F(0,T) + \alpha_1 \int_0^t e^{-\beta_1 (T-u)} dW_1(u) - \frac{1}{2} \int_0^t h(u,T) du$$
$$+ \alpha_2 \sin(\phi + 2\pi T) \int_0^t e^{-\beta_2 (T-u)} dW_2(u) + \alpha_3 \int_0^t g(u,T) dW_2(u) + \gamma W_2(t),$$

where

$$h(t,T) = \alpha_1^2 e^{-2\beta_1 (T-t)} + (\alpha_2 e^{-\beta_2 (T-t)} \sin(\phi + 2\pi T) + \alpha_3 e^{-\beta_3 (T-t)} \sin(\phi + 2\pi t) + \gamma)^2, \text{ and} \tag{1.7}$$

$$g(t,T) = e^{-\beta_3 (T-t)} \sin(\phi + 2\pi t),$$

so it is clear that

$$\ln F(t,T) \in N(\ln F(0,T) - \frac{A^0(t,T)}{2}, A^0(t,T)),$$

where $A^0(t,T)$ is given by the integral

$$A^0(t,T) = \int_0^t h(u,T)du. \tag{1.8}$$

Since futures prices are log-normally distributed, it is straightforward to derive the value of a futures call option. By general theory, the value at time t of a call option with exercise date T_0 and strike price K written on a futures contract maturing at time T_1, is given by (assuming deterministic interest rates)

$$\mathcal{C}(t;T_0,T_1) = e^{-r(T_0-t)}E_t^Q[(F(T_0,T_1) - K)^+],$$

where

$$E_t^Q[.] = E^Q[. \mid \mathcal{F}_t^W].$$

This can be written as

$$\mathcal{C}(t;T_0,T_1) = e^{-r(T_0-t)}E_t^Q(F(T_0,T_1)1\{F(T_0,T_1) > K\}) - Ke^{-r(T_0-t)}Q_t(F(T_0,T_1) > K), \tag{1.9}$$

where

$$Q_t(A) \triangleq E^Q[A \mid \mathcal{F}_t^W].$$

By defining

$$Z^t(T_0,T_1) \triangleq \ln \frac{F(T_0,T_1)}{F(t,T_1)},$$

the first term in (3.3) can be calculated in the following way:

$$E_t^Q[F(T_0,T_1)1\{F(T_0,T_1) > K\}] =$$

$$F(t,T_1)E_t^Q[e^{Z^t(T_0,T_1)}1\left\{Z^t(T_0,T_1) > \ln\frac{K}{F(t,T_1)}\right\}],$$

where $Z^t(T_0,T_1) \in N(-\frac{A^t(T_0,T_1)}{2}, A^t(T_0,T_1))$. The expectation above is an integral over the normal distribution, which after completing squares becomes

$$\int_{D_1-\sqrt{A^t(T_0,T_1)}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sqrt{A^t(T_0,T_1)})^2} dz,$$

where

$$D_1 = \frac{\ln\frac{F(t,T_1)}{K} + \frac{A^t(T_0,T_1)}{2}}{\sqrt{A^t(T_0,T_1)}}.$$

We recognize the integral as the probability that a $N(A^t(T_0,T_1),1)$-distributed variable is greater than $\sqrt{A^t(T_0,T_1)} - D_1$, which is given by $N(D_1)$.

The second term is immediate:

$$Q_t(F(T_0,T_1) > K) = Q_t\left(\frac{\ln\frac{F(T_0,T_1)}{F(t,S)} + \frac{A^t(T_0,T_1)}{2}}{\sqrt{A^t(T,S)}} > \frac{\ln\frac{K}{F(t,T_1)} + \frac{A^t(T_0,T_1)}{2}}{\sqrt{A^t(T_0,T_1)}}\right)$$
$$= N(D_2)$$

Proposition 4.1 *The price at time t of a European call option with exercise date T_0 written on a futures contract from the model (1.6) with maturity T_1 is given by*

$$C(t;T_0,T_1) = e^{-r(T_0-t)}[F(t,T_1)N(D_1) - KN(D_2], \qquad (1.10)$$

where

$$D_1 = \frac{\ln\frac{F(t,T_1)}{K} + \frac{A^t(T_0,T_1)}{2}}{\sqrt{A^t(T_0,T_1)}}$$
$$D_2 = D_1 - \sqrt{A^t(T_0,T_1)},$$

and $A^t(T_0,T_1)$ is the expression appearing in (2.9).

Proof. Follows from the above calculations. ■

Since we know the distribution of the entire futures curve, we can price derivatives that depend on the curve, such as swaptions, calender spread options, etc. In the next subsection we illustrate the procedure for the case of swaptions and calender spread options.

1.4.2 Swaptions

As mentioned in section 2, a swaption is a call option on a swap. Assume that the swaption has exercise date T_0, strike price M and that the individual futures contracts mature at times T_i, $i = n, n+1, ..N$, $T_i > T_0$ $\forall i$. From (1.5), the payoff from the swaption is given by

$$\mathcal{SWN}_n^N(T_0;M,T_0) = \max\{\sum_{i=n}^{N}(F(T_0,T_i) - M)e^{-r(T_i-T_0)}, 0\}.$$

With the same notation as in section 2, the price at time t of this contract is given by

$$\mathcal{SWN}_n^N(t;M,T_0) = e^{-r(T_0-t)}E_t^Q\left[\max\left(\sum_{i=n}^{N}(F(t,T_i)e^{Z^t(T_0,T_i)} - M)e^{-r(T_i-T_0)}, 0\right)\right]. \qquad (1.11)$$

The payoff is distributed as the sum of lognormal variables, so we will have to use a numerical routine to find the price. Clearly, the Z_is are correlated, and to be able to simulate the payoff, we would like to make them uncorrelated. For this, define

$$Z_i \triangleq Z^t(T_0, T_i),$$

$$Z \triangleq \begin{pmatrix} Z_n \\ \cdot \\ \cdot \\ \cdot \\ Z_N \end{pmatrix},$$

$$C = \begin{pmatrix} Cov(Z_n, Z_n) & \cdot\ \cdot\ \cdot & Cov(Z_n, Z_N) \\ Cov(Z_{n+1}, Z_n) & \cdot\ \cdot\ \cdot & Cov(Z_{n+1}, Z_N) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ Cov(Z_N, Z_n) & \cdot & Cov(Z_N, Z_N) \end{pmatrix},$$

$$\mu_i = -\frac{A^t(T_0, T_i)}{2},$$

and finally

$$\mu = \begin{pmatrix} \mu_n \\ \cdot \\ \cdot \\ \cdot \\ \mu_N \end{pmatrix}.$$

By Cholesky factorization,

$$Z = B\epsilon + \mu,$$

where

$$B = Chol(C),$$

and

$$\epsilon = \begin{pmatrix} \epsilon_n \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_N \end{pmatrix},$$

where

$$E[\epsilon_i \epsilon_j] = 0, \ i \neq j.$$

With M simulations, the price is now given by

$$SWN_n^N(t; M, T_0) = e^{-r(T_0-t)} \frac{1}{M} \sum_{k=1}^{M} \max\left(\sum_{i=n}^{N} \left(F(t, T_i) \exp\left(\sum_{j=n}^{N} b_{ij} \epsilon_j^{(k)} + \mu_i \right) - K \right) e^{-r(T_i - T_0)}, 0 \right), \tag{1.12}$$

where $\epsilon_j^{(k)}$ means drawing number k of ϵ_j, and b_{ij} denotes element (i,j) in the matrix B. The matrix C is calculated in the appendix.

1.4.3 Calender Spread Options

A *calender spread option* is a derivative that is designed to protect the holder from changes in the shape of the futures curve. The holder of such a contract obtains, at the exercise date, the difference between the futures price for two particular maturities. Formally, a calender spread option with exercise date T_0 and strike price K, written on two futures contracts with maturities T_1 and T_2, gives at time T_0 the payoff

$$\mathcal{CS}_{T_1}^{T_2}(T_0) = \max\{F(T_0, T_1) - F(T_0, T_2) - K, 0\}.$$

The value at time t of the contract is given by

$$\mathcal{CS}_{T_1}^{T_2}(t) = e^{-r(T_0-t)} E^Q[\max\{F(T_0, T_1) - F(T_0, T_2) - K, 0\}].$$

It is clear that this is the same type of expression as (1.11), so the same valuation method can be used. This time, though, the covariance matrix is only $2 * 2$, so the simulation will be quicker.

1.5 Calibrating the Model to Market Data

The standard pricing model for futures call options is the Black-76 model [7]. The Black model gives the following price at time t for a call option with exercise date T_0 and strike price K on a futures contract maturing at time T_1:

$$\mathcal{C}(t; T_0, T_1) = e^{-r(T_0-t)}[F(t, T_1)N(d_1) - KN(d_2], \tag{1.13}$$

where

$$d_1 = \frac{\ln \frac{F(t,T_1)}{K} + \frac{\sigma^2(T_0-t)}{2}}{\sigma\sqrt{T_0 - t}}, \text{ and}$$
$$d_2 = d_1 - \sigma\sqrt{T_0 - t}.$$

It is clear that the structure in (1.13) is the same as that in (1.10). The volatility σ from the Black-76 model, corresponds to

$$\sqrt{\frac{A^t(T_0, T_1)}{T_0 - t}}$$

from the model (1.6). This means that if we want to calibrate our model to the market prices of call options, calibrating

$$\sqrt{\frac{A^t(T_0, T_1)}{T_0 - t}}$$

to σ is equivalent to calibrating (1.10) to (1.13). Since implied volatility is what is most often quoted, this is what we choose to calibrate against. Two of the parameters, α_3 and β_3, measure t-dependence, so they need to be estimated separately. The procedure is as follows:

1) Use at least one year of historical data on the term structure volatility curve, and calibrate all eight parameters simultaneously.

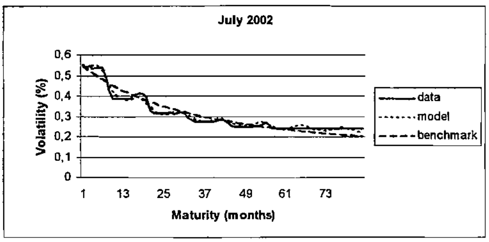
2) Keep the values for α_3 and β_3 and recalibrate the rest of the parameters to the current term structure.

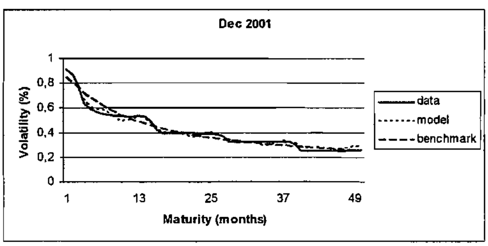
The reason for carrying out step 2 and for not using the values obtained in step 1 directly, is to make sure that the prices we obtain for various derivatives are consistent with the current volatility term structure, and not only with the historical term structure. The method above takes advantage of both historical and implied data.

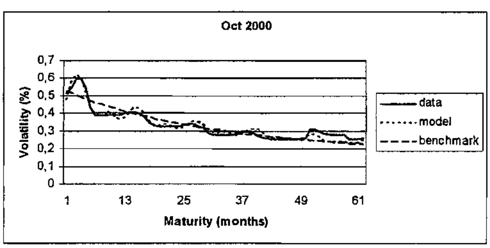
1.6 Empirical Results

We implemented the model and calibrated it against implied volatility from January 2000 to July 2002 simultaneously, using the first observed curve each month. More precisely, we minimized the sum of the mean squared errors from 32 calibrations. In so doing, the fit to each individual curve becomes much worse, but on the other hand, the estimates for the parameters that measure t-dependence, α_3 and β_3, now make sense. Once they are obtained, the model can be recalibrated to any given volatility curve. We choose to test the model on the volatility curve observed on three different dates: October 2000, December 2001 and July 2002. The results from these calibrations are shown on the next page. The mean squared error our model gives rise to varies between 50% and 25% of the mean squared error produced by the benchmark model. To see what impact this has on derivative prices, we used formula (11) to price seven different swaptions. By observing where the benchmark model deviates the most from the data, it is possible to anticipate where the price differences between our model and the benchmark model will be most pronounced. Note that the difference in price can go in both directions, since the benchmark model sometimes overestimates and sometimes underestimates the volatility. Take the July 2002 futures curve as an example. Nine months ahead in the graph, in March 2003, the benchmark model overestimates the volatility severely, so we would expect the benchmark model to overvalue swaptions with exercise in that month. The first table show that our hunch is confirmed; the swaption is about 8% cheaper in our model than in the benchmark model. If we now look at the graph for October 2000, the opposite happens. Here, the volatility increases initially, which of course is impossible for the benchmark model to capture. Hence, it is not surprising that the swaption maturing at the end of that month is about 6,5% more expensive in our model than in the benchmark model.

The final table shows the stability of the parameters over time. We cannot expect the parameters to stay completely constant over time, but we can at least hope that they do not vary in a *predictable* way, since that would suggest there is a pattern in the data that the model fails to capture. Luckily, no clear pattern can be seen.







Contract Initiated	Exercise Date	Swap Reset Dates						Price Difference	MSE	Benchmark
Jul 02	Oct 02	Oct 02	Nov 02	Dec 02	Jan 02	Feb 02	—	1,57%	0,01	0,04
- -	Mar 03	Mar 03	Apr 03	May 03	Jun 03	Jul 03	Aug 03	-8,10%	- -	- -
Dec 01	Dec 01	Dec 02	Jan 03	Feb 02	—	—	—	2,50%	0,02	0,06
- -	Mar 02	Mar 02	Apr 02	May 02	—	—	—	-4,60%	- -	- -
- -	Dec 02	Dec 02	Jan 03	Feb 03	—	—	—	4.15%	- -	- -
Oct 00	Nov 00	Nov 00	Dec 00	Jan 00	—	—	—	6.55 %	0,04	0,08
- -	Mar 02	Mar 01	Apr 01	May 01	—	—	—	-5,95%	- -	- -

Date of calibration	α_1	α_2	α_3	β_1	β_2	β_3	φ	γ
1 Jul 02	0,58	0,2	0,05	1,1	2,45	0,08	1,67	0,17
3 Jun 02	0,6	0,23	—	1,03	1,77	—	1,63	0,15
1 May 02	0,67	0,26	—	1,21	1,77	—	1,56	0,14
2 Apr 02	0,59	0,12	—	0,84	0,56	—	1,6	0,1
2 Mar 02	0,53	0,19	—	0,67	1,25	—	1,4	0,11
2 Feb 02	0,57	0,43	—	0,86	4	—	1,4	0,13
2 Jan 02	0,68	0,41	—	1,23	2,84	—	1,4	0,13
3 Dec 01	0,76	0,49	—	1,26	2,58	—	1,4	0,09
1 Nov 01	0,66	0,54	—	1,16	4	—	1,51	0,13
2 Oct 01	0,69	0,15	—	1,17	0,74	—	1,82	0,09
3 Sep 01	0,66	0,14	—	1,26	1,32	—	1,71	0,13
1 Aug 01	0,73	0,29	—	1,66	2,71	—	1,51	0,14
2 Jul 01	0,63	0,29	—	1,12	2,4	—	1,58	0,12
1 Jun 01	0,63	0,28	—	0,97	2,35	—	1,7	0,15
1 May 01	0,56	0,23	—	0,93	1,83	—	1,94	0,1
2 Apr 01	0,53	0,26	—	1,02	2,14	—	1,86	0,12
1 Mar 01	0,51	0,2	—	0,9	1,94	—	2	0,11
2 Feb 01	0,6	0,47	—	1,37	4	—	1,4	0,15
2 Jan 01	0,69	0,63	—	1,8	4	—	1,4	0,12
1 Dec 01	0,76	0,39	—	1,71	2,05	—	1,4	0,09
7 Nov 00	0,59	0,33	—	0,98	1,45	—	1,4	0,07
2 Oct 00	0,46	0,37	—	0,6	4	—	1,4	0,15
4 Sep 00	0,53	0,34	—	1,11	4	—	1,48	0,18
1 Aug 00	0,61	0,31	—	1,22	4	—	1,72	0,19
3 Jul 00	0,76	0,37	—	1,67	2,18	—	1,89	0,08
1 Jun 00	0,7	0,28	—	1,6	1,68	—	1,94	0,07
2 May 00	0,43	0,2	—	1,03	1,73	—	1,83	0,1
3 Apr 00	0,44	0,25	—	1,03	1,78	—	1,67	0,09
1 Mar 00	0,46	0,27	—	1,28	2,26	—	1,68	0,1
1 Feb 00	0,45	0,35	—	1,8	4	—	1,4	0,08
5 Jan 00	0,43	0,39	—	1,8	4	—	1,4	0,09

1.7 A Swap Market Model

In this section we present an alternative way of modelling swap prices. Let $t < T_n < T_N$, and let $K_n^N(t)$ be the par swap price (1.4) from section 2, i.e. let

$$K_n^N(t) = \frac{\sum_{i=n}^{N} F(t,T_i)e^{-r(T_i-t)}}{\sum_{i=n}^{N} e^{-r(T_i-t)}}.$$

The value at time t of a swap with swap price M is given by

$$SW_n^N(t,M) = \sum_{i=n}^{N} e^{-r(T_i-t)} * (K_n^N(t) - M).$$

By defining

$$p_n^N(t) = \sum_{i=n}^{N} e^{-r(T_i - t)},$$

the value of the swap can be written as

$$\mathcal{SW}_n^N(t, M) = p_n^N(t)(K_n^N(t) - M).$$

The option to enter the swap at time T_0 is worth

$$\mathcal{SWN}_n^N(T_0; M, T_0) = \max\{\mathcal{SW}_n^N(T_0, M), 0\}.$$

The object $p_n^N(t)$ can be seen as a portfolio of zero-coupon bonds, maturing at times $T_n, T_{n+1}, ...T_N$. It is clear that this portfolio has a strictly positive value; hence, we can use it as a numeraire. Let Q_n^N be the measure associated with the numeraire $p_n^N(t)$, i.e. the measure under which all assets scaled by $p_n^N(t)$ are martingales. Under this measure, the dynamics for $K_n^N(t)$ can be written on the form

$$Q_n^N: \qquad dK_n^N(t) = K_n^N(t)\sigma_n^N(t)dW_n^N(t),$$

where $W_n^N(t)$ is a Q_n^N–Wiener process. Now we can calculate the price for

$$\mathcal{SWN}_n^N(t; M, T_0)$$

in the following fashion:

$$\begin{aligned}\frac{\mathcal{SWN}_n^N(t; M, T_0)}{p_n^N(t)} &= E_t^{Q_n^N}\left[\frac{\mathcal{SWN}_n^N(T_0; M, T_0)}{p_n^N(T_0)}\right] = E_t^{Q_n^N}\left[\frac{\max\{(K_n^N(T_0) - M)p_n^N(T_0), 0\}}{p_n^N(T_0)}\right] \\ &= E_t^{Q_n^N}\left[\max\{K(T_0) - M, 0\}\right].\end{aligned}$$

Since $K_n^N(T_0)$ is lognormally distributed under Q_n^N, we arrive at a Black-Scholes type formula.

Proposition 7.1 *The value at time t of a swaption with exercise date T_0 and strike price M, written on futures contracts maturing at times T_i, $i = n, n+1, ..., N$ is given by*

$$\mathcal{SWN}_n^N(t; M, T_0) = p_n^N(t)\left(K_n^N(t)N(d_1) - MN(d_2)\right)$$

where

$$p_n^N(t) = \sum_{i=n}^{N} e^{-r(T_i - t)},$$

$$d_1 = \frac{\ln(\frac{K_n^N(t)}{M}) + \frac{\sigma_n^N(t)^2}{2}(T_0 - t)}{\frac{\sigma_n^N(t)}{2}\sqrt{T_0 - t}}, \text{ and}$$

$$d_2 = d_1 - \sigma_n^N(t)\sqrt{T_0 - t}.$$

$K_n^N(t)$ is the current swap price and $\sigma_n^N(t)$ is the swap price volatility that can be backed out from other swaptions.

