# Valuation and hedging of long-term asset-linked contracts

Henrik Andersson

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# Valuation and hedging of long-term asset-linked contracts

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Henrik Andersson
Karlstad, April 2003.

# Contents

## Preface

The five essays in this dissertation are all concerned with how commodity price uncertainty affects the valuation of real and financial assets. The price of the output is often the major source of uncertainty in basic industries and in the production of semi-manufactured goods and therefore also the most decisive factor in asset valuation. However, although price uncertainty is the general notion we have in mind, uncertainty is too vague a concept for any formal analysis. Uncertainty is therefore given a concrete form by different stochastic processes approximating the price process of the commodity. Valuation, hedging and how to distinguish between these processes are the issues studied in this dissertation.

When price uncertainty is represented by stochastic processes, the option framework is a natural environment for asset valuation. From the original focus on stock options, option theory has expanded enormously into a growth industry of which we still have not seen the final chapter. This dissertation moves along the borderline between the mainstream option literature, which is concerned with financial option contracts, and the research direction known as real options. Normally, financial option literature is concerned with stock, currency and interest rate options as these constitute the majority of options traded. Real option research, on the other hand, has more been directed towards finding new areas of application, rather than scrutinizing the underlying assumptions.

This dissertation contains five separate parts, all related to the valuation and hedging of real options and commodity-linked contracts. Essay 1, *An overview and summary of the dissertation*, summarises and combines the findings of the other four. The essays are put into context and the evidence of mean reverting commodity prices reviewed. We argue that mean reversion is much more predominant in commodity price series than econometric tests reveal. The possibility of pricing real and financial contracts in a way that is arbitrage-free is discussed with the benefit of the combined experience from the other studies. Generally, it is not possible to claim that a calculated value of a real asset is an arbitrage-free price that must hold in the market.

Essay 2, *Capital-budgeting in a situation with variable utilisation of capacity - an example from the pulp industry*, introduces the concept of option pricing in the context of real, as opposed to financial assets. A paper pulp plant is valued using a continuous cash flow and the extended version of the Feynman-Kac formula. A mean reverting process that allows the equilibrium price to increase over time due to inflation and other factors is developed and applied within this capital budgeting setting.

Essay 3, *The stochastic behaviour of commodity prices*, is an econometric study dealing with the stochastic properties of traded goods. Using a large database of close to tree-hundred different commodities, it is found that a random walk is the favoured process for the vast majority of commodities and that leptokurtosis is present in all series. Commodity time series are characterised by the fact that prices seldom change, implying that economic time runs slower than calendar time for commodities. Less news affect commodity prices in comparison to stock prices.

In response to the leptokurtosis in the time series, a stochastic volatility model is developed in essay 4, *A mean reverting stochastic volatility option-pricing model with an analytic solution*. The model, in combination with a new EGARCH model for parameter estimation, is tested on stock options traded on the Stockholm stock exchange. Of the different GARCH models tested, the proposed EGARCH model gives the highest value of the maximum likelihood objective function, and the stochastic volatility model provides a better fit to observed option prices than the Black-Scholes model.

Essay 5, *Hedging of long-term commodity-linked contracts*, tries to distinguish between different stochastic processes by studying the hedging error that different price processes give rise to. It is the possibility to create synthetically another contract with the same payment as the option that allows us to price the option in a way that is arbitrage-free. The difference in value between the option and the synthetic option, the hedging error, provides a direct measure of this ability. It is found, surprisingly, that the stochastic volatility model of essay 4 has the largest hedging error. The smallest hedging error is obtained for the mean reverting model of essay 2. Simpler mean reverting models are not found to possess any hedging ability at all. The frequent reference to such models in the real option literature is therefore rather surprising. Overall, however, the problem of parameter estimation makes the hedging errors very large and it is therefore unlikely that long-term commodity-linked contracts can be hedged in an effective way.

The five essays in this dissertation can be read separately and are all self-contained, even though they follow a logical sequence. The main focus is directed towards the stochastic processes approximating the price process of the goods. The main theoretical contributions are the mean reverting model in essay 2 and the stochastic volatility and EGARCH models of essay 4. There are also some practical applications to highlight, such as the detailed example of a real-option valuation in essay 2 and the use of implied volatility within a stochastic volatility framework in essay 4. Hedging tests, as in essay 5, can serve as tests of mean reversion since econometric tests unfortunately have low power.

List of studies:

Essay 1: An overview and summary of the dissertation

Essay 2: Capital budgeting in a situation with variable utilisation of capacity - an example from the pulp industry

Essay 3: The stochastic behaviour of commodity prices

Essay 4: A mean reverting stochastic volatility option-pricing model with an analytic solution

Essay 5: Hedging of long-term commodity-linked contracts

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# An overview and summary of the dissertation

## (Essay 1)

Henrik Andersson

**Abstract**

The study of asset pricing in general and option pricing in particular is a very technical subject. This is unfortunate, in that technical and mathematical issues often overshadow the main economic points. It is of course a cliché, but "not being able to see the wood for the trees", becomes especially true for option pricing. This first essay, 'An overview and summary of the dissertation', is used to summarise the results of the other four and with a minimum of mathematics discuss the combined findings and review the essays with the benefit of hindsight.

It is suggested that in line with economic intuition, mean reversion is much more predominant in commodity time series than econometric tests reveal, and that tests of hedging performance is a more efficient way of differentiating between mean reversion and a random walk. Notwithstanding the immense theoretical importance of the paradigm of arbitrage free prices, it is further argued that the no-arbitrage requirement does not provide tight bounds on commodity option prices other than possibly for contracts of short maturity. For real options and long-term commodity-linked contracts, option theory provides a tool for valuation but nothing more. The result should not be interpreted as an arbitrage free price.

*Keywords*: Arbitrage Free Prices, Commodity-linked Contracts, Mean Reversion, Real Options

*JEL Classification codes*: G13

# 1. A summary of the essays

One of the main advantages of using option-pricing techniques for asset pricing is that it allows profitable arbitrage opportunities to be explored if assets are mispriced. It thereby gives a very tangible meaning to theoretical pricing formulas as only one price can prevail in a well functioning capital market. However, option pricing and hedging is more difficult for real assets than for the stock and currency options constituting the majority of options traded. There are physical limitations to storing and transporting goods, markets are generally thin and less developed, and contracts are long-term. These characteristics all combine to make the study of option pricing on real assets both interesting and worthwhile.

Essay 1, *An overview and summary of the dissertation*, is used to summarise the findings of the other four and to further discuss some of the most crucial assumptions and also questions that the studies have given rise to. The existence of arbitrage-free commodity contracts is discussed as well as the convenience yield on commodities and its implication for asset pricing and hedging. The presence of mean reversion in the price series is argued and spot and futures markets are discussed. Suggestions for further research conclude.

Essay 2, *Capital-budgeting in a situation with variable utilisation of capacity - an example from the pulp industry*, provides a detailed example of how to value real assets by means of option pricing theory. Using the extended version of the Feynman-Kac formula and continuous cash flows, a paper-pulp plant is valued under different assumptions about the level of utilisation. Parameters are estimated and calculations provided in a step-by-step approach.

Comparisons are made under the hypotheses that the price process for pulp is either a geometric Brownian motion, or mean reverting. A new mean reverting model that allows the equilibrium price to increase over time is proposed and used in the comparison. The difference in plant value that results from using different stochastic processes overshadows any effect caused by different levels of utilisation. The operating characteristics, or real options, thereby become secondary to the specification of the price process for pulp. An overview of different real options concludes the study. These options are, however, of minor importance in the pulp industry.

The assumption found to be the most crucial in the second essay, the price process, is further studied in essay 3, *The stochastic behaviour of commodity prices*. With the objective of identifying the most appropriate process from a purely statistical viewpoint, this study takes a broader look at the price formation of commodities. Extracting from the Datastream commodity data base, time series data for close to three hundred different commodities are collected. A diversity of unit root[1] tests are applied in order to distinguish between a random walk (i.e. the discrete time version of the geometric Brownian motion) and mean reversion. As mean reversion is only present in 14 percent of the series, the random walk provides the best fit to the data.

The study also rejects normally distributed price shocks. Commodities do not differ from pure financial assets in this respect and the result holds independently of the price process assumed. Thicker tails are prevalent in all the series and the shocks are therefore better described by a stable Paretian distribution. The Fama and Roll (1971) technique is used to measure the characteristic exponent $\alpha$ of the Paretian distributions. As probability density functions are unknown for Paretian distributions, except in a few special cases, this class of distributions does not lend itself to standard statistical methods.

[1] See the introduction of essay 3 for the definition of a unit root.

One consolation, however, is that portfolio analysis does not differ substantially even though the market is stable Paretian.[2]

The fat tails have implications for option pricing, though, and this point is further discussed below. Another feature of commodity price formation is that the price seldom changes. News that will affect the price arrives less frequently for traded goods than for pure financial assets. The price formation is thereby slower and the same price can last for extended periods of time.

The fats tails, or in technical parlance, leptokurtosis, in the distribution of the price changes suggest that a stochastic volatility process could be beneficial in modelling commodity prices. This is not without predecessors: Stochastic volatility models have been successfully applied to stock option pricing, since stocks as well as most other securities exhibit leptokurtosis.

Essay 4, *A mean reverting stochastic volatility option-pricing model with an analytic solution*, develops analytic approximations to stochastic volatility European option prices, and derives a new variant of EGARCH for parameter estimation. The model thereby provides a consistent approach to the problem of option-pricing and parameter estimation. As stochastic volatility is frequently discussed in connection with stocks, it is natural to first of all test the performance for stock options. This test, performed on the 20 Swedish stocks with the largest number of option contracts outstanding, shows that the model works very well. The values of the maximum likelihood objective function are slightly higher for the EGARCH model than for the GARCH(1,1) model, which is the standard model for characterising heteroscedasticity, and the values are much higher than for a constant volatility.

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[2] See Fama (1965) for an analysis of portfolio theory in a stable Paretian market.

Comparing theoretical and observed option prices, the stochastic volatility model provides a better fit than the Black-Scholes model, but the agreement with observed prices is even so very poor. By using implied instead of historic volatility the fit is drastically improved. For the stochastic volatility model we introduce the novel technique of iteratively matching the initial and long-run variance with the prices of two traded options. Also for implied volatility, the stochastic volatility model provides a better fit to observed option prices.

Essay 5, *Hedging of long-term commodity-linked contracts*, brings together the previous essays by studying the hedging errors that different stochastic processes give rise to. Option pricing is based on the idea that an option can be replicated synthetically by constructing a portfolio whose payments match the payments of the option contract. As a direct consequence, the price of the option is uniquely determined by the value of the replicating portfolio. Otherwise arbitrage opportunities will occur. However, the argument hinges on the assumption that the stochastic processes of the assets included in the replicating portfolio are known.[3] It is thereby not too blunt a statement to say that the price of an option is a bet on these stochastic processes. The significant difference in pulp plant value between the geometric Brownian motion and the mean reverting process in essay 2 emphasised an important question. Which of the processes does best describe commodity price behaviour? This is a statistical issue, but the statistical tests in essay 3 were not really able to solve the question. Statistical tests of mean reversion, or more precisely, tests for the absence of a unit root, have very low power.

In spite of the fact that we were not able to reject a unit root even for the long time series used in essay 3, it is therefore unclear whether this depends on the low power of the tests or that commodity price series, contrary to intuition,

[3] Observe that it is enough to know the stochastic processes of the assets. The future prices of these assets, the realisations of the stochastic processes, are unknown.

follow random walks. Instead of solving this puzzle, essay 3 gave rise to an additional question. For all of the nearly 300 different commodities, normally distributed price changes were rejected in favour of leptokurtic distributions. This may imply that a stochastic volatility process approximates the price process in a better way. By mixing different normal distributions, the result will appear as leptokurtic in a time series.

The obvious answer to the previous questions is to test the hedging performance for option contracts based on the geometric Brownian motion, the mean reverting process and the stochastic volatility process. After all, the ability to hedge a contract is the ultimate test of whether any of these processes provides a workable assumption for asset pricing. However, the hedging tests are very cumbersome to perform and require substantial programming skills if option prices and hedge ratios must be calculated numerically. This, really, was the motivation behind the development of the stochastic volatility model in essay 4. For existing analytic approximations to stochastic volatility option-pricing models there is no simple way to estimate parameters, and when it is possible to estimate parameters, there is no analytic solution.

Essay 5 utilises the same commodity data set as essay 3 of close to 300 different commodities. The hedging ability is examined for a five-year put option, representing the option characteristics of a commodity-linked bond. The results are reasonably clear cut. The geometric Brownian motion is the most robust model and it works quite well as long as parameters are allowed to be sampled from the same set of data that the hedging performance is tested for. The tests also show that the inclusion of a convenience yield is crucial to the hedging performance. It cannot be omitted. All the commodities have a systematic risk (as measured by beta) that is almost zero. When estimating the convenience yield, it is therefore enough to set the required rate of return on the commodity equal to the risk-free rate.

Although the geometric Brownian motion gives a very robust model, it sometimes, although rarely, causes very large hedging errors. The other models, however, break down more frequently. In order to compare the different stochastic processes, the hedging performance is evaluated for the subset of series where the option prices identified by the different models are less than 200 percent apart. If several models identify prices of the same magnitude, we can be reasonably convinced that a fair price lies within this region. Sorted in this way, the mean reverting model gives very small hedging errors. However, the model must be carefully applied in order to perform well. In particular, a mean reverting model without the drift term suggested in essay 2 does not work and parameters must be estimated with care. Otherwise, hedging performance becomes very poor. The stochastic volatility model is also sensitive to parameter estimates and the hedging performance is worse than for the other two models. Apparently, stochastic volatility is not an important aspect of commodity price behaviour.

This is surprising given the presence of leptokurtosis in all the commodity time series and the lengths of the contracts, five years. Hull (2003, p. 459) reports that stochastic volatility is more important for longer contracts. Notwithstanding the normal explanation that a more advanced model has more parameters that can be misspecified, the most probable explanation is that the different GARCH models used to characterise the variance did not provide any substantial improvement over a constant variance. It may be the case that the leptokurtosis measured is just leptokurtosis and not a heteroskedastic variance in disguise and that stochastic volatility option-pricing models do not capture this leptokurtosis. Leptokurtosis is otherwise said to be the rationale for stochastic volatility models.

However, in an interesting although difficult paper by Popova and Ritchken (1998), it is shown theoretically that when the leptokurtosis is due to the fact that the distribution of price changes is stable Paretian, self-financing portfolios that replicate the contract cannot be found. The poor hedging performance of the stochastic volatility model is perhaps a consequence of this fact. If so, this is interesting since I know of no research in this direction.

When hedging performance is tested out of sample, as of course always is the case in reality, the hedging performance is drastically reduced. The mean absolute hedging error of around 3-5 percent when parameters are estimated within sample, increases to a magnitude of 50 percent when parameters are estimated out of sample and the results are much more dispersed. Again, the mean reverting model produces a smaller error.

The most important theoretical contributions in the essays and thereby in the dissertation are the mean reverting model in essay 2 and the stochastic volatility model in essay 4. The mean reverting model was developed as a response to the fact that no existing mean reverting model allows for a price that increases over time due to inflation and other factors. The importance of this characteristic was demonstrated in essay 5 where a mean reverting model with a fixed equilibrium price was not able to hedge the contract properly. It is therefore highly surprising that such simpler models are often recommended in the real-option literature as they provide an inadequate description of commodity price behaviour. Hull seems to have reached the same conclusion in the latest version "Options, Futures and Other Derivatives" (2003, 5$^{th}$.ed). At p. 681 he claims that "a realistic model" for energy and commodity prices is

$$d\ln S = [\theta(t) - a\ln S]dt + \sigma dz,$$

where "the $\theta(t)$ term captures seasonality and trends". In the notion of essay 2 equation (6.4), $\theta(t) = \eta(\gamma' + \omega t)$, and without this explicit dependence upon the passage of time, we were not able to hedge a long-term contract properly, i.e.

the process was misspecified. It is also worth stressing that the mean reverting model proposed is the continuous time equivalent of an autoregressive process with a constant drift, such as for example used in the Phillips-Perron test of a unit root. The model thereby bridges a gap between continuous and discrete time.

The second theoretical contribution is the EGARCH model proposed in essay 4, which in combination with an analytically approximated option-pricing formula simplifies the problem of stochastic volatility option pricing and parameter estimation considerably. Analytical option prices and maximum likelihood estimations via GARCH processes have been suggested before, but not in combination. This is new. In view of the enormous amount of contracts traded in the world option markets, this is perhaps the most interesting part of the dissertation and a field for future work.

There are also some practical applications to highlight. The valuation of a pulp plant in essay 2 is one such application. Detailed and realistic examples of real option valuation are not abundant in the literature, which is the reason for the somewhat textbook style of writing in essay 2. The use of a continuous cash flow is also rather unusual, since we are told in standard corporate finance courses to collect all cash flows at the end of each year. Using option valuation however, the Feynman-Kac formula makes it very convenient to model the cash flow as continuous.

The use of implied volatility in a stochastic volatility framework in essay 4 is also of much practical interest. The trading desks at several major banks are currently applying different, more or less advanced techniques to adjust the parameters of stochastic volatility models in a way that fits the implied volatility surface from traded options. The pricing formula in essay 4 makes it very easy to obtain two of the parameters. The instantaneous and the long-run average volatility can be iteratively backed out by using one option of the shortest maturity and one option with a longer maturity. Only three or four iterations are required for convergence. As exemplified in essay 4, we will now observe a much better fit to the data, as the "time-to-maturity-smile" has essentially disappeared.

Of practical interest is also the use of hedging tests to differentiate between a random walk in commodity prices and mean reversion. Existing statistical tests underestimate the true extent of mean reversion in the time series, and should therefore be complemented with tests of hedging performance. These tests will not only indicate the most appropriate process, they will also detail to what extent a derivative contract can be replicated and therefore uniquely priced.

# 2. Are commodity prices mean reverting or a random walk?

There are several interesting conclusions to draw from the results of essay 3, *The stochastic behaviour of commodity prices* and essay 5, *Hedging of commodity-linked contracts*. The recurring question of whether commodity prices are better modelled as a random walk or through a mean reverting process is probably never going to be completely resolved. The dissertation makes some progress on the issue, though, by having access to the prices of a couple of hundred different commodities compared to some dozens in previous research.

Normally, the question of a random walk, as the geometric Brownian motion becomes in discrete time, or mean reversion is studied by econometricians. The general conclusion is that it requires very long time series of data to reject a random walk and even so it is a difficult task due to the low power of the tests. As an example of this, see chapter 15 of Pindyck and Rubinfeld (1991). The access to a large data base of commodity prices made it interesting to study the problem once more, especially so since daily prices were available and the longest series had durations of 30 years. Judged a priori, the substantial number of observations ought to have been enough to detect mean reversion if it was present. The statistical tests in essay 3 found little evidence of mean reversion, however. The Phillips-Perron test, which was the most trustworthy of the different tests, rejected a unit root for only 14 percent of the series and other tests gave similar results.

Can this be taken as evidence that commodity prices follow a random walk? No, not formally because a random walk can still be rejected on the grounds of autocorrelation in the return, but that is a minor point in this context. The main interest is to see whether a random walk can be rejected in favour of mean reversion and the tests are therefore designed to be robust to the presence of autocorrelation. Leaving the question of autocorrelation aside, inferences are hampered by a number of other troublesome circumstances. The Phillips-Perron test has low power due to the asymmetric distribution of the test variable. Moreover, the KPSS-test used is only an indirect test of mean reversion. Statistical tests will therefore underestimate the true extent of mean reversion in the time series. Another reason for doubt is that although there are many observations the price seldom changes in many series, implying that the "economic length" of the series is shorter than the number of observations. More years of data are therefore required and previous studies have shown that a long time period is essential in order to reject a random walk.

The most serious objection to the result of a random walk is not technical, however, but economic. Intuitively, exceptionally high prices will attract new producers and therefore reduce the price. Unusually low prices will drive some producers out of business, something that tends to increase the price.

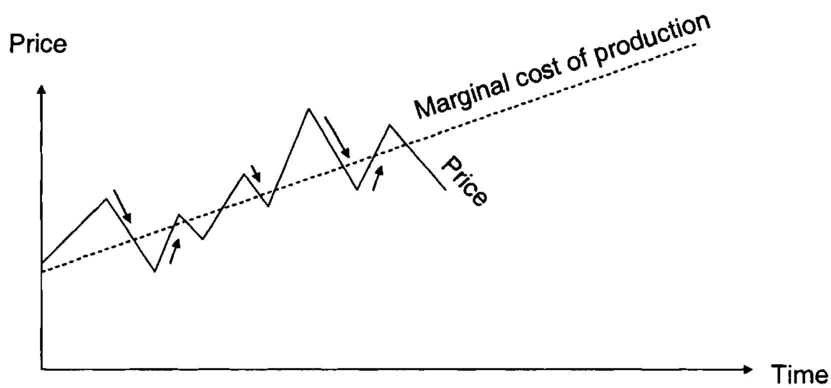


Figure 2.1 A mean reverting price process reverts toward the equilibrium price, the marginal cost of production.

Prices should therefore in the long-run revert towards the marginal cost of production, the broken line in figure 2.1. The marginal cost of production includes all marginal cost, independently whether these are production costs, overhead costs, or the cost of capital, i.e. a reasonable profit. If prices are, for example, above this equilibrium, the rate of production will increase due to the handsome profits that can be made and the price will eventually decrease thanks to the increased supply.

This intuition not only motivates mean reverting prices, it is also the fundamental process for how a market economy solves the production problem of what and how much the society should produce of each good. Economists may therefore be forgiven for not trusting the statistical results of a random walk completely. Many would probably argue that the economy seems to function quite nicely and in such a refined way that one may put some trust in the assumptions underlying a market economy, something also implying that commodity prices should be mean reverting.

Statistical tests are not the only way of differentiating between mean reversion and a random walk, however. Our interest is asset pricing, so why not test how well the respective processes perform for this objective? Essay 5 shows that notwithstanding the problems of finding an appropriate level of mean reversion, the mean reverting process results in a slightly smaller hedging error for option contracts than the geometric Brownian motion. As a test of hedging performance is not a statistical test, we are unable to attach confidence intervals to this result, but even so the indication is that mean reversion is much more common in commodity price series than the statistical tests reveal. Of the 150 series in table 4.2 of essay 5, 100 series had the smallest hedging error when the mean reverting process was used as an approximation of the price behaviour. The conclusion is therefore that mean reversion is much more common than what is revealed by econometric tests.

In one way this result is discouraging. We do not want to use different price processes for different commodities, just in the same way that we do not want to use separate price processes for the myriad of stocks listed on the stock exchanges. The fact that the geometric Brownian motion gives a smaller hedging error for some commodities and the mean reverting model for other commodities is therefore troublesome from a practical viewpoint.

However, it is not the price processes in themselves that cause large hedging errors. The tests show that both the geometric Brownian motion and the mean reverting process can describe commodity price behaviour quite well. Both processes resulted in small hedging errors when applied to an option-pricing model. But both processes were also extremely sensitive to how parameters were estimated. They only performed well when the parameters were known in advance, i.e. they were sampled from the actual outcome. When parameters were estimated from historical data both models performed poorly. This shows that it is not the assumed price process but instead parameter estimation, in combination with transaction costs, that are the most crucial aspects of hedging performance. This conclusion should carry over to stocks and currencies as well. Commodities are not unique in this respect. Option theorists spend a great deal of time worrying about the stochastic processes followed by different assets, a subject that also is the main issue in this dissertation. In view of the results in essay 5, one may wonder whether this really is advisable, since parameter estimates and also transaction costs will have the greatest influence on arbitrage free bounds for option prices.

The hedging performance was enhanced in that the largest errors were avoided when the models were compared to each other. When the models based on mean reversion and geometric Brownian motion gave theoretical option prices that were close to each other, the hedging errors were also smaller for both models. This is especially important for the mean reverting model as it is not very robust and sometimes gives unrealistic option prices and thereby large hedging errors. The reason is its dependence on the mean reversion level. As a mean reverting model, by definition, assumes that prices revert to the equilibrium price it is also extremely sensitive to estimates of the equilibrium price. This lack of robustness would make the model easily dismissible had it not been for one crucial fact: When the unrealistic situations are sorted out by only studying commodities where the option prices provided by the two models are of the same magnitude, the mean reverting model has a smaller error.

The recommendation is therefore to compare the option prices and hedging errors provided by the different models. When the models give approximately the same option prices, the mean reverting model is preferable as it is more exact. When the prices divert, the geometric Brownian motion provides a better hedging scheme because it is more robust and thereby has a smaller error. This recommendation not only holds for hedging of commodity-linked contracts, as in essay 5, but also in order to distinguish between mean reversion and a random walk. Due to the low power of the statistical tests, a comparison of the respective hedging errors in option prices seems to be a more relevant criterion and also the ultimate verdict of whether any of these processes provide a workable assumption for asset pricing.

It is therefore interesting to note that the hedging error for pulp options is smaller for the mean reverting process than for the geometric Brownian motion, 1.0 percent compared to 7.6 percent. Details are provided in appendix B of essay 5. The valuation of a pulp plant in essay 2 gave the net present value of the plant as significantly negative when prices were assumed to follow the geometric Brownian motion price process. Assuming mean reversion, the plant approximately broke even.

The results can therefore be said to support both company behaviour and academic beliefs. Academics are often doubtful to the expansion of pulp production capacity within the forest industry, and for good reasons: an ordinary discounted cash-flow analysis, which is congruent with the geometric Brownian motion, would show that the investment is not worthwhile. On the other hand, if we are to believe the result of essay 5, the pulp price is preferably modelled as mean reverting. In this case, an investment in extra capacity will break even, i.e. is a zero net present value investment. In this perspective, current expansion of pulp capacity is less surprising.

# 3. Arbitrage-free prices

Leaving the question of a random walk or mean reversion aside, the hedging exercise in essay 5 had a wider objective than merely to complement statistical tests. The ability to hedge a contract by creating an opposite position in the replicating portfolio is the core of option pricing. Without this ability, the price of an option is no longer an arbitrage free price, a price that must hold in a well functioning capital market. It is only an equilibrium price, which is a much weaker condition as it only represents "fair value" of the contract. If the arbitrage argument holds in practise, however, the "fair value" must also be the price observed in the market. Arbitrage arguments are in this sense much stronger than equilibrium results. It can be argued that models of market equilibrium, especially the Arbitrage Pricing Theory (APT), are also built around arbitrage opportunities, but this opportunity is indirect.[4] In order to explore such an arbitrage, the asset must be mispriced with respect to the entire capital market, something that is much more difficult to infer than to conclude that a derivative asset is mispriced with respect to an underlying asset.

There are several problems involved in claiming that the market price of a commodity derivative is an arbitrage-free price. First of all, the price process of the commodity must be known. All three processes tested in essay 5, the geometric Brownian motion; the mean reverting process; and the stochastic volatility process, could potentially be used to explain the price behaviour. Compared to other problems involved in a successful hedge, the mean absolute hedging errors of less than 6 percent that the processes in themselves gave rise to must be considered as relatively minor.[5] However, when the dispersion of hedging errors is large some investors may not feel comfortable by only

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[4] Fama (1993) provides a brief and accessible review of the assumptions underlying the Arbitrage Pricing Theory.
[5] Essay 5, section 4, stochastic volatility.

knowing the mean error. In addition they also want to know the maximum error to safeguard against the worst case. There is a theoretical argument stating that the hedging error is uncorrelated with the market return and that it is therefore enough to study the mean error, but this argument is only valid in the absence of transaction costs. The investors may therefore be right in worrying about the maximum error. And the maximum hedging error is substantially higher than the mean error.

The problem of choosing an appropriate price process was overshadowed in essay 5 by the problems of parameter estimation. This turned out to be a major obstacle for effective hedging. The models performed well as long as parameters were known, which means that they were estimated from the same sample as the tests were performed on. In reality, parameters are unknown, i.e. have to be estimated in advance. This produces a severe blow to the hedging performance. From previously being below 6 percent, the mean absolute hedging error now increases to 56 percent for the mean reverting model and 93 percent for the model based on the geometric Brownian motion.[6] These large errors make hedging of long-term commodity-linked contracts very difficult. It is not without taking a substantial risk that a financial institution can issue, for example, a put option of long maturity to a producer that wants to insure itself against a price decline. This risk makes the contract more expensive and increases the price above the theoretical value. This, in turn, severely limits the size of these markets, and it may explain why there is no standardised market for commodity-linked contracts and why these contracts are relatively rare despite the abundance of different contracts that characterises today's financial markets.

[6] Essay 5, section 5.

Even if it is impossible to hedge a commodity-linked contract in reality, this does not disqualify the real option approach to asset pricing. Remember that discounted cash flow analysis does not require that we should be able to hedge a contract, either. Industrial companies are not in the business of tailoring financial contracts to clients and do therefore not engage in any hedging activity. Instead, they simply want to know the (ex ante) value of an industrial investment. Should it be realised or not?

The real option approach makes it possible to value a derivative asset that has a contract payment that is non-linear in the underlying asset and thereby a risk that is different from the risk of the underlying asset. In truth, dynamic programming can also be used to calculate the value of an asymmetric contract, but being a mathematical tool it provides no guidance of what discount rate to use. (Dixit and Pindyck, 1994, chapters 5 and 6 and especially pages 147 and 185 elaborates on this difference.) When it is impossible to hedge a contract, the arbitrage-free price reduces to an equilibrium price. The lower the transaction costs and the more liquid the market, this equilibrium price will transform into an arbitrage-free price, but this is not really a requirement for the valuation of real assets. The valuation of the pulp plant in essay 2 is therefore valid independently of whether it is possible to hedge the contract or not. The possibility to hedge a real investment is of interest from a motivational viewpoint, in that the ability to actually device a hedging strategy can be used to convince managers about the correctness of an investment analysis. However, the hedge will never be carried out in reality.

The fact that the underlying assets in this dissertation are commodities is of major importance for the existence of arbitrage-free prices. Commodities are not pure financial assets whose only utility stems from the payments that the securities will generate in the future. Consumers of oil may find it practical to store some barrels in order not to risk any shortage that would cause inconvenience otherwise. This implies that some agents are willing to hold an asset, in this case oil, even though the expected return on the asset is lower than what can be motivated by a risk and return argument. In the words of McDonald and Siegel (1984), the "asset earns a below equilibrium rate of return". This shortfall is even more pronounced when the cost of physical storage is taken into account. In order to replicate the payments of the short put option synthetically, it is sometimes necessary to take a short position (i.e. borrow) in the underlying asset. This is all right for a financial contract where the yield is well defined. For a commodity contract, though, it is extremely difficult to construct a contract where the lender pays the party with the short position for the decreased storage cost (of the lender), and where the party with the short position compensates the lender for the lost convenience of physical storage. Rewriting equation (2.4) of essay 2,

$$\delta(t,S) = \mu - \frac{1}{dt} E\left[\frac{dS}{S}\right],$$

the convenience yield is equal to the equilibrium rate of return on an asset minus the expected price appreciation. However, departing from the geometric Brownian motion, $E\left[\frac{dS}{S}\right]$ is no longer constant, which also makes the convenience yield $\delta$ non-constant. As already argued, short positions specifying payments of the convenience yield to the holder of the long position are difficult to set up, and even more so when the convenience yield is non-constant and model dependent.

The severity of this problem is very dependent upon which stochastic process that best represents the price process. The geometric Brownian motion is the simplest process in this respect since the convenience yield is constant. For a mean reverting model, the convenience yield varies deterministically with the price. An example is equation (4.7) in essay 5. Stochastic convenience yield models are sometimes proposed in the option literature. See, for instance, Bjerksund (1991), and Schwartz and Smith (2000). In this case, the convenience yield varies stochastically over the life of the contract. As the convenience yield is an unobservable variable, this makes such a contract specification even more difficult to set up in practice.

The ability to replicate synthetically an option contract and thereby also obtain an arbitrage-free price is referred to by traders in the commodity markets as the issue of the fungibility of the asset.[7] For highly fungible assets, such as stocks, a derivative contract is readily replicated synthetically. For very delivery specific assets, such as live hogs, synthetic replication is much more cumbersome, to say the least. To physically trade hogs, there have to be appropriate facilities available and the hogs must be delivered at the facility at a certain point in time.

The problem of fungibility is avoided by having commodity derivatives written on the futures contract instead of on the commodity itself. A futures contract is a purely financial contract. The convenience yield is already priced in a futures contract, since there is no convenience yield attached to holding futures. It is in my view only when futures contracts are used as underlying assets, that it is meaningful to talk about arbitrage-free prices and then in relation to the futures. This does not disqualify the real option approach to asset pricing. In the case of real investments, there is no interest in actually replicating or hedging the contracts. This is possibly the reason why real option papers never base the Black-Scholes differential equation on futures contracts.

[7] Taleb (1997 chapter 12), provides a discussion about fungibility from a trader's point of view.

When it comes to commodity-linked financial contracts, the situation is more cumbersome. It is of course helpful if the contract is written in terms of a futures contract as the underlying asset. Then, there is no need to worry about the physical commodity at all. However, many of the contracts only make sense because they are related to the physical commodity, so this is not always contractually possible. But although a contract based on the physical commodity cannot be hedged, the value of the futures is closely linked to the physical commodity. For example, simple arbitrage argument allows us to determine a maximum price for the futures contract in terms of the commodity. See Hull 2003, section 3.12.

Futures and spot prices must also converge at maturity. Therefore, it is not more difficult to hedge a short-term commodity-linked contract than it is to hedge a contract where there is a purely financial underlying asset. In other words, a 3-month call option on aluminium is as readily hedged as a 3-month stock option. This is not to say that hedging is straightforward, though. The claim is only that there are no major differences and that it is meaningful to talk about arbitrage-free prices of short-term commodity-linked contracts.

Things become a little more involved for long-term commodity-linked contracts since the futures contracts are short-term. Hedging is therefore done "stack and roll", changing the underlying futures at regular intervals. This does not create any insurmountable problem since a dynamic hedging strategy requires that the weight of the underlying asset is changed anyway, implying that we can equally well change the underlying contract to another futures. But changing futures also changes the underlying futures process. If we want to connect the processes of these two futures contracts, we must use the price process of the physical commodity as the linkage. Although we have been able to circumnavigate the problem of contractually specifying a convenience yield by using futures contracts as the underlying, we are still left with the hedging uncertainty of not knowing the price process and the parameter settings of the price process for the physical commodity. As shown in essay 5, these problems are huge. Even if we accept mean errors as the reference measure, we must double the mean absolute hedging errors of 56 percent for the mean reverting model and 93 percent for the geometric Brownian motion to obtain the arbitrage free bound on option prices. In general, it is therefore not really meaningful to talk about arbitrage-free prices of long-term commodity-linked contracts.

## 4. Future research and a final word

For a dissertation that touches upon many different aspects of option pricing (market structure and interest rate derivatives are the only major subjects excluded), it is not very meaningful to talk about specific areas where research is beneficial. More research is, of course, called for in all areas. Let me nevertheless point to some specific questions that the completion of this dissertation has brought to surface.

The first such topic is a stochastic convenience yield. The economic rationale for a stochastic convenience yield is that the usefulness of owning and storing commodities changes over time. If supply is ample and cumulative storage levels are high, there is little point for a consumer to store some extra units. These are easily obtainable in the market, if need be, and the convenience yield is therefore low. On the other hand, if supply is scarce, the producer may well pay a premium for having physical ownership of the commodity. The convenience yield is high in this case.[8] As price and storage levels are negatively correlated, the argument also leads to the conclusion that convenience yield and price are positively correlated.

The importance of a correctly specified convenience yield was exemplified in essay 5 where a test without specifying any convenience yield at all resulted in a very poor hedging performance.[9] Parameter estimation is, however, very difficult for stochastic convenience yield models. The only accessible way is to infer parameters from futures prices. This renders the use of a large commodity database as in this dissertation impossible, since futures markets only exist for a fraction of all commodities, although it does exist for the most important ones.

[8] This is also qualitatively right for the convenience yield in a mean reverting model. See essay 5, section 4, mean reversion.
[9] See essay 5, section 4, the geometric Brownian motion.

A different set-up than the one in essay 5 is therefore required for testing the hedging performance of a stochastic convenience yield model.

Another area of some interest is the leptokurtosis that is present in all commodity time series. Are the fat tails due to the return distribution being stable Paretian, or is the return distribution normal with a stochastically changing volatility? The fact that the time series exhibit leptokurtosis also in subseries (not shown in essay 3) is often taken as evidence of a stable Paretian distribution, although it is very doubtful whether we are really able to dismiss the stochastic volatility hypothesis in this way. However, as previously accounted for, the poor hedging performance of the stochastic volatility model in essay 5 seems to suggest that the leptokurtosis really is due to the fact that the return distribution is stable Paretian. Further research is in any case warranted.

Though it did not work for commodities, perhaps the most interesting part of the dissertation is the stochastic volatility model developed in essay 4. The model was found to work quite well when tested on a set of option prices from the Stockholm stock exchange. The EGARCH model developed had the highest value of the maximum likelihood objective function of all the parameter estimation procedures tested. Also, the accompanying option pricing model gave prices more in line with observed market prices than the standard Black-Scholes model. More extensive comparisons on larger sets of data and for different markets are however required before any major conclusions can be drawn.

It is also potentially fruitful to develop a truly asymmetric GARCH-type parameter estimation technique. The EGARCH model developed was made symmetric by setting the asymmetry parameter to zero. This was necessitated first of all because the analytic call option formula required zero correlation between the price process and the variance process. Another reason was that the reduced form EGARCH model made it easy to find a stochastic volatility model matching it.

However, the symmetry requirement makes it impossible to capture the effect that a decline in stock prices normally is accompanied by an increase in volatility. Modelling correlation between the price process and the variance process also affects the option price substantially, which makes it possible to adjust the option pricing model to observed option prices in a better way. For example, it is sometimes argued that after the crash in October 1987, observed stock option prices imply a greater fear of a crash in the stock market than the prospect of a boom, "crash-o-phobia" in the words of Rubinstein (1994). This is difficult to model without introducing correlation between the stock price and the volatility as part of the model.

As a counterbalance against modelling such correlation stands the fact that this involves one more parameter with a substantial uncertainty about the correct value, and the observation that the GARCH(1.1) model seems to be able to explain heteroskedasticity just as well as the EGARCH model. Even if it should prove possible to build an asymmetric GARCH-type model for estimation of stochastic volatility parameters, it is therefore not certain that such a model would lead to option prices that are more correct. The "crash-o-phobia" hypothesis implies that the probability distribution implied by stock option prices is more skewed to the left than the realised stock return. A stochastic volatility model can therefore be of advantage in explaining observed stock option prices but less successful when it comes to hedging the contract. Perhaps this holds for commodities as well and is the reason why the hedging performance was so poor in essay 5.

One way to obtain correct option prices in the sense that the price is arbitrage-free in relation to other options (but not necessarily in relation to the underlying asset) is to use implied volatility surfaces. Judging from conference presentations etc., this seems to be a subject that commercial banks are very interested in right now. The procedure used for the stochastic volatility model in essay 4 made it possible to derive the instantaneous and the long-run variance rate implied by options traded. It should, however, also be possible to obtain the rest of the parameters in a systematic way. Present practice gives the impression of being more or less ad hoc.

Another aspect, worth some reflection, is how real option values are affected by different assumptions about the stochastic process. A mean reverting process often gives lower option values as the variance converges to a fixed number when the time goes to infinity. The variance of a random walk, on the other hand, grows linearly with the time-horizon. As option values increase with uncertainty, random walks often give higher option values. This result was obtained already in essay 2, and the issue becomes important for long-term contracts. However, the effect was rather small for the at-the-money options hedged in essay 5. The prices were on average only 6 percent lower for the mean reverting model and there were large individual differences.

Generally, it is fair to say that the existing literature on real options has mostly concentrated on inventing new areas of use for option theory, thereby neglecting the differences that exist between financial and real assets and overlooking the need for careful analysis of the assumptions made. This is all part of a natural process. New ideas will in due course be scrutinised, details filled in and reflections made. It is in this perspective that the present dissertation should be perceived. By studying the commodity price process from different angles, it contributes to the process of critical examination of how well option-pricing techniques work for real assets.

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# Capital budgeting in a situation with variable utilisation of capacity - an example from the pulp industry

## (Essay 2)

Henrik Andersson*

**Abstract**

It is well known that the profitability within the process industry is heavily dependent upon the degree of utilisation of the plants. Utilisation, in turn, is dependent upon the often very volatile market conditions for the commodity produced. This paper examines the implications for capital budgeting, dealing with a situation of changing levels of utilisation. A paper-pulp mill is chosen for the purpose of investigating whether, in this specific case, the variation of utilisation in response to changing market conditions affects plant value in any major way.

Comparing a fixed and a variable production rate (using the net present value rule and option pricing by means of the Feynman-Kac formula), it is found that the difference in value is considerable. However, an inappropriately specified price process may explain the difference. The geometric Brownian motion implicitly assumed in the net present value rule allows the price to decline to almost zero. In order to overcome this problem, a mean reverting price process allowing for drift in the equilibrium price level is developed and tested. The value of the ability to cut production is then found to be insignificant.

Based on the findings of this study, it is not worthwhile to model a variable utilisation of capacity. It is, however, of utmost importance to evaluate different assumptions about price behaviour, as this will affect results substantially.

Keywords: Capital Budgeting, Feynman-Kac, Mean Reversion, Real Options

JEL classification code: C60, G31, M21

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# 1. Introduction

### Background

As long as variable costs are non-zero, a company's profit will vary more when price varies and less when volume changes. A price increase does not invoke any extra costs, whereas an extra unit sold does. If the market is not perfect, in that a single actor can affect the market price acting alone, it makes sense for a producer to reduce production in order to reverse a price decline. This is especially so in the process industry where capacity expansion is slow. It takes time for the competitors to gain market shares. Besides, existing competitors are presumably equally interested in keeping the price up. Even without producers forming a cartel, which would violate antitrust laws, we may well observe behaviour where producers reduce production in times of heavy downward pressure on the price.

The latter may well be described as company policy for the major pulp and paper producer STORA, nowadays Stora Enso, who in their annual reports both -95 and -96 states that: *"Price changes have more than double the effect on income compared with volume changes, as a result of which STORA gives priority to maintaining prices in a weakening market compared with unchanged production volume."* [1]

This declaration finds support in the industry statistics. It is hardly surprising to find that during periods of high price, the utilisation of capacity has been high and vice versa. The change in aggregate production volume is in the order of 10-20%.[2]

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[1] STORA annual report 1996, p. 13.
[2] Source: Hansson & Partners data base Ecowin, "Swedish production paper and paperboard volume".

There are several good reasons to reduce pulp production in response to a price decline. One is to cut down on the storage levels of pulp. A price decline is normally preceded by an excessive supply compared to demand, resulting in high storage levels and a following price pressure.

Another reason for reducing pulp production is to put pressure on the price of pulpwood. Pulpwood is, of course, the major input to pulp production. By cutting production, an excess of pulpwood is created in the market and new price negotiations will commence with the forestry owners. A fall in the price of pulpwood will most certainly be the outcome of these negotiations. A comparison of pulp- and pulpwood prices shows a correlation of 0.74 for the period 1980-1996. An additional benefit of reducing production is the possibility to cut down on the most expensive or, when it comes to quality, inferior pulpwood first. Due to the amount of pulpwood consumed in a mill there are often logistic problems, getting access to the amount needed. The plants have to be supplied from forests further and further away, resulting in increasing costs. By reducing output, the marginal cost of pulpwood is also reduced.

**Research issue**

The question is if the willingness to decrease production in periods of low or declining price should affect the practice of capital budgeting? Obviously, it should, if the value of operating a variable production policy deviates substantially from using a fixed production rate.

However, establishing the plant value under a variable production policy is more difficult than it may first appear. Often, in practical capital budgeting, many different sources of cash flow are treated as equally risky and a single risk adjusted discount rate is used. As is pointed to in standard corporate finance textbooks, this is a simplification. Each cash flow should be considered separately and discounted with an interest rate appropriate to its systematic risk. Given a fixed production rate, valuation of cash flow resulting from sales of pulp is straightforward: Calculate, by means of the CAPM or any other market equilibrium model, the required rate of return for holding pulp and discount the cash flow accordingly.

Under a variable production rate, however, this procedure breaks down. Cash flow stemming from pulp sales now becomes a convex function of pulp price. During a recession, not only is the price low, the volume is also below normal. As a result, pulp price variation is no longer a measure of the risk of the cash flow. Luckily, option theory has been developed to deal with this situation, valuation of an arbitrary contract dependent upon an underlying asset.

In fact, the cash flow sometimes resembles that of an ordinary call option. Production is maintained as long as the market price exceeds costs and the cash flow is the difference between the two. If, on the other hand, costs exceed price, operation ceases and the resulting cash flow will be zero. The whole plant can then be valued as series of call options, expiring one at a time.[3]

### Outline of the study

In this paper the value of a fictitious, but realistic, pulp mill will be calculated using two techniques. First, a net present value calculation is applied on a plant capable of producing 400 000 tonnes of pulp annually. Thereafter, we will

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[3] Several authors have been credited for being the first to recognise the similarity between a call option and the cash flow from operating a plant. McDonald and Siegel (1985) is one of the best, but probably not the first example.

arrive at the same result using a real option technique. At this stage, a situation with varying utilisation of capacity will be introduced. Through this three-step approach, it is possible to isolate the effect of changing the scale of production from other effects, parameter settings, different assumptions etc., that may affect the result. Having introduced the option framework, further comparisons are made. This time by changing the stochastic process that the pulp price is assumed to follow.

The choice of pulp production as a case study was natural. It is both convenient and important. The convenience stems from pulp being a traded commodity with an established market price, thereby readily allowing derivative pricing. As a mature business, with not many options attached to production, modelling can be simplified without deviating too much from reality. Being a large part of the forestry industry, pulp production is also important to the Swedish economy. Of the total trade balance surplus of 131 billion SEK in 1997, 76 came from forestry industry products.[4] In Sweden, investments in the forestry sector have long-ranging economic consequences even outside the industry. Consequentially, even the procedure of capital budgeting has economic significance.

**Approach**

Let the stochastic way in which the pulp price evolves over time be described by an Ito process. As pulp is a traded commodity, the arbitrage free value of the plant *V(t,S)*, is a solution to the well known Black and Scholes differential equation:

$$\tfrac{1}{2}\sigma^2 S^2 V_{SS} + (r-\delta)SV_S + V_t - rV + \Pi(t,S) = 0,$$

[4] Source: The Swedish Forestry Industries Association. (www.forestindustries.se)

where $\Pi(t,S)$ is the flow of payments generated by the plant. A more rigorous treatment of the subject and a definition of the variables are given in section 2. As will be shown later, the above differential equation can be solved using a technique developed by Feynman-Kac. The solution is:

$$V(t_0,S) = e^{-r(T-t_0)} E^Q[V(T,S)] + \int_{t_0}^{T} e^{-r(t-t_0)} E^Q[\Pi(t,S)]\,dt.$$

The above formula seems rather complicated at first glance, but a more careful look should reveal the intuition. The value of the plant today, equals the present value of the salvage value plus the present value of all payments generated through operation of the plant. The derivation of the Feynman-Kac formula is provided in section 2.

Section 3 presents market and plant data as well as parameter estimates. In sections 4 and 5 plant value is established for a fixed- and variable production policy, respectively.

Under the assumption of a geometric Brownian motion, a net present value calculation can be seen as a special case of the Feynman-Kac formula. See appendix A for details. However, the formula is applicable to any Ito process, not only the geometric Brownian motion. As an alternative to the random walk, we will in section 6 model the pulp price as mean reverting,

$$dS = \eta(\gamma + \omega t - \ln S)Pdt + \sigma S dw.$$

The process features an interesting property in that the reversion price is allowed to increase over time, thus enabling inflation to be considered in a mean reverting model. This is an advantage over existing financial models, which treat the price level as fixed. The new model is applied to pulp data and plant value is calculated for both the fixed and variable production policies.

In addition to the variable production policy, section 7 contains an exposition of other real options. These are all minor in the pulp industry. Section 8 concludes the study. Longer derivations and repeated calculations are deferred to the appendicies.

## 2. Derivation of the Feynman-Kac formula

### The Black and Scholes differential equation

Assume a market with no taxes or transaction costs and where no agent has private information or can exercise market power. The latter assumptions are needed to ensure that the pulp price process is exogenous and we specify it as the Ito process

$$dS = \alpha(t,S)dt + \sigma(t,S)dw. \tag{2.1}$$

The change in price $dS$ is partly deterministic, specified by the term $\alpha(t,S)dt$, and partly stochastic. The stochastic behaviour is given by the volatility function $\sigma(t,S)$ and the Wiener-increment $dw$. We also assume the existence of a deterministic short rate of interest $r$. The short rate of interest will be held constant throughout this paper, as it will greatly simplify the notation, but all results hold as long as the short rate is a deterministic function of time.

To establish the value of the plant $V(t, S)$, consider the portfolio

$$\Phi = V(t,S) - S \cdot V_S(t,S), \tag{2.2}$$

where $V$ is the value of the plant.

$S \cdot V_S$ is the value of $V_S$ short positions in pulp.

$V_S$ denotes the partial derivative of $V(t, S)$ with respect to $S$.

The portfolio's return during the small increment of time $dt$ is,

$$d\Phi = dV + \Pi(t,S)dt - V_S dS - \delta S V_S dt, \tag{2.3}$$

where $dS$ is the change in plant value during $dt$.

$\Pi(t,S)$ is the profit flow generated by operating the plant during $dt$.

$V_S dS$ is the number of short positions multiplied by the price change.

$\delta S V_S dt$ is the payment that has to be made to the lender of pulp.

The term $\delta S V_S dt$ is worth some special attention and so is the notion of a short position. To create a short position in pulp, someone must be willing to lend the pulp, so that you can resell the borrowed pulp in the market and thereby establish the short position. The question is, why should anybody, presumably a paper producer holding pulp in inventory for later manufacturing, be willing to lend you the pulp?

To answer that question, another question has to be asked. Why does the lender hold an inventory of pulp in the first place? The expected price increase is not enough to motivate the inventory, so strictly on a financial basis, it should not exist. The reason for its existence is, of course, the convenience an inventory provides. Smoothing differences in supply and demand, avoiding local shortages, enhancing scheduling flexibility etc., the end goal being to disallow interruptions in paper production.

The inventory should, obviously, be large enough to serve its purpose. But, on the other hand, not too large, as the financial- and storage costs then would be excessive. Financial economists often refer to the difference between the required rate of return (if the commodity was seen as an investment object) and the expected price increase, as the marginal convenience yield net of storage costs, $\delta(t,S)$, or just convenience yield for short. As this is what the lender gives up, it is also what he should be compensated for. Thereby the term $-\delta S V_S dt$.

The required rate of return for holding pulp will be denoted by $\mu$. Generally, $\mu$ is allowed to be any deterministic function of time, but is in this paper held constant, since this is almost exclusively, the assumption made in practice.[5] The expected price increase is given by the Ito process and can be expressed as $\frac{1}{dt}E\left[\frac{dS}{S}\right]$, i.e. the expected percentage price change over the short time interval $dt$.[6]

Algebraically, the relation is

$$\mu = \frac{1}{dt}E\left[\frac{dS}{S}\right] + \delta(t,S), \qquad (2.4)$$

and the convenience yield $\delta(t,S)$ is often referred to as the rate of return shortfall, as it is the difference between the required return and the expected price increase, the name dividend yield is also used for financial assets. For a more thorough discussion on the topic, McDonald and Siegel (1984) is recommended.

Let us return to the derivation of the Black and Scholes differential equation. Applying Ito's lemma on $dV$ in (2.3) gives[7]

$$dV = V_t dt + V_S dS + \tfrac{1}{2} V_{SS} (dS)^2,$$

---

[5] Note that $\mu$ is independent of $S$. The required compensation for systematic risk may well vary over time but is, naturally, independent of the price of pulp.

[6] For the geometric Brownian motion $dS = \alpha S dt + \sigma S dw$, the expected price increase is just the constant $\alpha$. With the risk adjusted return $\mu$ held constant, the standard option textbook expression $\mu = \alpha + \delta$ is obtained, with $\delta$ being a constant.

[7] Ito's lemma can, for all practical reasons, be seen as an ordinary Taylor series expansion dropping all higher terms and noting that $(dw)^2 = dt$.

where

$$(dS)^2 = (\alpha\, dt + \sigma\, dz)^2 = \sigma^2 (dz)^2 = \sigma^2 dt.$$

The portfolio return of (2.3) becomes $d\Phi = (V_t - \delta S V_S + \frac{1}{2}\sigma^2 V_{SS} + \Pi)dt$. Note that this return is risk free, since the Wiener increment $dw$ is missing. Therefore, the portfolio's return must also equal $r\Phi dt$ in order to disallow arbitrage opportunities. We thereby get the equality

$$(V_t - \delta\, S V_S + \tfrac{1}{2}\sigma^2 V_{SS} + \Pi)dt = r(V - S V_S)dt.$$

This equality must be fulfilled for all times *dt*, giving the deterministic differential equation

$$V_t + (r - \delta) S V_S + \tfrac{1}{2}\sigma^2 V_{SS} + \Pi - rV = 0.$$

Writing explicitly all the variables suppressed in the derivation, the Black and Scholes differential equation becomes

$$\begin{aligned} &V_t(t,S) + (r - \mu + \frac{1}{dt} E\left[\frac{dS}{S}\right]) S V_S(t,S) + \\ &+ \tfrac{1}{2}\sigma^2(t,S) V_{SS}(t,S) + \Pi(t,S) - rV(t,S) = 0. \end{aligned} \tag{2.5}$$

This is a general equation for pricing of all contracts whose value is a function of another asset, the so-called underlying asset. In this case, the value of a pulp plant is a function of the market price of pulp. In the case of a stock option, the value of the option is a function of the stock price.

Depending on the type of asset or contract that is to be valued, parameters and boundary conditions change. In the case of the famous "Black and Scholes formula for a European call option on a non dividend paying stock", the dividend yield $\delta(t,S) = \mu - \frac{1}{dt}E\left[\frac{dS}{S}\right]$ is equal to zero. Further, a stock option gives the owner no profit flow before maturity, so $\Pi(t,S)$ must also be zero. Finally, the payment at maturity, *max{stock price – exercise price , 0}*, is the boundary condition that the differential equation must satisfy. [8]

**The Feynman - Kac formula**

The previous differential equation can be solved through an elegant statistical technique, which we will now go through. The resulting formula is called the Feynman-Kac formula after the originators.

Start by rewriting the Black and Scholes equation (2.5) as,

$$V_t(t,S) + \kappa(t,S)SV_S(t,S) + \tfrac{1}{2}\sigma^2(t,S)V_{SS}(t,S) + \Pi(t,S) - rV(t,S) = 0, \qquad (2.6)$$

and assume that a new variable, also called *S(t)*, follows the diffusion process

$$dS = \kappa(t,S)Sdt + \sigma(t,S)dv, \qquad (2.7)$$

where *dv* is the increment from (another) Wiener process. Note that $\kappa(t,S)$ is defined by (2.5) and (2.6) as $\kappa(t,S) = r - \mu + \frac{1}{dt}E\left[\frac{dS}{S}\right]$.

[8] Strictly speaking, in the pioneering Black and Scholes article of 1973 both $\delta$ and $\Pi$ were zero. The introduction of a dividend yield $\delta$ was made by Merton (1973). The extension to include a payment flow $\Pi(t,S)$ can be seen in Dixit and Pindyck (1994).

Applying Ito's lemma on a function $Z(t) = e^{-r(t-t_0)}V(t,S)$, we get the diffusion $dZ$ as

$$dZ = (Z_t + \kappa SZ_S + \tfrac{1}{2}\sigma^2 Z_{SS})dt + \sigma Z_S\, dv,$$

where $Z_t = -re^{-r(t-t_0)}V + e^{-r(t-t_0)}V_t$

$$Z_S = e^{-r(t-t_0)}V_S$$

$$Z_{SS} = e^{-r(t-t_0)}V_{SS}.$$

Substitution of the partial derivatives into the expression for $dZ$ gives

$$dZ = e^{-r(t-t_0)}(V_t + \kappa SV_S + \tfrac{1}{2}\sigma^2 V_{SS} - rV)dt + e^{-r(t-t_0)}\sigma V_S dv. \tag{2.8}$$

Now, let $V$ in equation (2.8) be a solution to the differential equation (2.6). This may seem questionable at a first glance but is perfectly in order. We are merely saying that the differential equation of (2.6), with a variable defined as in (2.7), has a solution along the lines of (2.8). Substituting (2.6) into (2.8) gives

$$dZ = -e^{-r(t-t_0)}\Pi(t,S)dt + e^{-r(t-t_0)}\sigma V_S dv.$$

This expression is no formal equation, but a representation of the integral equation

$$Z(T) = Z(t_0) - \int_{t_0}^{T} e^{-r(t-t_0)}\Pi(t,S)dt + \int_{t_0}^{T} e^{-r(t-t_0)}\sigma V_S(t)dv(t).$$

Taking the expected value, and noting that the expected value of a deterministic Ito-integral is zero, the expression becomes,

$$E^Q\left[Z(T)+\int_{t_0}^{T} e^{-r(t-t_0)}\Pi(t,S)dt\right]=Z(t_0)$$

$$E^Q\left[e^{-r(T-t_0)}V(T)+\int_{t_0}^{T} e^{-r(t-t_0)}\Pi(t,S)dt\right]=e^{-r(t_0-t_0)}V(t_0,S)$$

which gives the final form of the solution as

$$V(t_0,S)=e^{-r(T-t_0)}E^Q\left[V(T,S)\right]+\int_{t_0}^{T} e^{-r(t-t_0)}E^Q\left[\Pi(t,S)\right]dt. \tag{2.9}$$

The Feynman-Kac formula states that the value of an asset today, equals the discounted value of the expected payment at maturity, plus the discounted value of all expected payments that will be received before maturity. An amazingly simple solution to the partial differential equation earlier derived. However, note that the expectation should be computed for a price *S(t)* following the diffusion process (2.7), $dS=\kappa(t,S)Sdt+\sigma(t,S)dv$. Hence the notation $E^Q[...]$.

This is the price process that pulp would follow in a so-called risk neutral world, where investors do not require compensation for systematic risk. Rewriting (2.4), the expected price increase is the difference between the risk-adjusted return and the convenience yield,

$$\frac{1}{dt}E\left[\frac{dS}{S}\right]=\mu-\delta(t,S).$$

With the convenience yield $\delta(t,S)$ unchanged and total return decreasing from the risk-adjusted return $\mu$, to the riskless rate $r$ when no compensation for risk is required, the drift rate in a risk neutral world must be $r-\delta(t,S)$, which is equal to $\kappa(t,S)$[9].

The essence of a real option approach is therefore to pretend that investors are indifferent to risk and calculate the value under this assumption.[10] The result will be valid even when investors are not indifferent to risk.

**Option pricing and the capital asset pricing model**

The assumption of risk neutrality is just a computational trick, albeit a useful one, since it can be used to discount payments that do not fit into the framework of the capital asset pricing model.

CAPM is a one-period equilibrium model and the extension to a multiperiod setting is not easily made. Fama (1977) shows that discounting the expected future payments using a single risk adjusted rate of return requires the covariance with the market to be non-stochastic, i.e. the systematic risk of the payment is not allowed to change stochastically over time. Option pricing evades this problem through the creation of the instantaneous risk-free portfolio that can be used to replicate the payment. Even though the risk changes over time in a stochastic way, the portfolio can be maintained as risk-free by revising its composition, thus allowing valuation of the future payment.

---

[9] $r-\delta(t,S)=r-\mu+\frac{1}{dt}E\left[\frac{dS}{S}\right]=\kappa(t,S)$

[10] Real option literature is also concerned with the optimality problem of when to invest. The NPV criteria just says if an investment is good or bad and is not concerned with optimal timing.

The cost of this ability is the requirement of an underlying tradable asset, following a specified stochastic process, and the frequent updating of the portfolio. The advantage is, however, the ability to value any arbitrary contract as long as the above conditions are met. Specifically, the cash flow resulting from the simultaneous price and production size uncertainty can now be valued. When production size is altered as a response to changes in the market price, cash flow as a function of market price becomes convex. The risk will thereby change over time depending on the (stochastically changing) market price.

More generally, the risk will change for any payment not being a linear function of the underlying asset. Such a payment is referred to as an asymmetric payment, a terminology used by, for example, Trigeorgis and Mason (1987). In order for the CAPM to handle asymmetric payments, it would have to be updated instantaneously, which, in fact, was one of the ideas leading to the Black and Scholes differential equation, see Black 1989. In their original paper from 1973, Black and Scholes provide a derivation of their differential equation using the CAPM.

**The short rate of return**

In the derivation of the Feynman-Kac formula, a constant short rate of return was used. More generally, the short rate can be any deterministic function of time. This is also in accordance with the multiperiod CAPM since the discount rate used in a net present value calculation should, to quote Fama (1977), be "... known and non-stochastic, but the rates for the different periods preceding the realisation of the cash flow need not be the same ...".

If the short rate is not constant, but still deterministic, the discount factor $e^{-r(T-t_0)}$ should be substituted by $e^{-\int_{t_0}^{T} r(t)dt}$. Letting $\bar{r}$ denote the average risk-free short rate, the discount factor can be written as $e^{-\bar{r}(T-t_0)}$.

This highlights a subtle and often overlooked point. When, for practical purposes, a constant discount rate is used, it is the average short rate of return that should be used. Not the short rate presently observed in the money market.

## 3. Data gathering

### Model used

In order to value the pulp plant, it is necessary to state the assumptions more specifically. The derivation in section 2 was based on the very general Ito process (2.1). We now specify the price process as a geometric Brownian motion. This assumption is made partly for congruence and partly for convenience. As can be seen in appendix A, the net present value and the Feynman-Kac formula will give the same answer for any price process $dS = \alpha Sdt + \sigma(t,S)dw$. Specifying the volatility function $\sigma(t,S) = \sigma \cdot S$, where $\sigma$ is a constant, we arrive at the ordinary geometric Brownian motion. Observe, though, that the closer specification of the volatility function is not needed for congruence with the net present value calculation. It is only needed for the asymmetric payments, occurring when the production rate is altered in response to changing market conditions.

### Parameter estimation

In the growing literature about real options, parameter estimation is a problem that has been given surprisingly little coverage. It is unclear why. Either the problem is considered trivial or deemed as an applicational aspect rather than a theoretical problem. Whatever the reason, if the real option method is ever to gain acceptance in the business community, a reasonably simple procedure to estimate the parameters is a necessity. This is one reason why the geometric Brownian motion is suitable to start with. Following Björk (1998, p. 93), the geometric Brownian motion, $dS = \alpha Sdt + \sigma Sdw$, has the solution

$$\ln S_T - \ln S_0 = (\alpha - \tfrac{1}{2}\sigma^2)(T - t_0) + \sigma w(T - t_0).$$

Defining $\xi_T$ as the normally distributed variable

$$\xi_T = \ln \frac{S_T}{S_0} \sim N\left[\left(\alpha - \tfrac{1}{2}\sigma^2\right)(T - t_0), \sigma\sqrt{T - t_0}\right], \tag{3.1}$$

gives the discrete observations as

$$\xi_{t+1} = \ln \frac{S_{t+1}}{S_t} \sim N\left[\left(\alpha - \tfrac{1}{2}\sigma^2\right)\Delta t, \sigma\sqrt{\Delta t}\right].$$

Estimating parameters in the normal way, we have the mean as

$$\left(\alpha - \tfrac{1}{2}\sigma^2\right)\Delta t = \bar{\xi} = \frac{1}{n}\sum_t \xi_t\,,$$

and the standard deviation

$$\sigma\sqrt{\Delta t} = s = \sqrt{\frac{1}{n-1}\sum_t \left(\xi_t - \bar{\xi}\right)^2}\,.$$

The parameter estimates are then[11]

$$\sigma = \frac{s}{\sqrt{\Delta t}} \text{ and } \alpha = \frac{\bar{\xi}}{\Delta t} + \tfrac{1}{2}\sigma^2.$$

[11] Although these are the standard estimates, the estimate of $\alpha$ is not consistent. See Gourieroux and Jasiak (2001, chapter 12) and also appendix D in this paper for more details.

**Market data**

Parameters are based on continuously compounded historical market data for the period 1980-1996. Data have been collected quarterly and are detailed in appendix B. All data are related to Swedish crowns, although computations in U.S. dollars would yield almost identical results.

| Parameters | Comments |
|---|---|
| Risk free interest rate: $r = 6.4\%$ | Measured as the historical real rate plus an expected inflation of 2%.[12] |
| Expected price increase in pulp: $\alpha = 1.3\%$ | Measured as the historical real drift, −0.7% plus 2% expected inflation. |
| Standard deviation of pulp prices: $\sigma = 18.9\%$ | |
| Beta of pulp prices: $\beta = 0.16$ | The return on Affärsvärldens generalindex at the Stockholm Stock Exchange is used as a proxy for the market. |
| Market risk premium: $r_M = 8\%$ | This is an average during the 20th century, see Ibbotson and Sinquefield. |
| Risk adjusted discount rate: $\mu = 7.7\%$ | Through CAPM $(6.4 + 0.16 \cdot 8)$ |
| Rate of return shortfall: $\delta = 6.4\%$ | Defined as $\mu - \alpha$. |

By adopting the forecast (and objective) by the Swedish Riksbank of a future inflation rate of 2%, we are projecting a lower inflation than the one inherent in historical data. Therefore, the forecasted inflation rate is added to the historical real interest rate. When it comes to drift and diffusion of the pulp price, which

[12] The inflation forecast is made by the Swedish Riksbank. Since interest rates are continuously compounded, the nominal rate is obtained by just adding the real rate and the inflation.

are calculated in appendix B, it is interesting to note that diffusion is unchanged ($\sigma$ = 18.9%), independently of whether the pulp price is expressed as real or nominal. Furthermore, the difference between the real and the nominal drift rate equals the inflation rate, so the procedure to calculate the real drift rate and add the expected future inflation can be used. This is for practical applications quite important, as ambiguities are avoided.

There is another interesting observation in the above data. The systematic risk of pulp is very low, beta equals 0.16. Forestry companies have a much higher $\beta$, often above unity, although the major insecurity is the price of forestry products.[13] Even after adjusting for financial leverage, there is a huge gap.

One explanation may be that the stock market reacts faster than price changes. For example: If there is trustworthy information of an upcoming recession, the stock market will incorporate this information immediately. However, the pulp price will not decrease until the information has been proven true and an actual recession hits the market. In this scenario we would expect low correlation between the stock market and the pulp price, with a correspondingly low $\beta$. This is also what we observe in the above data.

The lack of correspondence between pulp $\beta$ and company $\beta$ presents a problem for capital budgeting. Using company $\beta$ to determine the riskiness of cash flow, as is often done, will give a very different result from using pulp $\beta$. A net present value calculation can be performed using whatever $\beta$ is preferred. A real option calculation, on the other hand, relies exclusively on

[13] Another major source of insecurity is the dollar exchange rate. The market price of forestry products is determined in dollars. However, at least for pulp, computations in dollar and using dollar stock markets, give similar results. Hence, it does not explain the deviation.

pulp $\beta$. Thus, one could not in general expect to be able to replicate a net present value calculation in a real option framework.

**Plant data**

The fictious, but realistic, plant used as a case study in this paper is capable of producing 400 000 tonnes of pulp per annum. The pulp is of the standard kraft traded: NBSK- Northern Bleached Softwood Kraft pulp, which is a sulphate pulp that is often specified as 90% dry. 90% dry means that it is storable and can be sold to papermills built separately. Otherwise it is quite common within the industry to place pulp- and papermill together. The advantage is that of less drying and transportation of the pulp. However, it also becomes more troublesome to separate the pulp process, which is the reason this study deals with a plant for market pulp only. Another reason to choose this kind of plant is that dried NBSK pulp is a standard commodity with an established price. For this commodity, there is also an operating futures market.

| Data | Comments |
|---|---|
| Today's price of pulp: SEK 4500 per tonne. | USD 600 × 7.50 SEK/USD = SEK 4500 |
| Milling capacity: 400 000 tonnes. | 400 000 tonnes is a reasonable size. Although there are some economies of scale in a larger mill, there will probably be logistic problems in receiving enough pulpwood. |
| Economic life: 30 Years. | Technically, a bit on the conservative side. However, the fact that there are not many older mills in operation today suggests that the economic life is around 30 years. |

| | |
|---|---|
| Investment costs: SEK 4 500 million. | Wood handling 300, Digesting, Screening and Washing 700, Bleach plant 500, Recovery boiler 800, Evaporation unit 400, Recaustizing 400, Logistics 400, Drying 1 000. |
| Cost of pulpwood: 30% of pulp price. | Generally speaking, when the price of pulp changes, so does the price of pulpwood. 30% is the average cost. |
| Other variable costs: SEK 1250 per tonne. | Chemicals, energy and transportation. |
| Maintenance: SEK 150 million the first 15 years, thereafter 250 million. | This is a simplification in order to reduce the amount of calculation needed. Presumably, maintenance costs follow a parabola. Costs are low when the machinery is new and when abandonment is close and higher in between. |
| Other fixed costs: SEK 300 million. | Whereof 70% are wages. |
| Salvage value: Zero. | Costs of site recovery and the value of being able to continue operation are minor. See Section 7, The expansion option, for a more detailed discussion. |

The costs have been obtained through interviews with industry representatives and should be seen as reasonable, but not necessarily true for any specific plant. Real costs will decrease in the future due to continuous productivity gains. According to the industry representatives, pulp has shown a long-term price decline of 1 % per annum in real terms. Naturally, even the costs of production have decreased, as there otherwise would be no producers left. The real price of pulp has for the 1980-96 period decreased with an average of 0.7 % per annum. We will use the same assumption for costs and therefore, with 2 % inflation, let the costs increase by 1.3% over time. Costs will be discounted at the risk-free rate since they are assumed to be quite stable and not correlated with market return.

In order to judge the realism of the data, a Profit and Loss Account can be helpful. Taking the initial price of 4500 SEK per tonne as given, the accounts for the first year will look like:

| | Year 1 (MSEK) |
|---|---|
| Pulp sales (400 000 tonnes) | 0.4·4500 |
| – Pulpwood cost (30% of sales) | – 0.3·0.4·4500 |
| – Other variable costs | – 0.4·1250 |
| – Maintenance | – 150 |
| – Other fixed costs | – 300 |
| – Depreciation (straight line, 30 years.) | – 150 |
| | 160 |

A small accounting profit can be expected for the first year. Also for subsequent years, a small profit can be expected. How small depends on the amount of maintenance needed and the depreciation method used. What the Profit and Loss Account fails to encompass, however, is the time value of money and the immense uncertainty of the pulp price. Let us therefore continue with a net present value calculation.

# 4. A fixed production policy

### The net present value

(All results are in MSEK)

$$PV(\,pulp\ sales)^{14} = \int_0^{30} e^{-0.077t}\left[0.4 \cdot 4500e^{0.013t}\right]dt = 24002$$

$$PV(\,pulpwood\ cost)^{15} = -\int_0^{30} e^{-0.077t}\left[0.3 \cdot 0.4 \cdot 4500e^{0.013t}\right]dt = -7201$$

$$PV(\,other\ variable\ costs)^{16} = -\int_0^{30} e^{-0.064t}\left[0.4 \cdot 1250e^{0.013t}\right]dt = -7681$$

$$PV(maintenance)^{17} = -\int_0^{15} e^{-0.064t}\left[150\, e^{0.013t}\right]dt - \int_{15}^{30} e^{-0.064t}\left[250\, e^{0.013t}\right]dt = -2792$$

$$PV(other\ fixed\ costs)^{18} = -\int_0^{30} e^{-0.064t}\left[300\, e^{0.013t}\right]dt = -4609$$

Added together, the present value of operating this plant is SEK 1719 million. Thus, operation of an existing plant is profitable. However, any new investment is out of question, since the owners would then have to pay the investment costs of SEK 4500 million as well.

The net present value is SEK – 2781 million.

[14] $4500e^{0.013t}$ is the expected price and 0.4 represent the 400 000 tonnes of pulp produced each year, thereby giving the result in MSEK. Pulp sales and pulpwood costs are discounted at the risk-adjusted discount rate appropriate to the risk of pulp. All other items are discounted at the risk free rate, as these costs are assumed uncorrelated with market return.

[15] Pulpwood cost is approximately 30% of pulp price.

[16] 400 000 tonnes of pulp per year, times a variable cost of $1250e^{0.013t}$ per tonne.

[17] Maintenance is 150 MSEK per year, for the first 15 years and thereafter 250 MSEK per year.

[18] Other fixed costs are 300 MSEK per year.

**The real option technique**

The net present value can also be obtained through the Feynman-Kac formula (2.9). Using this option technique, all values should be calculated as in a so-called risk neutral world. All costs (except pulpwood) are already discounted at the risk free rate, so there is no need to repeat the calculations here. Instead we demonstrate the technique by calculating the value of pulp sales.

Using the definition of expected value,

$$E^Q[\mathit{pulp\ sales}] = E^Q[0.4S(t)] = \int_{-\infty}^{\infty} (0.4S(t))\cdot\varphi(s)ds.$$

The price variable, $S(t)$, is here lognormally distributed. In order to work with the more familiar normal distribution, we use equation (3.1) to make the variable transformation

$$S(t) = S_0 e^{X(t)}, \text{ where } X(t) \sim N\left[\left(r-\delta-\tfrac{1}{2}\sigma^2\right)t, \sigma\sqrt{t}\right].$$

Denoting $X \sim N[a(t), b(t)]$, the probability density function $\varphi(x)$ is

$$\varphi(x) = \frac{1}{b(t)\sqrt{2\pi}} e^{-\frac{(x-a(t))^2}{2b^2(t)}},$$

and the numerical values of $a(t)$ and $b(t)$ are

$$a(t) = \left(r-\delta-\tfrac{1}{2}\sigma^2\right)t = -0.01786t$$
$$b(t) = \sigma\sqrt{t} = 0.189\sqrt{t}.$$

The expected pulp sales can be expressed as

$$E^Q[\mathit{pulp\ sales}] = \int_{-\infty}^{\infty} \left(0.4\cdot 4500e^x\right)\cdot\varphi(x)dx = \int_{-\infty}^{\infty} \frac{0.4\cdot 4500e^x}{b(t)\sqrt{2\pi}} e^{-\frac{(x-a(t))^2}{2b^2(t)}} dx,$$

which, using completion by squares, reduces to

$$E^Q[pulp\ sales] = 0.4 \cdot 4500 e^{a(t)+\frac{1}{2}b^2(t)}.$$

Discounting using the risk-free interest rate gives today's value as

$$V_0\,(\,pulp\ sales) = \int_0^{30} e^{-0.064t}\left[0.4 \cdot 4500 e^{a(t)+\frac{1}{2}b^2(t)}\right]dt = 24002\,.$$

This is the same value as was obtained by the present value calculation in the beginning of this section. The motive for repeating it is to show that there is no difference in assumptions so far and later differences in firm value are fully due to changes in the production rate. With all preliminaries behind us, it is now time to model such a situation.