1.8 A Finite Dimensional Realization

The futures model (1.6) is infinite dimensional in the sense that, for each value of the continuous variable T, it gives the dynamics of the futures price $F(t, T)$. It would be interesting to know if it is possible to write (1.6) as a finite dimensional state space model. The question is interesting from a theoretical as well as practical perspective. Simulating (1.6), would mean simulating an infinite dimensional system. If instead, we could express the model in terms of a finite dimensional Markovian system, we would be relieved of that task. It turns out that it is in fact possible to construct such a representation. The complete derivation is rather complicated, and is given in full detail in [6]. (For an introduction to the corresponding theory in a fixed income setting, see [2], and for more papers on the subject see e.g. [1], [3], [4], [5].)

Before we can state the result, we need to transform the model into a more convenient form. Let

$$r_t(x) \triangleq F(t, t + x),$$

and

$$y_t(x) \triangleq \ln r_t(x).$$

The dynamics for $y_t(x)$ are given by

$$dy_t(x) = \left(\frac{\partial}{\partial x} y_t(x) - \widehat{h}(t, x)\right) dt + \alpha_1 e^{-\beta_1 x} dW_1(t) \tag{1.14}$$

$$+ (\alpha_2 e^{-\beta_2 x} \sin(\phi + 2\pi(t + x)) + \alpha_3 e^{-\beta_3 x} \sin(\phi + 2\pi t) + \gamma) dW_2(t),$$

with initial term structure at t^*

$$y_{t^*}(x) = y^*. \tag{1.15}$$

The function $\widehat{h}(t,x)$ is given by

$$\widehat{h}(t,x) = h(t, t+x)$$

where

$$h(\cdot,\cdot)$$

is the function defined by (1.7). Now we can state the result.

Proposition 8.1 *Given the model (1.14)-(1.15), there exists a 6-dimensional stochastic process*

$$Z = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix}$$

and a function $G(Z_t, x)$ such that

$$y_t(x) = G(z_t, x); t \geq t^* \quad Q - a.s.,$$

where

$$G(Z,x) = y^*(z_0 + x) - \frac{1}{2}H(z_0, t^*, x) + \gamma z_1 + e^{-\beta_1 x} z_2$$

$$+ e^{-\beta_2 x} \sin(\phi + 2\pi(t^* + z_0 + x)) z_3$$

$$+ e^{-\beta_3 x} \sin(\phi + 2\pi(t^* + z_0)) z_4 + e^{-\beta_3 x} \cos(\phi + 2\pi(t^* + z_0)) z_5,$$

and

$$H(z_0, t^*, x) = \int_0^t \widehat{h}(t^* + s, x + t - s) ds.$$

Further, the dynamics of the state space variables z_i are given by

$$
\begin{aligned}
dz_0 &= dt \\
dz_1 &= dW_2 \\
dz_2 &= -\beta_1 z_2 dt + \alpha_1 dW_1 \\
dz_3 &= -\beta_2 z_3 dt + \alpha_2 dW_2 \\
dz_4 &= (2\pi z_5 - \beta_3 z_4)dt + \alpha_3 dW_2 \\
dz_5 &= -(\beta_3 z_5 + 2\pi z_4)dt.
\end{aligned}
$$

Proof. See [6] ∎

1.9 Appendix A

We begin by calculating the distribution of the futures price $F(t,T)$. From section 5, the integral we need to derive is

$$A^0(t,T) = \int_0^t h(u,T)du,$$

where

$$h(t,T) = \alpha_1^2 e^{-2\beta_1(T-t)} + (\alpha_2 e^{-\beta_2(T-t)} \sin(\phi+2\pi T) + \alpha_3 e^{-\beta_3(T-t)} \sin(\phi+2\pi t) + \gamma)^2.$$

After some tedious calculations, the result reads

$$
\begin{aligned}
&A^t(T,S)\\
&= \frac{\alpha_1^2}{2\beta_1}\left(e^{-2\beta_1(S-T)} - e^{-2\beta_1(S-t)}\right) + \frac{2\gamma\alpha_1}{\beta_1}\left(e^{-\beta_1(S-T)} - e^{-\beta_1(S-t)}\right) +\\
&\gamma^2(T-t) + \frac{\alpha_2^2\sin^2(\phi+2\pi S)}{2\beta_2}\left(e^{-2\beta_2(S-T)} - e^{-2\beta_2(S-t)}\right) +\\
&\frac{\alpha_3^2}{4\beta_3}\left(e^{-2\beta_3(S-T)} - e^{-2\beta_3(S-t)}\right) -\\
&\frac{\alpha_3^2\beta_3}{4(\beta_3^2+4\pi^2)}\left(e^{-2\beta_3(S-T)}\cos(2\phi+4\pi T) - e^{-2\beta_3(S-t)}\cos(2\phi+4\pi t)\right) -\\
&\frac{\alpha_3^2\pi}{2(\beta_3^2+4\pi^2)}\left(e^{-2\beta_3(S-T)}\sin(2\phi+4\pi T) - e^{-2\beta_3(S-t)}\sin(2\phi+4\pi t)\right) +\\
&\frac{2\alpha_2\alpha_3(\beta_2+\beta_3)\sin(\phi+2\pi S)}{4\pi^2+(\beta_2+\beta_3)^2}\left(e^{-(\beta_2+\beta_3)(S-T)}\sin(\phi+2\pi T) - e^{-(\beta_2+\beta_3)(S-t)}\sin(\phi+2\pi t)\right) -\\
&\frac{4\alpha_2\alpha_3\sin(\phi+2\pi S)}{4\pi^2+(\beta_2+\beta_3)^2}\left(e^{-(\beta_2+\beta_3)(S-T)}\cos(\phi+2\pi T) - e^{-(\beta_2+\beta_3)(S-t)}\cos(\phi+2\pi t)\right).
\end{aligned}
$$

1.10 Appendix B

Element (i, j) in the covariance matrix C, is given by

$$Cov(\ln(F(T,S_i)), \ln F(T,S_j)) =$$

$$Cov\left(\begin{array}{c} \int_t^T \alpha_1 e^{-\beta_1(S_i-u)}dW_1(u) + \alpha_2\sin(\phi+2\pi S_i)\int_t^T e^{-\beta_2(S_i-u)}dW_2(u) \\ +\alpha_3\int_t^T g(u,S_i)dW_2(u) + \gamma\int_t^T dW_2(u), \\ \int_t^T \alpha_1 e^{-\beta_1(S_j-u)}dW_1(u) + \alpha_2\sin(\phi+2\pi S_j)\int_t^T e^{-\beta_2(S_j-u)}dW_2(u) \\ +\alpha_3\int_t^T g(u,S_j)dW_2(u) + \gamma\int_t^T dW_2(u)) \end{array}\right)$$

$$= \frac{\alpha_1^2}{2\beta_1}\left(e^{-\beta_1(S_i+S_j-2T)} - e^{-\beta_1(S_i+S_j-2t)}\right) + \gamma^2(T-t)$$

$$+ \frac{\alpha_2^2\sin(\phi+2\pi S_i)\sin(\phi+2\pi S_j)}{2\beta_2}\left(e^{-\beta_2(S_i+S_j-2T)} - e^{-\beta_2(S_i+S_j-2t)}\right)$$

$$+\alpha_2\alpha_3\left(\sin(\phi+2\pi S_i)e^{-(\beta_2 S_i+\beta_3 S_j)} + \sin(\phi+2\pi S_j)e^{-(\beta_2 S_j+\beta_3 S_i)}\right) *$$

$$[\frac{\beta_2+\beta_3}{4\pi^2+(\beta_2+\beta_3)^2}\left(e^{(\beta_2+\beta_3)T}\sin(\phi+2\pi T) - e^{(\beta_2+\beta_3)t}\sin(\phi+2\pi t)\right)$$

$$- \frac{2\pi}{4\pi^2+(\beta_2+\beta_3)^2}\left(e^{(\beta_2+\beta_3)T}\cos(\phi+2\pi T) - e^{(\beta_2+\beta_3)t}\cos(\phi+2\pi t)\right)]$$

$$+ \frac{\gamma\alpha_2}{\beta_2}(e^{\beta_2 T} - e^{\beta_2 t})\left(\sin(\phi+2\pi S_i)e^{-\beta_2 S_i} + \sin(\phi+2\pi S_j)e^{-\beta_2 S_j}\right)$$

$$+ \gamma\alpha_3(e^{-\beta_3 S_i} + e^{-\beta_3 S_j})*$$

$$[\frac{\beta_3}{4\pi^2 + \beta_3^2}\left(e^{\beta_3 T}\sin(\phi + 2\pi T) - e^{\beta_3 t}\sin(\phi + 2\pi t)\right)$$

$$- \frac{2\pi}{4\pi^2 + \beta_3^2}\left(e^{\beta_3 T}\cos(\phi + 2\pi T) - e^{\beta_3 t}\cos(\phi + 2\pi t)\right)]$$

$$+ \frac{\alpha_3^2}{4\beta_3}\left(e^{-\beta_3(S_i+S_j-2T)} - e^{-\beta_3(S_i+S_j-2t)}\right)$$

$$+ \frac{\alpha_3^2 e^{-\beta_3(S_i+S_j)}}{2}*$$

$$[\frac{\pi}{4\pi^2 + \beta_3^2}\left(e^{2\beta_3 T}\sin(2\phi + 4\pi T) - e^{2\beta_3 t}\sin(2\phi + 4\pi t)\right)$$

$$+ \frac{\beta_3}{8\pi^2 + 2\beta_3^2}\left(\left(e^{2\beta_3 T}\cos(2\phi + 4\pi T\right) - e^{2\beta_3 t}\cos(2\phi + 4\pi t)\right)]$$

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2

Futures Pricing in a Hidden Markov Model of the Convenience Yield

2.1 Introduction

In interest rate theory, the concept of affine term structures was introduced in the mid-1990s (Brown and Schaefer [2], Duffie and Kan [3]), when it was discovered that short rate models admitting bond prices that are exponentially affine in the state variables are much more tractable and easier to handle computationally than other models. The same concept carries over to the futures market (Björk and Landén [1]). Recently, the concept of semi-affine term structures was introduced in the interest rate theory literature (Landén [5]). In this paper, we show how that concept, too, has a natural analog in the futures market. The set-up is very general: the asset on which the futures contract is based, is assumed to be driven by a $d-$dimensional Wiener process, and the drift and diffusion parameters are functions of an M-dimensional Markov process. The Markov process itself is driven by the same Wiener process as the underlying asset, as well as a marked point process. We provide sufficient and (in the time-homogenous case) necessary conditions on the drift and diffusion coefficients of the underlying asset for a semi-affine term structure to exist. For illustration, we carry out explicit calculations in two situations: the Gibson-Schwartz [4] model, and when X is a Markov chain on $\{1, 2, ..., N\}$ with intensity matrix $H = h_{ij}$. Next, we consider the case when the Markov process X is assumed to be unobservable. If X contains, e.g., a process for the convenience yield, this is a natural assumption. We analyze the two situations above, in this new unobservable framework. Finally, we compute the price of a European call option in the Gibson-Schwartz model, both when X is observable and when it is not.

2.2 The Model

The financial market is modelled by a filtered probability space $(\Omega, \mathbf{F}, \mathcal{F}, Q)$, $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$, where Q is the risk neutral martingale measure. On this space, the following adapted processes are defined:

- A d-dimensional Wiener process W
- A 1-dimensional stochastic process S
- A K-dimensional Markov process X

Here, S denotes the price of an underlying asset, and X denotes a vector of state variables. X could, e.g., contain a process for the convenience yield, the inflation rate, etc. Note that we do not assume that the market on which S is traded is idealized and frictionless. An example could be a commodity market or a market for a natural resource. We assume the following dynamics for S and X (under Q)

$$dS_t = S_t \mu_S(S_t, X_t)dt + S_t \sigma_S(S_t, X_t)dW_t \tag{2.1}$$

$$dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dW_t + \int_E \delta(t, X_{t-}, y)\mu(dt, dy). \tag{2.2}$$

Here, μ is a marked point process on a measurable Lusin mark space (E, ϵ). We will assume that the compensator v of μ can be written as $v(\omega; dt, dy) = \lambda(t, X_{t-}(\omega), dy)dt$.

The main objective of this paper is to compute futures prices on the market (2.1)-(2.2), with S as the underlying asset. This will be done in two situations: when we have full information, i.e., when we can observe both X and S, and when we can only observe S. If X represents, e.g., the convenience yield, it seems reasonable to treat it as unobservable. To formalize the two cases, we introduce an observation σ-algebra, $\mathcal{G}_t$. In the first situation, when we have full information, $\mathcal{G}_t$ is equal to $\mathcal{F}_t$, and in the second situation, the partially observable case, $\mathcal{G}_t$ is equal to $\mathcal{F}_t^S$. Before we continue we recall the following well-known facts.

Proposition 2.1 *Let $F(t,T)$ denote the futures price at time t, for delivery at time T of a futures contract written on S. Then $F(t,T)$ is given by*

$$F(t,T) = E^Q[S_T \mid \mathcal{G}_t] \tag{2.3}$$

$$F(t,t) = S_t. \tag{2.4}$$

Proof. See e.g. Björk [2]. ■

2.3 The Fully Observable Case

In this section, we consider the pricing of futures in the model (2.1)-(2.2) in the fully observable case, i.e., when $\mathcal{G}_t = \mathcal{F}_t$. It turns out that from a modelling point of view, the natural object to consider is not the spot price itself S, but rather the logarithm of the spot price, $\ln S$ [1]. We therefore introduce the variable

$$Z_t \equiv \ln S_t$$

and write (2.1) and (2.2) on the form

$$dZ_t = \mu_Z(Z_t, X_t)dt + \sigma_Z(Z_t, X_t)dW_t \tag{2.5}$$

$$dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dW_t + \int_E \delta(t, X_{t-}, y)\mu(dt, dy), \tag{2.6}$$

where $\mu_Z = \mu_S - \frac{1}{2}\sigma_S^2$, $\sigma_Z = \sigma_S$. Since (2.5)-(2.6) is a Markovian system, it follows from (2.3)-(2.4) that the futures price is of the form

$$F(t,T) = H(t, Z_t, X_t, T) \tag{2.7}$$

and in particular that

$$S_t = H(t, Z_t, X_t, t).$$

Now we are ready to present the fundamental PDE in this context.

Proposition 3.1 *The futures price function (2.7) satisfies the following partial differential equation*

$$\frac{\partial H}{\partial t}(t,z,x,T) + \mathcal{A}H(t,z,x,T) = 0 \tag{2.8}$$
$$H(T,z,x,T) = e^z,$$

where $\mathcal{A}$ is given by

$$\mathcal{A}H(t,z,x,T) = \mu_Z\frac{\partial H}{\partial z} + \sum_{i=1}^{K}\mu_X^i\frac{\partial H}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{K}(\sigma_X\sigma_X^*)_{ij}\frac{\partial^2 H}{\partial x_i \partial x_j} + \frac{1}{2}\sigma_Z\sigma_Z^*\frac{\partial^2 H}{\partial z^2}$$
$$+ \sum_{i=1}^{K}(\sigma_Z\sigma_X^*)_i\frac{\partial^2 H}{\partial z \partial x_i} + \int_E \left(H(t,z,x+\delta(t,x,y)) - H(t,z,x)\right)\lambda(t,x,dy). \tag{2.9}$$

Proof. Expand (2.7) by the Itô formula and use its martingale property. ■

It would be much easier to solve (2.8) if we had more information about the structure of H. The case when H is affine is treated in Björk and Landén [1]; we will now consider the case when H is *semi-affine*.

Definition 3.1 *A model is said to admit a* ***semi-affine term structure of futures prices*** *if futures prices can be written on the form*

$$F(t,T) = e^{A(t,X_t,T)+B(t,T)Z_t}, \tag{2.10}$$

where $A(t,x,T)$ and $B(t,T)$ are deterministic functions.

We note that we can write (2.10) as

$$F(t,T) = S_t^{B(t,T)} e^{A(t,X_t,T)}.$$

B is required to depend only on time, whereas A is allowed to depend on the state variables in a non-linear fashion. If we required A to be an affine function of the state variables, we would have an *affine* model. Below we show that a semi-affine term structure is guaranteed if we choose μ_Z and $\sigma_Z\sigma_Z^*$ semi-affine. Since a semi-affine term structure is restrictive enough to obtain tractable futures prices, the added flexibility it gives when choosing the coefficients in the factor model comes at a low cost.

The next proposition characterizes the class of semi-affine term structures.

Proposition 3.2 *Consider the model (2.5)-(2.6) and assume that the coefficients μ_Z and σ_Z are of the form*

$$\begin{aligned} \mu_Z(z,x) &= a_1(t,x) + a_2(t)z \\ \sigma_Z\sigma_Z^*(z,x) &= b_1(t,x) + b_2(t)z. \end{aligned} \tag{2.11}$$

Assume also that $\sigma_Z\sigma_X^*$ does not depend on Z. Then H from (2.7) can be written on the form

$$H(t,z,x,T) = e^{A(t,x,T)+B(t,T)z}, \tag{2.12}$$

where A and B satisfy

$$\begin{aligned} \frac{\partial B(t,T)}{\partial t} + a_2(t)B(t,T) + \frac{1}{2}b_2(t)B(t,T)^2 &= 0 \\ B(T,T) &= 1 \end{aligned} \tag{2.13}$$

$$\frac{\partial A(t,x,T)}{\partial t} + \sum_{i=1}^{k} \mu_X^i \frac{\partial A}{\partial x_i} + a_1(t,x)B(t,T) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}(\sigma_X\sigma_X^*)_{ij}\left(\frac{\partial^2 A}{\partial x_i \partial x_j} + \frac{\partial A}{\partial x_i}\frac{\partial A}{\partial x_j}\right) \tag{2.14}$$

$$+\frac{1}{2}B(t,T)^2 b_1(t,x) + \sum_{i=1}^{k}(\sigma_Z\sigma_X^*)_i B(t,T)\frac{\partial A}{\partial x_i} + \int_E \left(e^{\Delta A(t,x,T)} - 1\right)\lambda(t,x,dy) = 0$$

$$A(T, x, T) = 0,$$

and where

$$\Delta A(t, x, T) \equiv A(t, x + \delta(t, x, y), T) - A(t, x, T).$$

For the case when μ and σ are time-homogenous, and $B(t,T)$ is strictly monotone in t, *the reverse implication holds, i.e., H is semi-affine only if μ and σ are semi-affine.*

Proof. We begin by proving that assumption (2.11) leads to a semi-affine term structure. This is done by noting that

$$H(t, z, x, T) = \exp(A(t, x, T) + B(t, T)z),$$

where $A(t, x, T)$ and $B(t, T)$ solves (2.14) and (2.13) respectively, solves (2.8), which uniquely determines futures prices in this setting. Next, assume that H is of the form (2.12), and that μ_Z and $\sigma_Z \sigma_Z^*$ are time-homogenous. Then we can take two points in time, t_1 and t_2, insert (2.12) into (2.8), and after some manipulations, obtain the following structure for μ_Z and $\sigma_Z \sigma_Z^*$:

$$\begin{bmatrix} \mu_Z(z,x) \\ \sigma_Z \sigma_Z^*(z,x) \end{bmatrix} = -K_1^{-1}(K_2 + \frac{1}{2}K_3 + K_4 + K_5 + K_6 z), \tag{2.15}$$

where

$$K_1 = \begin{bmatrix} B(t_1,T) & \frac{1}{2}B^2(t_1,T) \\ B(t_2,T) & \frac{1}{2}B^2(t_2,T) \end{bmatrix}$$

$$K_2 = \begin{bmatrix} \frac{\partial A(t_1)}{\partial t} \\ \frac{\partial A(t_2)}{\partial t} \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^k \mu_X^i \frac{\partial A}{\partial x_i}(t_1) \\ \sum_{i=1}^k \mu_X^i \frac{\partial A}{\partial x_i}(t_2) \end{bmatrix}$$

$$K_3 = \begin{bmatrix} \sum_{i=1}^k \sum_{j=1}^k (\sigma_X \sigma_X^*)_{ij} (\frac{\partial^2 A}{\partial x_i \partial x_j} + \frac{\partial A}{\partial x_i}\frac{\partial A}{\partial x_j}) \\ \sum_{i=1}^k \sum_{j=1}^k (\sigma_X \sigma_X^{tr})_{ij} (\frac{\partial^2 A}{\partial x_i \partial x_j} + \frac{\partial A}{\partial x_i}\frac{\partial A}{\partial x_j}) \end{bmatrix}$$

$$K_4 = \begin{bmatrix} \sum_{i=1}^k (\sigma_Z \sigma_X^*)_i B(t,T) \frac{\partial A}{\partial x_i}(t_1) \\ \sum_{i=1}^k (\sigma_Z \sigma_X^*)_i B(t,T) \frac{\partial A}{\partial x_i}(t_2) \end{bmatrix}$$

$$K_5 = \begin{pmatrix} \int\limits_E \left(e^{\Delta A(t_1,x,T)} - 1\right) \lambda(t_1, x, dy) \\ \int\limits_E \left(e^{\Delta A(t_2,x,T)} - 1\right) \lambda(t_2, x, dy) \end{pmatrix}$$

$$K_6 = \begin{bmatrix} \frac{\partial B(t_1,T)}{\partial t} \\ \frac{\partial B(t_2,T)}{\partial t} \end{bmatrix}.$$

Expression (2.15) is affine in z, and the proof is complete. Note that the requirement that $B(t,T)$ is strictly monotone in t is exactly what is needed for K_1 to be invertible. ■

The requirement that $\sigma_Z \sigma_X^*$ does not depend on Z means that if σ_Z^i depends on Z, then X cannot be driven by W^i, i.e. $\sigma_X^i(t,z,x) = 0$.

Example 3.1 One model that falls within this framework is the following (Gibson and Schwartz [5]):

$$dS_t = S_t(r - X_t)dt + S_t\widehat{\sigma}_S d\widehat{W}_t^1 \tag{2.16}$$
$$dX_t = \alpha(\mu - X_t)dt + \widehat{\sigma}_X d\widehat{W}_t^2 \tag{2.17}$$
$$d\widehat{W}_t^1 d\widehat{W}_t^2 = \rho dt.$$

Note that we diverge from the set up in (2.1) - (2.2) in this example, since we chose to work with correlated Wiener processes. To express the model in terms of the notation (2.5) - (2.6), we can write

$$dS_t = S_t(r - X_t)dt + S_t\sigma_S dW$$
$$dX_t = \alpha(\mu - X_t)dt + \sigma_X dW,$$

where

$$\sigma_S = \begin{pmatrix} \sigma_S^1 & \sigma_S^2 \end{pmatrix}$$
$$\sigma_X = \begin{pmatrix} \sigma_X^1 & \sigma_X^2 \end{pmatrix}$$
$$W = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix}$$

$$\| \sigma_S \| = \widehat{\sigma}_S$$
$$\| \sigma_X \| = \widehat{\sigma}_X$$

and

$$\sigma_S \sigma_X^* = \| \sigma_S \| \| \sigma_X \| \rho.$$

If we use the notation $Z_t \equiv \ln S_t$, equation (2.16) becomes

$$dZ_t = (r - X_t - \frac{\sigma_S^2}{2})dt + \sigma_S dW_t^1.$$

Now it is clear that assumption (2.11) is fulfilled, and thus Proposition 3.1 applies. This means that the futures price is of the form (2.12). Further, we see that $B(t,T) \equiv 1$, so the futures price can be written as

$$F(T, X_t, S_t) = S_t e^{A(t,T,X_t)}, \tag{2.18}$$

where $A(t, x, T)$ solves

$$\frac{\partial A(t,x,T)}{\partial t} + \alpha(\mu - x)\frac{\partial A(t,x,T)}{\partial x} + r - x + \frac{1}{2}\sigma_X^2 \left(\frac{\partial^2 A(t,x,T)}{\partial x^2} + (\frac{\partial A(t,x,T)}{\partial x})^2\right) \tag{2.19}$$

$$+\rho\sigma_S\sigma_X\frac{\partial A(t,x,T)}{\partial x} = 0.$$

Since X_t is also an affine function, we know that $A(t, x, T)$ can be written on the form (Björk, Landén (2001))

$$A(t,x,T) = A_1(t,T) + A_2(t,T)x.$$

Inserting this into (2.19) yields the following system of equations:

$$\begin{aligned} \frac{\partial A_2(t,T)}{\partial t} - 1 - \alpha A_2(t,T) &= 0 \\ A_2(T,T) &= 0 \\ \frac{\partial A_1(t,T)}{\partial t} + r + \alpha\mu A_1(t,T) + \frac{1}{2}\sigma_x^2 A_2^2(t,T) + \sigma_s\sigma_x\rho A_2(t,T) &= 0 \\ A_1(T,T) &= 0. \end{aligned}$$

This can easily be solved, and the solution is given by

$$A_2(t,T) = \frac{e^{-\alpha(T-t)} - 1}{\alpha} \tag{2.20}$$

$$A_1(t,T) = \frac{1}{\alpha}\left(\mu + \frac{\rho\sigma_S\sigma_X}{\alpha} - \frac{\sigma_X^2}{\alpha^2}\right)(1 - e^{-\alpha(T-t)})$$

$$+\left(r - \mu + \frac{\sigma_X^2}{2\alpha^2} - \frac{\rho\sigma_S\sigma_X}{\alpha}\right)(T-t) + \frac{\sigma_X^2}{4\alpha^3}(1 - e^{-2\alpha(T-t)}).$$

Example 3.2 Consider the model

$$dZ_t = \mu_Z(Z_t, X_t)dt + \sigma_Z(Z_t, X_t)dW_t,$$

where X_t is a Markov chain on $\{1,2,...N\}$, with intensity matrix $H = h_{ij}$. By proposition 2.2, the futures price $H(t,z,x,T)$ satisfies (2.8). To determine the corresponding equation in this case, we need the infinitesimal generator for the Markov chain X_t. The calculation goes as follows:

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} E[f(X_{t+\Delta t}) - f(X_t) \mid X_t = i]$$

$$= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left[f(i)P(X_{t+\Delta t} = i \mid X_t = i) + \sum_{j \neq i} f(j)P(X_{t+\Delta t} = j \mid X_t = i) - f(i) \right]$$

$$= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left[f(i) \left(1 - \Delta t \sum_{j \neq i} h_{ij} \right) + \sum_{j \neq i} f(j) \left(h_{ij} \Delta t + o(\Delta t) \right) - f(i) \right]$$

$$= \sum_{j \neq i} h_{ij} \left(f(j) - f(i) \right) = \sum_{j=1}^{N} h_{ij} f(j).$$

From this we conclude that

$$E[df(X_t) \mid X_t = i] = \sum_{j=1}^{N} h_{ij} f(j) dt.$$

With this in mind, it is clear that equation (2.8) now reads

$$\sum_{j=1}^{N} h_{ij} H(t,z,j) + \frac{\partial H(t,z,i)}{\partial t} + \mu_Z \frac{\partial H(t,z,i)}{\partial z} + \frac{1}{2} \sigma_Z^2 \frac{\partial^2 H(t,z,i)}{\partial z^2} = 0 \qquad i = 1,...N.$$

Next, assume that μ_Z and σ_Z^2 are semi-affine, i.e., that they can be written on the form (2.11). Then the futures price is given by

$$F(t,T) = e^{A(t,X_t,T) + B(t,T) Z_t},$$

where $B(t,T)$ satisfies (2.13), and $A(t,x,T)$ satisfies, for $i = 1,...N$

$$\frac{\partial A(t,i,T)}{\partial t} + \sum_{j=1}^{N} h_{ij} e^{A(t,j,T) - A(t,i,T)} + a_1(t,i) B(t,T) + \tfrac{1}{2} b_1(t,i) B^2(t,T) = 0 \qquad A(T,i,T) = 0. \tag{2.21}$$

The equation for B,

$$\frac{\partial B(t,T)}{\partial t} + a_2(t) B(t,T) + \frac{1}{2} b_2(t) B(t,T)^2 = 0 \qquad B(T,T) = 1, \tag{2.22}$$

is a Riccati equation, but the equation for A is more complicated and can only be solved numerically.

2.4 The Partially Observable Case

When X is unobservable, the futures price process is given by

$$F(t,T) = E^Q[S_T \mid \mathcal{F}_t^S].$$

In order to use the results of the previous section, we note the following consequence of the law of iterated expectations:

$$F(t,T) = E^Q[S_T \mid \mathcal{F}_t^S] = E^Q[E^Q[S_T \mid \mathcal{F}_t] \mid \mathcal{F}_t^S] = E^Q[H(t,Z_t,X_t,T) \mid \mathcal{F}_t^S], \tag{2.23}$$

where $H(t,Z_t,X_t,T)$ is the function appearing in (2.8). Here H can be thought of as the arbitrage free futures price in a fictitious completely observable market. Because of (2.23), the partially observable case can be dealt with in two steps: first compute H using the results of the previous section, then take the result and run it through a filter. The structure of the particular filter we use is, of course, dependent on how we model the state variables. We will once again look at the examples from the previous section, but in the case when X is unobservable.

Example 4.1 Consider again the Gibson-Schwarts [5] model:

$$\begin{aligned} dS_t &= S_t(r - X_t)dt + S_t\sigma_S dW_t^1 \\ dX_t &= \alpha(\mu - X_t)dt + \sigma_X dW_t^2 \\ dW_t^1 dW_t^2 &= \rho dt. \end{aligned} \tag{2.24}$$

From (2.18) and (2.23), we know that the futures price in the partially observable case is given by

$$F(t,T) = E^Q[S_t \exp(A_1(t,T) + A_2(t,T)X_t) \mid \mathcal{F}_t^S], \tag{2.25}$$

where $A_1(t,T)$ and $A_2(t,T)$ are given by (2.20). Further, since $S_t \in \mathcal{F}_t^S$ and $\mathcal{L}(X_t \mid \mathcal{F}_t^S) = N(\cdot,\cdot)$ (which is not a trivial result, see Lipster and Shiryaev (1977)), we are led to the following proposition.

Proposition 4.1 *Consider the model (2.24) and define the processes $\pi_t[X]$ and $G_t[X]$ by*

$$\begin{aligned} \pi_t[X] &= E^Q[X \mid \mathcal{F}_t^S] \\ G_t[X] &= Var^Q[X \mid \mathcal{F}_t^S]. \end{aligned}$$

The futures price in the partially observable case is then given by

$$F(t,T) = S_t e^{A_1(t,T) + A_2(t,T)\pi_t[X] + \frac{1}{2}A_2^2(t,T)G_t[X]}, \tag{2.26}$$

where $A_1(t,T)$ and $A_2(t,T)$ are given by (2.20). Furthermore, the dynamics of $\pi_t[X]$ and $G_t[X]$ are given by the Kalman-Bucy filter

$$d\pi_t[X] = \alpha(\mu - \pi_t[X])dt + \left(\rho\frac{\sigma_x}{\sigma_s} - \frac{G_t[X]}{\sigma_s^2}\right)\left(dZ_t - \left(r - \frac{\sigma_s^2}{2} - \pi_t[X]\right)dt\right)$$
$$\frac{\partial G_t[X]}{\partial t} = \sigma_X^2 - 2\alpha G_t[X] - \left(\rho\sigma_X - \frac{G_t[X]}{\sigma_s}\right)^2.$$

Remark 4.1 *The equation for $G_t[X]$ is a Riccati equation, and can be solved explicitly. The solution is given by*

$$\begin{aligned} G_t[X] &= -d_1 \tan\left(d_1 \frac{t+C}{\sigma_s^2}\right) + d_2 \\ C &= \frac{\sigma_s^2}{d_1} \arctan\left(\frac{d_2 - d_3}{d_1}\right) \\ d_1 &= \sqrt{-\sigma_X^2\sigma_S^2 - \alpha^2\sigma_S^4 + 2\alpha\sigma_S^3\rho\sigma_X} \\ d_2 &= -\alpha\sigma_S^2 + \rho\sigma_X\sigma_S \\ d_3 &= \frac{\sigma_s^2}{2\alpha}. \end{aligned}$$

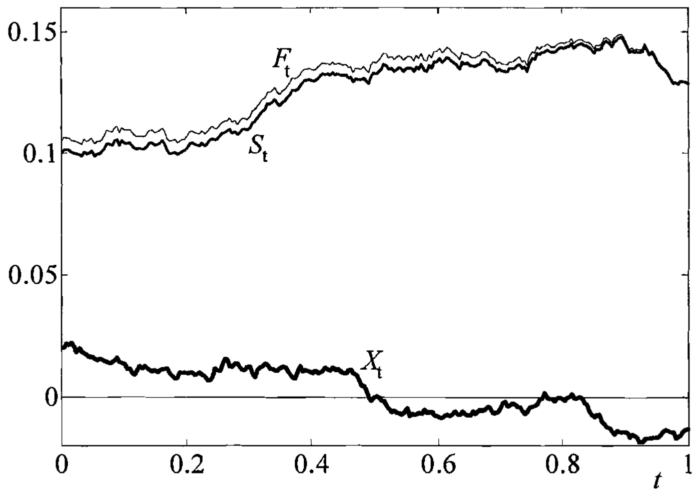
Proof. For $X \sim N(\cdot,\cdot)$, $E[e^X] = e^{E[X]+\frac{1}{2}V[X]}$, so since

$$\mathcal{L}(X_t \mid \mathcal{F}_t^S) = N(\pi_t[X], G_t[X])$$

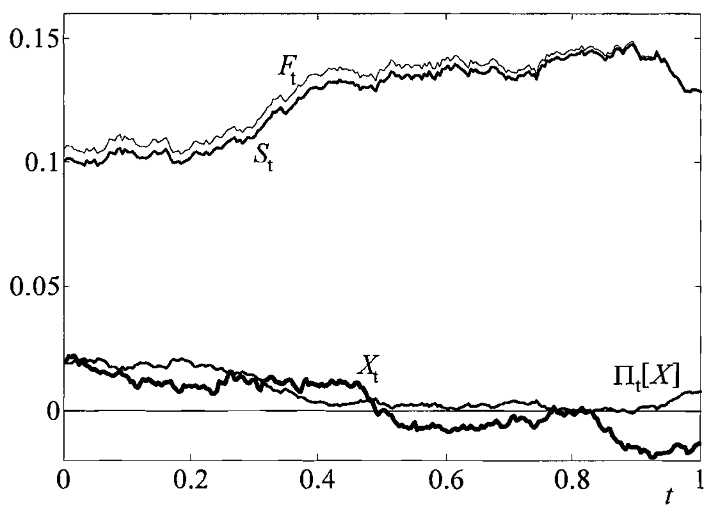
and $S_t \in \mathcal{F}_t^S$,

$$\begin{aligned} E^Q[S_t e^{A_1(t,T)+A_2(t,T)X_t} \mid \mathcal{F}_t^S] &= S_t e^{A_1(t,T)} E^Q[e^{A_2(t,T)X_t} \mid \mathcal{F}_t^S] \\ &= S_t e^{A_1(t,T)} e^{A_2(t,T)\pi_t[X]+\frac{1}{2}A_2^2(t,T)G_t[X]}. \end{aligned}$$

For details on the Kalman-Bucy filter, see e.g. Lipster and Shiryaev [7]. ■



The spot price S, the futures price F and the process X in the fully observable case. The total number of simulations is 250, $\rho = 0.1$, $\sigma_X = 0.02$, $\sigma_s = 0.15$, $\alpha = 0.2$, $\mu = 0.02$.



Here X is unobservable, so the futures price is now given by (26).

Example 4.2 Consider the following model:

$$dS_t = S_t(r - C(X_t))dt + S_t\sigma dW_t, \tag{2.27}$$

where X_t is a Markov chain on $\{1, 2, ..., N\}$ with intensity matrix $H = (h_{ij})$, and $C_i = C(i)$. We want to compute

$$\pi_t[F] = E^Q[F(t, S_t, X_t) \mid \mathcal{F}_t^S].$$

Since X_t can only assume the values $1, 2, ..., N$, we can write F as

$$F(t, S_t, X_t) = \sum_{1=1}^{N} F(t, S_t, i)\delta_i(t),$$

where $\delta_i(t)$ is defined by

$$\delta_i(t) = 1_{\{X_t=i\}}.$$

Further, since $F(t, S_t, i)$ is $\mathcal{F}_t^S$-measurable, we get the expression

$$\pi_t[F] = \sum_{i=1}^{N} F(t, S_t, i)\pi_t[\delta_i]. \tag{2.28}$$

To derive $F(t, S_t, i)$, we begin by noting that the set up here is the same as in example 3.3. In this case, μ_Z and σ_Z^2 are given by

$$\begin{aligned} \mu_Z &= r - i - \frac{\sigma^2}{2} \\ \sigma_Z^2 &= \sigma^2. \end{aligned}$$

Hence $F(t, S_t, i)$ satisfies (2.21) - (2.22) with

$$\begin{aligned} a_1(t, i) &= r - C_i - \frac{\sigma^2}{2} \\ a_2(t) &\equiv 0 \\ b_1(t, i) &= \sigma^2 \\ b_2(t) &\equiv 0. \end{aligned}$$

We immediately get that

$$B(t, T) \equiv 1,$$

so $F(t, S_t, i)$ is given by

$$F(t, S_t, i) = S_t e^{A(t,i,T)}, \tag{2.29}$$

where $A(t, i, T)$ satisfies

$$\begin{aligned} \tfrac{\partial A(t,i,T)}{\partial t} + r - C_i + \textstyle\sum_{j=1}^{N} h_{ij} e^{A(t,j,T) - A(t,i,T)} &= 0 \\ A(T, i, T) &= 0 \\ i &= 1, ...N. \end{aligned} \tag{2.30}$$

Finally, to be able to use (2.28), we need the dynamics of the filter estimate $\pi_t[\delta_i]$. These, together with the conclusions above, are given in the following proposition.

Proposition 4.2 *The futures price in the model (2.27), in the partially observable case, is given by (2.28), where $F(t, S_t, i)$ is given by (2.29), A satisfies (2.30), and the filter estimate $\pi_t[\delta_i]$ satisfies the Wonham filter equation*

$$d\pi_t[\delta_i] = \sum_{j=1}^{N} h_{ji}\pi_t[\delta_i]dt + \left\{ S_t(r-i)\pi_t[\delta_i] - \sum_{j=1}^{N} S_t(r-j)\pi_t[\delta_j]\pi_t[\delta_i] \right\} d\nu_t$$

$$d\nu_t = dS_t - \sum_{j=1}^{N} S_t(r-j)\pi_t[\delta_j]dt.$$

Proof. In Lipster and Shiryaev [7], the Wonham filter is treated in detail. ■

2.5 Options on Futures Prices

In this section we compute the value of a call option on a futures contract in the models (2.17) and (2.24).

First consider the model (2.17)

$$\begin{aligned} dS_t &= S_t(r - X_t)dt + S_t\sigma_S dW_t^1 \\ dX_t &= \alpha(\mu - X_t)dt + \sigma_X dW_t^2 \\ dW_t^1 dW_t^2 &= \rho dt. \end{aligned}$$

In this fully observable framework, the price at time $t = T_1$ of a futures contract written on S, and with maturity T, is given by

$$F(T_1, T) = E^Q[S_T \mid \mathcal{F}_{T_1}].$$

The aim of this section is to calculate the value of a European call option written on this futures contract. From standard theory, the general formula for the price at time t of a call option with strike price K and maturity T_1 on the above contract is given by

$$c(t, T_1, T, K, S_t) = e^{-r(T_1 - t)} E^Q[\max(F(T_1, T) - K, 0) \mid \mathcal{F}_t]. \qquad (2.31)$$

Now we can use the result from section 3

$$F(T_1, T) = S_{T_1} e^{A_1(T_1,T) + A_2(T_1,T)X_{T_1}},$$

where A_1 and A_2 in the fully observable case are given by (2.20). Inserting this into (2.31), we obtain

$$c(t, T_1, T, K, S_t) = e^{-r(T_1-t)} E^Q[\max(S_{T_1} e^{A_1(T_1,T)+A_2(T_1,T)X_{T_1}} - K, 0) \mid \mathcal{F}_t]. \tag{2.32}$$

Next, we want to know what S_{T_1} and X_{T_1} look like at time t :

$$\begin{aligned} S_{T_1} &= S_t e^{(r-\frac{\sigma_s^2}{2})(T_1-t)-\int_t^T X_u du+\sigma_s \int_t^{T_1} dW_u^s} \\ X_{T_1} &= X_t + \alpha\mu(T_1 - t) - \alpha \int_t^{T_1} X_u du + \sigma_x \int_t^{T_1} dW_u^x. \end{aligned}$$

If we insert this into (2.32), we can read off the distribution of $F(T_1, T) \mid \mathcal{F}_t$ as

$$\mathcal{L}(F(T_1, T) \mid \mathcal{F}_t) = \mathcal{L}(S_t e^{Y(t,T_1,T)} \mid \mathcal{F}_t),$$

where

$$\mathcal{L}(Y(t, T_1, T) \mid \mathcal{F}_t) = N(\mu_Y, \sigma_Y^2)$$

and

$$\begin{aligned} \mu_Y &= (r - \sigma_s^2 - \alpha\mu)(T_1 - T) + A_1(T_1, T) \\ &\quad + A_2(T_1, T)(X_t + \alpha\mu(T_1 - t) - \alpha^2\mu(T_1 - t)) \end{aligned} \tag{2.33}$$

$$\sigma_Y^2 = V(u) + \sigma_s^2(T_1 - t) + A_2^2(T_1, T)(\alpha^2 V(u) + \sigma_x^2(T_1 - t)) \tag{2.34}$$

$$V(u) = (\frac{\sigma_x}{\alpha})^2 \left(\frac{e^{\alpha^2 T_1} - e^{\alpha^2 t}}{\alpha^2} - (T_1 - t) \right).$$

The problem now looks very similar to the standard Black & Sholes case, the only difference being the mean and variance of the log-normally distributed asset on which the call option is written. Hence, we are not surprised by the shape of the following formula.