# 5. Modelling a variable production rate

The textbook rule of operation, "maximise the contribution to profit by utilising the plant to capacity as long as price exceeds variable cost", describes an ideal situation. It hinges upon many assumptions of which the most important are:

- No costs of stopping or starting production.
- There will be no market consequences if production is halted.
- Full competition is prevalent and no actor can affect the market price.
- True costs are known.

Since these assumptions are quite restrictive, the challenge facing the management is more complex than the simple textbook rule suggests. Of particular interest to this study is the willingness to decrease production in order to reverse a price decline. The company thereby believes that it has some discretion over market price development[19] or that other manufacturers will follow suit and decrease production.

This study is not trying to question the rationality in this behaviour, nor is the intent to find the optimal production policy. The purpose is to deduct whether it is necessary to detail different production policies when performing (advanced) capital budgeting.

It has been quite difficult to obtain a realistic production policy, even though industry statistics give a clear connection between market price of pulp and the production rate. Managers consider so much more than just the current market price. When deciding what production rate to choose, managers also consider

19 From a modelling point of view, we implicitly assume that the company does not have "too much" discretion over price, as the price process then would be endogenous, making valuation difficult.

market trends, as well as storage levels of pulp, both in the market[20] and in their own warehouses.

In spite of the difficulties associated with, a priori, specifying a production policy, the policy here specified is not unreasonable. It should, without any pretence of being optimal, give an appreciation of the magnitude of change in plant value that a variable production rate accounts for. We specify the production policy as follows:

Normally the plant operates at maximum capacity and it so continues as long as price stays above SEK 3500 (in the price level of year zero). The profit and loss account will show red figures even above this price, but due to competition in the marketplace and the contribution to profit, nothing will happen before the price decreases to SEK 3500. Then the company will react, trying to push the price upwards by cutting production. In the range SEK 3500-2600, production decreases linearly from 100% to 70%. If price is less than SEK 2600 the production is altogether shut down and also maintenance is stopped. This is approximately equivalent to saying that all work ceases when price is less than variable costs.[21]

**Price > 3500, full production**

An expected increase in price (and costs) of 1.3% per annum, requires the lower bound to be specified as $S(t) \geq 3500\, e^{0.013t}$. Using *X* as the stochastic variable

[20] Market storage levels are measured by the so called NORSCAN level. NORSCAN stands for North America and Scandinavia, and measures the amount of pulp warehoused in these regions.
[21] Only approximately equivalent, as pulpwood cost changes with the level of pulp price and there are semi-fixed maintenance costs to consider. Although the price of pulpwood mirrors that of pulp for reasonable price levels, it is unlikely that this will be the case if the pulp price is very low. Wood can be used for other purposes than pulp production. The figure SEK 2600 is obtained by adding the variable costs of SEK 1250, to the initial cost of pulpwood, which for a pulp price of SEK 4500, is $30\% \cdot 4500 = 1350$.

(since it is normally distributed) with $S(t) = 4500e^{X}$, the lower boundary for $X$ becomes:

$$X \geq \ln\frac{3500}{4500} + 0.013t.$$

The upper boundary of a normally distributed variable is, of course, infinity, but for computational convenience we confine the boundary to five standard deviations.

$$V_0\,(pulp\ sales) = \int_0^{30} e^{-0.064t} \int_{\ln\frac{3500}{4500}+0.013t}^{a(t)+5b(t)} (0.4 \cdot 4500e^{x}) \cdot \varphi(x)dxdt = 17629$$

$$V_0\,(pulpwood\ cost) = -\int_0^{30} e^{-0.064t} \int_{\ln\frac{3500}{4500}+0.013t}^{a(t)+5b(t)} (0.3 \cdot 0.4 \cdot 4500e^{x}) \cdot \varphi(x)dxdt = -5289$$

$$V_0\,(other\ variable\ costs) = -\int_0^{30} e^{-0.064t} \int_{\ln\frac{3500}{4500}+0.013t}^{a(t)+5b(t)} (0.4 \cdot 1250e^{0.013t}) \cdot \varphi(x)dxdt = -3918$$

$V_0\,(maintenance) =$

$$-\int_0^{15} e^{-0.064t} \int_{\ln\frac{3500}{4500}+0.013t}^{a(t)+5b(t)} (150e^{0.013t}) \cdot \varphi(x)dxdt - \int_{15}^{30} e^{-0.064t} \int_{\ln\frac{3500}{4500}+0.013t}^{a(t)+5b(t)} (250e^{0.013t}) \cdot \varphi(x)dxdt =$$

$= -1333$

$$V_0\,(other\ fixed\ costs) = -\int_0^{30} e^{-0.064t} \int_{\ln\frac{3500}{4500}+0.013t}^{a(t)+5b(t)} (300e^{0.013t}) \cdot \varphi(x)dxdt = -2351$$

**2600 < Price < 3500, reduced production**

As the price decreases, so does the production rate. Utilisation decreases linearly from 100% for a price of 3500 to only 70% when the price is 2600. Not only production and variable costs are reduced within this price range. Also maintenance is cut. It is possible to cut down on maintenance since maximum output is not an issue. Even if the plant is out of operation for a while, this is no major issue since it is possible to catch up on production later.

Denote the level of utilisation with *f(S)*. In nominal terms, utilisation changes linearly from 70% when the price equals $2600e^{0.013t}$ to 100% when the price is $3500e^{0.013t}$. Utilisation as a function of price will then be the straight line $f(S)=\frac{e^{-0.013t}}{3000}S-0.167$. Expressed in the variable *X* instead, the utilisation function becomes $f(X)=\frac{e^{-0.013t}}{3000}\cdot 4500e^{X}-0.167$.

The integration limits for the stochastic variable *X* are $\ln\frac{2600}{4500}+0.013t\le X\le\ln\frac{3500}{4500}+0.013t$.

$$V_0\,(pulp\ sales)=\int_0^{30}e^{-0.064t}\int_{\ln\frac{2600}{4500}+0.013t}^{\ln\frac{3500}{4500}+0.013t}f(x)\cdot\left(0.4\cdot 4500e^{x}\right)\cdot\varphi(x)\,dx\,dt=2750$$

$$V_0\,(pulpwood\ cost)=-\int_0^{30}e^{-0.064t}\int_{\ln\frac{2600}{4500}+0.013t}^{\ln\frac{3500}{4500}+0.013t}f(x)\cdot\left(0.3\cdot 0.4\cdot 4500e^{x}\right)\cdot\varphi(x)\,dx\,dt=-825$$

$$V_0\,(other\ variable\ costs) = -\int_0^{30} e^{-0.064t} \int_{\ln\frac{2600}{4500}+0.013t}^{\ln\frac{3500}{4500}+0.013t} f(x)\cdot\left(0.4\cdot 1250e^{0.013t}\right)\cdot\varphi(x)dxdt =$$

$= -\ 1116$

$V_0\,(maintenance) =$

$$-\int_0^{15} e^{-0.064t} \int_{\ln\frac{2600}{4500}+0.013t}^{\ln\frac{3500}{4500}+0.013t} f(x)\cdot\left(150e^{0.013t}\right)\cdot\varphi(x)dxdt -$$

$$-\int_{15}^{30} e^{-0.064t} \int_{\ln\frac{2600}{4500}+0.013t}^{\ln\frac{3500}{4500}+0.013t} f(x)\cdot\left(250e^{0.013t}\right)\cdot\varphi(x)dxdt$$

$= -\ 388$

$$V_0\,(other\ fixed\ costs) = -\int_0^{30} e^{-0.064t} \int_{\ln\frac{2600}{4500}+0.013t}^{\ln\frac{3500}{4500}+0.013t} \left(300e^{0.013t}\right)\cdot\varphi(x)dxdt = -\ 786$$

**Price < SEK 2600, no production**

Changing the price variable gives the upper integration limit

as $X \leq \ln\dfrac{2600}{4500} + 0.013t.$

The lower integration limit of minus infinity is for computational convenience confined to five standard deviations, $a(t) - 5b(t)$. Only fixed costs are present when the price is less than 2600 Swedish crowns.

$$V_0\,(\textit{other fixed costs}) = -\int_0^{30} e^{-0.064t} \int_{a(t)-5b(t)}^{\ln\frac{2600}{4500}+0.013t} \left(300e^{0.013t}\right)\cdot \varphi(x)dxdt = -\,1472$$

**Totally**

Added together, $V_0$ *(future cash flow)* = 2901 MSEK. In the case of a present value calculation, $PV$ = 1719 SEK. As the investment cost of 4500 SEK has to be subtracted in both cases, the investment is not worthwhile in either case. The difference between the two valuations, roughly 1200 MSEK, is the additional value of the specified production policy. This difference will increase the more volatile the market is, as the probability of a low price thereby increases, making loss cutting policies all the more important. On the other hand, a higher drift rate reduces the difference, as a low price then is less probable.[22] Hence, the differences are parameter dependent and should therefore also be dependent upon the price process specified.

There are two counteracting effects accounting for the difference in value. First and foremost the loss cutting procedure of closing the plant whenever price is less than variable costs. This increases the value by some 1250 MSEK.[23] Secondly, reducing production although price is above variable costs will result in a loss of contribution to profit by some 50 MSEK. Thus, from a capital budgeting point of view, the eagerness to restrain a price decline by decreasing production seems of less interest. The important thing is to stop production whenever price is less than variable costs.

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[22] For example, allow the plant to break even under the variable production scheme. This can be achieved by increasing $\sigma$ from 0.189 to 0.33, holding all other parameters constant. The difference between the two policies is now 2800 MSEK. If we instead adjust the drift rate $\alpha$, from 1.3% to 2.3%, the plant will also break even under the variable production scheme, but now the difference is only 500 MSEK.

[23] Changing the integration limits so that full production is sustained for a price exceeding SEK 2600 per tonne separates this effect.

Observe, however, that production is only stopped when the price declines below SEK 2600 per tonne (in year zero price level). As the price has never been this low, historically, it is easy to suspect the geometric Brownian motion price process of assigning relatively high probabilities to rather unlikely outcomes. The purpose of the next section is therefore to study the price process in more detail and to suggest an alternative.

# 6. Modelling the pulp price as mean reverting

The geometric Brownian motion assumed so far, is not uncontested as the model of how prices behave. It is definitely the most natural candidate, given its many advantages: The geometric Brownian motion is relatively easy to understand and has an explicit analytical solution. It is also congruent with net present value calculations and parameter estimation is fairly simple. Furthermore, as a model of stock price behaviour, a constant relative drift rate plus normally distributed noise conform nicely to how we (perhaps naively) would expect stock prices to behave.[24]

However, for the movements of commodity prices, there are some compelling arguments why the behaviour should not be modelled in this way. If the pulp price is exceptionally high, this will presumably attract new producers trying to profit from the situation, with a price decline as a result of the increased competition. Taking the other extreme, when price is below marginal cost, some producers will be forced out of the market, leaving the others struggling to increase the price.

Undoubtedly, a "truer" model of price movements should capture this equilibrium characteristic, called mean reversion. However, the geometric Brownian motion is not easy to disclaim empirically. Pindyck and Rubinfeld (1991, chapter 15) perform a unit root test, where they are only able to reject a random walk of copper and crude oil prices when more than 100 years of data are used. Even so, they fail to reject a random walk in the price of lumber.

[24] A constant relative drift rate represents the fact that some of the operating profits are reinvested and can be expected to earn the same return as existing funds. We could thus expect the stock price to appreciate over time, but have to add a diffusion term accounting for new information. This diffusion will be normally distributed (the central limit theorem) if it is caused by many independent pieces of news.

Other authors are not so sure. Schwartz (1997) found strong mean reversion in futures prices of copper and oil, with significant coefficients.[25]

It is of interest to calculate the value of the pulp plant under the assumption that the price is mean reverting. First and foremost because we earlier saw that a large part of the different results between a fixed and a variable production rate came from price levels that might (arguably) be unrealistically low. It is also important because careful analyses, allowing for mean reversion and other characteristics, should always be undertaken before committing capital to a major investment.

The simplest and most well known mean reverting process is the Ornstein-Uhlenbeck process, popularised by Vasicek (1977) as a model of how interest rates behave. It will also serve as a starting point for describing pulp price behaviour. The infinitesimal characteristic reveals the basic property of the process,

$$dS = \eta\left(\bar{S} - S\right)dt + \sigma dw. \tag{6.1}$$

If $\bar{S} < S$ this process exhibits a negative drift and when $\bar{S} > S$ the drift is positive. Hence, the (real) price of pulp is driven back to its long-term average $\bar{S}$ with a speed of reversion $\eta$.

[25] It is interesting to note that Schwartz fails to verify mean reversion in the price of gold futures. The coefficients are not significantly different from zero. Gold is often considered an investment asset rather than a commodity, and we would therefore be more tempted to model the price process as a geometric Brownian motion.

One of the characteristics of the Ornstein-Uhlenbeck process is that the price is normally distributed. Although this is computationally convenient, it lacks economic appeal in that negative prices thus are allowed. This is not so for the previously used geometric Brownian motion. There, negative prices are disallowed because it is the logarithm of the pulp price that is normally distributed. The pulp price is thereby, by definition, lognormally distributed.

Using the same logic, it is natural to suggest that the pulp price could be modelled by letting the logarithm of the price follow an Ornstein-Uhlenbeck process. This is also the approach taken by Schwartz (1997). Ekvall, Jennergren and Näslund (1995) make use of the same process for modelling the spot exchange rate and for valuing currency options.

However, in one important respect, commodity prices are different from the exchange rates underlying currency options. Commodity prices are subject to inflation. Whereas it is difficult to see why an exchange rate should exhibit any long-term drift, commodity prices do increase over time. Also, technological improvements may well lead to a drift rate separated from the rate of inflation, precluding a use of real discount rates.[26]

Therefore, this paper proposes a model where the nominal price is mean reverting. Instead of assuming the equilibrium price $\bar{S}$ to be a constant, we allow it be time dependent with $\bar{S}(t) = \bar{S}_0\, e^{\omega t}$.

Taking the logarithm of the equilibrium price, we get

[26] The use of real discount rates may also interfere with the ability to replicate the cash flow of the derivative asset. Cash flows are almost always nominal and trying to lock in an arbitrage profit from a mispricing of the real cash flow might be difficult.

$$\ln \bar{S}(t) = \gamma + \omega t \quad \text{with} \quad \gamma = \ln \bar{S}_0 . \tag{6.2}$$

The proposed model will then be

$$dS = \eta(\gamma + \omega t - \ln S)S\,dt + \sigma S\,dw. \tag{6.3}$$

This model will allow both mean reversion and an equilibrium price that increases over time, as well as disallow negative prices. By applying Ito's lemma to ln $S(t)$, as is done in appendix C, the process reduces to

$$d\ln S = \eta(\gamma' + \omega t - \ln S)dt + \sigma dw \quad \text{where} \quad \gamma' = \gamma - \frac{\sigma^2}{2\eta}. \tag{6.4}$$

Without drift, $\omega = 0$, the process (6.4) becomes an Ornstein-Uhlenbeck process of the type (6.1). The derivation in appendix C, shows that ln $S(T)$ is normally distributed with mean

$$E[\ln S(T)] = \left(\gamma' - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta(T-t_0)}\right) + \omega\left(T - t_0\, e^{-\eta(T-t_0)}\right) + \ln S(t_0) e^{-\eta(T-t_0)} \tag{6.5}$$

and variance

$$Var[\ln S(T)] = \frac{\sigma^2}{2\eta}\left(1 - e^{-2\eta(T-t_0)}\right). \tag{6.6}$$

**Parameter estimation**

There is no generally agreed way of parameter estimation for mean reverting processes and appendix D gives comments on different methods. The one used, inspired by Harvey (1989) pp. 481-82, is arguably the most straightforward and is congruent with the way the parameters of the geometric Brownian motion were estimated. For both types of stochastic processes, the logarithm of the price is normally distributed. In the case of the geometric Brownian motion, also the return is normally distributed, making parameter estimation straightforward. The additional $t$ term in the mean reverting process complicates matters, but since

$$\ln S_{t+1} = \left(\gamma' - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta\Delta t}\right) + \omega\,\Delta t + \omega\left(1 - e^{-\eta\Delta t}\right)t + e^{-\eta\Delta t}\ln S_t + + \sigma\sqrt{\frac{1 - e^{-2\eta\Delta t}}{2\eta}}\,\varepsilon_t, \tag{6.7}$$

where $\varepsilon_t \sim N(0,1)$, it is nevertheless possible to run the regression

$$\ln S_{t+1} = c_0 + c_1\,t + c_2 \ln S_t + s\,\varepsilon_t, \tag{6.8}$$

to estimate the parameters. The details are in appendix D, but the result follows immediately from the expressions of the mean and variance, equation (6.5) and (6.6).

Although the regression (6.8) looks straightforward, it nevertheless poses an inherent problem, that of multicollinearity. As prices tend to increase over time, $t$ and $\ln S_t$ will be correlated, affecting parameter estimations as a result. Harvey (1989) avoids this problem as he deals with a standard Ornstein-Uhlenbeck process, where the equilibrium price is not time dependent.

If price was constant in real terms, working with real price data would be a solution as time and price then would be uncorrelated.[27] However, the pulp price has decreased in real terms and the correlation between time and price is around 0.6 for both real and nominal data.

One typical remedy to the multicollinearity problem is to delete some of the collinear variables, thereby improving the precision of the remaining regression coefficients. See, for example, Canavos, 1984, p. 485.

Following this tradition, we delete time and use the logarithm of the real price as the only dependent variable in the regression. Formally, the regression run is the AR(1) process

$$\ln S_{t+1} = c_0 + c_2 \ln S_t + s\,\varepsilon_t \tag{6.9}$$

with

$$c_0 = \gamma'\left(1 - e^{-\eta \Delta t}\right)$$
$$c_2 = e^{-\eta \Delta t},$$

since $\omega$ in equation (6.7) is zero. As given by appendix D, the parameter estimations will be:

$$\eta = 0.25$$
$$\sigma = 0.19$$
$$\gamma = 8.56$$

[27] If $t$ and $\ln S_t$ were orthogonal, in the sense that $\mathrm{Corr}[t, \ln S_t] = 0$, the regression coefficients would be unaffected by the number of dependent variables used.

The next step is to give a separate estimation of $\omega$, the expected increase in the equilibrium price, as it was left out of the regression. It seems reasonable to give $\omega$ the same numerical value (1.3%) as the drift rate $\alpha$ of the geometric Brownian motion. Choosing another value, and thus different values for the drift of the geometric Brownian motion and the equilibrium drift of the mean reverting process, makes comparisons difficult.

The main difference between the two processes is shown in figure 6.1, where 95 percentage confidence intervals are depicted. Whereas the mean reverting process is confined to a price range which most people would conceive as reasonable, trajectories generated by the geometric Brownian motion can sometimes wander far off and away from any economic reality. This explains why the loss cutting procedure in section 5 had such an influence on plant value. It is possible that the price will wander below variable costs and stay there for extended periods of time.

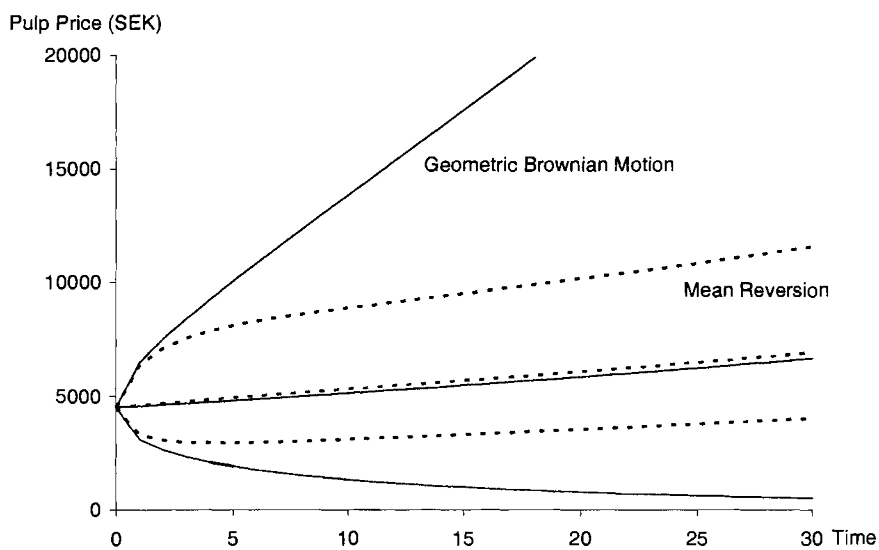


Figure 6.1 95 percentage confidence intervals and expected future pulp prices.

Figure 6.1 also points at the importance of choosing $\omega = 1.3\%$, as the expected value of the two processes thereby will be approximately the same. This congruence could not have been obtained with a less advanced mean reverting process. Such processes treat the real price as constant, whereas the pulp price has decreased in real terms.

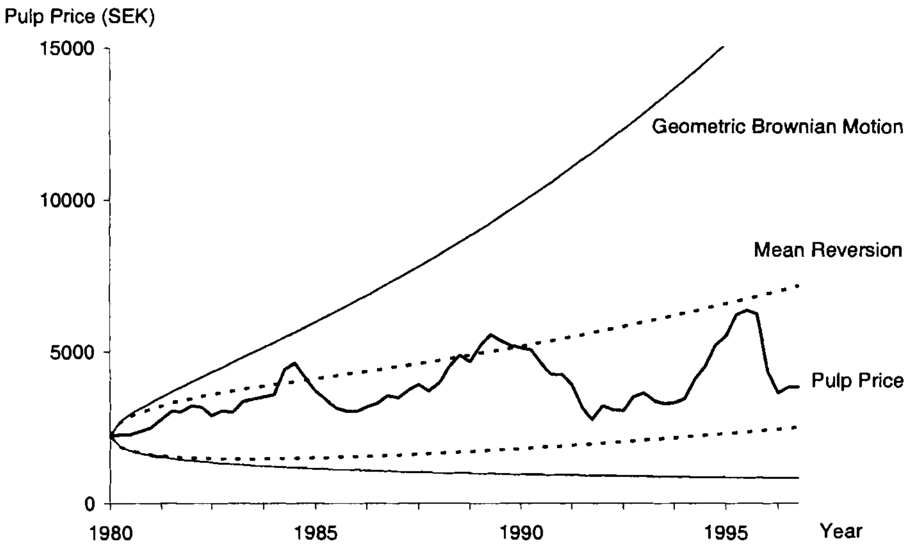


Figure 6.2 Historical pulp prices and 95 percentage confidence intervals.

Figure 6.2 shows how the price has evolved from 1980 to 1996. As prices tend to increase over time there is a difference in scale between figures 6.1 and 6.2. Judging from the graph, the pulp price seems to oscillate around a slowly increasing equilibrium. In other words, the pulp price gives the impression of being mean reverting. A diagram does not provide any conclusive evidence, however. It only gives one trajectory, one realisation of the price process. For each of the two explanatory price processes, there are an infinite number of other trajectories possible.

Also shown in figure 6.2 are 95 percentage confidence intervals. As parameters are estimated for the 1980-96 period, the confidence intervals are constructed within sample in figure 6.2. The most striking characteristic is the wide confidence interval for the geometric Brownian motion price process. Managers may question, and rightly so, why they should trust a valuation based on a price process that does not say anything about the price level 15 years from now, let alone 30 years. The proposed mean reverting process has a large advantage in this respect.

## The risk-neutral process

Following the derivation in section 2, we want the Black and Scholes differential equation (2.5) to be satisfied.

$$V_t(t,S)+(r-\mu+\frac{1}{dt}E\left[\frac{dS}{S}\right])SV_S(t,S)+\tfrac{1}{2}\sigma^2S^2V_{SS}(t,S)+\Pi(t,S)-rV(t,S)=0.$$

For the mean reverting process (6.3), the drift is

$$\frac{1}{dt}E\left[\frac{dS}{S}\right]=\eta(\gamma+\omega t-\ln S),$$

and the differential equation can therefore be written as

$$V_t+[r-\mu+\eta(\gamma+\omega t-\ln S)]SV_S+\tfrac{1}{2}\sigma^2S^2V_{SS}+\Pi-rV=0.$$

Substituting $\psi=\gamma+\frac{r-\mu}{\eta}$, in order to simplify the notation, gives the Black and Scholes differential equation as

$$V_t + \eta\left(\psi + \omega t - \ln S\right)SV_S + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + \Pi - rV = 0.$$

Thus, for the pricing of derivatives, the risk-neutral process

$$dS = \eta\left(\psi + \omega t - \ln S\right)S\,dt + \sigma\,S\,dv, \tag{6.10}$$

should be assumed and under this assumption the Feynman-Kac formula (2.7) will still apply.

Rather than working with the process (6.10), we substitute for $\ln S(t)$ instead as this will be a normally distributed variable, dictated by the process

$$d\ln S = \eta\left(\psi' + \omega t - \ln S\right)dt + \sigma dv \quad \text{where} \quad \psi' = \psi - \frac{\sigma^2}{2\eta}.$$

Numerically, the parameter $\psi'$ has the value

$$\psi' = \psi - \frac{\sigma^2}{2\eta} = \gamma + \frac{r-\mu}{\eta} - \frac{\sigma^2}{2\eta} = 8.56 + \frac{0.064 - 0.077}{0.25} - \frac{0.19^2}{2\cdot 0.25} = 8.44.$$

It is now possible to use the notation

$$S(t) = e^{X(t)},\; X(t) \sim N\left[a(t), b(t)\right], \tag{6.11}$$

where

$$a(t) = e^{-\eta t} \ln S_0 + \omega t + \left(1 - e^{-\eta t}\right)\left(\psi' - \frac{\omega}{\eta}\right)$$

$$= \left(\ln 4500\right)e^{-0.25t} + 0.013t + \left(8.44 - \frac{0.013}{0.25}\right)\left(1 - e^{-0.25t}\right)$$

and

$$b(t) = \sigma\left(\frac{1 - e^{-2\eta t}}{2\eta}\right)^{1/2} = 0.19\left(\frac{1 - e^{-2\cdot 0.25t}}{2\cdot 0.25}\right)^{1/2}.$$

The risk neutral process of $d \ln S$ and the parameters $a(t)$ and $b(t)$, follow immediately from their real world counterparts, (6.4)-(6.6). Note that the mean reverting process cannot be written in the form $S(t) = S_0 e^{X(t)}$. Contrary to the geometric Brownian motion, the starting value $S_0$ cannot be separated from the parameter $a(t)$. Calculations will therefore look a bit different, depending on the stochastic process used.

**Plant valuation**

As in section 4, we calculate today's value of future sales as

$$V_0\,(pulp\ sales) = \int_0^{30} e^{-0.064t} E^Q[pulp\ sales]dt = \int_0^{30} e^{-0.064t} E^Q[0.4 \cdot S(t)]dt.$$

The only difference being the mean reverting process specified in this section. The expected value of $S(t)$, parameterised as in (6.11), is obtained through completion by squares and the result is

$$E^Q[S(t)] = \int_{-\infty}^{\infty} \frac{e^x}{b(t)\sqrt{2\pi}} e^{-\frac{(x-a(t))^2}{2b^2(t)}} dx = e^{a(t)+\frac{1}{2}b^2(t)}.$$

Thus, today's value of future sales is

$$V_0(pulp\ sales) = \int_0^{30} e^{-0.064t}\left[0.4e^{a(t)+\frac{1}{2}b^2(t)}\right]dt = 28018.$$

With pulpwood costs as 30% of sales and the rest of the costs unaffected by the pulp price (and therefore the same as in section 4), $V_0$ *(future cash flow)* = 4531 MSEK. Taking the investment cost of 4500 MSEK into consideration, the value added is 31 million Swedish crowns. Nearly three billion higher than the net present value based on a geometric Brownian motion. The result is, however, very sensitive to changes in pulp price parameters and techniques of parameter estimation. It cannot alone be used as evidence that the pulp plant investment is actually worthwhile.

Using the variable production policy, described in section 5, $V_0$ (future cash flow) = 4499 MSEK. Calculations are shown in appendix E. The difference in value between the two production policies, 32 MSEK, can be divided into two parts. The ability to stop production whenever price is below variable costs increases the value with 9 MSEK. The policy to reduce production, in order to reverse a price decline, decreases the value by 41 MSEK.

That the difference in value between the two production policies is smaller than the 1200 MSEK encountered in sections 4 and 5 is not surprising. The band of probable pulp prices is narrower for the mean reverting process. The production policy therefore becomes less important since the probability of low pulp prices is small. Even so, it is surprising that the difference in value must be considered as negligible. Only for the geometric Brownian motion, there seems to be a benefit of modelling a variable production rate. However, the geometric Brownian motion is not suitable for long-lived projects, as the range of probable prices grows unbounded over time.

It is important to note that the assumption of a mean reverting process is not compatible with the use of a single risk adjusted discount rate. In order to demonstrate this incompatibility, let $e^{-(r+k)t}$ be the discount factor, with $r \cdot t$ as the risk free part and $k \cdot t$ as the risk-compensating part. The factor $k \cdot t$ depends on the systematic risk and not the total risk. However, if the total risk is bounded, as it is in a mean reverting model where the price eventually will stabilise around the long run equilibrium, we cannot allow the compensation for systematic risk $e^{-kt}$ to grow unbounded. $k$ must therefore decrease over time, with a higher discounted value as a result. Using the terminology of Laughton and Jacoby (1995), this is the risk-discounting effect of mean reversion. The risk-discounting effect explains why plant value is higher for the mean reverting process compared to the geometric Brownian motion.

## 7. Other options present in the pulp industry

The possibility to alter the scale of production in response to changing market conditions is just one option suggested in the real options literature. There are many more. Corporate finance textbooks usually detail three categories: Abandonment options, expansion options and timing options.[28]

### The abandonment option

The abandonment option comes in many disguises. From an outright abandonment of the investment, to the slight scaling of production that has been specified in this paper. There are stages in between, like a temporary shutdown.

In the pulp industry, a temporary shutdown is an alternative to decreasing the speed of production. A shutdown has the advantage of not lowering the process yield (which will be the effect if the plant is not running to full capacity), but instead there are costs associated with a restart. Unfortunately, shutdowns are not possible during wintertime; the plant would freeze. At least in Canada, U.S. and the Nordic countries where the majority of the world's softwood pulp is produced. During the summer, any extended shutdown will result in bacterial problems in the wet pulp. Longer shutdowns are rare in the pulp industry, since conservation of the processes, e.g. the recovery boiler, is quite expensive.

It is nevertheless not obvious that decreasing the speed of production, as is done in this paper, is preferable to a temporary shutdown. The preferences will differ from company to company. From a capital budgeting viewpoint, however, a lower capacity usage is much easier to model and that is the reason why it has

[28] See, for example, Ross, Westerfield, Jaffe, (1996), chapters 8 and 21. For an extensive survey, Trigeorgis (1996) chapter 1 is recommended.

been used in this paper. To build a model where shutdowns are incorporated is certainly possible, although much more complicated, since not only the present price of pulp matters, but also the status of the plant (in operation, stopped, shut down etc.), needs to be considered. It is however doubtful that such a model would bring any major improvement in accuracy compared to the calculations shown here, as the difference is largely technical. In both cases the result is a decrease in production.

**The expansion option**

The option to expand has to do with the competitive edge that operating the plant today renders you in the future. By producing today, you may be in a better position to make future investments. The establishment of market relations, technical know-how, established organisational procedures etc., sometimes summarise to a valuable option. It is an option because the decision to invest has not yet been made, and the opportunity to do so may to some extent be unknown.

What is the value of the expansion option for the pulp industry? Judging from the stock market not much. In stock valuation, it is common to talk about the Net Present Value of Growth Opportunities, as the difference between today's free cash flow generating capacity and the stock price. The NPVGO is the option to expand, branded under another name.

Even without any formal analysis, one can conclude that the low Price-Earnings ratios for most forestry companies in Sweden is indicative of very few growth opportunities. This conclusion is further supported by the fact that the market values have been less than book values for extended periods of time during the 1990:s.[29]

In addition, the industry representatives I have interviewed do not believe that there are any growth options attached to the production of the standard commodity considered here - Northern Bleached Softwood Kraft pulp. During the life of the mill, it is possible to trim production by 10%. However, it is not possible to increase production from 400 000 tonnes to, say, 600 000 tonnes. In this case it would be cheaper to build a new plant.

After 30 years, when the exemplified pulp mill has outlived its economic life, it is time to decide if a new investment should be made. Are we, thanks to the previous investment, in a better position than would otherwise have been the case? If so, this is an option because we have made no commitment to undertake it and the investment is 30 years ahead. The option is also a pulp derivative. The future investment will be carried through only if the pulp price is high enough to motivate it.

If the decision to build a new mill is made, it will probably be situated next to the old one. Thereby a smooth transition from the old facility to the new can be accomplished and, perhaps, a few stages from the old mill can be used in the new one. Also, already trained personnel will be employed, reducing the time needed to run-in the new mill.

What is the value of this option? Suppose that some of the logistics facilities can be used: pulpwood unloading, water supply and the purification plant, etc.

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[29] A lower market than book value can also be interpreted as showing that an investment has negative value. The future profits (i.e. market value) are not enough to cover the (previous) investment outlays (book value). Formally, this is the concept of Tobin's Q. See, for example, Dixit and Pindyck (1994).

The value of it would be 400 million SEK (in the price level of year zero). The running-in time is also reduced and an extra 0.1 million tonnes can be produced during the first year.[30]

Under the assumption that an investment only will occur if the pulp price (in year 30) exceeds SEK 4000 per ton, the value is:[31]

*$V_0$(expansion option, Brownian motion) =*

$$= e^{-0.064\cdot t} \int_{\ln\frac{4000}{4500}+0.013t}^{a(t)+5b(t)} (275e^{0.013t} + 0.07\cdot 4500e^{x})\cdot\varphi(x)dx = 40.7 \text{ MSEK}$$

*$V_0$(expansion option, mean reversion) =*

$$= e^{-0.064\cdot t} \int_{\ln(4000)+0.013t}^{a(t)+5b(t)} (275e^{0.013t} + 0.07e^{x})\cdot\varphi(x)dx = 88.4 \text{ MSEK}$$

It is notable that the option value under the assumption of the geometric Brownian motion is less than half that of the mean reverting process. The price process specification is of paramount importance also for the expansion option.

It may be argued that the above conditions are overly simplified. The decision to invest is likely to hinge not only on the pulp price in the year 30, but at the average price for the years preceding the investment decision. It seems reasonable to assume that managers will consider the history of prices, say the last ten years, before committing money to a new pulp-mill investment. The expansion option is thereby path dependent, i.e. it does not depend on the price

[30] Overall, the contribution to profit will be:
400 million SEK + 0.1 million tonnes · [$S$ (pulp price) – 0.3·$S$ (pulpwood) – 1250 (variable cost)] = 275 + 0.07· $S$.

[31] For the geometric Brownian motion, $a$, $b$, $\varphi$ and $x$ refers to the definitions in section 4. For the mean reverting process, the definitions are in section 6. $t$ is fixed as 30 years.

of pulp at a specific date, but on the whole trajectory of prices.[32] Such options are in the finance literature known as Asian options and valuation is, generally, quite hard work.

However, when the average is measured as the geometric average, analytical solutions are possible, as the geometric average of correlated lognormal variables also is lognormal.[33] In the case of the geometric Brownian motion, the distribution of the continuously observed geometric average is derived in appendix F, and letting the averaging period be between years 20-30, the value of the expansion option is,

$V_0$*(expansion option, Brownian motion )* = 41.2 MSEK.

Modelling path dependence only makes a negligible difference in value. As was the case with the variable utilisation of capacity, the exact formulation of the decision rule is subordinate to the price process assumed.

Apart form the option value of continued operation there is also the cost of disassembling the old plant. Today, such a disassembling would amount to around 500 million SEK, much of this due to asbestos decontamination. Even though asbestos is not used as a building material today and hence will be no problem in 30 years time, there is nothing indicating that disassembling will be cheaper in the future. On the contrary, thanks to the stricter environmental laws, a site recovery cost of 500 million SEK seems reasonable even in the

[32] Of course, one could certainly argue that any future decision, dependent on pulp price, is also path dependent. However, whilst a decision to cut production temporary is, at most, dependent upon the pulp prices for the last couple of months, a major investment decision is presumably hinging on the pulp price for the last couple of years.

[33] The geometric average is the product of all observations raised to the power of 1/n. The arithmetic average is the sum of all observations multiplied by 1/n.

future. The cost of site recovery is not dependent upon the price of pulp. Therefore, an ordinary present value calculation can be used, giving

$$PV(site\ recovery) = -e^{-0.064 \cdot t} \cdot 500 e^{0.013 \cdot t} = -108.3 \text{ MSEK.}$$

As the expansion option and the cost of site recovery are minor, the salvage value was for expositional clarity set to zero in section 3.

**The timing option**

Even if the present value of future payments exceeds the investment cost, it is not obvious that the investment should be made. Sometimes it is preferable to wait until a later date and the argument is as follows:

If you have an exclusive right to make an investment, for example, a concession to an oil field in the North Sea, the value of the concession will rise if the price of oil goes up. It is therefore not certain that extraction should commence immediately, even if the cash flow generated exceeds the investment cost. What makes the extraction profitable may very well be an anticipated increase in the price of crude oil. An increase in the price of crude oil will also affect the value of the concession positively. It may well be that the increase in the value of the concession makes it more profitable to keep it intact than to invest in an oilrig.

Note that this argument hinges on the existence of some sort of exclusive right. It is more profitable to keep the right to extract oil in the North See intact, than to pay the investment cost and actually start drilling, because the concession can be sold and give a higher return than the actual investment in an oilrig.

In many cases there is no exclusive right. Let us return to the pulp investment considered. This is, in everyday language, a right (i.e. an option) to invest. However, it is not an option in an economic sense. If you choose not to invest, your right to invest cannot be sold to a competitor. The competitors can invest themselves if they so wish. Therefore, the traditional net present value rule also leads to the optimal decision.[34]

It should be pointed out that allowing for imperfect markets makes the decision rule less clear cut. Even if there is no option value of waiting, in the sense that the right to invest can be sold, this does not necessarily mean that one should invest as soon as the net present value exceeds zero. Normally, if investing now means that the opportunity to invest in the future is gone, this opportunity has a market value, for example a patent or a concession. However, if markets are imperfect, waiting may have a value for you but nobody else. If available investment capital is scarce, for example, or if investing depletes an asset, it may serve to wait for the optimality condition to be satisfied, before investing.

To summarise the exposé in this section: Different operating options have little significance in the pulp industry. No surprise, really, as pulp is a standard commodity and the manufacturing of it a mature business. Changing the degree of utilisation captures the main optionlike characteristics of pulp production, even though this is secondary to the specification of the price process.

[34] Following Dixit and Pindyck (1994), the optimal decision rule is to invest when the value of the investment opportunity equals the value of the investment minus the investment cost, $F(S^*) = V(S^*) - I$. With no value of the investment opportunity, $F(S) = 0$, we get $V(S^*) = I$. It is optimal to invest when the price is such that the investment cost is covered.

# 8. Conclusions

Companies do not always behave as price takers. Sometimes they actively attempt to increase the price. One measure is to reduce supply through a reduction in plant utilisation. This behaviour strains the net present value criteria, since the price-risk no longer represents the riskiness of the cash flow. Option pricing methods can be used in the case where the output is a traded asset, but is it worth the trouble?

Judging from this study of the pulp industry, it is presumably not worthwhile to model a variable utilisation of capacity and it should not be given the highest of priorities. The difference between a fixed and variable production rate is crucially dependent upon the price process specified. The higher the probability of really low prices the bigger the difference. The geometric Brownian motion, as seen in figure 6.1, gives an extremely wide range and thereby a greater difference in value.

What is important, though, is to specify the price process carefully. Merely performing a net present value calculation is not enough. In fact, since the net present value is compatible with the geometric Brownian motion assumption, its ability to adequately represent the price process over a 30-year time span is appalling. Using the mean reverting process suggested here and applying the Feynman-Kac formula in order to evaluate capital investments therefore seems attractive. Modelling different production policies then becomes less interesting as the difference in value will be small. The problem of parameter estimation and the mathematics involved reduces the usefulness of the mean reverting process and the Feynman-Kac formula for everyday capital budgeting. It is however very suitable for major long-run investments.

## Appendix A - Congruence between present value and real option calculations

When the price *S* develops according to the process $dS = \alpha S dt + \sigma(t,S)dw$, where $\alpha$ is a constant, the Feynman-Kac formula will give the same answer as a present value calculation for all payments that are symmetric in *S*. The key to this congruence is the fact that the expected value of the above stochastic process is $E[S(t)] = S_0 e^{\alpha(T-t_0)}$. The stochastic term disappears, because the expected value of a deterministic Ito integral is zero.

Denote the future payment $f(S)$. That a payment is symmetric in *S*, is the same as saying that the payment is linear in *S*, i.e. $f(S) = aS$. The present value of $f(S)$ is,

$$PV = e^{-\mu(T-t_0)} E[f(S)] = e^{-\mu(T-t_0)} E[aS(t)] = e^{-\mu(T-t_0)} aS_0 e^{\alpha(T-t_0)}.$$

As is discussed in section 2, paper pulp is not an investment object but a commodity. The total return from holding the pulp can be subdivided into the expected increase in price $\alpha$ and the convenience yield $\delta$. Mathematically, $\mu = \alpha + \delta$, giving $-\mu + \alpha = -\delta$, and the present value as

$$PV = aS_0 e^{-\delta(T-t_0)}.$$

This is the same result as would have been obtained with the Feynman-Kac formula. With a drift rate of $r - \delta$, today's value of the future payment becomes

$$V_0 = e^{-r(T-t_0)} E^Q[f(S)] = e^{-r(T-t_0)} E^Q[aS(T)] = e^{-r(T-t_0)} aS_0 e^{(r-\delta)(T-t_0)} = aS_0 e^{-\delta(T-t_0)}.$$

# Appendix B - Market data, geometric Brownian motion

Market data are collected quarterly[35] for the 1980-96 period and is reproduced at the next page. Pulp prices have been obtained from OM Stockholm AB and refers to the market price for Northern Bleached Softwood Kraft pulp. The market price is established in U.S. dollars, and has been converted to Swedish crowns by the exchange rate given in "*Main Economic Indicator*", published by the OECD Statistics Directorate.

Inflation is measured by the Swedish Consumer Price Index, and the return on 3-month Treasury Bills serves as proxy for the instantaneous risk-free rate. The market portfolio is represented by "Affärsvärldens generalindex" at the Stockholm Stock Exchange. These data are obtained through the Hanson&Partner data base Ecowin, a macroeconomic data base.

All parameters are continuously compounded, and the drift and standard deviation of the geometric Brownian motion is calculated by means of the procedure described in section 2 of the main text. The numerical values of the parameters are:

$$\text{Real interest rate} = \frac{1}{n}\sum_t \left[r_t - \ln\left(\frac{CPI_t}{CPI_{t-4}}\right)\right] = 4.4\%$$

$$\text{Inflation} = 4\left[\frac{1}{n}\ln\left(\frac{CPI_n}{CPI_1}\right)\right] = 5.8\%$$

[35] Thereby the factor 4 in the formulas on this page.

$$\text{Beta of pulp prices} = \frac{Cov(pulp, market)}{Var(market)} = 0.164$$

Standard deviation of pulp prices,

$$\sigma = \sqrt{4\frac{1}{n}\sum_t \left[\ln\frac{Pulp_t}{Pulp_{t-1}} - \frac{1}{n}\ln\frac{Pulp_n}{Pulp_1}\right]^2} = 18.9\% \text{ (real data)}$$

$$= 18.9\% \text{ (nominal data)}$$

$$\text{Drift of pulp prices, } \alpha = 4\left[\frac{1}{n}\ln\left(\frac{Pulp_n}{Pulp_1}\right)\right] + \tfrac{1}{2}\sigma^2 = -0.7\% \text{ (real data)}$$

$$= 5.0\% \text{ (nominal data)}$$

Table B.1 Market data for pulp parameter estimations.

| Year | NBSK USD/tonne | F/X SEK/USD | CPI kr Jan 1980=100 | Interest rate 3 month | Stockholm share prices | NBSK SEK/tonne | NBSK REAL 1996 SEK/tonne |
|---|---|---|---|---|---|---|---|
| 80 | 500 | 4.457 | 97.1 | 0.105 | 104.4 | 2228 | 5850 |
| | 545 | 4.150 | 98.4 | 0.123 | 107.1 | 2262 | 5859 |
| | 545 | 4.163 | 102.6 | 0.125 | 103.8 | 2269 | 5637 |
| | 545 | 4.373 | 105.2 | 0.124 | 122.6 | 2383 | 5774 |
| 81 | 545 | 4.592 | 109.8 | 0.151 | 142.4 | 2503 | 5810 |
| | 545 | 5.085 | 111.6 | 0.135 | 167.2 | 2771 | 6330 |
| | 545 | 5.598 | 114.3 | 0.105 | 166.2 | 3051 | 6804 |
| | 545 | 5.571 | 114.9 | 0.090 | 192.3 | 3036 | 6736 |
| 82 | 545 | 5.951 | 119.3 | 0.128 | 185.1 | 3243 | 6930 |
| | 520 | 6.092 | 121.1 | 0.136 | 183.5 | 3168 | 6668 |
| | 460 | 6.290 | 122.9 | 0.145 | 202.9 | 2894 | 6002 |
| | 420 | 7.294 | 125.9 | 0.123 | 259.6 | 3064 | 6203 |
| 83 | 400 | 7.509 | 129.3 | 0.109 | 355.7 | 3004 | 5921 |
| | 440 | 7.642 | 131.8 | 0.112 | 367.1 | 3363 | 6503 |
| | 440 | 7.822 | 134.5 | 0.117 | 425.0 | 3442 | 6522 |
| | 440 | 8.001 | 137.5 | 0.117 | 430.3 | 3521 | 6526 |
| 84 | 465 | 7.716 | 140.9 | 0.108 | 476.1 | 3588 | 6491 |
| | 540 | 8.184 | 142.4 | 0.118 | 431.3 | 4419 | 7911 |
| | 540 | 8.584 | 144.8 | 0.134 | 406.1 | 4635 | 8160 |
| | 460 | 8.990 | 148.8 | 0.117 | 382.3 | 4135 | 7084 |

Table continued

| Year | NBSK USD/tonne | F/X SEK/USD | CPI kr Jan 1980=100 | Interest rate 3 month | Stockholm share prices | NBSK SEK/tonne | NBSK REAL 1996 SEK/tonne |
|---|---|---|---|---|---|---|---|
| 85 | 415 | 8.893 | 152.1 | 0.138 | 387.7 | 3691 | 6185 |
| | 390 | 8.804 | 153.9 | 0.163 | 367.4 | 3433 | 5687 |
| | 390 | 8.065 | 154.5 | 0.148 | 381.8 | 3145 | 5189 |
| | 400 | 7.616 | 157.1 | 0.124 | 479.7 | 3046 | 4943 |
| 86 | 415 | 7.322 | 158.7 | 0.108 | 574.5 | 3039 | 4881 |
| | 450 | 7.116 | 159.7 | 0.098 | 663.9 | 3202 | 5111 |
| | 480 | 6.903 | 161.3 | 0.087 | 703.1 | 3313 | 5236 |
| | 520 | 6.819 | 162.3 | 0.091 | 724.5 | 3546 | 5569 |
| 87 | 550 | 6.327 | 164.7 | 0.108 | 756.0 | 3480 | 5386 |
| | 585 | 6.388 | 164.9 | 0.088 | 803.0 | 3737 | 5777 |
| | 610 | 6.438 | 169.4 | 0.090 | 951.7 | 3927 | 5910 |
| | 635 | 5.848 | 170.7 | 0.091 | 667.5 | 3713 | 5545 |
| 88 | 680 | 5.878 | 173.7 | 0.094 | 790.5 | 3997 | 5865 |
| | 725 | 6.254 | 176.3 | 0.102 | 851.8 | 4534 | 6555 |
| | 760 | 6.434 | 178.8 | 0.104 | 907.2 | 4890 | 6971 |
| | 760 | 6.157 | 180.9 | 0.104 | 1013.8 | 4679 | 6593 |
| 89 | 810 | 6.425 | 184.7 | 0.114 | 1129.1 | 5204 | 7182 |
| | 840 | 6.648 | 187.9 | 0.116 | 1225.0 | 5584 | 7575 |
| | 840 | 6.409 | 190.2 | 0.116 | 1285.1 | 5384 | 7215 |
| | 840 | 6.227 | 192.8 | 0.123 | 1262.0 | 5231 | 6916 |
| 90 | 840 | 6.126 | 205.4 | 0.146 | 1142.2 | 5145 | 6386 |
| | 840 | 6.041 | 206.2 | 0.126 | 1309.7 | 5075 | 6273 |
| | 800 | 5.764 | 212.0 | 0.131 | 910.0 | 4611 | 5544 |
| | 750 | 5.698 | 213.9 | 0.144 | 870.0 | 4274 | 5093 |
| 91 | 700 | 6.091 | 225.8 | 0.121 | 1093.7 | 4263 | 4813 |
| | 600 | 6.546 | 227.0 | 0.106 | 1130.9 | 3927 | 4410 |
| | 520 | 6.067 | 229.2 | 0.103 | 1035.3 | 3155 | 3509 |
| | 500 | 5.529 | 230.8 | 0.136 | 917.6 | 2765 | 3053 |
| 92 | 540 | 5.977 | 231.3 | 0.117 | 999.9 | 3228 | 3557 |
| | 560 | 5.513 | 231.5 | 0.116 | 913.0 | 3087 | 3399 |
| | 580 | 5.292 | 234.6 | 0.201 | 696.7 | 3069 | 3335 |
| | 500 | 7.043 | 234.9 | 0.106 | 912.6 | 3521 | 3821 |
| 93 | 470 | 7.745 | 242.7 | 0.097 | 994.5 | 3640 | 3823 |
| | 440 | 7.707 | 242.3 | 0.083 | 1083.0 | 3391 | 3567 |
| | 410 | 8.041 | 244.5 | 0.077 | 1294.8 | 3297 | 3437 |
| | 400 | 8.304 | 244.3 | 0.071 | 1402.8 | 3321 | 3466 |
| 94 | 440 | 7.828 | 246.8 | 0.071 | 1403.6 | 3444 | 3557 |
| | 535 | 7.690 | 248.4 | 0.071 | 1372.4 | 4114 | 4222 |
| | 605 | 7.488 | 250.7 | 0.079 | 1412.4 | 4530 | 4606 |
| | 700 | 7.462 | 250.4 | 0.081 | 1470.8 | 5223 | 5317 |
| 95 | 750 | 7.372 | 253.3 | 0.087 | 1458.6 | 5529 | 5564 |
| | 858 | 7.268 | 255.3 | 0.093 | 1643.0 | 6236 | 6227 |
| | 925 | 6.905 | 256.2 | 0.088 | 1842.3 | 6387 | 6355 |
| | 940 | 6.658 | 256.0 | 0.084 | 1735.7 | 6259 | 6232 |
| 96 | 650 | 6.696 | 257.0 | 0.068 | 1898.0 | 4352 | 4317 |
| | 550 | 6.652 | 256.3 | 0.057 | 1981.6 | 3658 | 3638 |
| | 580 | 6.628 | 256.0 | 0.047 | 2091.3 | 3844 | 3828 |
| | 560 | 6.871 | 254.9 | 0.036 | 2402.9 | 3848 | 3848 |

## Appendix C - Derivation of the mean reverting process

Assume the commodity price to follow the mean reverting process

$$dS = \eta(\gamma + \omega t - \ln S)S\,dt + \sigma S\,dw. \tag{C.1}$$

Define $X(t) = \ln S(t)$ and apply Ito's lemma

$$\begin{aligned} dX &= \left[X_t + \eta(\gamma + \omega t - \ln S)S X_S + \tfrac{1}{2}\sigma^2 S^2 X_{SS}\right]dt + \sigma S X_S\,dw \\ &= \left[0 + \eta(\gamma + \omega t - \ln S)S\frac{1}{S} - \tfrac{1}{2}\sigma^2 S^2 \frac{1}{S^2}\right]dt + \sigma S \frac{1}{S}dw \\ &= \left[\eta(\gamma + \omega t - X) - \tfrac{1}{2}\sigma^2\right]dt + \sigma\,dw. \end{aligned}$$

The stochastic variable $X$ follows the process

$$dX = \eta(\gamma' + \omega t - X)dt + \sigma dw \quad \text{with} \quad \gamma' = \gamma - \frac{\sigma^2}{2\eta}, \tag{C.2}$$

which is an ordinary Ornstein-Uhlenbeck process if the equilibrium drift rate $\omega$ is zero. In order to write the process of $X(T)$ in explicit form, consider the function

$$f(t, X) = e^{-\eta(T-t)}X(t)$$

with the partial derivatives

$$f_t = \eta e^{-\eta(T-t)}X, \quad f_X = e^{-\eta(T-t)}, \quad f_{XX} = 0.$$

Applying Ito's lemma gives

$$\begin{aligned} df &= \left[f_t + \eta(\gamma' + \omega t - X) f_X + \tfrac{1}{2}\sigma^2 f_{XX}\right]dt + \sigma f_X dw \\ &= \left[\eta e^{-\eta(T-t)} X + \eta(\gamma' + \omega t - X) e^{-\eta(T-t)}\right]dt + \sigma e^{-\eta(T-t)} dw \\ &= \eta(\gamma' + \omega t) e^{-\eta(T-t)} dt + \sigma e^{-\eta(T-t)} dw. \end{aligned}$$

Formally, this diffusion process represents the integral equation

$$f(T) = f_{t_0} + \int_{t_0}^{T} \eta(\gamma' + \omega t) e^{-\eta(T-t)} dt + \int_{t_0}^{T} \sigma e^{-\eta(T-t)} dw(t)$$

$$X(T) e^{-\eta(T-T)} = X(t_0) e^{-\eta(T-t_0)} + \int_{t_0}^{T} \eta(\gamma' + \omega t) e^{-\eta(T-t)} dt + \int_{t_0}^{T} \sigma e^{-\eta(T-t)} dw(t)$$

$$X(T) = X(t_0) e^{-\eta(T-t_0)} + \int_{t_0}^{T} \eta(\gamma' + \omega t) e^{-\eta(T-t)} dt + \int_{t_0}^{T} \sigma e^{-\eta(T-t)} dw(t). \qquad \text{(C.3)}$$

In order to derive the distribution of $X(T)$, a useful lemma from stochastic calculus will be used:

If $h(t)$ is a deterministic function of time, and the process $Y(T)$ is defined as

$$Y(T) = \int_{t_0}^{T} h(t) dw(t),$$

then $Y(T)$ is normally distributed with zero mean and the variance

$$Var\left[Y(T)\right] = \int_{t_0}^{T} h^2(t) dt.$$

This result can, for example, be found in Björk (1998) p. 43.

Applied to the process $X$, we immediately get

$$E[X(T)] = X(t_0)e^{-\eta(T-t_0)} + \int_{t_0}^{T} (\gamma' + \omega t)\eta\, e^{-\eta(T-t)} dt.$$

Solving the integral through integration by parts,

$$E[X(T)] = X(t_0)e^{-\eta(T-t_0)} + \left[(\gamma' + \omega t)e^{-\eta(T-t)}\right]_{t_0}^{T} - \int_{t_0}^{T} \omega\, e^{-\eta(T-t)} dt$$

$$= X(t_0)e^{-\eta(T-t_0)} + \gamma' + \omega T - (\gamma' + \omega t_0)e^{-\eta(T-t_0)} - \left[\frac{\omega}{\eta}e^{-\eta(T-t)}\right]_{t_0}^{T}$$

$$= X(t_0)e^{-\eta(T-t_0)} + \omega\left(T - t_0 e^{-\eta(T-t_0)}\right) + \left(\gamma' - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta(T-t_0)}\right). \qquad \text{(C.4)}$$

The expression for the variance is directly obtained from the lemma above.

$$Var[X(T)] = \int_{t_0}^{T} \sigma^2 e^{-2\eta(T-t)} dt = \frac{\sigma^2}{2\eta}\left(1 - e^{-2\eta(T-t_0)}\right) \qquad \text{(C.5)}$$

**Summing up:**

The problem of explicitly solving the mean reverting process

$$dS = \eta(\gamma + \omega t - \ln S)S\,dt + \sigma S\,dw$$

can by defining $X(t) = \ln S(t)$ be reduced to solving the process

$$d\ln S = \eta(\gamma' + \omega t - \ln S)dt + \sigma dw \quad \text{where} \quad \gamma' = \gamma - \frac{\sigma^2}{2\eta}. \tag{C.6}$$

ln *S(T)* is then normally distributed with

$$E[\ln S(T)] = \left(\gamma' - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta(T-t_0)}\right) + \omega\left(T - t_0\, e^{-\eta(T-t_0)}\right) + \ln S(t_0) e^{-\eta(T-t_0)} \tag{C.7}$$

and

$$Var[\ln S(T)] = \frac{\sigma^2}{2\eta}\left(1 - e^{-2\eta(T-t_0)}\right). \tag{C.8}$$

Altogether, the integral equation of (C.3) can be written as

$$\begin{aligned} \ln S(T) = {} & \ln S_0 e^{-\eta(T-t_0)} + \omega\left(T - t_0\, e^{-\eta(T-t_0)}\right) + \\ & + \left(\gamma' - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta(T-t_0)}\right) + \sigma\sqrt{\frac{1 - e^{-2\eta(T-t_0)}}{2\eta}}\,\varepsilon, \end{aligned} \tag{C.9}$$

where $\varepsilon$ is a random drawing from a standardised normal distribution.

# Appendix D - Parameter estimation of the mean reverting process

There is no standard procedure for estimating the parameters of a continuous time mean reverting processes. One procedure, hinted at by Dixit and Pindyck (1994) p. 76, is to use the AR1 process,

$$\frac{S_{t+1} - S_t}{S_t} = (\gamma + \omega t)(1 - e^{-\eta \Delta t}) + (e^{-\eta \Delta t} - 1)\ln S_t + \sigma \varepsilon_{\Delta t}, \tag{D.1}$$

as the discrete time process on which to base parameter estimation. By using a Maclaurin expansion on the term $e^{-\eta \Delta t}$ and dropping all terms of $O(\Delta^2 t)$ and higher, the AR1 process converges to the mean reverting process of this paper,

$$dS = \eta(\gamma + \omega t - \ln S)S\,dt + \sigma S\,dw. \tag{D.2}$$

Unfortunately, running the AR1 process (D.1) on the available pulp data will result in a regression without any explanatory power. Presumably, dropping all terms of $O(\Delta^2 t)$ and higher is too crude a method for the quarterly data available.

Another approach is to follow Schwartz (1997) and apply the Kalman filter methodology. This approach suffers from two drawbacks. First of all, the recursive estimation procedure necessary is quite a complex task for non-statisticians. Secondly, the state variable (i.e. the pulp price) is usually unobservable when the Kalman filter is applied. As pulp prices are observable, the method seems less than ideal and a slight overkill.

Better then, is to follow Harvey (1989) and use the explicit solution to the continuous time process as the base for discrete time representation. Equation (C.9) gives the logarithm of the pulp price as,

$$\ln S_T = \ln S_0 e^{-\eta(T-t_0)} + \omega\left(T - t_0\, e^{-\eta(T-t_0)}\right) + \left(\gamma' - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta(T-t_0)}\right) + \sigma\sqrt{\frac{1-e^{-2\eta(T-t_0)}}{2\eta}}\,\varepsilon,$$

where $\varepsilon$ is a random drawing from a standardised normal distribution. Defining $\Delta t = t_{k+1} - t_k$ for a small time increment makes it possible to discretise the process as,

$$\begin{aligned}
\ln S(t_{k+1}) &= \ln S(t_k) e^{-\eta\Delta t} + \omega\left(t_{k+1} - t_k\, e^{-\eta\Delta t}\right) + \left(\gamma' - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta\Delta t}\right) + \sigma\sqrt{\frac{1-e^{-2\eta\Delta t}}{2\eta}}\,\varepsilon(t_k) \\
&= \left(\gamma' - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta\Delta t}\right) + \omega\left(\Delta t + t_k - t_k\, e^{-\eta\Delta t}\right) + e^{-\eta\Delta t}\ln S(t_k) + \sigma\sqrt{\frac{1-e^{-2\eta\Delta t}}{2\eta}}\,\varepsilon(t_k) \\
&= \left(\gamma' - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta\Delta t}\right) + \omega\,\Delta t + \omega\left(1 - e^{-\eta\Delta t}\right)t_k + e^{-\eta\Delta t}\ln S(t_k) + \sigma\sqrt{\frac{1-e^{-2\eta\Delta t}}{2\eta}}\,\varepsilon(t_k).
\end{aligned}$$

Parameters can then be estimated by running the regression,

$$\ln S_{t+1} = c_0 + c_1\, t + c_2 \ln S_t + s\varepsilon_t \tag{D.3}$$

with

$$\begin{aligned}
c_0 &= \left(\gamma' - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta\Delta t}\right) + \omega\Delta t \\
c_1 &= \left(1 - e^{-\eta\Delta t}\right) \\
c_2 &= e^{-\eta\Delta t} \\
s &= \sigma\sqrt{\frac{1-e^{-2\eta\Delta t}}{2\eta}}.
\end{aligned}$$

Unfortunately, the problem of multicollinarity makes this regression less trustworthy. One possible solution is to set the equilibrium drift rate $\omega = 0$, and run the following reduced regression on inflation adjusted prices.

$$\ln S_{t+1} = c_0 + c_2 \ln S_t + s\,\varepsilon_t \qquad \text{(D.4)}$$

The regression printout in table D.1 gives the intercept $c_0 = 0.50591$, the coefficient $c_2 = 0.94039$, and the random error $s = \sqrt{0.00886}$.

The parameter estimation of $\{\eta, \gamma, \gamma', \sigma\}$ will then be

$$\eta = -\frac{1}{\Delta t}\ln c_2 = -4\ln 0.94039 = 0.24584$$

$$\gamma' = \frac{c_0}{1-c_2} = \frac{0.50591}{1-0.94039} = 8.48700$$

$$\sigma = \sqrt{\frac{2\eta\, s^2}{1-e^{-2\eta\,\Delta t}}} = \sqrt{\frac{2\cdot 0.24584 \cdot 0.00886}{1-e^{-2\cdot 0.24584/4}}} = 0.19407$$

$$\gamma = \gamma' + \frac{\sigma^2}{2} = 8.48700 + \frac{0.19407^2}{2} = 8.56359.$$

The equilibrium drift rate $\omega$ is estimated separately and set equal to $\alpha$, the drift rate of the geometric Brownian motion.

Estimating parameters by means of a regression on real price data seems to work well in practice. However, it deserves to be mentioned that maximum likelihood estimations based on discrete points in time does not necessarily give consistent estimates of continuous time parameters. The variance of the estimates does not disappear when the number of observations tends to infinity. This problem applies both to the geometric Brownian motion and the mean reverting process and is due to correlation between the parameter estimates. It is a consequence of time aggregation. Gourieroux and Jasiak (2001, chapter 12) provide more details on the subject. They also discuss the Hansen and Scheinkman (1995) infinitesimal generator for moments of continuous time Markov processes, which can be used to obtain consistent estimates.

Another approach is due to Bibby and Sørensen (1995). They derive martingale estimating functions which are both consistent and asymptotically normal. Bibby and Sørensen only show this for the time-homogenous case, but Alaton, Djehiche and Stillberger (2002) provide an example of its use in a time-inhomogenous setting, as also would be the case for this paper. Further research on these new approaches is warranted. For example, it is unclear whether or not consistency is a problem for the small sample size used in this paper.

Table D.1 Regression printout for the mean reverting process, equation (D.4).

| Year | NBSK REAL 1996 SEK/tonne | ln S(t) | ln S(t+1) | Year | NBSK REAL 1996 SEK/tonne | ln S(t) | ln S(t+1) |
|---|---|---|---|---|---|---|---|
| 80 | 5850 | 8.6742 | 8.6758 | | 6971 | 8.8495 | 8.7938 |
| | 5859 | 8.6758 | 8.6371 | | 6593 | 8.7938 | 8.8793 |
| | 5637 | 8.6371 | 8.6612 | 89 | 7182 | 8.8793 | 8.9326 |
| | 5774 | 8.6612 | 8.6674 | | 7575 | 8.9326 | 8.8839 |
| 81 | 5810 | 8.6674 | 8.7530 | | 7215 | 8.8839 | 8.8415 |
| | 6330 | 8.7530 | 8.8252 | | 6916 | 8.8415 | 8.7618 |
| | 6804 | 8.8252 | 8.8152 | 90 | 6386 | 8.7618 | 8.7440 |
| | 6736 | 8.8152 | 8.8436 | | 6273 | 8.7440 | 8.6205 |
| 82 | 6930 | 8.8436 | 8.8051 | | 5544 | 8.6205 | 8.5356 |
| | 6668 | 8.8051 | 8.6998 | | 5093 | 8.5356 | 8.4790 |
| | 6002 | 8.6998 | 8.7328 | 91 | 4813 | 8.4790 | 8.3917 |
| | 6203 | 8.7328 | 8.6863 | | 4410 | 8.3917 | 8.1629 |
| 83 | 5921 | 8.6863 | 8.7801 | | 3509 | 8.1629 | 8.0240 |
| | 6503 | 8.7801 | 8.7830 | | 3053 | 8.0240 | 8.1766 |
| | 6522 | 8.7830 | 8.7836 | 92 | 3557 | 8.1766 | 8.1313 |
| | 6526 | 8.7836 | 8.7782 | | 3399 | 8.1313 | 8.1122 |
| 84 | 6491 | 8.7782 | 8.9760 | | 3335 | 8.1122 | 8.2483 |
| | 7911 | 8.9760 | 9.0070 | | 3821 | 8.2483 | 8.2489 |
| | 8160 | 9.0070 | 8.8656 | 93 | 3823 | 8.2489 | 8.1795 |
| | 7084 | 8.8656 | 8.7299 | | 3567 | 8.1795 | 8.1424 |
| 85 | 6185 | 8.7299 | 8.6459 | | 3437 | 8.1424 | 8.1506 |
| | 5687 | 8.6459 | 8.5543 | | 3466 | 8.1506 | 8.1767 |
| | 5189 | 8.5543 | 8.5056 | 94 | 3557 | 8.1767 | 8.3480 |
| | 4943 | 8.5056 | 8.4931 | | 4222 | 8.3480 | 8.4352 |
| 86 | 4881 | 8.4931 | 8.5392 | | 4606 | 8.4352 | 8.5787 |
| | 5111 | 8.5392 | 8.5633 | | 5317 | 8.5787 | 8.6241 |
| | 5236 | 8.5633 | 8.6250 | 95 | 5564 | 8.6241 | 8.7366 |
| | 5569 | 8.6250 | 8.5915 | | 6227 | 8.7366 | 8.7570 |
| 87 | 5386 | 8.5915 | 8.6616 | | 6355 | 8.7570 | 8.7374 |
| | 5777 | 8.6616 | 8.6843 | | 6232 | 8.7374 | 8.3702 |
| | 5910 | 8.6843 | 8.6207 | 96 | 4317 | 8.3702 | 8.1993 |
| | 5545 | 8.6207 | 8.6768 | | 3638 | 8.1993 | 8.2500 |
| 88 | 5865 | 8.6768 | 8.7880 | | 3828 | 8.2500 | 8.2552 |
| | 6555 | 8.7880 | 8.8495 | | 3848 | 8.2552 | |

| *Regression statistics* | |
|---|---|
| Multipel-R | 0.92867 |
| R-squared | 0.86243 |
| Adjusted R-squared | 0.86032 |
| Standarderror | 0.09412 |
| Observations | 67 |

ANOVA

| | *DF* | *SS* | *MS* | *F-value* |
|---|---|---|---|---|
| Regression | 1 | 3.6099 | 3.610 | 407.50 |
| Residual | 65 | 0.5758 | 0.009 | |
| Total | 66 | 4.1857 | | |

| | *Estimate* | *St.error* | *t-ratio* |
|---|---|---|---|
| Intercept | 0.50591 | 0.40040 | 1.26 |
| X-variabel 1 | 0.94039 | 0.04658 | 20.19 |

# Appendix E - Variable production rate under mean reversion

The risk-neutral process associated with the mean reverting process specified in section 6, is characterised by

$$\ln S(t) = X(t) \sim N[a(t), b(t)]$$

where $a(t) = (\ln 4500)e^{-0.25t} + 0.013t + \left(8.44 - \frac{0.013}{0.25}\right)\left(1 - e^{-0.25t}\right)$

$$b(t) = 0.19\left(\frac{1 - e^{-2 \cdot 0.25t}}{2 \cdot 0.25}\right)^{1/2}.$$

The probability density function is denoted by $\varphi(x) = \frac{1}{b(t)\sqrt{2\pi}} e^{-\frac{(x-a(t))^2}{2b^2(t)}}$.

**Price > 3500, full production**

An expected increase in price (and costs) of 1.3% per annum, requires the lower bound to be specified as $S(t) \geq 3500\, e^{0.013t}$. Using $X$ as the stochastic variable (since it is normally distributed) with $S(t) = e^X$, the lower boundary for $X$ becomes:

$$X \geq \ln 3500 + 0.013t.$$

The upper boundary of a normally distributed variable is, of course, infinity, but for computationally convenience we confine the boundary to five standard deviations.

$$V_0\,(pulp\ sales) = \int_0^{30} e^{-0.064t} \int_{\ln(3500)+0.013t}^{a(t)+5b(t)} (0.4 \cdot e^x) \cdot \varphi(x)dxdt = 24751$$

$$V_0\,(pulpwood\ cost) = -\int_0^{30} e^{-0.064t} \int_{\ln(3500)+0.013t}^{a(t)+5b(t)} (0.3 \cdot 0.4 \cdot e^x) \cdot \varphi(x)dxdt = -7425$$

$$V_0\,(other\ variable\ costs) = -\int_0^{30} e^{-0.064t} \int_{\ln(3500)+0.013t}^{a(t)+5b(t)} (0.4 \cdot 1250e^{0.013t}) \cdot \varphi(x)dxdt = -6345$$

$$V_0\,(maintenance) =$$

$$-\int_0^{15} e^{-0.064t} \int_{\ln(3500)+0.013t}^{a(t)+5b(t)} (150e^{0.013t}) \cdot \varphi(x)dxdt - \int_{15}^{30} e^{-0.064t} \int_{\ln(3500)+0.013t}^{a(t)+5b(t)} (250e^{0.013t}) \cdot \varphi(x)dxdt =$$

$$= -2295$$

$$V_0\,(other\ fixed\ costs) = -\int_0^{30} e^{-0.064t} \int_{\ln(3500)+0.013t}^{a(t)+5b(t)} (300e^{0.013t}) \cdot \varphi(x)dxdt = -3807$$

**2600 < Price < 3500, reduced production**

As the price decreases, so does the production rate. Utilisation decreases linearly from 100% for price of 3500 to only 70% for a price of 2600. Not only production and variable costs are reduced within this price range. Also maintenance is cut. It is possible to cut down on maintenance since maximum output is not an issue. Even if the plant is out of operation for a while, this is no major issue since it is possible to catch up on production later.

Denote the level of utilisation with *f(S)*. In nominal terms, utilisation changes linearly from 70% when the price equals $2600e^{0.013t}$ to 100% when the price is $3500e^{0.013t}$. Utilisation as a function of price will then be the straight line $f(S) = \frac{e^{-0.013t}}{3000} S - 0.167$. Expressed in the variable $X$ instead, the utilisation function becomes $f(X) = \frac{e^{-0.013t}}{3000} \cdot 4500e^{X} - 0.167$.

The integration limits for the stochastic variable $X$, is
$\ln 2600 + 0.013t \leq X \leq \ln 3500 + 0.013t$.

$$V_0\ (pulp\ sales) = \int_0^{30} e^{-0.064t} \int_{\ln(2600)+0.013t}^{\ln(3500)+0.013t} f(x) \cdot \left(0.4 \cdot e^{x}\right) \cdot \varphi(x) dx dt = 2644$$

$$V_0\ (pulpwood\ cost) = -\int_0^{30} e^{-0.064t} \int_{\ln(2600)+0.013t}^{\ln(3500)+0.013t} f(x) \cdot \left(0.3 \cdot 0.4 \cdot e^{x}\right) \cdot \varphi(x) dx dt = -793$$

$$V_0\ (other\ variable\ costs) = -\int_0^{30} e^{-0.064t} \int_{\ln(2600)+0.013t}^{\ln(3500)+0.013t} f(x) \cdot \left(0.4 \cdot 1250e^{0.013t}\right) \cdot \varphi(x) dx dt =$$

$$= -1043$$

$V_0$ *(maintenance)* =

$$-\int_0^{15} e^{-0.064t} \int_{\ln(2600)+0.013t}^{\ln(3500)+0.013t} f(x)\cdot\left(150e^{0.013t}\right)\cdot\varphi(x)dxdt$$
$$-\int_{15}^{30} e^{-0.064t} \int_{\ln(2600)+0.013t}^{\ln(3500)+0.013t} f(x)\cdot\left(250e^{0.013t}\right)\cdot\varphi(x)dxdt$$

$= -387$

$$V_0\,(other\ fixed\ costs) = -\int_0^{30} e^{-0.064t} \int_{\ln(2600)+0.013t}^{\ln(3500)+0.013t} \left(300e^{0.013t}\right)\cdot\varphi(x)dxdt = -710$$

**Price < SEK 2600, no production**

Changing the price variable gives the upper integration limit as $X \leq \ln 2600 + 0.013t$. The lower integration limit of minus infinity is for computational convenience confined to five standard deviations, $a(t) - 5b(t)$. Only fixed costs are present when the price is less than 2600 Swedish crowns and the present value becomes

$$V_0\,(other\ fixed\ costs) = -\int_0^{30} e^{-0.064t} \int_{a(t)-5b(t)}^{\ln(2600)+0.013t} \left(300e^{0.013t}\right)\cdot\varphi(x)dxdt = -91$$

**Totally**

Added together, $V_0$ *(future cash flow)* = 4499 MSEK. Changing the integration limits so that full production is sustained for a price exceeding SEK 2600 per ton, gives $V_0$ *(future cash flow)* = 4540 MSEK.

## Appendix F - The expansion option based on a price average

We are to value a contract (the expansion option in section 7) that pays SEK 590 million in the case that the geometric average of the pulp price between years 20 and 30 is above a specified price. Using risk-neutral valuation, we are to find the probability density function of the contract and given the geometric Brownian motion, $dS = (r-\delta)Sdt + \sigma\, Sdv$, this is analytically feasible.