Proposition 5.1 *The price at time t of a European call option with strike price K and maturity date T_1, on a futures contract with delivery date T, on the underlying asset S from (2.17), is in the fully observable case given by*

$$c(t, T_1, T, x, s) = e^{-r(T_1-t)} \left(s e^{\mu_Y + \frac{\sigma_Y^2}{2}} N(d_1) - K N(d_2) \right),$$

where

$$d_1 = \frac{\ln \frac{s}{K} + \sigma_Y^2 + \mu_Y}{\sigma_Y}$$
$$d_2 = d_1 - \sigma_Y$$

and μ_Y and σ_Y^2 are given by (2.33) and (2.34).

Proof. We want to solve

$$E^Q[\max(S_t e^{Y(t,T_1,T)} - K, 0) \mid \mathcal{F}_t],$$

where $Y(t,T_1,T) \sim N(\mu_Y, \sigma_Y^2)$. If we use a trick with indicator functions and the fact that S_t is $\mathcal{F}_t$-measurable, we get

$$E^Q[\max(S_t e^{Y(t,T_1,T)} - K, 0) \mid \mathcal{F}_t] = S_t E^Q[e^{Y(t,T_1,T)} 1_{\{Y(t,T_1,T) > \ln \frac{K}{S_t}\}} \mid \mathcal{F}_t]$$
$$-KP^Q[Y(t,T_1,T) > \ln \frac{K}{S_t} \mid \mathcal{F}_t].$$

The second term is immediate:

$$P^Q[Y(t,T_1,T) > \ln \frac{K}{S_t} \mid \mathcal{F}_t] = N(\frac{\ln \frac{K}{S_t} + \mu_Y}{\sigma_Y}).$$

For the first term, we have that

$$E^Q[e^{Y(t,T_1,T)} 1_{\{Y(t,T_1,T) > \ln \frac{K}{S_t}\}} \mid \mathcal{F}_t] = \int_{\ln \frac{K}{S_t}}^{\infty} e^y f_Y(y) dy,$$

where $f_Y(y)$ is the density function for a $N(\mu_Y, \sigma_Y^2)$-distributed variable. The integral is straightforward to solve. ■

Next, we want to find the price of a call option in the partially observable case, i.e., in the model (2.24), when we cannot observe X. Since we have already calculated the price in the fully observable case, this turns out to be quite simple.

Proposition 5.2 *The price at time t of a European call option with strike price K and maturity date T_1, on a futures contract with delivery date T, on the underlying asset S from (2.24), is in the partially observable case given by*

$$c^*(t, \pi_t[x], G_t[x], S_t) = e^{-r(T_1 - t)} \left(s e^{-\frac{\sigma_Y^2}{2}} \int_0^{\infty} e^{\mu_Y^x} N(d_1^x) f_X(x) dx - K \int_0^{\infty} N(d_2^x) f_X(x) dx \right), \tag{2.35}$$

where μ_Y and σ_Y^2 are given by (2.33) and (2.34); $f_X(x)$ is the density function of a $N(\pi_t[x], G_t[x])-$distributed variable, where $\pi_t[x]$ and $G_t[x]$ are the conditional mean and variance of X, defined in proposition 4.2.

Proof. It follows from the property (2.23) of iterated expectations that $c^*(t, T_1, T, S_t)$ is given by

$$c^*(t, T_1, T, S_t) = E^Q[c(t, T_1, T, X_t, S_t) \mid \mathcal{F}_t^S]. \tag{2.36}$$

Since we have already calculated the conditional distribution of $X \mid \mathcal{F}_t^S$, we can use this result to write (2.36) as

$$E^Q[c(t, T_1, T, X_t, S_t) \mid \mathcal{F}_t^S] = \int_0^\infty e^{-r(T_1-t)} \left(S_t e^{\mu_Y^x + \frac{\sigma_Y^2}{2}} N(d_1^x) - K N(d_2^x) \right) f_X(x) dx,$$

where $f_X(x)$ is the conditional density function of $X \mid \mathcal{F}_t^S$, which we found in section 4 to be $N(\pi_t[x], G_t[x])$. Since S_t is observable at time t, and σ_Y does not depend on x, we can move them outside the integral. To stress that μ_Y, d_1 and d_2 depend on x, we write μ_Y^x, d_1^x and d_2^x. ■

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3

A Note on the SABR Model

3.1 Introduction

One of the most heavily exploited shortcomings of the Black-Scholes model is the assumption of a constant volatility. Since there is a one-to-one correspondence between the price and volatility of a European call option, a way to test this assumption would be to plot the volatilities implied by the Black-Scholes model against strike prices of the underlying options. If the assumption of a constant volatility were correct, the result would be a straight line graph. However, prior to the 1987 stock market crash, the resulting pattern was a symmetric "smile," with the lowest volatility corresponding to the at-the-money strike price. After the crash, the heavy right tail became far less pronounced, turning the smile into a "smirk". This phenomenon is often attributed to "crashophobia", which sees investors assigning a higher probability to very low equity prices than to very high. Several attempts have been made to deal with the fact that volatility seems to vary systematically with strike price and the price of the underlying asset. In the early 1990s, local volatility models (LV models) pioneered by Derman [4] and Dupire [6] appeared. Instead of using a constant volatility, these models introduced a volatility function with a general Markovian structure, which made it possible to fit the volatility pattern observed in the market. However, out-of-sample tests were never done, and thus it was not clear what sort of dynamics these models implied for the volatility. In fact, it was shown in later studies [5] that although LV models can be made to fit data perfectly *in sample*, they can perform worse than a simple constant BS model *out of sample.* Perhaps a more appealing framework is that of the *stochastic volatility models* (SV models); see e.g., [1], [8], [9]. In these models, the volatility is itself a stochastic process, often normally or log-normally distributed. One drawback of this approach is that it introduces an additional source of randomness, and hence this framework no longer makes it possible to hedge a typical derivative using only the underlying asset and the risk free-asset. In other words, the market is no longer complete. In spite of this, these models have become very popular, and new models appear frequently. One of the latest contributions to the field of SV models is the *stochastic* $\alpha\beta\rho$ model - also known as the *SABR* model [7] - a widely used model in the financial industry. The main purpose of this paper is to test the model empirically to see what implications it may have.

3.2 The SABR Model

The SABR model is expressed in terms of the futures price of an underlying asset. Normally, the futures price is denoted by $F(t,T)$, where the first argument, t, refers to the current time, and the second argument, T, refers to the maturity of the contract. In the SABR model, however, the maturity of the contract is fixed, so the second argument is dropped. The model is specified as follows:

$$\begin{aligned} dF(t) &= \alpha(t)F(t)^{\beta}dW_1(t) \\ d\alpha(t) &= \nu\alpha(t)dW_2(t) \\ dW_1(t)dW_2(t) &= \rho dt. \\ \alpha(0) &= \alpha_0 \\ F(0) &= f. \end{aligned} \tag{3.1}$$

Here

$$\alpha_0, \beta, \nu \in R_+,\ \rho \in [-1,1],$$

and f is the futures price quoted today.

It is clear from the specification that we could never obtain, even for β equal to 0 or 1, a closed-form analytical solution for European call option prices within the model. Therefore a substantial contribution is made in [7], where the authors provide an approximation formula for that price. By using singular pertubation techniques, they manage to express the value of a European call option with exercise price K, exercise date T_0 and settlement date T_1 in terms of the Black-76 formula [3]

$$c = e^{-r(T_1-t)}\left(fN(d_1) - KN(d_2)\right),$$

where

$$d_{1,2} = \frac{\ln(\frac{f}{K}) \pm \frac{1}{2}\sigma_B^2(T_0-t)}{\sigma_B\sqrt{T_0-t}}$$

and the implied volatility is given by

$$\sigma_B = \frac{\alpha_0}{(fK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\ln^2(\frac{f}{K}) + \frac{(1-\beta)^4}{1920}\ln^4(\frac{f}{K}) + ...\right\}}\frac{z}{x(z)}$$
$$\cdot\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha_0}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]T_0 + ...\right\}. \tag{3.2}$$

Here

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\ln(\frac{f}{K}),$$

and $x(z)$ is defined by

$$x(z) = \ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}.$$

The above expansion can be carried out up to arbitrary order, but the authors of [7] claim that this is unnecessary, since the next error terms in the expansion would be too small to make a difference to accurate pricing. One of the objectives of this paper is to test this claim.

When estimating the parameters, β needs to be treated separately since it affects the smile curve in a way similar to ρ (see [7]). Typically, β is therefore chosen from a priori views of the futures price distribution, with $\beta = 0$ corresponding to a normal distribution, $\beta = 1$ a lognormal distribution, and $\beta = \frac{1}{2}$ corresponding to a non-central χ^2-distribution. In this analysis, we consider the case $\beta = 1$.

3.3 Testing the Approximation by Monte Carlo Simulation

3.3.1 The Simulation

Before analyzing the model any further, we need to check if the expansion presented above provides a good approximation of the correct price of the option in question. Such a test was never performed in (3.2). We use Monte Carlo simulation to calculate the "correct" price of a European call option within the SABR model. By standard arbitrage arguments, the value at time t of a European call option with exercise price K, exercise date T_0 and settlement date T_1 is given by

$$c = e^{-r(T_1-t)}E^Q[(F(T_0)-K)^+ \mid \mathcal{F}_t]. \tag{3.3}$$

By (3.1),

$$\begin{pmatrix} F(t+\triangle t) \\ \alpha(t+\triangle t) \end{pmatrix} = \begin{pmatrix} F(t) \\ \alpha(t) \end{pmatrix} + \begin{pmatrix} \alpha(t)F(t) & 0 \\ 0 & v\alpha(t) \end{pmatrix} \begin{pmatrix} \triangle W^1(t) \\ \triangle W^2(t) \end{pmatrix} \tag{3.4}$$

where

$$\begin{pmatrix} \triangle W^1(t) \\ \triangle W^2(t) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \triangle t \right).$$

For simulation purposes, it is easier to work with uncorrelated Wiener processes, so we decompose the Wiener increments as

$$\begin{pmatrix} \triangle W^1(t) \\ \triangle W^2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \tag{3.5}$$

where

$$D = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

is the Cholevski factorization of $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and ϵ_1 and ϵ_2 are independent normally distributed variables according to

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \triangle t \right).$$

Now we can compute the expectation (3.3) by simulating two vectors ϵ_1 and ϵ_2 of independent $N(0,1)$-distributed variables, plugging them into (3.5) and inserting the result into (3.4). This gives us a vector of simulated values for $F(t+\triangle t)$. Repeating the procedure $\frac{T_0}{\triangle t}$ times gives an estimation of $F(T_0)$. The expectation (3.3) can now be calculated simply by taking the arithmetic average of the payoff. This is our candidate for the correct value of a European call option within the SABR model. To reduce the variance in the simulation procedure, we use the sample paths for ϵ_1 and ϵ_2 in four different versions; $[\epsilon_1, \epsilon_2]$, $[-\epsilon_1, \epsilon_2]$, $[\epsilon_1, -\epsilon_2]$,and $[-\epsilon_1, -\epsilon_2]$. In each simulation we draw 2.5 million random numbers, corresponding to 10 million different sample paths, and we carry out the simulation 25 times to obtain a reliable value of the standard deviation. We take $r = 0.05$, $t = 0$, $T_0 = T_1 = 1$, $f = 100$, $K = 100$, $\beta = 1$, $\nu = 0.7$, $\rho = -0.3$, and $\alpha(0) = 0.25$.

3.3.2 Results

The result of the simulation was an average option price of 10.0721, with a standard deviation of 0.0238.The price suggested by the expansion was 10.1673, i.e., a difference of 0.9%. It is clear that the expansion indeed overestimates the price, since it is four standard deviations away from the simulated price. Whether or not a 0.9% overestimation is acceptable, depends entirely on the application at hand. For exotic products where margins are high, it would probably not pose a problem, but for standard instruments where fractions of a basis point can matter, the conclusion might be different.

3.4 The Data

The exchange we consider for the analysis is the Spanish *Mercado Espanol de Futuros Financieros sobre Renta Variable* (MEFF-RV), which is one of the most important forward and options exchanges in Europe. The market is very liquid; in fact the IBEX-35 Plus futures contract on which we base this analysis was the most actively traded futures contract in the world during 1994-1995 [10]. The data consists of minute-by-minute bid and ask prices of European call options written on the IBEX-35 Plus futures contract, as well as the futures price of the index. We consider two periods: one "normal" period extending from February 24 to March 21, and one very convulsive period from October 22 to October 30. All derivatives in the first period expire on March 21 1997; in the second period they expire on November 21, 1997. For the calibration, we use the arithmetical average of the bid and ask price of the options. Each day consists of about 700 observations of the futures price; for each futures price, call and put option prices are quoted for at most 25 different strike prices.

3.5 Calibration and Parameter Stability Tests

Given the price, c, of a European call option, the *implied Black's volatility* is the volatility that inserted into the Black-76 formula [3] gives back the price, c. If we compute the implied volatility for a set of different strike prices, holding everything else fixed, the result is referred to as the *implied volatility curve*, or the *smile curve*. In the SABR model, the implied volatility is given by the expression (3.2). If we decide on values for the parameters, α_0, ν, and ρ, in (3.2), we can compare the implied volatility curve given by the SABR model, to the curve backed out from market data on traded call options. Calibrating the SABR model to market data means choosing α_0, ν, and ρ, in such a way that the two curves lie as close to each other as possible (e.g., by minimizing the sum of the mean squared errors (MSEs)). The number of available strike prices for each option varied between 6 and 12 in the data set, and so with only 3 parameters to adjust, we cannot hope for a perfect fit.

The first issue we address, is the stability of the parameter estimates over time. For this procedure, two methods were applied. First, the model was calibrated at the time when the market opened on February. 24 and

October. 22, respectively, and then the resulting MSE was computed. Next, we saw how the MSE changed over time when the model was not recalibrated (see Figures 2 and 3). For comparison, a constant Black-Scholes model was also calibrated. At the time of the first calibration, the MSE for the BS model was 48 times larger than the MSE for the SABR model, but letting time run without recalibrating, and computing the average MSE for the whole period (19 days), the quotient was down to 1.6. Even more remarkable, when applying the same procedure for the second period, the BS model started out with an MSE that was 183 times that of the SABR model, but ended up (after 7 days) with an average fit that was nearly 10 percent *better*. We conclude from this observation that, during periods of high and changing volatility, striving for a perfect fit to the current volatility curve requires very frequent recalibration to be worthwhile. In another stability test, we recalibrated the model three times a day to see how the parameter estimates changed over time (see Figure 1 and Figures.4-7). Again, a constant BS model was also calibrated to see how the average volatility changed. As expected, the estimates of α_0 in the SABR model followed very closely those of σ in the constant volatility model (see Figure 1). Perhaps somewhat surprisingly, the parameters ρ and ν are more stable during the convulsive period than they are during the normal period (see Figure 4 - Figure 7). One reason for this could be that longer price series were available during the convulsive period (on average 9.2 compared to 6.5), and thus changes in slope and correlation were less sensitive to changes in the level of volatility. This can be seen in Figure 8, where two curves are depicted; the solid line is the implied volatility on the morning of February 24, and the dashed line shows the curve a few hours later. With seven observations, the slope parameter (Vega), in particular, becomes very sensitive even to relatively small changes in volatility. Implied volatility has changed by less than one percent for the first four strike prices, changing the parameter estimate of Vega from 0.75 to 1.84. Figure 9 also shows how Vega is the least stable parameter in the SABR model. Here, the average volatility changes by less than half a percent from noon to evening, changing the parameter estimate of Vega from 2.23 to 2.90.

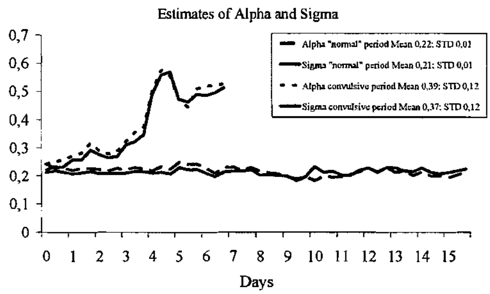

Figure 1

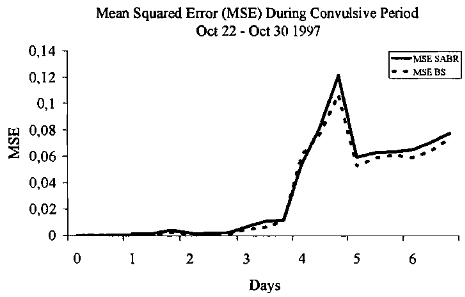

Figure 2

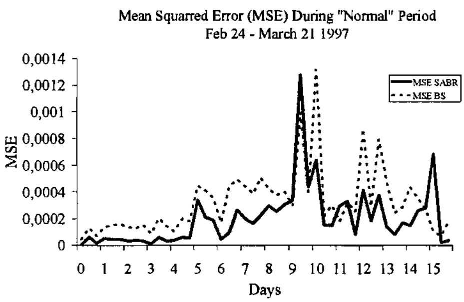

Figure 3

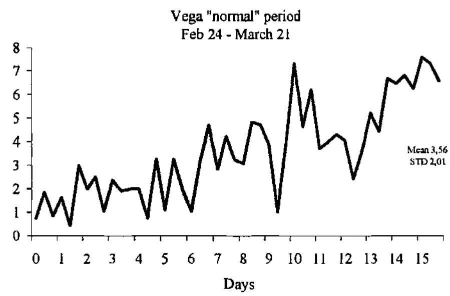

Figure 4

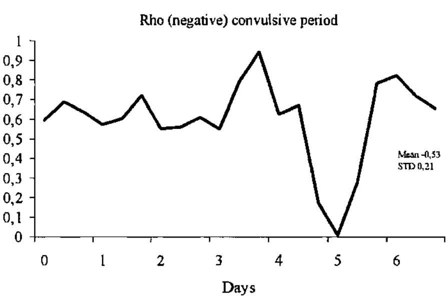

Figure 5

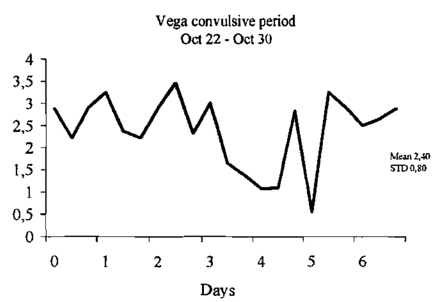

Figure 6

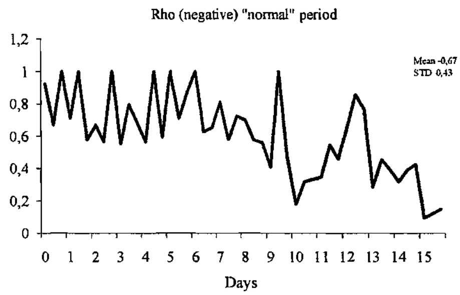

Figure 7

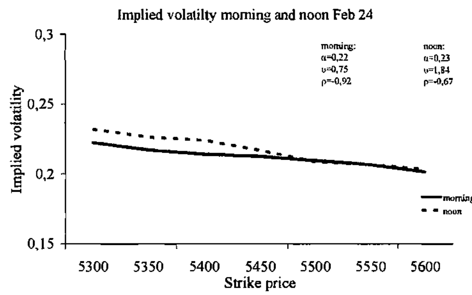

Figure 8

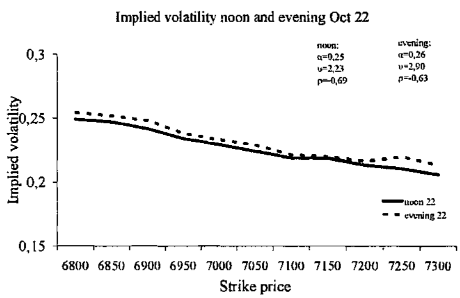

Figure 9

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4

Finite Dimensional Markovian Realizations of Futures Term Structure Models

4.1 Introduction

When studying the term structure of futures prices, there has traditionally been two main approaches in the literature; the state space approach and the Heath-Jarrow-Morton type approach. They can roughly be described as follows.