Letting the averaging period start at $t_1$ and the average to be observed $n$ times $\{t_2, t_3, t_4,\ldots, t_n, T\}$, the geometric average becomes

$$G_1(T) = [S(t_2)\cdot S(t_3)\cdot S(t_4)\cdot\ldots\cdot S(T)]^{1/n}.$$

Using the solution to the geometric Brownian motion,

$$S(t+1) = S(t)e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)\Delta t+\sigma\sqrt{\Delta t}\,\varepsilon}, \tag{F.1}$$

where $\varepsilon \sim N(0,1)$, we can through iteration express $G_1(T)$ as

$$\begin{aligned} G_1(T) = [&S(t_1)e^{(r-\delta-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\,\varepsilon_1} \\ &\cdot S(t_1)e^{(r-\delta-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\,\varepsilon_1}\cdot e^{(r-\delta-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\,\varepsilon_2} \\ &\cdot S(t_1)e^{(r-\delta-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\,\varepsilon_1}\cdot e^{(r-\delta-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\,\varepsilon_2}\cdot e^{(r-\delta-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\,\varepsilon_3} \\ &\vdots \\ &\cdot S(t_1)e^{(r-\delta-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\,\varepsilon_1}\cdot e^{(r-\delta-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\,\varepsilon_2}\cdot\ldots\cdot e^{(r-\delta-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\,\varepsilon_n}]^{1/n} \end{aligned}$$

Noting that there are $n$ rows and $n$ columns above makes it possible to shorten the notation to

$$G_1(T) = S(t_1)\left[ e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)\Delta t \sum_{i=1}^{n} i \;+\sigma\sqrt{\Delta t}\sum_{i=1}^{n} i\varepsilon_i} \right]^{1/n}.$$

Taking $1/n$ inside the parenthesis and writing $\Delta t = \frac{T-t_1}{n}$ gives the geometric average as

$$G_1(T) = S(t_1)\cdot e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)(T-t_1)\frac{1}{n^2}\sum_{i=1}^{n} i \;+\sigma(T-t_1)^{1/2}\frac{1}{n^{3/2}}\sum_{i=1}^{n} i\varepsilon_i}.$$

Except for notation, the derivation so far has followed Turnbull and Wakeman (1991). For analytical tractability and ease of application, we now allow for continuous observations by letting $n$ go to infinity.

Using the results,

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n} i \;= \lim_{n\to\infty} \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}$$

and[36]

$$\lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{1}{3},$$

the geometric average becomes

$$G_1(T) = S(t_1)\cdot e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)\left(\frac{T-t_1}{2}\right)+\sigma\left(\frac{T-t_1}{3}\right)^{1/2}\varepsilon}.$$

---

[36] $i$ and $n$ are squared in the summation below as it is the variance of the normal distribution that can be summed.

This result can also be found in Kemna and Vorst (1990), although they use another derivation.

To find the distribution of the contract, it is necessary to express $S(t_1)$ as a function of the price today as we want $t_1$ to be 20 years ahead. This can be done by writing $S(t_1)$ in the same form as equation (F.1), giving

$$G(T) = S_0 \cdot e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)\left(\frac{T+t_1-2t_0}{2}\right)+\sigma\left(\frac{T+2t_1-3t_0}{3}\right)^{1/2}\varepsilon}.$$

The contribution to profit if the next generation's pulp mill is built was estimated in section 7 as 400 million SEK from using existing facilities and an extra 0.1 million tonnes of pulp in the first year as running in time is reduced. Algebraically, the contribution to profit becomes 275 + 0.07$S$ million SEK and we can write the expansion option as

*$V_0$(expansion option, Brownian motion) =*

$$= e^{-0.064\cdot 30} \int_{cut\ off\ price}^{a+5b} (275e^{0.013\cdot 30} + 0.07 \cdot 4500e^{x}) \cdot \varphi(x)dx \qquad \text{(F.2)}$$

with the pulp price $S = 4500e^{x}$, where $X$ is normally distributed with probability density function $\varphi$, mean $a$ and standard deviation $b$.

$$\varphi(x) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2b^2}}$$

$$a = \left(r - \delta - \tfrac{1}{2}\sigma^2\right)\left(\frac{T + t_1 - 2t_0}{2}\right) = \left(0.064 - 0.064 - \tfrac{1}{2} \cdot 0.189^2\right)\left(\frac{30+20}{2}\right) = -0.447$$

$$b = \sigma\left(\frac{T + 2t_1 - 3t_0}{3}\right)^{1/2} = 0.189\left(\frac{30 + 2 \cdot 20}{3}\right)^{1/2} = 0.913.$$

Without averaging, the cut off price was set as SEK 4000 per tonne multiplied by the expected increase in spot price. Using the same logic, we set the cut off price to SEK 4000 per tonne multiplied by the expected increase in the average price (between the years 20-30).

Changing the drift rate in a risk-neutral world $r-\delta$, to the (spot) drift rate in the real world $\alpha$, gives $a = -0.122$. We therefore get the cut off price as,

$$\textit{Cut off price} = E[G(T)] = 4000 \cdot \int_{-\infty}^{\infty} \frac{e^x}{b\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2b^2}} dx = 4000 \cdot e^{a+\frac{1}{2}b^2} = 5374.$$

Switching back to the risk neutral world, the value of the expansion option F.2 becomes

*$V_0$(expansion option, Brownian motion) =*

$$= e^{-0.064 \cdot 30} \int_{\ln\left(\frac{5374}{4500}\right)}^{a+5b} (275e^{0.013 \cdot 30} + 0.07 \cdot 4500e^x) \cdot \varphi(x)dx = 41.2 \text{ MSEK}.$$

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# The stochastic behaviour of commodity prices

# (Essay 3)

Henrik Andersson*

## Abstract

Over the past thirty-five years, economists have shown a great deal of interest in the stochastic behaviour of financial asset prices. Mostly concentrating on stock prices, it has been shown that stock prices show signs of both random walk and stationarity. Returns are found to be leptokurtic, rather than normally distributed. The behaviour of commodity spot prices has attracted less interest, and studies have only been made of a few goods each time.

This paper takes a broader perspective, studying the market prices of almost three hundred different commodities from 1970 and onwards. Results are easily summarised. For most commodities, we are not able to reject a unit root, whereas normality is always rejected due to excess leptokurtosis. Commodities are, however, distinguished by rather infrequent price changes, questioning the use of continuous distributions in describing the stochastic behaviour.

*Keywords*: Commodities, Mean Reversion, Paretian Distributions, Unit Root tests.

*JEL classification code*: C22, G14

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## 1. Introduction

The question whether the return on financial assets follows a random walk and adheres to a normal distribution is of fundamental interest in financial economics. Consequently, it has also gained considerable interest since the early studies of Mandelbrot (1963) and Fama (1965). Letting $x(t)$ represent the natural logarithm of the price at time $t$, a random walk with drift is in discrete time[1] described by the time series

$$x(t) = \alpha + x(t-1) + u(t), \tag{1.1}$$

where $u(t)$ is normally distributed white noise, with mean 0 and variance $\sigma^2$. For a $k$ period random walk, (1.1) can be written as

$$x(t) = \alpha k + x(t-k) + \sum_{i=0}^{k-1} u(t-i), \tag{1.2}$$

where $\sum_{i=0}^{k-1} u(t-i)$ is $N(0, k\sigma^2)$. The perhaps cumbersome notation of equation (1.2) emphasises two important properties of a random walk. First of all that residuals are added for a $k$ period random walk, and secondly that the variance of a forecast grows linearly with the time-horizon.

Rewriting (1.1) as

$$r = x(t) - x(t-1) \sim N(\alpha, \sigma^2), \tag{1.3}$$

[1] The continuous time equivalent of (1.1) is the geometric Brownian motion. This is the standard assumption behind most option pricing models.

we see that the random walk implies a continuously compounded return $r$ that is normally distributed. Portfolio theory and models of capital market equilibrium typically assume the normal distribution, as assumptions of investor preferences for skewness and kurtosis are thereby avoided. The standard assumptions that investors prefer more to less and are risk-averse are enough to specify investor behaviour and with some additional assumptions also market equilibrium.

From a statistical viewpoint, the random walk is a rather special object. The coefficient of $x(t-1)$ is by definition equal to unity. In the jargon of statisticians, equation (1.1) thereby has a unit root. If the coefficient is less than one, the time series exhibits mean reversion. We now define this concept formally in equation (1.4), which shows a trend-stationary autoregressive process of the first order,

$$x(t) = \alpha + \beta t + \phi x(t-1) + u(t). \tag{1.4}$$

$x(t)$ can also be expressed as a function of $x(t-k)$ by iteratively substituting $x(t-1)$ for $x(t-2)$ and so forth. The result is,

$$\begin{aligned} x(t) = {} & \alpha(1+\phi+\ldots+\phi^{k-1}) + \\ & + \beta t(1+\phi+\ldots+\phi^{k-1}) - \beta(\phi+2\phi^2+\ldots+(k-1)\phi^{k-1}) + \\ & + \phi^k x(t-k) + u(t) + \phi u(t-1) + \ldots + \phi^{k-1}u(t-k+1). \end{aligned} \tag{1.5}$$

If $|\phi| < 1$, then as $k \to \infty$, equation (1.5) converges to,

$$x(t) = \frac{\alpha}{1-\phi} - \frac{\phi}{(1-\phi)^2}\beta + \frac{\beta}{1-\phi}t + \eta(t), \tag{1.6}$$

where $\eta(t) \sim N(0, \frac{\sigma^2}{1-\phi^2})$.

Note that the diffusion $\eta(t)$ is independent of the number of time-periods $k$. The distribution of $x(t)$ does not explode as time passes. Providing that $k$ is sufficiently large, the variance of a forecast over $k$+1 periods is approximately the same as the variance of a forecast over $k$ periods. This contrasts the previous random walk model where $\phi = 1$ and $\beta = 0$. For the random walk, the variance grows linearly with the number of time-periods $k$, as can be seen in expression (1.2). This difference forms the basis of the variance ratio test performed in section 3. It should also be noted that (1.4) is the discrete time equivalent of the mean reverting process developed in essay 2, equation (6.3). Comparing (1.4) with equation (6.8) of essay 2, we see that the structure is the same.

In economic terms, the difference between a random walk and a mean reverting process can be described by the effect that a price shock will have on prices far into the future. In the case of a random walk, the effect is permanent. As is revealed by expression (1.2), a price shock that occurs at time $t-k$ changes the expectation about prices at time $t$ by the same amount as the original price shock. If $x(t-k)$ changes by one unit, so does the expectation of $x(t)$. The other possibility is a mean reverting process. In this case, the effect of a shock at time $t-k$ gradually fades away. In the limit, expression (1.6), $x(t)$ is unaffected of what happened at time $t-k$. Figure 1.1 is trying to capture the spirit of this difference. Letting a large shock occur at time $t-k$, the random walk has not recovered at time $t$, and it will never adjust to the long-term growth path. The effect of a price shock is permanent. The mean reverting trajectory, on the other hand, adjusts to the long-term growth path and the shock has only a temporary effect on prices.

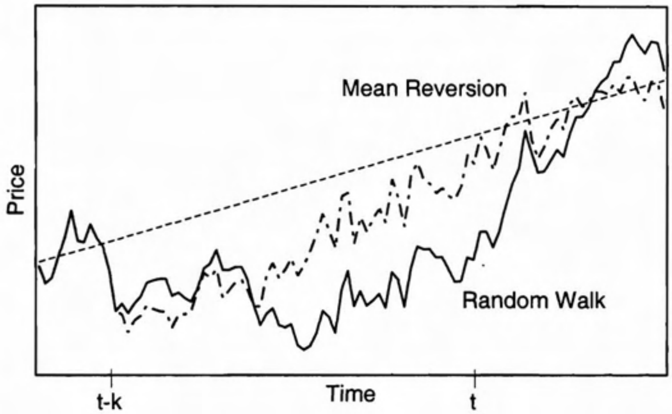


Figure 1.1 The mean reverting process will revert towards the long-term trend after a shock, whereas the effect on a random walk process is permanent.

There are some pros and cons of the respective processes when it comes to modelling asset prices. It is tempting to describe stock prices as random walks. First of all, if markets are efficient, future prospects of a stock are incorporated in the price. Thus, there is no gain from looking at the path of historical prices when predicting the stock price tomorrow. The path, or walk, is random. Apart from the stock price today, the only thing that affects the forecast of the stock price tomorrow is the drift and diffusion of the price process. The drift is captured by $\alpha$. Since not all earnings are distributed as dividends to the shareholders, the part that is retained will increase the value of the stock in the same way as a bank deposit will accrue interest over time. However, things are not always smooth. New information comes along that changes the prospects of the company. It may be a new order, changing market conditions, competitor announcements, etc. The stock price thus changes, motivating the stochastic term $u(t)$.

Similar economic arguments cannot be made for commodity prices. If the price of a commodity is subject to a positive shock, we expect this shock to be temporary. New companies will be attracted by the higher price, leading to increased supply and eventually a lower price. The shock to aggregate production volume may or may not be permanent, but the price will decline to the marginal cost of production where, once again, equilibrium is restored.

It may be clarifying to study what will happen to the stock price of a producer of the commodity. When the price shock occurs, the stock price increases accordingly as the producer's profits will increase. However, without any new price shock occurring, we can rationally expect the commodity price to decline to the marginal cost of production. When this actually happens, it should not affect the stock price as the lower commodity price already is anticipated. Thus, the effect of a commodity price shock is a permanent effect on the stock price. When the commodity price eventually comes down again, the stock price does not decline. Thus we may want to model different kinds of assets with different models. Mean reversion seems a reasonable assumption in the case of commodities, whereas for stock prices, a random walk is the basic assumption.[2]

Commodity prices can also be described as a mixture of a random walk and mean reversion. In this case a shock is not permanent, but neither will the price revert back to the pre-shock path. The papers by Fama and French (1988) and Cochrane (1988) emphasise such an interpretation. We will, however, abstain from following this route. In portfolio theory, stock prices are assumed to follow a random walk. Our primary interest is therefore to see if this is a workable assumption for commodities. Commodities and standardised industrial products are not a negligible fraction of world assets. Their stochastic behaviour is important in order to understand their relationship to financial markets in general. Secondly, also most option pricing models

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[2] In fact, stock prices cannot in equilibrium be mean reverting as this would imply that the expected return on stocks is sometimes negative. No one would invest in such a stock and the market price would therefore change.

assume a random walk, as pricing models of that type are more tractable from a mathematical point of view than their mean reverting counterparts.

So far, the assumption has been that price shocks, $u(t)$, are normally distributed. This hypothesis is nearly always rejected for stock prices. Fama (1965), among others, shows that the tails are "fatter" and the centre more peaked compared to the normal distribution. This is shown in figure 1.2.

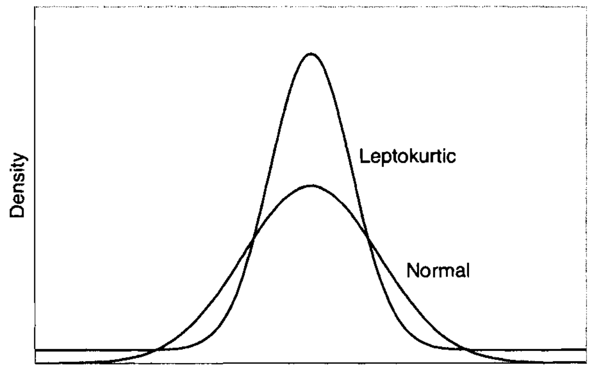


Figure 1.2 Leptokurtic and normal distributions.

The thicker tails and peaked top, or leptokurtosis, are consistent with a class of distributions known as stable Paretian distributions. The idea of a stable distribution is straightforward. If the sum of i.i.d. variables has the same distribution as the variables themselves, the distribution is said to be stable under addition. The example in case is the normal distribution. Adding normally distributed variables leaves us with a sum that is normally distributed. The family of stable distributions is known as Paretian distributions. Limiting ourselves to symmetric distributions, and following Fama and Roll (1968), the stable symmetric Paretian distribution may be defined by the characteristic function, $\phi_x(t)$, given as

$$\ln \phi_x(t) = \ln \int_{-\infty}^{\infty} e^{itx} dF(x) = i\delta t - \gamma |t|^\alpha = i\delta t - |ct|^\alpha, \tag{1.7}$$

where, $t$ is any real valued number,

$F$ is the cumulative density function,

$\alpha$ is the characteristic exponent $(0 < \alpha \leq 2)$,

$\delta$ is the location parameter,

$\gamma = c^\alpha$ is the scale parameter.

In the case that $\alpha = 2$, this distribution degenerates to the familiar normal distribution, with mean $\delta$ and $\gamma$ equal to one-half the variance. When $\alpha < 2$, the distribution exhibits the fatter tails often seen in economic time series. Also, when $\alpha < 2$, the second, third, fourth, and so on, moments do not exist. Referring to the second moment, it is often said that the variance is infinite. The concept of an infinite variance may seem strange to many people since the sample variance always exists. That the population variance sometimes does not exist is not very surprising, however. Variance is defined as an expectation. An infinite variance only means that the integral defining this expectation does not converge to a finite number.

Paretian distributions generalise the central limit theorem. If new information arrives more or less continuously and is, at least reasonably, i.i.d., central limit type of arguments assure that the cumulative effect of this new information will be a random variable that is stable under addition and converges to a limiting distribution. However, this distribution is only normal when shocks are of finite variance. Individual pieces of information do not introduce price changes that are "too large" and occur "too often". If, however, the shocks have infinite variance, the resulting distribution is stable Paretian with $\alpha < 2$. Unfortunately, probability density functions are, with a few exceptions, unknown for this class of distributions, severely limiting the statistical methods available in dealing with such processes.

In this paper, we will examine the price process of 280 different commodities where the time series are of varying length, starting somewhere between 1970 and 1994 and ending March 2000. The adherence to a random walk will be examined by performing four tests:

- Lo and MacKinlay variance ratio test.
- Phillips and Perron unit root test.
- Kwiatkowski et al. stationarity test.
- Fisher panel data test.

Normality is examined by studying:

- Skewness
- Kurtosis
- Characteristic exponent

In general, trustworthy commodity data are difficult to obtain. There are several reasons why this is so. Many markets are characterised by few and large sellers and buyers. Even in cases where there exists a marketplace, trading may be thin and not representative. Major deals are agreed upon in negotiations between the parties, and may or may not be reported. As in many cases there is no formal market place, records of the transactions are also difficult to obtain and less organised. In this study, prices are extracted from the Datastream commodity data base. This data base contains daily price quotes for some 1500 different price series. However, many of these series are closely related. For example, there are several different qualities and quotations of crude oil. As these time series are strongly correlated, only one series represents crude oil.

Some prices are quotes from organised markets, such as the London Metal Exchange, whereas other are import/export prices. All are, however, quotes of spot prices paid for delivery of the physical asset. In many cases, prices do not change very often. There can be months and sometimes even years in between. This inertia in price movements may reflect price controls or inadequately developed markets, rather than actual market price behaviour. In some cases even cartels. The series with the highest number of price changes is therefore used in cases where there are different time series available for the same commodity. Longer series are also selected rather than shorter time series. Indices and forward/futures prices are excluded and no series are shorter than six years. In all, 280 time series are examined, and most run between 1986-2000. The longest series starts in 1970 and the shortest in 1994.

All tests are performed on weekly and monthly sampled data even though daily data are available. The reason is the infrequent price changes in many series. Most tests assume a normally or at least continuously distributed error term, something that requires aggregation of data. However, it turned out that the results were almost identical and therefore independent of the sampling frequency. Only weekly sampled results are reported. The only exception is the characteristic exponent of the error term, which was sensitive to the sampling frequency.

The remainder of this paper is organised as follows. Section 2 reviews earlier studies of commodity prices. Section 3 contains tests of random walks. Tests for normality of price changes are made in section 4. Section 5 concludes by discussing the empirical results and summarises the evidence. The appendix details the respective commodities, the lengths of observations, and trade details.

## 2. Previous research

The broadest study to date of commodity price behaviour is made by Barkoulas, Labys and Onochie (1997). They survey 20 commodities, as well as one composite index. Using the Phillips-Perron (PP) test for a unit root, they are only able to reject the random walk (1.1) for one commodity, tea. If we are willing to accept a 10 percent significance level, the random walk is also rejected for copper and jute. Turning the null-hypothesis around, and using the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test, the authors are able to reject a stationary process for all commodities except copper, jute, wool and zinc. Thus, only for copper and jute is a random walk rejected and mean reversion not rejected. For the vast majority of commodities, empirical evidence favours a random walk. Barkoulas, Labys and Onochie also test for the presence of long-term memory by allowing processes to be of the fractionally integrated ARFIMA type. This line of studies is potentially important as there is nothing suggesting that commodity prices should be mean reverting in the short run, whereas some long-term reversal to the marginal cost of production ought to occur. We could therefore expect temporal dependence between distant observations even though there is no dependence in the short run. Using the Geweke and Porter-Hudak (GPH) test, the authors find signs of fractional integration, and thus rejecting the random walk, for five commodities: copper, gold, soybeans, tea and wool.

The random walk is more frequently rejected when longer time series of data are used. Pindyck and Rubinfeld (1992 p. 465) reject a random walk in the price of copper and crude oil, when more than 100 years of (annual) data are used. Even so, they fail to reject a random walk in the price of lumber. Schwartz (1997), on the other hand, finds strong mean reversion in futures prices of copper and oil, with significant coefficients. However, it is difficult to assess what significant means for the Kalman filter estimation used by

Schwartz. In particular, the Kalman filter may exhibit the same downward bias in the estimate of $\phi$ as the OLS does when the true process is a random walk. Nevertheless, it is interesting to note that Schwartz fails to verify mean reversion in the price of gold futures. Gold is often considered an investment asset rather than a commodity, and we are therefore more tempted to model the price process as a random walk. However, it should be pointed out that a failure to verify mean reversion does not necessarily imply a random walk. For example, single shifts in the mean produces a non-stationary process.

It also deserves to be mentioned that for a process to be a true random walk it is required that the error term is i.i.d. normal and in particular that there is no autocorrelation between different observations. This requirement is, however, of less interest when we try to distinguish between a random walk and mean reversion. The focus is on the coefficient $\phi$, and whether this is equal to unity, as it would be for a random walk. Many tests are therefore made robust to the presence of autocorrelation. Formally these are tests of a unit root, but as this term is unfamiliar to many people, they are often called tests of a random walk.

Moving on to the question about normality, Hudson, Leuthold and Sarassoro (1987) examine futures prices of wheat, soybeans and live cattle contracts for the period 1974-82. Taking the random walk for granted, they find a higher degree of normality than tests on earlier time periods have revealed. Using tests for kurtosis, normality is rejected for approximately 40 percent of the contracts. Independent price moves are rejected for 14 percent of the contracts and symmetry rejected for 11 percent of the contracts. However, using the Fama and Roll (1971) procedure to test for the characteristic exponent $\alpha$ of a stable Paretian distribution, normality is rejected for 75 percent of the contracts. In another test of futures prices, Wahab (1995) examines the behaviour of daily gold and silver futures prices between 1982-91. Using the Augmented Dickey-

Fuller test, a unit root is not rejected.[3] Returns are highly leptokurtic and negatively skewed.

Summing up the research accounted for in this section is easy. In most cases, we cannot reject a random walk. The return should therefore be stationary and although we are not always able to reject the normal hypothesis formally, leptokurtosis seems always present. Skewness, both to the left and right, are reported for some assets. Despite the rather unambiguous results, it is nevertheless of interest to perform a broad study of the behaviour of commodity spot prices. First and foremost because it has, to the best of my knowledge, never been performed before. Secondly, when the random walk is rejected, are price changes leptokurtic also in this case, or is the leptokurtosis an effect of model misspecification? Will the leptokurtosis disappear when the error term is taken from equation (1.4)? These are the questions that we will try to answer in sections 3 and 4.

[3] The Augmented Dickey-Fuller test is asymptotically equivalent to the Phillips-Perron test used in this paper.

## 3. Random walk tests

In trying to distinguish between the random walk (1.1) and the mean reverting process (1.4), a variance plot is a reasonable point of departure. As $Var[u(t+k)-u(t)]/k = \sigma^2$ under (1.1), and decreases towards 0 as $k \to \infty$ when (1.4) holds, a variance plot will give some idea of how the prices behave. The plot for energy related time series is shown in figure 3.1, where all variances are scaled by the initial variance (when $k = 1$), in order to graph all time series in one figure.

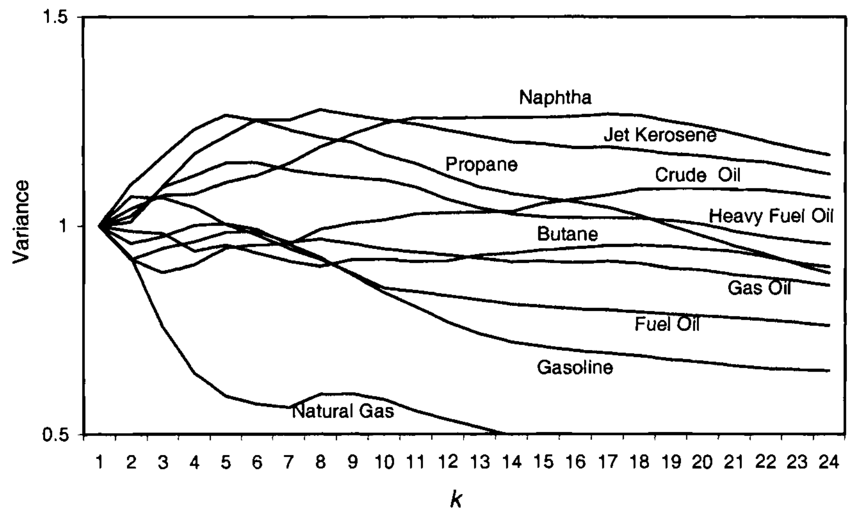


Figure 3.1 $1/k$ times the variance of the $k^{\text{th}}$ difference in log commodity prices.

Only Natural Gas, Gasoline and Fuel Oil exhibit the downward sloping curve characterising a mean reverting process. A random walk, ideally straight lines in the plot, is probably favoured for the majority of series even though distinct inferences are difficult to draw. An interesting feature of figure 3.1 is the increasing slopes that characterise some of the time series for the first lags. This is due to positive serial correlation. Equation (1.2) emphasises that the residuals are added for a random walk. However, it is only when the residuals are i.i.d. that $Var[u(t+k)-u(t)]/k=\sigma^2$. When there is a linear dependence between the residuals, $Var[u(t+k)-u(t)]/k=k\sigma^2$. The slope in the variance plot is therefore sensitive to the degree of serial correlation. Although it may pose a problem in individual time series, the average serial correlation in the error term $u(t)$ is quite small. This is independent of whether (1.1) or (1.4) is assumed to be the right model. As (1.1) is a special case of (1.4), only serial correlation of (1.4) is reported in table 3 at the end of the paper. A quick summary is given beneath.

| | Average serial correlation | | |
|---|---|---|---|
| corr[$u(t)$,$u(t-k)$] | k = 1 | k = 2 | k = 4 |
| Weekly data | 0.07 | 0.06 | 0.05 |

We use the Lo and MacKinlay (1988) variance ratio test, in its simplest form, to formalise the variance plot made.[4] Showed in the first columns of table 2, it rejects a random walk for approximately 40 percent of the price series. However, only when the statistics are significantly negative is the random walk rejected in favour of mean reversion. This is so for only 10 percent of the series. The rest of the rejections are due to the positive slopes caused by positive serial correlation.

[4] In its simplest form, the Lo and MacKinlay variance ratio test is nothing more than a formalisation of the variance plot with the purpose of formally rejecting the null hypothesis of a random walk. There is also a version of the variance ratio test where serial correlation and heteroskedasticity are corrected for. Obviously, this version does not match the plot, where serial correlation will add the variance from previous lags. We therefore save this correction for the Phillips-Perron test.

An obvious alternative to studying the variance of $u(t)$ is to concentrate on the mean reversion coefficient $\phi$ directly. This is more difficult than it may seem. Assume the true process to be the random walk of equation (1.1). To ascertain that the autoregressive parameter $\phi$ is indeed unity, we may want to run regression (1.4), to distinguish between $\phi = 1$ and $\phi < 1$. In an ordinary least squares regression, the estimator $\hat{\phi}$ converges in distribution (or law) and is asymptotically normal when $\phi < 1$,

$$\sqrt{T}(\hat{\phi} - \phi) \xrightarrow{L} N(0, 1 - \phi^2).$$

Unfortunately the asymptotic distribution collapses to a point mass if $\phi = 1$. It is possible to scale the test statistics with $T$ instead of $\sqrt{T}$, but this will lead to an asymptotic distribution that is no longer symmetric, but skewed to the right. As a consequence, the estimator $\hat{\phi}$ will in most cases be less than one even though the true value is $\phi = 1$. This downward bias is severe. Hamilton (1994 p. 493) reports that when $\phi = 1$ and $\beta = 0$, ninety-five percent of the estimates $\hat{\phi}$ will be less than one. This explains the low power of tests to distinguish between a random walk and a mean reverting process. Notwithstanding the difficulties, it is possible to test for a unit root by studying the asymptotic behaviour of the estimator. The most common test to distinguish between the processes (1.1) and (1.4) is the Phillips-Perron test (for example detailed in Hamilton 1994 p. 514), where (1.1) is the null hypothesis. The Phillips-Perron test is a generalisation of the original Dickey-Fuller test where the error terms are allowed to be autocorrelated and possibly heteroskedastic. Errors do not necessarily have to be normal as long as the fourth moment is finite. As many financial time series have leptokurtic error terms this requirement is often not satisfied. However, as the sampling moments always exist the violation may not be too severe. At least to the author's knowledge, no paper addresses this problem.

An alternative to the traditional unit root tests is a test developed by Kwiatkowski, Phillips, Schmidt and Shin (1992). Their test allows the alternative null hypothesis of a stationary process to be tested. If prices are mean reverting, the growth path will settle down around the long-term deterministic trend, as in figure 1.1, and deviations from this trend are only temporary. The idea behind the KPSS test is not to test for mean reversion directly, but to see if a deterministic trend can be found in the data. If prices are indeed mean reverting, they will eventually revert towards a long-term trend and it may therefore be possible to detect mean reversion by discerning such a trend in the data. Formally, the KPSS test the following time series process,

$$x_t = \beta t + r_t + \varepsilon_t, \tag{3.1}$$

where $r_t$ is a constant. The alternative hypothesis is

$$r_t = r_{t-1} + u_t. \tag{3.2}$$

The null hypothesis is thus that the process is trend-stationary vs. the process contains a unit root. This is tested in the following way:

$$H_0 : \frac{\sigma_u^2}{\sigma_\varepsilon^2} = 0 \quad \text{vs.} \quad H_1 : \frac{\sigma_u^2}{\sigma_\varepsilon^2} > 0.$$

If the null hypothesis is true, $u_t$ will be a constant (e.g. 0). Then the process is trend-stationary. If the alternative is true, $r_t$ follows a random walk and $x_t$ is non-stationary.[5]

The appealing feature of the KPSS test is that, in combination with the PP test, it discerns four separate cases:

1) Rejection of the mean reversion hypothesis, and no rejection of the random walk hypothesis. (Rejection of $H_0$ in the KPSS test, but not in the PP test.) We thus conclude that the random walk is the best description of commodity spot prices.

2) Rejection of the random walk hypothesis, and no rejection of the mean reversion hypothesis. (Rejection of $H_0$ in the PP test, but not in the KPSS test.) We thus conclude that mean reversion is the best description.

3) Neither null hypothesis is rejected, in which case we infer that the data set is not sufficient to distinguish between the two processes or that the tests are not powerful enough.

4) Both null hypotheses are rejected. In this case neither the random walk nor the mean reversion is an adequate description of commodity prices.

The results of the PP and KPSS tests are detailed in table 2. Both tests are robust to the presence of serial correlation, when the number of lagged residuals included in the test-statistics is chosen properly. The catch is to choose the number of lags properly. Too few will not free the test of problems caused by serial correlation, whereas too many give a very insensitive test. In practice, the

[5] The test statistic is given in the explanation of table 2.

lag window $l$, the order of serial correlation allowed in constructing the statistics, is between 4 and 12.[6] Table 2 reports the results for $l = \{4, 8, 12\}$. For $l = 12$, significant values are underlined. The lag window used is the Bartlett (Newey and West) window.

Allowing for no autocorrelation, $l = 0$, the Phillips-Perron test rejects the random walk for 34 series (not shown in table 2). The statistics for these price series are below –21.4, the critical value at the 5 percent significance level. Choosing $l = 12$ as the statistic of primary interest, the PP test rejects a random walk in 39 of the 280 price series, 14 percent.

Combining the PP and KPSS tests works well in general:

- The mean reversion hypothesis is rejected and the random walk not rejected for 222 of the 280 series (case 1).
- The random walk hypothesis is rejected and the mean reversion not rejected for 14 series (case 2).
- Data are not sufficient to distinguish between the hypotheses for 19 series (case 3).
- Both null hypotheses are rejected for 25 series (case 4).

Thus, for the majority of commodities, a random walk seems to be the preferred process. The examples shown in figure 3.2 are fairly representative for the behaviour of the series in case 1. There are some marked peaks, but just from viewing the graphs it is difficult to see that these series are examples of random walks. If mean reversion is slow, deviations from a long-term trend can persist

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[6] Observe that the order of serial correlation allowed is smaller than the number of lagged residuals included in the test. For example, $l = 12$ and 800 observations means that 20 lagged residuals are included.

for a long time. Gold, which is arguably more an investment asset than a commodity, is the series with a test statistic most in favour of a random walk. The test statistic for the PP test is –4.43 and the critical value is –21.4 at a 5 percent significance level. Steers provides the opposite example, the test statistic is –13.96. Although a random walk cannot be rejected in either case, the difference is perhaps also visible in figure 3.2.

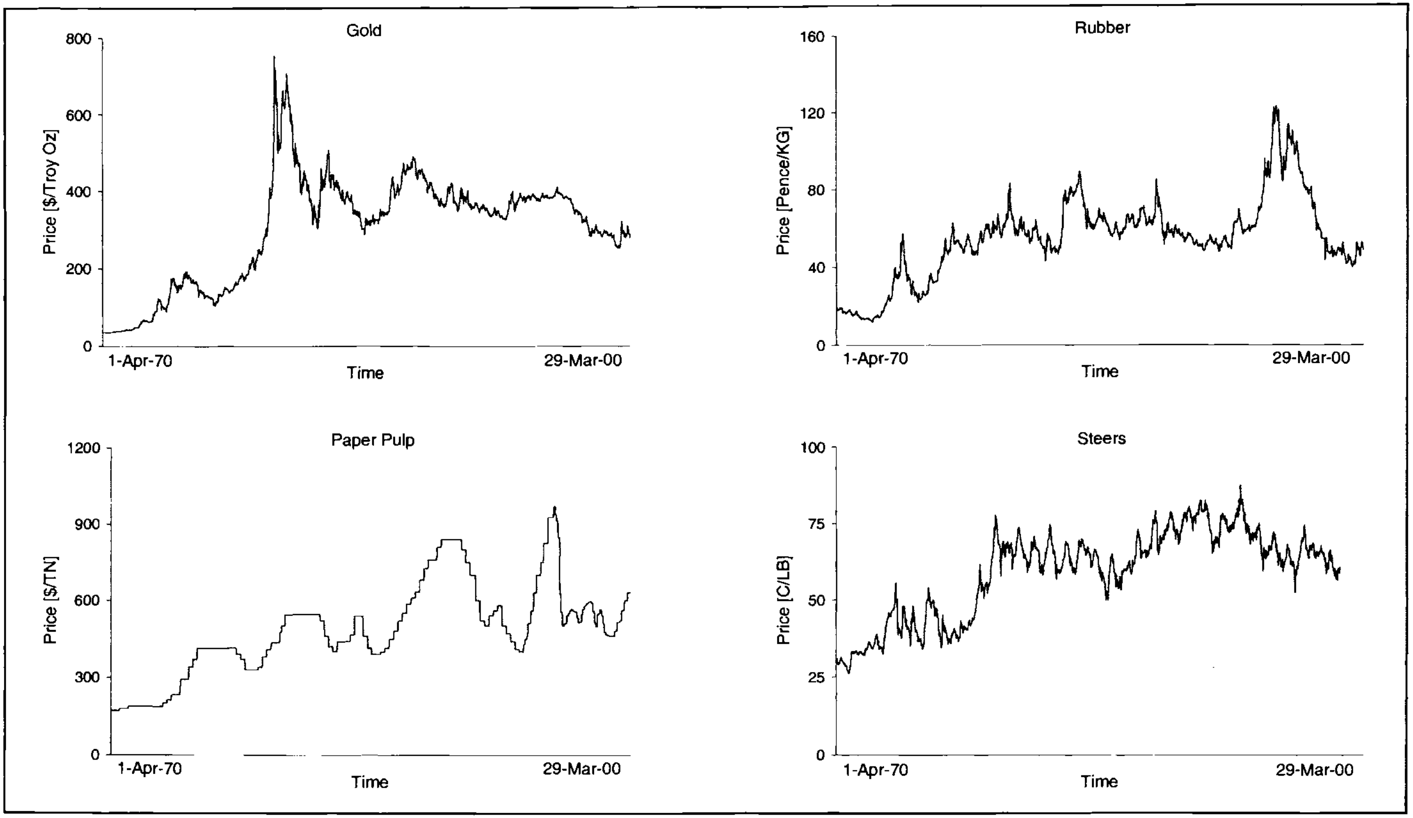


Figure 3.2 Examples of individual time series where mean reversion is rejected and a random walk not rejected.

Generally, the PP test is the most trustworthy of the tests. Sadly (as the null hypothesis of stationarity is very attractive), there are several reasons to be suspicious about the KPSS test. The sum of squared errors is on average thirty times higher for the trend stationary regression (3.1), compared to the AR1 regression in (1.4), and has practically no power. This is perhaps not so surprising, as deviations from the deterministic trend are persistent when mean reversion is slow. Remember, equation (3.1) is not tested to see if it is a good description of price data. We know it is not. (3.1) is only tested to see if it can give a hint on mean reversion. Even so, it feels shaky to trust a test based on a regression with such a low power.[7] Also, rejection of the mean reversion hypothesis is critically dependent upon the lag window *l*.

Another reason for suspicion is that the KPSS test rejects mean reversion for some series that the Variance plot and the LM and PP tests identify as strongly mean reverting. Most notably this is the case for Natural Gas, Pork Bellies, Sri Lanka Tea and Barley. The failure of the KPSS test to accept mean reversion for these series casts doubt on the other results as well. One explanation is probably parameter shifts.

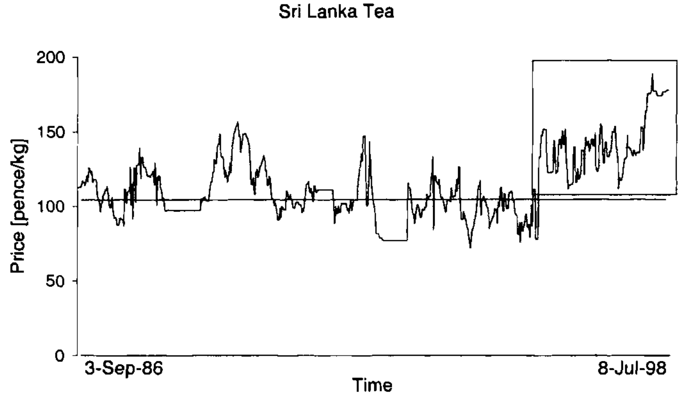


Figure 3.3 A parameter shift may cause problems for the KPSS test.

[7] It should be pointed out, however, that the test worked well in simulations, even when data were generated by (1.4) and the sum of squared errors was large.

For example, a single shift in the price level $r$ in formula (3.1) produces a nonstationary process. Figure 3.3 provides one such example. When a shorter time series is chosen, excluding the last years which are marked by the rectangle in figure 3.3, the KPSS does not reject stationarity.

The lack of sensitivity in time series unit root tests has led a number of researchers to develop panel data tests for unit roots. In a clearly written and interesting paper that compares different panel data tests, Maddala and Wu (1999) suggest the use of a test that they attribute to Ronald Fisher. He derived it in the 1930's. The idea behind this so called Fisher test is to test for the significance that results from $N$ independent tests of a hypothesis. There are different versions of this "meta test". The one proposed by Maddala and Wu is to sum the logarithms of the observed significance levels ($p$-values) from $N$ independent time series. Denoting the significance levels $p_i (i = 1,2,\ldots,N)$, the distribution of a variable $\lambda$ where

$$\lambda = -2\sum_{i=1}^{N} \ln p_i \tag{3.3}$$

is $\chi^2$ with $2N$ degrees of freedom. Independently of the null hypothesis used, with a Fisher test, we can therefore measure the probability $p_\lambda$ that the null is rejected. Using this procedure on the Augmented Dickey-Fuller test of individual time series, Maddala and Wu claim that the Fisher test is superior to other panel data tests suggested in the literature.

There are several appealing features of the Fisher test. First of all it does not require a balanced panel. The number of observations does not have to be the same for all the series. This is important since the lack of power in unit root tests requires the use of as many observations as possible. To maximise the number of observations some of the 280 time series start in 1970 and some in 1994.[8] A balanced panel is therefore not feasible. The Fisher test allows us to pool the information from the individual tests and in this way study commodities as a group. Just in the same way as we do not want to model individual stocks with different price processes, we do not want to model different commodities with different processes. Viewing the commodity data as an (unbalanced) panel of data are therefore appropriate.

The Fisher test also has the advantage of being applicable to both the PP test and the KPSS test, i.e. it is possible to use both a random walk and a stationary process as the null hypotheses. Monte Carlo simulation is used to determine the *p*-values for the respective tests. Assuming the null hypothesis to be true, 100 000 trajectories are simulated. Each trajectory is based on 500 time-steps. Individual series vary in length between 285 and 1566 observations. Using this many observations, the test statistics of the PP and the KPSS tests are relatively insensitive to the number of observations. A simulation based on 500 time-steps is therefore an acceptable approximation.[9]

The test statistics are also independent of parameter settings and serial correlation. It is therefore possible to set $\alpha$ in (1.1) and $\beta$ in (3.1), the respective null hypotheses, equal to zero and to sample independent price moves. After having simulated the 100 000 trajectories, the probability (*p*-value) of having a test statistic greater than a critical value is calculated. This boils down to actually counting the number of simulations that have given test statistics greater than the critical value.

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[8] Most of the series start in 1986. Details of the respective time series are found in the appendix.

[9] In fact, in the original KPSS paper of Kwiatkowski et al. (1992), the authors use the asymptotic critical values on a data set with 62 observations. However, this might be stretching asymptotic theory too far.

The procedure is reasonably exact. For the KPSS test, for example, Kwiatkowski et al. (1992) report the critical values {0.119, 0.146, 0.215} for the significance levels 10 %, 5% and 1% respectively. The critical values obtained by the Monte-Carlo simulations are {0.119, 0.147, 0.215}. Table 2 reports the *p*-values for the respective commodity under each test.

Concentrating first of all on the KPSS test, stationarity was rejected for 247 of the 280 series in the individual tests. As can be expected, also the Fisher test based on the KPSS test rejects stationarity (at any significance level).

Turning the null hypothesis around and using a random walk as the null, a Fisher test performed on the PP test rejects a random walk at any significance level. This strong rejection is interesting. One of the merits of testing a panel of data is the ability of finding out whether rejections of the null hypothesis in tests of individual time series are due to chance or not. With a significance level of 5%, 5 series out of 100 should be rejected due to chance.

The strong rejection of a random walk in the Fisher test on the PP test seems to indicate that the rejections of a random walk for 39 commodities in the PP tests were not entirely due to chance. Some series are probably mean reverting although these series are much fewer than can be expected from economic reasoning. The Fisher test therefore complements the results of the individual tests. A random walk is the best description for a clear majority of the commodity price series, but not for all of them.

## 4. Normality tests of price shocks

The purpose of this section is to study the distribution of price changes, and test whether the normal distribution is an appropriate description of the error term $u(t)$. Table 3 reports skewness and excess kurtosis when $u(t)$ is sampled weekly and generated by (1.4). The result of the skewness test is quite unambiguous. The hypothesis of a symmetric distribution holds for 86 percent of the series independently of whether data are sampled weekly or monthly. If anything, the distribution of $u(t)$ is somewhat skewed to the right, as approximately two-thirds of the series have positive skewness even though it is not significant. In all price series, a significant excess kurtosis is apparent. Using monthly sampled data (not shown), the kurtosis is much smaller but normality is still rejected in all but sixteen cases. Similar results are obtained when the error term is generated by the random walk (1.1). Thus, the leptokurtosis is not a result of model misspecification.

Rejecting the normal distribution in favour of the more general symmetric stable Paretian distribution (1.7), the procedure developed by Fama and Roll (1971) is used to estimate the characteristic exponent $\alpha$. The estimation is based on the observation that independently of how fat the tails are, the probability density function of the stable Paretian distribution crosses the probability density function of the normal distribution at approximately the same place (figure 1.2). Using this point as a benchmark, we can estimate the "variance" and thereafter measure the "fatness" of the tails.

In technical terms: The values corresponding to some fractiles of the standardised distribution are almost insensitive to $\alpha$. In particular, the 0.72 fractile corresponds to the interval 0.824 - 0.830 when $1 \le \alpha \le 2$. Given the symmetry, the scale parameter $c$ can be estimated as $\hat{c} = \frac{\hat{x}_{0.72} - \hat{x}_{0.28}}{2 \cdot 0.827}$, where

$\hat{x}_y$ refers to the *y*:th fractile of the sample. Once *c* has been estimated, the test statistic $\hat{z} = \frac{\hat{x}_f - \hat{x}_{1-f}}{2\hat{c}}$ can be used to estimate $\alpha$ because the distance between the fractiles *f* and 1–*f* is uniquely determined by $\alpha$, when the other parameters are known. In our case we choose the fractile *f*=0.96. Table 2 in Fama and Roll (1968) is then used to find $\alpha$. Estimations are shown in table 3 and summarised beneath:

Table 1. Summary statistics for the characteristic exponent $\alpha$. The subset refers to series where price changes occur at least every fortnight.

| | All series weekly sampled | Subset of series weekly sampled | Subset of series monthly sampled |
|---|---|---|---|
| $1.0 < \alpha$ | 130 | 13 | 0 |
| $1.0 < \alpha < 1.2$ | 31 | 11 | 4 |
| $1.2 < \alpha < 1.4$ | 29 | 21 | 10 |
| $1.4 < \alpha < 1.6$ | 33 | 29 | 33 |
| $1.6 < \alpha < 1.8$ | 23 | 16 | 29 |
| $1.8 < \alpha < 2.0$ | 6 | 1 | 10 |
| $2.0 = \alpha$ | 8 | 0 | 5 |
| $2.0 < \alpha$ | 20 | 0 | 0 |
| Total no. of series | 280 | 91 | 91 |

The results in the first column of table 1 make very little sense. An $\alpha$ greater than two is not possible. Also, $\alpha$ less than one, which among other things implies that the mean does not exist, is very rare. The problem arises because of the infrequent price changes that many commodities exhibit. Sometimes the price stays the same for months or even years. Of course, when price changes are far between, no model based on a continuous distribution will adequately describe the price shocks. This seems to be a minor problem when it comes to tests for random walks. The results are similar independently of the sampling frequency and the particular distribution of $u(t)$. However, when measuring kurtosis and the characteristic exponent of $u(t)$, the mismatch of using a continuous distribution is obvious. The problem disappears if sticky price series are dismissed. In the second and third column of table 1, only series where the price changes at least every fortnight are included, as the price shocks in these series can more likely be represented by a continuous distribution.

In the second column, where observations are gathered weekly, i.e. prices change at least every second observation, the problem of $\alpha > 2$ has disappeared. Moving on to column three, where the observations are gathered monthly, no series has an $\alpha$ less than one and for the majority of series $1.2 < \alpha < 1.8$. In this case, a continuous distribution is the most appropriate description of a price shock.

# 5. Economic interpretation and summary

The stochastic behaviour of commodity spot prices does not differ substantially from financial assets. Data on almost three hundred different commodities give very little support for the economic intuition that commodity prices should be mean reverting. Earlier studies on less extensive data sets are thus confirmed. There may be several reasons why we don't observe many price series that are mean reverting. First of all it is difficult to detect, statistically. Especially if the mean reversion is slow. This is true both for "pure" processes and, of course, even more so if the true process is a combination of mean reversion and a random walk. There may also be economic explanations, for example technological change. Shocks that occur due to technological change have a permanent effect on prices. Technological changes do not revert and restore prices at the original equilibrium. Strong entry and exit barriers will also thwart mean reversion since it stops potential producers from profiting from the changing prices. The most important barrier is probably time. It may take years to build a production facility, and there is a reluctance to abandon an existing one. Output will thereby remain constant over extended periods of time and random price moves will prevail.

The fact that price changes are leptokurtic is not surprising. Stock price changes have repeatedly been shown to exhibit leptokurtosis and there are no convincing arguments why commodity price changes should behave differently. Economically, this is nothing strange. It merely reflects the fact that some news are so important that the price will change more than what can be represented by a normal distribution.

Commodities do stand out in one important respect. The flow of information is slow, leading to infrequent price changes. When the time horizon is short, price changes are preferably represented by a discrete distribution. For longer periods, however, a stable Paretian distribution provides a better fit as the sum of i.i.d. discretely distributed variables adds up to a stable distribution.

The economic implication of the slow flow of information is probably not very important for portfolio theory. Even though all assets, including commodities, constitute the market portfolio, we find in practice that commodities are not much held in managed portfolios. Also, equilibrium models, such as the CAPM, are one-shot models. It is doubtful whether the planning horizon of households is so short that commodity price shocks should be modelled as discretely distributed.

The infrequent price changes may affect the ability to hedge and synthetically replicate a commodity option contract, however. Even though the hedging parameters of the standard Black-Scholes model are quite robust, the model is clearly misspecified when shocks are discretely distributed. For contracts where frequent updating of the hedged position is expected, the Poisson distribution may therefore be preferable.

The presence of leptokurtosis in all time series also suggests that stochastic volatility models should be useful for pricing of commodity options. A volatility that is stochastically changing over time will make the return distribution appear leptokurtic although it actually consists of many different normal distributions.

The most interesting question from an economic point of view is perhaps if commodities can be considered as a homogenous group of assets or whether individual differences in the respective markets make such a characterisation meaningless. The results here are encouraging. Commodities are best characterised by a random walk price process, and price changes are always leptokurtic with a fairly small amount of positive skewness and positive serial correlation. There are few exceptions to this rule. However, the exceptions that exist are probably not due to chance. The Fisher test performed on the PP test indicated that a few series are probably mean reverting. These series are very few, however. By and large, commodities can be considered as a group and their characteristics do not differ much from purely financial assets.

## Table 2. Tests of mean reversion

| | Lo-MacKinlay Variance Ratio Test | | | Phillips-Perron Unit Root Test | | | | Kwiatkowski-Phillips-Schmidt-Shin Trend Stationarity Test | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (5% Critical Values) | (*t* statistic 1.96) | | | | (-21.4) | | *p*-value | | (0.146) | | *p*-value |
| | *k* = 4 | *k* = 8 | *k* = 12 | *l* = 4 | *l* = 8 | *l* = 12 | (*l* = 12) | *l* = 4 | *l* = 8 | *l* = 12 | (*l* = 12) |
| Acid Oils-Fish | 1.89 | 2.02 | 1.41 | -7.17 | -7.44 | -6.79 | 0.68 | 0.49 | 0.26 | 0.19 | 0.02 |
| Acid Oils-Hard | 1.08 | 0.54 | -0.09 | -9.95 | -10.06 | -10.18 | 0.42 | 0.78 | 0.41 | 0.29 | 0.001 |
| Acid Oils-Soft | 2.29 | 0.85 | -0.12 | -10.52 | -9.59 | -8.72 | 0.53 | 1.02 | 0.54 | 0.38 | 0.0005 |
| Acid Oils-Soya... | 1.84 | 1.71 | 1.09 | -7.33 | -7.34 | -6.71 | 0.68 | 1.16 | 0.61 | 0.44 | 0.0005 |
| Acrylonitrile | 12.39 | 16.74 | 18.25 | -6.67 | -9.72 | -11.42 | 0.34 | 0.48 | 0.25 | 0.18 | 0.02 |
| Alloy Steel Scrap | 6.30 | 8.20 | 8.58 | -6.24 | -7.82 | -8.36 | 0.55 | 0.96 | 0.49 | 0.34 | 0.001 |
| Almonds | -0.68 | -0.64 | 0.01 | -7.12 | -7.93 | -8.18 | 0.57 | 0.91 | 0.47 | 0.33 | 0.001 |
| Aluminium | 0.66 | -0.83 | -0.97 | -12.62 | -13.33 | -13.85 | 0.23 | 0.46 | 0.25 | 0.18 | 0.02 |
| Ammonia | 9.44 | 11.90 | 11.25 | -14.36 | -16.96 | -15.47 | 0.17 | 0.76 | 0.40 | 0.30 | 0.001 |
| Aniseed Oil | -1.11 | -0.60 | -0.04 | -6.94 | -7.89 | -8.67 | 0.53 | 1.09 | 0.56 | 0.40 | 0.0005 |
| Antimony | 18.14 | 19.15 | 18.03 | -5.34 | -6.48 | -6.69 | 0.69 | 0.94 | 0.48 | 0.33 | 0.001 |
| Antimony Ore | 1.09 | 0.81 | 1.64 | -3.10 | -3.92 | -4.58 | 0.85 | 0.88 | 0.45 | 0.31 | 0.001 |
| Apricot Kernals | -0.29 | -0.21 | -0.13 | -5.05 | -5.25 | -6.19 | 0.73 | 0.70 | 0.39 | 0.27 | 0.005 |
| Arabic Gum | 0.39 | 0.99 | 1.24 | -3.05 | -3.20 | -3.30 | 0.93 | 1.70 | 0.93 | 0.65 | 0.0005 |
| Arsenic | 4.66 | 3.30 | 1.94 | -27.43 | -24.48 | -22.41 | 0.04 | 0.38 | 0.23 | 0.18 | 0.02 |
| Balsam-Canadian | 0.04 | 0.18 | 0.22 | -18.06 | -18.80 | -18.98 | 0.09 | 0.59 | 0.31 | 0.24 | 0.005 |
| Balsam-Copaiba | 0.16 | 0.24 | -0.52 | -29.17 | -28.24 | -27.08 | 0.02 | 0.25 | 0.14 | 0.11 | 0.13 |
| Balsam-Peru | 0.12 | 1.08 | 1.55 | -6.31 | -7.51 | -8.70 | 0.53 | 0.42 | 0.23 | 0.17 | 0.03 |
| Balsam-Tolu | -1.59 | -1.28 | -0.97 | -12.02 | -12.55 | -12.98 | 0.26 | 0.93 | 0.49 | 0.35 | 0.0005 |
| Barley | -2.67 | -2.84 | -2.70 | -27.84 | -29.57 | -31.85 | 0.005 | 1.16 | 0.63 | 0.47 | 0.0005 |
| Bay Oil | -0.49 | -0.20 | -0.28 | -11.99 | -12.26 | -12.21 | 0.30 | 1.42 | 0.75 | 0.55 | 0.0005 |
| Beans-Black Eye | -0.05 | 0.41 | 0.34 | -7.01 | -7.35 | -7.92 | 0.59 | 0.63 | 0.35 | 0.25 | 0.005 |
| Beans-Butter | -0.26 | -0.71 | -0.65 | -8.71 | -8.95 | -9.18 | 0.49 | 0.66 | 0.37 | 0.26 | 0.005 |
| Beans-Dark Red | -0.42 | -0.42 | -0.15 | -16.35 | -17.93 | -19.58 | 0.08 | 0.38 | 0.22 | 0.16 | 0.04 |
| Beans-Haricot | 0.39 | -0.59 | -0.48 | -14.73 | -15.25 | -16.11 | 0.15 | 0.67 | 0.38 | 0.28 | 0.001 |
| Beeswax | -1.46 | -2.04 | -1.72 | -5.19 | -4.25 | -4.27 | 0.87 | 0.92 | 0.52 | 0.37 | 0.0005 |
| Benzene | 9.40 | 9.09 | 7.04 | -35.31 | -35.01 | -33.35 | 0.005 | 0.23 | 0.14 | 0.11 | 0.13 |
| Benzoin | -0.26 | -0.16 | -0.65 | -47.61 | -44.76 | -39.27 | 0.001 | 0.90 | 0.54 | 0.42 | 0.0005 |
| Bismuth | 9.17 | 8.80 | 7.90 | -8.63 | -9.38 | -9.78 | 0.45 | 0.70 | 0.36 | 0.25 | 0.005 |
| Black Cohosh | -0.12 | -0.09 | -0.10 | -5.81 | -5.93 | -6.08 | 0.74 | 1.30 | 0.67 | 0.46 | 0.0005 |
| Brass Scrap | 0.89 | 1.73 | 1.51 | -5.35 | -5.61 | -6.10 | 0.74 | 1.21 | 0.62 | 0.42 | 0.0005 |
| Brazils | 0.42 | -2.19 | -2.61 | -25.20 | -21.48 | -21.25 | 0.05 | 0.81 | 0.44 | 0.32 | 0.001 |
| Butadiene | 14.78 | 18.05 | 17.48 | -17.67 | -21.34 | -21.95 | 0.05 | 0.23 | 0.13 | 0.10 | 0.16 |
| Butane | -0.72 | -0.71 | -0.49 | -12.19 | -12.93 | -13.84 | 0.23 | 0.26 | 0.14 | 0.10 | 0.16 |
| Cadmium | 8.76 | 7.09 | 5.94 | -4.59 | -4.67 | -4.86 | 0.83 | 1.19 | 0.60 | 0.41 | 0.0005 |
| Camphor | -0.46 | -0.05 | 0.22 | -2.74 | -2.87 | -2.95 | 0.94 | 2.16 | 1.09 | 0.78 | 0.0005 |
| Camphor Oil | -1.71 | -1.72 | -1.94 | -24.69 | -24.02 | -24.28 | 0.03 | 0.45 | 0.24 | 0.18 | 0.02 |
| Cananga Oil | -0.81 | -0.91 | -1.00 | -9.70 | -9.06 | -8.58 | 0.54 | 0.67 | 0.35 | 0.25 | 0.005 |
| Candelilla | -0.62 | -0.60 | -0.97 | -24.32 | -24.26 | -24.48 | 0.03 | 0.43 | 0.23 | 0.17 | 0.03 |
| Caprolactam | 0.04 | 0.11 | 0.54 | -10.00 | -11.18 | -12.37 | 0.29 | 0.39 | 0.20 | 0.14 | 0.06 |
| Caraway Seed | -0.18 | -0.56 | -0.74 | -4.78 | -4.46 | -4.37 | 0.86 | 1.34 | 0.74 | 0.51 | 0.0005 |
| Carbonated Wool | -0.04 | 0.22 | -0.03 | -10.97 | -11.22 | -11.03 | 0.37 | 0.85 | 0.48 | 0.34 | 0.0005 |
| Cardomoms | 1.83 | 1.73 | 1.80 | -5.30 | -6.27 | -6.82 | 0.68 | 0.40 | 0.21 | 0.15 | 0.05 |
| Carnuba | 2.62 | 3.55 | 3.55 | -7.17 | -8.00 | -8.27 | 0.56 | 0.61 | 0.31 | 0.22 | 0.01 |
| Cascara | 0.20 | 0.57 | 0.60 | -5.74 | -5.97 | -6.27 | 0.72 | 1.75 | 0.89 | 0.64 | 0.0005 |
| Cashew Kernal | -2.81 | -3.21 | -2.97 | -14.52 | -13.99 | -14.83 | 0.19 | 0.43 | 0.23 | 0.16 | 0.04 |
| Cassia Lignea | -0.47 | -1.04 | -0.78 | -13.67 | -14.77 | -14.62 | 0.20 | 0.81 | 0.46 | 0.33 | 0.001 |
| Cassia Oil | -0.77 | -1.54 | -1.57 | -6.59 | -6.22 | -6.36 | 0.72 | 1.34 | 0.68 | 0.50 | 0.0005 |
| Cattle | -4.42 | -2.77 | -2.04 | -9.26 | -10.47 | -10.79 | 0.38 | 1.32 | 0.74 | 0.52 | 0.0005 |
| Caustic Soda | 7.90 | 11.31 | 13.42 | -8.59 | -11.42 | -13.21 | 0.25 | 0.39 | 0.20 | 0.15 | 0.05 |
| Cedarwood Oil | -3.02 | -2.67 | -2.52 | -15.82 | -15.61 | -16.49 | 0.14 | 0.54 | 0.28 | 0.20 | 0.01 |
| Celery Seed | 1.17 | 2.91 | 3.15 | -10.81 | -12.69 | -13.75 | 0.23 | 0.28 | 0.15 | 0.11 | 0.13 |
| Cement | 0.04 | -2.41 | -1.88 | -12.65 | -13.04 | -14.23 | 0.21 | 1.05 | 0.55 | 0.40 | 0.0005 |
| Chicken | -3.27 | -4.07 | -3.73 | -72.33 | -78.49 | -85.21 | 5E-05 | 0.24 | 0.14 | 0.11 | 0.13 |
| Chickpeas | -0.41 | -0.46 | -0.26 | -5.39 | -5.60 | -5.87 | 0.76 | 0.64 | 0.35 | 0.25 | 0.005 |
| Chilles | 0.66 | 1.55 | 1.55 | -5.02 | -5.49 | -5.72 | 0.77 | 0.46 | 0.26 | 0.18 | 0.02 |
| Chlorine | -2.81 | -3.34 | -3.51 | -6.74 | -5.52 | -6.03 | 0.74 | 1.41 | 0.72 | 0.51 | 0.0005 |
| Chrome | 2.08 | 1.95 | 2.07 | -6.02 | -6.82 | -7.59 | 0.61 | 0.75 | 0.42 | 0.29 | 0.001 |
| Chromite | 1.53 | 2.70 | 2.84 | -9.09 | -10.77 | -12.17 | 0.30 | 0.66 | 0.35 | 0.24 | 0.005 |
| Cinnamon Bark | -1.13 | -0.86 | -0.96 | -23.28 | -23.51 | -21.88 | 0.05 | 0.68 | 0.40 | 0.30 | 0.001 |
| Cinnamon Leaf Oil | 0.35 | 1.15 | 1.60 | -10.20 | -11.64 | -11.98 | 0.31 | 0.44 | 0.23 | 0.17 | 0.03 |
| Citronella Oil | 0.07 | 1.13 | 1.24 | -4.46 | -4.91 | -5.63 | 0.78 | 0.79 | 0.40 | 0.29 | 0.001 |
| Citrus Pulp | 0.99 | 0.02 | -0.05 | -20.20 | -20.91 | -21.40 | 0.05 | 0.54 | 0.29 | 0.21 | 0.01 |
| Clove Leaf Oil | -1.75 | -2.21 | -1.63 | -7.68 | -8.10 | -8.64 | 0.53 | 0.69 | 0.38 | 0.27 | 0.005 |
| Cloves | 2.70 | 4.58 | 4.33 | -1.55 | -1.92 | -1.49 | 0.98 | 1.77 | 0.92 | 0.66 | 0.0005 |

**Table 2. Tests of mean reversion**

| | Lo-MacKinlay Variance Ratio Test | | | Phillips-Perron Unit Root Test | | | | Kwiatkowski-Phillips-Schmidt-Shin Trend Stationarity Test | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (5% Critical Values) | (*t* statistic 1.96) | | | | (-21.4) | | *p*-value | | (0.146) | | *p*-value |
| | *k* = 4 | *k* = 8 | *k* = 12 | *l* = 4 | *l* = 8 | *l* = 12 | (*l* = 12) | *l* = 4 | *l* = 8 | *l* = 12 | (*l* = 12) |
| Cobalt | 7.11 | 5.64 | 4.16 | -10.83 | -10.45 | -9.99 | 0.43 | 0.92 | 0.52 | 0.38 | 0.0005 |
| Cochineal | 0.28 | 0.24 | 0.21 | -5.57 | -5.72 | -5.86 | 0.75 | 0.80 | 0.45 | 0.32 | 0.001 |
| Cocoa-Brazil | -2.48 | -2.45 | -1.96 | -11.26 | -12.01 | -12.41 | 0.29 | 1.04 | 0.54 | 0.39 | 0.0005 |
| Cocoa-Ivory Coast | -1.98 | -1.76 | -1.23 | -8.52 | -9.05 | -9.25 | 0.49 | 1.64 | 0.84 | 0.58 | 0.0005 |
| Coconut | 1.51 | 1.86 | 1.97 | -15.34 | -17.54 | -18.15 | 0.01 | 0.39 | 0.23 | 0.17 | 0.03 |
| Coconut Fibres | -1.34 | -0.28 | 0.59 | -6.23 | -7.86 | -8.88 | 0.51 | 1.22 | 0.63 | 0.45 | 0.0005 |
| Coconut Oil | 0.68 | 0.41 | 0.47 | -12.01 | -12.73 | -13.70 | 0.23 | 0.96 | 0.53 | 0.37 | 0.0005 |
| Coffee-Brazil | 2.45 | 1.66 | 2.31 | -9.75 | -11.31 | -12.03 | 0.31 | 1.33 | 0.72 | 0.51 | 0.0005 |
| Coffee-Columbia | 1.52 | 1.23 | 1.68 | -11.17 | -12.31 | -12.73 | 0.27 | 1.01 | 0.53 | 0.38 | 0.0005 |
| Coir Yarn | -0.18 | -0.15 | -0.18 | -11.48 | -11.04 | -10.50 | 0.40 | 1.66 | 0.87 | 0.65 | 0.0005 |
| Columbite | 0.00 | 0.05 | -0.15 | -7.24 | -7.38 | -7.43 | 0.63 | 1.47 | 0.77 | 0.53 | 0.0005 |
| Copper | 0.76 | 0.36 | -0.01 | -12.93 | -13.13 | -13.20 | 0.25 | 1.45 | 0.80 | 0.56 | 0.0005 |
| Copra | 1.35 | 1.43 | 1.15 | -12.11 | -12.57 | -13.68 | 0.23 | 0.96 | 0.52 | 0.37 | 0.0005 |
| Copra Meal | 2.05 | 2.37 | 1.89 | -8.23 | -8.87 | -8.37 | 0.55 | 0.46 | 0.27 | 0.20 | 0.01 |
| Coriander Seed | 1.30 | 1.39 | 1.39 | -7.05 | -7.66 | -8.81 | 0.52 | 0.36 | 0.19 | 0.14 | 0.06 |
| Corn | 2.20 | 3.36 | 3.77 | -14.56 | -16.77 | -17.39 | 0.12 | 0.71 | 0.39 | 0.28 | 0.005 |
| Cotton | 0.48 | 0.08 | -0.33 | -27.75 | -28.14 | -28.27 | 0.01 | 0.62 | 0.36 | 0.26 | 0.005 |
| Cottonseed Oil | 0.24 | 0.39 | 0.25 | -18.97 | -19.80 | -19.06 | 0.08 | 0.25 | 0.14 | 0.11 | 0.13 |
| Coumarin Oil | -2.39 | -1.77 | -1.12 | -4.63 | -5.00 | -5.54 | 0.78 | 1.62 | 0.82 | 0.58 | 0.0005 |
| Crude Oil | -1.42 | -0.09 | 0.22 | -21.03 | -23.75 | -25.69 | 0.02 | 0.21 | 0.11 | 0.08 | 0.27 |
| Cummin Seed | -1.33 | -1.23 | -1.41 | -14.43 | -13.51 | -12.43 | 0.29 | 0.60 | 0.34 | 0.25 | 0.005 |
| Dimethyl Ter. | 0.15 | 0.21 | 0.66 | -3.44 | -3.77 | -4.79 | 0.84 | 1.13 | 0.58 | 0.39 | 0.0005 |
| Drawn Text. Yarn | 0.04 | 1.19 | 1.60 | -10.88 | -12.50 | -13.20 | 0.25 | 0.58 | 0.30 | 0.22 | 0.01 |
| Eggs | 4.91 | -3.56 | -3.25 | -58.85 | -55.81 | -59.01 | 5E-04 | 0.32 | 0.18 | 0.13 | 0.08 |
| Elm Bark Powder | -0.15 | -1.58 | -1.69 | -13.27 | -12.14 | -12.35 | 0.29 | 0.70 | 0.37 | 0.26 | 0.005 |
| Ethanol | 0.19 | 0.81 | 1.69 | -0.16 | -0.35 | -0.47 | 0.995 | 1.79 | 0.91 | 0.62 | 0.0005 |
| Ethylene | 15.85 | 18.57 | 17.59 | -11.76 | -15.15 | -16.92 | 0.13 | 0.35 | 0.19 | 0.14 | 0.06 |
| Ethylene Di-Chloride | 9.99 | 13.20 | 14.58 | -7.64 | -11.12 | -12.46 | 0.29 | 0.51 | 0.26 | 0.19 | 0.02 |
| Ethylene Glycol | 0.03 | 0.77 | 1.76 | -6.90 | -8.49 | -10.11 | 0.42 | 0.39 | 0.20 | 0.14 | 0.06 |
| Eucalyptus Oil | 0.07 | 0.87 | 0.96 | -9.91 | -10.55 | -10.52 | 0.40 | 0.71 | 0.37 | 0.26 | 0.005 |
| Fennel Seed | -3.31 | -4.12 | -3.73 | -28.13 | -30.00 | -33.88 | 0.003 | 0.38 | 0.21 | 0.15 | 0.05 |
| Fenugreek Seed | 1.29 | 1.87 | 1.51 | -14.45 | -15.51 | -15.87 | 0.15 | 0.38 | 0.22 | 0.16 | 0.04 |
| Ferro-Chrome | 3.69 | 6.17 | 7.08 | -6.02 | -8.02 | -9.73 | 0.45 | 0.51 | 0.26 | 0.18 | 0.02 |
| Ferro-Manganese | -0.03 | 0.00 | -0.06 | -4.09 | -4.04 | -3.98 | 0.89 | 2.25 | 1.14 | 0.77 | 0.0005 |
| Ferro-Molybdenum | 16.48 | 18.45 | 18.16 | -8.11 | -10.60 | -10.10 | 0.43 | 0.63 | 0.33 | 0.24 | 0.005 |
| Ferro-Silicon | 0.34 | 0.39 | 0.43 | -8.28 | -8.79 | -9.16 | 0.49 | 1.80 | 0.93 | 0.64 | 0.0005 |
| Ferro-Titanium | 1.86 | 3.50 | 4.40 | -6.65 | -8.27 | -8.84 | 0.52 | 0.59 | 0.30 | 0.21 | 0.01 |
| Ferro-Tungsten | 5.80 | 6.33 | 5.45 | -8.43 | -9.11 | -8.04 | 0.58 | 0.66 | 0.37 | 0.26 | 0.005 |
| Ferrous Scrap | 1.93 | 3.02 | 2.52 | -6.58 | -6.86 | -6.89 | 0.67 | 0.75 | 0.39 | 0.26 | 0.005 |
| Ferro-Vanadium | 10.51 | 12.77 | 12.89 | -7.75 | -9.84 | -10.58 | 0.39 | 0.60 | 0.31 | 0.21 | 0.01 |
| Fish Meal | 2.08 | 1.67 | 1.53 | -14.95 | -16.00 | -16.75 | 0.13 | 0.77 | 0.44 | 0.32 | 0.001 |
| Fish Oil | 3.14 | 4.57 | 4.13 | -6.39 | -7.24 | -7.15 | 0.65 | 0.64 | 0.36 | 0.26 | 0.005 |
| Freight Route 1 | 8.85 | 4.66 | 2.66 | -20.42 | -17.85 | -16.24 | 0.14 | 0.51 | 0.30 | 0.22 | 0.01 |
| Freight Route 10 | 12.79 | 11.86 | 9.61 | -11.22 | -11.77 | -11.85 | 0.32 | 0.45 | 0.26 | 0.19 | 0.02 |
| Freight Route 1A | 10.47 | 6.20 | 3.95 | -12.28 | -10.86 | -9.59 | 0.47 | 0.79 | 0.42 | 0.29 | 0.001 |
| Freight Route 2 | 4.09 | 1.15 | 0.44 | -11.52 | -10.67 | -11.38 | 0.35 | 0.81 | 0.46 | 0.33 | 0.001 |
| Freight Route 3 | 10.88 | 7.05 | 5.15 | -16.51 | -16.23 | -15.95 | 0.15 | 0.56 | 0.33 | 0.24 | 0.005 |
| Freight Route 3A | 8.79 | 4.70 | 2.92 | -16.22 | -14.91 | -12.91 | 0.26 | 0.76 | 0.41 | 0.30 | 0.001 |
| Freight Route 7 | 11.78 | 10.24 | 8.24 | -12.33 | -12.94 | -12.83 | 0.27 | 0.52 | 0.30 | 0.22 | 0.01 |
| Freight Route 9 | 11.05 | 6.53 | 4.55 | -21.37 | -19.99 | -18.08 | 0.10 | 0.45 | 0.26 | 0.20 | 0.01 |
| Fuel Oil | 0.75 | -0.90 | -1.46 | -25.89 | -25.29 | -26.09 | 0.02 | 0.73 | 0.41 | 0.30 | 0.001 |
| Gallium | 1.69 | 1.16 | 0.73 | -4.56 | -4.47 | -4.53 | 0.85 | 0.81 | 0.41 | 0.28 | 0.005 |
| Gas Oil | -0.56 | -0.30 | -0.51 | -21.12 | -21.65 | -22.37 | 0.04 | 0.50 | 0.27 | 0.20 | 0.01 |
| Gasoline | 0.00 | -0.75 | -1.76 | -27.98 | -24.21 | -24.70 | 0.03 | 0.25 | 0.14 | 0.10 | 0.16 |
| Genetian Root | -0.22 | -0.11 | -0.17 | -4.32 | -4.26 | -4.18 | 0.88 | 2.08 | 1.05 | 0.76 | 0.0005 |
| Geranium Oil | -1.32 | -0.79 | -0.91 | -7.74 | -7.34 | -6.91 | 0.67 | 0.98 | 0.50 | 0.35 | 0.0005 |
| Germanium | 6.68 | 8.33 | 7.83 | -5.38 | -6.08 | -6.19 | 0.73 | 0.76 | 0.42 | 0.30 | 0.001 |
| Germanium Dioxide | 7.83 | 9.91 | 11.08 | -3.65 | -4.67 | -5.07 | 0.81 | 0.78 | 0.43 | 0.30 | 0.001 |
| Ginger | -2.73 | -1.90 | -1.84 | -25.26 | -26.26 | -28.05 | 0.01 | 0.50 | 0.27 | 0.20 | 0.01 |
| Ginger Oil | -3.13 | -3.28 | -2.77 | -20.45 | -20.93 | -20.75 | 0.06 | 0.87 | 0.46 | 0.34 | 0.0005 |
| Gold | 1.23 | 2.27 | 2.26 | -4.38 | -4.41 | -4.43 | 0.86 | 3.70 | 1.87 | 1.26 | 0.0005 |
| Grapefruit Juice | -0.57 | -0.35 | 0.09 | 1.40 | 1.48 | 1.54 | 0.98 | 0.88 | 0.50 | 0.36 | 0.0005 |
| Groundnut Oil | 1.16 | 2.30 | 2.70 | -6.70 | -8.24 | -8.50 | 0.54 | 0.78 | 0.40 | 0.29 | 0.001 |
| Gunmetal Scrap | 0.90 | 1.49 | 2.00 | -3.73 | -4.40 | -5.10 | 0.81 | 0.99 | 0.50 | 0.34 | 0.0005 |
| Hazelnut Kernals | -2.43 | -1.82 | -1.73 | -14.29 | -14.76 | -15.03 | 0.18 | 0.97 | 0.51 | 0.37 | 0.0005 |