1. In the state space approach we start by modeling the dynamics of a finite dimensional Markovian state vector process Z. This is typically done in the form of a system of stochastic differential equations (henceforth SDEs) under a risk neutral martingale measure Q. The underlying asset price S is then specified as $S_t = h(Z_t)$ for some deterministic function h, and very often S is one of the components of Z in which case h is just a coordinate function. The futures price at t for delivery at T is then given by standard theory as

$$F(t,T) = E^Q \left[S_T | \mathcal{F}_t \right]. \tag{4.1}$$

   From the Markovian structure it follows that we can write $F(t,T) = H(t, Z_t, T)$ for some deterministic function H, and H can finally be determined as the solution of a parabolic partial differential equation. Examples of this approach can be found in [1], [5], [16], and [24].

2. In the HJM type approach, we do not have an exogenously given finite dimensional state vector process, Z. Instead, we view the entire term structure of futures prices (or equivalently the entire futures

[1]This paper is co-authored with Tomas Björk and Camilla Landén.

price curve) as the primitive object, and model all futures prices simultaneously. Observing that, because of (4.1), all futures prices are Q martingales and restricting ourselves to nonnegative underlying claims, we can model futures prices as

$$dF(t,T) = F(t,T)\sigma_F(t,T)dW_t, \tag{4.2}$$

where, for each fixed maturity T, the volatility $\sigma_F(t,T)$ is some exogenously given vector valued adapted process, W is Q-Wiener, and we use the observed term structure of today as the initial condition. This approach was introduced in [11] (referring to the unpublished paper [23]), and has also been used (with some variations) in [5] and [21].

These approaches have obvious counterparts in interest rate theory; the first corresponds to multifactor models (including short rate models), whereas the second corresponds to the HJM framework for modelling forward rates. As is the case in interest rate theory, a multifactor state space model will, in a trivial way, generate an HJM type model. In the other direction, however, there is, in general, no implication. The reason for this is that an HJM model of the form (4.2) is an *infinite dimensional* system of SDEs (one equation for each T), and it is obviously only for very particular choices of the volatility structure $\sigma_F(t,T)$ that such an infinite system can be represented by a state space model. The purpose of the present paper is precisely to investigate under what conditions on the volatility structure $\sigma_F(t,T)$ the inherently infinite dimensional model (4.2) can be represented in terms of a finite dimensional state space model. In such a case we say that the model (4.2) admits a Finite Dimensional Realization (henceforth FDR). The corresponding FDR problem for interest rate models was more or less completely solved by geometric arguments in a series of papers [4], [6], [7], [14], and the main technical tool is the Lie algebra theory developed in [6]. For an overview of the theory see [3]. In the present paper we adapt the geometric theory developed in the papers above to solve the FDR problem for futures prices. The structure and results of the paper are as follows:

- In Section 2 we present the probabilistic setup and formulate the precise problem under study.
- In Section 3 we give a very general necessary and sufficient condition for the existence of an FDR. By specializing to the cases of deterministic volatility structures and "deterministic direction" volatility structures we obtain more concrete results.
- Section 4 is devoted to a brief discussion on so called invariant manifolds, which provide a detailed description of the possible shapes of the futures price curve, that can be produced by a given model.

- In Section 5 we provide an algorithm which, given the *existence* of an FDR, will actually *construct* a concrete state space realization.
- Time varying models (which thus allow for seasonality effects) are treated in Section 6. As an illustration of the theory we study, in some detail, a concrete model for gas futures.
- We finish the main part of the paper in Section 7 by clarifying exactly under what conditions a given HJM-type model admits a realization where the spot price acts as the single underlying factor.
- In Appendix A, we have collected, for the convenience of the reader, the basic definitions and results from differential geometry that are needed for the present paper.

4.2 Basics

We consider a financial market living on a given filtered probability space $\{\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\}_{t\geq 0}\}$ carrying an m-dimensional Wiener process, W. The main assets to be considered are futures contracts written on a given underlying asset. Let $F(t,T)$ denote the futures price at time t of a futures contract with delivery date T. A simple arbitrage argument then yields the following relationship between the futures prices and the induced spot price, S,

$$S(t) = F(t,t). \tag{4.3}$$

We assume that the market is arbitrage free in the sense that the probability measure Q is a risk-neutral martingale measure. From standard theory (see e.g. [2]) we have the following well-known result:

$$F(t,T) = E^Q[S(T)|\mathcal{F}_t]. \tag{4.4}$$

From equation (4.4) it is clear that for every fixed T the futures price process is a Q-martingale. Thus, considering only nonnegative claims, we may assume that the futures prices to have dynamics of the following form under Q

$$dF(t,T) = F(t,T)\sigma_F(t,T)dW_t, \tag{4.5}$$

where for each T, the volatility $\sigma_F(t,T)$ is an exogenously given adapted m-dimensional row vector process. The main purpose of this paper is to characterize the volatilities σ_F for which the solution of the infinite dimensional SDE (4.5), i.e. the SDE for the futures prices, possesses a finite dimensional realization (FDR). However, before giving the precise definition of an FDR of the futures prices model (4.5), it is necessary to rewrite (4.5) on a form which is more convenient for our purposes, and we start by reparameterizing it. We thus choose to parameterize the futures prices in terms of t and x, where x denotes time **to** maturity in contrast to T, which denotes time **of** maturity (compare with the Musiela parameterization of the forward rates, [9] and [22]). Therefore, let $F_0(t,x)$ be defined by

$$F_0(t,x) = F(t,t+x). \tag{4.6}$$

It is then relatively easy to see that the process F_0 will have the following induced dynamics

$$dF_0(t,x) = \frac{\partial}{\partial x}F_0(t,x)dt + F_0(t,x)\sigma_0(t,x)dW_t, \tag{4.7}$$

where

$$\sigma_0(t,x) = \sigma_F(t,t+x). \tag{4.8}$$

It turns out that the analysis is considerably simplified if, instead of working with the process F_0 defined in (4.6), we work with the process q where q is defined by

$$q(t,x) = \ln F_0(t,x). \tag{4.9}$$

Using the Itô formula on (4.5) or (4.7), we obtain the SDE for q as

$$dq(t,x) = \left\{\frac{\partial}{\partial x}q(t,x) - \frac{1}{2}\|\sigma_0(t,x)\|^2\right\}dt + \sigma_0(t,x)dW_t. \tag{4.10}$$

There are two ways in which we can view the system 4.10:

- We can view (4.10) as a coupled system of infinitely many scalar SDEs (one for each fixed x).
- Alternatively we can view (4.10) as a *single* SDE, describing the dynamics of an *infinite dimensional* object. The infinite dimensional object is of course the futures price curve $x \longmapsto q_t(x)$.

The second interpretation turns out to much more fruitful for our purposes, so from now on we will denote the entire futures price curve at time t, i.e. the curve $x \longmapsto q_t(x)$, by the symbol q_t, and we can thus view q_t as a point or vector in an infinite dimensional function space $\mathcal{H}$. It then remains to specify a suitable function space $\mathcal{H}$, and it turns out that it can in fact be chosen as a Hilbert space. See [6] or [14] for details. In order to have a Markovian structure (albeit infinite dimensional) we make the following assumption.

Assumption 2.1 *We assume that the futures price volatility process is of the form*

$$\sigma_0(t,x) = \sigma(q_t,x), \tag{4.11}$$

where $\sigma : H \times R_+ \longrightarrow R^m$ is an exogenously given mapping.

Under this assumption, each component σ_i (for $i = 1, \ldots, m$) is a mapping $\sigma_i : \mathcal{H} \times R_+ \to R$, i.e. a point $q \in \mathcal{H}$ and a nonnegative real number $x \in R_+$ will be mapped into the real number $\sigma_i(q, x)$. We may however also, and more profitably, view σ_i as a mapping between function spaces. More precisely: a point $q \in \mathcal{H}$ is mapped to the function $x \longmapsto \sigma_i(q, x)$. We will in fact assume that this function is a member of $\mathcal{H}$.

Assumption 2.2... *We assume that the futures price volatility mapping is of the form $\sigma : H \to H^m$, where each component σ_i is a smooth vector field on H.*

We can now write the futures price equation (4.10) more compactly as

$$dq_t = \left\{ \mathbf{A}q_t - \frac{1}{2}D(q_t) \right\} dt + \sigma(q_t)dW_t \tag{4.12}$$

where the vector fields $\mathbf{A}$ and d are given by

$$\mathbf{A} = \frac{\partial}{\partial x}, \tag{4.13}$$

$$D(q) = \|\sigma(q)\|^2_{R^m} = \sum_{i=1}^{m} \sigma_i^2(q). \tag{4.14}$$

Finally, because of the geometrical ideas behind the results we will use, we need the Stratonovich form of the SDE (4.12). This is given by

$$dq_t = \left\{ \mathbf{A}q_t - \frac{1}{2}D(q_t) - \frac{1}{2}\sigma_q'(q)\sigma(q) \right\} dt + \sigma(q_t) \circ dW_t \tag{4.15}$$

where $\sigma_q'(q)\sigma(q)$ denotes the Frechet derivative $\sigma_q'(q)$ operating on $\sigma(q)$.We can now specify exactly what we mean by a finite dimensional realization of (the logarithm of) the futures prices generated by a volatility σ. To this end fix a volatility $\sigma : \mathcal{H} \times R_+ \to R^m$. We then have the following definition.

Definition 2.1 *We say that the SDE (4.15) has a generic (local) d-dimensional realization at a given point $q^0 \in H$ if, for each initial point q_0 near q^0, there exists a point $z_0 \in R^d$, smooth vector fields $a, b_1, \ldots, b_m$ on some open subset Z of R^d and a smooth (submanifold) map $G : Z \to H$, such that q has the local representation*

$$q_t = G(Z_t), \quad a.s. \tag{4.16}$$

where Z is the strong solution of the d-dimensional Stratonovich SDE

$$\begin{cases} dZ_t &= a(Z_t)dt + b(Z_t) \circ dW_t, \\ Z_0 &= z_0. \end{cases} \tag{4.17}$$

and where the driving Wiener process W in (4.17) is the same as in (4.15). The term "local" above means that the representation is assumed to hold for all t with $0 \le t \le \tau(q^0)$, a.s. where, for each $q^0 \in H$, $\tau(q^0)$ is a strictly positive stopping time.

In slightly more pedestrian terms we can view the "output map" G above as a mapping $G : \mathcal{H} \times R_+ \to R$, so (4.16) can also be written as

$$q_t(x) = G(Z_t, x).$$

Our main problems are now as follows.

- What are the necessary and sufficient conditions on the volatility structure σ for the existence of a generic finite dimensional realizations?
- Suppose you know, from abstract theory, that an FDR exists, how do you construct a concrete realization?

These problems will be solved in the next sections.

Remark 2.1 *We have, for convenience, chosen to model the futures prices in terms of the log price q and the corresponding volatility $\sigma(q, x)$. We could of course also have chosen to model $F_0(t, x)$ and a corresponding volatility $\sigma_0(F_0, x)$. The volatilities correspond as follows*

$$\sigma_0(F_0, x) = \sigma(e^q, x), \tag{4.18}$$

It is important to note that it may well happen that the SDE generated by

$\{\mu, \sigma\}$ has a finite dimensional realization for a particular choice of initial point q^0, whereas no finite dimensional realization exists for points close to q^0. We say that such a system has a non-generic or "accidental" finite dimensional realization at q^0. If, on the other hand, the system has a finite dimensional realization for all points in a neighborhood of q^0 we say that the system has a generic finite dimensional realization at r^o. The existence of a non-generic realization is of course of very limited value, since the situation is structurally unstable. We note that our Lie algebraic result above guarantees the existence of a generic finite dimensional realization. In the sequel we will thus suppress the adjective "generic".

4.3 Conditions For the Existence of a Finite Dimensional Realization

In this section we give necessary and sufficient conditions for the term structure of futures prices to possess a finite dimensional realization. As it turns out, our present problem falls perfectly within the abstract framework developed in [6]. We need a simple nondegeneracy condition.

Assumption 3.1 *Define the futures price drift vector field* μ *by*

$$\mu(q) = \mathbf{A}q_t - \frac{1}{2}D(q_t) - \frac{1}{2}\sigma_q'(q)\sigma(q), \tag{4.19}$$

with A *and* D *as above. We assume that the dimension of the Lie algebra* $\{\mu, \sigma_1, \ldots, \sigma_m\}_{LA}$ *is constant near the initial curve* q^0.

We now have the following fundamental result.

Theorem 3.1 *Take as given the volatility mapping* $\sigma = (\sigma_1, \ldots, \sigma_m)$ *as well as an initial curve* $q^0 \in H$. *Then the futures price model generated by* σ *generically admits a finite dimensional realization at* q^0, *if and only if*

$$dim\{\mu, \sigma_1, \ldots, \sigma_m\}_{LA} < \infty$$

in a neighborhood of q^0, *where* μ *is given by equation (4.19).*

Proof. Follows directly from Theorem 3.2. in [6]. ∎

This result solves completely, but in very abstract terms, the existence problem for finite dimensional realizations. It is, however, not at all clear what the Lie algebra conditions mean in concrete terms for the structure of σ. To obtain more concrete results we will therefore, in the next two sections, apply the abstract theorem to the special cases when the volatility is deterministic and when it has a constant direction. The arguments and results in these sections are very similar to those in [6].

4.3.1 Deterministic Volatility

We start by considering the simplest case possible, i.e. when the volatility σ is deterministic and we can write

$$\sigma(q, x) = \sigma(x). \tag{4.20}$$

In geometric terms this means that each vector field σ_i is a constant vector field. As mentioned above, the realization problems that we study for futures price models have earlier been studied for forward interest rate models in [6]. In the present paper, the basic equation under study is the forward price equation (4.15) study, whereas the focus in [6] is on the HJM forward rate equation (with Musiela parameterization and on Stratonovich form). The main difference between the two settings lies in the drift μ. In the HJM forward rate framework the drift is given by

$$\mu(r) = \frac{\partial}{\partial x}r + \sigma(r)\mathbf{H}\sigma(r)^* - \frac{1}{2}\sigma_r'(r)[\sigma(r)].$$

Here r denotes a generic forward rate curve $x \longmapsto r(x)$, $\sigma(r,x)$ denotes the forward rate volatility, $*$ denotes transpose, and $\mathbf{H}\sigma$ is defined by

$$\mathbf{H}\sigma(r,x) = \int_0^x \sigma(r,s)ds.$$

Thus the main difference between the forward rate equation and the forward price equation is that the drift term $\sigma(r)\mathbf{H}\sigma(r)^*$, which is related to the HJM drift condition for forward rates, is replaced by the term $D(q)$ in the futures price equation. As a consequence, the arguments in the present paper are often more or less parallel to the corresponding arguments in [6]. In order to make the present paper self contained, we will however often give full arguments rather than referring the reader to parallel arguments in [6].

We may now state the main result for deterministic futures price volatilities.

Proposition 3.1 *Assume that the volatility structure is of the form (4.20). Then the model possesses a finite dimensional realization if and only if the function space*

$$span\{\frac{\partial^k \sigma_i}{\partial x^k};: i = 1, \ldots, m;: k = 0, 1, \ldots\}$$

is finite dimensional.

Proof. For simplicity of notation we start by considering only the case with a scalar Wiener process so, referring to Theorem 3.1, we compute the Lie algebra $\mathcal{L}(\mu,\sigma)$. Since the vector field σ is constant we have $\sigma' = 0$. Thus there is now Stratonovich correction term, so from (15) we have

$$\mu(r) = \mathbf{A}q - \frac{1}{2}D,$$

where, as before, $\mathbf{A} = \frac{\partial}{\partial x}$.and the vector field D is defined by $D(q,x) = \sigma^2(x)$ is a *constant* vector field. The Frechet derivatives are trivial in this case. Since $\mathbf{A}$ is linear and D is constant, we obtain

$$\mu' = \mathbf{A}$$
$$\sigma' = 0.$$

Thus the Lie bracket $[\mu, \sigma]$ is given by

$$[\mu, \sigma] = \mathbf{A}\sigma,$$

and in the same way, we have

$$[\mu, [\mu, \sigma]] = \mathbf{A}^2\sigma.$$

Continuing in the same way, it is easily seen that the relevant Lie algebra $\mathcal{L}$ is given by

$$\mathcal{L} = \{\mu, \sigma\}_{LA} = span\{\mu, \sigma, \mathbf{A}\sigma, \mathbf{A}^2\sigma, ...\}$$

It is thus clear that $\mathcal{L}$ is finite dimensional (at each point q) if, and only if, the function space

$$\mathcal{L} = \{\mu, \sigma\}_{LA} = span\{\mathbf{A}^n\sigma; n = 0, 1, 2, ...\}$$

is finite dimensional. Recalling that $\mathbf{A} = \frac{\partial}{\partial x}$ finishes the proof, and the argument easily carries over to the case of several driving Wiener processes. ∎

This result is stated in somewhat abstract terms, but we can easily obtain a much more concrete formulation. To do this we need the concept of a quasi exponential function.

Definition 3.1 *A scalar real valued function of a real variable x is called quasi exponential if it has the form*

$$f(x) = \sum_i e^{\lambda_i x} + \sum_j e^{\alpha_j x}\left[p_j(x)\cos(\omega_j x) + q_j(x)\sin(\omega_j x)\right], \qquad (4.21)$$

where $\lambda_i, \alpha_j, \omega_j$ are real numbers, whereas p_j and q_j are real polynomials.

We recall the following well known facts about quasi exponential functions.

Lemma 3.1 *The following hold for the quasi-exponential functions.*

- *A function is QE if and only if it is a component of the solution of a vector valued linear ODE (ordinary differential equation) with constant coefficients.*
- *A function is QE if and only if it can be written as $f(x) = ce^{Ax}b$.*
- *If f is QE, then f' is QE.*
- *If f is QE, then its primitive function is QE.*
- *If f and g are QE, then fg is QE. In particular, if f is QE, then so is f^2.*

We now have the following very explicit formulation of Proposition 3.1.

Corollary 3.1 *In the case of deterministic volatilities, the futures price equation possesses an FDR if and only if each component $\sigma_i(x)$ for $i = 1, \ldots, m$ is a quasi exponential function.*

Proof. From Proposition3.1 we know that the existence of an FDR is equivalent to the condition that

$$span\{\frac{\partial^k \sigma_i}{\partial x^k};: i = 1, \ldots, m;: k = 0, 1, \ldots\}$$

is finite dimensional. This condition, on the other hand, is equivalent to to existence of linear relations between the various components of σ and their derivatives. This, in turn, is equivalent to saying that each σ_i is a component of the solution of a vector valued linear ODE with constant coefficients, and thus σ_i is quasi exponential. ∎

Remark 3.1. *Since the volatility $\sigma(q, x)$ does not depend upon q, we have from remark 2.1 that $\sigma(x) = \sigma_0(x)$, and thus the conditions on σ in Proposition 3.1 apply also to σ_0.*

4.3.2 Constant Direction Volatility

The next simplest volatility structure you can consider is a volatility, where each component is of the form of the form

$$\sigma_i(q, x) = \varphi_i(q)\lambda_i(x). \tag{4.22}$$

Here λ_i is a constant vector field (it does not depend on q) whereas φ_i is a smooth scalar field, i.e. a mapping $\varphi : \mathcal{H} \to R$. As a vector field we thus see that σ_i has "constant direction" (namely the direction of λ_i) but that the length of σ_i is modulated by the scalar field φ_i.