## Table 2. Tests of mean reversion

| | Lo-MacKinlay Variance Ratio Test | | | Phillips-Perron Unit Root Test | | | | Kwiatkowski-Phillips-Schmidt-Shin Trend Stationarity Test | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (5% Critical Values) | ($t$ statistic 1.96) | | | | (-21.4) | | $p$-value | | (0.146) | | $p$-value |
| | $k = 4$ | $k = 8$ | $k = 12$ | $l = 4$ | $l = 8$ | $l = 12$ | ($l = 12$) | $l = 4$ | $l = 8$ | $l = 12$ | ($l = 12$) |
| Heavy Fuel Oil | 1.69 | 1.09 | 0.43 | -29.15 | -28.57 | -29.51 | 0.01 | 0.19 | 0.11 | 0.08 | 0.27 |
| Heliotropin | -1.60 | -0.06 | 0.76 | -6.51 | -8.02 | -9.57 | 0.46 | 0.46 | 0.24 | 0.17 | 0.03 |
| Hogs | 1.19 | 0.34 | -0.29 | -31.17 | -30.41 | -29.96 | 0.01 | 0.41 | 0.23 | 0.17 | 0.03 |
| Hydrastis Roots | -0.05 | -0.11 | -1.12 | -15.77 | -14.09 | -14.08 | 0.22 | 0.80 | 0.43 | 0.30 | 0.001 |
| Indium Ingots | 18.81 | 22.97 | 24.29 | -4.27 | -6.28 | -7.43 | 0.63 | 0.76 | 0.39 | 0.26 | 0.005 |
| Ipecuanha Root | 0.75 | 1.02 | 1.53 | -3.30 | -3.82 | -4.31 | 0.87 | 1.10 | 0.56 | 0.39 | 0.0005 |
| Iridium | 3.21 | 4.71 | 5.04 | -0.47 | -0.68 | -0.80 | 0.99 | 1.90 | 0.96 | 0.68 | 0.0005 |
| Jet Kerosene | 2.42 | 2.49 | 1.61 | -21.10 | -21.15 | -21.32 | 0.05 | 0.45 | 0.25 | 0.19 | 0.02 |
| Jute | 0.80 | 1.58 | 1.96 | -12.41 | -13.72 | -14.16 | 0.21 | 0.42 | 0.22 | 0.16 | 0.04 |
| Kola Nuts | -1.62 | -0.89 | -0.53 | -55.76 | -62.57 | -63.60 | 5E-04 | 0.08 | 0.05 | 0.04 | 0.72 |
| Lard | -0.49 | -0.91 | -0.91 | -9.16 | -8.80 | -7.59 | 0.61 | 1.51 | 0.79 | 0.58 | 0.0005 |
| Lavender Spike Oil | -1.39 | -1.21 | -1.68 | -21.66 | -20.04 | -19.69 | 0.07 | 0.43 | 0.23 | 0.17 | 0.03 |
| Lead | -0.94 | -1.03 | -1.02 | -22.50 | -22.55 | -22.49 | 0.04 | 0.38 | 0.22 | 0.15 | 0.05 |
| Lead Ore | 0.40 | 0.36 | 0.18 | -15.36 | -15.38 | -14.48 | 0.20 | 0.95 | 0.50 | 0.34 | 0.0005 |
| Lemon Grass Oil | -1.24 | -0.28 | -0.04 | -7.01 | -7.38 | -7.31 | 0.64 | 1.00 | 0.56 | 0.39 | 0.0005 |
| Lentils | 0.97 | 0.50 | -0.18 | -10.61 | -9.54 | -8.36 | 0.55 | 1.25 | 0.71 | 0.50 | 0.0005 |
| Lime Oil | -1.39 | -1.21 | -1.68 | -8.82 | -8.08 | -9.00 | 0.51 | 0.65 | 0.33 | 0.23 | 0.01 |
| Linseed Meal | -0.64 | -0.44 | -0.83 | -23.46 | -23.25 | -23.81 | 0.03 | 0.25 | 0.15 | 0.11 | 0.13 |
| Linseed Oil | 3.53 | 3.52 | 3.34 | -6.70 | -7.44 | -7.62 | 0.61 | 0.47 | 0.24 | 0.18 | 0.02 |
| Lithium Ore | 0.02 | 0.05 | 0.07 | -4.33 | -4.33 | -4.34 | 0.87 | 2.31 | 1.17 | 0.80 | 0.0005 |
| Litsea Cubesa Oil | -1.46 | -2.51 | -2.44 | -6.05 | -7.44 | -8.10 | 0.57 | 1.06 | 0.54 | 0.38 | 0.0005 |
| Mace | -9.11 | -6.69 | -5.79 | -212.20 | -315.83 | -414.21 | 5E-04 | 1.08 | 0.63 | 0.45 | 0.0005 |
| Magnesium | 3.65 | 6.10 | 7.69 | -2.38 | -3.58 | -4.54 | 0.85 | 1.20 | 0.61 | 0.41 | 0.0005 |
| Maize | -0.51 | -0.30 | -0.66 | -22.29 | -22.19 | -22.89 | 0.04 | 0.72 | 0.39 | 0.29 | 0.001 |
| Maize/Corn Oil | 2.89 | 4.11 | 4.03 | -10.91 | -12.58 | -13.50 | 0.24 | 0.59 | 0.33 | 0.24 | 0.005 |
| Manganese | 7.86 | 7.50 | 7.04 | -5.54 | -6.40 | -6.36 | 0.71 | 0.65 | 0.33 | 0.23 | 0.01 |
| Manganese Ore | 1.76 | 3.27 | 3.36 | -1.90 | -2.11 | -2.22 | 0.96 | 2.15 | 1.08 | 0.73 | 0.0005 |
| Meat and Bone Meal | 0.50 | 1.38 | 1.49 | -5.20 | -5.96 | -6.10 | 0.74 | 0.57 | 0.30 | 0.21 | 0.01 |
| MEG | 8.08 | 9.91 | 11.05 | -7.06 | -9.47 | -10.40 | 0.41 | 0.51 | 0.26 | 0.19 | 0.02 |
| Menthol Oil | 0.67 | 3.33 | 3.97 | -7.18 | -9.05 | -9.80 | 0.45 | 0.53 | 0.27 | 0.19 | 0.02 |
| Mercury | 13.44 | 16.62 | 15.83 | -6.05 | -7.38 | -7.96 | 0.58 | 1.16 | 0.59 | 0.41 | 0.0005 |
| Methanol | 12.88 | 13.68 | 13.22 | -11.87 | -14.63 | -16.18 | 0.15 | 0.45 | 0.23 | 0.17 | 0.03 |
| Millet | -0.82 | -0.79 | -0.74 | -3.31 | -3.19 | -3.32 | 0.92 | 1.56 | 0.86 | 0.60 | 0.0005 |
| Molybdenite | 6.04 | 8.38 | 8.04 | -14.92 | -17.92 | -20.81 | 0.06 | 0.29 | 0.15 | 0.11 | 0.13 |
| Molybdenum | 16.90 | 18.67 | 18.93 | -10.40 | -14.00 | -15.16 | 0.17 | 0.54 | 0.28 | 0.20 | 0.01 |
| Musk | -0.42 | -1.16 | -1.04 | -13.00 | -13.48 | -14.05 | 0.22 | 0.84 | 0.44 | 0.32 | 0.001 |
| Naphtha | 1.06 | 1.69 | 1.81 | -18.92 | -21.37 | -21.67 | 0.05 | 0.41 | 0.22 | 0.17 | 0.03 |
| Natural Gas | -3.20 | -2.31 | -2.10 | -30.23 | -35.59 | -36.32 | 0.003 | 0.37 | 0.23 | 0.19 | 0.02 |
| Nickel | 2.73 | 3.68 | 3.65 | -9.09 | -10.12 | -10.63 | 0.39 | 0.90 | 0.49 | 0.34 | 0.0005 |
| Nutmeg | -0.59 | 0.16 | 0.25 | -8.65 | -8.74 | -8.10 | 0.57 | 0.79 | 0.46 | 0.33 | 0.001 |
| Nux Vomica | -0.09 | -0.11 | -0.13 | -6.02 | -6.07 | -6.11 | 0.73 | 0.94 | 0.52 | 0.36 | 0.0005 |
| Nylon Yarn | 0.03 | 0.58 | 1.11 | -7.40 | -8.51 | -9.25 | 0.49 | 0.40 | 0.21 | 0.15 | 0.05 |
| Oats | -0.11 | -0.74 | -0.47 | -14.95 | -15.74 | -15.86 | 0.15 | 0.31 | 0.17 | 0.12 | 0.1 |
| Olive Oil | 2.67 | 2.79 | 3.50 | -4.64 | -5.18 | -5.64 | 0.77 | 0.55 | 0.31 | 0.22 | 0.01 |
| Orange Juice | -2.69 | -2.25 | -1.87 | -11.26 | -12.46 | -13.72 | 0.23 | 0.43 | 0.24 | 0.18 | 0.02 |
| Orange Pera | -1.79 | -0.73 | -0.01 | -3.39 | -3.85 | -4.52 | 0.85 | 1.13 | 0.57 | 0.39 | 0.0005 |
| Palladium | -0.02 | -0.99 | -0.82 | -5.85 | -5.41 | -5.89 | 0.75 | 0.97 | 0.55 | 0.39 | 0.0005 |
| Palm Kernal Cake | -1.10 | -2.14 | -2.19 | -15.12 | -14.02 | -14.93 | 0.18 | 0.73 | 0.39 | 0.27 | 0.005 |
| Palm Kernal Oil | 2.38 | 1.87 | 0.04 | -23.20 | -21.64 | -21.67 | 0.05 | 0.40 | 0.22 | 0.16 | 0.04 |
| Palm Oil | 1.36 | 1.14 | 0.40 | -13.19 | -12.43 | -12.86 | 0.27 | 0.66 | 0.34 | 0.24 | 0.005 |
| Paper Pulp | -0.07 | 1.75 | 2.87 | -6.01 | -7.50 | -8.69 | 0.53 | 2.30 | 1.16 | 0.79 | 0.0005 |
| Paper, Corrugated | 0.03 | 0.82 | 1.06 | -8.10 | -8.93 | -8.91 | 0.51 | 0.64 | 0.35 | 0.25 | 0.005 |
| Patchouli Oil | -0.35 | 0.25 | 0.72 | -1.55 | -1.91 | -2.20 | 0.97 | 0.77 | 0.43 | 0.30 | 0.001 |
| Peanuts | 0.63 | 1.22 | 1.25 | -19.24 | -20.66 | -21.31 | 0.05 | 0.37 | 0.20 | 0.15 | 0.05 |
| Pepper-Black | -1.04 | 0.20 | 0.47 | -11.69 | -11.81 | -11.25 | 0.35 | 0.84 | 0.48 | 0.34 | 0.0005 |
| Peppermint Oil | 2.13 | 4.32 | 5.01 | -10.20 | -12.96 | -14.02 | 0.22 | 0.73 | 0.38 | 0.27 | 0.005 |
| Pepper-White | -0.31 | 0.06 | -0.09 | -4.11 | -3.94 | -3.90 | 0.89 | 2.16 | 1.10 | 0.79 | 0.0005 |
| PET-Bottle Gde | 3.33 | 6.12 | 7.45 | -4.23 | -6.25 | -8.10 | 0.57 | 0.58 | 0.30 | 0.20 | 0.01 |
| Petitgrain Oil | -0.74 | 0.37 | 1.20 | -7.71 | -8.67 | -9.13 | 0.50 | 0.93 | 0.48 | 0.35 | 0.0005 |
| PET-Staple Fibre | 0.03 | 3.07 | 4.29 | -5.30 | -7.07 | -8.12 | 0.57 | 0.64 | 0.33 | 0.23 | 0.001 |
| PGE | 6.48 | 7.26 | 6.80 | -18.32 | -20.97 | -21.24 | 0.05 | 0.20 | 0.12 | 0.09 | 0.21 |
| Phosphor Copper | 1.37 | 1.45 | 1.12 | -2.90 | -2.83 | -2.87 | 0.94 | 1.69 | 0.86 | 0.59 | 0.0005 |
| Phosphor Tin | -0.82 | -0.57 | -0.37 | -11.54 | -11.72 | -11.85 | 0.32 | 0.64 | 0.33 | 0.23 | 0.01 |
| Pigs Live Weight | 1.84 | 0.65 | 0.02 | -16.07 | -15.74 | -15.77 | 0.16 | 0.77 | 0.44 | 0.32 | 0.001 |
| Pimento | 6.13 | 6.52 | 5.99 | 4.12 | 4.14 | 4.05 | 0.88 | 1.52 | 0.80 | 0.58 | 0.0005 |

## Table 2. Tests of mean reversion

| | Lo-MacKinlay Variance Ratio Test | | | Phillips-Perron Unit Root Test | | | | Kwiatkowski-Phillips-Schmidt-Shin Trend Stationarity Test | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (5% Critical Values) | ($t$ statistic 1.96) | | | | (-21.4) | | $p$-value | | (0.146) | | $p$-value |
| | $k$ = 4 | $k$ = 8 | $k$ = 12 | $l$ = 4 | $l$ = 8 | $l$ = 12 | ($l$ = 12) | $l$ = 4 | $l$ = 8 | $l$ = 12 | ($l$ = 12) |
| Pimento Leaf Oil | -0.06 | -0.06 | -1.08 | -25.73 | -23.38 | -22.04 | 0.05 | 0.53 | 0.36 | 0.30 | 0.001 |
| Pineapple Juice | 0.37 | -0.03 | -0.01 | -5.17 | -4.75 | -4.53 | 0.85 | 0.69 | 0.40 | 0.29 | 0.001 |
| Platinum | -1.65 | -2.29 | -2.45 | -18.83 | -18.11 | -18.44 | 0.09 | 0.48 | 0.27 | 0.19 | 0.02 |
| Polyacryloinitrile | -0.01 | -0.21 | 0.10 | -7.15 | -7.56 | -8.53 | 0.54 | 1.03 | 0.53 | 0.37 | 0.0005 |
| Polye HDPE-Inj | 5.13 | 10.20 | 12.65 | -5.21 | -8.42 | -10.14 | 0.42 | 0.77 | 0.40 | 0.29 | 0.001 |
| Polye LDPE-GP Film | 14.10 | 18.45 | 18.78 | -10.18 | -13.65 | -14.57 | 0.20 | 0.70 | 0.37 | 0.27 | 0.005 |
| Polye LLDPE | 11.95 | 16.43 | 18.17 | -6.66 | -9.68 | -11.36 | 0.35 | 0.46 | 0.26 | 0.19 | 0.02 |
| Polystyrene-Hips | 9.35 | 12.56 | 13.53 | -6.34 | -8.61 | -9.66 | 0.46 | 0.78 | 0.41 | 0.30 | 0.001 |
| Poppyseed | 0.03 | 0.10 | 0.02 | -23.91 | -25.68 | -25.20 | 0.02 | 0.29 | 0.16 | 0.12 | 0.10 |
| Pork Bellies | -4.67 | -5.09 | -4.09 | -28.48 | -32.09 | -33.86 | 0.003 | 0.95 | 0.54 | 0.39 | 0.0005 |
| PP Copolymer | 8.13 | 11.69 | 13.28 | -7.74 | -11.31 | -12.92 | 0.26 | 0.95 | 0.50 | 0.35 | 0.0005 |
| PP Homopolymer Inj | 10.38 | 12.37 | 11.26 | -5.59 | -7.00 | -6.93 | 0.67 | 0.99 | 0.51 | 0.35 | 0.0005 |
| Propane | 3.48 | 2.02 | 0.89 | -30.51 | -28.78 | -27.13 | 0.02 | 0.35 | 0.19 | 0.14 | 0.06 |
| Propylene (C) | 8.36 | 9.58 | 9.97 | -5.65 | -8.05 | -9.26 | 0.48 | 0.43 | 0.24 | 0.17 | 0.03 |
| Propylene (P) | 13.00 | 14.53 | 14.49 | -8.34 | -11.70 | -13.49 | 0.24 | 0.40 | 0.21 | 0.15 | 0.05 |
| Propylene (R) | 0.24 | 4.81 | 6.66 | -4.41 | -7.43 | -9.43 | 0.47 | 0.39 | 0.20 | 0.14 | 0.06 |
| Propylene Oxide | 2.85 | 3.02 | 3.03 | -4.19 | -4.58 | -4.93 | 0.83 | 0.86 | 0.44 | 0.30 | 0.001 |
| PTA | 0.04 | 2.84 | 4.20 | -4.53 | -6.10 | -7.22 | 0.64 | 0.92 | 0.47 | 0.33 | 0.001 |
| PVC | 8.37 | 12.69 | 12.80 | -4.47 | -6.33 | -6.64 | 0.69 | 0.96 | 0.49 | 0.34 | 0.0005 |
| Rapeseed | -0.16 | -0.23 | -0.64 | -12.04 | -11.82 | -11.17 | 0.36 | 0.69 | 0.40 | 0.30 | 0.001 |
| Rapeseed Meal | 1.76 | 1.43 | 1.44 | -16.82 | -18.31 | -17.55 | 0.16 | 0.75 | 0.40 | 0.29 | 0.001 |
| Rapeseed Oil | -0.59 | 0.16 | -0.18 | -19.58 | -20.22 | -18.48 | 0.09 | 0.39 | 0.21 | 0.15 | 0.05 |
| Rayon, Staple Fibre | 0.02 | 1.42 | 2.62 | -5.37 | -6.78 | -8.35 | 0.56 | 0.32 | 0.18 | 0.13 | 0.08 |
| Rhodium | 0.84 | 2.59 | 3.31 | -2.52 | -3.38 | -3.58 | 0.91 | 1.42 | 0.72 | 0.49 | 0.0005 |
| Rice | 1.81 | 3.02 | 2.75 | -5.02 | -5.40 | -5.57 | 0.78 | 1.36 | 0.75 | 0.53 | 0.0005 |
| Rice/Bran | -0.70 | -1.27 | -1.88 | -18.78 | -16.59 | -16.26 | 0.14 | 0.97 | 0.55 | 0.40 | 0.0005 |
| Rubber | 2.66 | 2.82 | 2.48 | -7.47 | -7.51 | -7.39 | 0.63 | 2.49 | 1.27 | 0.86 | 0.0005 |
| Ruthenium | 3.17 | 3.76 | 3.44 | -1.20 | -1.58 | -1.87 | 0.97 | 1.69 | 0.86 | 0.61 | 0.0005 |
| Saffron | -3.01 | -2.89 | -2.57 | -12.13 | -12.00 | -12.72 | 0.27 | 0.83 | 0.43 | 0.30 | 0.001 |
| Sandalwood Oil | -1.34 | -0.80 | -0.49 | -11.16 | -12.20 | -13.52 | 0.24 | 0.83 | 0.44 | 0.31 | 0.001 |
| Sasparilla Root | -0.02 | -0.01 | 0.00 | -6.50 | -6.57 | -6.62 | 0.69 | 1.36 | 0.75 | 0.53 | 0.0005 |
| Sassafras Oil | 0.72 | 0.84 | 1.10 | -9.40 | -10.37 | -11.42 | 0.34 | 0.28 | 0.16 | 0.11 | 0.13 |
| Selenium | 4.91 | 4.63 | 4.68 | -7.80 | -9.10 | -10.25 | 0.42 | 0.90 | 0.46 | 0.32 | 0.001 |
| Sesame Seed | -0.02 | -0.01 | -0.01 | -11.45 | -11.47 | -11.27 | 0.35 | 0.83 | 0.47 | 0.34 | 0.0005 |
| Sheep | -0.01 | 2.08 | 2.67 | -15.28 | -19.58 | -21.07 | 0.06 | 0.31 | 0.19 | 0.15 | 0.05 |
| Silico-Manganese | 0.59 | 0.12 | -0.35 | -9.96 | -9.55 | -9.68 | 0.45 | 1.32 | 0.68 | 0.47 | 0.0005 |
| Silicon | 1.39 | 2.22 | 2.54 | -3.00 | -3.52 | -3.69 | 0.91 | 1.10 | 0.61 | 0.43 | 0.0005 |
| Silver | 0.91 | 1.42 | 1.13 | -7.81 | -7.80 | -7.74 | 0.60 | 3.04 | 1.54 | 1.04 | 0.0005 |
| Sisal | 0.50 | -0.35 | -0.15 | -5.08 | -5.46 | -5.83 | 0.76 | 0.71 | 0.37 | 0.26 | 0.005 |
| Soda Ash | 0.06 | 0.36 | 0.82 | -2.27 | -2.24 | -2.17 | 0.97 | 1.05 | 0.59 | 0.42 | 0.0005 |
| Soyabean Oil | 2.28 | 2.80 | 2.56 | -16.77 | -17.31 | -16.54 | 0.14 | 0.36 | 0.20 | 0.14 | 0.06 |
| Soyabeans | -0.69 | -0.31 | -0.13 | -17.37 | -18.48 | -18.58 | 0.09 | 0.39 | 0.22 | 0.15 | 0.05 |
| Soyameal | 0.60 | 0.42 | 0.40 | -6.37 | -6.54 | -6.73 | 0.68 | 0.74 | 0.41 | 0.29 | 0.001 |
| Spearmint Oil | 0.30 | 0.39 | 0.15 | -5.61 | -5.79 | -6.20 | 0.73 | 0.73 | 0.37 | 0.26 | 0.005 |
| Steel,Galv. Coils | 0.86 | 2.76 | 4.06 | -3.98 | -5.22 | -6.24 | 0.73 | 0.99 | 0.50 | 0.34 | 0.0005 |
| Steel,Merchant Bars | -0.21 | 0.74 | 0.83 | -4.95 | -5.15 | -5.43 | 0.79 | 1.64 | 0.83 | 0.57 | 0.0005 |
| Steers | 1.84 | 2.50 | 2.13 | -17.07 | -16.78 | -13.96 | 0.22 | 3.06 | 1.58 | 1.09 | 0.0005 |
| Styrene | 13.08 | 13.35 | 13.30 | -11.11 | -14.22 | -16.15 | 0.15 | 0.37 | 0.19 | 0.14 | 0.06 |
| Sugar | 0.70 | 1.68 | 1.56 | -12.64 | -13.40 | -14.37 | 0.20 | 0.58 | 0.30 | 0.20 | 0.01 |
| Sunflowerseed | -0.03 | 0.24 | 0.24 | -8.72 | -9.11 | -9.27 | 0.48 | 0.68 | 0.36 | 0.25 | 0.005 |
| Sunflowerseed Meal | 0.23 | 0.26 | 0.09 | -15.34 | -15.55 | -16.09 | 0.15 | 0.77 | 0.44 | 0.32 | 0.001 |
| Sunflowerseed Oil | -0.01 | 0.05 | -0.02 | -7.72 | -7.74 | -7.50 | 0.62 | 0.93 | 0.52 | 0.37 | 0.0005 |
| Tallow | 2.76 | 2.28 | 1.89 | -10.89 | -11.59 | -11.44 | 0.34 | 1.14 | 0.59 | 0.43 | 0.0005 |
| Tallow, extra fancy | 5.85 | 5.77 | 4.21 | -6.41 | -6.44 | -5.56 | 0.78 | 1.25 | 0.66 | 0.47 | 0.0005 |
| Tantalite Ore | -0.06 | -0.16 | -0.19 | -19.52 | -20.49 | -19.98 | 0.07 | 0.56 | 0.33 | 0.24 | 0.005 |
| Tapioca | 0.00 | -1.22 | -2.28 | -20.63 | -16.48 | -16.76 | 0.13 | 0.27 | 0.16 | 0.11 | 0.13 |
| Tea-Kenya | 1.25 | 0.07 | -0.37 | -26.79 | -26.52 | -26.94 | 0.02 | 0.37 | 0.22 | 0.16 | 0.04 |
| Tea-Sri Lanka | -4.44 | -4.06 | -3.58 | -34.74 | -37.36 | -40.58 | 0.001 | 0.94 | 0.55 | 0.38 | 0.0005 |
| Terpineol | -0.49 | -1.60 | -1.60 | -8.10 | -7.59 | -7.92 | 0.59 | 0.98 | 0.51 | 0.35 | 0.0005 |
| Tin | -0.06 | -0.91 | -0.66 | -20.68 | -22.41 | -21.48 | 0.05 | 0.34 | 0.19 | 0.14 | 0.06 |
| Titanium | -0.15 | -1.76 | -2.52 | -36.21 | -31.50 | -32.58 | 0.005 | 0.60 | 0.33 | 0.23 | 0.01 |
| Titanium Dioxide | -0.39 | 0.76 | 1.65 | -3.41 | -4.26 | -5.28 | 0.80 | 0.94 | 0.48 | 0.33 | 0.001 |
| Titanium Ore | -0.86 | -0.49 | -0.55 | -12.82 | -12.72 | -12.98 | 0.26 | 0.81 | 0.42 | 0.29 | 0.001 |
| Titanium Scrap | 3.38 | 4.75 | 5.24 | -5.80 | -7.03 | -7.54 | 0.62 | 0.81 | 0.41 | 0.28 | 0.005 |
| Toluene | 5.15 | 4.60 | 2.94 | -15.80 | -15.17 | -14.59 | 0.20 | 0.23 | 0.13 | 0.09 | 0.21 |

## Table 2. Tests of mean reversion

| | Lo-MacKinlay Variance Ratio Test | | | Phillips-Perron Unit Root Test | | | | Kwiatkowski-Phillips-Schmidt-Shin Trend Stationarity Test | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (5% Critical Values) | (*t* statistic 1.96) | | | (-21.4) | | | *p*-value | (0.146) | | | *p*-value |
| | *k* = 4 | *k* = 8 | *k* = 12 | *l* = 4 | *l* = 8 | *l* = 12 | (*l* = 12) | *l* = 4 | *l* = 8 | *l* = 12 | (*l* = 12) |
| Tonquin | -1.73 | -2.24 | -1.52 | -20.03 | -22.67 | -24.62 | 0.03 | 0.24 | 0.14 | 0.10 | 0.16 |
| Tung Oil | 1.33 | 1.25 | 1.57 | -6.68 | -7.57 | -7.77 | 0.60 | 0.71 | 0.37 | 0.26 | 0.005 |
| Tungsten | 9.67 | 13.81 | 16.09 | -2.83 | -4.47 | -5.63 | 0.77 | 0.72 | 0.37 | 0.25 | 0.005 |
| Tungsten Ore | -0.83 | -0.49 | -0.38 | -11.09 | -12.75 | -14.00 | 0.22 | 0.27 | 0.14 | 0.10 | 0.16 |
| Turmeric | -0.24 | -0.86 | -1.29 | -20.43 | -19.11 | -18.98 | 0.09 | 0.45 | 0.24 | 0.18 | 0.03 |
| Valerian Root | 1.11 | 1.75 | 2.21 | -5.43 | -6.91 | -7.35 | 0.64 | 0.32 | 0.17 | 0.12 | 0.10 |
| Walnuts | -1.99 | -2.12 | -2.63 | -28.29 | -24.71 | -25.00 | 0.02 | 0.76 | 0.41 | 0.30 | 0.001 |
| Vanadium | 10.04 | 11.95 | 10.77 | -7.38 | -8.97 | -8.07 | 0.57 | 0.64 | 0.36 | 0.25 | 0.005 |
| VCM | 6.40 | 10.58 | 11.51 | -5.67 | -8.21 | -9.34 | 0.48 | 0.46 | 0.24 | 0.17 | 0.03 |
| Vetivert Oil | 0.22 | -0.66 | -0.69 | -4.17 | -3.84 | -3.90 | 0.89 | 0.93 | 0.47 | 0.34 | 0.0005 |
| Wheat-No.2 Hard | -0.09 | 0.87 | 1.03 | -11.85 | -12.73 | -12.58 | 0.28 | 0.73 | 0.40 | 0.28 | 0.005 |
| Wheat-No.2 Soft | -0.03 | 0.09 | 0.24 | -11.86 | -12.62 | -13.21 | 0.25 | 0.83 | 0.43 | 0.30 | 0.001 |
| Wheat-Spring | -3.77 | -2.18 | -1.60 | -17.22 | -18.54 | -18.87 | 0.09 | 0.89 | 0.50 | 0.35 | 0.0005 |
| Wild Cherry Bark | -0.12 | -2.75 | -2.70 | -6.93 | -5.96 | -6.21 | 0.73 | 0.95 | 0.49 | 0.34 | 0.0005 |
| Witch Hazel Leaves | -0.62 | -0.38 | -0.08 | -4.82 | -5.09 | -5.30 | 0.80 | 1.31 | 0.67 | 0.48 | 0.0005 |
| Wooltops | 5.33 | 5.00 | 4.70 | -3.91 | -4.45 | -5.14 | 0.81 | 3.17 | 1.61 | 1.09 | 0.0005 |
| Xylenes | 6.77 | 5.06 | 2.81 | -8.41 | -7.08 | -6.01 | 0.74 | 0.48 | 0.28 | 0.21 | 0.01 |
| Zinc | 0.67 | 0.64 | 0.50 | -14.40 | -14.75 | -14.25 | 0.21 | 0.72 | 0.41 | 0.30 | 0.001 |
| Zinc Sulphide | -0.83 | -0.49 | -0.38 | -13.80 | -14.61 | -15.35 | 0.17 | 1.35 | 0.71 | 0.50 | 0.0005 |
| Zircon | -0.48 | -0.22 | 0.35 | -6.52 | -6.77 | -6.85 | 0.68 | 0.87 | 0.45 | 0.31 | 0.001 |
| $H_0$ accepted: | 178 | 161 | 163 | 246 | 243 | 241 | | 1 | 14 | 33 | |
| $H_0$ rejected: | 102 | 119 | 117 | 34 | 37 | 39 | | 279 | 266 | 247 | |

Notes: - 280 series.

- The test statistic for the variance ratio test is $\hat{z} = \sqrt{T}(\hat{\sigma}^2(k)/\hat{\sigma}^2 - 1)/\sqrt{2(2k-1)(k-1)/3k}$.

  $k$ is the k:th difference, $x(t)$–$x(t-k)$, in equation (1.2).
- The Phillips-Perron unit root test tests the null hypothesis of equation (1.1) against equation (1.4). Parameters are obtained by OLS on (1.4) and the test statistic is

  $\hat{z} = T(\hat{\phi} - 1) - \frac{1}{2}(T^2\hat{\sigma}_\phi^2 / \hat{\sigma}_{u(T-k)}^2)(\hat{\lambda}^2(l) - \hat{\sigma}_{u(T)}^2)$. $\hat{\lambda}^2(l)$is the Newey -West estimator and $l$ is the order of serial correlation allowed in constructing the Newey-West estimator.
- In the Kwiatkowski et al. stationarity test, parameters are obtained by OLS on (3.1), and the test statistic is $\hat{z} = (T^{-2}/\hat{\lambda}^2(l))\sum s_t^2$, where $s_t$ is the partial sum process $s_t = \sum_{i=1}^{t} u(t)$.
- For $l = 12$, significant results in the PP and the KPSS tests are underlined.
- $p$-values are based on 100 000 Monte Carlo simulations.

## Table 3. Distribution of the error term

| | Observations | | Serial Correlation | | | Skewness | | Kurtosis | | Exponent | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | No. of obs. | No. of price changes | Corr[u(t),u(t-k)] $k$ =1 | $k$ =2 | $k$ =4 | Skew-ness | $t$ stat. | Excess Kurtosis | $t$ stat. | Characteristic exponent $z$ | alpha |
| (5% Critical Values) | | | | | | | (1.96) | | (1.96) | | |
| Acid Oils-Fish | 709 | 230 | 0.05 | 0.05 | 0.06 | -0.43 | -0.17 | 10.2 | 55.6 | 36.55 | <1.0 |
| Acid Oils-Hard | 709 | 223 | 0.06 | -0.01 | 0.02 | 1.92 | 0.79 | 27.0 | 147.0 | 13.12 | <1.0 |
| Acid Oils-Soft | 709 | 226 | 0.07 | 0.07 | 0.01 | -0.34 | -0.14 | 11.5 | 62.5 | 16.02 | <1.0 |
| Acid Oils-Soya... | 709 | 265 | 0.05 | 0.05 | 0.05 | 0.65 | 0.27 | 5.5 | 29.8 | 20.76 | <1.0 |
| Acrylonitrile | 709 | 323 | 0.29 | 0.32 | 0.27 | 1.78 | 0.73 | 14.9 | 81.3 | 24.13 | <1.0 |
| Alloy Steel Scrap | 806 | 330 | 0.13 | 0.15 | 0.10 | -0.03 | -0.01 | 9.5 | 55.5 | 26.35 | <1.0 |
| Almonds | 709 | 225 | 0.01 | -0.07 | -0.06 | 1.87 | 0.77 | 34.6 | 188.5 | 10.01 | <1.0 |
| Aluminium | 1053 | 1050 | -0.01 | 0.05 | -0.14 | -0.73 | -0.30 | 17.6 | 117.0 | 3.00 | 1.6-1.7 |
| Ammonia | 709 | 356 | 0.23 | 0.21 | 0.20 | -0.08 | -0.03 | 3.7 | 20.0 | 14.86 | <1.0 |
| Aniseed Oil | 709 | 173 | -0.01 | -0.03 | 0.03 | -0.16 | -0.06 | 30.2 | 164.5 | 12.63 | <1.0 |
| Antimony | 806 | 527 | 0.49 | 0.36 | 0.17 | 1.23 | 0.50 | 11.6 | 67.4 | 8.33 | <1.0 |
| Antimony Ore | 806 | 68 | 0.04 | 0.02 | -0.01 | -3.07 | -1.26 | 106.0 | 615.6 | 25.71 | <1.0 |
| Apricot Kernals | 537 | 67 | 0.01 | 0.01 | 0.03 | 8.07 | 3.31 | 111.5 | 529.4 | 4.60 | 1.2-1.3 |
| Arabic Gum | 537 | 113 | 0.02 | 0.02 | 0.03 | -2.52 | -1.03 | 57.9 | 275.0 | 8.58 | <1.0 |
| Arsenic | 524 | 73 | 0.21 | 0.08 | 0.00 | -0.05 | -0.02 | 33.1 | 157.3 | 5.35 | 1.1-1.2 |
| Balsam-Canadian | 709 | 23 | 0.02 | 0.02 | 0.02 | 3.47 | 1.42 | 135.8 | 740.1 | 1.37 | >2.0 |
| Balsam-Copaiba | 709 | 28 | 0.03 | 0.03 | 0.03 | -1.38 | -0.57 | 90.3 | 492.2 | 2.15 | >2.0 |
| Balsam-Peru | 537 | 20 | 0.00 | 0.00 | 0.00 | 8.22 | 3.37 | 117.0 | 555.6 | 1.23 | >2.0 |
| Balsam-Tolu | 709 | 23 | -0.04 | 0.01 | 0.03 | 5.66 | 2.32 | 76.4 | 416.4 | 5.59 | 1.1-1.2 |
| Barley | 709 | 291 | -0.04 | 0.00 | -0.04 | -1.58 | -0.65 | 34.7 | 189.2 | 5.60 | 1.1-1.2 |
| Bay Oil | 709 | 221 | 0.01 | 0.11 | 0.10 | 5.93 | 2.43 | 124.6 | 679.2 | 4.35 | 1.3-1.4 |
| Beans-Black Eye | 537 | 75 | 0.00 | 0.00 | 0.04 | 3.09 | 1.27 | 62.4 | 296.4 | 21.81 | <1.0 |
| Beans-Butter | 537 | 68 | 0.01 | 0.00 | -0.13 | 0.85 | 0.35 | 63.1 | 299.6 | 13.93 | <1.0 |
| Beans-Dark Red | 537 | 77 | -0.01 | 0.05 | 0.05 | -0.03 | -0.01 | 31.6 | 150.2 | 11.99 | <1.0 |
| Beans-Haricot | 537 | 63 | -0.01 | 0.14 | 0.05 | -0.90 | -0.37 | 32.3 | 153.2 | 9.66 | <1.0 |
| Beeswax | 285 | 26 | 0.07 | -0.22 | 0.02 | 0.17 | 0.07 | 42.3 | 146.9 | 5.24 | 1.1-1.2 |
| Benzene | 709 | 621 | 0.27 | 0.24 | 0.05 | 0.24 | 0.10 | 6.8 | 36.9 | 3.81 | 1.4-1.5 |
| Benzoin | 709 | 34 | 0.03 | 0.04 | 0.03 | 6.90 | 2.82 | 321.2 | 1750.9 | 1.11 | >2.0 |
| Bismuth | 806 | 371 | 0.31 | 0.11 | 0.06 | 2.63 | 1.08 | 34.7 | 201.4 | 17.11 | <1.0 |
| Black Cohosh | 465 | 16 | 0.01 | 0.01 | 0.01 | 18.49 | 7.58 | 373.9 | 1653.0 | 1.18 | >2.0 |
| Brass Scrap | 806 | 510 | 0.03 | 0.02 | 0.07 | 0.77 | 0.31 | 16.4 | 95.1 | 5.77 | 1.1-1.2 |
| Brazils | 709 | 190 | 0.05 | 0.05 | -0.24 | 1.61 | 0.66 | 120.6 | 657.1 | 2.73 | 1.7-1.8 |
| Butadiene | 709 | 340 | 0.37 | 0.34 | 0.25 | -0.22 | -0.09 | 10.4 | 56.6 | 11.57 | <1.0 |
| Butane | 465 | 435 | 0.00 | 0.03 | 0.11 | 0.33 | 0.13 | 7.3 | 32.5 | 3.29 | 1.5-1.6 |
| Cadmium | 806 | 424 | 0.31 | 0.11 | 0.03 | 2.67 | 1.09 | 22.4 | 130.0 | 13.31 | <1.0 |
| Camphor | 709 | 29 | 0.01 | -0.05 | 0.04 | 18.10 | 7.41 | 442.4 | 2411.0 | 1.43 | >2.0 |
| Camphor Oil | 709 | 229 | -0.02 | 0.03 | 0.02 | 1.61 | 0.66 | 28.5 | 155.6 | 7.08 | 1.0-1.1 |
| Cananga Oil | 709 | 256 | 0.00 | -0.02 | 0.04 | 9.58 | 3.92 | 171.5 | 934.6 | 8.81 | <1.0 |
| Candelilla | 709 | 30 | 0.05 | 0.01 | 0.13 | 1.78 | 0.73 | 81.9 | 446.3 | 1.78 | >2.0 |
| Caprolactam | 709 | 81 | 0.01 | 0.01 | 0.06 | 2.61 | 1.07 | 41.0 | 223.4 | 12.53 | <1.0 |
| Caraway Seed | 537 | 61 | 0.02 | -0.03 | 0.00 | 8.30 | 3.40 | 142.8 | 677.9 | 8.88 | <1.0 |
| Carbonated Wool | 537 | 71 | 0.02 | 0.02 | 0.05 | -2.47 | -1.01 | 34.5 | 163.7 | 12.96 | <1.0 |
| Cardomoms | 709 | 52 | 0.04 | 0.09 | 0.05 | -0.52 | -0.21 | 30.0 | 163.4 | 12.80 | <1.0 |
| Carnuba | 709 | 128 | 0.03 | 0.08 | 0.01 | -0.28 | -0.11 | 40.3 | 219.9 | 22.22 | <1.0 |
| Cascara | 709 | 25 | 0.01 | 0.00 | 0.01 | 6.07 | 2.49 | 159.4 | 868.7 | 1.41 | >2.0 |
| Cashew Kernal | 709 | 216 | -0.09 | 0.07 | -0.01 | -5.41 | -2.21 | 93.0 | 507.1 | 6.19 | 1.1-1.2 |
| Cassia Lignea | 537 | 152 | 0.16 | 0.07 | 0.13 | -2.97 | -1.22 | 34.9 | 165.6 | 2.84 | 1.7-1.8 |
| Cassia Oil | 709 | 194 | 0.01 | 0.00 | -0.03 | -1.17 | -0.48 | 28.7 | 156.5 | 7.63 | 1.0-1.1 |
| Cattle | 537 | 533 | -0.27 | 0.11 | 0.01 | -1.34 | -0.55 | 15.4 | 73.2 | 3.51 | 1.4-1.5 |
| Caustic Soda | 709 | 310 | 0.23 | 0.11 | 0.16 | 3.07 | 1.26 | 39.9 | 217.6 | 9.84 | <1.0 |
| Cedarwood Oil | 709 | 260 | -0.01 | -0.13 | 0.01 | 0.19 | 0.08 | 21.3 | 116.1 | 13.09 | <1.0 |
| Celery Seed | 709 | 67 | 0.05 | 0.02 | 0.07 | 2.57 | 1.05 | 38.0 | 207.1 | 10.98 | <1.0 |
| Cement | 709 | 50 | 0.01 | 0.01 | -0.19 | -0.86 | -0.35 | 62.5 | 340.8 | 5.83 | 1.1-1.2 |
| Chicken | 806 | 801 | 0.19 | -0.27 | 0.08 | -0.21 | -0.09 | 1.9 | 10.9 | 3.28 | 1.5-1.6 |
| Chickpeas | 537 | 77 | -0.01 | 0.00 | -0.03 | 1.32 | 0.54 | 48.2 | 228.8 | 20.21 | <1.0 |
| Chilles | 537 | 17 | 0.00 | 0.01 | 0.08 | -2.11 | -0.87 | 72.6 | 344.8 | 2.55 | 1.9-2.0 |
| Chlorine | 709 | 651 | -0.16 | 0.04 | -0.13 | -0.65 | -0.27 | 11.3 | 61.5 | 5.89 | 1.1-1.2 |
| Chrome | 537 | 67 | 0.01 | 0.22 | 0.02 | 3.41 | 1.40 | 30.9 | 146.7 | 6.33 | 1.0-1.1 |
| Chromite | 806 | 23 | 0.01 | 0.01 | 0.01 | 6.91 | 2.83 | 130.4 | 757.3 | 11.63 | <1.0 |
| Cinnamon Bark | 537 | 17 | 0.06 | -0.02 | 0.11 | -7.98 | -3.27 | 169.5 | 804.9 | 1.85 | >2.0 |
| Cinnamon Leaf Oil | 709 | 236 | -0.01 | 0.05 | 0.02 | 2.26 | 0.93 | 34.9 | 190.0 | 14.24 | <1.0 |
| Citronella Oil | 709 | 285 | 0.01 | 0.03 | 0.08 | 0.55 | 0.22 | 17.7 | 96.6 | 23.42 | <1.0 |
| Citrus Pulp | 709 | 497 | 0.06 | 0.06 | 0.00 | -0.79 | -0.33 | 11.5 | 62.9 | 5.14 | 1.2-1.3 |
| Clove Leaf Oil | 537 | 201 | -0.08 | 0.04 | -0.05 | -0.61 | -0.25 | 20.3 | 96.5 | 7.66 | 1.0-1.1 |
| Cloves | 709 | 153 | 0.04 | 0.07 | 0.13 | 0.78 | 0.32 | 22.6 | 123.2 | 494.38 | <1.0 |

## Table 3. Distribution of the error term

| | Observations | | Serial Correlation | | | Skewness | | Kurtosis | | Exponent | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | No. of obs. | No. of price changes | Corr[u(t),u(t-k)] *k* =1 | *k* =2 | *k* =4 | Skew-ness | *t* stat. | Excess Kurtosis | *t* stat. | Characteristic exponent *z* | alpha |
| (5% Critical Values) | | | | | | | (1.96) | | (1.96) | | |
| Cobalt | 537 | 346 | 0.32 | 0.07 | 0.03 | 3.08 | 1.26 | 24.5 | 116.5 | 6.07 | 1.1-1.2 |
| Cochineal | 537 | 15 | 0.01 | 0.03 | 0.00 | -17.58 | -7.20 | 391.1 | 1856.8 | 3.30 | 1.5-1.6 |
| Cocoa-Brazil | 709 | 615 | -0.04 | -0.08 | -0.01 | 0.06 | 0.03 | 3.9 | 21.1 | 3.37 | 1.5-1.6 |
| Cocoa-Ivory Coast | 806 | 795 | -0.02 | -0.03 | 0.04 | 0.17 | 0.07 | 1.1 | 6.2 | 2.84 | 1.7-1.8 |
| Coconut | 537 | 114 | 0.04 | 0.08 | 0.05 | 1.16 | 0.47 | 42.3 | 200.7 | 15.75 | <1.0 |
| Coconut Fibres | 709 | 60 | -0.06 | -0.01 | -0.01 | 3.00 | 1.23 | 51.1 | 278.5 | 15.10 | <1.0 |
| Coconut Oil | 1053 | 962 | -0.02 | 0.06 | -0.03 | 0.09 | 0.04 | 3.0 | 19.8 | 3.35 | 1.5-1.6 |
| Coffee-Brazil | 1053 | 781 | 0.01 | 0.11 | -0.05 | 1.03 | 0.42 | 9.7 | 64.4 | 5.79 | 1.1-1.2 |
| Coffee-Columbia | 709 | 570 | 0.03 | 0.04 | 0.00 | 0.22 | 0.09 | 6.4 | 34.8 | 4.38 | 1.3-1.4 |
| Coir Yarn | 709 | 20 | -0.01 | 0.03 | 0.01 | -0.59 | -0.24 | 184.2 | 1003.9 | 2.64 | 1.8-1.9 |
| Columbite | 806 | 10 | 0.01 | 0.01 | 0.01 | -2.42 | -0.99 | 236.2 | 1372.3 | 1.75 | >2.0 |
| Copper | 1053 | 1048 | 0.00 | 0.11 | 0.02 | 0.25 | 0.10 | 2.6 | 17.5 | 3.06 | 1.6-1.7 |
| Copra | 1053 | 823 | 0.04 | 0.04 | 0.07 | 0.32 | 0.13 | 4.5 | 29.9 | 4.03 | 1.3-1.4 |
| Copra Meal | 285 | 201 | 0.08 | 0.09 | 0.07 | -0.22 | -0.09 | 2.4 | 8.2 | 6.16 | 1.1-1.2 |
| Coriander Seed | 709 | 78 | 0.07 | 0.02 | 0.00 | 0.78 | 0.32 | 66.6 | 363.2 | 8.24 | <1.0 |
| Corn | 1053 | 1017 | 0.05 | 0.05 | 0.09 | -0.09 | -0.04 | 4.7 | 31.1 | 3.22 | 1.5-1.6 |
| Cotton | 1053 | 1016 | -0.01 | 0.04 | 0.00 | -10.50 | -4.30 | 235.1 | 1560.4 | 3.23 | 1.5-1.6 |
| Cottonseed Oil | 537 | 369 | -0.03 | 0.10 | 0.05 | -0.69 | -0.28 | 7.4 | 35.0 | 4.83 | 1.2-1.3 |
| Coumarin Oil | 709 | 101 | -0.03 | -0.09 | 0.02 | 2.12 | 0.87 | 50.7 | 276.2 | 13.90 | <1.0 |
| Crude Oil | 806 | 788 | -0.07 | 0.01 | 0.08 | -0.24 | -0.10 | 6.1 | 35.4 | 3.36 | 1.5-1.6 |
| Cummin Seed | 537 | 157 | 0.01 | -0.09 | 0.07 | 1.86 | 0.76 | 25.4 | 120.7 | 12.79 | <1.0 |
| Dimethyl Ter. | 465 | 38 | 0.01 | 0.00 | 0.01 | -3.43 | -1.40 | 64.4 | 284.6 | 4.54 | 1.2-1.3 |
| Drawn Text. Yarn | 709 | 88 | 0.01 | 0.01 | 0.07 | 0.58 | 0.24 | 34.1 | 186.0 | 26.68 | <1.0 |
| Eggs | 806 | 572 | 0.41 | -0.03 | -0.31 | 0.03 | 0.01 | 1.7 | 9.8 | 3.66 | 1.4-1.5 |
| Elm Bark Powder | 465 | 19 | 0.09 | 0.08 | -0.07 | 7.69 | 3.15 | 94.5 | 417.9 | 4.84 | 1.2-1.3 |
| Ethanol | 465 | 72 | 0.01 | -0.02 | -0.04 | 0.49 | 0.20 | 23.0 | 101.7 | 184.59 | <1.0 |
| Ethylene | 709 | 387 | 0.39 | 0.37 | 0.24 | 0.14 | 0.06 | 12.2 | 66.5 | 18.12 | <1.0 |
| Ethylene Di-Chloride | 709 | 308 | 0.23 | 0.25 | 0.18 | -1.55 | -0.63 | 18.0 | 98.0 | 27.58 | <1.0 |
| Ethylene Glycol | 709 | 94 | 0.00 | 0.00 | 0.03 | -2.03 | -0.83 | 60.4 | 329.5 | 15.12 | <1.0 |
| Eucalyptus Oil | 709 | 332 | -0.02 | 0.04 | 0.03 | 0.63 | 0.26 | 11.4 | 62.1 | 13.07 | <1.0 |
| Fennel Seed | 709 | 59 | 0.04 | -0.22 | -0.08 | -1.06 | -0.43 | 95.5 | 520.3 | 5.49 | 1.1-1.2 |
| Fenugreek Seed | 537 | 69 | 0.04 | 0.04 | 0.01 | 0.75 | 0.31 | 24.4 | 115.7 | 18.13 | <1.0 |
| Ferro-Chrome | 806 | 129 | 0.06 | 0.13 | 0.06 | -1.63 | -0.67 | 45.0 | 261.4 | 39.84 | <1.0 |
| Ferro-Manganese | 806 | 84 | 0.00 | 0.00 | 0.00 | 15.09 | 6.18 | 283.8 | 1648.6 | 2.99 | 1.6-1.7 |
| Ferro-Molybdenum | 709 | 403 | 0.42 | 0.40 | 0.22 | 2.45 | 1.00 | 21.2 | 115.4 | 12.75 | <1.0 |
| Ferro-Silicon | 806 | 102 | 0.01 | 0.03 | 0.00 | 15.00 | 6.14 | 287.1 | 1667.8 | 3.57 | 1.4-1.5 |
| Ferro-Titanium | 709 | 187 | 0.05 | 0.06 | 0.06 | 1.38 | 0.56 | 28.4 | 154.9 | 17.59 | <1.0 |
| Ferro-Tungsten | 537 | 148 | 0.15 | 0.21 | 0.10 | 3.35 | 1.37 | 28.6 | 135.8 | 16.41 | <1.0 |
| Ferrous Scrap | 806 | 336 | 0.08 | 0.02 | 0.12 | -0.26 | -0.11 | 15.9 | 92.1 | 23.86 | <1.0 |
| Ferro-Vanadium | 806 | 421 | 0.33 | 0.29 | 0.16 | 3.43 | 1.40 | 35.3 | 205.2 | 16.81 | <1.0 |
| Fish Meal | 537 | 161 | 0.13 | 0.14 | 0.05 | 2.17 | 0.89 | 29.1 | 138.0 | 4.34 | 1.3-1.4 |
| Fish Oil | 537 | 287 | 0.08 | 0.07 | 0.08 | -1.15 | -0.47 | 21.8 | 103.6 | 16.43 | <1.0 |
| Freight Route 1 | 520 | 509 | 0.41 | 0.14 | -0.11 | 0.40 | 0.16 | 2.5 | 11.7 | 3.26 | 1.5-1.6 |
| Freight Route 10 | 520 | 490 | 0.50 | 0.23 | 0.13 | 1.03 | 0.42 | 5.5 | 25.7 | 4.76 | 1.2-1.3 |
| Freight Route 1A | 445 | 420 | 0.53 | 0.15 | -0.11 | 0.56 | 0.23 | 1.3 | 5.6 | 3.09 | 1.6-1.7 |
| Freight Route 2 | 520 | 504 | 0.30 | -0.04 | -0.06 | -0.05 | -0.02 | 1.3 | 6.1 | 2.80 | 1.7-1.8 |
| Freight Route 3 | 520 | 505 | 0.48 | 0.19 | -0.10 | 0.27 | 0.11 | 3.3 | 15.4 | 3.93 | 1.3-1.4 |
| Freight Route 3A | 445 | 418 | 0.46 | 0.14 | -0.12 | 0.83 | 0.34 | 4.3 | 18.6 | 3.37 | 1.5-1.6 |
| Freight Route 7 | 520 | 497 | 0.48 | 0.20 | 0.08 | 0.19 | 0.08 | 2.9 | 13.3 | 3.97 | 1.3-1.4 |
| Freight Route 9 | 520 | 478 | 0.53 | 0.16 | -0.11 | 1.29 | 0.53 | 7.9 | 36.6 | 4.94 | 1.2-1.3 |
| Fuel Oil | 1053 | 1012 | 0.09 | -0.02 | -0.05 | 0.27 | 0.11 | 9.4 | 62.1 | 3.48 | 1.4-1.5 |
| Gallium | 465 | 52 | 0.06 | 0.19 | 0.09 | 4.38 | 1.80 | 59.1 | 261.2 | 3.02 | 1.6-1.7 |
| Gas Oil | 709 | 683 | -0.07 | 0.10 | 0.04 | 0.29 | 0.12 | 14.4 | 78.2 | 2.81 | 1.7-1.8 |
| Gasoline | 806 | 799 | -0.03 | 0.06 | -0.01 | -0.05 | -0.02 | 2.2 | 12.7 | 3.06 | 1.6-1.7 |
| Genetian Root | 709 | 9 | 0.01 | 0.01 | 0.03 | 14.30 | 5.85 | 233.3 | 1271.9 | 1.46 | >2.0 |
| Geranium Oil | 709 | 310 | -0.02 | -0.04 | 0.02 | 2.26 | 0.93 | 38.8 | 211.5 | 11.68 | <1.0 |
| Germanium | 537 | 79 | 0.13 | 0.20 | 0.07 | 8.77 | 3.59 | 138.4 | 657.2 | 8.76 | <1.0 |
| Germanium Dioxide | 537 | 92 | 0.20 | 0.23 | 0.11 | 1.84 | 0.75 | 23.5 | 111.5 | 13.62 | <1.0 |
| Ginger | 709 | 143 | -0.10 | -0.02 | 0.00 | 1.11 | 0.45 | 18.6 | 101.1 | 8.67 | <1.0 |
| Ginger Oil | 709 | 264 | -0.01 | -0.11 | -0.01 | -1.17 | -0.48 | 32.7 | 178.3 | 6.05 | 1.1-1.2 |
| Gold | 1566 | 1529 | -0.01 | 0.03 | 0.01 | 0.69 | 0.28 | 8.1 | 65.9 | 4.06 | 1.3-1.4 |
| Grapefruit Juice | 285 | 24 | 0.01 | -0.09 | -0.06 | -2.29 | -0.94 | 30.1 | 104.4 | 47.22 | <1.0 |
| Groundnut Oil | 709 | 541 | 0.00 | 0.09 | 0.02 | 1.46 | 0.60 | 14.1 | 76.7 | 4.97 | 1.2-1.3 |
| Gunmetal Scrap | 806 | 469 | 0.06 | -0.01 | 0.04 | -0.46 | -0.19 | 4.0 | 23.3 | 9.62 | <1.0 |
| Hazelnut Kernals | 709 | 226 | -0.06 | -0.06 | 0.01 | 0.71 | 0.29 | 33.0 | 179.8 | 6.82 | 1.0-1.1 |

## Table 3. Distribution of the error term

| | Observations | | Serial Correlation | | | Skewness | | Kurtosis | | Exponent | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | No. of obs. | No. of price changes | Corr[u(t),u(t-k)] $k$ =1 | $k$ =2 | $k$ =4 | Skew-ness | $t$ stat. | Excess Kurtosis | $t$ stat. | Characteristic exponent $z$ | alpha |
| (5% Critical Values) | | | | | | | (1.96) | | (1.96) | | |
| Heavy Fuel Oil | 709 | 656 | 0.05 | 0.11 | 0.04 | -0.04 | -0.02 | 5.1 | 27.6 | 3.51 | 1.4-1.5 |
| Heliotropin | 709 | 277 | -0.07 | 0.00 | 0.11 | 0.04 | 0.02 | 11.6 | 63.4 | 29.74 | <1.0 |
| Hogs | 784 | 724 | 0.04 | 0.11 | 0.04 | -1.28 | -0.52 | 19.5 | 111.6 | 2.91 | 1.6-1.7 |
| Hydrastis Roots | 465 | 34 | 0.06 | 0.06 | 0.05 | 6.85 | 2.81 | 110.2 | 487.1 | 2.09 | >2.0 |
| Indium Ingots | 806 | 257 | 0.44 | 0.41 | 0.27 | 2.42 | 0.99 | 14.8 | 85.7 | 81.94 | <1.0 |
| Ipecuanha Root | 465 | 30 | 0.00 | 0.06 | 0.00 | -3.00 | -1.23 | 65.8 | 290.8 | 2.53 | 1.9-2.0 |
| Iridium | 709 | 213 | 0.00 | 0.16 | 0.08 | 1.24 | 0.51 | 28.0 | 152.4 | 504.14 | <1.0 |
| Jet Kerosene | 709 | 669 | 0.02 | 0.13 | 0.00 | 1.19 | 0.49 | 17.7 | 96.3 | 3.60 | 1.4-1.5 |
| Jute | 709 | 159 | 0.04 | 0.00 | 0.10 | 1.32 | 0.54 | 42.2 | 230.2 | 11.71 | <1.0 |
| Kola Nuts | 709 | 18 | 0.03 | -0.05 | 0.02 | 7.96 | 3.26 | 245.2 | 1336.2 | 8.84 | <1.0 |
| Lard | 709 | 162 | 0.05 | -0.10 | 0.06 | -1.18 | -0.48 | 51.2 | 279.1 | 10.06 | <1.0 |
| Lavender Spike Oil | 709 | 219 | 0.06 | 0.13 | 0.15 | 4.94 | 2.02 | 116.0 | 632.1 | 2.39 | >2.0 |
| Lead | 1053 | 1042 | -0.03 | -0.01 | -0.02 | 0.29 | 0.12 | 4.3 | 28.7 | 2.94 | 1.6-1.7 |
| Lead Ore | 806 | 18 | 0.05 | 0.02 | 0.03 | 6.01 | 2.46 | 120.8 | 702.1 | 2.87 | 1.7-1.8 |
| Lemon Grass Oil | 537 | 107 | -0.06 | 0.01 | 0.10 | 0.82 | 0.34 | 24.3 | 115.6 | 13.50 | <1.0 |
| Lentils | 537 | 91 | 0.04 | 0.05 | 0.03 | 1.90 | 0.78 | 24.7 | 117.1 | 12.49 | <1.0 |
| Lime Oil | 709 | 87 | -0.05 | 0.00 | 0.00 | 0.12 | 0.05 | 27.6 | 150.4 | 15.67 | <1.0 |
| Linseed Meal | 537 | 214 | 0.00 | 0.09 | 0.09 | 0.01 | 0.00 | 13.4 | 63.6 | 5.57 | 1.1-1.2 |
| Linseed Oil | 709 | 212 | 0.09 | 0.08 | -0.02 | 0.13 | 0.05 | 20.4 | 111.1 | 29.02 | <1.0 |
| Lithium Ore | 806 | 8 | 0.00 | 0.00 | 0.00 | 19.07 | 7.80 | 556.2 | 3231.5 | 1.88 | >2.0 |
| Litsea Cubesa Oil | 709 | 319 | 0.01 | 0.17 | 0.14 | 0.02 | 0.01 | 26.4 | 144.0 | 16.88 | <1.0 |
| Mace | 537 | 75 | -0.06 | 0.27 | 0.27 | -4.47 | -1.83 | 144.2 | 684.8 | 1.40 | >2.0 |
| Magnesium | 464 | 158 | 0.08 | 0.14 | 0.12 | 1.40 | 0.57 | 11.4 | 50.5 | 78.62 | <1.0 |
| Maize | 709 | 529 | 0.05 | -0.04 | 0.07 | 0.30 | 0.12 | 3.7 | 20.3 | 4.14 | 1.3-1.4 |
| Maize/Corn Oil | 537 | 193 | 0.08 | 0.13 | 0.17 | 0.79 | 0.32 | 8.4 | 39.8 | 20.57 | <1.0 |
| Manganese | 806 | 277 | 0.21 | 0.16 | 0.04 | -0.43 | -0.18 | 14.8 | 85.8 | 24.81 | <1.0 |
| Manganese Ore | 806 | 38 | -0.01 | 0.13 | 0.10 | 2.75 | 1.13 | 118.4 | 688.0 | 2.58 | 1.8-1.9 |
| Meat and Bone Meal | 709 | 185 | 0.00 | 0.03 | 0.08 | 0.10 | 0.04 | 28.3 | 154.5 | 23.07 | <1.0 |
| MEG | 709 | 257 | 0.21 | 0.18 | 0.11 | 2.08 | 0.85 | 14.1 | 76.9 | 27.23 | <1.0 |
| Menthol Oil | 709 | 374 | -0.01 | 0.07 | 0.22 | 2.88 | 1.18 | 25.3 | 137.9 | 16.51 | <1.0 |
| Mercury | 806 | 298 | 0.29 | 0.32 | 0.23 | 0.99 | 0.41 | 12.2 | 71.0 | 26.27 | <1.0 |
| Methanol | 709 | 442 | 0.39 | 0.25 | 0.17 | 0.52 | 0.21 | 11.7 | 63.9 | 7.26 | 1.0-1.1 |
| Millet | 537 | 58 | -0.04 | -0.01 | 0.01 | -3.22 | -1.32 | 67.8 | 321.9 | 17.27 | <1.0 |
| Molybdenite | 806 | 81 | 0.14 | 0.14 | 0.24 | 4.35 | 1.78 | 122.9 | 714.0 | 19.95 | <1.0 |
| Molybdenum | 806 | 517 | 0.46 | 0.29 | 0.16 | 2.35 | 0.96 | 21.9 | 127.0 | 8.75 | <1.0 |
| Musk | 709 | 199 | 0.05 | -0.02 | -0.04 | 1.39 | 0.57 | 42.3 | 230.4 | 7.61 | 1.0-1.1 |
| Naphtha | 709 | 679 | 0.05 | 0.05 | 0.06 | 0.92 | 0.38 | 8.5 | 46.1 | 3.08 | 1.6-1.7 |
| Natural Gas | 285 | 276 | -0.01 | -0.15 | 0.07 | 1.67 | 0.68 | 22.4 | 77.6 | 3.64 | 1.4-1.5 |
| Nickel | 1053 | 1049 | 0.02 | 0.12 | 0.07 | 1.30 | 0.53 | 12.2 | 81.1 | 3.37 | 1.5-1.6 |
| Nutmeg | 285 | 80 | 0.11 | 0.25 | 0.21 | 0.11 | 0.05 | 15.8 | 54.7 | 2.98 | 1.6-1.7 |
| Nux Vomica | 537 | 3 | 0.02 | 0.02 | 0.02 | -19.85 | -8.13 | 425.3 | 2019.0 | 0.83 | >2.0 |
| Nylon Yarn | 709 | 81 | 0.01 | 0.01 | 0.05 | 1.10 | 0.45 | 81.9 | 446.4 | 10.21 | <1.0 |
| Oats | 1053 | 909 | 0.04 | -0.01 | 0.02 | 0.29 | 0.12 | 3.5 | 23.1 | 3.63 | 1.4-1.5 |
| Olive Oil | 285 | 51 | 0.07 | 0.16 | -0.08 | 0.15 | 0.06 | 15.8 | 54.7 | 10.86 | <1.0 |
| Orange Juice | 285 | 18 | 0.02 | -0.15 | 0.03 | 0.63 | 0.26 | 42.2 | 146.4 | 7.58 | 1.0-1.1 |
| Orange Pera | 465 | 80 | -0.12 | 0.03 | 0.05 | 4.84 | 1.98 | 75.2 | 332.6 | 45.74 | <1.0 |
| Palladium | 537 | 516 | 0.09 | -0.03 | 0.03 | 0.19 | 0.08 | 7.1 | 33.7 | 3.44 | 1.5-1.6 |
| Palm Kernal Cake | 465 | 269 | 0.09 | -0.11 | -0.03 | 0.01 | 0.00 | 26.8 | 118.5 | 5.09 | 1.2-1.3 |
| Palm Kernal Oil | 709 | 574 | 0.06 | 0.06 | 0.02 | 0.81 | 0.33 | 9.0 | 49.0 | 4.11 | 1.3-1.4 |
| Palm Oil | 806 | 724 | 0.00 | 0.11 | 0.06 | -0.23 | -0.09 | 3.3 | 19.1 | 3.26 | 1.5-1.6 |
| Paper Pulp | 1566 | 108 | 0.02 | 0.00 | 0.04 | 0.24 | 0.10 | 4.6 | 37.6 | 3.39 | 1.5-1.6 |
| Paper, Corrugated | 537 | 39 | 0.01 | 0.01 | 0.03 | -1.08 | -0.44 | 28.7 | 136.1 | 16.77 | <1.0 |
| Patchouli Oil | 285 | 53 | 0.05 | -0.11 | -0.01 | 3.71 | 1.53 | 69.2 | 240.0 | 19.24 | <1.0 |
| Peanuts | 709 | 245 | 0.05 | -0.08 | 0.01 | 0.14 | 0.06 | 37.8 | 205.8 | 10.25 | <1.0 |
| Pepper-Black | 537 | 127 | -0.05 | 0.13 | 0.09 | -3.28 | -1.35 | 58.9 | 279.6 | 3.67 | 1.4-1.5 |
| Peppermint Oil | 709 | 340 | 0.01 | 0.11 | 0.16 | 2.34 | 0.96 | 28.9 | 157.6 | 13.65 | <1.0 |
| Pepper-White | 709 | 274 | -0.03 | 0.02 | 0.03 | 3.23 | 1.32 | 46.0 | 250.9 | 12.15 | <1.0 |
| PET-Bottle Gde | 465 | 82 | 0.05 | 0.13 | 0.17 | 0.33 | 0.14 | 16.2 | 71.5 | 30.00 | <1.0 |
| Petitgrain Oil | 709 | 292 | -0.06 | 0.01 | -0.01 | 1.79 | 0.73 | 13.8 | 75.4 | 17.01 | <1.0 |
| PET-Staple Fibre | 709 | 72 | 0.00 | 0.00 | 0.27 | -0.87 | -0.35 | 29.0 | 158.0 | 18.18 | <1.0 |
| PGE | 537 | 255 | 0.17 | 0.24 | 0.11 | 2.00 | 0.82 | 12.3 | 58.2 | 8.10 | <1.0 |
| Phosphor Copper | 806 | 329 | 0.03 | 0.01 | -0.01 | -2.05 | -0.84 | 34.2 | 198.9 | 30.56 | <1.0 |
| Phosphor Tin | 806 | 161 | -0.01 | -0.01 | 0.04 | -19.16 | -7.84 | 479.9 | 2788.2 | 17.17 | <1.0 |
| Pigs Live Weight | 537 | 533 | 0.19 | -0.06 | -0.01 | 0.22 | 0.09 | 2.6 | 12.1 | 3.28 | 1.5-1.6 |
| Pimento | 709 | 133 | 0.09 | 0.21 | 0.10 | 1.75 | 0.72 | 38.4 | 209.1 | 32.03 | <1.0 |

## Table 3. Distribution of the error term

| | Observations | | Serial Correlation | | | Skewness | | Kurtosis | | Exponent | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | No. of obs. | No. of price changes | Corr[u(t),u(t-k)] $k$ =1 | $k$ =2 | $k$ =4 | Skew-ness | $t$ stat. | Excess Kurtosis | $t$ stat. | Characteristic exponent $z$ | alpha |
| (5% Critical Values) | | | | | | | (1.96) | | (1.96) | | |
| Pimento Leaf Oil | 285 | 23 | 0.02 | 0.01 | -0.01 | 7.23 | 2.97 | 94.0 | 326.3 | 3.89 | 1.3-1.4 |
| Pineapple Juice | 285 | 31 | 0.01 | 0.04 | 0.00 | -0.95 | -0.39 | 46.7 | 162.0 | 10.47 | <1.0 |
| Platinum | 1053 | 1041 | 0.00 | -0.06 | -0.03 | -0.42 | -0.17 | 9.8 | 65.1 | 3.66 | 1.4-1.5 |
| Polyacryloinitrile | 709 | 41 | 0.00 | 0.00 | -0.02 | -1.09 | -0.44 | 58.6 | 319.2 | 2.89 | 1.6-1.7 |
| Polye HDPE-Inj | 709 | 251 | 0.06 | 0.19 | 0.26 | 0.59 | 0.24 | 23.8 | 129.9 | 36.00 | <1.0 |
| Polye LDPE-GP Film | 709 | 320 | 0.35 | 0.32 | 0.32 | 0.10 | 0.04 | 5.9 | 31.9 | 19.89 | <1.0 |
| Polye LLDPE | 537 | 147 | 0.33 | 0.31 | 0.29 | 0.42 | 0.17 | 12.2 | 57.8 | 43.23 | <1.0 |
| Polystyrene-Hips | 709 | 321 | 0.28 | 0.15 | 0.23 | 1.91 | 0.78 | 26.0 | 141.7 | 23.95 | <1.0 |
| Poppyseed | 465 | 13 | 0.07 | 0.07 | 0.08 | 9.06 | 3.71 | 146.9 | 649.3 | 1.14 | >2.0 |
| Pork Bellies | 1053 | 994 | -0.10 | -0.02 | -0.08 | 0.00 | 0.00 | 0.9 | 6.1 | 2.77 | 1.7-1.8 |
| PP Copolymer | 709 | 218 | 0.21 | 0.15 | 0.21 | 0.29 | 0.12 | 10.3 | 55.9 | 21.54 | <1.0 |
| PP Homopolymer Inj | 465 | 220 | 0.34 | 0.26 | 0.20 | 0.16 | 0.07 | 7.0 | 30.8 | 17.89 | <1.0 |
| Propane | 806 | 739 | 0.11 | 0.07 | 0.00 | 1.08 | 0.44 | 14.4 | 83.8 | 4.00 | 1.3-1.4 |
| Propylene (C) | 537 | 300 | 0.25 | 0.23 | 0.13 | -1.48 | -0.61 | 22.4 | 106.4 | 29.94 | <1.0 |
| Propylene (P) | 709 | 441 | 0.35 | 0.30 | 0.21 | 0.76 | 0.31 | 5.1 | 27.9 | 6.37 | 1.0-1.1 |
| Propylene (R) | 465 | 98 | 0.01 | 0.00 | 0.41 | 1.19 | 0.49 | 10.8 | 47.8 | 34.87 | <1.0 |
| Propylene Oxide | 465 | 75 | 0.06 | 0.16 | 0.03 | 1.69 | 0.69 | 19.0 | 83.9 | 12.31 | <1.0 |
| PTA | 709 | 96 | 0.00 | 0.00 | 0.22 | -3.22 | -1.32 | 44.5 | 242.5 | 37.23 | <1.0 |
| PVC | 465 | 229 | 0.27 | 0.19 | 0.36 | 0.13 | 0.05 | 6.5 | 28.9 | 23.32 | <1.0 |
| Rapeseed | 285 | 215 | 0.06 | 0.03 | 0.06 | -3.08 | -1.26 | 20.9 | 72.7 | 4.14 | 1.3-1.4 |
| Rapeseed Meal | 709 | 448 | 0.05 | 0.05 | 0.02 | 0.93 | 0.38 | 29.5 | 160.6 | 5.68 | 1.1-1.2 |
| Rapeseed Oil | 709 | 528 | -0.02 | -0.01 | 0.06 | 1.99 | 0.81 | 19.5 | 106.3 | 3.99 | 1.3-1.4 |
| Rayon, Staple Fibre | 537 | 21 | 0.01 | 0.01 | 0.17 | -1.94 | -0.79 | 55.8 | 265.1 | 1.51 | >2.0 |
| Rhodium | 806 | 592 | 0.02 | -0.05 | 0.06 | 3.24 | 1.33 | 39.7 | 230.6 | 11.16 | <1.0 |
| Rice | 537 | 117 | 0.04 | 0.07 | 0.15 | -1.39 | -0.57 | 22.4 | 106.2 | 34.24 | <1.0 |
| Rice/Bran | 537 | 181 | 0.02 | -0.02 | -0.02 | 0.55 | 0.23 | 14.4 | 68.4 | 8.43 | <1.0 |
| Rubber | 1566 | 1336 | 0.06 | 0.03 | 0.04 | 0.09 | 0.04 | 4.0 | 32.2 | 3.69 | 1.4-1.5 |
| Ruthenium | 709 | 190 | 0.06 | 0.06 | -0.01 | 0.84 | 0.35 | 24.5 | 133.3 | 347.08 | <1.0 |
| Saffron | 465 | 5 | 0.08 | -0.16 | 0.07 | -9.45 | -3.88 | 142.6 | 630.3 | 0.83 | >2.0 |
| Sandalwood Oil | 465 | 66 | 0.04 | 0.02 | 0.14 | 1.86 | 0.76 | 20.0 | 88.3 | 5.37 | 1.1-1.2 |
| Sasparilla Root | 537 | 5 | 0.02 | 0.02 | 0.02 | 20.23 | 8.29 | 446.8 | 2121.5 | 0.96 | >2.0 |
| Sassafras Oil | 537 | 122 | 0.04 | 0.01 | 0.06 | 1.11 | 0.45 | 17.0 | 80.8 | 14.35 | <1.0 |
| Selenium | 806 | 259 | 0.20 | 0.06 | 0.06 | -1.68 | -0.69 | 34.3 | 199.3 | 12.53 | <1.0 |
| Sesame Seed | 537 | 24 | 0.00 | 0.00 | 0.00 | 21.74 | 8.91 | 492.1 | 2336.6 | 11.22 | <1.0 |
| Sheep | 285 | 283 | -0.06 | 0.12 | 0.24 | -0.43 | -0.18 | 1.2 | 4.3 | 3.03 | 1.6-1.7 |
| Silico-Manganese | 806 | 98 | 0.01 | 0.03 | -0.02 | 3.80 | 1.56 | 74.1 | 430.3 | 5.99 | 1.1-1.2 |
| Silicon | 537 | 123 | 0.03 | 0.07 | 0.07 | 2.76 | 1.13 | 43.8 | 207.8 | 54.65 | <1.0 |
| Silver | 1566 | 1560 | 0.02 | 0.00 | 0.04 | 0.24 | 0.10 | 4.6 | 37.6 | 3.39 | 1.5-1.6 |
| Sisal | 709 | 98 | 0.03 | -0.01 | -0.08 | 0.37 | 0.15 | 32.8 | 178.5 | 18.72 | <1.0 |
| Soda Ash | 285 | 12 | 0.01 | 0.01 | 0.01 | 0.96 | 0.39 | 43.3 | 150.2 | 1.32 | >2.0 |
| Soyabean Oil | 1053 | 1039 | 0.00 | -0.01 | 0.05 | -0.15 | -0.06 | 2.7 | 17.7 | 3.03 | 1.6-1.7 |
| Soyabeans | 1053 | 1029 | -0.04 | 0.06 | 0.10 | 0.02 | 0.01 | 2.8 | 18.5 | 3.06 | 1.6-1.7 |
| Soyameal | 537 | 397 | 0.00 | 0.08 | 0.01 | -1.00 | -0.41 | 7.0 | 33.1 | 4.36 | 1.3-1.4 |
| Spearmint Oil | 709 | 277 | -0.02 | 0.02 | -0.05 | -2.43 | -0.99 | 46.6 | 254.1 | 7.46 | 1.0-1.1 |
| Steel,Galv. Coils | 806 | 120 | -0.01 | 0.08 | 0.08 | 1.31 | 0.54 | 22.6 | 131.4 | 30.10 | <1.0 |
| Steel,Merchant Bars | 806 | 157 | -0.02 | 0.01 | 0.06 | 4.55 | 1.86 | 79.7 | 462.9 | 22.66 | <1.0 |
| Steers | 1566 | 1386 | 0.00 | 0.09 | 0.05 | 0.15 | 0.06 | 3.0 | 23.9 | 3.31 | 1.5-1.6 |
| Styrene | 709 | 482 | 0.40 | 0.26 | 0.13 | -0.06 | -0.02 | 5.3 | 28.8 | 7.57 | 1.0-1.1 |
| Sugar | 1566 | 1537 | 0.02 | 0.00 | 0.04 | 0.24 | 0.10 | 4.6 | 37.6 | 3.39 | 1.5-1.6 |
| Sunflowerseed | 465 | 12 | 0.05 | 0.06 | 0.05 | 10.64 | 4.36 | 141.0 | 623.2 | 1.52 | >2.0 |
| Sunflowerseed Meal | 537 | 306 | 0.03 | -0.03 | 0.00 | -0.90 | -0.37 | 12.0 | 57.0 | 7.27 | 1.0-1.1 |
| Sunflowerseed Oil | 537 | 394 | 0.02 | 0.01 | 0.03 | 1.10 | 0.45 | 11.0 | 52.4 | 5.11 | 1.2-1.3 |
| Tallow | 709 | 297 | 0.08 | 0.07 | 0.00 | -12.15 | -4.97 | 244.6 | 1333.2 | 6.77 | 1.0-1.1 |
| Tallow, extra fancy | 465 | 260 | 0.18 | 0.17 | 0.07 | -0.24 | -0.10 | 4.6 | 20.3 | 10.78 | <1.0 |
| Tantalite Ore | 537 | 24 | 0.01 | 0.09 | 0.03 | 0.81 | 0.33 | 69.2 | 328.5 | 2.02 | >2.0 |
| Tapioca | 537 | 314 | 0.08 | 0.12 | 0.07 | -0.25 | -0.10 | 6.9 | 32.9 | 3.72 | 1.4-1.5 |
| Tea-Kenya | 619 | 447 | 0.09 | 0.09 | -0.08 | -1.25 | -0.51 | 12.4 | 63.0 | 4.99 | 1.2-1.3 |
| Tea-Sri Lanka | 619 | 396 | -0.14 | 0.05 | 0.00 | 0.52 | 0.21 | 9.7 | 49.6 | 4.53 | 1.2-1.3 |
| Terpineol | 709 | 244 | 0.02 | -0.02 | -0.10 | 0.11 | 0.05 | 15.0 | 81.7 | 13.78 | <1.0 |
| Tin | 537 | 528 | 0.03 | 0.03 | 0.03 | -0.41 | -0.17 | 3.7 | 17.4 | 3.35 | 1.5-1.6 |
| Titanium | 806 | 54 | 0.00 | 0.03 | 0.03 | -7.98 | -3.27 | 214.5 | 1245.8 | 4.38 | 1.3-1.4 |
| Titanium Dioxide | 465 | 60 | -0.05 | 0.04 | 0.07 | -0.21 | -0.09 | 27.4 | 121.1 | 17.63 | <1.0 |
| Titanium Ore | 806 | 40 | -0.02 | 0.01 | 0.01 | -0.74 | -0.30 | 56.9 | 330.7 | 2.57 | 1.9-2.0 |
| Titanium Scrap | 806 | 155 | 0.10 | 0.04 | 0.09 | 0.99 | 0.40 | 28.4 | 165.3 | 51.93 | <1.0 |
| Toluene | 465 | 260 | 0.20 | 0.14 | 0.02 | -0.47 | -0.19 | 5.9 | 26.0 | 8.02 | <1.0 |

**Table 3. Distribution of the error term**

| | Observations | | Serial Correlation | | | Skewness | | Kurtosis | | Exponent | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | No. of obs. | No. of price changes | Corr[u(t),u(t-k)] $k=1$ | $k=2$ | $k=4$ | Skew-ness | $t$ stat. | Excess Kurtosis | $t$ stat. | Characteristic exponent $z$ | alpha |
| (5% Critical Values) | | | | | | | (1.96) | | (1.96) | | |
| Tonquin | 537 | 21 | 0.13 | 0.01 | 0.11 | 7.64 | 3.13 | 97.0 | 460.5 | 2.35 | >2.0 |
| Tung Oil | 709 | 120 | 0.02 | 0.03 | -0.06 | 0.03 | 0.01 | 74.3 | 404.9 | 10.48 | <1.0 |
| Tungsten | 465 | 118 | 0.33 | 0.23 | 0.31 | 0.77 | 0.32 | 9.4 | 41.6 | 65.07 | <1.0 |
| Tungsten Ore | 806 | 218 | 0.05 | 0.12 | 0.07 | -1.17 | -0.48 | 20.4 | 118.5 | 17.43 | <1.0 |
| Turmeric | 709 | 119 | 0.05 | -0.03 | 0.03 | 1.30 | 0.53 | 24.7 | 134.4 | 8.05 | <1.0 |
| Valerian Root | 709 | 26 | 0.00 | 0.06 | 0.01 | 3.10 | 1.27 | 192.8 | 1051.0 | 12.10 | <1.0 |
| Walnuts | 709 | 134 | -0.05 | 0.05 | 0.09 | 1.44 | 0.59 | 33.8 | 184.4 | 4.20 | 1.3-1.4 |
| Vanadium | 537 | 200 | 0.34 | 0.19 | 0.24 | 0.74 | 0.30 | 18.7 | 88.6 | 18.99 | <1.0 |
| VCM | 709 | 250 | 0.16 | 0.11 | 0.24 | -0.88 | -0.36 | 11.1 | 60.4 | 45.90 | <1.0 |
| Vetivert Oil | 709 | 177 | 0.01 | 0.02 | 0.00 | 2.19 | 0.90 | 51.8 | 282.2 | 21.62 | <1.0 |
| Wheat-No.2 Hard | 1053 | 1036 | 0.00 | -0.01 | 0.05 | -0.15 | -0.06 | 2.7 | 17.7 | 3.03 | 1.6-1.7 |
| Wheat-No.2 Soft | 806 | 783 | 0.01 | -0.01 | -0.01 | -0.51 | -0.21 | 5.8 | 33.8 | 2.69 | 1.8-1.9 |
| Wheat-Spring | 1053 | 1040 | -0.15 | -0.01 | 0.01 | -0.11 | -0.05 | 9.6 | 64.0 | 3.15 | 1.5-1.6 |
| Wild Cherry Bark | 709 | 25 | 0.01 | 0.01 | -0.20 | 2.61 | 1.07 | 147.8 | 805.8 | 2.28 | >2.0 |
| Witch Hazel Leaves | 709 | 29 | -0.02 | 0.01 | 0.01 | 18.13 | 7.42 | 423.6 | 2308.9 | 1.71 | >2.0 |
| Wooltops | 1566 | 1025 | 0.12 | 0.06 | 0.07 | 0.55 | 0.23 | 17.1 | 138.1 | 6.07 | 1.1-1.2 |
| Xylenes | 285 | 216 | 0.31 | 0.29 | 0.04 | -0.09 | -0.04 | 3.4 | 11.6 | 4.94 | 1.2-1.3 |
| Zinc | 537 | 530 | -0.02 | 0.11 | 0.04 | -0.87 | -0.36 | 6.0 | 28.3 | 3.06 | 1.6-1.7 |
| Zinc Sulphide | 806 | 30 | -0.04 | 0.03 | 0.01 | -1.84 | -0.75 | 118.7 | 689.6 | 2.04 | >2.0 |
| Zircon | 465 | 25 | 0.02 | 0.02 | 0.01 | 0.26 | 0.11 | 81.7 | 361.3 | 1.57 | >2.0 |
| $H_0$ accepted: | | | | | | | 242 | | 0 | | |

Notes: - 280 series.