For the case when there is only a scalar driving Wiener process we obtain the following proposition, where we use the notation

$$\Phi(q) = \varphi^2(q), \tag{4.23}$$

and where $\Phi''(q)[\lambda; \lambda]$ denotes the second order Frechet derivative of Φ evaluated at the point q, operating on the pair $(\lambda, \lambda) \in \mathcal{H}$. Notice the difference between the pair $[\lambda; \lambda]$ with a semicolon), and the Lie bracket $[\lambda, \lambda]$ with a comma.

Proposition 3.2 *Assume that the Wiener process is scalar, that $\varphi(q) \neq 0$ for all $q \in H$, and that $\Phi''(q)[\lambda; \lambda] \neq 0$ for all $q \in H$. Then the futures* *price model has a finite dimensional realization if and only if λ is a quasi-exponential function, whereas φ is allowed to be any smooth field.*

Proof. In this case the drift vector field μ is given by

$$\mu(q) = \mathbf{F}r - \frac{1}{2}\varphi^2(q)L - \frac{1}{2}\varphi'(q)[\lambda]\varphi(q)\lambda, \tag{4.24}$$

where $\varphi'(q)[\lambda]$ denotes the Frechet derivative of φ evaluated at q and acting on the vector λ. The constant vector field $L \in \mathcal{H}$ is given by

$$L(x) = \lambda^2(x).$$

We now want to prove that the Lie algebra generated by

$$\begin{aligned}
\mu(q) &= \mathbf{F}q - \frac{1}{2}\varphi^2(q)L - \frac{1}{2}\varphi'(q)[\lambda]\varphi(q)\lambda,\\
\sigma(q) &= \varphi(q)\lambda,
\end{aligned}$$

is finite dimensional. Under the assumption that $\varphi(q) \neq 0$ for all q, we can use Lemma A.1, to see that the Lie algebra is in fact generated by the simpler system of vector fields

$$\begin{aligned}
f_0(q) &= \mathbf{F}q - \frac{1}{2}\Phi(q)L,\\
f_1(q) &= \lambda,
\end{aligned}$$

where we have used the notation

$$\Phi(q) = \varphi^2(q).$$

Since the field f_1 is constant, it has zero Frechet derivative, so the first Lie bracket is easily computed as

$$[f_0, f_1](q) = \mathbf{F}\lambda + \Phi'(q)[\lambda]L.$$

The bracket $[[f_0, f_1], f_1]$ is easily obtained as

$$[[f_0, f_1], f_1](q) = \Phi''(q)[\lambda; \lambda]L.$$

We may again use Lemma A.1 to see that the Lie algebra is generated by the following vector fields

$$\begin{aligned}
f_0(q) &= \mathbf{F}q,\\
f_1(q) &= \lambda,\\
f_3(q) &= \mathbf{F}\lambda,\\
f_4(q) &= L.
\end{aligned}$$

Of these vector fields, all but f_0 are constant, so all brackets are easy. After elementary calculations we see that in fact

$$\{\mu, \sigma\}_{LA} = span\,\{\mathbf{F}, \mathbf{F}^n\lambda,\ \mathbf{F}^n D;\ n = 0, 1, \ldots\}.$$

From this expression it follows immediately that a necessary condition for the Lie algebra to be finite dimensional is that the vector space spanned by $\{\mathbf{F}^n\lambda;\ n \geq 0\}$ is finite dimensional. This occurs if and only if λ is quasi-exponential If, on the other hand, λ is quasi-exponential, then we know from Lemma 3.1, that also $L = \lambda^2$ is quasi-exponential. Thus the space $\{\mathbf{F}^n L;\ n = 0, 1, \ldots\}$ is also finite dimensional, and we are finished. ∎

We may also restate this result in terms of $F_0(t,x)$ and a volatility structure of the form $\sigma_0(F_0, x) = \varphi_0(F_0)\lambda(x)$ where λ is the same constant field as in (4.22), and

$$\varphi_0(F_0) = \varphi(\ln F_0).$$

Proposition 3.2 can now be rewritten in these new terms. If we let

$$\Phi_0(F_0) = \varphi_0^2(F_0),$$

we have the following proposition.

Proposition 3.3 *Assume that $\varphi_0(F_0) \neq 0$ for all $F_0 \in H$, and that the condition $\Phi_0''(F_0)[\lambda;\lambda] \neq 0$ is satisfied for all $F_0 \in H$. Then the futures price model has a finite dimensional realization if and only if λ is a quasi-exponential function, whereas φ_0 is allowed to be any smooth field.*

All this was done for the special case of a *scalar* Wiener process. For the general case when there are m driving Wiener process, and each volatility is a constant direction volatility, i.e.

$$\sigma_i(q, x) = \varphi_i(q)\lambda_i(x), \qquad 1, \ldots, m, \tag{4.25}$$

we are unable to obtain necessary *and* sufficient conditions (apart, of course, from the abstract one in Theorem 3.1. We can however easily obtain the following *sufficient* conditions for the existence of a finite dimensional realization.

Proposition 3.4 *Under the assumption that $\varphi_i(q) \neq 0$ for all $q \in H$ and $i = 1, \ldots, m$, a sufficient condition for the volatility structure (4.25) to have a finite dimensional realization is that all the functions λ_i $i = 1, \ldots, m$ are quasi-exponential functions. The vector fields $\varphi_1, \ldots, \varphi_m$ are allowed to be any smooth fields.*

Proof. In this case the drift vector field is given by

$$\mu(q) = Ar - \frac{1}{2}\sum_{i=1}^{m}\varphi_i^2(q)L_i - \frac{1}{2}\sum_{i=1}^{m}\varphi_i'(q)[\lambda_i]\varphi_i(q)\lambda_i,$$

where

$$L_i(x) = \lambda_i^2(x), ...i = 1, ..., m.$$

We thus have to study the Lie algebra generated by the vector fields

$$\mu(r) = Ar - \frac{1}{2}\sum_{i=1}^{m} \varphi_i^2(r)L_i - \frac{1}{2}\sum_{i=1}^{m} \varphi_i'(r)[\lambda_i]\varphi_i(r)\lambda_i,$$

$$\sigma_1(r) = \varphi_i(r)\lambda_1,$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\sigma_m(r) = \varphi_m(r)\lambda_m.$$

Under the assumptions of the proposition, we can perform Gaussian elimination to see that the Lie algebra is in fact generated by the much simpler system of vector fields

$$f_0(q) = \mathbf{A}q - \frac{1}{2}\sum_{i=1}^{m} \Phi_i(q)L_i,$$

$$f_1(q) = \lambda_1$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$f_m(q) = \lambda_m,$$

where we have used the notation

$$\Phi_i(q) = \varphi_i^2(q).$$

This clearly shows that the Lie algebra $\{\mu, \sigma\}_{LA}$ is in fact included in the algebra $\{\mathbf{A}q, L_1, ..., L_m, \lambda_1, ...\lambda_m\}_{LA}$. For this Lie algebra, we have (see the proof of Proposition 3.1) the trivial relation

$$\{\mathbf{A}q, L_1, ..., L_m, \lambda_1, ...\lambda_m\}_{LA} = span\{Aq, A^n L_i, A^n \lambda_i; i = 1, ..., m; \quad n \geqslant 0\}.$$

Finally, we may argue as in the proof of Proposition 3.2 to see that

$$span\{Aq, A^n L_i, A^n \lambda_i; i = 1, ..., m; \quad n \geqslant 0\}$$

is finite dimensional if $\lambda_1, ...\lambda_m$ are quasi exponential. ∎

Remark 3.2 *In terms of modelling F_0 and σ_0 this means that if we consider a volatility structure of the form*

$$\sigma_{0i}(F_0, x) = \varphi_{0i}(F_0)\lambda_i(x), \quad 1, \ldots, m,$$

Then, under the nondegeneracy condition $\Phi_0''(F_0)[\lambda;\lambda] \neq 0$, we see that a sufficient condition for the existence of an FDR is that $\lambda_1, \dots, \lambda_M$ are quasi exponential.

4.4 Invariant Manifolds

Consider a general SDE

$$\begin{cases} dq_t &= \mu(q_t)dt + \sigma(q_t) \circ dW_t, \\ q_0 &= q^0. \end{cases} \tag{4.26}$$

on the space $\mathcal{H}$.

A key concept when dealing with FDRs is the idea of an invariant manifold.

Definition 4.1 *A submanifold G in H is said to be (locally) invariant under the action of the SDE (4.26), if for every choice of $q^0 \in G$ we have $q_t \in G$ for $0 \leq t \leq \tau(q^0)$, where τ is a strictly positive stopping time.*

To understand this concept, suppose for example that the SDE under study is the futures price equation (4.15) with an initial point q^0, and suppose for simplicity that we can (which is often the case) disregard the prefix "local" in the definition. Then, in more pedestrian terms, the invariant manifold $\mathcal{G}$ is a finite dimensional family of (potential) futures price curves, and in fact all futures price curves that will ever be produced by your model will in fact belong to $\mathcal{G}$.

From a more theoretical perspective, the importance of the invariant manifolds stems from the following central result which was proved in [4].

Theorem 4.1 (Björk and Christensen) *Consider the SDE (4.26). There will exist an FDR for this equation if and only if, for each fixed initial point q^0, there exists a finite dimensional invariant manifold (typically depending upon the choice of q^0) which contains the point q^0.*

For any given model admitting an FDR, and for a given initial point q^0, it is of great importance to understand what the invariant manifold looks like.

In [6] a very concrete parameterization of the invariant manifold of an SDE is given, and we need the following definition to describe the parameterization.

Definition 4.2 *Let f be a smooth vector field on H, and let y be a fixed point in H. Consider the ODE*

$$\begin{cases} \dfrac{dy_t}{dt} &= f(y_t), \\ y_0 &= y. \end{cases}$$

We denote the solution y_t as $y_t = e^{ft}y$.

The invariant manifold can now be computed using the following construction from [6].

Theorem 4.2 (Björk and Svensson) *Assume that the Lie algebra $\{\mu, \sigma\}_{LA}$ is spanned by the smooth vector fields $f_1, \ldots, f_d$. Then, for the initial point q^0, all curves produced by the model will belong to the induced tangential manifold G, which can be parameterized as $G = Im[G]$, where*

$$G(z_1, \ldots, z_d) = e^{f_d z_d} \ldots e^{f_1 z_1} q^0,$$

and where the operator $e^{f_i z_i}$ is given in Definition 4.2.

4.4.1 An Example

As a simple example of an invariant manifold let us find the invariant manifold of the SDE (4.15) with deterministic volatility given by

$$\sigma(x) = \sigma_0 e^{-cx}, \tag{4.27}$$

where c and σ_0 are scalar constants There is thus only one driving Wiener process. This volatility is obviously quasi exponential so the condition in Corollary 3.1 is therefore satisfied. Thus there exists an FDR and from Theorem 4.1 we see that there exists an invariant manifold. We now go on to compute the invariant manifold, and to this end we note that if we let λ denote the constant vector field defined by $\lambda(x) = e^{-cx}$, we easily obtain

$$\{\mu, \sigma\}_{LA} = span\{\mu, \lambda\}.$$

Next we have to compute the operators $\exp\{\mu t\}$ and $\exp\{\lambda t\}$. Since for this model the Frechet derivative with respect to q of the volatility is zero, i.e. $\sigma'_q(q, x) = 0$, we obtain the following expression for μ from (4.19)

$$\mu(q, x) = \frac{\partial q}{\partial x}(x) - \frac{1}{2}\sigma^2(x).$$

Define the constant field h by $h(x) = \sigma^2(x)$. Then $\exp\{\mu t\}$ is obtained as the solution to

$$\frac{dq}{dt} = \mathbf{A}q - \frac{1}{2}h,$$

where $\mathbf{A} = \frac{\partial}{\partial x}$. The solution to this linear equation is

$$q_t = e^{\mathbf{A}t}q_0 - \frac{1}{2}\int_0^t e^{\mathbf{A}(t-s)}h ds,$$

where the operator $e^{\mathbf{A}t}$ is left translation (see e.g. [3]), i.e. for any continuous real valued function f we have

$$e^{\mathbf{A}t}f(x) = f(x+t).$$

Thus we have that

$$(e^{\mu t}q_0)(x) = q_0(x+t) - \frac{1}{2}H(t,x),$$

where

$$\begin{aligned} H(t,x) &= \int_0^t e^{\mathbf{A}(t-s)}h(x)ds = \sigma_0^2 \int_0^t \lambda(x+t-s)ds \\ &= \sigma_0^2 \int_0^t e^{-2c(x+t-s)}ds = \frac{\sigma_0^2 e^{-2cx}}{2c}(1-e^{-2ct}). \end{aligned}$$

Since the vector field λ is constant, the corresponding ODE is trivial, and we have

$$(e^{\lambda t}q_0) = q_0 + \lambda t.$$

The parameterization of the invariant manifold generated by the initial curve q_0 is therefore given as

$$G(z_0,z_1)(x) = q_0(x+z_0) - \frac{\sigma_0^2 e^{-2cx}}{4c}(1-e^{-2cz_0}) + e^{-cx}z_1. \tag{4.28}$$

4.5 Construction of Finite Dimensional Realizations

In this section we describe a method for how to construct a concrete finite dimensional realization of the futures prices, when such a realization exists. As before we will actually be looking at the logarithm of the futures prices, but the results can of course be translated to the futures prices themselves. We basically follow the methodology in [7].

4.5.1 The Construction Algorithm

Consider a volatility $\sigma : \mathcal{H} \times R_+ \to R^m$ for which $\{\mu,\sigma\}_{LA} < \infty$, that is consider a volatility such that the futures prices generated by this volatility can generically be realized by means of a finite dimensional SDE. Then a finite dimensional realization can be constructed in the following way:

- Choose a finite number of vector fields $f_1,\ldots,f_d$ which span $\{\mu,\sigma\}_{LA}$.
- Compute the invariant manifold $G(z_1,\ldots,z_d)$ using Theorem 4.2.

- We now have that $q = G(Z)$. Make the following *Ansatz* for the dynamics of the state space variables Z

$$dZ = a(Z)dt + b(Z) \circ dW_t.$$

It must then (see Appendix A) hold that

$$G_\star a = \mu, \qquad G_\star b = \sigma. \tag{4.29}$$

Use the equations in (4.29) to solve for the vector fields a and b.

4.5.2 An Example

As a simple example of how to construct a realization consider again the deterministic volatility given by

$$\sigma(x) = \sigma_0 e^{-cx}, \tag{4.30}$$

where σ_0 and c are scalar constants. In Section 4.1 we showed that the Lie algebra is spanned by μ and λ where $\lambda(x) = \exp(-cx)$, and we also computed invariant manifold. The first two steps in the algorithm given in the previous section have thus been completed. To obtain a finite dimensional realization it remains to find the dynamics of the state space variables. This means solving the equations (4.29) and since, in this case the Lie algebra is two dimensional, we look for a two dimensional realization of the form

$$\begin{aligned} dz_0 &= a_0(z)dt + b_0(z) \circ dW_t \\ dz_1 &= a_1(z)dt + b_1(z) \circ dW_t, \end{aligned}$$

where $z = (z_0, z_1)$. We therefore need the Frechet derivative G' of G. Since G is just a function of the two real variables z_0 and z_1, the Frechet derivative $G'(z)$ is in this case given by the standard Jacobian

$$G'(z_0, z_1) = \left[\frac{\partial G(z_0, z_1)}{\partial z_0}, \frac{\partial G(z_0, z_1)}{\partial z_1} \right]$$

Thus, using the expression (4.28) for G, we see that for any $(\alpha_0, \alpha_1) \in R^2$ we have

$$G'(z_0, z_1) \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} (x) = \frac{\partial}{\partial x} q_0(x + z_0)\alpha_0 - \frac{1}{2} h(x + z_0)\alpha_0 + \lambda_i(x)\alpha_1,$$

where h is the constant field given by

$$h(x) = \sigma^2(x).$$

Recall that for this model μ is given by

$$\mu(q) = \mathbf{A}q - \frac{1}{2}h.$$

If we use that $q = G(z)$ we can obtain an expression for $\mathbf{A}q$, and the equation $G_\star a = \mu$ then reads

$$\begin{aligned} & \mathbf{A}q_0(x+z_0)a_0 - \frac{1}{2}h(x+z_0)a_0 + \lambda(x)a_1 \\ = \quad & \mathbf{A}q_0(x+z_0) - \frac{1}{2}h(x+z_0) + \mathbf{A}\lambda(x)z_1. \end{aligned}$$

Since this equality is to hold for all x, and a is not allowed to depend on x it is possible to identify a. Using that $\mathbf{A}\lambda = -c\lambda$ we obtain

$$\begin{aligned} a_0 &= 1, \\ a_1 &= -cz_1, \end{aligned}$$

From $G_\star b(z)(x) = \sigma(x)$ we obtain the equation

$$\begin{aligned} & \mathbf{A}q_0(x+z_0)b_0 - \frac{1}{2}h(x+z_0)b_0 + \lambda(x)b_1 \\ = \quad & \sigma_0\lambda(x), \end{aligned}$$

Therefore we have that

$$\begin{aligned} b_0 &= 0, \\ b_1 &= \sigma_0. \end{aligned}$$

The dynamics of the state variables are thus given by

$$\begin{cases} dZ_0(t) &= dt, \\ dZ_1(t) &= -cZ_1(t)dt + \sigma_0 \circ dW_t. \end{cases}$$

Since σ_0 is a constant, the Itô-dynamics will look the same, and we have thus proved the following proposition.

Proposition 5.1 *Given the initial forward rate curve q_0 the system generated by the volatility of equation (4.30) has a finite dimensional realization given by*

$$q_t = G(Z_t),$$

where G is given by

$$G(z_0, z_1)(x) = q_0(x+z_0) - \frac{{\sigma_0}^2 e^{-2cx}}{4c}(1 - e^{-2cz_0}) + e^{-cx}z_1.$$

and the dynamics of the state space variables Z are given by

$$\begin{cases} dZ_0(t) &= dt, \\ dZ_1(t) &= -cZ_1(t)dt + \sigma dW_t. \end{cases}$$

The interpretation of the state variable Z_1 in the parameterization in

Proposition 5.1 is not clear. There is however considerable freedom in choosing the state variables. Suppose we would like the spot price log return $R = q(0)$ to be the state variable instead. We have that

$$R = q(0) = G(Z_0, Z_1)(0) = q_0(Z_0) - \frac{{\sigma_0}^2}{4c}(1 - e^{-2cZ_0}) + Z_1,$$

and thus

$$Z_1 = R - q_0(Z_0) + \frac{{\sigma_0}^2}{4c}(1 - e^{-2cZ_0}).$$

The parameterization of the invariant manifold in terms of R is then given by

$$\begin{aligned} q(x) &= \tilde{G}(Z_0, R)(x) \\ &= q_0(x + Z_0) - e^{-cx} q_0(Z_0) + \frac{{\sigma_0}^2 e^{-2cx}}{4c}(1 - e^{-2cZ_0})(1 - e^{-cx}) + e^{-cx} R. \end{aligned}$$

Using Itô's formula the dynamics of R can be found to be

$$dR(t) = \{\alpha[Z_0(t)] - cR(t)\}\, dt + \sigma_0 dW_t,$$

where

$$\alpha(z) = q_0'(z) + cq_0(z) - \frac{{\sigma_0}^2}{4}(1 + e^{-2cz}).$$

We may now want to change state variables to the spot price S. This is easily done since the parameterization of the invariant manifold with the spot price as state variable is

$$q(x) = \bar{G}(Z_0, S)(x) = \tilde{G}(Z_0, \ln S)(x).$$

The dynamics of the spot price are given by

$$dS(t) = \left(\alpha[Z_0(t)] + \frac{{\sigma_0}^2}{2} - c \ln S(t)\right) S(t)dt + \sigma_0 S(t) dW_t.$$

We thus see that we have obtained a well known model for the spot price. It is basically a standard Black-Scholes model with two modifications.