- Corr [u(t),u(t–k)] refers to the correlation of the error terms in regression (1.4). Monthly and semi-monthly refers to the sampling frequency of the price series.
- The sample estimate of Skewness is $\hat{b} = \frac{T}{(T-1)(T-2)} \sum [\frac{u(t)-\bar{u}}{s}]^3$. The $t$ statistic is $\hat{b} / s_b$, where $s_b = [\frac{6T(T-1)}{(T-2)(T+1)(T+3)}]^{1/2}$.
- The sample estimate of excess Kurtosis is $\hat{\kappa} = \frac{T(T+1)}{(T-1)(T-2)(T-3)} \sum [\frac{u(t)-\bar{u}}{s}]^4 - \frac{3(T-1)^2}{(T-2)(T-3)}$, where $s$ is the sample standard deviation. The $t$ statistic is $\hat{\kappa} / s_k$ where $s_\kappa = [\frac{24T(T-1)^2}{(T-3)(T-2)(T+3)(T+5)}]^{1/2}$.
- The Characteristic exponent $\alpha$ is estimated as in Fama and Roll (1971). The statistic is $\hat{z} = 0.827 \cdot \frac{\hat{x}_{0.96} - \hat{x}_{0.04}}{\hat{x}_{0.72} - \hat{x}_{0.28}}$, and $\alpha$ is found by searching table 2 in Fama and Roll (1968).

# Appendix - Commodity trade details

| Name | Series Start | Trade Details | Category |
|---|---|---|---|
| Acid Oils-Fish | 1986 | Acid Oils-Fish, Ex Tank Liverpool, £/tonne | Oilseeds, Oils&Fats |
| Acid Oils-Hard | 1986 | Acid Oils-Mixed Hard, Ex Tank Liverpool, £/tonne | Oilseeds, Oils&Fats |
| Acid Oils-Soft | 1986 | Acid Oils-Mixed Soft, Ex Tank Liverpool, £/tonne | Oilseeds, Oils&Fats |
| Acid Oils-Soya... | 1986 | Acid Oils-Soya/Sunflower/Maize, Ex Tank Liverpool, £/tonne | Oilseeds, Oils&Fats |
| Acrylonitrile | 1986 | Acrylonitrile,Europe Spot, CIF $/MT | Chemicals |
| Alloy Steel Scrap | 1984 | Eur. Alloy Steel Scrap 18/8S, $/TN | Metals |
| Almonds | 1986 | Almonds Shelled-CIF 25/27 Non Pareil, CIF $/tonne | Exotics: Edible Nuts |
| Aluminium | 1980 | LME-Aluminium 99.7% Cash £/TN Exp'd - A.M. Official | Metals |
| Ammonia | 1986 | Ammonia,Europe Spot CFR Med $/MT | Chemicals |
| Aniseed Oil | 1986 | Aniseed Oil (Star)- China Spot $/KG | Exotics: Essential Oils |
| Antimony | 1984 | Antimony Regulus Min 99.65% $/TN | Metals |
| Antimony Ore | 1984 | Ore, Antimony Clean Sulphide, $/TN | Metals |
| Apricot Kernals | 1989 | Apricot Kernals-C&F Chinese £/tonne | Exotics: Edible Nuts |
| Arabic Gum | 1989 | Arabic Gum, Sudanese Kordofan, FOB $/tonne | Exotics: Waxes&Gums |
| Arsenic | 1989 | Arsenic Metal 99%, Free Market, $/LB | Metals |
| Balsam-Canadian | 1986 | Balsam-Canadian, CIF £/KG | Exotics: Crude Drugs |
| Balsam-Copaiba | 1986 | Balsam-Copaiba, CIF £/KG | Exotics: Crude Drugs |
| Balsam-Peru | 1989 | Balsam-Peru, CIF £/KG | Exotics: Crude Drugs |
| Balsam-Tolu | 1986 | Balsam-Tolu, CIF £/KG | Exotics: Crude Drugs |
| Barley | 1986 | Barley-English Feed, Export, FOB East Cost £/tonne | Grains |
| Bay Oil | 1986 | Bay Oil-West Indies Spot $/KG | Exotics: Essential Oils |
| Beans-Black Eye | 1989 | Black Eye No.1 Beans, ex store UK, £/tonne | Seeds&Pulses |
| Beans-Butter | 1989 | Butter Beans Cal. No.1 cif UK, $/tonne | Seeds&Pulses |
| Beans-Dark Red | 1989 | Dark Red Kidney, Recleaned, Polished, ex store UK, £/tonne | Seeds&Pulses |
| Beans-Haricot | 1989 | US No.1 Haricot Beans, Ex-Store UK, £/tonne | Seeds&Pulses |
| Beeswax | 1994 | Beeswax, C&F Tanzanian $/TN | Exotics: Waxes&Gums |
| Benzene | 1986 | Benzene,US Gulf Spot FOB Barges $/GAL | Chemicals |
| Benzoin | 1986 | Benzoin-CIF Leopard £/KG | Exotics: Crude Drugs |
| Bismuth | 1984 | Bismuth 99.99% Free Market UK $/LB | Metals |
| Black Cohosh | 1991 | Black Cohosh-CIF £/KG | Exotics: Crude Drugs |
| Brass Scrap | 1984 | UK Scrap,Brass Cuttings £/TN | Metals |
| Brazils | 1986 | Brazils Medium, ex store UK, $/LB | Exotics: Edible Nuts |
| Butadiene | 1986 | Butadiene,Spot FOB Rotterdam $/MT | Chemicals |
| Butane | 1991 | Butane,Normal,Mont. Belvieu,C/GALLON | Energy |
| Cadmium | 1984 | Cadmium 99.95%, Free Market UK, C/LB | Metals |
| Camphor | 1986 | Camphor Natural BP, CIF $/KG | Exotics: Crude Drugs |
| Camphor Oil | 1986 | Camphor Oil-CIF Chinese $/KG | Exotics: Essential Oils |
| Cananga Oil | 1986 | Cananga Oil, CIF Java $/KG | Exotics: Essential Oils |
| Candelilla | 1986 | Candelilla, FOB Mexican $/tonne | Exotics: Waxes&Gums |
| Caprolactam | 1986 | Caprolactam CPL, CIF Import Price $/TN | Chemicals |
| Caraway Seed | 1989 | Caraway Seed, Dutch ex store, $/tonne | Exotics: Spices |
| Carbonated Wool | 1989 | Carbonated Wool No.64, Taiwan, $/KG | Fibres |
| Cardomoms | 1986 | Cardomoms, Guatemalan Sundried, CIF $/MT | Exotics: Spices |
| Carnuba | 1986 | Carnauba, Prime Yellow, FOB Brazil, $/tonne | Exotics: Waxes&Gums |
| Cascara | 1986 | Cascara Bark, CIF £/KG | Exotics: Crude Drugs |
| Cashew Kernal | 1986 | Cashew Kernal 320s, Delv/Monthly, £/tonne | Exotics: Edible Nuts |
| Cassia Lignea | 1989 | Cassia Lignea, Chinese Broken, CIF $/tonne | Exotics: Spices |
| Cassia Oil | 1986 | Cassia Oil, Chinese fwd, $/KG | Exotics: Essential Oils |
| Cattle | 1989 | Cattle Live Weight Pence/KG. | Livestock |
| Caustic Soda | 1986 | Caustic Soda, Liquid, FOB US-Gulf, Spot C/LB | Chemicals |
| Cedarwood Oil | 1986 | Cedarwood Oil, Chinese, Spot $/KG | Exotics: Essential Oils |
| Celery Seed | 1986 | Celery Seed, India ASTA, CIF $/TN | Exotics: Spices |
| Cement | 1986 | Taiwan-Cement North Area TW$/50KG | Build |
| Chicken | 1984 | Brolier Chickens, Dressed 'A' (NY), C/LB | Livestock |
| Chickpeas | 1989 | Chickpeas-Turkish 1%, Ex-Store UK £/MT | Seeds&Pulses |
| Chilles | 1989 | Chillies, Malawi, CIF $/tonne | Exotics: Spices |
| Chlorine | 1986 | Chlorine,Europe Spot (Domestic) C/LB | Chemicals |
| Chrome | 1989 | Chrome Metal 99%, Free Mkt, $/LB | Metals |
| Chromite | 1984 | Transvaal Friably Lumpy Chromite $/tonne | Metals |
| Cinnamon Bark | 1989 | Cinnamon Bark, Madagascar, CIF $/TN | Exotics: Spices |
| Cinnamon Leaf Oil | 1986 | Cinnamon Leaf Oil, Sri Lankan, Spot $/KG | Exotics: Essential Oils |
| Citronella Oil | 1986 | Citronella Oil, Chinese, CIF $/KG | Exotics: Essential Oils |
| Citrus Pulp | 1986 | Citrus Pulp Pellets, US, CIF Rotterdam $/tonne | Feeds |
| Clove Leaf Oil | 1989 | Clove Leaf Oil, C&F Madagascar $/KG | Exotics: Essential Oils |
| Cloves | 1986 | Cloves, Madagascar, CIF $/TN | Exotics: Spices |

| Name | Series Start | Trade Details | Category |
|---|---|---|---|
| Cobalt | 1989 | Cobalt,Broken Cath. 99.8%, Free Mkt, $/LB | Metals |
| Cochineal | 1989 | Cochineal, Spot £/KG | Exotics: Crude Drugs |
| Cocoa-Brazil | 1986 | Cocoa, Brazil Liquor, USNH Spot $/tonne | Seeds&Pulses |
| Cocoa-Ivory Coast | 1984 | Cocoa, Ivory Coast, $/MT | Seeds&Pulses |
| Coconut | 1989 | Desiccated Coconut, Philippine Macaroon Med. Grade, CIF Eu. $/TN | Exotics: Edible Nuts |
| Coconut Fibres | 1986 | Coconut Fibres, S.Lankan Mattres Faq (baled coir), $/tonne | Fibres |
| Coconut Oil | 1980 | Coconut Oil, Philippines, $/MT | Oilseeds, Oils&Fats |
| Coffee-Brazil | 1980 | Coffee, Brazilian, (NY) Cents/LB | Seeds&Pulses |
| Coffee-Columbia | 1986 | Coffee, Colombian mild arabicas, Cents/LB | Seeds&Pulses |
| Coir Yarn | 1986 | Coir Yarn, Indian Weaving 5, £/tonne | Fibres |
| Columbite | 1984 | Columbite Min 65% Cb2O5+Ta2O5, $/LB | Metals |
| Copper | 1980 | LME Copper High Grade, Cash £/tonne | Metals |
| Copra | 1980 | Copra, Philippines, $/tonne | Oilseeds, Oils&Fats |
| Copra Meal | 1994 | Copra Meal, Phi/Ind. Expellers, Afloat $/tonne | Feeds |
| Coriander Seed | 1986 | Coriander Seed, Moroccan, CIF £/tonne | Exotics: Spices |
| Corn | 1980 | Corn, No.2 Yellow, Cents/Bushel | Grains |
| Cotton | 1980 | Cotton,1 1/16Str, Low -Mid, Memphis C/LB | Fibres |
| Cottonseed Oil | 1989 | Cottonseed Oil, US Crude, FOB Missisippi C/LB | Oilseeds, Oils&Fats |
| Coumarin Oil | 1986 | Coumarin Oil, Chinese, Spot $/KG | Exotics: Essential Oils |
| Crude Oil | 1984 | Brent Crude Oil, Current Month, FOB $/BBL | Energy |
| Cummin Seed | 1989 | Cummin Seed, Iranian, CIF $/tonne | Exotics: Spices |
| Dimethyl Ter. | 1991 | Dimethyl Terephthalate. (Liq.) Cont., NWE DM/MT | Chemicals |
| Drawn Text. Yarn | 1986 | Drawn Text.Yarn 150D, Local TW/KG | Fibres |
| Eggs | 1984 | Eggs. Large White, (CHGO) Cents/Dozen | Seeds&Pulses |
| Elm Bark Powder | 1991 | Slippery Elm Bark Powder, Spot £/KG | Exotics: Crude Drugs |
| Ethanol | 1991 | Ethanol,Europe Spot Rdam T1 Bk $/HLT | Chemicals |
| Ethylene | 1986 | Ethylene,Europe Spot CIF NWE $/MT | Chemicals |
| Ethylene Di-Chloride | 1986 | Ethylene Di-Chloride W.Eur Spot C/LB | Chemicals |
| Ethylene Glycol | 1986 | Ethylene Glycol (OUCC), Taiwan, US$/TN | Chemicals |
| Eucalyptus Oil | 1986 | Eucalyptus Oil, Chinese, CIF $/KG | Exotics: Essential Oils |
| Fennel Seed | 1986 | Fennel Seed, Indian, CIF $/tonne | Exotics: Spices |
| Fenugreek Seed | 1989 | Fenugrek Seed, Indian, CIF $/tonne | Exotics: Spices |
| Ferro-Chrome | 1984 | Ferro-Chrome, 6-8% C 60% Cr, $/LB | Metals |
| Ferro-Manganese | 1984 | Ferro-Manganese, 78% Mn, $/TN | Metals |
| Ferro-Molybdenum | 1986 | Ferro-Molybdenum, 65-70% Mo, $/KG | Metals |
| Ferro-Silicon | 1984 | Ferro-Silicon,75% Si Lumpy, $/TN | Metals |
| Ferro-Titanium | 1986 | Ferro-Titanium, 70 % (4.5% Al), $/KG | Metals |
| Ferro-Tungsten | 1989 | Ferro-Tungsten,75 % W, R'Dam, $/KG | Metals |
| Ferrous Scrap | 1984 | A' Basic Ferrous Scrap Index, UK, Price lindex | Metals |
| Ferro-Vanadium | 1984 | Ferro-Vanadium,70 -80% V, $/KG | Metals |
| Fish Meal | 1989 | Fish Meal, Chilean Pellets/Meal, ex store UK, £/tonne | Feeds |
| Fish Oil | 1989 | Fish Oil, Crude. (promt) CIF Duty Paid R'DAM, £/tonne | Oilseeds, Oils&Fats |
| Freight Route 1 | 1989 | Baltic Freight Index , Route 1 - Dead Weight - Price Index | Freight |
| Freight Route 10 | 1989 | Baltic Freight Index , Route 10 - Dead Weight - Price Index | Freight |
| Freight Route 1A | 1991 | Baltic Freight Index , Route 1A - Dead Weight - Price Index | Freight |
| Freight Route 2 | 1989 | Baltic Freight Index , Route 2 - Dead Weight - Price Index | Freight |
| Freight Route 3 | 1989 | Baltic Freight Index , Route 3 - Dead Weight - Price Index | Freight |
| Freight Route 3A | 1991 | Baltic Freight Index , Route 3A - Dead Weight - Price Index | Freight |
| Freight Route 7 | 1989 | Baltic Freight Index , Route 7 - Dead Weight - Price Index | Freight |
| Freight Route 9 | 1989 | Baltic Freight Index , Route 9 - Dead Weight - Price Index | Freight |
| Fuel Oil | 1980 | Fuel Oil, No.2 (New York), C/Gallon | Energy |
| Gallium | 1991 | Gallium, Ingots 99.9%, $/KG | Metals |
| Gas Oil | 1986 | Gas Oil-EEC, CIF Cargos NWE U$/MT | Energy |
| Gasoline | 1984 | Gasoline,Unld. Reg. Non-Oxy,NY,C/Gallon | Energy |
| Genetian Root | 1986 | Genetian Root Cut, Spot £/KG | Exotics: Crude Drugs |
| Geranium Oil | 1986 | Geranium Oil, Chinese, CIF $/KG | Exotics: Essential Oils |
| Germanium | 1989 | Germanium, Metal 50 ohms/cm, $/KG | Metals |
| Germanium Dioxide | 1989 | Germanium, Dioxide 99.99%, $/KG | Metals |
| Ginger | 1986 | Ginger, Cochin, CIF $/tonne | Exotics: Spices |
| Ginger Oil | 1986 | Ginger Oil, Chinese, CIF $/KG | Exotics: Essential Oils |
| Gold | 1970 | Gold, Bullioon, $/Troy oz | Metals |
| Grapefruit Juice | 1994 | Fruit Juices, White Grapefruit, European, C&F Rotterdam $/tonne | Seeds&Pulses |
| Groundnut Oil | 1986 | Groundnut Oil, Any Origin, refined, ex tank Rotterdam, £/MT | Oilseeds, Oils&Fats |
| Gunmetal Scrap | 1984 | UK Scrap,Gunmetal Commercial, £/TN | Metals |
| Hazelnut Kernals | 1986 | Hazelnut Kernals, Turkish Levants Standard 1, ex store UK duty paid | Exotics: Edible Nuts |

| Name | Series Start | Trade Details | Category |
|---|---|---|---|
| Heavy Fuel Oil | 1986 | Heavy Fuel Oil 3.5%, CIF NWE $/MT | Energy |
| Heliotropin | 1986 | Heliotropin, Chinese, CIF $/KG | Exotics: Essential Oils |
| Hogs | 1984 | Hogs, Live Weight, Omaha, Avg.cwt., C/LB | Livestock |
| Hydrastis Roots | 1991 | Hydrastis Roots, Spot £/KG | Exotics: Crude Drugs |
| Indium Ingots | 1984 | Indium Ingots, Min 99.97%, $/KG | Metals |
| Ipecuanha Root | 1991 | Ipecacuanha Root Whole, Costa Rican, Spot £/KG | Exotics: Crude Drugs |
| Iridium | 1986 | Iridium, Min 99.9%, Free market $/Troy OZ | Metals |
| Jet Kerosene | 1986 | Jet Kerosene, Barges, FOB NWE $/MT | Energy |
| Jute | 1986 | Jute, Bangladesh BTD, FOB Mongla $/tonne | Fibres |
| Kola Nuts | 1986 | Kola Nuts, West Africa, CIF £/KG | Exotics: Edible Nuts |
| Lard | 1986 | Lard, EC packers, delivered UK, Boxed, £/tonne | Oilseeds, Oils&Fats |
| Lavender Spike Oil | 1986 | Lavender Spike Oil, Spot $/KG | Exotics: Essential Oils |
| Lead | 1980 | LME, Lead, Cash £/tonne | Metals |
| Lead Ore | 1984 | Ore, 70/80% Pb Lead Concentrate $/TN | Metals |
| Lemon Grass Oil | 1989 | Lemon Grass Oil, Cochin, CIF $/KG | Exotics: Essential Oils |
| Lentils | 1989 | Lentils, Turkish Red, CIF UK $/TN | Seeds&Pulses |
| Lime Oil | 1986 | Lime Oil, Mexican, CIF $/KG | Exotics: Essential Oils |
| Linseed Meal | 1989 | Linseed Meal, Arg/Uru 37/38%, CIF Ghent, £/tonne | Feeds |
| Linseed Oil | 1986 | Linseed/Flax, Linseed Oil, Crude, any origin, ex tank R'DAM, $/tonne | Oilseeds, Oils&Fats |
| Lithium Ore | 1984 | Lithium Ores Petalite, $/TN | Metals |
| Litsea Cubesa Oil | 1986 | Litsea Cubesa Oil, Chinese, CIF $/KG | Exotics: Essential Oils |
| Mace | 1989 | Mace, East Indian Broken No2, CIF $/tonne | Exotics: Spices |
| Magnesium | 1991 | Magnesium, 99.9% CIS $/TN | Metals |
| Maize | 1986 | Maize, Gluten Free, US Pellets 23/24, CIF Rotterdam $/tonne | Feeds |
| Maize/Corn Oil | 1989 | Maize/Corn Oil, Crude US, FOB Midwest, C/LB | Oilseeds, Oils&Fats |
| Manganese | 1984 | Manganese, Electro Flake 99.7%, $/TN | Metals |
| Manganese Ore | 1984 | Manganese Ore, 48/ 50% Mn, $/TN | Metals |
| Meat and Bone Meal | 1986 | Meat and Bone Meal, 45/48% protein, Eng/Irish, £/tonne | Feeds |
| MEG | 1986 | Mono Ethylene Glycol, US Gulf Spot FOB C/LB | Chemicals |
| Menthol Oil | 1986 | Menthol Oil, Chinese, Spot $/KG | Exotics: Essential Oils |
| Mercury | 1984 | Mercury, Min 99.99 %, $/Flask | Metals |
| Methanol | 1986 | Methanol, US Gulf Barges Domestic C/GAL | Chemicals |
| Millet | 1989 | Millet, N.American 99.9% Purity, Ex-Store UK £/tonne | Seeds&Pulses |
| Molybdenite | 1984 | Ore, Molybdenite Conc, CIF $/LB | Metals |
| Molybdenum | 1984 | Molybdenum Drum. Oxide, Euro $/LB | Metals |
| Musk | 1986 | Musk, Chinese, Spot Xylol $/KG | Exotics: Essential Oils |
| Naphtha | 1986 | Naphtha, Europé, CIF $/MT | Energy |
| Natural Gas | 1994 | Natural Gas, Henry Hub, $/MMBTU | Energy |
| Nickel | 1980 | LME-Nickel Cash US$/MT - A.M. OFFICIAL | Metals |
| Nutmeg | 1994 | 'Nutmeg, Indonesia sound shrivels, Spot $/tonne | Exotics: Spices |
| Nux Vomica | 1989 | Nux Vomica, Spot £/KG | Exotics: Crude Drugs |
| Nylon Yarn | 1986 | Nylon Yarn 70D, Taiwan, TW/KG | Fibres |
| Oats | 1980 | Oats, No.2 Milling, Minneapolis, C/BU | Grains |
| Olive Oil | 1994 | Olive Oil, X.virgin<1%FFA, ex tank UK, £/tonne | Oilseeds, Oils&Fats |
| Orange Juice | 1994 | Fruit Juices, Orange juice, European, C&F Rotterdam $/tonne | Seeds&Pulses |
| Orange Pera | 1991 | Orange Pera, Brazilian, Spot $/KG | Exotics: Essential Oils |
| Palladium | 1989 | Palladium, P.M. Fixing, $/troy oz | Metals |
| Palm Kernal Cake | 1991 | Palm Kernal Cake, Ind./Mal.21/23%, ex store Avonmouth, £/tonne | Feeds |
| Palm Kernal Oil | 1986 | Palm Kernal Oil, Mal. CIF R'dam, $/tonne | Oilseeds, Oils&Fats |
| Palm Oil | 1984 | Palm Oil, (Malaysian), $/MT | Oilseeds, Oils&Fats |
| Paper Pulp | 1970 | NBSK Pulp (CIF W. Europe) $/TN | Paper |
| Paper, Corrugated | 1989 | Corrugated Paper, Medium, TW/TN | Paper |
| Patchouli Oil | 1994 | Patchouli Oil, Indonesian, Spot $/KG | Exotics: Essential Oils |
| Peanuts | 1986 | Peanuts/Groundnuts, Argentine 40/50, CIF $/tonne | Exotics: Edible Nuts |
| Pepper-Black | 1989 | Black Pepper, Brazil, Spot $/tonne | Exotics: Spices |
| Peppermint Oil | 1986 | Peppermint Oil, Chinese, Spot $/KG | Exotics: Essential Oils |
| Pepper-White | 1986 | White Pepper, Sawarak, Faq shipment $/tonne | Exotics: Spices |
| PET-Bottle Gde | 1991 | PET (Bottle Gde),NWE Spot FD DM/KG | Chemicals |
| Petitgrain Oil | 1986 | Petitgrain Oil, FOB Paraguay $/KG | Exotics: Essential Oils |
| PET-Staple Fibre | 1986 | PET Staple Fibre, Taiwan, Export, TW/KG | Fibres |
| PGE | 1989 | Propylene Glycol Ether, Methanol Based, Spot FD NWE DM/MT | Chemicals |
| Phosphor Copper | 1984 | Phosphor Copper, 15% P, UK £/TN | Metals |
| Phosphor Tin | 1984 | Phosphor Tin, 5% P, UK £/TN | Metals |
| Pigs Live Weight | 1989 | Pigs, Live Weight, Pence/KG. | Livestock |
| Pimento | 1986 | Pimento, Jamaican, Spot $/tonne | Exotics: Spices |

| Name | Series Start | Trade Details | Category |
|---|---|---|---|
| Pimento Leaf Oil | 1994 | Pimento Leaf Oil, Spot $/KG | Exotics: Essential Oils |
| Pineapple Juice | 1994 | Fruit Juices, Pineapple juice, European, C&F Rotterdam $/tonne | Seeds&Pulses |
| Platinum | 1980 | Platinum, London Free Market, $/Troy oz | Metals |
| Polyacryloinitrile | 1986 | Polyacrylonitrile, Fibre Yarn GM, Staple, Taiwan, TW/KG | Fibres |
| Polye HDPE-Inj | 1986 | Polyethylene HDPE-Inj, NWE Spot FD DM/KG | Chemicals |
| Polye LDPE-GP Film | 1986 | Polyethylene LDPE-GP Film, Spt FD NWE DM/KG | Chemicals |
| Polye LLDPE | 1989 | Polyethylene LLDPE,Europe Spot HEX DM/KG | Chemicals |
| Polystyrene-Hips | 1986 | Polystyrene-Hips,NWE Spot FD DM/KG | Chemicals |
| Poppyseed | 1991 | Poppyseed, Euro Blue, ex store UK, £/tonne | Exotics: Spices |
| Pork Bellies | 1980 | Pork Bellies, 12-14 Lbs, (Mid-US), C/LB | Livestock |
| PP Copolymer | 1986 | Polypropylene Copolymer,Spot FD NWE DM/KG | Chemicals |
| PP Homopolymer Inj | 1991 | Polypropylene Homopolymer Inj, Spot H.Kong $/MT | Chemicals |
| Propane | 1984 | Propane, M.Belvieu, C/Gallon | Energy |
| Propylene (C) | 1989 | Propylene (C-Grade),Spot CIF NWE C/MT | Chemicals |
| Propylene (P) | 1986 | Propylene (P-Grade),Spot CIF NWE DM/MT | Chemicals |
| Propylene (R) | 1991 | Propylene (R-Grade),US-Gulf Contract C/LB | Chemicals |
| Propylene Oxide | 1991 | Propylene Oxide,Spot FD NWE DM/MT | Chemicals |
| PTA | 1986 | Terephthalic Acid ,Contract Price $/TN | Chemicals |
| PVC | 1991 | PVC,Spot Hong Kong $/MT | Chemicals |
| Rapeseed | 1994 | Rapeseed, UK Delivered. Erith,Buyer £/MT | Oilseeds, Oils&Fats |
| Rapeseed Meal | 1986 | Rapeseed Meal, UK produced HP 37%, single-low, resell Erith, £/MT | Feeds |
| Rapeseed Oil | 1986 | Rapeseed Oil, Crude Dutch/EC, FOB Mill, DM/MT | Oilseeds, Oils&Fats |
| Rayon, Staple Fibre | 1989 | Rayon, Staple Fibre, Taiwan Local price, TW/KG | Fibres |
| Rhodium | 1984 | Rhodium, 99.9%, Europe $/Troy OZ | Metals |
| Rice | 1989 | Rice, Italien No.1, Milled round grain, £/TN | Grains |
| Rice/Bran | 1989 | Rice bran, India/Pak. ex Avonmouth/Newport, £/tonne | Feeds |
| Rubber | 1970 | Rubber, LONDON, Cash Pence/KG | Seeds&Pulses |
| Ruthenium | 1986 | Ruthenium, 99.9%, Europe $/Troy OZ | Metals |
| Saffron | 1991 | Saffron, Spanish, CIF London £/KG | Exotics: Spices |
| Sandalwood Oil | 1991 | Sandalwood Oil, E Indian, CIF $/KG | Exotics: Essential Oils |
| Sasparilla Root | 1989 | Sasparilla Roots, Whole, CIF £/KG | Exotics: Crude Drugs |
| Sassafras Oil | 1989 | Sassafras Oil, Chinese, CIF $/KG | Exotics: Essential Oils |
| Selenium | 1984 | Selenium, 99.5%, Free Market $/LB | Metals |
| Sesame Seed | 1989 | Sesame Seed, Guatemalan hulled 99.9% Purity, pence/LB | Exotics: Spices |
| Sheep | 1994 | Sheep, Live Weight, Pence/KG | Livestock |
| Silico-Manganese | 1984 | Silico-Manganese, Lumpy, $/TN | Metals |
| Silicon | 1989 | Silicon, US Free Market, C/LB | Metals |
| Silver | 1970 | Silver, FIX LBM Cash C/Troy oz | Metals |
| Sisal | 1986 | Sisal, Brazil No.2, FOB Bahia, $/tonne | Fibres |
| Soda Ash | 1994 | Soda Ash, Dense, USG Spot FOB Blk $/MT | Chemicals |
| Soyabean Oil | 1980 | Soyabean Oil, Crude Decatur, C/LB | Oilseeds, Oils&Fats |
| Soyabeans | 1980 | Soyabeans, No.1 Yellow, C/Bushel | Oilseeds, Oils&Fats |
| Soyameal | 1989 | Soyameal, Argentine pellets 45%, CIF R'dam $/tonne | Feeds |
| Spearmint Oil | 1986 | Spearmint Oil, Chinese 80%, CIF $/KG | Exotics: Essential Oils |
| Steel,Galvanised Coi | 1984 | Steel,Galvanised Coils $/TN | Metals |
| Steel,Merchant Bars | 1984 | Steel,Merchant Bars $/TN | Metals |
| Steers | 1970 | Steers, Omaha, (CME), Choice avg.cwt., C/LB | Livestock |
| Styrene | 1986 | Styrene, T2, Spot FOB Rdam $/MT | Chemicals |
| Sugar | 1970 | Raw Sugar, London, CIF $/MT | Seeds&Pulses |
| Sunflowerseed | 1991 | Sunflowerseed, US hulled, ex store UK, £/tonne | Exotics: Spices |
| Sunflowerseed Meal | 1989 | Sunflowerseed Meal, Arg/Uru pellets 34/35%, ex-store Liverpool, £/MT | Feeds |
| Sunflowerseed Oil | 1989 | Sunflowerseed Oil, RBD, ex tank UK, Broker price, £/MT | Oilseeds, Oils&Fats |
| Tallow | 1986 | Tallow, UK Grade 6, Basis 15% FFA, Delivered UK, £/tonne | Oilseeds, Oils&Fats |
| Tallow, extra fancy | 1991 | Tallow, Extra Fancy, inedible in bulk, FOB US-Gulf $/TN | Oilseeds, Oils&Fats |
| Tantalite Ore | 1989 | Tantalite Ore, 25/40%, CIF NWE $/LB | Metals |
| Tapioca | 1989 | Tapioca, Thai Hard Pellets, Spot, FOB R'dam, DM/100KG | Feeds |
| Tea-Kenya | 1986 | Tea, London Auction, Kenya dead, pence/KG | Seeds&Pulses |
| Tea-Sri Lanka | 1986 | Tea, London Auction, Sri Lanka dead, pence/KG | Seeds&Pulses |
| Terpineol | 1986 | Terpineol M.U. Chinese, CIF $/KG | Exotics: Essential Oils |
| Tin | 1989 | Tin 99.85%, LME, Cash A.M. Official $/MT | Metals |
| Titanium | 1984 | Titanium, Americas Sponge, $/LB | Metals |
| Titanium Dioxide | 1991 | Titanium Dioxide, Spot FD NWE DM/KG | Chemicals |
| Titanium Ore | 1984 | MB-Titanium Rutile Conc 95% $/TN | Metals |
| Titanium Scrap | 1984 | Scrap,Titanium Turnings (<0.2% Sn), UK, £/LB | Metals |
| Toluene | 1991 | Toluene,Spot CFR NE Asia $/MT | Chemicals |

| Name | Series Start | Trade Details | Category |
|---|---|---|---|
| Tonquin | 1989 | Tonquin, Spot £/KG | Exotics: Crude Drugs |
| Tung Oil | 1986 | Tung Oil, any origin, ex tank R'dam, $/tonne | Oilseeds, Oils&Fats |
| Tungsten | 1991 | Tungsten, APT, Euro Free Market $/TN | Metals |
| Tungsten Ore | 1984 | Tungsten Ore Min 65% WO3, $/TN | Metals |
| Turmeric | 1986 | Turmeric, Madras Fingers, CIF $/tonne | Exotics: Spices |
| Valerian Root | 1986 | Valerian Root, Whole, European, Spot, £/KG | Exotics: Crude Drugs |
| Walnuts | 1986 | Walnuts, Indian, Light Amber Broken, ex store UK duty paid, £/tonne | Exotics: Edible Nuts |
| Vanadium | 1989 | Vanadium, Pentoxide 98%, $/LB | Metals |
| VCM | 1986 | Vinyl Chloride Monomer,USG Spot FOB C/LB | Chemicals |
| Vetivert Oil | 1986 | Vetivert Oil, Indonesian, CIF $/KG | Exotics: Essential Oils |
| Wheat-No.2 Hard | 1980 | Wheat, No.2 Hard (Kansas), C/BU | Grains |
| Wheat-No.2 Soft | 1984 | Wheat, No.2, Soft Red (Chicago), C/BU | Grains |
| Wheat-Spring | 1980 | Wheat, Spring, 14%-Pro, (Minn) C/BU | Grains |
| Wild Cherry Bark | 1986 | Wild Cherry Bark, Spot £/KG | Exotics: Crude Drugs |
| Witch Hazel Leaves | 1986 | Witch Hazel Leaves, Spot £/KG | Exotics: Crude Drugs |
| Wooltops | 1970 | Wooltops, London, Pence/KG | Fibres |
| Xylenes | 1994 | Xylenes, USG Spot FOB Barges $/GAL | Chemicals |
| Zinc | 1989 | Zinc 99.995%, LME-SHG, Cash A.M. Official $/MT | Metals |
| Zinc Sulphide | 1984 | Ore, Zinc Sulphide 49/ 55% Zn Conc, $/TN | Metals |
| Zircon | 1991 | Zircon, Foundary Grade, Bulk, FOB A$/TN | Metals |

Notes: - 280 series.

- All series are extracted from the Datastream commodity data base and end March 2000.

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# A mean reverting stochastic volatility option-pricing model with an analytic solution

## (Essay 4)

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### Abstract

In this paper we derive a closed form approximation to a stochastic volatility option-pricing model and propose a variant of EGARCH for parameter estimation. The model thereby provides a consistent approach to the problem of option pricing and parameter estimation. Using Swedish stocks, the model provides a good fit to the heteroscedasticity prevalent in the time series. The stochastic volatility model also prices options on the underlying stock more accurately than the traditional Black-Scholes formula. This result holds for both historic and implied volatility. A large part of the volatility smile that is observed for options of different maturities and exercise prices is thereby explained.

*Keywords*: Stochastic Volatility, EGARCH, Option-Pricing

*JEL classification code*: C13, G13

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# 1. Introduction

One of the assumptions behind the Black-Scholes option pricing model that is clearly violated in practice is that of constant volatility. Not only empirical data, in form of fat tails in the return distribution, but also economic theory support models where the volatility of stock returns changes in response to investment and financing decisions made within the company. Any changes in operating or financial leverage that change the cash flow to shareholders in a systematic way will also affect the volatility. Macroeconomic events may also affect volatility, either through changes in the discount factor or their effect on company profits. There are also many cyclical effects that will combine to make the volatility appear stochastic. For instance, volatility is higher on Mondays than other weekdays simply because more information has been assembled during weekends. Regular releases of economic statistics may also affect volatility on these days. Models that allow for stochastic volatility have been around since the late eighties. It has been noted by several authors that the stochastic volatility model

$$dS = \alpha S dt + \sqrt{V} S dw$$
$$dV = \kappa(\gamma - V)dt + \sigma V dz,$$

is consistent with the GARCH(1,1) estimation of the parameters $\{\kappa, \gamma, \sigma\}$.[1] The problem is, however, that there exists no closed form solution for option prices based on these assumptions. In this paper, we instead propose the model

$$dS = \alpha S dt + \sqrt{V} S dw \tag{1.1}$$
$$dV = \kappa(\gamma + \frac{\sigma^2}{2\kappa} - \ln V)V dt + \sigma V dz, \tag{1.2}$$

[1] The idea to use GARCH, generalised autoregressive conditional heteroscedasticity, for parameter estimation derives from Engle and Lee (1996), but can also be traced to Nelson (1990). For a particularly simple exposition, see Veronesi (2000). In a similar fashion, Duan (1995) assumes that the stock price variance follows a GARCH process and develops corresponding option pricing formulas.

where the Wiener processes $dw$ and $dz$ are uncorrelated. This model is characterised by the standard assumption that the stock price $S$ follows a geometric Brownian motion, with the additional twist that the volatility parameter $\sqrt{V}$ is stochastic. The stochastic properties are determined in terms of the variance $V$, which is lognormal and mean reverting. For this model, it is possible to obtain an analytic approximation to the European option price and use a new variant of Exponential-GARCH for parameter estimation. By applying Ito's lemma to the log of the variance in equation (1.2) we obtain the Ornstein-Uhlenbeck process

$$d\ln V = \kappa(\gamma - \ln V)dt + \sigma dz. \tag{1.3}$$

As the Ornstein-Uhlenbeck process can be solved explicitly it is possible to derive a closed form approximation to the option pricing problem in continuous time. Parameters in the model have to be estimated from discrete time data, however. The discrete time equivalents of (1.1) and (1.3) are,[2]

$$y_t = \sqrt{V_t \Delta t}\, w_t \tag{1.4}$$

$$\ln V_{t+1} = a_0 + a_1 \ln V_t + a_2\, \varepsilon_t. \tag{1.5}$$

The processes (1.4) and (1.5) are in econometrics known as the lognormal stochastic volatility SV(1,1) model. Unfortunately, there is no standardised way of estimating the parameters of SV models, and therefore no standardised way of estimating the parameters of (1.3) either. The principal disadvantage of SV models is that the likelihood function is difficult to derive and maximise. For the GARCH family of processes, on the other hand, maximum likelihood estimation is straightforward. Our suggestion is therefore to match the moments of (1.5) to the moments of the EGARCH model[3]

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[2] $y_t$ is the change in the asset price $S$ when the drift has been eliminated, see section 5 for further details. Equation (5.5), expresses $\{a_0, a_1, a_2\}$ in terms of the original continuous time parameters.

[3] The variance is denoted $\sigma_t^2$ instead of $V_t$ in order to separate the processes.

$$\ln \sigma_{t+1}^2 = \psi_0 \ln \bar{\sigma}^2 + (\psi_1 + \psi_2) \ln \sigma_t^2 + \psi_2 \ln(\varepsilon_t^2 + 1). \tag{1.6}$$

The difficult problem of estimating stochastic volatility parameters is thereby reduced to estimating a GARCH model. The idea of matching the moments of some difficult function to the moments of a well-known distribution has previously been successfully applied to the pricing of arithmetic Asian options. See Turnbull and Wakeman (1991) and Milevsky and Posner (1998). It has, however, also been used by Engle and Lee (1996) to estimate volatility processes, an idea that is further developed in this paper.

For the standard Black-Scholes formula, the closest conformity between observed and theoretical option prices are obtained when implied volatility is used to price the options. The use of implied volatility can be extended to include stochastic volatility models as well. Through iteratively backing out the instantaneous variance rate $V_0$ and the long-run average variance rate $\gamma$ from the pricing formula, using the prices of two traded options as inputs, we are able to provide an even closer fit to observed option prices.

Thus, this paper contributes to the existing literature in three ways. First of all it derives an analytical approximation to European option prices based on the model (1.1) and (1.2). Previous studies have relied on numerical methods. Secondly, it provides parameter estimates based on a new variant of EGARCH. Initial tests indicate that this new variant is accurate, robust and easy to apply. The third contribution is to combine implied volatility with a stochastic volatility option-pricing model.

The paper is structured as follows. Section 2 briefly reviews some of the previous studies made of stochastic volatility option-pricing models. These models are based on both option theory and econometrics, and are presented with the objective of acquainting the reader with the different issues of stochastic volatility that researchers have focused on. Section 3 presents the new model and accounts for the approximations made. The accuracy of the approximations is tested in section 4, by means of Monte Carlo simulations. The issue of parameter estimation is dealt with in section 5 where the new EGARCH model is developed.

Theoretical option prices for the 20 most actively traded stocks on the Stockholm stock exchange are compared to the actual market prices in section 6. Comparisons are made between the Black-Scholes and the stochastic volatility option-pricing model using both historic and implied volatility. Section 7 concludes by discussing the findings and their implications. All longer derivations are deferred to the appendices.

## 2. Theoretical preliminaries and previous research

Models that allow for stochastic volatility are nothing new in the world of option pricing. Sparked by several articles in 1987, papers have regularly appeared in finance journals. A common theme in most papers is that the models are based on the geometric Brownian motion, as in the Black-Scholes model, but with the extension that the volatility parameter in itself is stochastic. Scott (1987) accounts for empirical evidence that volatility changes are not independent over time but instead strongly autocorrelated and possibly mean reverting. One possibility to incorporate these characteristics is by letting the volatility parameter in the stochastic process also be stochastic (and mean reverting). This would lead to a so-called mixture-of-normals model, as represented by equation (1.4), $y_t = \sqrt{V_t \Delta t}\, w_t$. Shocks to the return are caused by both the lognormal variable $V_t$ and the normal variable $w_t$. The shocks are therefore more "fat tailed" than the normal distribution. This leptokurtosis, in technical parlance, is prevalent in most financial time series. See, for example, Fama (1965) and Kon (1984).

By letting volatility be stochastic we introduce two sources of uncertainty, the Wiener processes $dw$ and $dz$. This complicates option pricing. It is no longer possible to form a riskless hedge by combining a bond and the underlying asset. A third contract is needed to price the option. However, the price of this third contract is precisely what we are trying to determine. The simplest way around this problem is to follow Hull and White (1987). They were the first to provide a closed form option pricing formula by assuming that both the asset and the volatility follow geometric Brownian motions that are independent of each other. They also assumed that volatility has no systematic risk in order to disallow risk preferences entering into the problem. This is equivalent to saying that the market price of volatility risk is zero.

Under these assumptions, Hull and White show that the price of a European option is the Black-Scholes price integrated over the mean variance rate. Thereafter they utilise a Taylor series expansion to approximate this integral. Hull and White (1988) relax some of the assumptions in their previous article by allowing volatility to be mean reverting and instantaneously correlated with the underlying asset. Furthermore they derive a set of differential equations that the pricing bias (compared to the Black-Scholes price) must obey. A series solution to this bias is provided and Monte Carlo simulations show that the solution is quite exact, at least for short-term options.

Allowing for mean reversion and instantaneous correlation is a marked improvement over their 1987 paper, although it is difficult to infer how parameters should be estimated in the model. It is possible that the approach generalises to other volatility processes, for which parameter values are easier to infer. However, the complexity of the differential equations makes such a generalisation difficult to derive.

Heston (1993) generalises Hull and White (1988) in that he allows the interest rate to be stochastic. His approach is somewhat different, utilising characteristic functions of the state price densities. The resulting formula is in integral form.

Stein and Stein (1991) criticise the Taylor series expansion made by Hull and White, claiming that it is not clear that a series expansions around the point where volatility is non-stochastic works when the volatility is highly stochastic. The alternative that Stein and Stein propose is to express the terminal stock price in integral form and thereafter calculate the option price as an expectation over the terminal stock price. The double integral that results is a much simpler numerical problem than solving a partial differential equation, but calling the solution analytic, as Stein and Stein do, might be an exaggeration. As an

Ornstein-Uhlenbeck process governs the volatility, there is also a possibility, however small, of volatility becoming negative.

Melino and Turnbull (1990) study foreign currency options on the Canada - U.S. exchange rate at the Philadelphia Stock Exchange. The B-S model constantly overestimates call options on this particular exchange rate by around 15 percent, but applying a stochastic volatility model eliminates most of this bias. The differences are striking although the study is somewhat open to the criticism of data dredging, since some parameters are chosen as to provide the best fit. In their approach, mean reverting processes govern both the exchange rate and the volatility, and option prices are determined by solving the underlying partial differential equation numerically. To estimate the parameters, they follow Hansen's (1982) generalised method of moments and choose among a number of different moment candidates. One advantage of the generalised method of moments is that it is possible to determine the standard errors of the estimates. There is only weak correlation between the Canada - U.S. exchange rate and the volatility processes, and not much evidence that the exchange rate is mean reverting. Volatility, on the other hand, is stochastic and quickly mean reverting.

These are encouraging results, suggesting that the model proposed in this paper can be used to price currency options. Less encouraging is the fact that option prices are quite sensitive to the market price of volatility risk. The market price of volatility risk must be zero for the risk neutrality assumption to hold and the analytical approximation to work. The best fit to observed prices are obtained when the market price of volatility risk $\lambda$ is slightly negative, $-0.1$ or $-0.2$. One may well question the realism of having a market price for bearing volatility risk that is negative, but this objection is never discussed in the paper.

Chesney and Scott (1989) examine options on the dollar/Swiss franc exchange rate traded in Geneva. Contrary to Melino and Turnbull they find a best fit to observed option prices when $\lambda$ is zero or slightly positive. An even better fit is obtained when the B-S model is used together with implied volatility from other traded options and the volatility is revised daily. (Using the Black-Scholes model with a constant historical volatility results in a very bad fit to observed option prices.) There may be several reasons why the B-S model with an implied volatility that is revised daily works so well. One suspicion is of course that market makers and participants determine option prices in this way. Another reason might be that the B-S model serves as a good proxy, whatever the true process of the volatility is.

Melino and Turnbull (1995) focus on hedging of long-term currency options. Long-term options rarely trade and in the case they do, the market is very illiquid. An option writer is thereby forced to hedge his position by creating an (opposite) synthetic option by constructing a replicating portfolio whose payment match the payment of the original option. In an idealised world where trading can take place continuously and the stochastic process followed by the exchange rate is known, there is no difference in value between the synthetic option and the original option. The mismatch between the two option values thereby becomes a measure of how well the hedging strategy works when the replicating portfolio is revised at discrete intervals and the true stochastic process is unknown.

Through a number of simulations, Melino and Turnbull conclude that discrete-time revision of the portfolio does not in itself cause any substantial hedging error, not even for long-term options. They also report that the common practice of using the short-term implied volatility from traded options to price long-term non-traded options does not work very well when volatility is stochastic. This is in contradiction to Chesney and Scott (1989). Pricing errors are of the magnitude 15 percent for five-year options, whereas the mean hedging error can be as large as 50 percent of the option value.

That a misspecified model causes mispricing is no surprise in itself, what is alarming is the magnitude of the error. The root of the problem is that with only one underlying asset, the exchange rate, we are unable to hedge against changes in the volatility. Option traders would phrase this as only delta hedging being possible, not vega hedging. Introducing another option as a second underlying asset makes vega hedging feasible. However, using the vega of the constant volatility model does not work. In fact, the mean hedging error increases to 500 percent. The message from Melino and Turnbull is clear. Hedging must be based on a stochastic volatility model when volatility is stochastic. They do not examine if delta hedging is enough or whether another option must be added to the hedging-scheme, but Hull and White (1988) report that another option must be included and vega hedged, in order to achieve adequate hedging accuracy.

Several authors have also attacked the subject of stochastic volatility from an econometric viewpoint. Scott (1987) and Wiggins (1987) use the method of moments to estimate the parameters of the stochastic volatility model. Wiggins applies the procedure on eight American stocks and two indices, using daily observations between July 1962 and December 1984. He finds a higher persistence of volatility shocks, i.e. a slower mean reversion, for the indices compared to individual stocks and also a lower "volatility of volatility". Neither of these effects seem unreasonable from an economic point of view. Not just volatility, but also changes in volatility should be diversified in larger portfolios, and macroeconomic uncertainty, regarding business cycles for example, might be more persistent than firm-specific uncertainty. Wiggins also finds that parameter estimates are sensitive to the sampling interval of the data observations. This result is quite displeasing but may partly be explained by the fact that volatility is by necessity assumed constant within the sampling period. Interestingly, he only detects significant correlation between the asset and volatility processes for the indices. This is interesting in its own right, since the analytic solution in this paper assumes independence between the processes.

Harvey, Ruiz and Shephard (1994), suggest that the estimation should be set up as a quasi-maximum likelihood problem. The Kalman filter can then be applied to compute the constants. The technique can be applied independently of whether the log of the variance process is a random walk or mean reverting and the authors use exchange rates as a demonstration.

Andersen, Chung and Sørensen (1999) propose the efficient method of moments as the base of parameter estimations. The efficient method of moments is an extension of the generalised method of moments that approaches the efficiency of maximum likelihood for large samples. The main advantage of this method seems to be its robustness. Quite often, the generalised method of moment fails to converge to the correct solution and sometimes it does not converge at all. Anderson, Chung and Sørensen report that for a sample size of 500 observations, they encountered this failure in 30 percent of their simulations. The efficient method of moments, by comparison, did not encompass any problems.

The above approaches to parameter estimation have all been variants of stochastic volatility (SV) models, where the volatility is an unobserved and serially correlated process. SV models mostly utilise the unconditional moments of the process. An alternative is to condition the variance on the realised return and the variance estimate of the previous period. This is what GARCH or generalised autoregressive conditional heteroscedasticity models are all about. The GARCH(1,1) model

$$\sigma_{t+1}^2 = c_0\overline{\sigma}^2 + c_1\sigma_t^2 + c_2\, y_t^2$$

is closely related to forecast techniques based on exponential smoothing. The variance forecast for the next period depends on the variance and the realised return $y_t$ in this period. The last observation thereby has the greatest influence on the forecast for the next period and the influence decreases geometrically for observations further distant. The principal advantages of GARCH are that it allows for time-varying volatility and volatility clustering (the phenomenon that some periods exhibit a volatility that is persistently higher or lower than the average) and that the likelihood function is easy to derive and maximise.

The family of GARCH models has during the last decade become the most popular way to model changing, i.e. heteroscedastic, volatility. This success is mostly explained by its ability to characterise volatility changes. It is therefore easy to be indulgent towards the rather questionable assumption that variance is a deterministic function of historical returns and not an independent stochastic process as in the SV models.

Another reason for its success is the number of statistical software packages that incorporates routines for GARCH estimation. A recent paper by Brooks, Burke and Persand (2001) provides an overview of such packages. Generalised autoregressive conditional heteroscedasticity, or GARCH in short, was developed in the mid eighties and has since then seen a number of extensions. Exponential GARCH, EGARCH, was introduced by Nelson (1991) as a way of modelling non-symmetric shocks to volatility.

$$\ln\sigma_{t+1}^2 = \psi_0 \ln\overline{\sigma}^2 + (\psi_1 + \psi_2)\ln\sigma_t^2 + \psi_2\left[\left|\varepsilon_t\right| - \chi\,\varepsilon_t\right] \qquad (2.1)$$

The asymmetry is determined by the parameter $\chi$. When $\chi \neq 0$, not only the size, but also the sign of $\varepsilon_t$ has importance. Provided that $\chi$ is positive, a negative shock will increase the variance more than a positive shock. Let $\varepsilon_t$ be the (normalised) return on stocks. A price decline would be accompanied by increasing volatility and a price increase with lower volatility. This is also something that has been observed in the stock market, (Hamilton, 1994, p. 668).[4]

Nevertheless, later studies show no particular advantage of EGARCH over GARCH, not even when heteroscedasticity is asymmetric. Both models do a very good job in explaining volatility. Bracker and Smith (1999) examine the volatility in the copper futures market. EGARCH and GARCH do an equally good job of explaining the leptokurtic and negatively skewed return. For the same futures series, Asymmetric GARCH and a random walk do not work. Gokcan (2000) reaches much the same conclusions when he examines stock indices from emerging markets. Although the return is negatively skewed, GARCH performs slightly better than EGARCH, both for estimations and forecasts. Both models work well, however.

In order to make EGARCH compatible to the stochastic volatility model (1.1) and (1.3), we have to make some alterations from the original EGARCH model of Nelson (1991). The innovations must first of all be symmetric, $\chi = 0$. In view of how well the original GARCH model works, this should be a minor restriction and translates into the requirement of independent Wiener processes. The second alteration is the distribution of the innovations. Nelson suggested using the absolute value of the generalised error distribution GED for the error term, but in this paper we follow the common practice to use the normal

[4] Hamilton, page 672, also provides a short overview over some of the earlier GARCH-studies and has a short discussion of different approaches.

distribution. The absolute value $|\varepsilon_t|$ is substituted for the log of the displaced $\chi^2$ distribution, $\ln(\varepsilon_t^2+1)$, in order to obtain a model that is compatible with the proposed continuous time model. The addition of 1 is needed to ensure that the whole expression is positive and to avoid taking the logarithm of zero when $\varepsilon_t = 0$. We thus end up with the model,

$$\ln\sigma_{t+1}^2 = \psi_0 \ln\bar{\sigma}^2 + (\psi_1 + \psi_2)\ln\sigma_t^2 + \psi_2 \ln(\varepsilon_t^2 + 1). \tag{2.2}$$

That the resulting EGARCH model actually is compatible with the proposed continuous time model is shown in section 5.

For some reason, option papers and econometric papers diverge on whether it is the volatility or the variance that should be modelled explicitly. Option papers on stochastic volatility mostly model the volatility, whereas econometric papers model the variance. There should be no particular preference for any of these two approaches. If the variance is mean reverting, then also the volatility will tend towards its long-run average. In this paper, we explicitly model the variance as mean reverting, as that is the idea derived from using GARCH(1,1) for stochastic volatility option pricing. To the best of the author's knowledge, no one has tried to value options analytically based on the model (1.1)-(1.2) and used EGARCH for parameter estimation. That is the purpose of the present paper.

# 3. The model

Starting with the model (1.1) and (1.2),

$$dS = \alpha S dt + \sqrt{V} S dw$$
$$dV = \kappa(\gamma + \frac{\sigma^2}{2\kappa} - \ln V) V dt + \sigma V dz,$$

we invoke the normal assumptions from the Black-Scholes analysis, where the market is arbitrage-free and there are no transaction costs or restrictions on borrowing. Trading can take place continuously and the short-term interest rate is deterministic. In addition, it is also assumed that the Wiener processes driving the diffusions are uncorrelated with each other and that changes in the variance rate have no systematic risk.

Under these assumptions, Hull and White (1987) show that the European call option price is the Black-Scholes price integrated over the mean variance rate during the life of the option,

$$f(S_0, V_0) = \int C(\bar{V}) h(\bar{V}) d\bar{V}. \tag{3.1}$$

$\bar{V}$ denotes the mean variance rate, $\bar{V} = \frac{1}{T}\int_0^T V_t \, dt$, and $h(\bar{V})$ is the probability density function of $\bar{V}$. $C(\bar{V})$ is the well known Black-Scholes formula for the price of a European call option paying a continuous dividend yield.

$$C(\bar{V}) = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) \tag{3.2}$$
$$d_1 = \frac{\ln(S_0/K) + (r - \delta + \bar{V}/2)T}{\sqrt{\bar{V}T}}$$
$$d_2 = d_1 - \sqrt{\bar{V}T}$$

Still following the analysis by Hull and White, we expand $C(\bar{V})$ in a Taylor series expansion of $\bar{V}$ around its expected value $E[\bar{V}]$ to obtain

$$f(S_0,V_0) = C(E[\bar{V}]) + \frac{1}{2}\frac{\partial^2 C}{\partial \bar{V}^2}\bigg|_{E[\bar{V}]} Var[\bar{V}] + \frac{1}{6}\frac{\partial^3 C}{\partial \bar{V}^3}\bigg|_{E[\bar{V}]} Skew[\bar{V}] + \dots \quad (3.3)$$

It is shown in the following section that the series in (3.3) converges quickly except when options are deep-out-of-the-money and of short maturity. Only the first two terms are required for achieving adequate accuracy in the call option price $f$. This greatly simplifies the problem of finding a robust analytic approximation to equation (3.3).

Obtaining the moments of $\bar{V}$ is much more cumbersome for equation (1.2) than for the geometric Brownian motion assumed by Hull and White. This might be the reason why no one has explored this route before. Starting with the moments of $V$, the first two moments are:

$$E[V_t] = \exp[\gamma + b + ae^{-\kappa t} - be^{-2\kappa t}] \quad (3.4)$$

$$\begin{aligned} E[V_t \cdot V_u | u \geq t] = \exp[2(\gamma + b) + a(e^{-\kappa t} + e^{-\kappa u}) + \\ + b\left(2e^{-\kappa(u-t)} - 2e^{-\kappa(u+t)} - e^{-2\kappa t} - e^{-2\kappa u}\right)] \end{aligned} \quad (3.5)$$

where $a = \ln V_0 - \gamma$ , $b = \frac{\sigma^2}{4\kappa}$, and derivations are provided in appendix A.

It is not possible to analytically obtain the moments of the continuously sampled average volatility, $E[\bar{V}^n]$. Such calculations would involve integration of the exponential function E[$V_t$], where the exponent itself contains exponential functions. The most natural way around this problem is to approximate $E[\bar{V}^n]$ with the discretely sampled average volatility, as integrals thereby are avoided. An alternative approach is to approximate the exponential functions in the exponent of equations (3.4) and (3.5) with a Taylor series expansion and thereafter calculate $E[\bar{V}^n]$ analytically. In order to achieve a robust estimate, we apply a combination of the two. The first moment $E[\bar{V}]$, is the most decisive factor of the call option price (3.3) and numerical trials reveal that a Taylor expansion of higher order works better than a discretely sampled average. An adequate degree of accuracy is achieved when a fifth order expansion is used. Extensive mathematical details are in appendix A, but the result is simple enough.

$$
\begin{aligned}
E[\bar{V}] &= E[\frac{1}{T}\int_0^T V_t\,dt] \\
&\approx \frac{e^{Y+b}}{T}[T + \frac{a}{\kappa}(1-e^{-\kappa T}) + \frac{\frac{1}{2}a^2 - b}{2\kappa}(1-e^{-2\kappa T}) + \frac{\frac{1}{6}a^3 - ab}{3\kappa}(1-e^{-3\kappa T}) + \\
&+ \frac{\frac{1}{24}a^4 + \frac{1}{2}b^2 - \frac{1}{2}a^2 b}{4\kappa}(1-e^{-4\kappa T}) + \frac{\frac{1}{120}a^5 + \frac{1}{2}ab^2 - \frac{1}{6}a^3 b}{5\kappa}(1-e^{-5\kappa T})]
\end{aligned}
\qquad (3.6)
$$

The series expansion procedure does not work for finding $E[\bar{V}^2]$, however. Numerical tests show that terms to the sixth or seventh order must be included to achieve a robust estimate and this would involve hundreds of terms. The second moment is instead approximated with a discretely sampled average. This works surprisingly well. Not more than five discrete points in time are needed to achieve an acceptable precision. Taking the average of five points, we get:

$$
\begin{aligned}
E[\bar{V}^2] &= E[(\frac{1}{T}\int_0^T V_t\,dt)^2] \\
&\approx E\left[\left(\frac{V_{t1}+V_{t2}+V_{t3}+V_{t4}+V_{t5}}{5}\right)^2\right] \\
&= \frac{1}{5^2}\left[\sum_{i=1}^{5} E(V_{ti}^2) + 2\sum_{i=1}^{4}\sum_{j=1}^{5} E(V_{ti}V_{tj})\right].
\end{aligned} \tag{3.7}
$$

$E[\bar{V}^n]$ is also easily obtained through numerical integration.[5]

As expected volatility in a mean reverting model is a well-behaved function that converges smoothly from initial to long-run volatility, the accuracy can be improved by choosing the times $t_1$ - $t_5$ properly. Dividing the time to maturity into five intervals, and averaging over the mid-point in each interval, provides a better estimate of average volatility than simply choosing the end points and three points in the middle. This means choosing

$$
\{t_1 = 0.1T\,, t_2 = 0.3T\,, t_3 = 0.5T\,, t_4 = 0.7T\,, t_5 = 0.9T\}. \tag{3.8}
$$

Substituting $Var[\bar{V}]$ in formula (3.3) for the moments of (3.6) and (3.7), and the notation $\frac{\partial^2 C}{\partial \bar{V}^2}$ for the expression derived in appendix B, the value of a European call option given the stochastic volatility model (1.1) and (1.2) becomes

$$
f(S_0,V_0) \approx C(E[\bar{V}]) + \frac{1}{2}\frac{S_0 e^{-\delta T}\sqrt{T}N'(d_1)(d_1 d_2 - 1)}{4E[\bar{V}]^{3/2}}(E[\bar{V}^2] - E[\bar{V}]^2). \tag{3.9}
$$

[5] The first moment is calculated by means of the integral (A.5) and the expected value (A.4). For the second moment, integral (A.8) is used together with the expectation (A.13).

The value of a European put option, under the same circumstances, can be obtained through the usual put-call parity. The simple arbitrage condition leading to the put-call parity requires no assumptions about probability distributions of underlying assets. It therefore applies to any option pricing model. Letting $g(S_0, V_0)$ denote the value of a European put option we have

$$g(S_0, V_0) = f(S_0, V_0) + e^{-rT} K - e^{-\delta T} S_0. \tag{3.10}$$

## 4. Model Accuracy

In order to test the accuracy of the approximations made, we compare formulas (3.3) and (3.9) to Monte-Carlo simulations. The results are shown in tables 4.1-4.3. Before commenting on the results, however, we elaborate on how the volatility parameters $\{V_0, \gamma, \kappa, \sigma\}$ have been chosen, since there seems to be some confusion over what constitutes reasonable parameter settings for stochastic volatility models. Solving equation (1.3) explicitly gives the log of the variance as

$$\ln V_t = e^{-\kappa t} \ln V_0 + (1 - e^{-\kappa t})\gamma + \sqrt{\frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)}\,\varepsilon_t. \tag{4.1}$$

Letting $t \to \infty$ implies that $\ln V_t \sim N(\gamma, \sigma \frac{1}{\sqrt{2\kappa}})$. Note that the standard deviation of $\ln V_t$ (when $t \to \infty$) depends both on the diffusion parameter $\sigma$ and on the speed of reversion $\kappa$. These two variables must therefore be estimated simultaneously. Specifying a confidence interval for the volatility with lower limit $l$ and upper limit $u$ is equivalent to specifying the variance as $2\ln l \le \ln V_t \le 2\ln u$. A 95-percent confidence interval is given by $\gamma \pm 1.96 \cdot \frac{\sigma}{\sqrt{2\kappa}}$, giving the lower and upper limits of the confidence interval as

$$l = e^{\frac{1}{2}\left(\gamma - 1.96 \cdot \frac{\sigma}{\sqrt{2\kappa}}\right)}$$
$$u = e^{\frac{1}{2}\left(\gamma + 1.96 \cdot \frac{\sigma}{\sqrt{2\kappa}}\right)}.$$

To find the mean of the interval, we note that equation (4.1) implies that the expected value of the variance is

$E[V_t] = \exp[e^{-\kappa t}\ln V_0 + (1-e^{-\kappa t})\gamma + \frac{\sigma^2}{4\kappa}(1-e^{-2\kappa t})]$. Taking the square root to get the volatility and letting $t \to \infty$, gives the mean of the confidence interval as $e^{\frac{1}{2}(\gamma+\frac{\sigma^2}{4\kappa})}$.

In order to test a stochastic volatility model, we need to choose parameter settings that cover the worst case, namely that volatility is highly stochastic. If the confidence interval is very small there is no need for a stochastic volatility model, and by letting $\sigma \to 0$ the model converges to the Black-Scholes model.[6] The objective behind the chosen parameters has been to ensure a wide but reasonable range of probable volatilities. Letting the initial and long-run volatility be either 20% or 40% (i.e. the variance either 0.04 or 0.16) and setting $\{\kappa,\sigma\}$ equal to values identified as reasonable by the EGARCH model in sections 5 and 6 results in the chosen parameter settings. The wide volatility interval chosen is also a response to Stein and Stein's (1991) doubt that a Taylor series expansion works when volatility is highly stochastic.

When performing the Monte Carlo simulations, there is no need to simulate both the variance $V$ and the stock price $S$. Since $V$ and $S$ are uncorrelated, only the mean of $V$ is required to price the option. It is enough, then, to simulate a trajectory of $V$ and use the mean of this trajectory as input to the Black-Scholes formula (3.2). The accuracy of Monte-Carlo simulations can be greatly enhanced by utilizing the antithetic technique. The idea is to simulate two trajectories of $V$ using the dependent random draws $\varepsilon$ and $-\varepsilon$. We thereby obtain two different call prices and save the average of these two, before commencing the next simulation. What makes the antithetic technique work is that the average call price obtained in this way has much less variance than two

[6] When the evolution of $V$ becomes deterministic, the original Black-Scholes formula is correct when the average volatility is used as input. There is no need for a stochastic volatility model.

independent samples. When $V(\varepsilon)$ is above the median of $V$, $V(-\varepsilon)$ will be below, and vice versa. The resulting error in the average call price is therefore considerably smaller.

It is generally more important to simulate many paths, than to simulate small time increments in each path. In the Monte-Carlo simulations, 100 000 independent trajectories have been used to simulate the price of the European call options. Each trajectory is based on 50 - 200 discrete time increments in order to simulate a continuous path. The larger number of time increments is used for options with ten years to maturity. One of the advantages of Monte Carlo simulation is that the remaining pricing error can be easily computed. If $\mu$ and $\sigma$ denote the mean and standard deviation of the option prices obtained in the simulations, then $\mu \pm 1.96\sigma / 100000^{1/2}$ becomes the upper and lower price of the 95-percentage confidence interval. The results are quite exact. In fact, the widest confidence interval is only 0.19 percent of the call price. For this reason only the mean of the simulations is reported in tables 4.1 - 4.3.

A Monte Carlo simulation is acceptable in the Over-The-Counter markets, where there may be hours, and sometimes even days before the contract is initiated. On the trading floor, however, Monte Carlo simulation is not an alternative as it is too time consuming. Faster techniques are a prerequisite. The second rightmost column in the tables shows the percentage pricing error when the price is calculated by the Taylor series expansion in equation (3.3). Only the first two terms of the series have been used and the moments of $\bar{V}$ have been obtained through numerical integration.[7] The tables are sorted in descending order of the pricing error in the Taylor expansion.

---

[7] The first moment is calculated by means of the integral (A.5) and the expected value (A.4). For the second moment, integral (A.8) is used together with the expectation (A.13).