- Instead of having a Wiener process with constant drift as the log return, we now have a mean reverting log return.

- The drift term $\alpha[Z_0(t)]$ allows us to fit the model to the given initial term structure of futures prices.

As far as the log return dynamics are concerned, we thus have a close resemblance to the Hull-White extension of the Vasicek short rate model.

4.6 Time-Varying Systems

So far we have only considered homogeneous systems. In this section we introduce the adjustments needed for the theory to be applicable to time varying systems. Consider the following system of equations for the logarithm of the futures prices

$$\begin{cases} dq_t &= \mu(q_t,t)dt + \sigma(q_t,t) \circ dW_t, \\ q_s &= q^0. \end{cases} \tag{4.31}$$

The volatility is now of the form $\sigma : \mathcal{H} \times R \times R_+ \longrightarrow R^m$. The drift μ is still given by the expression in (4.19), except that there is now an explicit time dependence. The definition of a realization for this SDE is given below.

Definition 6.1 *We say that the SDE (4.31) has a (local) d-dimensional realization at (s, q^0), if there exists a point $z_s \in R^d$, smooth vector fields $a, b_1, \ldots, b_m$ on some open subset Z of R^d, and a smooth (submanifold) map $G : Z \longrightarrow H$, such that q has the local representation*

$$q_t = G(Z_t), \qquad t \geq s,$$

where Z is the solution of the d-dimensional Stratonovich SDE

$$\begin{cases} dZ_t &= a(Z_t)dt + b(Z_t) \circ dW_t, \\ Z_s &= z_s. \end{cases}$$

The way to handle the explicit time dependence is to enlarge the state space to include running time as a state variable.

Definition 6.2 *Define the following extended objects.*

$$\begin{aligned} \widehat{\mathcal{H}} &= \mathcal{H} \times R, \\ \hat{r} &= \begin{bmatrix} q \\ t \end{bmatrix}, \\ \hat{\mu}(\hat{q}) &= \begin{bmatrix} \mu(q,t) \\ 1 \end{bmatrix}, \\ \hat{\sigma}(\hat{q}) &= \begin{bmatrix} \sigma(q,t) \\ 0 \end{bmatrix}. \end{aligned}$$

We have the following theorem from [6].

Theorem 6.1 (Björk and Svensson) *The time varying system (4.31) has a finite dimensional realization if and only if*

$$dim\{\hat{\mu}, \hat{\sigma}\}_{LA} < \infty.$$

4.6.1 An Example

In this section we study a real life model for gas futures from the point of view of existence and construction of a finite dimensional realization. The model, introduced in [8] and developed for the US natural gas market, is defined as

$$\frac{dF(t,T)}{F(t,T)} = \alpha_1 e^{-\beta_1(T-t)} dW_1(t) \tag{4.32}$$
$$+ \left[\alpha_2 e^{-\beta_2(T-t)} \sin(\phi + 2\pi T) + \alpha_3 e^{-\beta_3(T-t)} \sin(\phi + 2\pi t) + \gamma\right] dW_2(t).$$

For ease of exposition, we carry out the calculations for a slightly more general model, after which we apply the results to our particular case. Consider therefor the model

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} \sigma_i(t,T) dW_i(t) \tag{4.33}$$

or, using the Musiela parametrization

$$r_t(x) \triangleq F(t, t+x)$$

$$dr_t(x) = \frac{\partial}{\partial x} r_t(x) dt + r_t(x) \sum_{i=1}^{n} \sigma_i(t,x) dW_i(t) \tag{4.34}$$

As above we consider log prices, i.e.

$$q_t(x) \triangleq \ln r_t(x)$$

The dynamics of this process are

$$dq_t(x) = \left(\frac{\partial}{\partial x} q_t(x) - \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2(t,x) \right) dt + \sum_{i=1}^{n} \sigma_i(t,x) dW_i(t) \tag{4.35}$$

The dynamics on Stratonovich form are

$$dq_t(x) = \left(\frac{\partial}{\partial x} q_t(x) - \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2(t,x) \right) dt + \sum_{i=1}^{n} \sigma_i(t,x) \circ dW_i(t) \tag{4.36}$$

We use the notation

$$\mathbf{A} \triangleq \frac{\partial}{\partial x}$$
$$\mathbf{H} \triangleq \frac{\partial}{\partial t}$$

Further, define

$$\widehat{q} \triangleq \begin{bmatrix} q \\ t \end{bmatrix}$$

$$\widehat{\mu} \triangleq \begin{bmatrix} \mathbf{A}q - \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(t) \\ \\ 1 \end{bmatrix}$$

$$\widehat{\sigma} \triangleq \begin{bmatrix} \sigma(t) \\ \\ 0 \end{bmatrix}$$

where

$$\sigma(t) = [\sigma_1(t) \; ... \; \sigma_n(t)] \, .$$

By theorem 6.1, (4.33) admits a finite dimensional realization, if and only if

$$\dim \{\widehat{\mu}, \widehat{\sigma}\}_{LA} < \infty.$$

To see when this condition holds, we compute the Lie-brackets

$$[\widehat{\mu}, \widehat{\sigma}_i] = \begin{bmatrix} (\mathbf{A} - \mathbf{H})\sigma_i(t) \\ 0 \end{bmatrix}, i = 1, ..., n$$

$$[\widehat{\sigma}_i, \widehat{\sigma}_j] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, i, j = 1, ...n$$

$$[\widehat{\mu}, [\widehat{\mu}, \widehat{\sigma}_i]] = \begin{bmatrix} (\mathbf{A} - \mathbf{H})^2\sigma_i(t) \\ 0 \end{bmatrix}, i = 1, ...n$$

From the above calculations, the following proposition follows.

Proposition 6.1 *The Lie-algebra generated by $\widehat{\mu}$ and $\widehat{\sigma}$ is given by*

$$\{\widehat{\mu}, \widehat{\sigma}\}_{LA} = span \left\{ \begin{bmatrix} \frac{\partial}{\partial x}q_t(x) - \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(t,x) \\ \\ 1 \end{bmatrix}, \begin{bmatrix} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^k \sigma_i(t,x) \\ \\ 0 \end{bmatrix}; i = 0, ..., n; k = 1, 2, ... \right\}$$

To see if the model (1.6) admits a finite dimensional realization, we must hence calculate

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^k \sigma_i(t,x)$$

for the two components (now written on Musiela form)

$$\sigma_1(t,x) = \alpha_1 e^{-\beta_1 x}$$
$$\sigma_2(t,x) = \alpha_2 e^{-\beta_2 x}\sin(\phi + 2\pi(t+x)) + \alpha_3 e^{-\beta_3 x}\sin(\phi + 2\pi t) + \gamma$$

and hope that there is a $k < \infty$ (hopefully not too large), such that the vectors

$$\left\{\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^k \sigma_i(t,x), i = 1,2; k = 1,2...\right\}$$

are linearly dependent. Starting with the first component

$$\sigma_1(t,x) = \alpha_1 e^{-\beta_1 x}$$

and introducing the notation

$$G \triangleq \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)$$

we see that

$$G\alpha_1 e^{-\beta_1 x} = -\beta_1 \alpha_1 e^{-\beta_1 x} = -\beta_1 \sigma_1(t,x)$$

so it is clear that with $k = 1$, we already obtain linearly dependent terms. Continuing with the second volatility factor we get

$$G^0\sigma_2(t,x) = \alpha_2 e^{-\beta_2 x}\sin(\phi + 2\pi(t+x)) + \alpha_3 e^{-\beta_3 x}\sin(\phi + 2\pi t) + \gamma$$

$$G^1\sigma_2(t,x) = -\alpha_2\beta_2 e^{-\beta_2 x}\sin(\phi + 2\pi(t+x)) - \alpha_3\beta_3 e^{-\beta_3 x}\sin(\phi + 2\pi t) - 2\pi e^{-\beta_3 x}\cos(\phi + 2\pi t)$$

$$G^2\sigma_2(t,x) = \alpha_2\beta_2^2 e^{-\beta_2 x}\sin(\phi + 2\pi(t+x)) + (\alpha_3\beta_3^2 - 4\pi^2)e^{-\beta_3 x}\sin(\phi + 2\pi t) + 4\pi\alpha_3 e^{-\beta_3 x}\cos(\phi + 2\pi t)$$

Taking higher order derivatives of G will only result in linear combinations of the terms represented in the three equations above. We conclude that the four terms needed to generate $G^k\sigma_2$ for all k are

$$\gamma,\ e^{-\beta_2 x}\sin(\phi + 2\pi(t+x)),\ e^{-\beta_3 x}\sin(\phi + 2\pi t),\ \text{and } e^{-\beta_3 x}\cos(\phi + 2\pi t).$$

By Lemma 3.1 in [6], we can choose the vector fields simply as

$$\widehat{f_0} = \left[\begin{array}{c} \mathbf{A}q_t(x) - \frac{1}{2}\left(\alpha_1^2 e^{-2\beta_1 x} + \left(\alpha_2 e^{-\beta_2 x}\sin(\phi + 2\pi(t+x)) + \alpha_3 e^{-\beta_3 x}\sin(\phi + 2\pi t) + \gamma\right)^2\right) \\ 1 \end{array} \right]$$

$$\widehat{f_1} = \left[\begin{array}{c} \gamma \\ 0 \end{array} \right]$$

$$\widehat{f_2} = \left[\begin{array}{c} e^{-\beta_1 x} \\ 0 \end{array} \right]$$

$$\widehat{f_3} = \left[\begin{array}{c} e^{-\beta_2 x}\sin(\phi + 2\pi(t+x) \\ 0 \end{array} \right]$$

$$\widehat{f_4} = \left[\begin{array}{c} e^{-\beta_3 x}\sin(\phi + 2\pi t) \\ 0 \end{array} \right]$$

$$\widehat{f_5} = \left[\begin{array}{c} e^{-\beta_3 x}\cos(\phi + 2\pi t) \\ 0 \end{array} \right].$$

The differential equations we need to solve are the following (with starting values $q(0) = q^*$ and $u(0) = u^*$ throughout):

$$\begin{array}{lcl} \frac{dq}{dt} = \mathbf{A}q - \frac{1}{2}h(u,x) & & q(t) = q^*(t+x) - \frac{1}{2}h(u^*+t,x) \\ \frac{du}{dt} = 1 & \Longrightarrow & u(t) = u^* + t \\[2ex] \frac{dq}{dt} = \gamma & & q(t) = q^* + \gamma t \\ \frac{du}{dt} = 0 & \Longrightarrow & u(t) = u^* \\[2ex] \frac{dq}{dt} = e^{-\beta_1 x} & & q(t) = q^* + e^{-\beta_1 x}t \\ \frac{du}{dt} = 0 & \Longrightarrow & u(t) = u^* \\[2ex] \frac{dq}{dt} = e^{-\beta_2 x}\sin(\phi + 2\pi(u+x) & & q(t) = q^* + e^{-\beta_2 x}\sin(\phi + 2\pi(u^*+x))t \\ \frac{du}{dt} = 0 & \Longrightarrow & u(t) = u^* \\[2ex] \frac{dq}{dt} = e^{-\beta_3 x}\sin(\phi + 2\pi u) & & q(t) = q^* + e^{-\beta_3 x}\sin(\phi + 2\pi u^*)t \\ \frac{du}{dt} = 0 & \Longrightarrow & u(t) = u^* \\[2ex] \frac{dq}{dt} = e^{-\beta_3 x}\cos(\phi + 2\pi u^*) & & q(t) = q^* + e^{-\beta_3 x}\cos(\phi + 2\pi u^*)t \\ \frac{du}{dt} = 0 & \Longrightarrow & u(t) = u^* \end{array}$$

where

$$h(u,x) = \alpha_1^2 e^{-2\beta_1 x} + \left(\alpha_2 e^{-\beta_2 x}\sin(\phi + 2\pi(u+x)) + \alpha_3 e^{-\beta_3 x}\sin(\phi + 2\pi u) + \gamma\right)^2,$$

$$q_t = \mathcal{G}(z_t),$$

and

$$\begin{bmatrix} q_t \\ t \end{bmatrix} = \widehat{\mathcal{G}}(z_t) = \begin{bmatrix} \mathcal{G}_0(z_t) \\ \mathcal{G}_1(z_t) \end{bmatrix}.$$

The interesting part is, of course, $\mathcal{G}_0$ which we can now write down as

$$\mathcal{G}_0(z_i, x; i = 0, ..., 5) = q^*(z_0 + x) - h(z_0 + u^*, x) + \gamma z_1 + e^{-\beta_1 x} z_2 +$$

$$e^{-\beta_2 x} \sin(\phi + 2\pi(u^* + z_0 + x)) z_3$$

$$+ e^{-\beta_3 x} \sin(\phi + 2\pi(u^* + z_0)) z_4 + e^{-\beta_3 x} \cos(\phi + 2\pi(u^* + z_0)) z_5$$

We need to impose some dynamics for the z_i's :

$$dz_i(t) = a_i(z)dt + b_i(z)dW_1(t) + c_i(z)dW_2(t); i = 0, ..., 5,$$

where

$$z \triangleq \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix}.$$

By Itô's formula

$$dq = \sum_{i=0}^{5} \frac{\partial \mathcal{G}}{\partial z_i} dz_i,$$

so we need the partial derivatives of $\mathcal{G}_0$ with respect to z_i for $i = 0, ..., 5$.

$$\begin{aligned}
\frac{\partial \mathcal{G}_0}{\partial z_0} &= \mathbf{A}q^*(z_0+x) - \frac{\partial}{\partial z_0}h(z_0+u^*,x) + 2\pi e^{-\beta_2 x}\cos(\phi+2\pi(u^*+z_0+x))z_3 \\
&\quad + 2\pi e^{-\beta_3 x}\cos(\phi+2\pi(u^*+z_0))z_4 - 2\pi e^{-\beta_3 x}\sin(\phi+2\pi(u^*+z_0))z_5 \\
\frac{\partial \mathcal{G}_0}{\partial z_1} &= \gamma \\
\frac{\partial \mathcal{G}_0}{\partial z_2} &= e^{-\beta_1 x} \\
\frac{\partial \mathcal{G}_0}{\partial z_3} &= e^{-\beta_2 x}\sin(\phi+2\pi(u^*+x+z_0)) \\
\frac{\partial \mathcal{G}_0}{\partial z_4} &= e^{-\beta_3 x}\sin(\phi+2\pi(u^*+z_0)) \\
\frac{\partial \mathcal{G}_0}{\partial z_5} &= e^{-\beta_3 x}\cos(\phi+2\pi(u^*+z_0)).
\end{aligned}$$

After some reshuffling, the dynamics of q now take the form:

$$dq = \left\{ \begin{array}{c} a_0 \left(\begin{array}{c} \mathbf{A}q^*(z_0+x) - \frac{\partial}{\partial z_0}h(z_0+u^*,x) + 2\pi e^{-\beta_2 x}\cos(\phi+2\pi(u^*+z_0+x))z_3 \\ +2\pi e^{-\beta_3 x}\cos(\phi+2\pi(u^*+z_0))z_4 - 2\pi e^{-\beta_3 x}\sin(\phi+2\pi(u^*+z_0))z_5 \end{array} \right) \\ +a_1\gamma + a_2 e^{-\beta_1 x} + a_3 e^{-\beta_2 x}\sin(\phi+2\pi(u^*+x+z_0)) \\ +a_4 e^{-\beta_3 x}\sin(\phi+2\pi(u^*+z_0)) + a_5 e^{-\beta_3 x}\cos(\phi+2\pi(u^*+z_0)) \end{array} \right\} dt$$

$$+ \left\{ \begin{array}{c} b_0 \left(\begin{array}{c} \mathbf{A}q^*(z_0+x) - \frac{\partial}{\partial z_0}h(z_0+u^*,x) + 2\pi e^{-\beta_2 x}\cos(\phi+2\pi(u^*+z_0+x))z_3 \\ +2\pi e^{-\beta_3 x}\cos(\phi+2\pi(u^*+z_0))z_4 - 2\pi e^{-\beta_3 x}\sin(\phi+2\pi(u^*+z_0))z_5 \end{array} \right) \\ +b_1\gamma + b_2 e^{-\beta_1 x} + b_3 e^{-\beta_2 x}\sin(\phi+2\pi(u^*+x+z_0)) \\ +b_4 e^{-\beta_3 x}\sin(\phi+2\pi(u^*+z_0)) + b_5 e^{-\beta_3 x}\cos(\phi+2\pi(u^*+z_0)) \end{array} \right\} dW_1$$

$$+ \left\{ \begin{array}{c} c_0 \left(\begin{array}{c} \mathbf{A}q^*(z_0+x) - \frac{\partial}{\partial z_0}h(z_0+u^*,x) + 2\pi e^{-\beta_2 x}\cos(\phi+2\pi(u^*+z_0+x))z_3 \\ +2\pi e^{-\beta_3 x}\cos(\phi+2\pi(u^*+z_0))z_4 - 2\pi e^{-\beta_3 x}\sin(\phi+2\pi(u^*+z_0))z_5 \end{array} \right) \\ +c_1\gamma + c_2 e^{-\beta_1 x} + c_3 e^{-\beta_2 x}\sin(\phi+2\pi(u^*+x+z_0)) \\ +c_4 e^{-\beta_3 x}\sin(\phi+2\pi(u^*+z_0)) + c_5 e^{-\beta_3 x}\cos(\phi+2\pi(u^*+z_0)) \end{array} \right\} dW_2.$$

This must be equal to the dynamics we started out with, namely

$$\left\{\begin{array}{c} \mathbf{A}q^*(z_0+x) - \frac{\partial}{\partial x}h(z_0+u^*,x) - \beta_1 e^{-\beta_1 x}z_2 - \beta_2 e^{-\beta_2 x}\sin(\phi+2\pi(u^*+x+z_0))z_3 \\ +2\pi e^{-\beta_2 x}\cos(\phi+2\pi(u^*+x+z_0))z_3 \\ -\beta_3 e^{-\beta_3 x}\sin(\phi+2\pi(u^*+z_0))z_4 - \beta_3 e^{-\beta_3 x}\cos(\phi+2\pi(u^*+z_0))z_5 \\ -\frac{1}{2}\left(\alpha_1^2 e^{-2\beta_1 x} + \left(\alpha_2 e^{-\beta_2 x}\sin(\phi+2\pi(u^*+x+z_0)) + \alpha_3 e^{-\beta_3 x}\sin(\phi+2\pi(u^*+z_0))+\gamma\right)^2\right) \end{array}\right\} dt$$

$$+\,\alpha_1 e^{-\beta_1 x}dW_1 + \left(\alpha_2 e^{-\beta_2 x}\sin(\phi+2\pi(t+x)) + \alpha_3 e^{-\beta_3 x}\sin(\phi+2\pi t)+\gamma\right)dW_2$$

We want to find coefficients $a_i(z)$, $b_i(z)$, $c_i(z); i = 0, ...5$ that solves this equation. We know that at least one solution exists, and that any solution will do. By mere inspection, it is clear that the following is a solution:

$$a = \begin{bmatrix} 1 \\ 0 \\ -\beta_1 z_2 \\ -\beta_2 z_3 \\ 2\pi z_5 - \beta_3 z_4 \\ -\beta_3 z_5 - 2\pi z_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \alpha_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \alpha_2 \\ \alpha_3 \\ 0 \end{bmatrix},$$

so the dynamics for the state space process become

$$\begin{aligned} dz_0 &= dt \\ dz_1 &= dW_2 \\ dz_2 &= -\beta_1 z_2 dt + \alpha_1 dW_1 \\ dz_3 &= -\beta_2 z_3 dt + \alpha_2 dW_2 \\ dz_4 &= (2\pi z_5 - \beta_3 z_4)dt + \alpha_3 dW_2 \\ dz_5 &= -(\beta_3 z_5 + 2\pi z_4)dt. \end{aligned}$$

4.7 Spot Price Realizations

n factor models of the term structure of futures prices a considerable amount of attention has been given to models where, apart from running time t, the spot price $S(t) = F(t,t) = e^{q_t(0)}$ is the only state variable. See [5] for references. In the framework of the HJM type futures price models that we are studying, this raises an interesting inverse question, namely what the volatility structure $\sigma(q,x)$ must look like in order to guarantee the existence of a realization with the spot price as the only nontrivial factor or, equivalently, to guarantee that the induced spot price is Markovian. In interest rate theory this problem corresponds to the question about which HJM models can be realized as short rate models, and the problem was solved in [19]. See also [6] and [10]. In the present study we follow the ideas in [6].