Table 4.1 Accuracy tests (call option prices, time to maturity: 10 years).

| Parameter settings | | | | | Limiting distribution of volatility Confidence interval 95% | | | Monte Carlo | Taylor Expansion | Analytic Approx. |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | Price | % | % |
| $K$ | $\kappa$ | $\gamma$ | $\sigma$ | $V_0$ | Lower | Mean | Upper | | deviation | deviation |
| 150 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 42.73 | -1.03 | -1.36 |
| 140 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 44.84 | -0.93 | -1.23 |
| 130 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 47.12 | -0.85 | -1.11 |
| 120 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 49.58 | -0.74 | -0.97 |
| 110 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 52.23 | -0.64 | -0.84 |
| 150 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 30.46 | -0.62 | -0.85 |
| 140 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 33.14 | -0.53 | -0.72 |
| 100 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 55.10 | -0.53 | -0.69 |
| 130 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 36.08 | -0.44 | -0.59 |
| 90 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 58.21 | -0.42 | -0.55 |
| 150 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 39.39 | -0.35 | -0.55 |
| 120 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 39.30 | -0.34 | -0.46 |
| 80 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 61.58 | -0.32 | -0.41 |
| 140 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 41.64 | -0.32 | -0.49 |
| 130 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 44.07 | -0.27 | -0.43 |
| 110 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 42.83 | -0.25 | -0.34 |
| 120 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 46.70 | -0.22 | -0.35 |
| 150 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 32.36 | 0.20 | 0.78 |
| 110 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 49.55 | -0.19 | -0.30 |
| 150 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 29.77 | 0.19 | 0.21 |
| 140 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 34.92 | 0.18 | 0.67 |
| 140 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 32.46 | 0.16 | 0.19 |
| 130 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 37.71 | 0.16 | 0.57 |
| 100 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 46.66 | -0.15 | -0.20 |
| 100 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 52.64 | -0.14 | -0.23 |
| 130 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 35.41 | 0.14 | 0.16 |
| 120 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 40.76 | 0.13 | 0.45 |
| 120 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 38.63 | 0.13 | 0.14 |
| 110 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 44.08 | 0.11 | 0.35 |
| 150 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 48.13 | -0.11 | -0.41 |
| 140 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 50.00 | -0.11 | -0.38 |
| 110 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 42.16 | 0.10 | 0.11 |
| 90 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 55.99 | -0.10 | -0.16 |
| 130 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 52.01 | -0.09 | -0.34 |
| 100 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 47.69 | 0.09 | 0.25 |
| 120 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 54.15 | -0.08 | -0.30 |
| 100 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 46.00 | 0.08 | 0.09 |
| 150 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 29.09 | 0.08 | 0.20 |
| 150 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 51.57 | -0.08 | -0.95 |
| 110 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 56.46 | -0.07 | -0.27 |
| 150 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 50.69 | -0.07 | -0.49 |
| 90 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 51.61 | 0.07 | 0.16 |
| 140 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 31.81 | 0.07 | 0.18 |
| 140 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 53.30 | -0.07 | -0.87 |
| 140 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 52.46 | -0.07 | -0.46 |
| 90 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 50.82 | -0.07 | -0.09 |
| 100 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 58.94 | -0.07 | -0.23 |
| 90 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 50.17 | 0.07 | 0.07 |
| 130 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 34.81 | 0.06 | 0.15 |
| 120 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 38.08 | 0.06 | 0.13 |

Table continued

| Parameter settings | | | | | Limiting distribution of volatility Confidence interval 95% | | | Monte Carlo | Taylor Expansion | Analytic Approx. |
|---|---|---|---|---|---|---|---|---|---|---|
| $K$ | $\kappa$ | $\gamma$ | $\sigma$ | $V_0$ | Lower | Mean | Upper | Price | % deviation | % deviation |
| 130 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 55.15 | -0.06 | -0.79 |
| 110 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 41.67 | 0.06 | 0.10 |
| 130 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 54.35 | -0.06 | -0.41 |
| 120 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 57.13 | -0.06 | -0.72 |
| 90 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 61.62 | -0.06 | -0.19 |
| 120 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 56.37 | -0.06 | -0.37 |
| 80 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 55.86 | 0.06 | 0.08 |
| 100 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 45.58 | 0.05 | 0.08 |
| 90 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 49.82 | 0.05 | 0.06 |
| 110 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 58.53 | -0.05 | -0.33 |
| 90 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 63.97 | -0.05 | -0.47 |
| 80 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 54.69 | 0.05 | 0.05 |
| 80 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 64.52 | -0.05 | -0.15 |
| 100 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 61.52 | -0.05 | -0.55 |
| 80 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 54.41 | 0.05 | 0.04 |
| 110 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 59.24 | -0.05 | -0.63 |
| 80 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 59.61 | -0.05 | -0.09 |
| 100 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 60.85 | -0.04 | -0.28 |
| 90 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 63.35 | -0.04 | -0.24 |
| 80 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 66.61 | -0.04 | -0.38 |
| 150 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 29.60 | -0.03 | -0.46 |
| 80 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 66.06 | -0.03 | -0.19 |
| 140 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 32.30 | -0.03 | -0.38 |
| 130 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 35.26 | -0.02 | -0.31 |
| 150 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 29.29 | -0.02 | -0.18 |
| 150 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 29.03 | -0.02 | -0.22 |
| 120 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 38.50 | -0.02 | -0.23 |
| 140 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 31.76 | -0.02 | -0.18 |
| 110 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 42.04 | -0.02 | -0.16 |
| 140 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 32.01 | -0.02 | -0.15 |
| 130 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 35.00 | -0.02 | -0.12 |
| 120 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 38.04 | -0.02 | -0.11 |
| 130 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 34.76 | -0.02 | -0.15 |
| 100 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 45.90 | -0.01 | -0.09 |
| 110 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 41.63 | -0.01 | -0.07 |
| 110 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 41.84 | -0.01 | -0.06 |
| 120 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 38.27 | -0.01 | -0.09 |
| 90 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 50.09 | -0.01 | -0.04 |
| 100 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 45.74 | -0.01 | -0.04 |
| 100 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 45.54 | -0.01 | -0.04 |
| 90 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 49.79 | -0.01 | -0.01 |
| 80 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 54.62 | -0.01 | 0.01 |
| 90 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 49.97 | -0.01 | -0.02 |
| 80 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 54.38 | 0.00 | 0.01 |
| 80 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 54.54 | 0.00 | 0.00 |
| 80 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 55.30 | 0.00 | 0.00 |

Notes:
- Stock price $S$ = 100, Time to maturity $t$ = 10, Interest rate: $r$ = 0.05
- The Monte Carlo simulation uses the antithetic technique with 100 000 paths and 200 steps in each path.
- Maximum width of Monte Carlo confidence interval: 0.06 percent
- The Taylor series expansion utilises the two first terms of the series.
  Moments are obtained through numerical integration.
- The analytic approximation also uses the first two terms of the Taylor series.
  Moments are analytically approximated.

Concentrating on table 4.1, the pricing error from the Taylor series expansion is never larger than 1.03 percent, measured as the deviation from the mean of the confidence interval given by the Monte Carlo simulation. For most parameter settings the difference is considerably smaller. For long-maturity options, in this case 10 years, the Taylor series converges rapidly, and only two terms are required to obtain an adequate accuracy of the option price.

Numerical integration is, however, not a realistic alternative for many pocket calculators since integration routines are quite slow. A fully analytic approximation is sometimes a necessity. The rightmost column gives the call price as calculated by the formula (3.9) where the first and second moments of $\bar{V}$ are approximated as in equations (3.6)-(3.8). The maximum pricing error here increases to 1.36 percent. The biggest differences occur for deep-out-of-the-money options. Out-of-the-money options have no intrinsic value in that they cannot be immediately exercised. When the intrinsic value decreases, the size of the error caused by the approximation is more or less unchanged, whereas the decreasing option price increases the relative error of the approximation. A larger variance and deviations between current and long-term variance will amplify the error, something that is quite natural as it is the variance process that is approximated. For some reason, the biggest pricing errors occur when initial variance $V_0$ is small and the average long-run variance $\gamma$ is high, a pattern that is to be repeated for options of shorter maturity.

Table 4.2 Accuracy tests (call option prices, time to maturity: 1 year).

| Parameter settings | | | | | Limiting distribution of volatility Confidence interval 95% | | | Monte Carlo | Taylor Expansion | Analytic Approx. |
|---|---|---|---|---|---|---|---|---|---|---|
| $K$ | $\kappa$ | $\gamma$ | $\sigma$ | $V_0$ | Lower | Mean | Upper | Price | % deviation | % deviation |
| 140 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 1.03 | 2.31 | -1.09 |
| 140 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 1.33 | 1.92 | 0.41 |
| 130 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 2.00 | 1.75 | 1.85 |
| 130 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 2.40 | 1.45 | 1.78 |
| 120 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 3.72 | 1.19 | 2.67 |
| 120 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 4.22 | 1.02 | 2.06 |
| 110 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 6.58 | 0.81 | 2.25 |
| 110 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 7.13 | 0.68 | 1.67 |
| 150 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 4.79 | -0.59 | -0.84 |
| 150 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 4.84 | -0.51 | -0.91 |
| 150 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 1.09 | -0.51 | -0.42 |
| 140 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 1.84 | -0.51 | -0.51 |
| 140 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 6.27 | -0.49 | -0.97 |
| 130 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 3.07 | -0.48 | -0.48 |
| 100 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 10.96 | 0.46 | 1.27 |
| 150 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 3.03 | 0.43 | 0.48 |
| 140 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 6.34 | -0.43 | -1.13 |
| 100 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 11.49 | 0.41 | 0.99 |
| 130 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 8.20 | -0.41 | -0.98 |
| 140 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 4.26 | 0.39 | 0.41 |
| 120 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 5.03 | -0.36 | -0.38 |
| 130 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 8.29 | -0.35 | -1.19 |
| 120 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 10.69 | -0.32 | -0.88 |
| 130 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 5.98 | 0.31 | 0.33 |
| 120 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 10.79 | -0.28 | -1.09 |
| 130 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 2.17 | -0.27 | -0.59 |
| 110 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 8.01 | -0.27 | -0.28 |
| 120 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 3.90 | -0.24 | -0.60 |
| 120 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 8.32 | 0.23 | 0.26 |
| 110 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 13.99 | -0.23 | -0.93 |
| 90 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 17.08 | 0.21 | 0.41 |
| 110 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 13.87 | -0.21 | -0.69 |
| 90 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 17.49 | 0.21 | 0.37 |
| 100 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 17.90 | -0.19 | -0.55 |
| 110 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 6.76 | -0.18 | -0.44 |
| 110 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 11.47 | 0.18 | 0.19 |
| 100 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 18.01 | -0.17 | -0.69 |
| 100 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 12.32 | -0.16 | -0.17 |
| 90 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 22.89 | -0.13 | -0.34 |
| 100 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 15.62 | 0.12 | 0.13 |
| 90 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 22.98 | -0.11 | -0.43 |
| 120 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 3.40 | -0.11 | 0.33 |
| 100 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 11.13 | -0.10 | -0.26 |
| 110 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 6.19 | -0.10 | 0.22 |
| 150 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 4.37 | 0.09 | -0.14 |
| 90 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 18.15 | -0.09 | -0.09 |
| 80 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 25.03 | 0.08 | 0.03 |
| 90 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 20.89 | 0.08 | 0.08 |
| 80 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 24.79 | 0.08 | -0.01 |
| 80 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 28.91 | -0.08 | -0.16 |
| 80 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 28.97 | -0.08 | -0.20 |
| 110 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 6.39 | -0.08 | -0.40 |

Table continued

| Parameter settings | | | | | Limiting distribution of volatility Confidence interval 95% | | | Monte Carlo | Taylor Expansion | Analytic Approx. |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | Price | % | % |
| $K$ | $\kappa$ | $\gamma$ | $\sigma$ | $V_0$ | Lower | Mean | Upper | | deviation | deviation |
| 130 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 1.75 | -0.07 | -0.02 |
| 120 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 3.57 | -0.07 | -0.41 |
| 130 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 1.89 | -0.06 | -0.06 |
| 120 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 3.36 | -0.06 | -0.09 |
| 130 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 1.79 | -0.05 | 0.45 |
| 100 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 10.60 | -0.05 | 0.13 |
| 80 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 27.36 | 0.05 | 0.05 |
| 110 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 6.16 | -0.04 | -0.07 |
| 90 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 17.23 | -0.04 | -0.10 |
| 140 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 5.78 | 0.04 | -0.20 |
| 80 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 25.45 | -0.04 | -0.03 |
| 140 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 1.18 | -0.03 | -0.29 |
| 100 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 10.56 | -0.03 | -0.05 |
| 100 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 10.63 | -0.02 | -0.14 |
| 100 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 10.78 | -0.02 | -0.20 |
| 90 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 16.83 | -0.01 | 0.06 |
| 130 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 1.77 | -0.01 | -0.01 |
| 110 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 6.23 | -0.01 | -0.24 |
| 120 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 10.10 | -0.01 | -0.20 |
| 90 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 16.80 | -0.01 | -0.01 |
| 90 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 16.83 | -0.01 | -0.04 |
| 90 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 16.96 | -0.01 | -0.05 |
| 80 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 24.69 | 0.01 | 0.02 |
| 80 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 24.74 | -0.01 | 0.02 |
| 80 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 28.54 | 0.01 | -0.03 |
| 110 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 13.28 | 0.00 | -0.15 |
| 80 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 24.66 | 0.00 | 0.01 |
| 80 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 24.90 | 0.00 | -0.02 |
| 120 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 3.41 | 0.00 | -0.24 |
| 100 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 17.34 | 0.00 | -0.11 |
| 90 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 22.40 | 0.00 | -0.07 |
| 130 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 7.65 | 0.00 | -0.21 |
| 80 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 24.66 | 0.00 | 0.00 |

Notes:
- Stock price $S$ = 100, Time to maturity $t$ = 1, Interest rate: $r$ = 0.05
- The Monte Carlo simulation uses the antithetic technique with 100 000 paths and 100 steps in each path.
- Maximum width of Monte Carlo confidence interval: 0.19 percent
- The Taylor series expansion utilises the two first terms of the series.
  Moments are obtained through numerical integration.
- The analytic approximation also uses the first two terms of the Taylor series.
  Moments are analytically approximated.

A shorter maturity amplifies the pricing error. Tables 4.2 and 4.3 report results when time to maturity is 1 year and 0.25 years, respectively. Errors of the Taylor expansion are listed in descending order. When options go deeper and deeper out of money and there is a big difference between initial and long-run variance, the percentage error increases without bounds. On the other hand, the price goes to zero so the dollar difference is quite small and the options do not trade anyway. For this reason, and because all option pricing models are extremely sensitive to the choice of volatility parameters when pricing deep-out-of-the-money options of short maturity, prices less than \$1 are not reported. (The stock price is \$100.) This boundary is of course arbitrary, but as options often trade in multiples of \$ $\frac{1}{8}$ or \$ $\frac{1}{16}$ pricing errors due to tick size also becomes important.

Maximum pricing errors are of the order 2-3 percent both for 1 year and 0.25 year options and on the whole somewhat larger for the truly analytical approximation. Whether or not an error of this magnitude is acceptable is as always dependent on the application. It should be kept in mind though, that the errors reported are worst-case scenarios. The volatility is highly stochastic, reverts slowly to the mean and there is a large deviation between actual and long run volatility. In the majority of cases, the model performs much better. However, it seems wiser to report worst-case outcomes rather than the average errors. Inevitably, extreme events are bound to occur and these are the occasions when a stochastic volatility model is most required.

In relation to the differences between observed market prices and the B-S price in section 6, which are on average between 10-60 percent, the approximation error seems acceptable. The B-S errors, however, will be considered later on. First we must introduce the method for parameter estimation. This is the subject of the next section.

Table 4.3 Accuracy tests (call option prices, time to maturity: 0.25 years).

| Parameter settings | | | | | Limiting distribution of volatility Confidence interval 95% | | | Monte Carlo | Taylor Expansion | Analytic Approx. |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | Price | % | % |
| $K$ | $\kappa$ | $\gamma$ | $\sigma$ | $V_0$ | Lower | Mean | Upper | | deviation | deviation |
| 110 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 1.81 | 2.14 | 2.80 |
| 120 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 1.47 | -1.47 | -1.40 |
| 110 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 2.37 | 1.40 | 1.61 |
| 120 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 1.91 | -1.24 | -1.35 |
| 110 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 3.42 | -1.02 | -1.17 |
| 100 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 5.39 | 0.88 | 1.34 |
| 110 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 4.06 | -0.79 | -1.01 |
| 100 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 6.04 | 0.64 | 0.77 |
| 100 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 7.19 | -0.54 | -0.63 |
| 100 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 7.88 | -0.43 | -0.58 |
| 110 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 1.40 | -0.28 | -0.28 |
| 120 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 2.10 | 0.28 | 0.28 |
| 90 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 12.09 | 0.23 | 0.24 |
| 90 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 12.51 | 0.21 | 0.21 |
| 90 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 13.33 | -0.20 | -0.23 |
| 90 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 13.83 | -0.19 | -0.23 |
| 110 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 4.31 | 0.17 | 0.17 |
| 110 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 1.30 | -0.16 | -0.23 |
| 110 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 1.26 | -0.12 | -0.67 |
| 100 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 4.88 | -0.10 | -0.10 |
| 100 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 8.15 | 0.10 | 0.10 |
| 100 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 4.74 | -0.07 | -0.16 |
| 80 | 20.0 | ln0.16 | 2.00 | 0.04 | 0.29 | 0.41 | 0.55 | 21.77 | -0.06 | -0.05 |
| 110 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 1.26 | -0.05 | -0.13 |
| 100 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 4.71 | -0.05 | -0.28 |
| 110 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 1.21 | -0.05 | -0.05 |
| 80 | 10.0 | ln0.04 | 2.00 | 0.16 | 0.13 | 0.21 | 0.31 | 21.22 | 0.04 | 0.03 |
| 90 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 14.03 | 0.04 | 0.04 |
| 80 | 10.0 | ln0.16 | 2.00 | 0.04 | 0.26 | 0.42 | 0.62 | 21.55 | -0.04 | -0.03 |
| 110 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 4.60 | 0.04 | -0.22 |
| 120 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 2.33 | 0.04 | -0.26 |
| 80 | 20.0 | ln0.04 | 2.00 | 0.16 | 0.15 | 0.21 | 0.27 | 21.10 | 0.03 | -0.02 |
| 100 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 4.69 | -0.02 | -0.09 |
| 100 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 8.46 | 0.02 | -0.13 |
| 90 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 11.81 | -0.02 | -0.02 |
| 100 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 4.63 | -0.02 | -0.02 |
| 130 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 1.11 | 0.01 | -0.17 |
| 80 | 0.7 | ln0.04 | 0.50 | 0.16 | 0.13 | 0.21 | 0.30 | 21.87 | 0.01 | 0.01 |
| 90 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 14.27 | 0.01 | -0.05 |
| 100 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 4.64 | -0.01 | 0.34 |
| 90 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 11.72 | -0.01 | -0.03 |
| 90 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 11.76 | -0.01 | -0.01 |
| 80 | 0.2 | ln0.04 | 0.50 | 0.16 | 0.09 | 0.23 | 0.43 | 22.00 | 0.01 | -0.01 |
| 90 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 11.69 | 0.00 | 0.04 |
| 110 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 1.21 | 0.00 | 0.00 |
| 80 | 0.7 | ln0.16 | 0.50 | 0.04 | 0.26 | 0.42 | 0.61 | 21.04 | 0.00 | 0.00 |
| 90 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 11.72 | 0.00 | 0.00 |
| 90 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 11.69 | 0.00 | 0.00 |
| 80 | 0.7 | ln0.04 | 0.50 | 0.04 | 0.13 | 0.21 | 0.30 | 21.03 | 0.00 | 0.00 |

Table continued

| Parameter settings | | | | | Limiting distribution of volatility Confidence interval 95% | | | Monte Carlo Price | Taylor Expansion % deviation | Analytic Approx. % deviation |
|---|---|---|---|---|---|---|---|---|---|---|
| $K$ | $\kappa$ | $\gamma$ | $\sigma$ | $V_0$ | Lower | Mean | Upper | | | |
| 80 | 20.0 | ln0.04 | 2.00 | 0.04 | 0.15 | 0.21 | 0.27 | 21.03 | 0.00 | 0.00 |
| 80 | 0.2 | ln0.04 | 0.50 | 0.04 | 0.09 | 0.23 | 0.43 | 21.03 | 0.00 | 0.00 |
| 80 | 0.2 | ln0.16 | 0.50 | 0.04 | 0.18 | 0.47 | 0.87 | 21.03 | 0.00 | 0.00 |
| 80 | 10.0 | ln0.04 | 2.00 | 0.04 | 0.13 | 0.21 | 0.31 | 21.04 | 0.00 | 0.01 |

Notes:

- Stock price $S$ = 100, Time to maturity $t$ = 0.25, Interest rate: $r$ = 0.05
- The Monte Carlo simulation uses the antithetic technique with 100 000 paths and 50 steps in each path.
- Maximum width of Monte Carlo confidence interval: 0.14 percent
- The Taylor series expansion utilises the two first terms of the series. Moments are obtained through numerical integration.
- The analytic approximation also uses the first two terms of the Taylor series. Moments are analytically approximated.

# 5. Parameter estimations

A significant cost of a stochastic volatility model is the accuracy of parameter estimations. In the B-S model, parameter estimation is technically a relatively straightforward problem. For a stochastic volatility model, however, the problem of appropriate parameter settings is a much more pressing concern. In our case four parameters are required, compared to one for the B-S model. The increasing number of parameters has two consequences. First of all it can decrease the accuracy of the estimation, an effect that in turn may cause realistic models to perform worse than simpler ones. The second problem is the increasing complexity, which makes it difficult to choose between different statistical methods and to apply them correctly. This is especially so for the SV models which constitute the discrete time equivalent of the stochastic volatility continuous time models. GARCH type models, on the other hand, are considerably easier to deal with and we therefore develop such a model.

Consider the model

$$y_t = \sigma_t \varepsilon_t \tag{5.1}$$

$$\ln \sigma_{t+1}^2 = \psi_0 \ln \bar{\sigma}^2 + \psi_1 \ln \sigma_t^2 + \psi_2 \ln(y_t^2 + \sigma_t^2), \tag{5.2}$$

where $\varepsilon_t$ is standard normal and $\bar{\sigma}^2$ is the long-run average variance. The updating equation (5.2) may look rather strange but the intuition will be apparent shortly. Substituting (5.1) into (5.2) gives

$$\ln \sigma_{t+1}^2 = \psi_0 \ln \bar{\sigma}^2 + \psi_1 \ln \sigma_t^2 + \psi_2 \ln(\sigma_t^2(\varepsilon_t^2 + 1)).$$

This expression can be further simplified into

$$\ln \sigma_{t+1}^2 = \psi_0 \ln \bar{\sigma}^2 + (\psi_1 + \psi_2) \ln \sigma_t^2 + \psi_2 \ln(\varepsilon_t^2 + 1). \tag{5.3}$$

Thus, the logarithm of the variance follows an autoregressive process of the first order. The variance is updated in each period by weighting the long-run average variance, the variance of the previous period and new innovations. Equation (5.3), which is the same as equation (2.2), is the new Exponential GARCH model. Like other GARCH models there is no need for the weights to sum to one. What is required, though, is that all the weights are less than unity. Otherwise the variance will explode. It is also required that $\psi_0 > 0$ if we are to assign some weight to the long run average variance and that $\psi_2 > 0$ for the updating equation to work properly. $\psi_1$ might be slightly negative in the (unlikely) situation that the previously observed variance has very little influence over the variance in the next period, but $0 < \psi_1 + \psi_2 < 1$ in order for the process to work properly and the continuous time estimation to work.

The key insight is that (5.3) looks quite similar in structure to the discrete time version of the Ornstein-Uhlenbeck process (4.1). Once this has been realised, the rest is just tedious algebra. Following Engle and Lee (1996), the idea is to match the mean and variance of the GARCH and the continuous time model. The theoretical justification for such matching is that as the time intervals get shorter, the first two moments of the GARCH model must converge to the continuous time process. As all innovations are normally distributed, the first two moments are sufficient for this purpose.

The geometric Brownian motion of (1.1) implies that the stock return in discrete time is lognormally distributed with $\Delta \ln S_t = (\alpha - \frac{1}{2}V_t)\Delta t + \sqrt{V_t \Delta t}\, w_t$, where $w_t \sim N(0,1)$. Let $(\alpha - \frac{1}{2}V_t)\Delta t$ be estimated as the sample mean of

$\Delta \ln S_t$, and define a new variable $y_t$ as $y_t \equiv \Delta \ln S_t - (\alpha - \frac{1}{2}V_t)\Delta t$. The process $y_t$ thereby becomes a martingale, as it has no drift,

$$y_t = \sqrt{V_t \Delta t}\, w_t. \tag{5.4}$$

The discrete time version of the Ornstein-Uhlenbeck process is

$$\ln V_{t+1} = (1 - e^{-\kappa\Delta t})\gamma + e^{-\kappa\Delta t} \ln V_t + \sigma\sqrt{\frac{1-e^{-2\kappa\Delta t}}{2\kappa}}\,\varepsilon_t. \tag{5.5}$$

Comparing equations (5.1) to (5.4) and (5.3) to (5.5), the only difference is that no time interval between the observations has been specified for the EGARCH model and that the innovations are different. In the continuous time model the innovations are standard normal, whereas in the EGARCH model the random variable is the log of a demeaned chi-squared distribution. The parameters of the continuous time process (which are needed for pricing the options) can now be determined by matching the mean and variance of the respective distributions. This is done in appendix C and we here merely state the result.

$$\kappa = -\frac{\ln(\psi_1 + \psi_2)}{\Delta t} \tag{5.7}$$

$$\gamma = \frac{\psi_0 \ln \bar{\sigma}^2 + 0.53345\psi_2}{1 - (\psi_1 + \psi_2)} - \ln \Delta t$$

$$\sigma^2 = \frac{2 \cdot 0.26834\kappa\psi_2^2}{1 - (\psi_1 + \psi_2)^2}$$

There is one more parameter to determine, the initial variance $V_0$. This is straightforward. Using maximum likelihood estimation to select the parameters of equation (5.2), $V_0$ is given as the terminal variance of the estimation time series $\sigma_T^2$ divided by $\Delta t$ to annualise the variance,

$$V_0 = \frac{\sigma_T^2}{\Delta t}. \tag{5.8}$$

If the estimation period does not continue to the day when the option contract is written, $\sigma_T^2$ can be updated by means of equation (5.2) to give the initial variance.

Market participants often prefer to pay less attention to historic volatility and instead concentrate on what is called implied volatility. Given the price of a plain vanilla call/put option, it is possible to extract the volatility from the B-S formula that is consistent with the observed market price. This implied volatility is then used to price other contracts that are less liquid and perhaps have more complicated payment structure.

One of the reasons that implied volatility is so often used in practice is that it provides option prices that are consistent with other option prices. If the pricing of a new derivative is based on historic volatility, it is open to the criticism that a model that is unable to price a traded option accurately is not trustworthy for pricing derivatives that are not traded. More technically, by using a historic volatility different from the implied volatility, there is an inherent arbitrage opportunity available between the traded option and the new derivative. By basing the price of the new derivative on the implied volatility of the traded option, this kind of problem is avoided. It is a well-known characteristic of option markets that the implied volatility varies over time to maturity and exercise prices. The volatility of equity options normally increases with time to maturity and decreases with the exercise price.

The introduction of stochastic volatility models alleviates these so called volatility smiles, but they do not disappear completely. At least not when the stock and the variance processes are assumed to be independent and the market price of volatility risk zero, see Melino and Turnbull (1990).

The idea to use implied volatility can, however, be further developed in a stochastic volatility model. It is possible, to some extent, to calibrate the model after the volatility smile. There are four parameters to estimate.

- The initial volatility $V_0$.
- The long-run average volatility $\gamma$.
- The speed of reversion $\kappa$.
- The diffusion $\sigma$.

Using historical data to estimate $\kappa$ and $\sigma$, e.g. by means of the EGARCH model proposed here, leaves $V_0$ and $\gamma$ to be determined from observed option prices. One simple iterative procedure is to start with the implied B-S variance from the most liquid option with short maturity and use this as $V_0$. A long-term liquid option is then used for backing out the first iteration of $\gamma$. The value of $\gamma$ is then used to extract $V_0$ from the near maturity option and so forth until the values of $\gamma$ and $V_0$ are consistent with the two option prices. This novel procedure is applied in the next section and only three or four iterations are needed to get consistency. For options that mature in-between the two benchmark options it is now possible to use these implied $V_0$ and $\gamma$ to price the options. It should be stressed, however, that it is the options with the shortest and longest time to maturity that should be used as benchmark options. As the implied values of $V_0$ and $\gamma$ are very rough approximations of the initial and long-run variances, it is not possible to extrapolate this procedure to cover shorter or longer time horizons. Only interpolation is feasible.

# 6. An application to Swedish stock market data

We test the stochastic volatility model on options traded on the Stockholm stock exchange. The twenty stocks for which option trading is most frequent are included in the test. Daily stock price quotes between 2nd of January 1996 and 21st of May 2001 are used for parameter estimation. Based on these estimations, theoretical option prices are compared to observed prices on the 22nd of May 2001.

Maximum likelihood estimation can be used for all GARCH processes. This is one of the main advantages of GARCH models. It is thus possible to compare the values of the maximum likelihood objective function. In particular, we like to compare the EGARCH model to the standard GARCH(1,1) model. GARCH(1,1) has during the last decade become a standard procedure for updating and characterising heteroscedasticity. By comparing the two models it is possible to obtain an indication of how well the proposed EGARCH model fits the data. However, differences in value are difficult to transform into a statement about how much better or worse the performance of the proposed model is. Therefore, the Exponential Weighted Moving Average and constant volatility are also included in the test. EWMA can be seen as a simplified version of GARCH(1,1) where no weight is given to the long-run average variance.

Following equation (5.1), the stock return after the drift has been eliminated is denoted as $y_t$,

$$y_t = \sigma_t \varepsilon_t . \tag{6.1}$$

Given that $\varepsilon_t$ is standard normal, the probability distribution of $y_t$ conditional on the volatility parameter $\sigma_t$ is also normal. The parameters of $y_t$ (which of course only is $\sigma_t$ since there is no drift) can then be estimated by the maximum log likelihood. For $T$ observations, we thus want to maximise

$$\sum_{t=1}^{T}(-\ln\sigma_t^2 - \frac{y_t^2}{\sigma_t^2}). \tag{6.2}$$

However, $\sigma_t$ is not generally believed to be constant, or there would be no need for modifying the Black-Scholes formula in the first place. Three updating schemes are tested.

$$\text{GARCH(1,1): } \sigma_{t+1}^2 = c_0\bar{\sigma}^2 + c_1\sigma_t^2 + c_2\, y_t^2 \tag{6.3}$$

$$\text{EWMA: } \sigma_{t+1}^2 = \lambda\sigma_t^2 + (1-\lambda)\, y_t^2 \tag{6.4}$$

$$\text{New EGARCH: } \ln\sigma_{t+1}^2 = \psi_0 \ln\bar{\sigma}^2 + \psi_1 \ln\sigma_t^2 + \psi_2 \ln(y_t^2 + \sigma_t^2) \tag{6.5}$$

The parameters are chosen so that the value of the maximum likelihood objective function (6.2) is maximised. The results are reported in table 6.1, where the values are presented as deviations from the benchmark GARCH(1,1) model.

Table 6.1 Values of the maximum likelihood objective function (6.2) for different heteroscedasticity models.

| Company | Maximum likelihood score | | | |
|---|---|---|---|---|
| | GARCH(1,1) Equation (6.3) | Constant $\sigma^2$ | EWMA Equation (6.4) | New EGARCH Equation (6.5) |
| | Score | Difference | Difference | Difference |
| ABB | 9419 | -299 | -30 | 7 |
| Astra Zeneca | 9233 | -71 | -17 | 28 |
| Atlas Copco A | 9013 | -147 | -16 | 34 |
| Autoliv | 9218 | -60 | -60 | 24 |
| Electrolux B | 8821 | -174 | -49 | 10 |
| Ericsson B | 8124 | -265 | -30 | 14 |
| Föreningssparbanken | 8981 | -63 | -63 | 10 |
| Hennes&Mauritz B | 8567 | -141 | -96 | -13 |
| Holmen B | 9008 | -141 | -63 | 11 |
| Investor B | 9657 | -112 | -67 | -2 |
| Nokia SDR | 8112 | -190 | -12 | 42 |
| Pharmacia-Upjohn | 8995 | -44 | -44 | -18 |
| Sandvik | 9331 | -140 | -10 | 19 |
| SCA B | 9443 | -159 | -70 | 13 |
| SE-Banken A | 9047 | -157 | -66 | 10 |
| Skandia | 8320 | -271 | -23 | 25 |
| Stora Enso R | 8678 | -147 | -16 | 9 |
| Svenska Handelsbanken A | 9371 | -49 | -28 | 13 |
| Trelleborg B | 9471 | -154 | -59 | 21 |
| Volvo B | 9186 | -89 | -33 | 8 |
| Average difference: | | -144 | -43 | 13 |

Notes:
- Column 2 reports the ML values of equation (6.2) when the GARCH(1,1) model in (6.3) is used.
- Columns 3-5 report the difference in value, compared to GARCH(1,1), for the respective models.
- Data are daily closing prices on the Stockholm stock exchange between 2nd Jan 1996 and 21st Maj 2001.
- The abbreviations A,B,R in the company names refers to the specific class of stock voting rights.
- The initial and the long-run average variance are set as the sample variance in the GARCH(1,1) and EGARCH estimations.

Results are rather unambiguous. A constant volatility model cannot adequately capture the stochastic properties of the stock return. The values are always lower than for the GARCH(1,1) model. The differences are smallest for Pharmacia-Upjohn and Svenska Handelsbanken and it is possible to argue that the constant volatility assumption will suffice for these stocks. The difference in value, –44 and –49 units of measure respectively, are of the same magnitude as the Exponentially Weighted Moving Average model. EWMA is often perceived to be a reasonable model for heteroscedasticity, at least for practical applications. The average difference between EWMA and GARCH(1,1) is –43 units of measure. As EWMA is a special case of GARCH(1,1), the value of the maximum likelihood objective function is also lower. The difference being that EWMA assigns no weight to the long-run average variance rate. In other words, EWMA assumes no mean reversion in the variance process.

The rightmost column in table 6.1 is the most interesting. The proposed EGARCH model exhibits the highest values for seventeen of the twenty stocks and thus works better than the benchmark model. The three exceptions are Hennes&Mauritz, Investor and Pharmacia-Upjohn. The values of the EGARCH model are on average 13 units higher than for the GARCH model. This is greatly encouraging and slightly surprising. The motivation for the proposed specification was its compatibility with the continuous time model, not its properties as a stand-alone econometric model. A priori, there is no reason to believe that it should perform any better, or worse for that matter, than the standard GARCH model. The fact that it does perform slightly better indicates that the new EGARCH model might be applicable in other contexts than continuous time approximations. Further econometric research is therefore warranted.

The proposed EGARCH model also seems to be more robust than the GARCH(1,1) model. Hull (2000, chapter 15) reports that the solver routine in Microsoft's Excel spreadsheet has difficulties in finding the optimal solution of the GARCH(1,1) model when all three parameters are being estimated simultaneously.[8] He therefore recommends the variance targeting technique, which basically means setting $c_0 = 1 - c_1 - c_2$. Thus there are only two parameters to estimate, which enhance the robustness of the model.[9] The problems described in Hull are confirmed for the tested Swedish stocks. When

[8] The solver routine is based on Newton-Raphson iterations.

[9] The maximum likelihood values for the variance targeting technique are not reported, as they are only slightly smaller than for the unrestricted estimation. For most series the difference is less than one unit.

all three parameters are estimated simultaneously, the parameters must be restricted between zero and one, in order to stop the likelihood maximisation from diverging. This constraint is never binding in optimum but must nevertheless be present for the maximisation to work well. These difficulties are not encountered for the EGARCH model. All parameters were estimated without any restrictions to parameter values. For some reason, working with the log variance in (6.5), gives an easier maximisation problem than working with the variance directly as in (6.3). This is for practical purposes and daily use an important characteristic of the new EGARCH model.

Having argued that the new EGARCH model is easy to apply and captures the heteroscedasticity in stock returns quite well, the next step is to compare the observed option prices at the Stockholm stock exchange to the prices predicted by the stochastic volatility model (using parameter estimates from the EGARCH model) and the standard Black-Scholes formula. Table 6.2 depicts the parameter values given by the maximum likelihood estimation of the EGARCH model and the formulas in (5.7). 30-day historical volatility has been used in the Black-Scholes formula. The choice of 30-days is of course arbitrary, but motivated on the grounds that the major Swedish newspapers calculate Black-Scholes theoretical option prices based on 30-days volatility and report these prices together with the quoted price. 30-days is thus some sort of standard. The return on Swedish treasury bills with the same maturity as the options is used as the proxy for the risk-free rate.

Table 6.2 Parameter estimations for the option pricing models

| Company | Black-Scholes | Stochastic volatility | | | |
|---|---|---|---|---|---|
| | Volatility (30 days) | $\kappa$ | $\gamma$ | $\sigma$ | $V_0$ |
| ABB | 0.397 | 4.92 | -1.726 | 1.795 | 0.148 |
| Astra Zeneca | 0.269 | 5.27 | -2.103 | 1.277 | 0.077 |
| Atlas Copco A | 0.424 | 5.54 | -1.764 | 1.784 | 0.177 |
| Autoliv | 0.451 | 19.43 | -2.008 | 2.468 | 0.133 |
| Electrolux B | 0.437 | 18.56 | -1.742 | 2.501 | 0.138 |
| Ericsson B | 0.891 | 7.79 | -1.163 | 2.303 | 0.336 |
| Föreningssparbanken | 0.384 | 51.72 | -2.079 | 2.764 | 0.083 |
| Hennes&Mauritz B | 0.488 | 21.37 | -1.437 | 2.668 | 0.106 |
| Holmen B | 0.348 | 43.11 | -1.997 | 4.059 | 0.163 |
| Investor B | 0.326 | 19.43 | -2.457 | 1.929 | 0.076 |
| Nokia SDR | 0.704 | 2.79 | -0.841 | 1.317 | 0.353 |
| Pharmacia-Upjohn | 0.376 | 20.52 | -2.032 | 1.267 | 0.134 |
| Sandvik | 0.394 | 3.37 | -2.040 | 1.212 | 0.153 |
| SCA B | 0.283 | 34.25 | -2.404 | 3.476 | 0.107 |
| SE-Banken A | 0.390 | 18.67 | -2.023 | 2.481 | 0.062 |
| Skandia | 0.734 | 2.83 | -0.946 | 1.157 | 0.369 |
| Stora Enso R | 0.270 | 3.90 | -1.517 | 1.178 | 0.132 |
| Svenska Handelsbanken A | 0.309 | 55.01 | -2.386 | 2.299 | 0.068 |
| Trelleborg | 0.257 | 14.22 | -2.268 | 2.173 | 0.098 |
| Volvo | 0.407 | 15.34 | -2.087 | 1.732 | 0.105 |

Notes:
- Data are daily closing prices on the Stockholm stock exchange between 2nd Jan 1996 and 21st Maj 2001.
- The abbreviations A,B,R in the company names refers to the specific class of stock voting rights.

Individual stock options traded on the Stockholm stock exchange are American and can therefore be exercised prior to expiry whereas the pricing formulas only apply to European options. However, early exercise of call options is never optimal on non-dividend paying stocks. It is therefore possible to compare the market and the theoretical prices of call options as none of the stocks pay dividend before option expiry. Table 6.3 provides the comparison between the different prices. Historical volatility is one thing and expected future volatility (required to price options) quite another. As can be expected there is a substantial deviation between theoretical and actually observed market prices. In particular, the stock markets have during the first half of 2001 been very volatile and also experienced a major decline. The expected future volatility implicit in the option prices is much lower than the historic volatility.

Table 6.3 Call option prices (historic volatility).

| | Exercise Price SEK | Market Price SEK | Black-Scholes % deviation | Stochastic Volatility % deviation |
|---|---|---|---|---|
| ABB | | | | |
| August | 180 | 17.7 | 37.2 | 40.4 |
| | 187 | 14.2 | 41.5 | 45.6 |
| | 195 | 10.0 | 59.4 | 65.4 |
| | 202 | 7.3 | 77.2 | 85.7 |
| | 210 | 4.8 | 108.6 | 121.8 |
| November | 170 | 30.0 | 24.0 | 28.5 |
| | 200 | 13.5 | 55.9 | 67.3 |
| | 210 | 9.8 | 75.3 | 91.5 |
| Astra Zeneca | | | | |
| July | 470 | 42.2 | 14.7 | 19.8 |
| | 510 | 19.2 | 20.2 | 35.0 |
| | 540 | 9.0 | 26.8 | 56.6 |
| | 570 | 3.5 | 41.7 | 100.9 |
| September | 490 | 42.0 | 7.7 | 21.1 |
| | 510 | 29.0 | 18.6 | 39.3 |
| | 540 | 20.7 | 5.3 | 34.3 |
| | 570 | 12.0 | 9.3 | 54.8 |
| Atlas Copco A | | | | |
| August | 200 | 35.0 | 8.5 | 9.5 |
| | 220 | 21.5 | 17.0 | 18.3 |
| | 230 | 16.5 | 21.3 | 23.0 |
| | 240 | 11.0 | 42.5 | 45.3 |
| | 250 | 7.5 | 61.7 | 65.9 |
| November | 240 | 19.7 | 26.1 | 29.1 |
| Autoliv | | | | |
| June | 180 | 32.0 | -4.8 | -6.4 |
| | 200 | 13.0 | 14.9 | 3.5 |
| | 210 | 5.8 | 64.0 | 34.6 |
| | 220 | 2.8 | 101.1 | 45.1 |
| September | 220 | 10.2 | 79.5 | 44.3 |
| March | 160 | 53.7 | 19.3 | 13.2 |
| Electrolux B | | | | |
| July | 150 | 20.0 | 0.2 | -2.3 |
| | 160 | 13.5 | 3.8 | -0.7 |
| | 170 | 6.8 | 38.8 | 29.6 |
| | 180 | 4.5 | 33.6 | 21.6 |
| October | 170 | 14.0 | 20.9 | 17.0 |
| | 180 | 10.0 | 32.0 | 26.8 |
| | 190 | 7.5 | 36.0 | 29.7 |
| January | 150 | 30.5 | 4.9 | 3.4 |
| Ericsson B | | | | |
| June | 50 | 25.0 | 1.4 | 0.6 |
| | 55 | 20.5 | 1.1 | -1.3 |
| | 60 | 15.5 | 6.2 | -0.1 |
| | 65 | 11.7 | 8.5 | -4.9 |
| | 70 | 7.5 | 26.7 | -0.9 |
| | 75 | 4.5 | 53.6 | 2.0 |
| | 80 | 2.5 | 100.0 | 7.8 |
| | 85 | 1.4 | 142.1 | 2.1 |
| | 90 | 0.7 | 253.8 | 13.8 |
| | 95 | 0.4 | 337.1 | 5.7 |
| August | 55 | 22.2 | 9.3 | -1.5 |
| | 60 | 19.0 | 10.6 | -5.6 |
| | 65 | 15.0 | 20.7 | -3.7 |
| | 70 | 11.0 | 41.2 | 4.0 |
| | 75 | 8.5 | 56.4 | 5.2 |
| August | 80 | 6.5 | 74.6 | 6.3 |
| | 85 | 5.3 | 84.1 | 0.9 |
| | 90 | 3.5 | 134.9 | 15.3 |
| | 95 | 2.5 | 179.4 | 22.8 |
| | 100 | 1.8 | 238.8 | 33.5 |
| October | 105 | 1.3 | 302.4 | 42.5 |
| | 65 | 17.0 | 27.9 | -0.2 |
| | 70 | 14.5 | 34.0 | -1.9 |
| | 75 | 11.8 | 47.1 | 0.5 |
| Ericsson B | | | | |
| | 80 | 8.8 | 77.1 | 12.7 |
| | 85 | 7.5 | 84.7 | 8.9 |
| | 90 | 6.5 | 90.5 | 4.0 |
| February | 40 | 37.2 | 8.8 | 1.2 |
| | 60 | 22.7 | 29.0 | 3.5 |
| | 70 | 17.0 | 47.2 | 7.4 |
| | 80 | 13.5 | 59.1 | 4.8 |
| | 90 | 9.0 | 105.6 | 21.8 |
| | 100 | 7.5 | 113.3 | 13.6 |
| | 110 | 5.3 | 164.6 | 27.0 |
| | 130 | 3.8 | 182.4 | 11.2 |
| | 150 | 1.3 | 529.2 | 107.7 |
| | 165 | 1.0 | 580.0 | 99.0 |
| Föreningssparbanken | | | | |
| July | 130 | 6.3 | 17.4 | 6.2 |
| | 135 | 4.5 | 19.1 | 4.0 |
| October | 125 | 13.5 | 11.0 | 4.2 |
| Hennes&Mauritz B | | | | |
| July | 140 | 42.0 | 0.6 | 0.0 |
| | 180 | 10.5 | 39.1 | 29.0 |
| | 190 | 6.3 | 67.2 | 50.2 |
| | 200 | 3.4 | 114.1 | 86.5 |
| | 210 | 2.1 | 141.5 | 103.9 |
| September | 180 | 16.2 | 31.5 | 28.6 |
| | 190 | 12.0 | 42.5 | 38.6 |
| | 200 | 9.0 | 51.3 | 46.3 |
| | 210 | 5.8 | 87.0 | 80.3 |
| November | 200 | 13.2 | 38.0 | 36.6 |
| | 210 | 9.3 | 63.9 | 62.3 |
| | 220 | 8.0 | 57.3 | 55.9 |
| January | 180 | 25.2 | 20.6 | 20.6 |
| | 190 | 20.0 | 31.5 | 31.4 |
| | 200 | 16.7 | 35.8 | 35.9 |
| Holmen B | | | | |
| July | 208 | 9.5 | 29.1 | 42.4 |
| | 228 | 3.0 | 68.3 | 106.7 |
| Investor B | | | | |
| June | 135 | 5.5 | 5.5 | -4.4 |
| | 140 | 2.9 | 17.6 | -1.0 |
| September | 140 | 7.5 | 30.3 | 17.2 |
| | 150 | 3.5 | 73.7 | 47.7 |
| Nokia SDR | | | | |
| June | 240 | 118.0 | -0.1 | -0.2 |
| | 250 | 111.0 | -2.6 | -2.8 |
| | 260 | 99.0 | -0.5 | -0.9 |
| | 280 | 79.0 | 1.3 | 0.0 |
| | 290 | 71.0 | 0.3 | -1.6 |
| June | 300 | 59.7 | 5.4 | 2.3 |
| | 310 | 53.0 | 3.9 | -0.4 |
| | 330 | 35.0 | 17.3 | 8.5 |
| | 350 | 22.0 | 34.3 | 18.2 |
| | 370 | 12.0 | 71.0 | 41.0 |
| | 390 | 6.3 | 120.6 | 68.2 |
| | 410 | 3.0 | 199.3 | 108.3 |
| August | 280 | 89.0 | 5.7 | 2.0 |
| | 290 | 81.0 | 7.7 | 3.1 |
| | 300 | 74.5 | 8.3 | 2.8 |
| | 310 | 66.2 | 12.6 | 5.8 |
| | 330 | 51.2 | 29.4 | 13.7 |
| | 350 | 39.5 | 35.0 | 21.5 |
| | 370 | 30.7 | 45.8 | 27.8 |
| | 390 | 25.0 | 49.4 | 27.6 |
| | 410 | 18.0 | 72.6 | 43.3 |
| | 430 | 13.0 | 98.0 | 59.8 |
| | 450 | 10.0 | 112.6 | 66.9 |

| | Exercise Price SEK | Market Price SEK | Black-Scholes % deviation | Stochastic Volatility % deviation |
|---|---|---|---|---|
| Nokia SDR | | | | |
| August | 490 | 5.8 | 150.3 | 86.6 |
| October | 310 | 75.0 | 18.2 | 11.6 |
| | 370 | 43.5 | 40.1 | 26.5 |
| | 390 | 38.0 | 41.1 | 25.4 |
| February | 370 | 59.5 | 41.4 | 31.6 |
| | 410 | 44.0 | 60.7 | 47.2 |
| | 450 | 34.0 | 75.1 | 57.9 |
| | 540 | 20.5 | 98.5 | 74.2 |
| | 600 | 11.0 | 188.9 | 150.7 |
| Pharmacia-Upjohn | | | | |
| June | 490 | 32.2 | 11.6 | 10.3 |
| | 510 | 21.2 | 11.3 | 8.8 |
| | 540 | 8.5 | 28.8 | 23.2 |
| | 570 | 2.8 | 56.4 | 45.5 |
| September | 470 | 66.5 | 10.6 | 9.0 |
| | 510 | 43.5 | 16.8 | 13.9 |
| | 540 | 27.2 | 37.8 | 33.1 |
| | 570 | 20.0 | 35.4 | 29.3 |
| Sandvik | | | | |
| June | 220 | 16.2 | -1.1 | -1.5 |
| | 230 | 8.5 | 18.9 | 17.9 |
| | 240 | 4.3 | 38.4 | 36.5 |
| | 250 | 2.0 | 57.5 | 55.0 |
| September | 230 | 18.5 | 23.7 | 21.6 |
| | 240 | 14.0 | 31.9 | 29.0 |
| | 250 | 10.0 | 47.4 | 43.6 |
| SCA B | | | | |
| August | 230 | 16.5 | 4.8 | 13.0 |
| | 240 | 10.0 | 22.1 | 36.4 |
| | 250 | 7.0 | 18.7 | 38.6 |
| | 260 | 3.8 | 45.3 | 79.5 |
| SE-Banken A | | | | |
| June | 100 | 7.0 | -6.4 | -18.4 |
| | 105 | 3.8 | 1.9 | -24.0 |
| | 110 | 1.9 | 8.1 | -35.7 |
| August | 100 | 9.8 | 7.1 | -1.5 |
| | 105 | 6.8 | 17.0 | 3.7 |
| | 110 | 4.3 | 37.6 | 16.9 |
| Skandia | | | | |
| July | 100 | 28.2 | 11.3 | 6.6 |
| | 110 | 18.5 | 32.2 | 22.3 |
| | 115 | 17.5 | 22.4 | 10.7 |
| | 120 | 12.2 | 53.0 | 34.9 |
| Skandia | | | | |
| July | 125 | 10.0 | 61.8 | 38.8 |
| | 130 | 7.0 | 99.6 | 66.0 |
| | 135 | 5.0 | 140.4 | 93.4 |
| | 140 | 3.9 | 163.8 | 105.1 |
| | 145 | 2.4 | 266.3 | 174.2 |
| | 150 | 1.5 | 398.7 | 259.3 |
| September | 105 | 28.0 | 18.7 | 10.1 |
| | 110 | 24.2 | 25.4 | 14.6 |
| | 115 | 21.5 | 28.7 | 15.7 |
| | 120 | 18.0 | 40.1 | 23.6 |
| | 125 | 15.0 | 52.9 | 32.4 |
| | 130 | 12.5 | 66.6 | 41.6 |
| | 135 | 10.0 | 89.1 | 57.4 |
| November | 135 | 14.0 | 68.9 | 42.9 |
| January | 100 | 35.5 | 22.9 | 14.3 |
| | 130 | 18.2 | 64.6 | 42.8 |
| | 150 | 11.0 | 112.4 | 74.9 |
| Stora Enso R | | | | |
| June | 110 | 14.0 | -17.4 | -12.9 |
| | 120 | 6.3 | -35.7 | -14.7 |
| | 125 | 3.3 | -42.5 | -4.0 |
| September | 125 | 8.8 | -25.8 | 20.6 |
| | 135 | 5.5 | -40.9 | 27.6 |
| Svenska Handelsbanken A | | | | |
| June | 160 | 3.0 | -11.7 | -18.7 |
| September | 170 | 4.0 | 46.3 | 42.0 |
| Trelleborg B | | | | |
| July | 70 | 15.7 | 2.0 | 3.4 |
| | 80 | 7.0 | 3.1 | 13.9 |
| | 85 | 4.8 | -14.9 | 4.8 |
| | 90 | 2.3 | -12.4 | 27.6 |
| January | 90 | 7.0 | -11.7 | 17.7 |
| Volvo B | | | | |
| July | 170 | 8.3 | 54.3 | 34.4 |
| | 180 | 4.8 | 75.6 | 41.1 |
| | 190 | 2.2 | 137.3 | 72.7 |
| September | 170 | 12.0 | 51.0 | 33.5 |
| | 180 | 9.0 | 52.3 | 28.4 |
| | 190 | 5.3 | 94.1 | 55.0 |
| November | 170 | 15.5 | 41.4 | 25.7 |
| January | 195 | 8.5 | 86.7 | 53.8 |
| Mean Absolute Deviation (%) | | | 59.2 | 32.0 |

Notes:
- Option prices reported are closing call option prices at the Stockholm stock exchange 22nd of May 2001.
- 195 different contracts in 20 different companies are used.

As can be seen in table 6.3, the theoretical models overprice the majority of the 195 contracts examined. For all 195 contracts depicted in table 6.3, the mean absolute deviation between the Black-Scholes and the observed market prices is an astounding 59 percent. The stochastic volatility model fares better, the mean absolute deviation is 32 percent.

These huge pricing errors are of course unsatisfactory, but both the absolute size of the errors and the difference between the two theoretical models should be attributed to the divergence between historic and expected future volatility. If nothing else, the errors serve as a reminder that these two concepts are not the same thing. The indication is however, that the stochastic volatility option pricing model works better than the Black-Scholes model. This should come as no surprise since the EGARCH model captures the volatility behaviour better than the constant volatility assumption.

Historic volatility is not the premiere technique used by option analysts and market traders. Implied volatility plays a much more significant role. Implied volatility is calculated for the most liquid option contract of each stock, which without exception is the at-the-money option of the shortest maturity. The implied volatility is then used as input to the Black-Scholes formula for pricing other options on the same stock. In table 6.4, the Black-Scholes column, near maturity options are priced in this way. The mean absolute deviation for 59 contracts is 5.9 percent. Thus, this procedure provides a much better fit between theoretical and observed option prices.

Table 6.4 Call option prices (implied volatility, near maturity contracts).

| | Exercise Price SEK | Market Price SEK | Black-Scholes % deviation | Stochastic Volatility % deviation |
|---|---|---|---|---|
| ABB | | | | |
| August | 180 | 17.7 | 11.6 | 12.0 |
| | 187 | 14.2 | 5.1 | 5.2 |
| | 195 | 10.0 | 3.7 | 3.6 |
| | 210 | 4.8 | -2.9 | -2.1 |
| Astra Zeneca | | | | |
| July | 470 | 42.2 | 8.7 | 8.9 |
| | 540 | 9.0 | -11.6 | -10.8 |
| | 570 | 3.5 | -23.1 | -19.7 |
| Autoliv | | | | |
| June | 180 | 32.0 | -7.6 | -7.6 |
| | 200 | 13.0 | -9.3 | -9.2 |
| | 220 | 2.8 | -17.8 | -17.1 |
| Electrolux B | | | | |
| July | 150 | 20.0 | -6.5 | -6.5 |
| | 160 | 13.5 | -8.2 | -8.4 |
| | 170 | 6.8 | 14.2 | 13.8 |
| Ericsson B | | | | |
| June | 50 | 25.0 | 0.6 | 0.6 |
| | 55 | 20.5 | -1.5 | -1.4 |
| | 60 | 15.5 | -0.5 | -0.3 |
| | 65 | 11.7 | -5.6 | -5.4 |
| | 70 | 7.5 | -2.1 | -2.0 |
| | 80 | 2.5 | 4.1 | 4.1 |
| | 85 | 1.4 | -5.0 | -2.9 |
| | 90 | 0.7 | 0.0 | 6.2 |
| | 95 | 0.4 | -14.3 | -2.9 |
| Hennes&Mauritz B | | | | |
| July | 140 | 42.0 | -2.0 | -1.9 |
| | 180 | 10.5 | -1.3 | -1.4 |
| | 200 | 3.4 | 4.1 | 5.0 |
| | 210 | 2.1 | -8.3 | -5.4 |
| Investor B | | | | |
| June | 135 | 5.5 | -3.6 | -3.8 |
| Nokia SDR | | | | |
| June | 240 | 118.0 | -0.3 | -0.3 |
| | 250 | 111.0 | -3.0 | -3.0 |
| Nokia SDR | | | | |
| June | 260 | 99.0 | -1.1 | -1.3 |
| | 280 | 79.0 | -1.2 | -1.2 |
| | 290 | 71.0 | -3.7 | -3.6 |
| | 300 | 59.7 | -1.2 | -1.1 |
| | 310 | 53.0 | -5.5 | -5.5 |
| | 330 | 35.0 | -2.9 | -2.9 |
| | 350 | 22.0 | -3.9 | -4.0 |
| | 390 | 6.3 | -0.6 | -0.2 |
| | 410 | 3.0 | -2.0 | -0.3 |
| Pharmacia-Upjohn | | | | |
| June | 490 | 32.2 | 5.5 | 5.5 |
| | 540 | 8.5 | 2.7 | 2.7 |
| | 570 | 2.8 | 4.4 | 5.1 |
| Sandvik | | | | |
| June | 220 | 16.2 | -10.0 | -9.9 |
| | 230 | 8.5 | -1.3 | -1.3 |
| | 250 | 2.0 | -6.0 | -5.0 |
| SE-Banken A | | | | |
| June | 100 | 7.0 | -7.3 | -7.1 |
| | 110 | 1.9 | 5.4 | 5.4 |
| Skandia | | | | |
| June | 100 | 28.2 | -0.4 | -0.2 |
| | 110 | 18.5 | 4.8 | 5.0 |
| | 115 | 17.5 | -11.0 | -10.9 |
| | 120 | 12.2 | 0.2 | 0.2 |
| | 125 | 10.0 | -6.4 | -6.4 |
| | 135 | 5.0 | 2.2 | 2.4 |
| | 140 | 3.9 | -6.4 | -5.9 |
| | 145 | 2.4 | 6.3 | 7.5 |
| | 150 | 1.5 | 16.7 | 18.7 |
| Stora Enso R | | | | |
| June | 110 | 14.0 | -8.8 | -8.7 |
| | 125 | 3.3 | 24.0 | 24.0 |
| Volvo | | | | |
| June | 170 | 8.3 | 10.7 | 10.4 |
| | 180 | 4.8 | 0.0 | -0.2 |
| Mean Absolute Deviation (%) | | | 5.9 | 5.7 |

Notes:
- Option prices reported are closing call option prices at the Stockholm stock exchange 22nd of May 2001.
- 59 different contracts in 14 different companies are used.

It is possible to use implied volatility for the stochastic volatility model as well. In this case the most liquid contract near maturity is used to determine $V_0$ and the most liquid contract of the longest traded maturity is used to determine $\gamma$. (Historic EGARCH estimates are used for $\kappa$ and $\sigma$.) An iterative procedure is required as was described in the end of the previous section. Only 14 of the 20 companies have more than two liquid call option contracts of different maturity. The implied volatility sample is therefore smaller than the historic volatility sample. For the near maturity options in table 6.4, the differences between

Black-Scholes and stochastic volatility prices are negligible. The mean absolute deviation for the stochastic volatility model is 5.7 percent. More interesting, though, is that the differences increase with increasing maturity.

Table 6.5 Call option prices (implied volatility, longer maturity contracts).

| | Exercise Price SEK | Market Price SEK | Black-Scholes % deviation | Stochastic Volatility % deviation |
|---|---|---|---|---|
| ABB | | | | |
| November | 170 | 30.0 | 4.8 | 6.6 |
| | 200 | 13.5 | -3.7 | 1.0 |
| Astra Zeneca | | | | |
| September | 490 | 42.0 | -4.4 | 5.2 |
| | 510 | 29.0 | -0.5 | 14.7 |
| | 570 | 12.0 | -28.8 | 2.0 |
| Autoliv | | | | |
| March | 160 | 53.7 | 6.1 | 6.4 |
| Electrolux B | | | | |
| October | 170 | 14.0 | 1.8 | 9.1 |
| | 180 | 10.0 | 5.7 | 15.8 |
| | 190 | 7.5 | 2.8 | 15.6 |
| Ericsson B | | | | |
| August | 55 | 22.2 | -2.4 | -2.4 |
| | 60 | 19.0 | -6.8 | -7.1 |
| | 65 | 15.0 | -5.3 | -5.9 |
| | 70 | 11.0 | 1.8 | 0.5 |
| | 75 | 8.5 | 2.2 | 0.5 |
| | 80 | 6.5 | 2.3 | 0.0 |
| | 85 | 5.3 | -4.2 | -6.7 |
| | 90 | 3.5 | 7.1 | 4.6 |
| | 95 | 2.5 | 11.2 | 9.2 |
| | 100 | 1.8 | 16.6 | 16.0 |
| | 105 | 1.3 | 18.4 | 20.8 |
| October | 65 | 17.0 | -2.2 | -3.5 |
| | 70 | 14.5 | -4.3 | -6.2 |
| | 75 | 11.8 | -2.5 | -5.0 |
| | 80 | 8.8 | 8.5 | 4.9 |
| | 85 | 7.5 | 3.9 | 0.0 |
| | 90 | 6.5 | -2.0 | -6.2 |
| February | 40 | 37.2 | 0.3 | 0.2 |
| | 70 | 17.0 | 4.6 | 1.9 |
| | 80 | 13.5 | 1.0 | -2.7 |
| | 90 | 9.0 | 15.7 | 10.2 |
| | 100 | 7.5 | 6.0 | 0.1 |
| | 110 | 5.3 | 15.4 | 8.6 |
| | 130 | 3.8 | -5.3 | -11.2 |
| | 150 | 1.3 | 61.5 | 53.8 |
| | 165 | 1.0 | 43.0 | 39.0 |
| Hennes&Mauritz B | | | | |
| September | 180 | 16.2 | -6.0 | 1.8 |
| | 190 | 12.0 | -8.8 | 1.8 |
| | 200 | 9.0 | -14.8 | -1.2 |
| | 210 | 5.8 | -8.7 | 10.6 |
| November | 200 | 13.2 | -17.7 | -2.7 |
| | 210 | 9.3 | -12.3 | 8.0 |
| | 220 | 8.0 | -25.3 | -3.8 |
| January | 180 | 25.2 | -12.8 | -2.5 |
| | 190 | 20.0 | -11.8 | 1.6 |

| | Exercise Price SEK | Market Price SEK | Black-Scholes % deviation | Stochastic Volatility % deviation |
|---|---|---|---|---|
| Investor B | | | | |
| September | 150 | 3.5 | 42.0 | 13.4 |
| Nokia SDR | | | | |
| August | 280 | 89.0 | -5.0 | -4.3 |
| | 290 | 81.0 | -5.7 | -4.9 |
| | 300 | 74.5 | -7.8 | -6.9 |
| | 310 | 66.2 | -7.1 | -6.1 |
| | 330 | 51.2 | -5.4 | -4.0 |
| | 350 | 39.5 | -5.3 | -3.4 |
| | 370 | 30.7 | -7.6 | -5.1 |
| | 390 | 25.0 | -15.5 | -12.3 |
| | 410 | 18.0 | -14.0 | -9.5 |
| | 430 | 13.0 | -14.0 | -7.8 |
| | 450 | 10.0 | -20.3 | -12.4 |
| | 490 | 5.8 | -32.0 | -19.7 |
| October | 310 | 75.0 | -5.6 | -3.6 |
| | 370 | 43.5 | -8.9 | -5.0 |
| | 390 | 38.0 | -15.5 | -10.9 |
| February | 410 | 44.0 | -4.7 | 3.2 |
| | 450 | 34.0 | -8.6 | 1.8 |
| | 540 | 20.5 | -23.9 | -7.8 |
| | 600 | 11.0 | -10.8 | 16.5 |
| Pharmacia-Upjohn | | | | |
| September | 470 | 66.5 | 3.8 | 1.4 |
| | 510 | 43.5 | 4.5 | 0.0 |
| | 540 | 27.2 | 17.6 | 10.4 |
| SE-Banken A | | | | |
| August | 100 | 9.8 | 5.8 | -4.0 |
| | 110 | 4.3 | 34.8 | 11.1 |
| Sandvik | | | | |
| September | 230 | 18.5 | 3.0 | 0.0 |
| | 250 | 10.0 | 9.3 | 4.0 |
| Skandia | | | | |
| September | 105 | 28.0 | -7.1 | -5.1 |
| | 110 | 24.2 | -7.9 | -5.1 |
| | 115 | 21.5 | -12.0 | -8.6 |
| | 120 | 18.0 | -11.7 | -7.4 |
| | 125 | 15.0 | -11.9 | -6.5 |
| | 130 | 12.5 | -13.0 | -6.2 |
| | 135 | 10.0 | -11.2 | -2.6 |
| November | 135 | 14.0 | -17.2 | -4.6 |
| January | 130 | 18.2 | -10.7 | 4.4 |
| | 150 | 11.0 | -14.9 | 10.2 |
| Stora Enso R | | | | |
| September | 135 | 5.5 | 45.5 | -3.6 |
| Volvo | | | | |
| September | 170 | 12.0 | 8.3 | 7.5 |
| | 180 | 9.0 | -6.1 | -7.1 |
| | 190 | 5.3 | -0.2 | -1.5 |
| November | 170 | 15.5 | 1.6 | 0.8 |
| Mean Absolute Deviation (%) | | | 11.0 | 7.1 |

Notes:
- Option prices reported are closing call option prices at the Stockholm stock exchange 22nd of May 2001.
- 86 different contracts in 14 different companies are used.

Table 6.5 depicts the 86 contracts whose maturity is longer than the shortest maturity traded. The pricing error measured as the mean absolute deviation is 11.0 percent for the Black-Scholes model, whereas it is only 7.1 percent for the stochastic volatility model. The systematic overpricing that occurred when historic volatility was used has also disappeared. The mean error, which is not reported in the table, is –1.3 percent for the B-S model and 1.2 percent for the stochastic volatility option-pricing model. These values are not much different from zero and also of different signs.

Thus, for this sample, the stochastic volatility model is preferable independently whether historic or implied volatility is used. Implied volatility, however, provides a much better fit to observed option prices.

# 7. Summary and concluding discussions

The models of Hull and White (1987) and (1988) are, to date, the only stochastic volatility models providing a fully analytical pricing formula for European options. The model suggested in this paper is based on the same ideas as Hull and White (1987), but is characterised by a mean reverting variance component, where the logarithm of the variance follows an Ornstein-Uhlenbeck process. Under the conditions that the stock and the variance processes are uncorrelated and changes in the variance are not correlated with market return, the price of a European option is the Black-Scholes price integrated over the mean variance rate. The integral can be approximated by a Taylor series expansion, and Monte Carlo simulations show that only two terms of the series are needed to achieve a pricing error that for reasonable option prices is less than two percent. Errors are the largest for deep out-of-the-money options. To obtain an analytical pricing formula, it is possible to further approximate the first two moments of the mean variance rate, the first moment by a Taylor series expansion and the second moment through a discrete time average. Also this second approximation is reasonably accurate and robust for a wide variety of parameter settings. The pricing errors are of the same magnitude as the first approximation.

This paper suggests using GARCH for parameter estimation and develops a variant of EGARCH that is consistent with the continuous time model. The idea is to match the mean and variance of the continuous time process to that of the EGARCH process. As the time intervals gets infinitesimally short, the moments of the EGARCH process must converge to that of the continuous time process and since a Wiener process is driving the diffusion only the first two moments are required. The basic advantage of this model is that maximum likelihood estimation of the parameters is possible. The procedure is easily implemented in a standard program as for example Microsoft's Excel spreadsheet.

Initial tests of the accuracy of the stochastic volatility model are encouraging. The EGARCH model suggested gives a slightly higher value of the maximum likelihood objective function than the standard GARCH(1,1) model. Comparisons between the Black-Scholes formula and the stochastic volatility model show that the prices given by the stochastic volatility model are more in line with observed marked prices. Using historic volatility the differences between the models are striking. The mispricing is 59 percent and 32 percent respectively. For implied volatility, the novel technique of iteratively matching the initial volatility to the price of near maturity options and long-run average volatility to the price of longer maturity options was used. (Historic estimates were used for the speed of mean reversion and the diffusion parameters.) For longer maturity options the stochastic volatility model provided a better estimate. The average pricing error was 7 percent compared to 11 percent for the Black-Scholes formula. However, further empirical tests are required in order to determine the merits of the model. The cross-sectional test in this paper only includes the Swedish option market. Nevertheless, it seems reasonable to assume the model will perform well also in other markets and for different periods of time.

For implied volatility there is some hope that a deterministic volatility model would provide a god fit to observed option prices. Matching iteratively the initial volatility to the price of near maturity options and the long-run average volatility to the price of longer maturity options is of course possible for deterministic volatility models as well. In a Black-Scholes framework, it is the average volatility that is required to price the option. A linear volatility model could therefore be used as long as we are careful not to price options of shorter or longer maturity than the ones traded.

Furthermore, it might be possible to use implied volatility to estimate more parameters than just the initial and long-run average volatility. The speed of mean reversion; the diffusion parameters; the correlation factor and the market price of volatility risk can potentially also be determined from observed option prices. However, such methods have not yet been developed and the difficulties involved in solving simultaneous non-linear equations are substantial. Progress is further hampered by fact that numerical solutions are mostly required to solve stochastic volatility option-pricing models.

## Appendix A - The mean reverting stochastic volatility process

Let the variance be defined by the stochastic process

$$dV = \kappa(\gamma + \frac{\sigma^2}{2\kappa} - \ln V)Vdt + \sigma Vdz. \tag{A.1}$$

Applying Ito's lemma to the function ln $V_t$ reduces equation (A.1) to the Ornstein-Uhlenbeck process,

$$d\ln V = \kappa(\gamma - \ln V)dt + \sigma dz. \tag{A.2}$$

The mean and variance of the Ornstein-Uhlenbeck process are

$$\begin{cases} E[\ln V_t] = e^{-\kappa t}\ln V_0 + (1 - e^{-\kappa t})\gamma \\ Var[\ln V_t] = \dfrac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}). \end{cases} \tag{A.3}$$

As ln $V_t$ is normally distributed, $V_t$ becomes lognormally distributed with mean

$$E[V_t] = \exp\left[e^{-\kappa t}\ln V_0 + (1 - e^{-\kappa t})\gamma + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa t})\right]. \tag{A.4}$$

**First moment of the average**

Define

$$\begin{cases} a = \ln V_0 - \gamma \\ b = \dfrac{\sigma^2}{4\kappa} \\ y = ae^{-\kappa t} - be^{-2\kappa t} \end{cases}$$

It is now possible to simplify the notation of equation (A.4) into the much shorter expression $E[V_t] = e^{\gamma+b+y}$. However, in order to calculate the first moment of the average volatility,

$$E[\bar{V}] = E\left[\frac{1}{T}\int_0^T V_t dt\right] = \frac{1}{T}\int_0^T E[V_t]dt, \qquad \text{(A.5)}$$

it is necessary to approximate $e^{\gamma+b+y}$, as $y$ itself is an exponential function. Applying a Taylor series expansion to the factor $e^y$ and expanding it to the 5th order results in a rather exact approximation.

$$E[V_t] = e^{\gamma+b}(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} + \ldots)$$

where

$$\begin{aligned} y^2 &= a^2 e^{-2\kappa t} - 2abe^{-3\kappa t} + b^2 e^{-4\kappa t} \\ y^3 &= a^3 e^{-3\kappa t} - 3a^2 be^{-4\kappa t} + 3ab^2 e^{-5\kappa t} + O(e^{-6\kappa t}) \\ y^4 &= a^4 e^{-4\kappa t} - 4a^3 be^{-5\kappa t} + O(e^{-6\kappa t}) \\ y^5 &= a^5 e^{-5\kappa t} + O(e^{-6\kappa t}) \end{aligned}$$

Skipping everything of order $O(e^{-6\kappa t})$ and higher gives

$$\begin{aligned} E[V_t] \approx e^{\gamma+b}[1 + ae^{-\kappa t} + (\tfrac{1}{2}a^2 - b)e^{-2\kappa t} + (\tfrac{1}{6}a^3 - ab)e^{-3\kappa t} + \\ + (\tfrac{1}{24}a^4 - \tfrac{1}{2}a^2 b + \tfrac{1}{2}b^2)e^{-4\kappa t} + (\tfrac{1}{120}a^5 - \tfrac{1}{6}a^3 b + \tfrac{1}{2}ab^2)e^{-5\kappa t}]. \end{aligned} \qquad \text{(A.6)}$$

The advantage of (A.6) compared to (A.4) is that (A.6) is possible to integrate. By doing so, we obtain the first moment of the average volatility.

$$
\begin{aligned}
E[\bar{V}] &= \frac{1}{T}\int_0^T E[V_t]dt \\
&\approx \frac{1}{T}\int_0^T e^{\gamma+b}\begin{bmatrix} 1+ae^{-\kappa t}+(\tfrac{1}{2}a^2-b)e^{-2\kappa t}+(\tfrac{1}{6}a^3-ab)e^{-3\kappa t}+ \\ +(\tfrac{1}{24}a^4-\tfrac{1}{2}a^2b+\tfrac{1}{2}b^2)e^{-4\kappa t}+(\tfrac{1}{120}a^5+\tfrac{1}{2}ab^2-\tfrac{1}{6}a^3b)e^{-5\kappa t}\end{bmatrix}dt \\
&= \frac{e^{\gamma+b}}{T}[T+\frac{a}{\kappa}(1-e^{-\kappa T})+\frac{\tfrac{1}{2}a^2-b}{2\kappa}(1-e^{-2\kappa T})+\frac{\tfrac{1}{6}a^3-ab}{3\kappa}(1-e^{-3\kappa T})+ \\
&\quad +\frac{\tfrac{1}{24}a^4+\tfrac{1}{2}b^2-\tfrac{1}{2}a^2b}{4\kappa}(1-e^{-4\kappa T})+\frac{\tfrac{1}{120}a^5+\tfrac{1}{2}ab^2-\tfrac{1}{6}a^3b}{5\kappa}(1-e^{-5\kappa T})]
\end{aligned}
\tag{A.7}
$$

**Second moment of the average volatility**

The second moment is a bit more cumbersome. The first step is to change the order of integration, so that the expectation can be carried out inside the integrals. The problem thereby reduces to that of solving a deterministic double integral. The processes $V(t)$ and $V(u)$ are identical and there are two possibilities, $u \geq t$ or

$t > u$. We solve for the case when $u \geq t$ and multiply the result by two.

$$
\begin{aligned}
E\left[\left(\frac{1}{T}\int_0^T V_t\,dt\right)^2\right] &= E\left[\frac{1}{T^2}\int_0^T V_t\,dt\int_0^T V_u\,du\right] \\
&= E\left[\frac{1}{T^2}\int_0^T\int_0^T V_tV_u\,dtdu\right] \\
&= \frac{1}{T^2}\int_0^T\int_0^T E[V_tV_u]dtdu \\
&= \frac{2}{T^2}\int_0^T\int_0^u E[V_tV_u|u\geq t]dtdu
\end{aligned}
\tag{A.8}
$$

The next quest is to find $E[V_tV_u|u \geq t]$. Equation (A.3) gives

$$V_t = \exp[\ln V_t]$$
$$= \exp\left[e^{-\kappa t}\ln V_0 + (1-e^{-\kappa t})\gamma + \sqrt{\frac{\sigma^2}{2\kappa}(1-e^{-2\kappa t})}\ \varepsilon_1\right], \qquad \text{(A.9)}$$

and therefore

$$V_t V_u = \exp\left[\begin{array}{l} e^{-\kappa t}\ln V_0 + (1-e^{-\kappa t})\gamma + \sqrt{\frac{\sigma^2}{2\kappa}(1-e^{-2\kappa t})}\ \varepsilon_1 + \\ + e^{-\kappa(u-t)}\ln V_t + (1-e^{-\kappa(u-t)})\gamma + \sqrt{\frac{\sigma^2}{2\kappa}(1-e^{-2\kappa(u-t)})}\ \varepsilon_2 \end{array}\right]. \qquad \text{(A.10)}$$

Notice that when $u = t$, expression (A.10) reduces to $V_t^2$. Substituting $\ln V_t$ for the expression within the brackets of (A.9) and using the fact that $\{\varepsilon_1, \varepsilon_2\}$ are i.i.d. standard normal results in some algebra that boils down to

$$V_t V_u = \exp\left[\begin{array}{l} (e^{-\kappa t} + e^{-\kappa u})\ln V_0 + (2 - e^{-\kappa t} - e^{-\kappa u})\gamma + \\ + \sqrt{\frac{\sigma^2}{2\kappa}(1+e^{-\kappa(u-t)})^2(1-e^{-2\kappa t}) + \frac{\sigma^2}{2\kappa}(1-e^{-2\kappa(u-t)})}\ \varepsilon \end{array}\right], \qquad \text{(A.11)}$$

where $\varepsilon$ is a drawing from a standardised normal distribution. The expected value of $V_t V_u$ is

$$E[V_t V_u | u \geq t] = \exp\left[\begin{array}{l} (e^{-\kappa t} + e^{-\kappa u})\ln V_0 + (2 - e^{-\kappa t} - e^{-\kappa u})\gamma + \\ + \frac{\sigma^2}{4\kappa}\left((1+e^{-\kappa(u-t)})^2(1-e^{-2\kappa t}) + 1 - e^{-2\kappa(u-t)}\right) \end{array}\right]. \qquad \text{(A.12)}$$

which simplifies into

$$E[V_t V_u | u \geq t] = \exp[2(\gamma + b) + a(e^{-\kappa t} + e^{-\kappa u}) + \\ + b\left(2e^{-\kappa(u-t)} - 2e^{-\kappa(u+t)} - e^{-2\kappa t} - e^{-2\kappa u}\right)]. \tag{A.13}$$

In the same way as for the first moment, we now want to approximate equation (A.13) through a Taylor series expansion. Sadly, this is not feasible. Numerical tests show that terms to the sixth or seventh order in the Taylor expansion must be included if adequate accuracy is to be obtained. This would result in hundreds of terms. The situation is not hopeless, however, as a discrete time approximation works quite well,

$$\begin{aligned} E[\bar{V}^2] &\approx E[(\frac{1}{T}\int_0^T V_t dt)^2] \\ &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} V_{ti}\right)^2\right] \\ &= \frac{1}{n^2}\left[\sum_{i=1}^{n} E(V_{ti})\right]^2 \\ &= \frac{1}{n^2}\left[\sum_{i=1}^{n} E(V_{ti}^2) + 2\sum_{i=1}^{n-1}\sum_{j=1}^{n} E(V_{ti}V_{tj})\right]. \end{aligned} \tag{A.14}$$

Formula (A.14) can also be represented as a triangular matrix where all cross terms are multiplied by two. Normally, only five discrete points in time are needed to achieve an acceptable precision.

# Appendix B - The second derivative of the B-S formula with respect to the variance

Given the Black-Scholes formula (3.2), we want to derive the second derivative of the call option price with respect to the variance rate, $\frac{\partial^2 C}{\partial V^2}$. The derivation is included for completeness since it to the best of the author's knowledge has not been published elsewhere, although the result is stated in Hull and White (1987).

In (3.2) the notation $\bar{V}$ was used in order to stress that it is the average variance rate that is needed to price a stochastic volatility option in the extended Black-Scholes framework of this paper. However, for the purpose of this derivation we use the simpler notation $V$ for the variance instead. The B-S formula is

$$C(V) = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) \tag{B.1}$$

$$d_1 = \frac{\ln(S_0/K) + (r-\delta)T + \frac{1}{2}VT}{\sqrt{VT}}$$

$$d_2 = \frac{\ln(S_0/K) + (r-\delta)T - \frac{1}{2}VT}{\sqrt{VT}}.$$

The first step is to simplify the notation of $d_1$ and $d_2$ in order to avoid using the product or the quotient differentiation rules more than once. Splitting $d_1$ and $d_2$ into two terms and simplifying the constant give

$$d_1 = dV^{-1/2} + \tfrac{1}{2}T^{1/2}V^{1/2} \tag{B.2}$$

$$d_2 = dV^{-1/2} - \tfrac{1}{2}T^{1/2}V^{1/2} \tag{B.3}$$

where $d = \dfrac{\ln(S_0 / K) + (r - \delta)T}{\sqrt{T}}$.

The product of $d_1$ and $d_2$ is also needed. Using the conjugate rule we get

$$d_1 \cdot d_2 = d^2 V^{-1} - \tfrac{1}{4} TV . \tag{B.4}$$

Taking the first partial derivatives of $d_1$ and $d_2$ with respect to the variance gives

$$\frac{\partial d_1}{\partial V} = -\tfrac{1}{2} dV^{-3/2} + \tfrac{1}{4} T^{1/2} V^{-1/2}$$

$$\left(\frac{\partial d_1}{\partial V}\right)^2 = \tfrac{1}{4} d^2 V^{-3} - \tfrac{1}{4} dT^{1/2} V^{-2} + \tfrac{1}{16} TV^{-1} \tag{B.5}$$

$$\frac{\partial d_2}{\partial V} = -\tfrac{1}{2} dV^{-3/2} - \tfrac{1}{4} T^{1/2} V^{-1/2}$$

$$\left(\frac{\partial d_2}{\partial V}\right)^2 = \tfrac{1}{4} d^2 V^{-3} + \tfrac{1}{4} dT^{1/2} V^{-2} + \tfrac{1}{16} TV^{-1} . \tag{B.6}$$

The second derivatives are

$$\frac{\partial^2 d_1}{\partial V^2} = \tfrac{3}{4} dV^{-5/2} - \tfrac{1}{8} T^{1/2} V^{-3/2} \tag{B.7}$$

$$\frac{\partial^2 d_2}{\partial V^2} = \tfrac{3}{4} dV^{-5/2} + \tfrac{1}{8} T^{1/2} V^{-3/2} . \tag{B.8}$$

We also need to differentiate $N(x)$.

$$N'(x) = \tfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$N''(x) = -x \cdot \tfrac{1}{\sqrt{2\pi}} e^{-x^2/2} = -xN'(x) \tag{B.9}$$

The last result needed before starting to differentiate the Black-Scholes formula is to express $N'(d_2)$ in terms of $N'(d_1)$.

$$\begin{aligned} N'(d_2) &= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_1 - \sqrt{TV})^2} \\ &= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_1^2 - 2d_1\sqrt{TV} + TV)} \\ &= N'(d_1) e^{d_1\sqrt{TV} - \frac{1}{2}TV} \\ &= N'(d_1) e^{\ln S_0/K + (r-\delta)T + \frac{1}{2}TV - \frac{1}{2}TV} \\ &= N'(d_1) \frac{S_0 e^{-\delta T}}{Ke^{-rT}} \end{aligned} \tag{B.10}$$

We are now in a position to actually differentiate the Black-Scholes formula.

$$\frac{\partial C}{\partial V} = S_0 e^{-\delta T} N'(d_1) \frac{\partial d_1}{\partial V} - Ke^{-rT} N'(d_2) \frac{\partial d_2}{\partial V}$$

$$\begin{aligned} \frac{\partial^2 C}{\partial V^2} = S_0 e^{-\delta T} &\left[ N''(d_1) \left( \frac{\partial d_1}{\partial V} \right)^2 + N'(d_1) \frac{\partial^2 d_1}{\partial V^2} \right] + \\ &+ Ke^{-rT} \left[ -N''(d_2) \left( \frac{\partial d_2}{\partial V} \right)^2 - N'(d_2) \frac{\partial^2 d_2}{\partial V^2} \right] \end{aligned}$$

Use equation (B.9),

$$\frac{\partial^2 C}{\partial V^2} = S_0 e^{-\delta T} N'(d_1)\left[-d_1\left(\frac{\partial d_1}{\partial V}\right)^2 + \frac{\partial^2 d_1}{\partial V^2}\right] + Ke^{-rT} N'(d_2)\left[d_2\left(\frac{\partial d_2}{\partial V}\right)^2 - \frac{\partial^2 d_2}{\partial V^2}\right].$$

Use equation (B.10),

$$\frac{\partial^2 C}{\partial V^2} = S_0 e^{-\delta T} N'(d_1)\left[-d_1\left(\frac{\partial d_1}{\partial V}\right)^2 + \frac{\partial^2 d_1}{\partial V^2} + d_2\left(\frac{\partial d_2}{\partial V}\right)^2 - \frac{\partial^2 d_2}{\partial V^2}\right].$$

Use the partial derivatives (B.5)-(B.8),

$$\frac{\partial^2 C}{\partial V^2} = S_0 e^{-\delta T} N'(d_1)[-d_1\left(\tfrac{1}{4}d^2 V^{-3} - \tfrac{1}{4}dT^{1/2}V^{-2} + \tfrac{1}{16}TV^{-1}\right) + \tfrac{3}{4}dV^{-5/2} - \tfrac{1}{8}T^{1/2}V^{-3/2}$$
$$+ d_2\left(\tfrac{1}{4}d^2 V^{-3} + \tfrac{1}{4}dT^{1/2}V^{-2} + \tfrac{1}{16}TV^{-1}\right) - \tfrac{3}{4}dV^{-5/2} - \tfrac{1}{8}T^{1/2}V^{-3/2}].$$

Within the bracket, the terms $\tfrac{3}{4}dV^{-5/2}$ cancel. By taking $\tfrac{1}{4}T^{1/2}V^{-3/2}$ outside the bracket we get

$$\frac{\partial^2 C}{\partial V^2} = \frac{S_0 e^{-\delta T} T^{1/2} N'(d_1)}{4V^{3/2}}[-d_1\left(d^2 T^{-1/2}V^{-3/2} - dV^{-1/2} + \tfrac{1}{4}T^{1/2}V^{1/2}\right) +$$
$$+ d_2\left(d^2 T^{-1/2}V^{-3/2} + dV^{-1/2} + \tfrac{1}{4}T^{1/2}V^{1/2}\right) - 1].$$

Substituting $d_1$ and $d_2$ with the expressions in (B.2) and (B.3) gives

$$\frac{\partial^2 C}{\partial V^2} = \frac{S_0 e^{-\delta T} T^{1/2} N'(d_1)}{4V^{3/2}} \cdot$$
$$\cdot [-\left(dV^{-1/2} + \tfrac{1}{2}T^{1/2}V^{1/2}\right)\left(d^2T^{-1/2}V^{-3/2} - dV^{-1/2} + \tfrac{1}{4}T^{1/2}V^{1/2}\right) +$$
$$+ \left(dV^{-1/2} - \tfrac{1}{2}T^{1/2}V^{1/2}\right)\left(d^2T^{-1/2}V^{-3/2} + dV^{-1/2} + \tfrac{1}{4}T^{1/2}V^{1/2}\right) - 1].$$

The expression within the bracket is lengthy, but simple in structure. It has the form

$$-(A+B)(C-D+E)+(A-B)(C+D+E)-1.$$

Simplifying gives

$$\begin{aligned} -(A+B)(C-D+E)+(A-B)(C+D+E)-1 &= 2(AD-BC-BE)-1 \\ &= 2(d^2V^{-1} - \tfrac{1}{2}d^2V^{-1} - \tfrac{1}{8}TV) - 1 \\ &= d^2V^{-1} - \tfrac{1}{4}TV - 1. \end{aligned}$$

The second derivative becomes

$$\frac{\partial^2 C}{\partial V^2} = \frac{S_0 e^{-\delta T} T^{1/2} N'(d_1)}{4V^{3/2}} [d^2V^{-1} - \tfrac{1}{4}TV - 1].$$

The final formula is obtained by using (B.4),

$$\frac{\partial^2 C}{\partial V^2} = \frac{S_0 e^{-\delta T} T^{1/2} N'(d_1)}{4V^{3/2}} [d_1 d_2 - 1]. \tag{B.11}$$

Formula (B.11) is part of the call price formula (3.9).

## Appendix C - Parameter estimation by means of EGARCH

The task is to match the mean and variance of the EGARCH model (5.1) and (5.3) in the main text to the continuous time model of (5.4) and (5.5). The equations are here repeated as (C.1) - (C.4).

$$y_t = \sigma_t \varepsilon_t \tag{C.1}$$

$$\ln \sigma_{t+1}^2 = \psi_0 \ln \overline{\sigma}^2 + (\psi_1 + \psi_2) \ln \sigma_t^2 + \psi_2 \ln(\varepsilon_t^2 + 1) \tag{C.2}$$

$$y_t = \Delta \ln S_t - (\alpha - \tfrac{1}{2} V_t)\Delta t = \sqrt{V_t \Delta t}\, w_t \tag{C.3}$$

$$\ln V_{t+1} = (1 - e^{-\kappa\Delta t})\gamma + e^{-\kappa\Delta t} \ln V_t + \sigma \sqrt{\frac{1 - e^{-2\kappa\Delta t}}{2\kappa}}\, \varepsilon_t \tag{C.4}$$

As the innovations $\{\varepsilon_t, w_t\}$ are standard normal, equations (C.1) and (C.3) imply that

$$\sigma_t^2 = V_t \Delta t\,. \tag{C.5}$$

Substitution of equation (C.5) into (C.4) gives

$$\ln \frac{\sigma_{t+1}^2}{\Delta t} = (1 - e^{-\kappa\Delta t})\gamma + e^{-\kappa\Delta t} \ln \frac{\sigma_t^2}{\Delta t} + \sigma \sqrt{\frac{1 - e^{-2k\Delta t}}{2\kappa}}\, \varepsilon_t\,,$$

which can be further simplified into

$$\ln \sigma_{t+1}^2 = (1 - e^{-\kappa\Delta t})(\gamma + \ln \Delta t) + e^{-\kappa\Delta t} \ln \sigma_t^2 + \sigma \sqrt{\frac{1 - e^{-2k\Delta t}}{2\kappa}}\, \varepsilon_t\,. \tag{C.6}$$

Equations (C.2) and (C.6) are now compatible except for the different distribution of the innovations. In order to match the mean and variance of these equations, we need the following results obtained by numerical integration.

$$E[\ln(\varepsilon_t^2+1)] = \int_{-\infty}^{\infty} \frac{\ln(x^2+1)}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \sqrt{2/\pi}\int_0^{\infty} \ln(x^2+1)\cdot e^{-\frac{x^2}{2}} dx = 0.53345 \qquad \text{(C.7)}$$

$$E[[\ln(\varepsilon_t^2+1)]^2] = \int_{-\infty}^{\infty} \frac{[\ln(x^2+1)]^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \sqrt{2/\pi}\int_0^{\infty} \ln[(x^2+1)]^2\cdot e^{-\frac{x^2}{2}} dx = 0.55291$$

$$\begin{aligned} Var[\ln(\varepsilon_t^2+1)] &= E[[\ln(\varepsilon_t^2+1)]^2] - E[\ln(\varepsilon_t^2+1)]^2 \\ &= 0.55291 - 0.53345^2 = 0.26834 \end{aligned} \qquad \text{(C.8)}$$

Using (C.7) to match the mean of (C.2) and (C.6) gives the equality

$$\psi_0 \ln \bar{\sigma}^2 + (\psi_1+\psi_2)\ln\sigma_t^2 + 0.53345\psi_2 = (1-e^{-\kappa\Delta t})(\gamma + \ln \Delta t) + e^{-\kappa\Delta t}\ln\sigma_t^2.$$

In order for this equality to hold for all values of $\ln\sigma_t^2$, it is required that both the constant terms and the coefficients of $\ln\sigma_t^2$ are the same on both sides of the equality. This gives two equations.

$$\begin{cases} \psi_0 \ln\bar{\sigma}^2 + 0.53345\psi_2 = (1-e^{-\kappa\Delta t})(\gamma+\ln\Delta t) \\ \psi_1 + \psi_2 = e^{-\kappa\Delta t} \end{cases}$$

Solving the first equation for $\gamma$ and the second for $\kappa$, we get

$$\begin{cases} \gamma = \dfrac{\psi_0 \ln \bar{\sigma}^2 + 0.53345\psi_2}{1-(\psi_1+\psi_2)} - \ln \Delta t \\ \kappa = -\dfrac{\ln(\psi_1+\psi_2)}{\Delta t}. \end{cases} \tag{C.9}$$

Using (C.8) to match the variances of (C.2) and (C.6) gives

$$0.26834\psi_2^2 = \frac{\sigma^2}{2\kappa}(1-e^{-2\kappa\Delta t}),$$

which allows the third and final parameter to be expressed as

$$\sigma^2 = \frac{2 \cdot 0.26834\kappa\psi_2^2}{1-(\psi_1+\psi_2)^2}. \tag{C.10}$$

The formulas (C.9) and (C.10) appear in the main text.

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# Hedging of long-term commodity-linked contracts

# (Essay 5)

Henrik Andersson

## Abstract

In this paper we research the hedging error that comes about through synthetic replication of long-term commodity-linked contracts. An extensive data set of close to 300 different commodities has been used in the study. Focusing on the price process, the standard geometric Brownian motion provides the most robust pricing formulas and hedging recipes. Hedging errors are however smaller for a new mean reverting process, supporting the intuition that commodity prices are mean reverting. In this way, hedging tests can act as a complement to purely statistical tests of mean reversion since such tests unfortunately have low power. A stochastic volatility model is also tested but is found to work poorly. Hedging errors are the largest for this model. The issue of choosing an appropriate price process is overshadowed by the problem of parameter estimation. Historic volatility may be very different from the actual volatility and this deviation introduces a substantial error that makes the hedging of long-term commodity-linked contracts very difficult.

*Keywords*: Commodity-Linked Contracts, Hedging, Mean Reversion, Stochastic Volatility

*JEL Classification codes*: C13, G13, G24

# 1. Introduction

Commodity-linked contracts are financial securities where the payments are linked to the price of one or several commodities. For example, companies within the oil industry have issued bonds where the repayment is linked to the price of oil. If the price is high, the oil industry is profitable and the companies are therefore able to pay a higher return on the bonds. If the price of oil is low, the contracts stipulate a smaller and thereby less burdensome repayment of the bonds. Using this type of exotic financing, the cost of debt financing is related to the prosperity of the company. When income is high, so is the ability to service the bonds.

Issuing a bond that is linked to the price of a commodity can make much sense for the issuing company. It may ensure survival even when times are tough and there is a major decline in commodity prices. A straight loan will be charged a relatively high interest rate due to the credit risk of the issuing company. However, the number of basis points charged above the risk-free rate decreases if the straight loan is combined with some option kickers that allow investors to benefit from an increase in commodity prices. Debt payments are therefore smaller, something that becomes especially important for companies experiencing rapid growth and in investment-heavy industry sectors such as the extraction of raw materials. A smaller debt burden is also important for developing countries, where a large part of the national income is depending on the export of commodities.

As always when financial securities are at issue, there are two questions that overshadow all others, namely how to price and hedge the contract. In one way the hedging concern is more pressing than the pricing problem. Of course it helps to have a trustworthy model that provides a theoretical price of the security, but it is always the market that determines the price, one way or the other. How to hedge the risks involved in the often very complicated financial arrangement, however, needs some direct support from theory. And financial theory has it that pricing and hedging are inextricably interlinked.

The commodity-linked bond above is distinguished from a straight bond by the options included. Option pricing is based on the idea that the option can be replicated synthetically by constructing a portfolio whose payments match the payments of the option contract. As a direct consequence, the price of the option is uniquely determined by the value of the so-called replicating portfolio. Otherwise arbitrage opportunities would exist. It is thus possible for an investor, at least in theory, to take a long position in the commodity-linked bond and hedge the commodity price risk by replicating the value of a short position. Hedging and replication are two sides of the same coin and it is the possibility to replicate a contract that gives the arbitrage-free price.

However, the argument hinges on the assumption that the stochastic processes of the assets included in the replicating portfolio are known. It is thereby not too blunt a statement to say that the price of an option is a bet on these stochastic processes. Long-term contracts are especially sensitive to this bet, as the difference in value between various assumptions becomes substantial. The pulp plant in essay 2 provides one example of this sensitivity.

Hedging of commodity contracts involve some specific difficulties. The only asset generally available is the commodity itself. Sometimes futures contracts exist. The wide variety of option contracts of different maturities and exercise prices that exists for financial assets is not available. In discrete time this implies that only delta-hedging can be performed. Hedging of convexity (gamma), time (theta) and volatility (vega) require multiple assets, typically other options. Being obliged to resort to delta-hedging severely limits the effectiveness of hedging, making the choice of an appropriate stochastic process all the more important. Long-term contracts, as in the present case, also make static hedging impossible. There are no long-term contracts available in the market to perform a static hedge. Dynamic hedging is therefore a necessity.

This essay studies to what extent long-term commodity-linked contracts can be hedged. The purpose is to identify the price process that most closely approximates the data. Three different processes and several estimation procedures are tested. The first candidate stochastic process is the ordinary geometric Brownian motion

$$dS = \alpha S dt + \sigma S dw, \tag{1.1}$$

as it is the standard assumption in derivative pricing and the process on which the Black-Scholes formula is based. It has the advantage of being simple, but not simplistic, and is also quite robust when it comes to hedging. The geometric Brownian motion is the benchmark model and it normally works reasonably well.

The second candidate is the mean reverting model developed in essay 2,

$$dS = \eta(\gamma + \omega t - \ln S)S dt + \sigma S dw. \tag{1.2}$$

This model allows the price to revert towards an equilibrium price $\gamma$ that is increasing over time $\omega t$. Allowing for a drift in the equilibrium price is an innovation compared to existing mean reverting models. It is therefore of extra interest to study the performance, both in comparison to models that does not allow for drift and to the standard geometric Brownian motion.

The mean reversion is modelled in the log price $\ln S$ in order to make the price process lognormal and therefore non-negative. There are good theoretical reasons to believe that commodity prices are mean reverting. In the long run, prices of standardised goods should revert to the marginal cost of production. But mean reversion is difficult to detect statistically. The Phillips-Perron test applied in essay 3 to the discrete time version of (1.2) only supports mean reversion in a few cases. However, given the statistical difficulties present in detecting mean reversion, studying the hedging performance is especially interesting. After all, the ability to hedge a contract is the ultimate test of whether or not mean reversion is a workable assumption for asset pricing. When different stochastic processes give rise to different option values, the correct process and the correct price is the one that is possible to hedge.

Apart from mean reversion, there is another characteristic that we want to test in order to determine its relative importance: stochastic volatility. As was detailed in essay 3, leptokurtosis, or fat tails, is prevalent in all commodity return time series. Leptokurtosis provides a challenge for option pricing. If the fat tails are due to the fact that the distribution of price changes is stable Paretian, option prices cannot be calculated in the normal way. Self-financing portfolios that replicate the contract cannot be found. This is proved by Popova and Ritchken (1998). An alternative explanation for the measured leptokurtosis is that it stems from a mixture of normal distributions. Such a characterisation is less tractable in that the distribution is not stable over time and a lot more parameters are required to describe the price shock. However, the advantage is that stochastic volatility option pricing models have been developed to deal with this situation. It is therefore interesting to investigate hedging efficiency when stochastic volatility is accounted for. We test the stochastic volatility model (1.3) and (1.4), by using the approximations and parameter estimation procedures developed in essay 4,

$$dS = \alpha S dt + \sqrt{V} S dw \tag{1.3}$$

$$dV = \kappa(\gamma + \frac{\sigma^2}{2\kappa} - \ln V)V dt + \sigma V dz. \tag{1.4}$$

In this model, the commodity follows a geometric Brownian motion where the volatility parameter is stochastic and determined by a mean reverting process.

The contract used in the study is the put option of a zero-coupon commodity-linked bond. The bond is repaid in full if the price of the commodity is above a certain threshold level and the repayment decreases linearly when the commodity price is below the threshold. Consequently, the investors' side of the contract can be decomposed into a straight bond and a short put option. (We choose to study the investors' side of the contract. The issuing company is long a put option on the commodity it produces and thereby already hedged.) The straight bond is not of any separate interest but the put option is. It is a simple but appropriate contract for testing the hedging performance. The option payment is non-linear in the commodity price, implying that a buy-and-hold hedging strategy is not possible. At the same time the contract is simple enough not to divert the focus away from the stochastic processes we want to highlight. If the contract becomes too complex, this complexity and the choice of hedging methodology will probably overshadow the differences that the use of different stochastic processes give rise to.

280 price series of different commodities are collected from the Datastream commodity data base. The data have been carefully screened for consistency. 5-year contracts are being hedged. The tests are conducted on daily sampled price quotations. This is as close we can get to the continuously sampled observations specified by the model. 280 price series provide a sufficiently large sample for making inferences about which stochastic process best approximates the price series in terms of its ability to hedge option contracts. It

is also the largest sample we could reasonably hope for. There are only a limited number of commodities available in the world markets.

The paper is organised as follows. Section 2 describes some commodity-linked contracts that have been issued and accounts for earlier research concerning hedging efficiency. Section 3 gives the mathematical foundation for option replication and hedging. Tests are carried out in sections 4 and 5. In section 4, the parameters are estimated from within the sample in order to minimise the problem of parameter misspecification. This is probably the best way to identify the most appropriate price process, but as a feasible hedging scheme it is unrealistic. Section 5 therefore proceeds by testing the best procedures identified in section 4 for parameters estimated out of sample. Section 6 summarises. European option prices for the mean reverting process (1.2) are derived in appendix A. Details of the hedging schemes are provided appendix B. The commodities included are listed in the appendix of essay 3.

## 2. Market preliminaries and previous research

### Contract specification

In this essay, a commodity-linked contract refers to any financial contract that has a payment dependent on the price of a commodity. Sometimes, more narrow definitions are seen in the literature. For example, the phrase 'commodity-linked bonds' has been used to describe the specific contract of a bond whose principal may be exchanged for a predetermined amount of the commodity at the discretion of the bond-holder. We will use 'commodity-linked' in a wider sense.

Often, commodity-linked contracts are tailored to provide financing for a specific investment, so called project financing. Finnerty (1996, p. 2), defines project financing as "the raising of funds to finance an economically separable capital investment project in which the providers of the funds look primarily to the cash flow from the project as the source of funds to service their loans and provide a return on their equity invested in the project". The most important characteristic is that the project is a separate entity. When this is the case, more or less advanced financial engineering techniques can be applied to obtain financing for the entity. Mineral extraction and energy production are among the projects that are the easiest to separate.

In 1972, BP (British Petroleum) obtained bank financing to develop a North Sea oil field. The $945 million provided was the largest industrial loan ever at that time. The repayment of the loan was connected to the price of oil. The contract called for a certain amount of oil to be delivered to the financing consortium. At the same time, another BP subsidiary repurchased the oil at a fixed price. The loan can therefore be described as an advanced payment for future deliveries of oil. The investors bore no price risk, their only risk were that the reserves of the field were inadequate.[1]

To finance the multibillion dollar construction of the Trans-Alaska Pipeline in the seventies, a wide variety of financing vehicles were used. Some were commodity-linked. Sohio - Standard Oil of Ohio, the largest partner in the consortium, received $175 million in form of an advanced payment for natural gas and crude oil deliveries to the Columbia Gas Company. The Internal Revenue Service agreed to treat the agreement as a loan despite the language of the agreement. Taxation were thereby postponed and Sohio obtained off-balance sheet financing, that did not affect other commitments made about debt and seniority. Another $300 million were obtained in a similar fashion from other companies. $665 million were raised through revenue bonds on the operation of the Valdez Marine Terminal.[2]

Valuation of the above contracts is not very difficult as long as the interest rate and the convenience yield of having physical ownership of the oil are treated as fixed. After all, the alternative to a forward delivery of oil is to purchase the oil immediately and store it to the contract date. From the eighties and onwards, however, option features have come to dominate the contracts.

Continuing on the oil theme, Chase investment bank in London arranged in November 1989 a deal for the benefit of the Algerian state-owned oil company Sonatrach. In this case $100 million was lent from a variety of creditors. The low interest rate charged, LIBOR + 1%, reflected some options embedded in the contract. The rate on straight debt would have been LIBOR + 3-4%, according to market estimates quoted by Millman (1991). If the market price of

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[1] In Brealey&Myers, "*Principles of corporate finance*", 5th.ed., chapter 24, McGraw-Hill (1996).

[2] In Philips P., Groth J., Richards M., "Financing the Alaskan project: The experience at Sohio", *Financial Management*, pp: 7-16 (Autumn 1979).

oil rose above a certain level, Sonatrach was to pay an additional amount of money to the investors. If the market price of oil declined below a certain level, Chase was to pay the investors an additional amount of money as compensation for the increased credit risk of the loan. As a compensation for the deal, Chase was paid a series of call options from Sonatrach and hedged itself in the short-term oil derivatives market.[3]

Magma Copper Company has issued 10-year debentures where the coupon payments are linked to the price of copper.[4] The annual coupon, paid quarterly, varies between 12 percent and 21 percent depending on the price of copper. In effect, the bondholders' position can be decomposed in two ways. Either as a 10-year 12 percent coupon bond with 40 capped call options on copper, or as a 21 percent coupon bond in combination with 40 short put options with a floor equivalent to 12 percent. The options are, one at a time, maturing in each quarter over the next ten years.

Commodity swaps are sometimes employed as a way of obtaining capital. In 1989, Banque Paribas arranged a deal where Mexicana de Cobre, a Mexican copper mining company, received $210 million in financing from syndicated banks.[5] Some of the copper produced was sold to a Belgian copper mining company. Payments for the copper were deposited in an account in New York. These payments, which were floating in the sense that they depended on the market price of copper, were then swapped through a standard commodity swap into fixed payments. The fixed payments went to the syndicated banks for servicing the 3-year loan. Mexicana de Cobre was thus able to obtain international financing at a time when the Latin American markets were not sufficiently developed to effect a deal of this size and investors were reluctant

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[3] In O'Brian, "*Global financial management*", chapter 7, Wiley (1996).
[4] In Smithson C., Chew D., "The uses of hybrid debt in managing corporate risk", *Journal of Applied Corporate Finance*, pp. 79-89 (1992).
[5] In O'Brian, "*Global financial management*", chapter 7, Wiley (1996).

to invest in Latin-America. It was, in fact, the first commercial loan to a Mexican company since 1982.

Commodity-linked contracts also appear in the seemingly ever-increasing market for asset backed commercial paper (ABCP). An asset backed commercial paper is much like securitization of mortgages except that other things than housing-loans are being pooled and issued to investors: Trade receivables; equipment loans; high-yield bonds etc. At the end of the year 2000 the value of outstanding ABCP was a stunning $642 billion in the US alone.[6] Relating to commodities, collateralised trade receivables for oil are the most common. It has been put forward that commodity receivables in general and oil receivables especially, have less default risk due to the national importance of the assets for many countries.

Asset backed commercial paper also exist for other commodities than oil. In June 2000, Glencore, one of the world's largest privately owned companies, closed a 5-year $1.2 billion trade receivables securitization. The deal is backed by trade receivables from Glencore's normal business of oil, mineral and metal trading.[7] In November 2000, ABN Amro arranged an $150 million ABCP for Hanjin shipping, the 4th largest container firm in the world. The deal is backed by trade receivables for bills of lading and terminal operation fees.[8]

Arguably, the biggest importance of commodity-linked contracts is for developing and less industrialised countries. In a report from the World Bank, Varangis and Larson (1996), say that for as many as 36 developing countries, primary commodity exports stand for more than 50 percent of total export

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[6] This figure is quoted from O'Malley, Maza and Capecci (2001).
[7] Newsflash from *Asset Finance International*, London (June 2000).
[8] Newsflash from *Euroweek*, London (Nov 24. 2000).

revenues. For several countries, one single commodity accounts for more than 80 percent.

Many companies in these countries have trouble finding enough financing because the domestic capital market is not sufficiently developed. International investors are also reluctant to supply adequate means due to previous experiences. Debt service in form of commodities may provide a "harder currency" and a more liquid asset than the local means of exchange. Governments keen on putting a lid on debt services, dividend payments, and conversion of the local currency are also less likely to interfere in the export of goods. For companies in the industrialised world, commodity-linked contracts may provide a lower effective rate on debt payments and can in some ways be seen as providing risk-capital to the company because repayment is higher when the company is prosperous and lower during bad times. In this way, it can also help to alleviate principal agent problems between shareholders and bondholders. Tax reasons and off-balance sheet financing (whatever advantages the latter may provide) are also quoted reasons for the commodity linkage.

Pricing of commodity-linked contracts would be easy had there been a market for long-term commodity options. The contracts could then be decomposed into separate option positions and each position valued in accordance with the price set by the option market. Given that no such market exists, however, one is forced to rely on arbitrage arguments built around a replicating portfolio. Being able to replicate the position not only provides an arbitrage-free price. It also means that financial intermediaries, banks and investment banks, can act basically as brokers, hedging the position and earning the transaction fees. In the earlier example of the Sonatrach deal, Chase investment bank took on a considerable price risk in that it was to compensate the debtors for the increased credit risk when the oil price was low. In return, Chase received a series of call options on oil from Sonatrach. Thus Chase had negative payments when the oil price was low and positive when the oil price was high. Corresponding short futures contracts can to some extent offset the option positions, but there is also a time discrepancy - futures markets are short-term. More sophisticated hedging is therefore required.

**Previous research**

There are two crucial aspects of option hedging. The first one is that the assumed price process is an adequate description of actual asset price behaviour, and the second is the presence of transaction costs.[9] Research has for some reason concentrated almost exclusively on the thorny question of hedging in the presence of transaction costs, taken the price process as given. The premier exception is Melino and Turnbull (1995). They assume that a stochastic volatility model governs the exchange rate and that both the exchange rate and the volatility are mean reverting. Short-term options are assumed to be traded and prices set in accordance with this model. The B-S implied volatility from these options is then used to hedge long-term option contracts. Thus there is a model misspecification causing a hedging error. The exchange rate exhibits stochastic volatility while the replicating portfolio is built around the assumption that the volatility is constant. Simulations show that the misspecification on average causes the portfolio value to deviate from the contract payment by close to 50 percent for 5-year contracts. The pricing error is less severe, around 12-15 percent depending on the exercise price.

It is thus possible, and perhaps not even unlikely, for a financial institution to offer approximately the right price on an OTC-contract, but not to be able to hedge the market exposure. There is, of course, nothing unique in that

[9] Actually, we will see in section 5 that there is a third crucial aspect as well, parameter estimation. This issue overshadows the price process errors and possibly the transaction costs as well, at least when historical data are "naively" applied to the future.

situation. Many strategies can give a correct price, for example pricing the contract in line with what other institutions charge, but that provides no guidance in hedging the exposure. What the Melino and Turnbull study adds is the observation that a slightly misspecified theoretical model is not much help in hedging either. Pricing formulas appear to be relatively robust whereas hedging recipes are not. The root of the misfortune is that with only one underlying asset in the portfolio, a foreign bond in this case, it is impossible to hedge against changes in both the exchange rate and the volatility. One more asset is required. An asset whose value changes when the volatility changes, i.e. a derivative. The weight of the foreign bond in such a portfolio would be vastly different from the weight in the one underlying asset portfolio, explaining why the hedging performance is so poor.

On a more positive note however, the simulations by Melino and Turnbull also show, repeatedly, that the hedging performance is not much affected by infrequent revision of the portfolio. The performance does not deteriorate substantially if revision is carried out every fortnight instead of daily.

It should come as no surprise that Black-Scholes, in their 1972 paper, were the first to test the hedging ability. They studied whether it was possible to earn an excess return by writing and hedging options that the model has identified as overvalued, and buying and hedging options that were undervalued?[10] The answer was affirmative, if transaction costs were ignored. However, using theoretical option prices instead of market prices, the excess return was significantly negative. These results can be interpreted as saying that the

[10] Fisher Black (1989) gives an interesting account of how they tried to earn some real money on their formula. Organised option trading had not started at the time of their break-through, but warrants were available. Having identified a warrant that was cheap in comparison to the theoretical value, Black, Scholes and Merton invested some of their own money. However, there was a good reason for the deviation. The company soon became the target for a tender offer and the warrants fell sharply. The moral is that the market may know something that a formula does not.

market is not always right, but neither is the model. Market mispricing can occur. Excluding transaction costs, profits can be made. But the model does not work perfectly. Using theoretical option prices, Black and Scholes were not able to replicate the payoff exactly, and for the chosen options the difference was negative.

Boyle and Emanuel (1980) study the return distribution of a hedged portfolio. The continuous time models normally used to describe the underlying asset process require a continuous readjustment of the hedging portfolio. This is impossible as asset prices are only observable during daytime when the exchanges are open and then restricted to certain discrete values, tick-sizes. The presence of transaction costs also makes a continuous readjustment immensely expensive, in the limit infinitely so. Discrete time approximations are therefore a prerequisite, but then the replicating portfolio is no longer risk-free. The worse the hedge and the longer the time between rebalancing, the more pressing is the problem.

However, changes in the value of the hedging portfolio are uncorrelated with the market return. The proof of this appears as a footnote in the original Black and Scholes article of 1973. The hedging portfolio should therefore earn the risk-free rate of return even with discrete-time hedging. The difference being that the return is not certain, it is the expected return on the portfolio that should be equal to the risk-free rate.

The hedged portfolio in the Boyle and Emanuel paper is constructed by going long one call option and shorting the required number of stocks to make the hedge riskless. The initial cost of setting up the portfolio is borrowed at the risk-free rate. The expected return of this strategy is zero but the distribution is skewed to the right and leptokurtic. The value will be negative 68 percent of the time. A positive return is only experienced for relatively large price movements. The effect should not be exaggerated, however. As shown by Melino and Turnbull (1995), the percentage deviation is quite small if the model is correctly specified. Discrete-time revisions do not in themselves lead to substantial mishedging of long-term contracts.

Discrete-time revisions may lead to cash flow problems, however, as the misfortune of Metallgesellshaft (MG) shows. In 1991, their North-American subsidiary - MG Refining and Marketing, started to market 5 and 10-year contracts for delivery of heating oil and gasoline at a fixed price. The risk borne by Metallgesellshaft was hedged by purchasing short-term futures contracts on crude oil and gasoline and rolling these contracts forward. The interest for such contracts were huge and in the end of 1993, MG Refining and Marketing had long-term contracts for the delivery of more than 100 million barrels of oil outstanding. With such an amount of contracts outstanding it seems a correct decision of MG to use the liquid market of short-term futures contracts rather than longer-term futures contracts. By using the illiquid market of longer-term futures, MG would have run the risk of turning the market against them. But the use of short-term futures contracts presented MG with a cash flow problem. The crude oil price did steadily decline during the second half of 1993. The price fell from $18 in the beginning of June to $13.5 in mid December, causing large losses in the long futures contracts. These losses were somewhat offset by the gains on the fixed delivery contracts. There is, however, an academic debate whether Metallgesellshaft was overly hedged or whether the hedge ratio was appropriate.[11]

The big difference, however, between the losses on the futures contracts and the gains on the long-term fixed contracts was that the losses on the futures contracts were realised immediately. This is not only because of the market-to-market settlement process for futures, as some commentators have indicated,

[11] See Culp and Miller (1995 a,b); Mello and Parsons (1995); Pirrong (1997).

but also due to the fact that the futures contracts were short-term. Whether the losses were realised daily or monthly did not make any major difference in this case of steady declining oil prices. The cash-flow constraint could have been avoided by using long-term forwards, if such had been available. The long-term fixed delivery contracts would then have been hedged by means of another long-term contract with no intermediary payments.

From a theoretical viewpoint, however, it is important to realise that the cash-flow problems would also have been avoided if the hedging strategy had been self-financing, as is always the case in the artificial framework of a continuous time set-up. Then the fixed long-term contract could have been hedged by shorting the self-financing replicating portfolio. To be fair, this would not have been a simple problem though, not even in continuous time. The major risk involved in a hedging strategy of buying futures contracts to offset the risk of the short position in the fixed long-term contracts is not the price risk, but the risk of changing levels of convenience yields and interest rates. The sheer size of the positions taken by Metallgesellschaft implies that these risks cannot be ignored.

The mishaps of Metallgesellschaft has spurred a debate whether it is at all possible to hedge forward contracts of long maturity using short term futures. Most academics seem to agree that this indeed is possible. Culp and Miller (1995 a,b) argue that the basis risk, i.e. the risk that the difference between the spot and futures contract prices widens or narrows, is small compared to the risk of changes in the spot price. This is perhaps a questionable argument, hinging on that the volatility of the convenience yield is small. Brennan and Crew (1995) provide more convincing evidence in that they actually test the hedging efficiency of different models, allowing for a stochastic convenience yield, using price data from the NYMEX light oil futures contract. Although the small set of data does not allow for any statistical conclusions to be made, errors are rather small for the 24 month contracts hedged and in the magnitude of a few dollars per barrel.

To hedge options are more difficult than to price options, though. Perhaps the most classic article on option hedging is Leland (1985). The presentation is also unusually clear. When transaction costs are included, stocks have to be bought at a higher price and sold at a lower price than otherwise. Buying at a higher price and selling at a lower price can be seen as an increase in the volatility. Higher volatility increases option prices. What Leland does is to change the volatility parameter in the B-S model so that the expected payoff of the replicating portfolio inclusive of transaction costs is equal to the payoff from the option.

That it is enough to replicate the expected payoff is motivated by the fact that the hedging error is uncorrelated with the stock price. This was also the case for discrete time hedging in the absence of transaction costs. When the error is uncorrelated with the market, it has no systematic risk. Investors, supposedly, do not care whether the outcome is deterministic or stochastic. In other words, it is enough to have the expected payoff from the replicating portfolio equal to the option payoff. Normally the hedging error is correlated with the stock price and therefore with the market. For example, if a written in-the-money call option is hedged by going long the stock, the hedge ratio, delta, is close to one. Thus, an increase in the stock price does not cause as much transaction costs as a decrease. In this case the hedging error is negatively correlated with the market. Having the hedging error uncorrelated with the market is the only way of not having to consider the maximum replication error.

Taking transaction costs into consideration, there no longer exists a unique arbitrage-free price, but instead a range of possible prices. In Leland's model, this range is wider for out-of-the-money options. The fact that the B-S model prices out-of-the-money options less accurately therefore does not necessarily indicate a flaw in the model.

Boyle and Vorst (1992) abandon the continuous time framework. If hedging has to be done in discrete time, why not use a discrete time model? They extend the Cox, Ross and Rubinstein (1979) binomial model to include transaction costs. Like Leland, they assume that the transaction costs are proportional to the stock price and extend Merton's (1990, section 14.2) analysis of the one-step binomial model. By setting up the one-step binomial model and, which is the crucial part, proving that the number of shares in the replicating portfolio is higher in the up node, Boyle and Vorst get a linear equation system in two unknowns. This procedure is repeated iteratively in much the same style as the original binomial model. The difference is that a long call is a little bit more expensive to replicate since you have to buy stock that is worth less than what you pay due to the transaction costs. Similarly, for a short call, the stock you short is worth more than the proceeds. Thus there are different portfolios depending on whether it is a long or a short position that is to be hedged. For this reason, the binomial formulas that exist for several periods do not apply when transaction costs are included.

When the number of periods is large, the binomial approach converges to a B-S type formula with an increased variance rate similar to Leland (1985). The difference is that the range of possible premiums is a little wider using the Boyle and Vorst discrete time technique, since their hedging portfolio is self-financing inclusive of transaction costs. Leland's continuous time model, by contrast, is not self-financing since the portfolio weights are not revised continuously. Instead, the average hedging error is zero so the portfolio on average replicates the payment of the option.

The papers by Leland (1985) and Boyle and Vorst (1992) use time-based hedging. To rebalance the portfolio at fixed intervals in time whether or not this is optimal in some sense entails unnecessary transaction costs. This is not really a problem in these two papers since the assumption is that the transaction costs are proportional to the amount of stocks traded. Having a larger amount of stocks traded less frequently is not much gain when transaction costs are proportionate. (There is some gain, though, as price movements can be reversed.) In reality, however, most of the costs associated with rebalancing are fixed. The administrative work in establishing how to rebalance, and the fee for executing the buy or sell orders are mainly fixed costs.

Edirisinghe, Naik and Uppal (1993) take the research a little further by allowing for a move-based strategy where the portfolio is rebalanced only when the price has changed by a certain amount. The non-linear programming model in this paper allows for both fixed and variable costs as well as position limits in trading and lot-size constraints. Two such constraints are the indivisibility of treasury bills and also futures contracts in the case that these are used. Futures contracts are normally specified in quite large sums. The Edirisinghe, Naik and Uppal approach is very general and the programming difficulties, as can be imagined, substantial. The idea is to minimise the initial cost of a self-financing portfolio whose terminal payoff is at least as large as the derivative's. The derivative can here be any position and is not limited to plain vanilla options. This line of attack therefore seems suitable for large financial institutions that have a portfolio of derivative contracts to hedge and not just individual options. These large institutions can probably also afford the cost of programming.

Assuming binomial stock price movements, the first step is to allow for only proportional trading costs. The problem can then be recast in linear form. Since the replicating strategy is path-dependent, the number of constraints and variables is exponential in relation to the trading frequency. An approximation in the form of a new linear program is therefore developed by assuming that the lowest cost replicating strategy is path-independent. The motivation for this assumption is weak but empirical tests show that it works quite well. However, so does the Boyle and Vorst approach which is more straightforward, although it does not work for option combinations.

When transaction costs are large and volatility high, the optimal strategy is to create a too large position in the stock initially, since this will reduce trading in later periods. As can be conjectured, it is not optimal to revise the portfolio in every period. The total amount of trading is dependent on the degree of non-linearity of the payment. The closer we are to replicate the "hockey stick" payment curve of an option, the more trading is required. Super-replication of the payment is therefore optimal. The value of the replicating portfolio is not allowed to be smaller than the option payment but may well be larger. The higher cost of super-replication is then balanced against the cost of replicating the payment exactly.

Allowing for fixed costs and trading constraints, which really is the contribution of the Edirisinghe, Naik and Uppal paper, requires dynamic programming. The problem is solved recursively in two stages. In the first stage, the portfolio weights that closest replicate the expected payoff for a given level of initial wealth is found. In the second stage, the minimum initial wealth is determined. The drawback is that the mathematics get rather involved. This approach is therefore less tractable in practice.

Hoggard, Whalley and Wilmott (1994) extend on Leland's (1985) work by deriving the differential equation that is valid in the presence of proportional transaction costs. It is thereby feasible to judge the impact of transaction costs, using the Leland approach, on portfolios of options rather than just plain vanilla options. However, it is not clear that the hedging error is uncorrelated with the market portfolio so that we are justified in replicating the expected hedging error, rather than considering the maximum hedging error. This seems to be a weak spot in their analysis.

Toft (1996) derives some analytic results regarding the expectation and variance of the hedging error, both for time-based and move-based strategies. The hedging problem can thereby be recast as a cost versus risk problem. That is, there is a choice between the low cost strategy of rebalancing infrequently and facing the risk of unfavourable price moves, versus the more costly strategy of rebalancing more frequently, but also more closely replicating the payoff.

Based on the geometric Brownian motion, Toft derives formulas for the period-by-period expected replication error, and for the time zero $t_0$ expected replication error at a future time $T$. By summing over all rebalancing points he obtains the total expected replication error. The same exercise is then repeated in the presence of commissions. The variance of the cash flow from the replicating strategy is also studied. Formulas are, however, lengthy so approximations are provided instead. The variance of the cash flow at time $T$, conditional on the information at time 0 is derived. The variances can then be aggregated to give the total variance if the cash flows of the hedging strategy are uncorrelated over time, a requirement that can only be fulfilled when the expected hedging error net of transaction costs is zero.

Toft further argues that the suboptimality of time-based strategies (i.e. to rebalance at fixed intervals in time) is not very serious in practice. The first argument he makes is that option-traders are mostly required to close their books at the end of each trading day for risk management purposes. This can be translated as an institutional requirement for daily rebalancing.

The second argument in favour of time-based rebalancing is that in some cases it actually works better than the alternative of move-based strategies. Toft finds this to be the case when the underlying asset volatility is low, transaction costs are large and more precise replication is required. Relatively low volatility and high transaction costs characterise commodities compared to many individual stocks. Surprisingly, we can therefore expect a time-based strategy to work at least as well as a much more complicated move-based one.

Figlewski (1989) provides some reasonable examples where the arbitrage bounds on 3-month option prices vary between 30% and 120% of the theoretical option price. In these cases it is impossible to replicate the payoff, even for market makers. The wide bound are due to a combination of two factors, transaction costs and the volatility parameter. For a one-month contract, there are roughly 20 trading days. With so few observations, the realised volatility (needed for accurate option replication in discrete time) can be very different from the true volatility of the underlying distribution. As a remedy, Figlewski recommends using too high volatility estimates rather than too low, on the basis that this has less influence on the hedging performance.

The importance of transaction costs for valuation and hedging of long-maturity contracts is difficult to judge. On the one hand, longer contracts lead to a larger number of revisions of the portfolio. On the other hand, the value of the contract also increases, so the transaction cost as a percentage of the price may well decrease, see Swidler and Diltz (1992). In Etzioni (1986), transaction costs are negligible in comparison to the amount of money being hedged. Even so, in many cases the transaction costs are substantial and the bound on prices so broad that it is meaningless to talk about an arbitrage-free price.[12] However, by comparing different price process, we get theoretical guidelines for what the equilibrium price should be. Even if we cannot create a reasonably tight arbitrage-free bound on the prices of option contracts, we can at least find the price that would prevail in the absence of market imperfections. If there exists a liquid market where many actors with homogenous expectations take both long and short positions, price spreads will diminish and the market price will approach the equilibrium price.

[12] In an interesting paper by Neuberger (1994), it is shown that the assumption of a pure jump process can lead to a tight arbitrage-free bound on the derivative price inclusive of transaction costs.

## 3. Option pricing and hedging

Normally, the Black-Scholes differential equation for the price of a derivative is derived by creating a risk-free portfolio consisting of the derivative and the underlying asset. Apart from being the original derivation by B-S, this derivation has the advantage of being the easiest mathematically and it is therefore included in most textbooks on option pricing.

Logically, however, it is more intuitive to create a portfolio $V$ that does not include the derivative but has the same value at the maturity of the derivative contract, $V(T) = \Phi(T)$. Then, it is natural to infer that two assets with the same pay-off in the future also must have the same value today. If the portfolio value is, say, less than the price of the derivative, $V(t) < \Phi(t)$, the (expensive) derivative contract is shorted and a long position is taken in the (cheap) portfolio. Arbitrage arguments thus force the value of the derivative to be the same as the value of the portfolio and since the value of the portfolio is known, the pricing problem of the derivative is solved. During the process we have also obtained a recipe for hedging the derivative by creating an off-setting position in the replicating portfolio.

Another advantage of this derivation is that it does not require the derivative to be a traded asset since it is not part of the portfolio. This is especially important for the sometimes thin and illiquid market for commodity derivatives. The only requirement is that when the derivative actually trades, it must do so at the arbitrage-free price, which of course is the point of the whole exercise. There is also a third advantage of basing the derivation on a replicating portfolio rather than a risk-free portfolio. With a risk-free portfolio it is assumed that the derivative follows an Ito-process whereas with the replicating portfolio we see that the derivative must indeed follow an Ito process when the underlying asset follows an Ito process.

One restriction that has to be placed on the portfolio is that is has to be "self-financing", i.e. it is not allowed to withdraw or insert money into it. Obviously, if there are random cash flows occurring over time, the portfolio no longer replicates the contract.[13]

It is further assumed that the asset *S*(*t*) is a traded security and it may therefore be part of the replicating portfolio. The price of the asset evolves according to the Ito process

$$dS = \alpha(t,S)Sdt + \sigma(t,S)Sdw. \tag{3.1}$$

There is also a dividend process *D*(*t*) associated with holding the asset,

$$dD = \delta(t,S)Sdt. \tag{3.2}$$

The dividend yield $\delta(t,S)$ is allowed to be any deterministic function of time and asset price and is the difference between the risk-adjusted return $\mu$ and the expected drift in prices,

$$\delta(t,S) = \mu - \frac{1}{dt}E\left[\frac{dS}{S}\right].$$

Put in another different way, the expected return from holding the asset comes in two parts, the expected increase in price and the dividend. The dividend yield $\delta(t,S)$ is for commodities equal to the convenience yield, the "dividend" or utility received by having physical ownership of the commodity.

[13] Deterministic cash flows in the replicating portfolio causes no problem, however. As the cash flows are deterministic the present value of these are known and can be added or subtracted from the portfolio today. Such deterministic cash flows are called a consumption process.

Apart from the asset, the market also consists of a risk-free bond $B(t)$,

$$\begin{cases} dB = rBdt \\ B(T) = 1, \end{cases} \tag{3.3}$$

where the short rate $r$ is assumed to be constant.

Having introduced the securities, the next step is to develop a portfolio,

$$V(t) = h^B(t,S)B(t) + h^S(t,S)S(t), \tag{3.4}$$

that has a terminal value equal to the value of the derivative contract, $V(T) = \Phi(T)$. $h^B(t,S)$ and $h^S(t,S)$, the number of bonds and assets held in the portfolio, are called the portfolio strategy. It is understood that $h^B(t,S)$ and $h^S(t,S)$ can and will change over time as long as there is no inflow or outflow of money to the portfolio. For brevity, the time and asset price dependence of $\{h^B, h^S, \alpha, \delta, \sigma\}$ is suppressed in the rest of the derivation.

The portfolio $V$ evolves over time according to the dynamics,

$$dV = h^B dB + h^S dS + h^S \delta S dt. \tag{3.5}$$

In words, the change in portfolio value consists of three terms. The first term is the number of bonds held times the change in bond value. The second term denotes the number of assets held times the change in asset price, and the third term is the dividend flow over the time interval $dt$. Over the short time interval $dt$ the number of assets and bonds, $h^B$ and $h^S$, is unchanged. The change in portfolio value therefore stems from the change in bond price, asset price and the cash dividend payment. At the end of the time interval, $h^B$ and $h^S$ changes instantaneously and a new time interval $dt$ commences.

For the portfolio strategy to be self-financing, no inflow or outflow of money is allowed. Only a different allotment between the number of bonds and assets are allowed and also necessary since the cash dividend payment either uses up or contributes some money. The value of the portfolio must be the same before and after the instantaneous change in portfolio weights. Although the intuition of the process (3.5) is straightforward, it is rather technical to actually prove that $dV$ is the process followed by a self-financing portfolio and interested readers are advised to consult Björk (1998, chapter 5) for details.

Substituting (3.2)-(3.4) into (3.5) gives the dynamics of the replicating portfolio as

$$dV = (rh^B B + \alpha h^S S + \delta h^S S)dt + \sigma h^S S dw. \tag{3.6}$$

Rather than working with the number of assets in the portfolio, $h^S$ and $h^B$, it is easier to work with the relative portfolio weights

$$\begin{cases} u^S = \dfrac{h^S S}{V} \\ u^B = \dfrac{h^B B}{V}. \end{cases} \tag{3.7}$$

This implies that the value process (3.6) can be rewritten as

$$dV = (ru^B + \alpha u^S + \delta u^S)V dt + \sigma u^S V dw. \tag{3.8}$$

The process (3.8) together with the requirement that the portfolio weights should sum to one,

$$u^B + u^S = 1, \tag{3.9}$$

can now be used as the definition of a self-financing portfolio. As long as the weight sum to one, every relative portfolio $u^S$ and $u^B$ is a relative portfolio for a self-financing portfolio strategy.

The next step is to describe how the value of the contract evolves over time, *F(t,S)*. As the payment is solely dependent upon the underlying asset *S(T)*, it seems reasonable that the price process is also some function of the asset price. The fact that the asset follows an Ito process makes it possible to describe the price process of the derivative through Ito's lemma,

$$\begin{cases} dF = (F_t + \alpha S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS})dt + \sigma S F_S dw \\ F(T) = \Phi(T). \end{cases} \tag{3.10}$$

The idea is now to find the portfolio weights $u^S$ and $u^B$ so that the processes (3.8) and (3.10) are identical. If this is possible, the condition of no arbitrage implies that the value of the contract is indeed determined by (3.10), it is no longer just a reasonable supposition. Otherwise, $u^S$ and $u^B$ would form a portfolio that has the same pay-off as the derivative but a different price, causing arbitrage opportunities.

Define

$$\begin{cases} u^S = \dfrac{SF_S}{F} \\ u^B = \dfrac{F_t - \delta SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS}}{rF}. \end{cases} \tag{3.11}$$

Substituting (3.11) into the value process for the portfolio (3.8) gives,

$$\begin{aligned} dV &= (ru^B + \alpha u^S + \delta u^S)Vdt + \sigma u^S Vdw \\ &= \left( r\frac{F_t - \delta SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS}}{rF} + \alpha\frac{SF_S}{F} + \delta\frac{SF_S}{F} \right)Vdt + \sigma\frac{SF_S}{F}Vdw \\ &= (F_t + \alpha SF_S + \tfrac{1}{2}\sigma^2 S^2 F_{SS})\frac{V}{F}dt + \sigma SF_S\frac{V}{F}dw. \end{aligned} \tag{3.12}$$

The process for the replicating portfolio (3.12) is identical to the price process for the derivative (3.10) if $V(t) = F(t)$ for all $t$. Thus, if $V(0) = F(0)$ we will end up with $V(T) = F(T) = \Phi(T)$, i.e. the portfolio replicates the payment of the derivative.

However, the problem is not quite solved yet. The portfolio weights $u^S$ and $u^B$ in (3.11) are defined in terms of the function $F(t)$. (3.10) implies that $F$ is an Ito process and (3.12) that it is equal to $V(t)$ for all $t$, but apart from these characteristics $F$ is not defined. The requirement $u^B + u^S = 1$ will give the definition of $F$. Inserting (3.11) into (3.9) gives

$$\begin{aligned} &u^B + u^S = 1 \\ &\frac{F_t - \delta SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS}}{rF} + \frac{SF_S}{F} = 1 \\ &F_t + (r - \delta)SF_S + \tfrac{1}{2}\sigma^2 S^2 F_{SS} - rF = 0. \end{aligned} \tag{3.13}$$

Thus, $F$ is the solution to the well-known Black-Scholes differential equation with the boundary condition $F(T) = \Phi(T)$. If we instead of the relative portfolio weights $u^S$ and $u^B$ want to work with the actual number of securities , $h^S$ and $h^B$, (3.7) is rewritten by using (3.9), (3.11) and $V = F$ to obtain

$$\begin{cases} h^S = u^S \dfrac{V}{S} = \dfrac{SF_S}{F}\dfrac{V}{S} = F_S \\ h^B = u^B \dfrac{V}{B} = (1-u^S)\dfrac{V}{B} = (1-\dfrac{SF_S}{F})\dfrac{V}{B} = \dfrac{F-SF_S}{B}. \end{cases} \tag{3.14}$$

To summarise: The replicating portfolio is created by holding a position in the underlying asset and a risk-free bond. The number of assets and bonds required are,

$$\begin{cases} h^S = F_S \\ h^B = \dfrac{F-SF_S}{B}, \end{cases} \tag{3.15}$$

where $F$ is the solution to the Black-Scholes differential equation,

$$\begin{cases} F_t + (r-\delta)SF_S + \tfrac{1}{2}\sigma^2 S^2 F_{SS} - rF = 0 \\ F(T) = \Phi(T). \end{cases} \tag{3.16}$$

It is important to stress that $h^S$ and $h^B$ have to be updated after every time interval $dt$. Thus the replication strategy and the resulting arbitrage-free price is only valid when transaction costs are zero. When the time between the updates gets infinitesimally short, the transaction costs increase without bounds. However, infrequent revision of the portfolio is not enough to introduce any substantial errors. In practice then, it is the size of the transaction costs together with the validity of the assumed price process and the accuracy of parameter estimation that determine how narrow the arbitrage-free price interval can be.

Disregarding the transaction costs, the validity of the price process, can be measured by the deviation between $V(T)$ and $\Phi(T)$, i.e. the extent to which the portfolio replicates the contract payment. If the price process is correctly specified, this deviation will be minor (but still different from zero due to discrete time revision of the portfolio). Testing different price processes and parameter estimation techniques is the purpose of the next section.

# 4. Hedging performance - parameter estimates within sample

In order to explore the ability to hedge a long-term commodity-linked contract, the debenture issued by the Magma Copper Company is taken as a point of departure. In this contract, the coupon rate varied depending on the price of copper. A lower price resulted in a lower coupon rate. Keeping things simple, we get the same characteristics of smaller debt payments in times of lower prices by assuming a zero-coupon bond with a short put option on the commodity attached to the bond. This put option is the object of the study. A producer that goes long a put option is insured against a price decline in the output. However, the investor buying the bond and thereby writing the put option is faced with the problem of hedging the short position. It is this hedging ability we explore. In essence, this is similar to the problem studied by Brennan and Crew (1995), where they research the possibility for a financial institution to hedge a long forward position by shorting futures. Although the option is more difficult to value and hedge, it serves the same purpose of protecting the commodity producer against a price decline.

The short put option also surfaces in another context. An unsecured bond can be decomposed into a default-free bond and a short put option on the value of the company. The risk of default is in many cases of mineral extraction and power generation directly connected to the price of the underlying commodity. In project financing it is therefore common that banks and other investors require the output to be sold in advance, preferably at a fixed price, before committing any money to the project. These delivery contracts do not always exist[14], however, and when they do not, the value of the short put option becomes considerable.

[14] For an example, see Arnold (2002, exhibit 11.22).

To hedge a short put option the investor is forced to short the underlying commodity. This is something that is rarely done in practice and the deals must be negotiated as there is no standardised procedure.[15] Physically borrowing a commodity from a storage facility and returning it there afterwards can only be done by paying a substantial fee for administration and transportation. The lender of the commodity must also be compensated for the lost convenience of physical storage; the increased risk of stockouts and the unability to trade the commodity to someone that is willing to pay a premium for immediate delivery or a specific quality.

Specifying this lost convenience or convenience yield, as it is normally given as a percentage of the commodity price, in a way that is acceptable to both parties is the biggest obstacle to shorting the actual commodity. The parties must not just agree on the convenience yield but also on the storage cost. In practice, the futures contract on the commodity is shorted instead. A futures is a purely financial contract and therefore readily shorted. Neither is there a convenience yield attached to the ownership of the futures, implying that the pricing of the convenience yield is already included in the futures price.

For these reasons, most traded commodity options are written on the futures contract as the underlying asset, not on the asset itself. Hull (2003, page 283) also makes the argument that since options and futures mostly trade on the same exchange, pricing and hedging are facilitated when futures contracts are used as underlying assets. This probably makes the market more efficient.

The use of futures contracts, however, would make the price process of the underlying asset more difficult to study. There are only a limited number of

[15] One such occasion that made it to the headlines of the newspapers in Sweden was in 1980 when the city of Växjö agreed to lend oil from its war and emergency supply and the investor did not return it.

commodities in the world and futures contracts are not traded on all of them. In the preparation of this study, spot prices of 280 commodities were obtained but less than 70 futures price series of considerably shorter length. Results regarding the best price process would therefore be much weaker using the futures series. Another reason why futures series are less suitable for studying the price process is that futures contracts are short term so that different futures contracts have to be used for hedging of a long-term contract. This complicates matters. Finally, if futures prices are studied, it is really the futures price process that is studied. The futures process can of course be connected to the price process for the underlying commodity, but this seems hardly worth the effort. Going the other way, using the spot price series to create theoretical futures prices also seems pointless. In order to study the price process we therefore short the underlying asset itself rather than the futures contract, fully aware that this in most situations is unrealistic as a practical set-up.

More specifically, the contract to be hedged is a five-year at-the-money European put option. The option is initiated at the 1st of April 1995 and maturing at the 31st of March 2000. The length of the contract is chosen purely for empirical reasons. By not using a longer contract, say 10-years, we are able to make use of some additional commodity time series and should therefore get more trustworthy results. An at-the-money option is chosen because the convexity (gamma) is highest for at-the-money options. Options that are sufficiently deep in or out-of-the-money will be more or less linear in the asset price regardless of the assumed price process. This implies that the differences between competing stochastic processes are most significant for at-the-money options. Of course, there is no guarantee that the option remains at-the-money once initiated, but one can at least start there. The choice of a put option, finally, is made for expositional clarity. A producer that goes long a put option is insured against a price decline in the output. The financial institution writing the put option, however, is faced with the problem of hedging the short position. It is this hedging ability that we explore.

Altogether, there are 280 time series for different commodities available. Data are extracted from the Datastream commodity data base. In this data base, there are around 1500 different commodity price series of different length. However, all of them are not complete and for some commodities many different series exist. The most complete series is chosen for each commodity. The price seldom changes in some series and it is doubtful whether these series really convey the market sentiment. Instead they might reflect a price determined by the producer. Series with many price changes are therefore preferred to series with fewer price changes. Series from established exchanges, for example the London Metal Exchange, are preferred to prices quoted in major ports on the ground that the former series reflect the market sentiment better and are probably subject to less errors.

Some series are also heavily correlated with each other. This is particularly so for different qualities of crude oil. Only one series has been chosen in these cases. However, only close substitutes have been screened for correlation. Due to the large number of price series examined, we can expect some correlation between totally unrelated commodities just by chance. Such correlation has not been considered. In all there are 280 price series used in the study. These series have been screened for errors in the first differences. If the price changes more than 100 percent and is back to the previous price within a week, the outliers are deleted on the basis that these are probably data errors. In case of any doubt, the series is left unaltered.

The market price of different commodities is denoted in different currencies, most frequently U.S. dollars and British pounds. Hedging will be performed in the currency of denomination, as exchange rate fluctuations thereby will have no direct effect on the hedging performance.[16] Such fluctuations may otherwise overshadow the effect of changes in commodity prices. Especially so when price changes are infrequent. Apart from U.S. dollars and British pounds, deutschemarks, Taiwanese won and Australian dollars are also used. Freight charges for standardised maritime routes are not denominated in any currency but as indices. These series are treated as if they were denominated in U.S. dollars, as this is the standard currency for freight charges. The appendix in essay 3 provides details about the respective commodities.

**The geometric Brownian motion, GBM**

We try to replicate synthetically the 5 year at-the-money short European put option by constructing a portfolio with the same payoff. Using the geometric Brownian motion of (1.1), the price of the option, *F(t,S)* in the notation of the differential equation (3.16), is the ordinary Black-Scholes price on an asset paying a continuous dividend yield,

$$p = Ke^{-r(T-t_0)}N(-d_2) - Se^{-\delta(T-t_0)}N(-d_1) \tag{4.1}$$

where $d_1$ and $d_2$ are given by

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)(T - t_0)}{\sigma\sqrt{T - t_0}}$$

$$d_2 = d_1 - \sigma\sqrt{T - t_0}.$$

The number of assets to short, $F_S$ from equation (3.15) is equal to $e^{-\delta(T-t_0)}[N(d_1) - 1]$, see Hull (2003, p. 305). The proceeds from the short sales

[16] There will of course be an indirect effect, stemming from how currency fluctuations affect supply and demand and thereby price of the commodity.

minus the convenience yield paid are invested at the risk-fee rate. The position is updated daily. The value of the position at maturity, plus the end value of the premium from writing the option minus the payment of the option, gives the hedging error. The percentage hedging error is obtained by dividing the present value of the hedging error by the option premium. We use the present value of the hedging error since the time value of money becomes significant for 5-year contracts. A more detailed explanation is given in appendix B.

In section 4, all parameters are estimated from the same commodity time series that the hedging test is conducted upon. This procedure is of course the severest form of data dredging imaginable, but the advantages are several. If the hedging procedure does not work even when parameters are estimated from the actual outcome, it will never work. This allows us to rule out a number of cases. It also provides a best case. The hedging error can never be smaller than when parameters are estimated from the actual outcomes.

Parameters are estimated in the standard way. The geometric Brownian motion of (1.1) implies that $x_{t+1} = \ln \frac{S_{t+1}}{S_t} \sim N\left[(\alpha - \frac{1}{2}\sigma^2)\Delta t, \sigma\sqrt{\Delta t}\right]$. Defining $\bar{x} = \frac{1}{n}\sum_t x_t$, the diffusion $\sigma$ is estimated from the standard deviation of the geometric Brownian motion $\sigma\sqrt{\Delta t} = \sqrt{\frac{1}{n}\sum_t (x_t - \bar{x})^2}$ and the drift rate $\alpha$ from the expression for the mean, $\bar{x} = \left(\alpha - \frac{1}{2}\sigma^2\right)\Delta t$.

For pricing and hedging of options we also need the risk-free rate $r$ which is estimated as the average of the continuously compounded rate on treasury bills during the period. The convenience yield $\delta(t, S)$ is the difference between the risk-adjusted return $\mu$ and the drift in commodity prices,

$$\delta(t,S) = \mu - \frac{1}{dt} E\left[\frac{dS}{S}\right]. \tag{4.2}$$

For the geometric Brownian motion $\frac{1}{dt} E\left[\frac{dS}{S}\right] = \alpha$, so the convenience yield becomes constant in a Black-Scholes world, $\delta = \mu - \alpha$.

To estimate $\mu$, the CAPM is used with the market risk-premium set to 8%. However $\beta$, the sensitivity of commodity price changes to changes in the market portfolio is almost zero. (The market portfolio is approximated by the major stock market index in the respective currencies.) The average $\beta$ for the 280 different commodities is 0.002 and for only 16 series is $\beta$ significantly different from zero at a 95 percent confidence level. Also in absolute terms is $\beta$ small and only for 13 series outside the region –0.1 - 0.1.

Measurement of $\beta$ for commodities is subject to the same type of measurement error as $\beta$ on small company shares, though. As the price of many commodities does not change very often, the price changes have a very low correlation with changes in the market portfolio. The true $\beta$ is therefore underestimated when the daily price quotes of this study are used. However, using longer time series and monthly price quotes does not change the result very much. $\beta$ is still small. It is therefore of interest to test the whether the simplifying assumption that $\mu = r$ would adversely affect the hedging performance.

Another case to examine is what would happen when the convenience yield $\delta$ is set to zero. There are large practical problems of constructing a contract where the "lender" of the commodity pays the party with the short position for the decreased storage cost (of the lender), and where the party with the short position compensates the lender for the lost convenience of physical storage. It is therefore interesting to examine the hedging performance if the convenience yield is set equal to zero. The commodity is then equivalent to a stock that pays no dividend.

Formally, there are three hypotheses to test:

$H_0 : \delta = \mu - \alpha.$

$H_1 : \delta = r - \alpha.$

$H_2 : \delta = 0.$

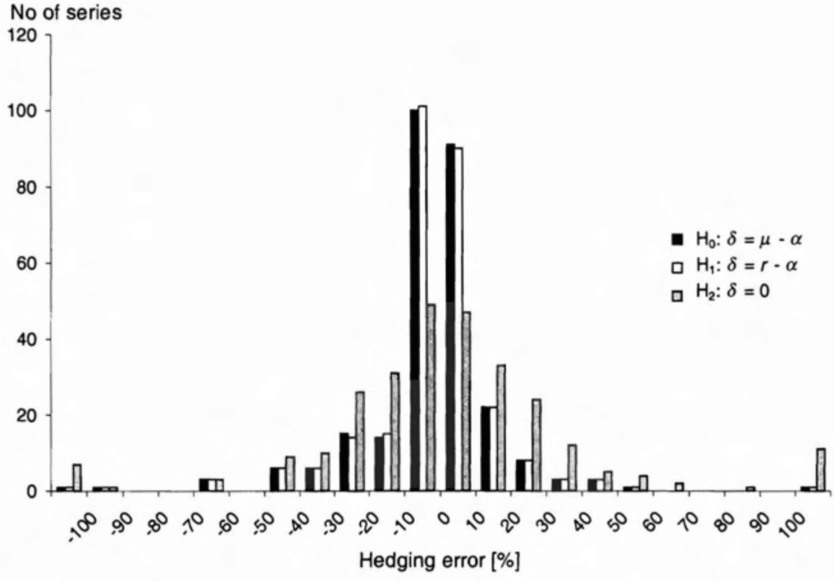


Figure 4.1 Geometric Brownian motion hedging errors for three different assumptions about the convenience yield.

Figure 4.1 shows that the convenience yield cannot be ignored. The hypothesis $H_2$: $\delta = 0$ results in a substantially worse hedging performance than the other two hypotheses. Clearly, the convenience yield plays a significant part in the hedging of commodity contracts. In hypothesis $H_1 : \delta = r - \alpha$, the hypothesis is that since the systematic risk in commodity prices is almost zero, $\beta \approx 0$, it will not make much difference to the hedging performance if the risk-adjusted required return on the commodity is substituted for the risk-free rate. Figure 4.1 reveals that this logic is indeed true. The differences between $H_0$ and $H_1$ are very small. The table actually exaggerates the differences. If an error is ever so little outside the ten percent interval in the histogram, it falls into the neighbouring column.

The error distribution also appears to be reasonably symmetric around zero. If anything, a negative outcome seems a little more probable and this is presumably due to how the portfolios are constructed. As can be expected from a large sample, some commodities have appreciated very much in price. To model a drift in commodity prices that is substantially higher than the required rate of return is not economically viable as it creates arbitrage opportunities. The convenience yield is therefore set to zero in this case. It can be argued that this is not realistic either since the convenience yield can be negative when supply of the commodity is ample and the storage costs high. Zero is chosen as a breakpoint to avoid any discussion regarding how negative the convenience yield can be. This choice may cause some negative skewness in the hedging distribution, as the short put option is not fully hedged for these commodities.

Figure 4.1 is easily summarised. The convenience yield cannot be ignored, but it is enough to set $\delta = r - \alpha$. There is no need to measure the required rate of return through the CAPM or any other equilibrium model since the systematic risk in the commodity return is almost zero. We therefore proceed by testing only the hypothesis $H_1$ and try to characterise the hedging error further.

In particular, we are interested in how much leptokurtosis, i.e. fat tails, and infrequent price changes affect the hedging performance. Both these two characteristics are common in commodity return price series, as was detailed in essay 3. Both characteristics also seem to suggest that commodity price changes should be described by jump processes, but jumps are more difficult to work with than the Wiener processes mostly used in continuous-time finance. How well do Wiener processes explain the price changes? Table 4.1 tries to answer the question by sorting the hedging errors from the 280 series after different criteria. The first sort is made for the number of price changes. When the price seldom changes, Wiener processes can be expected to perform worse than when price changes are more frequent. Daily data for 5 years give approximately 1300 data points. Table 4.1 reveals that the hedging error indeed decreases when price changes are more frequent. However, apart from the first two rows with less than 100 price changes, the difference is perhaps smaller than could have been anticipated. The average absolute hedging error is between five and seven percent when the series with fewer than 100 price changes have been excluded. The indication is that Wiener processes are quite robust and work reasonably as long as price changes are not very rare.