As before it is easier to work in terms of q defined in (4.9) than with the futures prices themselves. The problem is thus to determine when the SDE (4.15) has a two-dimensional realization in terms of time and $q_t(0)$. To simplify the notation we define R as

$$R(t) = q_t(0). \tag{4.37}$$

In order to avoid degenerate cases we need a basic assumption.

Assumption 7.1 *We assume that we have only one scalar driving Wiener process, i.e. that $m = 1$.*

Theorem 7.1 *Assume that the model is not deterministic, and consider a given time invariant volatility $\sigma(q,x)$. Then there exists a realization of (4.15) with time and q as state variables if and only if the Lie bracket vector field $[\mu,\sigma]$ is parallel to the vector field σ, i.e. if and only if there exists a scalar field $\alpha(q)$ such that the following relation holds (locally) for all q.*

$$[\mu,\sigma](q) = \alpha(q)\sigma(q). \tag{4.38}$$

where μ is the drift of q, i.e.

$$\mu(t,x) = \frac{\partial}{\partial x}q(t,x) - \frac{1}{2}\sigma^2(q,x) - \frac{1}{2}\sigma'_q(q,x)[\sigma(q)]. \tag{4.39}$$

If the volatility is time varying of the form $\sigma(q,t,x)$, then (4.38) is replaced with

$$\mu'_q(q,t)[\sigma(q,t)] - \sigma'_q(q,t)[\mu(q,t)] - \sigma'_t(q,t)\alpha(q,t)\sigma(q,t).$$

Proof. The proof is identical to the corresponding proof in [6]. To give the reader an idea of the technique, we provide the necessity part of the proof. Assume thus that there exists a two dimensional realization where running time is one of the state variables. Then it must have the following form (where z_1 is time and $z_2 = q$)

$$\begin{aligned} dz_1 &= 1 \cdot dt + 0 \circ dW, \\ dz_2 &= a(z)dt + b(z) \circ dW, \\ q_t &= G(z_t). \end{aligned}$$

In vector notation this reads

$$dz = A(z)dt + B(z) \circ dW,$$

where the vector fields A and B are given by

$$A(z) = \begin{bmatrix} 1 \\ a(z) \end{bmatrix}, \quad B(z) = \begin{bmatrix} 0 \\ b(z) \end{bmatrix}.$$

The Frechet derivatives (Jacobians) are easily obtained as follows, where subscript denotes partial derivatives.

$$A'(z) = \begin{bmatrix} 0 & 0 \\ a_1(z) & a_2(z) \end{bmatrix}, \quad B'(z) = \begin{bmatrix} 0 & 0 \\ b_1(z) & b_2(z) \end{bmatrix}.$$

Thus the Lie bracket is given by

$$[A, B] = A'B - B'A = \begin{bmatrix} 0 \\ a_2 b - b_2 a - b_1 \end{bmatrix}.$$

It is now clear by inspection that we have $[A, B]//B$, where $//$ denotes parallel. On the other hand, because of the relation $q = G(z)$, we also have $\mu = G_\star A$ and $\sigma = G_\star B$. Using the fact that the bracket is preserved under smooth mappings (see Appendix A) we obtain

$$[\mu, \sigma] = [G_\star A, G_\star B] = G_\star [A, B].$$

Since $[A, B]//B$ we thus have $G_\star [A, B] //G_\star B$, but $G_\star B = \sigma$ and we are finished. The sufficiency part is slightly trickier, but can be done. See [6]. ∎

Remark 7.1 *We will write the condition (4.38) as*

$$[\mu, \sigma] \parallel \sigma,$$

i.e. we will use $\parallel$ to denote that two fields are parallel.

Remark 7.2 *Note that our aim is to have a spot price realization of the futures price model, i.e. a realization in terms of time and S_t. However if the SDE for q can be realized in terms of time and R we will indeed also have a spot price realization, since the futures prices are given by*

$$F(t, T) = F_0(t, T - t) = e^{q(t, T-t)},$$

and the spot price is related to R in the following way

$$S(t) = e^{q_t(0)} = e^{R_t}.$$

Also note that all results concerning the log futures price volatility $\sigma(q, x)$ can easily be transformed into a corresponding futures price volatility $\sigma_0(F_0, x)$ by using the relation in Remark 2.1.

In the following two sections we apply Theorem 7.1 to the special cases studied previously: deterministic and constant direction volatility.

4.7.1 Deterministic Volatility

In this section we consider a deterministic, but time varying volatility

$$\sigma(q,t,x) = \sigma(t,x). \tag{4.40}$$

Using (4.19) we obtain

$$\mu(q,t,x) = \frac{\partial}{\partial x} q(t,x) - \frac{1}{2}\sigma^2(t,x).$$

From the above we see that $\mu'_q = \partial/\partial x$ and $\sigma'_q = 0$. The condition for time varying volatilities in Theorem 7.1 now reads

$$\sigma'_x(t,x) - \sigma'_t(t,x) = \alpha(q,t)\sigma(t,x).$$

Therefore we must have that $\alpha(q,t) = \alpha(t)$. Let $g(t,x) = \ln\sigma(t,x)$. The equation can then be written as

$$g'_x(t,x) - g'_t(t,x) = \alpha(t).$$

Finally, letting $h(t,x) = g'_x(t,x)$ and taking derivatives with respect to x we can write

$$h'_x(t,x) - h'_t(t,x) = 0.$$

The solution to this equation is $h(t,x) = \lambda(t+x)$, where λ is an arbitrary function. Going back to σ we have

$$\sigma(t,x) = c(t)e^{\int_t^{t+x}\lambda(u)du},$$

where λ and c are functions, with $c > 0$.

Proposition 7.1 *Assume that the volatility is deterministic. i.e. $\sigma = \sigma(t,x)$. Then there exists a realization in terms of R if and only if σ is of the form*

$$\sigma(t,x) = c(t)e^{\int_t^{t+x}\lambda(u)du},$$

where λ and c are some functions, with $c > 0$.

Again we have that $\sigma = \sigma_0$ when they are deterministic and the above proposition can be rewritten as

Proposition 7.2 *Assume that the futures price volatilities are deterministic. i.e. $\sigma_0 = \sigma_0(t,x)$. Then there exists a spot price realization if and only if σ_0 is of the form*

$$\sigma_0(t,x) = c(t)e^{\int_t^{t+x}\lambda(u)du},$$

where λ and c are some functions, with $c > 0$.

Remark 7.3 *The reader will recognize this result from the corresponding result for short rate realizations in [10] (where completely different techniques were used).*

4.7.2 Constant Direction Volatility

Now we go on to consider a time invariant constant direction volatility, i.e. a volatility of the form

$$\sigma(q, x) = \varphi(q)\lambda(x).$$

For this model we obtain from equation (4.19)

$$\mu(q, x) = \frac{\partial}{\partial x} q(x) - \frac{1}{2}\left\{\varphi^2(q)\lambda^2(x) + \varphi_q'(q)[\lambda]\varphi(q)\lambda(x)\right\}.$$

Assume that $\varphi(q) \neq 0$ for all $q \in \mathcal{H}$. Then $[\mu, \sigma] \parallel \sigma$ if and only if $[f_0, f_1] \parallel \lambda$ where

$$\begin{cases} f_0(q, x) &= \dfrac{\partial q}{\partial x}(x) - \dfrac{1}{2}\varphi^2(q)\lambda^2(x) \\ f_1(q, x) &= \lambda(x), \end{cases}$$

We have that

$$[f_0, f_1](q) = \frac{\partial \lambda}{\partial x} - \varphi_q'(q)[\lambda]\varphi(q)\lambda^2,$$

and therefore there exists a realization in terms of R if and only if there exists a scalar field $c(q)$ such that

$$\frac{\partial \lambda}{\partial x}(x) - \varphi_q'(q)[\lambda]\varphi(q)\lambda^2(x) + c(q)\lambda(x) = 0, \tag{4.41}$$

for all q. Now specialize to the case that φ only depends on $q_t(0) = R_t$, i.e.

$$\sigma(q, x) = \varphi(R)\lambda(x). \tag{4.42}$$

Equation (4.41) then becomes

$$\frac{\partial \lambda}{\partial x}(x) - \varphi_R'(R)\lambda(0)\varphi(R)\lambda^2(x) + \gamma(R)\lambda(x) = 0.$$

Assume that $\lambda(0) = 1$ and set $x = 0$ to obtain that

$$\gamma(R) = \varphi_R'(R)\varphi(R) - \frac{\partial \lambda}{\partial x}(0).$$

Insert this to obtain the equation

$$\varphi_R'(R)\varphi(R) = \frac{\lambda'(0)\lambda(x) - \lambda'(x)}{\lambda(x) - \lambda^2(x)}.$$

Since the left hand side only depends on R and the right hand side only depends on x both sides must equal a constant. If we denote the constant by $a/2$ we have the following proposition.

Proposition 7.3 *Assume that the volatility is of the form*

$\sigma(q,x) = \varphi(R)\lambda(x)$, and that $\lambda(0) = 1$. Then there exists a realization in terms of R if and only if φ^2 is affine in R i.e. of the form

$$\varphi^2(R) = aR + b,$$

where a and b are constants, and λ is the solution to the Riccati equation

$$\frac{a}{2}\lambda^2 + \left(\lambda'(0) - \frac{a}{2}\right)\lambda - \lambda' = 0.$$

Since for this case we have that

$$\sigma_0(S,x) = \varphi_0(S)\lambda(x),$$

where λ is the same constant field as in (4.42), and

$$\varphi_0(S) = \varphi(\ln S).$$

we can rewrite Proposition 7.3 as follows:

Proposition 7.4 *Assume that the volatility is of the form $\sigma(q,x) = \varphi(R)\lambda(x)$, and that $\lambda(0) = 1$. Then there exists a realization in terms of R if and only if φ^2 is affine in R i.e. of the form*

$$\bar{\varphi}^2(S) = a\ln S + b,$$

where a and b are constants, and λ is the solution to the Riccati equation

$$\frac{a}{2}\lambda^2 + \left(\lambda'(0) - \frac{a}{2}\right)\lambda - \lambda' = 0.$$

4.7.3 Generic Spot Price Models

We will now consider the case when the volatility $\sigma(q)$ only depends on R, i.e. when we with a slight abuse of notation can write

$$\sigma(q,x) = \sigma(R,x), \tag{4.43}$$

where σ on the right hand side is a smooth function of the two real variables R and x.

It turns out that the condition $[\mu,\sigma] \parallel \sigma$ is rather restrictive for this case, and we have the following result.

Proposition 7.5 *Assume that the volatility only depends on R as in (4.43), then there exists a realization in terms of R (and time) if and only if σ factors as*

$$\sigma(R,x) = \varphi(R)\lambda(x). \tag{4.44}$$

Proof. For this model, we can compute the drift to be

$$\mu(q,x) = \frac{\partial}{\partial x}q(x) - \frac{1}{2}\sigma^2(R,x) - \frac{1}{2}\sigma'(R,x)g(R),$$

where $g(R) = \sigma(R,0)$ and $\sigma'(R) = \frac{\partial\sigma}{\partial R}$. We now compute the Lie Bracket $[\mu,\sigma] = \mu'_q[\sigma(q)] - \sigma'_q[\mu(q)]$. We have

$$\mu'_q[\sigma(q)](x) = \sigma'_x(R,x) - \sigma(R,x)\sigma'_R(R,x)g(R) - \frac{1}{2}\{\sigma''_R(R,x)g(R) + \sigma'_R(R,x)g'(R)\}g(R),$$

and

$$\sigma'_q[\mu(q)](x) = \sigma'_R(R,x)\frac{\partial}{\partial x}q(0) - \frac{1}{2}\sigma'_R(R,x)g^2(R) - \frac{1}{2}\sigma'_R(R,x)g'(R)g(R),$$

and thus

$$[\mu,\sigma](x) = \sigma'_R(R,x) - \sigma(R,x)\sigma'_R(R,x)g(R) - \frac{1}{2}\{\sigma''_R(R,x) - \sigma'_R(R,x)\}g^2(R) - \sigma'_R(R,x)\frac{\partial}{\partial x}q(0).$$

Applying Theorem 7.1, we know that a necessary and sufficient condition for a realization in terms of R (and time) to exist is that $[\mu,\sigma] \parallel \sigma$. If we let $z = \frac{\partial q(0)}{\partial x}$, we find that there must exist a function $\alpha(z,R)$ such that

$$\sigma'_R(R,x) - \sigma(R,x)\sigma'_R(R,x)g(R) - \frac{1}{2}\{\sigma''_R(R,x) - \sigma'_R(R,x)\}g^2(R) - \sigma'_R(R,x)z = \alpha(z,R)\sigma(R,x)$$

is satisfied for all z and R. Take the derivative with respect to z to obtain

$$-\sigma'_R(R,x) = \alpha'_z(z,R)\sigma(R,x).$$

From this it follows that $\alpha'_z(z,R) = f(R)$ and we must have

$$\sigma'_R(R,x) + f(R)\sigma(R,x) = 0.$$

This is an ODE for $\sigma(R,x)$ with the solution

$$\sigma(R,x) = \exp\left\{\int_0^R f(u)du\right\}\sigma(0,x),$$

and we see the volatility factors as in (4.44). ∎

Since we for this case have that $\sigma_0(S,x) = \sigma(\ln S, x)$ we can rewrite the proposition in the following way.

Proposition 7.6 *Assume that the futures price volatility only depends on S, i.e. $\sigma_0(F_0,x) = \sigma_0(S,x)$, then there exists a spot price realization if and only if σ_0 factors as*

$$\sigma_0(S,x) = \varphi(S)\lambda(x). \tag{4.45}$$

4.7.4 All Generic Spot Price Models Are Affine

The following three spot price models are well known from the literature.

$$dS_t = \alpha(t)S_t dt + \sigma S_t dW_t, \tag{4.46}$$

$$dS_t = (\alpha(t) - \ln S_t)S_t dt + \sigma S_t dW_t, \tag{4.47}$$

$$dS_t = [a_1(t) + a_2(t)\ln S_t]S_t dt + S_t\sqrt{k_0 + k_1 \ln S_t}\; dW_t. \tag{4.48}$$

Since all these models (and in fact only these, see [5]) will generate exponentially affine futures prices, the models will be referred as the **affine spot price models**. In interest rate theory they correspond to the Ho-Lee model, the Hull-White extension of the Vasicek model, and the Hull-White extension of the CIR model respectively. Because of the affine structure one can easily (by solving a simple system of ODEs, see [5]) compute the induced futures price volatilities. The structural result is as follows.

Lemma 7.1 *The futures price volatilities generated by the affine models (4.46)-(4.48) all have the structure*

$$\varphi_0^2(S) = a \ln S + b,$$

where a and b are constants, and λ is the solution to the Riccati equation

$$\frac{a}{2}\lambda^2 + \left(\lambda'(0) - \frac{a}{2}\right)\lambda - \lambda' = 0.$$

More precisely, the following hold.

- *For the model (4.46) it holds that $a = 0$ and thus $\lambda(x) = \lambda(0)$ for all x, i.e. λ is constant.*
- *For the model (4.47) it holds that $a/2 = \lambda(0)$ so λ is an exponential function.*
- *For the model (4.48) it holds that $a/2 \neq \lambda(0)$, so λ solves a Riccati equation.*

We recognize the structure above from Proposition 7.4, and we are now ready to state and prove the main theorem of this section.

Theorem 7.2 *Assume that the futures price volatilities are of the form*

$$\sigma_0(F_0, x) = \sigma_0(S, x).$$

Then the model has a generic spot price realization if and only if it is affine.

Proof. This follows from proposition 7.4, Proposition 7.6, and Lemma 7.1. ∎

The word "generic" is very important in the statement above (see Remark 2.2 for more details). To understand the geometric picture one can think of the following program.

1. Choose an arbitrary spot price model, say of the form

$$dS_t = a(t, S_t)dt + b(S_t)dW_t$$

with a fixed initial point S_0.

2. Solve the associated PDE in order to compute futures prices. This will also produce:
   - An initial futures price curve $\hat{q}^o(x)$.
   - Time invariant futures price volatilities of the form $\sigma_0(S, x)$.

3. Now forget about the underlying spot price model, and take the futures price volatility structure $\sigma_0(S, x)$ as exogenously given in the futures price equation (4.7).

4. Initiate the futures price equation (4.7) with an arbitrary initial forward rate curve $q^o(x)$.

The question is now whether the thus constructed futures price model will have a spot price realization. Obviously, if you choose the initial futures price curve q^o as $q^o = \hat{q}^o$, then you are back where you started, and everything is OK. If, however, you choose another initial forward rate curve q^o, then it is no longer clear that the price will be Markovian (or rather, satisfy a scalar SDE). What the theorem above says, is that only the models listed above will be generated by a spot price model for all initial points in a neighborhood of q^o. If you take another model then a generic choice of the initial futures price curve will produce a futures price process which is not generated by a scalar (time dependent) SDE for the spot price.

Appendix A
Some Background Facts

In this Appendix we recall from basic concepts and results from differential geometry. For more details see [6].

Consider a real Hilbert space $\mathcal{H}$. By an n-dimensional **distribution** we mean a mapping F which, to every r in an open subset V of $\mathcal{H}$, associates an n-dimensional subspace $F(r) \subseteq \mathcal{H}$. A collection $f_1, \dots, f_n$ of vector fields on $\mathcal{H}$ **generates** (or spans) F if it holds that $span\{f_1(r), \dots, f_n(r)\} = F(r)$ for every r, where *span* denotes the linear hull over the real field. The distribution is **smooth** if there exists smooth (i.e. C^∞) vector fields $f_1, \dots, f_n$ spanning F. If F and G are distributions and $G(r) \subseteq F(r)$ for all r we say that F **contains** G, and we write $G \subseteq F$. The **dimension** of a distribution F is defined pointwise as $dimF(r)$.

Let f and g be smooth vector fields on U. Their **Lie bracket** is the vector field

$$[f, g](r) = f'(r)g(r) - g'(r)f(r),$$

where $f'(r)$ denotes the Frechet derivative of f at r, and correspondingly for g. We will sometimes write $f'(r)[g(r)]$ instead of $f'(r)g(r)$ to emphasize that the Frechet derivative is operating on g. A distribution F is called **involutive** if for all smooth vector fields f and g in F, their lie bracket also lies in F, i.e.

$$[f, g](r) \in F(r) \qquad \forall r.$$

We are now ready to define the concept of a Lie algebra which will play a central role in what follows.

Definition A.1 *Let F be a smooth distribution on H. The Lie algebra generated by F, denoted by $\{F\}_{LA}$, is defined as the minimal (under inclusion) involutive distribution containing F.*

When trying to determine a concrete Lie algebra the following observations often come in handy.

Lemma A.1 *Consider the vector fields $f_1, \ldots, f_k$ as given. Then the Lie algebra $\{f_1, \ldots, f_k\}_{LA}$ remains unchanged under the following operations.*

- *The vector field f_i may be replaced by αf_i, where α is any smooth nonzero scalar field.*
- *The vector field f_i may be replaced by*

$$f_i + \sum_{j \neq i} \alpha_j f_j,$$

where $\alpha_1, \ldots, \alpha_k$ are any smooth scalar fields.

Let F be a distribution on $\mathcal{H}$ and let $\varphi : \mathcal{H} \to \mathcal{K}$ be a diffeomorphism between the two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Then we can define a new distribution $\varphi_\star F$ on $\mathcal{K}$ by

$$(\varphi_\star F)(\varphi(r)) = \varphi'(r)F(r).$$

Similarly, for any smooth vector field $f \in C^\infty(U, X)$, we define the field $\varphi_\star f$ by $\varphi_\star f = \left(\varphi' \circ \varphi^{-1}\right)\left(f \circ \varphi^{-1}\right)$. By a straightforward calculation one verifies easily that

$$\varphi_\star [f, g] = [\varphi_\star f, \varphi_\star g] .$$

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