Table 4.1 Hedging errors for the geometric Brownian motion, hypothesis $H_1$.

| | No. of series | Mean error | Mean absolute error |
|---|---|---|---|
| All series | 280 | -0.02 | 0.11 |
| Series sorted after the number of price changes | | | |
| 0-49 price changes | 106 | -0.08 | 0.18 |
| 50-99 price changes | 62 | 0.01 | 0.10 |
| 100-299 price changes | 48 | 0.03 | 0.07 |
| 300-999 price changes | 20 | 0.00 | 0.05 |
| >1000 price changes | 44 | 0.01 | 0.06 |
| Sorted after the ratio largest price change / volatility | | | |
| 1:st change / vol >1.50 | 54 | -0.13 | 0.27 |
| 1.50> 1:st change / vol >1.10 | 64 | -0.02 | 0.11 |
| 1.10> 1:st change / vol >0.80 | 79 | 0.03 | 0.08 |
| 0.80> 1:st change / vol >0.50 | 48 | 0.01 | 0.05 |
| 0.50> 1:st change / vol | 35 | -0.01 | 0.04 |
| Sorted after the ratio largest price change / volatility when there are more than 50 price changes | | | |
| 1:st change / vol >1.50 | 10 | -0.05 | 0.14 |
| 1.50> 1:st change / vol >1.10 | 21 | 0.00 | 0.13 |
| 1.10> 1:st change / vol >0.80 | 60 | 0.03 | 0.08 |
| 0.80> 1:st change / vol >0.50 | 48 | 0.01 | 0.05 |
| 0.50> 1:st change / vol | 35 | -0.01 | 0.04 |
| Sorted after convenience yield | | | |
| $\delta = 0$ | 85 | -0.07 | 0.22 |
| $\delta > 0$ | 195 | 0.00 | 0.02 |

The next characteristic to examine is jumps in the price process. It is of course possible to sort the series by the size of the price changes but this would be a blunt measure of jumps as highly volatile series have larger price changes. Instead we sort the series after the ratio (largest absolute price change / volatility). This ratio is similar to the Studentized Range of measuring normality or leptokurtosis. For the Studentized Range, the difference between the biggest and the smallest return is divided by the volatility.[17] However, not

[17] For a description of the Studentized Range, see Fama (1976, chapter 1).

requiring jumps to be symmetric, the largest absolute price change is instead divided by the volatility.

The second category of table 4.1 shows this ratio. For the 54 price series where the largest absolute price change is more than 1.5 times the volatility, the hedging performance is poor in absolute terms and the average error is negative. The mean absolute hedging error is 27 percent for these series. When the jump size decreases, the hedging performance is improved.

The third sort is a combination of the previous two. The sort is still made after the ratio (largest absolute price change / volatility), but in this case all series having fewer than 50 price changes are excluded. This provides a clear improvement when the ratio exceeds 1.5. The absolute hedging error decreases from 27 percent to 14 percent. However, there are only ten series in the latter category so one should be cautious with the interpretations. For the rest of the series, there is not much change.

Let us summarise the results thus far. These are encouraging in that the geometric Brownian motion and the Gaussian price changes of the Wiener process actually work reasonably well even when the price seldom changes and the time series exhibit jumps. The hedging performance does not break down completely when the geometric Brownian motion provides an inadequate description of price behaviour. The differences can be economically very important, though. A hedging error of, say, 5 percent or 15 percent may be crucial to a financial institution, even though the differences are not that big in mathematical terms. A disappointing result is that the smallest average hedging error is around 5 percent. Given that parameters are estimated within the sample and that transaction costs are excluded, this hedging error is quite large.

The last sort in table 4.1 provides an explanation for this large error. Here the sort is made for the convenience yield. For a positive convenience yield the mean absolute hedging error is only two percent, whereas the error is twenty-two percent for the series where the convenience yield is set to zero. The root of the problem is the drift in prices. We saw in figure 4.1 that having no convenience yield at all, and the drift rate in a risk-neutral world therefore equal to the risk-free rate, resulted in very poor hedging. The same problem appears again when the drift in commodity prices is very high. An example will serve as an illustration. The price of white grapefruit juice (juice is actually one of the biggest commodity markets in the world) has increased by around twelve percent per annum between 1995 and 2000. With $\beta = 0.01$, the expected return should be around the risk-free rate of six percent per year. Is the storage cost six percent higher than the convenience yield of storing the juice, motivating an expected drift rate of twelve percent? Maybe, but it is more likely that the price increase of twelve percent reflects and increased demand for white grapefruit juice. However, setting the expected drift rate to six percent when the realised drift rate becomes twelve percent results in a very bad hedge.

**Mean reversion**

The next step is to test the hedging ability when the mean reverting process (1.2),

$$dS = \eta(\gamma + \omega t - \ln S)Sdt + \sigma Sdw,$$

is used to describe the price behaviour. If we believe in the mechanics of a market economy, prices of standardised goods should in the long-run revert towards the marginal cost of production as a result of competition among the producers. European option prices for mean reverting commodity prices are derived in appendix A and reproduced here as formulas (4.3)-(4.4),

$$c = e^{-r(T-t_0)+a+\frac{1}{2}b^2} N(d_3) - Ke^{-r(T-t_0)} N(d_4) \tag{4.3}$$

$$p = Ke^{-r(T-t_0)} N(-d_4) - e^{-r(T-t_0)+a+\frac{1}{2}b^2} N(-d_3) \tag{4.4}$$

where

$$\begin{cases} d_3 = \dfrac{a + b^2 - \ln K}{b} \\ d_4 = d_2 - b \\ a = e^{-\eta(T-t_0)} \ln S_0 + \omega(T - t_0) + \left(1 - e^{-\eta(T-t_0)}\right)\left(\gamma + \dfrac{r-\mu}{\eta} - \dfrac{\sigma^2}{2\eta} - \dfrac{\omega}{\eta}\right) \\ b = \sigma\left(\dfrac{1 - e^{-2\eta(T-t_0)}}{2\eta}\right)^{1/2}. \end{cases}$$

The formulas for the delta of the call and the put options are

$$\frac{\partial c}{\partial S} = \frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S} N(d_3) \tag{4.5}$$

$$\frac{\partial p}{\partial S} = \frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S} [N(d_3) - 1]. \tag{4.6}$$

Note the close similarity in structure between these formulas and the ordinary B-S formulas. This similarity is due to the fact that both the geometric Brownian motion and the mean reverting process are lognormally distributed. The formulas are not identical because of the convenience yield, see appendix A. Building on the results for the geometric Brownian motion it is assumed that the systematic risk of commodity prices is zero so that $\mu = r$. The mean reverting process has an expected drift of

$$\frac{1}{dt}E\left[\frac{dS}{S}\right]=\eta(\gamma+\omega t-\ln S),$$

giving the convenience yield

$$\delta(t,S)=r-\eta(\gamma+\omega t-\ln S). \tag{4.7}$$

The convenience yield is high when the price is above the equilibrium price, $(\ln S>\gamma+\omega t)$, and low when price is below the equilibrium price. This is qualitatively right. A price above the equilibrium price means that the demand is high or alternatively that the production is small. These are the situations when there is a higher probability of stockouts and therefore a higher convenience yield. In the same way, a low price suggests ample supply relative to demand. The convenience yield is therefore small as there is little benefit in storing an extra unit of the goods.

Given the purpose of hedging a short put option, we need the theoretical price of the put option, formula (4.4), and the first derivative with respect to the commodity, formula (4.6). The latter determines how many assets to short in the portfolio. The convenience yield (4.7) is also required as it is the cash flow that the lender of the asset receives as a compensation for the lost convenience of physical storage. Hedging details are provided in appendix B. For parameter estimates, the discrete time version of the mean reverting process (1.2) is

$$\ln S_{t+1}=\left(\gamma-\frac{\sigma^2}{2\eta}-\frac{\omega}{\eta}\right)\left(1-e^{-\eta\Delta t}\right)+\omega\Delta t+\omega\left(1-e^{-\eta\Delta t}\right)t+e^{-\eta\Delta t}\ln S_t+\sigma\sqrt{\frac{1-e^{-2\eta\Delta t}}{2\eta}}\,\varepsilon_t.$$

The formula is derived in essay 2, appendices C and D. It is thus possible to run the regression,

$$\ln S_{t+1} = c_0 + c_1 t + c_2 \ln S_t + s\varepsilon_t, \tag{4.8}$$

with
$$c_0 = \left(\gamma - \frac{\sigma^2}{2\eta} - \frac{\omega}{\eta}\right)\left(1 - e^{-\eta\Delta t}\right) + \omega\Delta t$$
$$c_1 = \left(1 - e^{-\eta\Delta t}\right)$$
$$c_2 = e^{-\eta\Delta t}$$
$$s = \sigma\sqrt{\frac{1 - e^{-2\eta\Delta t}}{2\eta}},$$

in order to estimate the parameters.

Also for the mean reverting process, there are three cases that we want to test. All three are related to the drift term $\omega t$ in (1.2). This term was suggested in essay 2 to overcome a flaw in existing mean reverting models that these do not allow for the equilibrium price to increase over time. The first question to ask is whether this term is at all important or if it can be skipped without any loss of hedging performance? In hypothesis $H_0$, we therefore set $c_1 = 0$ in equation (4.8), implying that $\omega = 0$ in the model. The second question is how to estimate $\omega$ when $\omega \neq 0$? As both $t$ and $S_t$ increase over time these two explanatory variables will be collinear, something that might affect the precision of the regression (4.8). The trend of the price series is therefore removed by eliminating the drift rate $\alpha$, as identified by the geometric Brownian motion, and the regression is run with $c_1 = 0$ in order to avoid the collinearity. After having estimated the parameters in this way, the options are in hypothesis $H_1$ priced and hedged by setting $\omega = \alpha$. We can thereby keep the trend of the mean reverting model without having to incorporate time as a dependent variable in the estimation procedure. In the case that the historical drift in price has been greater than the required return, we set $\alpha = r$ in order to avoid the economically unjustifiable model of having an equilibrium drift rate higher than the required rate of return.[18] In addition, hypothesis $H_2$ is the full regression (4.8), where $c_1$ in the regression is equal to $\omega\left(1-e^{-\eta\Delta t}\right)$ in the model. In all, there are three hypotheses to test formally.

$H_0$ : $c_1 = 0$ in the regression and $\omega = 0$ in the model.
$H_1$ : $c_1 = 0$ in the regression and $\omega = \alpha$ from the geometric Brownian motion.
$H_2$ : Full regression, $c_1 = \omega\left(1-e^{-\eta\Delta t}\right)$.

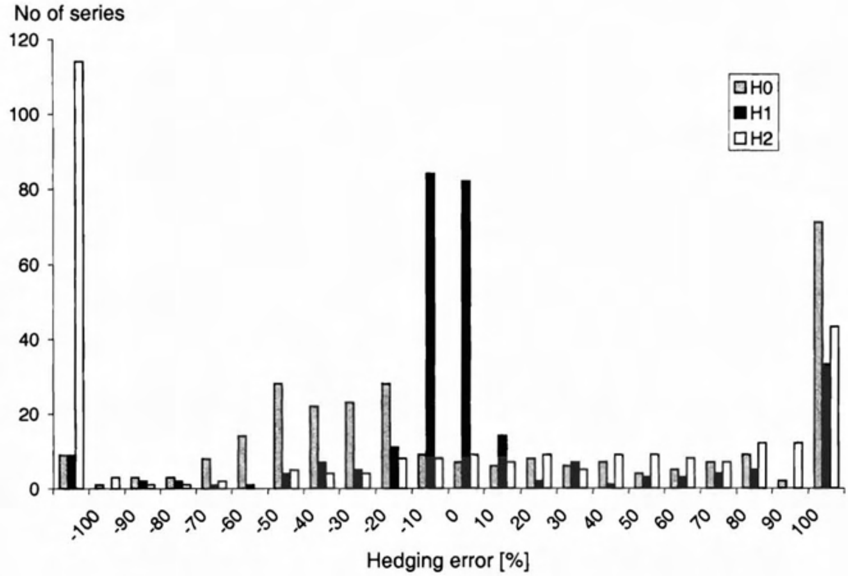


Figure 4.2 Hedging errors for three different assumptions about the equilibrium drift in the mean reverting model.

[18] The required rate of return $\mu$ is set equal to the risk-free rate of return $r$ since $\beta$ is very close to zero. The hedging performance for the geometric Brownian motion did not deteriorate under this simplified assumption.

The hedging results for the 280 time series are shown in figure 4.2. The inability of the hypotheses $H_0$ and $H_2$ to hedge the contract is striking. Starting with the first hypothesis, $H_0 : \omega = 0$, the errors are not centred around zero. The error distribution is skewed to the left and widely dispersed. In both $H_0$ and $H_1$, the parameters are estimated from the reduced regression where $c_1$ is set to zero. The difference lies in the fact that in $H_0$ also $\omega = 0$. The equilibrium price is constant and there is no drift in commodity prices over time. However, we know from real life that price levels change over time. Inflation, technology improvements and trends in the demand are factors that may contribute to a drift in the equilibrium price.

Standard mean reverting processes without drift therefore provide an inadequate description of the commodity price process, as is revealed by the poor hedging performance of $H_0$. Yet, to the best of my knowledge, no mean reverting process suggested in the option literature contains a drift term[19]. It is difficult to understand why this problem has received so little attention when the hedging performance is so poor.

Hypothesis $H_1$ works much better, however. Here $\omega = \alpha$, the drift in the equilibrium price of the mean reverting process is set equal to the drift in the geometric Brownian motion. The hedging error in hypothesis $H_1$ is much smaller and centred around zero.

For all the hypotheses tested, however, there exists a large number of series for which the hedging procedure clearly does not work. Hedging errors are even larger than 100 percent. The root of the problem is that a mean reverting process gives a very narrow range of probable future prices. This is in the very nature of the model. Prices revert towards the equilibrium level and the

[19] For some different mean reverting processes in a real option framework, see Dixit and Pindyck (1994, pages 74 and 161), Schwartz (1997), and Bjerksund and Ekern (1995).

difference between mean reversion and a random walk is well illustrated by diagram 6.1 in essay 2. The narrow range of probable future prices has two important consequences. It makes the model very sensitive to the estimate of the (log) equilibrium price $\gamma$ and it means that for exercise prices outside the narrow range of probable future prices, the option price becomes either zero or the difference between the exercise price and the spot price. If the price is zero, the percentage hedging error becomes infinite. The mean reverting model is therefore much less robust than the geometric Brownian motion. The problem is especially severe for the hypothesis $H_2$ where the full regression is used for parameter estimation and the multicollinearity problem between $t$ and $\ln S_t$ appears. As figure 4.2 reveals, hypothesis $H_2$ is not at all useful for hedging and pricing of derivative contracts. It is possible that multicollinearity is not the only reason why hypothesis $H_2$ breaks down. Additional explanatory variables give a flatter maximum likelihood function and therefore less precision in each estimate. It is possible that this problem affects the hedging performance. It should also be mentioned that independently of whether the full or the reduced regression is used, the estimates of $\sigma$ and $\eta$ are approximately the same. It is the estimates of $\omega$ and especially $\gamma$ that vary substantially.

It must be concluded that the only hypothesis that really works is $H_1$, running a regression without any time-dependence and set $\omega$ equal to the drift in the geometric Brownian motion. Even so, the method is not very robust and breaks down for a number of series. But how is the precision when the method does not break down? Economically, we know that a five-year at-the-money put option is not worthless. From real life, we also know never to trust the result of only one model. The results from several models are often compared before any major conclusion is drawn. We therefore proceed by examining the hedging performance of the mean reverting model when the put price seems reasonable. Reasonable is here defined as a price that lies within the region of 50-200% of the Black-Scholes price.[20]

The test can more generally be described as follows: There are two option pricing models available but we do not know which model to trust. Now and then the models give unreasonable option prices. Therefore, only commodities where the option prices as given by the two models are within 50-200 percent of each other are studied.

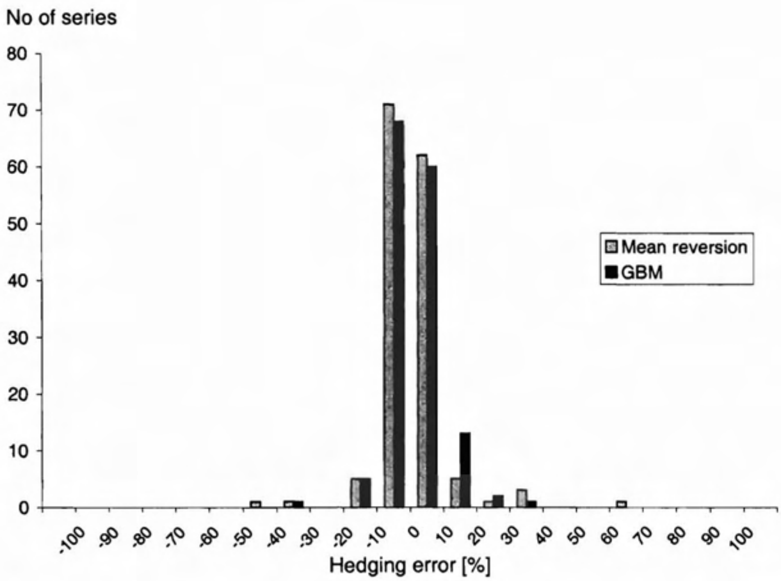


Figure 4.3 Histogram of the hedging errors for the mean reverting process and the geometric Brownian motion.

The process that most accurately describes the data are the process whose accompanying option pricing model provides the least hedging error. The histogram in figure 4.3 reveals that the hedging errors now are considerably

[20] This also implies that the B-S price is 50-200% of the mean reversion price.

smaller. Apart from a few outliers, the errors are small and symmetrically distributed around zero. More details are provided in Table 4.2.

Table 4.2 Hedging errors for the mean reverting process and the geometric Brownian motion.

| | No. of series | Mean reversion<br>Mean absolute error | GBM<br>Mean absolute error |
|---|---|---|---|
| All series | 150 | 4.2% | 5.0% |
| Excluding the five series with the largest hedging error for the respecitve process | 142 | 2.5% | 4.0% |
| Series sorted after the number of price changes | | | |
| 0-49 price changes | 47 | 4.2% | 6.0% |
| 50-299 price changes | 72 | 4.4% | 5.0% |
| >300 price changes | 31 | 3.7% | 3.3% |
| Sorted after the ratio largest price change / volatility | | | |
| 1:st change / vol >1.10 | 56 | 5.1% | 6.4% |
| 1.10> 1:st change / vol >0.80 | 47 | 3.3% | 5.1% |
| 0.80> 1:st change / vol | 47 | 4.0% | 3.1% |
| Sorted after convenience yield | | | |
| $\delta = 0$ | 11 | 13.2% | 9.9% |
| $\delta > 0$ | 139 | 3.5% | 4.6% |

- The mean reversion process provided the smallest mean absolute error for 100 out of the total 150 series.

For all the 150 series where the option prices identified by the two processes are within the range 50-200 percent of each other, the mean reverting process gives the smallest mean absolute error, 4.2 percent compared to 5.0 percent for the geometric Brownian motion. Excluding the five series with the worst hedging error tilts the comparison even more in favour of the mean reverting process. The mean absolute hedging error is 2.5 percent compared to 4.0 percent for the geometric Brownian motion. The conclusion is therefore that the mean reverting model provides a less robust but slightly more exact hedging scheme.

It is also interesting to note that the differences are not much affected by infrequent price changes or jumps in the price process. Infrequent and large price swings do increase the hedging error but the effect is fairly small. Surprisingly, the geometric Brownian motion seems slightly more sensitive to these characteristics. It should be noted, however, that the 130 series where the option prices are outside the region of 50-200 percent of each other are characterised by few price changes and jumps in the price processes. Thus, these characteristics often cause the models to break down, especially the mean reverting model. One interesting question is how to interpret a situation where two models give vastly different prices, but both have very small hedging errors. For example, for the time series for bulk trade in PVC, polyvinylchloride, the option price given by the mean reverting model was $447 and for the geometric Brownian motion $223. In both cases, however, the hedging error is around $1. Thus, we are able to replicate the price in two ways. However, in a well functioning market, only one price can prevail, the cheaper. On average, the mean reverting model prices the 5-year at-the-money option 6 percent lower than the geometric Brownian motion, even though there are large individual differences.

**Stochastic volatility**

The third and final process to assess is a stochastic volatility process. Stochastic volatility models are normally considered superior for stock options and it is therefore interesting to test it for commodity options as well. Although much discussed for stock options, stochastic volatility is a subject that is rarely mentioned in connection with commodity derivatives or in the real option literature. There are several possible reasons for this neglect. One is that commodity option markets are much smaller than stock and foreign exchange option markets. Research has therefore been concentrated to these markets. Another reason is that the issue is probably overshadowed by mean reversion and convenience yield concerns. Given the success in explaining stock option prices, it is of interest to test the hedging performance for a stochastic volatility model also on commodities. However, a stochastic volatility model is much more complex to estimate, price and hedge than a model based on only one source of randomness. Thus, there are many potential causes why it might not work, although most people would probably argue that volatility does change over time.

A numerical approximation for the option price based on the model (1.3)-(1.4) was developed in essay 4 together with an EGARCH model for parameter estimation.

$$y_t = \sigma_t \varepsilon_t \tag{4.9}$$

$$\ln \sigma_{t+1}^2 = \psi_0 \ln \bar{\sigma}^2 + \psi_1 \ln \sigma_t^2 + \psi_2 \ln(y_t^2 + \sigma_t^2) \tag{4.10}$$

In the model, the variance $\sigma_{t+1}^2$ is updated by weighting the average variance and the previous variance estimate together with the realised return $y_t$.[21] Matching the moments of the EGARCH and the mean reverting model give the continuous time parameter estimates as

$$\kappa = -\frac{\ln(\psi_1 + \psi_2)}{\Delta t} \tag{4.11}$$

$$\gamma = \frac{\psi_0 \ln \bar{\sigma}^2 + 0.53345\alpha}{1-(\psi_1 + \psi_2)} - \ln \Delta t$$

$$\sigma^2 = \frac{2 \cdot 0.26834 \kappa \psi_2^2}{1-(\psi_1 + \psi_2)^2}$$

$$V_0 = \frac{\sigma_T^2}{\Delta t}.$$

[21] $y_t$ is the return when the drift has been eliminated.

The first three estimates are derived in essay 4, appendix C. The fourth estimate, $V_0 = \frac{\sigma_T^2}{\Delta t}$, gives the instantaneous variance at the initiation of the option contract as the terminal variance of the EGARCH parameter estimation process. As the EGARCH process is in daily units, it is annualised through division by $\Delta t$, the number of trading days per annum.[22]

The difference between the stock market data of essay 4 and the commodity data examined here is striking. In essay 4 we applied maximum likelihood estimation to 20 Swedish stocks and used several different techniques: GARCH(1,1) with both unrestricted and variance targeting estimation; the Exponentially Weighted Moving Average; and also a constant volatility. Using the Excel's problem solver and Newton-Raphson iterations to maximise the likelihood worked perfectly well in all cases. For all series, the value of the maximum likelihood objective function of the unrestricted GARCH(1,1) model was higher than for the variance targeting technique, which in turn was higher than for the EWMA and the constant volatility models.

As the latter three are restricted versions of the first one, the clear ordering is exactly as can be expected and a strong indication that the maximisation procedure works well. The model of interest in the comparison, the EGARCH model, had a slightly higher ML-value than the unrestricted GARCH model, and therefore the highest value of all the models.

Repeating the same procedure for the commodity data does not work nearly as well. The root of the problem is that Newton-Raphson iterations only identify a local maximum, which may or may not be the global maximum. In order to find the truly global maximum, it is necessary to compute ML-values for all

[22] Hull (2003, section 12.12) recommends the use of trading days rather than calendar days for variance estimation.

parameter settings and choose the highest value as the global maximum, a procedure that calls for considerably more computer power and programming skills. The issue of global or local maximum becomes acute for those commodity price series that have very few price changes. The ML-functions seem to have multiple maxima close to each other. Not even the Exponentially Weighted Moving Average, with only one parameter, is trustworthy. To alleviate this problem, all series with fewer than 50 price changes during the 5 year period are excluded. However, neither for the remaining series are the results as unambiguous as for the stock data of essay 4. It is a well known problem with ML-estimations that the more explanatory variables there are, the "flatter" becomes the ML-surface and the more difficult it is to find the optimum. This problem becomes apparent for the EGARCH model and the unrestricted GARCH(1,1) model where there are three explaining variables.

For many of the series, the EGARCH model identifies a deterministic variance as the best description of price data, i.e. $\psi_2$ is zero. The stochastic volatility model thereby reduces to the Black-Scholes model.[23] Of the 280 price series, 106 are excluded since they have fewer than 50 price changes. For another 64 series, a constant volatility best describes the data. This leaves 110 different commodities.

Some of the continuous time estimates are clearly not economically viable, with a long-run average variance of several hundred percent or a speed of reversion $\kappa$ that is negative. These series are not separated from the rest. As for the case of mean reversion, we instead define reasonable as an option price within the range 50-200% of the Black-Scholes price. However, the series with doubtful

[23] Strictly speaking, $\psi_1 \neq 0$ and $\psi_2 = 0$ give rise to a situation where the variance is deterministically increasing or decreasing over time. Although this can be handled within the B-S framework, it seems unrealistic to forecast the future variance by blindly extrapolating an historical deterministic trend. It is therefore better to assume that the variance is constant whenever $\psi_2 = 0$ and use the standard B-S model.

estimates cause problems for the option pricing formula. The analytic approximation is not very robust for parameter settings when the time to maturity approaches zero. For parameter settings that seemed suspicious to start with, the approximation breaks down quite frequently. To remedy this problem, we instead measure the hedging performance after four of the five years have passed. This leaves us 75 series where the price identified by the stochastic volatility model is within the 50-200% range of the B-S price and for which the hedging performance is tested.

Hedging within a stochastic volatility framework is more complicated than when there is a single Wiener process driving the stochastics. The mathematics gets rather complex and since multiple assets are required also the practice is more cumbersome. The logic is straightforward, however, so there is no need to get bogged down in the mathematics. In the case of a single Wiener process it is only the price of the asset that changes stochastically. The only interesting information is how the option price changes when the price of the underlying asset changes. Within a continuous time framework all changes are infinitesimally small so only the first derivative, delta, is needed. Higher order derivatives are not required. The hedging portfolio can therefore be constructed by the short put option and shorting a number of the assets equal to delta. This will immunise the portfolio to changes in the asset price.

The same logic carries over to a situation where also the volatility is stochastic, but in order to hedge volatility changes another asset is needed, an asset whose value changes when the volatility changes, i.e. another option. We choose a 10-year put option as this second underlying asset. The idea is to hold a portfolio consisting of the 5-year and the 10-year options, plus the commodity in such numbers that the sensitivity of the portfolio to changes in the asset price and volatility is zero. This boils down to solving two simultaneous equations, one for the asset price sensitivity and one for the volatility. Since the expressions for the sensitivities would be very complicated, these are numerically approximated instead.[24] The choice of a 10-year option as one of the underlying assets is an artificial construct. It is, however, convenient to have an asset that has a longer life than 5-years to avoid a rolling hedge. More details about hedging of the stochastic volatility process is given in appendix B.

Even though the focus of this essay is the price process, let us briefly consider the practical problems that a stochastic volatility model causes for hedging of commodity-linked contracts. The introduction of an option as the second underlying asset causes problems since option markets only exist for some commodities. The option contracts traded are also short-term, necessitating a rolling hedge. In the case of only price uncertainty it is enough to know the sensitivity of the option price to movements in the asset price, in order to create a successful hedge. When another option is required we must know the sensitivity to asset price and volatility movements as well as the price of the other option. Depending on how misspecified the model is compared to the real prices and the real sensitivities, the hedging error may be larger than for a constant volatility model. A stochastic volatility model may also give unrealistic hedge-ratios. Appendix B provides an example. In order to hedge a short put option worth \$1158, one should borrow \$108399 and go long in both the asset and a 10-year put option. It is of course possible to find other asset combinations that hedge the contract equally well and do not require this huge amount of borrowing. Still the example illustrates a problem with stochastic volatility models, namely that the hedging positions can get very large.

[24] The pricing formula is expressed in terms of the variance, so we therefore hedge the variance rather than the volatility. This is a minor point. Stochastic volatility is the accepted name, independently of whether the volatility or the variance is modelled.

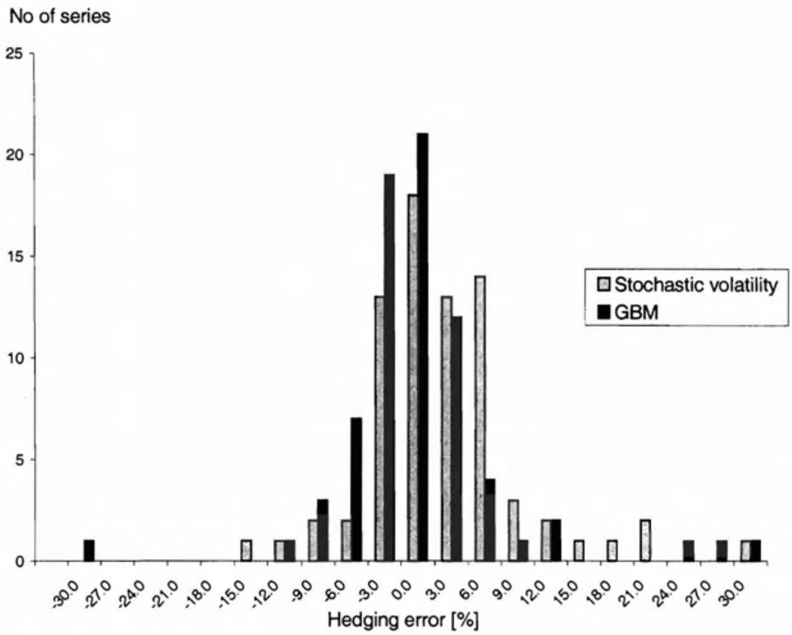


Figure 4.4 Hedging errors for the stochastic volatility process and the geometric Brownian motion.

Returning to the tests, figure 4.4 reveals no major differences in hedging ability between the stochastic volatility process and the geometric Brownian motion. Both models are able to hedge the processes reasonably well but the geometric Brownian motion, i.e. Black-Scholes, works slightly better. The mean absolute error is on average 5 percent compared to 6 percent for the stochastic volatility model. The hedging error for the geometric Brownian motion is also less skewed to the right than for the stochastic volatility model.

The next step is to compare the hedging errors of all three models. Series for which the models provide unreasonable option prices are kept out of the comparison by excluding all series where the highest option price is more than 200 percent higher than the lowest option price. This makes it possible to compare the hedging errors of 40 price series. The mean reverting model is slightly superior and the results are depicted in figure 4.5.

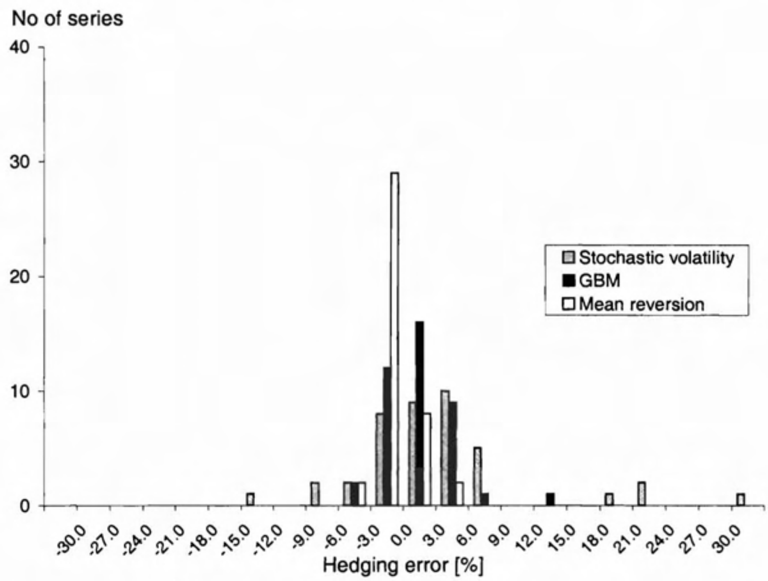


Figure 4.5 Hedging errors for all three processes.

The largest hedging error for the mean reverting model is only 5 percent and the mean absolute error is 0.7 percent. By comparison, the mean absolute error for the geometric Brownian motion is 2.5 percent and for the stochastic volatility 6.5 percent. The stochastic volatility model does the worst in this test and it is also the most complicated model, by far.

To sum up: The geometric Brownian motion is the most robust model and it works quite well. Since all the commodities have a systematic risk (as measured by beta) that is almost zero we assume the risk-free rate to be the required rate of return on commodities; an assumption that does not affect the hedging performance. The tests also show that the inclusion of a convenience yield in the model is crucial. It cannot simply be excluded.

Although the geometric Brownian motion is a very robust model, it sometimes, although rarely, gives very large hedging errors and the option prices also seem suspicious. The other models, however, break down more frequently. In order to compare the different stochastic processes we only test the hedging performance for commodities where the option prices identified by the different models are within the range 50-200 percent of the Black-Scholes price. If several models identify prices of the same magnitude we can be reasonably convinced that a fair price lies within this region. Sorted in this way, the mean reverting model gives very small hedging errors. However, the mean reverting model must be carefully applied in order to work well. In particular, a simplified model without the drift term introduced in essay 2 does not work. Neither does the full model if the regression made in order to estimate the parameters include time as a dependent parameter. In those cases, hedging performance is very poor.

The stochastic volatility model is also sensitive to parameter estimates. However, since the hedging performance is worse than for the other two models there is no need to further explore this issue of stochastic volatility. Instead we examine how much the hedging performance will deteriorate when parameters are estimated out of sample as, of course, always is the case in reality.

# 5. Hedging performance - parameter estimates out of sample

Although the previous tests have been very useful in identifying appropriate price processes and estimation procedures, they all share one serious drawback. Parameter estimates are made from the same sample the hedging performance is tested upon. In reality we have to use historical data (or some other method) to estimate future parameter values. Naturally, this hampers the hedging performance, but how severe is the effect? Building on the results from the previous tests, only two hedging procedures are of interest. First of all, the Black-Scholes model with a constant convenience yield and a required rate of return equal to the risk-free rate. This was hypothesis $H_1$ in figure 4.1. The hedging tests showed that the convenience yield cannot be excluded but that it is sufficient to set the expected return equal to the risk-free rate. This latter simplification was also motivated by the fact that $\beta$ was not significantly different from zero for most commodities. The second candidate is the mean reverting model where the regression to estimate parameters is made without drift and the drift of the geometric Brownian motion is used as the equilibrium drift. This was hypothesis $H_1$ in the mean reversion test of figure 4.2. This estimation procedure was necessary since the mean reverting process without drift was an inadequate description of the price behaviour and that the introduction of a trend complicated parameter estimation so much that the estimates became unreliable. The stochastic volatility model is skipped altogether as the hedging performance was worse than for the other two models.

The contract is the same as previously, an at-the-money five-year European put option. The option is initiated at the 1$^{st}$ of April 1995 and maturing at the 31$^{st}$ of March 2000. Parameters are estimated from the four years of data preceding the initiation of the option contract. Identifying a reasonable convenience yield and mean reversion level requires a fairly long time series of data. Demands and thereby also prices of commodities are probably somewhat related to the business cycle, which in turn changes quite slowly. Four years are chosen on the basis of still having a substantial number of different commodities available. Even so, the sample of different commodities is a little smaller, 263 compared to 280 in the previous case.

In order to rebalance the hedging portfolio, delta, the sensitivity of the option price with regards to the commodity price is required. This raises the question of whether or not to update the parameters during the length of the contract. The convenience yield and the mean reversion parameters are left unchanged. These parameters affect the amount of money paid by the party shorting the underlying commodity and should therefore be determined in advance.[25]

The volatility estimate is updated using a rolling 4-year average. The return volatility used for the geometric Brownian motion is indistinguishable from the volatility estimate given by the autoregressive model for the mean reverting process. Using the same estimate for both processes saves the trouble of having to repeat the regression and calculate the extended regression statistics for every trading day. The simplification can also be motivated theoretically. Excluding the time dependence in the regression (4.8), the price process is mean reverting if the constant $c_2$ in regression (4.8) is less than unity and the price process is a random walk if the constant is equal to unity. (The introduction of essay 3 provides more details.) However, since $c_2$ in any case is very close to unity, the volatility of the autoregression becomes almost identical to the return volatility.

The procedure of using a rolling 4-year volatility for updating the hedging portfolio results in a slightly smaller hedging error than holding the volatility constant. As volatility shocks average out over such a long period of time it

[25] It is, of course, possible for a short option contract to stipulate the algorithm for how the convenience yield should be calculated, rather than the absolute level of convenience yield. However, we do not consider this possibility.

would have been preferable to also test the hedging performance using a rolling average of one or two years. Unfortunately, this is not possible. Using a shorter average result in unreliable estimates, since the prices of many commodities seldom change.

As was also the case when parameters were estimated within the sample, the geometric Brownian motion is the most robust model. The mean reverting model breaks down in a number of cases and identifies an option price of zero. The reason is the narrow range of probable future prices concentrated around the mean reversion level. The histogram in figure 5.1 shows the hedging errors when mean reversion option prices are within the range of 50-200 percent of the Black-Scholes price.

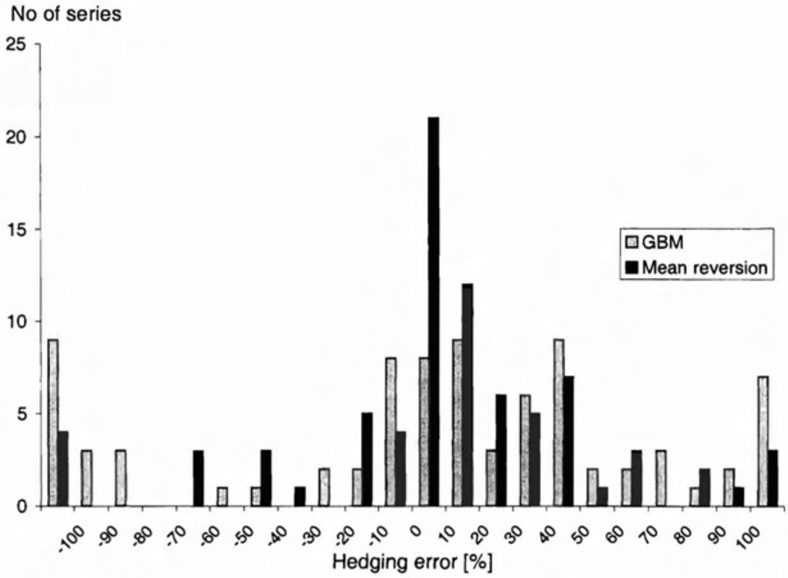


Figure 5.1 Hedging errors when parameters are estimated out of sample.

Using parameter estimates out of sample increases the hedging errors dramatically. Errors are sometimes larger than 100 percent of the option price and distributed over the whole scale, from large and negative hedging errors to large and positive. The mean reverting model appears to fare better, or less worse, in the comparison. The mean absolute hedging error is 56 percent compared to 93 percent for the geometric Brownian motion. These large errors are partly due to just one series. Without this series, the mean absolute error decreases to 39 and 67 percent respectively. The big difference between the two processes should be put in context, however. For only 81 out of 263 different commodities in this sample are the option prices within the region 50-200 percent of each other and thereby included in the text. Among the remaining series there are many examples where the mean reverting process gives unrealistic option prices and hedging schemes. The main results therefore carry over from the previous section. The mean reverting process gives smaller hedging errors, but the geometric Brownian motion is the much more robust process.

The size of the errors is discouraging but in the light of essay 4, perhaps not very surprising. In essay 4, the pricing errors for stock options were of the same magnitude, between 30-80%, when historical volatility was used to price the options. It is possible that the size of the hedging errors for commodity options are of the same order as pricing errors for options on shares of stock, just because historic volatility is not very useful in predicting the future volatility.

# 6. A summarising discussion

This study explores to what extent long-term commodity-linked contracts can be hedged and replicated synthetically. The ability to replicate an asset price has important consequences. From an economic point of view it leads to fair market prices and therefore better functioning capital markets. From a business viewpoint, it facilitates the valuation of assets, both real and financial. It also provides hedging and risk-insurance capabilities to banks and other financial institutions selling tailor-made financial contracts and facing the problem of managing the risk exposure.

The hedging performance for option pricing models based on three different stochastic processes has been tested: First of all the standard geometric Brownian motion, secondly the mean reverting process developed in essay 2, and finally the stochastic volatility process approximated in essay 4. The model based on stochastic volatility is easily dismissed. It has the largest hedging errors, it is not very robust, and for many series a constant volatility gives the highest value of the maximum likelihood objective function. The overall impression is that stochastic volatility is not an important issue for commodity options. This is surprising given its ability to explain observed stock option prices. Of the other two option pricing models, the one based on the geometric Brownian motion, i.e. the standard Black-Scholes model, is the most robust but the mean reverting model provides a smaller hedging error in most cases.

Estimating parameters from the same set of data that the tests are conducted upon reduces the hedging error substantially. The smaller level of "noise" allows for different aspects of hedging to be explored. We are thus able to confirm the importance of the convenience yield parameter for achieving hedging efficiency. It cannot simply be excluded. The convenience yield is the difference between the required rate of return and the expected price appreciation of the commodity. However, equalising the required rate of return to the risk-free rate will have no effect on the hedging performance. Since $\beta$ is not significantly different from zero, this simplification can also be motivated theoretically. Commodity price levels change over time, if for no other reasons simply because of the general inflation. A model that does not capture this drift is inadequate and using it should result in large hedging errors. That the real-option literature frequently suggests such models is therefore rather surprising. The mean reverting model developed in essay 2 and used here takes account of the drift in prices by introducing time as a dependent variable.

According to our tests, a model without drift is not able to hedge the contracts properly. Although this is not surprising, it is nevertheless slightly at odds with the results of Brennan and Crew (1995). Using a mean reverting model without drift (but including stochastic volatility), they are able to successfully hedge a forward contract by rolling futures contracts of shorter maturity. In addition, for the same model, Schwartz (1997) observes a god fit to the volatility term structure for futures contracts.[26] The reason for the importance of a drift term in the model in the present study is probably to find in the length of the contracts. Brennan and Crew and also Schwartz do not use longer than 2-year contracts, whereas the contract in the present study had 5-years to maturity. The added complexity of an option compared to futures might also explain the worse result in our tests.

It lies in the very nature of a mean reverting model that it gives a very narrow range of probable future prices. The model thereby becomes very sensitive to the estimation of the mean reversion price, something that quite frequently

[26] Assuming the standard geometric Brownian motion, the futures price is independent of the volatility. For all other price process, however, the futures price is a function of the volatility. A volatility term structure measures the implied volatility as a function of the time to maturity.

results in unrealistic option prices and hedging schemes. To overcome this problem we have compared the theoretical option prices given by the mean reverting model to the Black-Scholes option prices. The hedging errors obtained when the prices given by the respective models are within the range 50-200 percent of each other are studied. Such a comparison should be performed as no single model can be trusted blindfolded. In this case the mean reverting model has a slightly smaller hedging error than the B-S model. The results thus confirm the advantages of the mean reverting model developed in essay 2. The results also support the intuition that commodity prices are mean reverting and in effect, we suggest that hedging tests can work as a complement to pure statistical tests of mean reversion that so unfortunately have very low power. In table 4.2, the assumption of mean reversion in the price series resulted in the smallest hedging errors for 100 of the 150 series.

The fact that price changes are infrequent and sometimes quite large seems to suggest that the stochastic behaviour of commodities should be modelled using jump processes rather than Wiener processes as building blocks. However, the hedging errors for the affected series were not substantially larger than for other series. This indicates that infrequent and large price changes are more of an empirical problem. They make it very difficult to estimate the true volatility and mean reversion level correctly. This is a problem that will occur for jump processes as well, implying that such models will not provide any significant benefit over existing models after all.

What this study has shown, as far as the processes are concerned, is that both the geometric Brownian motion and the mean reverting model have the potential of hedging long-term commodity linked contracts. Hedging errors were only a few percent when parameters were estimated within sample. However, the large errors that occurred when parameters were estimated out of sample clearly show that the issue of adequate parameter estimation is crucial to the hedging ability. In general, we should not expect to be able to hedge commodity-linked contracts.

Things might look a bit brighter, though, for the most frequently used commodities. The examples of commodity-linked contracts in section 2 were related to oil and metals. For these commodities, there exist reasonably liquid markets for short-term option contracts. This means that we have the added possibility of using implied data from traded options to infer parameter estimates and hopefully improve hedging performance. Option prices do implicitly reveal the market's choice of parameter settings and these are, by definition, forward looking. However, liquid option markets are short-term, and there is no guarantee that short-term parameter settings should be the same as long-term. The reason that implied volatility is so popular in practice is that it prices a non-traded option in line with the traded option from which the volatility parameter is implied. There is no guarantee, however, that the implied volatility is correct. Both options might be equally mispriced. This "luxury" does not work for hedging the option. Here it is important that the implied estimate also is the correct one.

Another possibility that exists when there are short-term option markets available is to engage in gamma hedging. Hedging also the convexity of the option price with regards to the asset price should improve the precision, but it is unclear how much. The reason that gamma needs to be monitored is that trading is not continuous and asset prices do not change in infinitesimally small sizes. However, it was not the daily rebalancing that caused the large hedging errors, it was the parameter estimates. It is doubtful whether gamma hedging really is a remedy against bad estimates.

The best way for a financial institution to mitigate the hedging problem is probably to make the commodity-linked contract more "forward-like" than "option-like". This makes it possible to apply a buy-and-hold hedging strategy or a "stack and roll" hedge. The fact that option features have dominated the contracts during the last two decades is no real reason for continuing writing contracts in this way. After all, a forward contract provides all the insurance that an industrial company would need. Linking forward contracts to the issue of bonds, instead of making the repayment of the bonds dependent of the commodity price, makes no major difference for the producer. However, the reduced problem of hedging a forward contract means that financial institutions can provide such a contract at a much more attractive price. A return to specifying commodity-linked contracts in a more "forward-like" manner may therefore very well be a good future direction. The possibility to detach the forward from the bond may also enhance the popularity for this type of contracts.

# Appendix A - Option prices and delta for the mean reverting process

Given the price process (1.2),

$$dS = \eta(\gamma + \omega t - \ln S)Sdt + \sigma Sdw, \tag{A.1}$$

this appendix derives the European option prices and the respective deltas when the underlying asset $S$ pays a continuous convenience yield. When there is no convenience yield, the Black-Scholes differential equation is independent of whether the drift in asset prices is mean reverting or not. In this case, the ordinary B-S formulas for options on a non-dividend paying stock apply. When there is a dividend or convenience yield $\delta$ attached to the asset $S$, the situation gets a little more involved. The B-S differential equation is derived for the geometric Brownian motion in (1.1),

$$dS = \alpha Sdt + \sigma Sdw. \tag{A.2}$$

Given that there is a dividend or convenience yield attached to the ownership of the asset the B-S differential equation for a derivative $F$ becomes

$$F_t + (r - \delta)SF_S + \tfrac{1}{2}\sigma^2 S^2 F_{SS} - rF = 0. \tag{A.3}$$

$\delta$ is the difference between the required rate of return on the asset and the expected drift in prices. As the geometric Brownian motion (A.2) normally is assumed, this is written $\delta = \mu - \alpha$. Allowing for other price processes, the more general expression is

$$\delta(t, S) = \mu - \frac{1}{dt} E\left[\frac{dS}{S}\right].$$

For the mean reverting process (A.1), we have

$$\frac{1}{dt}E\left[\frac{dS}{S}\right]=\eta(\gamma+\omega t-\ln S),$$

giving the differential equation

$$F_t+[r-\mu+\eta(\gamma+\omega t-\ln S)]SF_S+\tfrac{1}{2}\sigma^2S^2F_{SS}-rF=0. \tag{A.4}$$

Using Feynman-Kac representation, the value of the derivative contract can be calculated as an expectation when the price process is assumed to follow the so called risk-neutral process,

$$dS=[r-\mu+\eta(\gamma+\omega t-\ln S)]Sdt+\sigma Sdw. \tag{A.5}$$

Solving (A.5) gives the risk-neutral price process as

$$S(t)=e^{X(t)},\quad X(t)\sim N[a(t),b(t)], \tag{A.6}$$

where[1]

$$a(t)=e^{-\eta t}\ln S_0+\omega t+\left(1-e^{-\eta t}\right)\left(\gamma+\frac{r-\mu}{\eta}-\frac{\sigma^2}{2\eta}-\frac{\omega}{\eta}\right) \tag{A.7}$$

$$b(t)=\sigma\left(\frac{1-e^{-2\eta t}}{2\eta}\right)^{1/2}. \tag{A.8}$$

---

[1] In essay 2, equation (6.11), the simplification $\psi'=\gamma+\frac{r-\mu}{\eta}-\frac{\sigma^2}{2\eta}$ is used in the expression for $a(t)$.

The value of the derivative contract becomes

$$F(t_0, S) = e^{-r(T-t_0)} E^Q[F(T, S(T))]. \tag{A.9}$$

So far, the derivation of option pricing formulas for the price process (A.1) has been informal. More details are provided in essay 2, section 6. Being more specific about the contract, the value of a European call option is

$$
\begin{aligned}
c &= e^{-r(T-t_0)} \int_{-\infty}^{\infty} \max\{e^x - K, 0\} f_X(x)dx \\
&= e^{-r(T-t_0)} \int_{\ln K}^{\infty} (e^x - K) f_X(x)dx = \\
&= e^{-r(T-t_0)} \int_{\ln K}^{\infty} \frac{e^x}{b\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2b^2}} dx \quad - Ke^{-r(T-t_0)} \int_{\ln K}^{\infty} f_X(x)dx \\
&= e^{-r(T-t_0)+a+\frac{1}{2}b^2} \int_{\ln K}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{(x-[a+b^2])^2}{2b^2}} dx \quad - Ke^{-r(T-t_0)} \int_{\ln K}^{\infty} f_X(x)dx \\
&= e^{-r(T-t_0)+a+\frac{1}{2}b^2} \left(1 - N\left(\frac{\ln K - [a+b^2]}{b}\right)\right) \quad - Ke^{-r(T-t_0)} \left(1 - N\left(\frac{\ln K - a}{b}\right)\right) \\
&= e^{-r(T-t_0)+a+\frac{1}{2}b^2} N\left(\frac{[a+b^2] - \ln K}{b}\right) \quad - Ke^{-r(T-t_0)} N\left(\frac{a - \ln K}{b}\right) \\
&= \left| \textit{Define} \quad d_3 = \frac{[a+b^2] - \ln K}{b} \quad \textit{and} \quad d_4 = d_3 - b \right| \\
&= e^{-r(T-t_0)+a+\frac{1}{2}b^2} N(d_3) \quad - Ke^{-r(T-t_0)} N(d_4).
\end{aligned}
$$

Similarly, for a European put option

$$
\begin{aligned}
p &= e^{-r(T-t_0)} \int_{-\infty}^{\infty} \max\{K - e^x, 0\} f_X(x)dx \\
&= Ke^{-r(T-t_0)} \int_{-\infty}^{\ln K} f_X(x)dx \quad - e^{-r(T-t_0)} \int_{-\infty}^{\ln K} \frac{e^x}{b\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2b^2}} dx \\
&= Ke^{-r(T-t_0)} \int_{-\infty}^{\ln K} f_X(x)dx \quad - e^{-r(T-t_0)+a+\frac{1}{2}b^2} \int_{-\infty}^{\ln K} \frac{1}{b\sqrt{2\pi}} e^{-\frac{(x-[a+b^2])^2}{2b^2}} dx \\
&= Ke^{-r(T-t_0)} N\left(\frac{\ln K - a}{b}\right) \quad - e^{-r(T-t_0)+a+\frac{1}{2}b^2} N\left(\frac{\ln K - [a+b^2]}{b}\right) \\
&= Ke^{-r(T-t_0)} N(-d_4) \quad - e^{-r(T-t_0)+a+\frac{1}{2}b^2} N(-d_3).
\end{aligned}
$$

The delta for the European call option is slightly more complicated to derive. We use $N'$ to denote the probability density function of the standardised normal distribution, and suppress the index indicating time zero on the asset price $S_0$ as there should be no risk of confusion.

$$\frac{\partial c}{\partial S} = e^{-r(T-t_0)+a+\frac{1}{2}b^2} N\left(\frac{[a+b^2]-\ln K}{b}\right)\frac{\partial a}{\partial S} +$$

$$+ e^{-r(T-t_0)+a+\frac{1}{2}b^2} N'\left(\frac{[a+b^2]-\ln K}{b}\right)\frac{1}{b}\frac{\partial a}{\partial S} -$$

$$- Ke^{-r(T-t_0)} N'\left(\frac{a-\ln K}{b}\right)\frac{1}{b}\frac{\partial a}{\partial S}$$

$$= \left| \frac{\partial a}{\partial S} = \frac{\partial}{\partial S}\left(e^{-\eta(T-t_0)} \ln S\right) = \frac{e^{-\eta(T-t_0)}}{S} \right|$$

$$= \frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S} N\left(\frac{[a+b^2]-\ln K}{b}\right) +$$

$$+ \frac{1}{b^2\sqrt{2\pi}}\frac{\partial a}{\partial S} e^{-r(T-t_0)}\left[ e^{a+\frac{1}{2}b^2} e^{-\frac{1}{2}\left(\frac{a+b^2-\ln K}{b}\right)^2} - Ke^{-\frac{1}{2}\left(\frac{a-\ln K}{b}\right)^2} \right]$$

$$= \left| a + \tfrac{1}{2}b^2 - \frac{1}{2}\left(\frac{a+b^2-\ln K}{b}\right)^2 = -\frac{1}{2}\left(\frac{a-\ln K}{b}\right)^2 + \ln K \right|$$

$$= \frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S} N\left(\frac{[a+b^2]-\ln K}{b}\right) +$$

$$+ \frac{1}{b^2\sqrt{2\pi}}\frac{\partial a}{\partial S} e^{-r(T-t_0)}\left[ Ke^{-\frac{1}{2}\left(\frac{a-\ln K}{b}\right)^2} - Ke^{-\frac{1}{2}\left(\frac{a-\ln K}{b}\right)^2} \right]$$

$$= \frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S} N(d_3)$$

In the same way, the delta for the European put is

$$\frac{\partial p}{\partial S} = -Ke^{-r(T-t_0)}N'\left(\frac{\ln K - a}{b}\right)\frac{1}{b}\frac{\partial a}{\partial S} -$$
$$- e^{-r(T-t_0)+a+\frac{1}{2}b^2}N\left(\frac{\ln K - [a+b^2]}{b}\right)\frac{\partial a}{\partial S} +$$
$$+ e^{-r(T-t_0)+a+\frac{1}{2}b^2}N'\left(\frac{\ln K - [a+b^2]}{b}\right)\frac{1}{b}\frac{\partial a}{\partial S}$$

$$= -\frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S}N\left(\frac{\ln K - [a+b^2]}{b}\right) +$$
$$+ \frac{1}{b^2\sqrt{2\pi}}\frac{\partial a}{\partial S}e^{-r(T-t_0)}\left(e^{a+\frac{1}{2}b^2}e^{-\frac{1}{2}\left(\frac{\ln K - [a+b^2]}{b}\right)^2} - Ke^{-\frac{1}{2}\left(\frac{\ln K - a}{b}\right)^2}\right)$$

$$= -\frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S}N\left(\frac{\ln K - [a+b^2]}{b}\right) +$$
$$+ \frac{1}{b^2\sqrt{2\pi}}\frac{\partial a}{\partial S}e^{-r(T-t_0)}\left(Ke^{-\frac{1}{2}\left(\frac{\ln K - a}{b}\right)^2} - Ke^{-\frac{1}{2}\left(\frac{\ln K - a}{b}\right)^2}\right)$$

$$= \frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S}\left[N(d_3) - 1\right].$$

To summarise, given the mean reverting price process (A.1), the prices of European call and put options, respectively, on an asset paying a convenience or dividend yield are,

$$c = e^{-r(T-t_0)+a+\frac{1}{2}b^2} N(d_3) - Ke^{-r(T-t_0)} N(d_4) \tag{A.10}$$

$$p = Ke^{-r(T-t_0)} N(-d_4) - e^{-r(T-t_0)+a+\frac{1}{2}b^2} N(-d_3) \tag{A.11}$$

where

$$\begin{cases} d_3 = \dfrac{a+b^2-\ln K}{b} \\ d_4 = d_3 - b \\ a = e^{-\eta(T-t_0)} \ln S_0 + \omega(T-t_0) + \left(1-e^{-\eta(T-t_0)}\right)\left(\gamma + \dfrac{r-\mu}{\eta} - \dfrac{\sigma^2}{2\eta} - \dfrac{\omega}{\eta}\right) \\ b = \sigma\left(\dfrac{1-e^{-2\eta(T-t_0)}}{2\eta}\right)^{1/2}. \end{cases}$$

The formulas for the delta of the call and the put options are

$$\frac{\partial c}{\partial S} = \frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S} N(d_3) \tag{A.12}$$

$$\frac{\partial p}{\partial S} = \frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S} \left[N(d_3) - 1\right]. \tag{A.13}$$

# Appendix B - Hedging in detail

The derivation in section 3 showed that the value of a derivative contract could be replicated by constructing a portfolio of the underlying asset and a bond, where the number of assets and bonds were continuously rebalanced. More specifically, $h^S$ and $h^B$ in equation 3.15 gave the number of assets and bonds required as,

$$\begin{cases} h^S = F_S \\ h^B = \dfrac{F - SF_S}{B}. \end{cases} \tag{B.1}$$

In order to hedge a short put option, $-F$ in the formula, we need to take the opposite position in the replicating portfolio. The opposite position of $-F$ is, of course, just $F$. The number of assets required in the portfolio is $F_S$, which is also quite commonly denoted as delta $\Delta$. For the Black-Scholes model, (i.e. the geometric Brownian motion), Hull (2003, page 305) provides the formula,

$$F_S = e^{-\delta(T-t_0)}[N(d_1) - 1]. \tag{B.2}$$

In case of the mean reverting process, the delta of a European put option is derived in appendix A, equation A.13,

$$F_S = \frac{e^{-(r+\eta)(T-t_0)+a+\frac{1}{2}b^2}}{S}\left[N(d_3) - 1\right]. \tag{B.3}$$

In both cases, $F_S$ is negative. The partial derivative of a put option with respect to the underlying asset is always negative. This translates into the intuition that the put option price decreases when the asset price increases. In order to hedge a short put option, it is therefore necessary to short the underlying asset.

As the asset shorted in this case is a commodity, the party lending the commodity must be compensated for the loss of the convenience yield. The formula for the convenience yield is $\delta(t,S) = \mu - \frac{1}{dt} E\left[\frac{dS}{S}\right]$. Given that the systematic risk of commodity prices is zero, this boils down to

$$\delta = r - \alpha \tag{B.4}$$

$$\delta(t,S) = r - \eta(\gamma + \omega t - \ln S), \tag{B.5}$$

for the geometric Brownian motion and the mean reverting process respectively.

In addition to giving the number of assets required in the replicating portfolio, equation (B.1) also provides the number of bonds necessary. The value of the bond position is quite simply the difference between the value of the put option and the value of the short stock position, $h^B B = F - SF_S$. However, rather than working with the number of bonds and the respective bond prices, it is more convenient to place the required amount in an overnight account yielding the risk-free rate.

As time passes, the value of the short commodity position and the money in the overnight account will differ from the value of the short put option. First of all, because the exact price process and parameter settings are unknown. Secondly, the replicating portfolio is only self-financing in continuous time. Revising the position daily is as close to continuous updating as it is possible to get, but it will not be self-financing. The value of the overnight account is allowed to vary in order to keep the required short commodity position. Any mishedging will therefore show up in the overnight account.

At maturity the differences are measured. On the positive side is the value of the proceeds from issuing the put option, which over time has yielded the average short term rate, and the amount in the overnight account. These positive positions are counterbalanced by the eventual payment to the owner of the put option and the closing of the short asset position. When the option matures out-of-the-money there is no payment connected to the put option and the short asset position has presumably been closed earlier. Only the proceeds from writing the put option is guaranteed to have any value at maturity. The balance of the overnight account can be either positive or negative, depending on the particular path taken by the commodity prices.

The value of the combined position should be zero at maturity if the hedging has been successful. Any large deviation, be it positive or negative, indicates a failure to hedge the contract properly. The present value of the combined position, discounted at the average short-term rate, is divided by the proceeds from writing the option. The percentage error thus achieved is referred to as the hedging error and depicted in the figures and tables of sections 4 and 5.

**The geometric Brownian motion**

Let us exemplify the procedure by choosing one particular commodity: paper pulp in order to connect to the valuation of the pulp plant in essay 2. Here the B-S model has identified a price of $158 for a 5-year at-the-money put option for the delivery of one tonne of NBSK-Northern Bleached Softwood Kraft pulp.

Delta for the 5-year put option, $F_S$ in the notation used in equation (B.1), is –0.3590 at initiation of the contract. This means that 0.3590 tonnes of pulp has to be shorted, giving an immediate inflow of $296.191. The numbers can be seen in the first data row of table B.1. Observe, though, that the computer program uses more digits in the calculations in order to avoid round off errors.

The \$296.191 is deposited in an overnight account yielding the short term rate 6.1 percent. This is actually the three-month yield on Treasury bills but it is used as a proxy for the overnight risk-free rate. In order to short the pulp we must also pay the convenience yield to the lender of the pulp which amounts to \$0.101.[2]

At the end of the next trading day, the pulp price is unchanged, but due to the passage of time a small change in the portfolio is made. Another tiny fraction of the asset is shorted, providing a small cash inflow, and increasing the amount in the overnight account to $296.191 + 0.070 - 0.101 + 0.063 = \$296.22$. Continuing in this fashion and updating the portfolio daily gives the value of the overnight account as \$600.88 at option maturity. The short pulp position is now closed and the option exercised. The combined position for the party taking the short option position and hedging the risk now looks the following:

| | |
|---|---|
| Value of overnight account | 600.9 |
| Repayment of short asset position: $-0.9998 \cdot 630$ | −629.8 |
| Option payment: $-\max\{630-825,0\}$ | −195.0 |
| End value of option premium[3]: $158e^{0.055 \cdot 5}$ | 208.2 |
| Hedging error | \$ −15.7 |

The \$15.7 hedging error has to be related to the size of the position. In order to obtain the percentage hedging error, the present value of the hedging error is divided by the option premium.

| | |
|---|---|
| Percentage hedging error | |
| Geometric Brownian motion: $-15.7e^{-0.055 \cdot 5} / 158$ | −7.6% |

[2] The convenience yield in dollars is equal to $SF_S(e^{\delta/260}-1)$, where 260 is the number of trading days and $\delta$ is defined in (B.4).

[3] The discount rate 5.5% is the average overnight rate.

Table B.1 Hedging error for the geometric Brownian motion.

| Date | NBSK Pulp CIF W. Europe [U$/TN] | Interest rate Cont. comp. [%] | Time to maturity [years] | Delta | Additional assets to short | Payments from assets shorted [$] | Overnight account | Interest payments (next day) | Payment of net convenience yield (next day) |
|---|---|---|---|---|---|---|---|---|---|
| 3-Apr-95 | 825 | 0.061 | 5.00 | -0.3590 | 0.3590 | 296.191 | 296.19 | 0.070 | -0.101 |
| 4-Apr-95 | 825 | 0.061 | 4.99 | -0.3591 | 0.0001 | 0.063 | 296.22 | 0.070 | -0.101 |
| 5-Apr-95 | 825 | 0.061 | 4.99 | -0.3592 | 0.0001 | 0.063 | 296.26 | 0.070 | -0.101 |
| 6-Apr-95 | 825 | 0.061 | 4.99 | -0.3592 | 0.0001 | 0.063 | 296.29 | 0.070 | -0.101 |
| 7-Apr-95 | 825 | 0.061 | 4.99 | -0.3593 | 0.0001 | 0.063 | 296.32 | 0.069 | -0.101 |
| 10-Apr-95 | 825 | 0.061 | 4.98 | -0.3596 | 0.0002 | 0.190 | 296.48 | 0.070 | -0.101 |
| 11-Apr-95 | 825 | 0.061 | 4.98 | -0.3596 | 0.0001 | 0.063 | 296.51 | 0.070 | -0.101 |
| 12-Apr-95 | 825 | 0.061 | 4.97 | -0.3597 | 0.0001 | 0.063 | 296.54 | 0.069 | -0.101 |
| : | : | : | : | : | : | : | : | : | : |
| : | : | : | : | : | : | : | : | : | : |
| : | : | : | : | : | : | : | : | : | : |
| 24-Mar-00 | 630 | 0.061 | 0.02 | -0.9983 | 0.0002 | 0.152 | 600.33 | 0.142 | -0.214 |
| 27-Mar-00 | 630 | 0.062 | 0.01 | -0.9990 | 0.0007 | 0.457 | 600.71 | 0.143 | -0.214 |
| 28-Mar-00 | 630 | 0.062 | 0.01 | -0.9993 | 0.0002 | 0.152 | 600.79 | 0.143 | -0.214 |
| 29-Mar-00 | 630 | 0.062 | 0.01 | -0.9995 | 0.0002 | 0.152 | 600.87 | 0.143 | -0.214 |
| 30-Mar-00 | 630 | 0.062 | 0.00 | -0.9998 | 0.0002 | 0.152 | 600.95 | 0.143 | -0.214 |
| 31-Mar-00 | 630 | | | | | | 600.88 | | |

**The mean reverting process**

Hedging of the mean reverting process is more complicated than for the geometric Brownian motion. The parameter estimation procedure is more cumbersome since the estimation has to be performed on drift adjusted prices. Secondly, also the actual hedging procedure is a little more difficult.

Equation (4.4) identifies the price of the put option as \$199 and the hedging procedure is detailed in table B.2. With an initial delta of –0.0153, this number of assets should be shorted, giving an inflow of \$12.605. The amount is placed in an overnight account yielding an interest of \$0.003. Given the short position in the underlying asset, a convenience yield has to be paid to the lender of the asset. The size of the convenience yield in percentage terms is given by equation (B.5) and the dollar amount is 0.011. When the position is updated the next day, notation becomes important. Notice first of all the formula (4.4) for the put option price,

$$p = Ke^{-r(T-t_0)}N(-d_4) - e^{-r(T-t_0)+a+\frac{1}{2}b^2}N(-d_3).$$

As usual, the factor $(T-t_0)$ denotes the time to maturity. However for a time dependent process like (1.2),

$$dS = \eta(\gamma + \omega t - \ln S)Sdt + \sigma S dw,$$

not only the difference $(T-t_0)$, but also the actual numbers of $T$ and $t_0$ becomes important. $T$ should be kept constant at 5 years and $t_0$ denotes running time in this case. An alternative, in order to keep the normal intuition that only the difference $(T-t_0)$ matters, is to adjust gamma so that $\gamma = \gamma_0 + \omega s$, where $s$ is the time that has elapsed since initiation of the contract. It was felt that this alternative is less prone to errors since the time dependence in this case is more visible. Thereby the column "updated gamma".

At the end of the next day, 04-apr-95, the price is unchanged, but due to the passage of time a tiny amount of the asset is shorted providing \$0.026. The amount in the overnight account is therefore $12.605 + 0.003 - 0.011 + 0.026 =$ \$12.62. The procedure is repeated until maturity when the amount in the overnight account is \$564.42. The total position for the party taking the short option position and hedging the risk is:

| | |
|---|---|
| Value of overnight account | 564.4 |
| Repayment of short asset position: $-0.9976 \cdot 630$ | −628.5 |
| Option payment: $-\max\{630-825,0\}$ | −195.0 |
| End value of option premium: $199e^{0.055 \cdot 5}$ | 261.8 |
| Hedging error | \$ 2.7 |

The percentage hedging error is given by the present value of the hedging error divided by the option premium.

| | |
|---|---|
| Percentage hedging error | |
| Mean reverting process: $2.7e^{-0.055 \cdot 5} / 199$ | 1.0% |

In essay 2, there was a big difference in the value of the pulp plant depending on whether the price of pulp followed a geometric Brownian motion or the mean reverting process. It is interesting to note that the hedging test in this appendix identifies the mean reverting process as the best description of the actual price process for pulp. The hedging error under this assumption is 1 percent as opposed to 7.6 percent for the geometric Brownian motion.

Table B.2 Hedging error for the mean reverting process.

| Date | NBSK Pulp CIF W. Europe [U$/TN] | Interest rate Cont. comp. [%] | Time to maturity [years] | Updated gamma | Delta | Additional assets to short | Payments from assets shorted [$] | Overnight account | Interest payments (next day) | Payment of net convenience yield (next day) |
|---|---|---|---|---|---|---|---|---|---|---|
| 3-Apr-95 | 825 | 0.061 | 5.00 | 6.4617 | -0.0153 | 0.0153 | 12.605 | 12.60 | 0.003 | -0.011 |
| 4-Apr-95 | 825 | 0.061 | 4.99 | 6.4617 | -0.0153 | 0.0000 | 0.026 | 12.62 | 0.003 | -0.011 |
| 5-Apr-95 | 825 | 0.061 | 4.99 | 6.4616 | -0.0153 | 0.0000 | 0.026 | 12.64 | 0.003 | -0.011 |
| 6-Apr-95 | 825 | 0.061 | 4.99 | 6.4615 | -0.0154 | 0.0000 | 0.026 | 12.66 | 0.003 | -0.011 |
| 7-Apr-95 | 825 | 0.061 | 4.99 | 6.4614 | -0.0154 | 0.0000 | 0.027 | 12.68 | 0.003 | -0.011 |
| 10-Apr-95 | 825 | 0.061 | 4.98 | 6.4611 | -0.0155 | 0.0001 | 0.080 | 12.75 | 0.003 | -0.012 |
| 11-Apr-95 | 825 | 0.061 | 4.98 | 6.4610 | -0.0155 | 0.0000 | 0.027 | 12.77 | 0.003 | -0.012 |
| 12-Apr-95 | 825 | 0.061 | 4.97 | 6.4609 | -0.0156 | 0.0000 | 0.027 | 12.79 | 0.003 | -0.012 |
| : | : | : | : | | | | | | | |
| : | : | : | : | | | | | | | |
| : | : | : | : | | | | | | | |
| 24-Mar-00 | 630 | 0.061 | 0.02 | 6.2952 | -0.9836 | 0.0023 | 1.463 | 556.83 | 0.132 | -0.383 |
| 27-Mar-00 | 630 | 0.062 | 0.01 | 6.2949 | -0.9906 | 0.0070 | 4.411 | 560.99 | 0.133 | -0.386 |
| 28-Mar-00 | 630 | 0.062 | 0.01 | 6.2948 | -0.9929 | 0.0023 | 1.478 | 562.21 | 0.133 | -0.387 |
| 29-Mar-00 | 630 | 0.062 | 0.01 | 6.2947 | -0.9953 | 0.0024 | 1.482 | 563.44 | 0.134 | -0.389 |
| 30-Mar-00 | 630 | 0.062 | 0.00 | 6.2946 | -0.9976 | 0.0024 | 1.486 | 564.67 | 0.134 | -0.390 |
| 31-Mar-00 | 630 | | | | | | | 564.42 | | |

**The stochastic volatility process**

Hedging of the stochastic volatility process is more complicated than in the case of a single underlying stochastic process. When both the asset price and the variance are stochastic, there are two sources of risk and two underlying assets are therefore needed to hedge these risks. In addition, one of the assets must be sensitive to variance changes, i.e. another option has to be used. We therefore hedge the 5-year option by creating a portfolio consisting of the underlying asset, a 10-year option and a risk-free position in an overnight account. In this case, it is not possible to use paper pulp to exemplify the hedging procedure. The EGARCH parameter estimation model provides the highest ML-value for a constant variance, and this reduces the stochastic volatility process to the ordinary geometric Brownian motion and the pricing model to the Black-Scholes model. The hedging procedure is instead exemplified for magnesium, a completely arbitrary choice.

Given that the approximate pricing-formulas in essay 4 are quite long and cumbersome we calculate the sensitivities to changes in asset prices and variance numerically for both the 5 and 10-year options. Table B.3 shows the hedging scheme. At initiation, 03-Apr-95, the price of one tonne of magnesium is \$3875. The stochastic volatility model identifies the price of the 5 and 10-year option as \$1158 and \$1381.6, respectively. Given are also the sensitivities delta and "vega"; how much the option price changes when asset price and variance change one unit. "Vega" is put within quotation marks because vega normally denotes the sensitivity to changes in the volatility rate, not the variance rate.

Simultaneously matching delta and "vega" of the 5-year option to that of a portfolio consisting of the underlying asset and the 10-year option requires an equation system of two equations to be solved.

Delta-matching: $-0.207 \cdot X(p_{10}) + 1 \cdot X(S) = -0.401$

"Vega"-matching: $1.158 \cdot X(p_{10}) + 0 \cdot X(S) = 58.254$

$\Rightarrow X(S) = 10.035 \quad , \quad X(p_{10}) = 50.313$

$X(S)$ and $X(p_{10})$ denote the number of assets and 10-year put options required to obtain a delta and "vega" neutral portfolio. The delta for the 10-year put option is $-0.207$, delta for the asset is 1, and in the same way for "vega".

In order to hedge the price and variance risk of the short 5-year put option it is required to long ten tonnes of magnesium and fifty 10-year put options. Thus it is necessary to borrow $10.035 \cdot 3875 + 50.313 \cdot 1381 = \$108399$ overnight. The negative sign in table B.3 indicates borrowing. This huge amount of borrowing is, of course, unsatisfactory in a real setup and can partly be avoided by changing the option used as the second underlying asset. Still it exemplifies a general problem with stochastic volatility models; the hedging position can become very large.

The interest payment of the \$108399 overnight loan will be –\$25.6. Thanks to the long asset position, a convenience yield of \$22.3 is earned. The next day, the price of magnesium is unchanged but due to the passage of time, a small change in the portfolio is required. The number of 10-year options and assets to hold now becomes:

$\Rightarrow X(S) = 10.115 \quad , \quad X(p_{10}) = 50.677.$

Updating the portfolio requires additional borrowing of

$(10.035 - 10.115) \cdot 3875 + (50.313 - 50.677) \cdot 1381.6 = -\$812.$

The amount in the overnight account becomes

$$-108399 - 25.6 + 22.3 - 812.2 = -\$109215.$$

The hedging procedure is repeated for 4-years. The amount in the overnight account is now positive $2448 because "vega" for the 10-year option is now larger, in absolute terms, than "vega" for the 5-year option. The reason we do not want to measure the hedging error for the full 5-years is that the analytic pricing approximation breaks down for some short-maturity contracts. This problem appears because the EGARCH parameter estimation models sometimes provide unreasonable parameter estimates. The unreasonable parameter estimates are in turn due to the fact that GARCH models in general seem less appropriate for commodity data. The problems are most frequent for the series where the prices do not change very often.

The total position for the party taking the short 5-year option position and hedging the risk is:

| | |
|---|---|
| Value of overnight account | 2448 |
| Repayment of short asset position: $-0.906 \cdot 2300$ | –2085 |
| Repayment of short 10-year option: $-0.113 \cdot 1850.3$ | –209 |
| Value of the short 5-year option | –1687 |
| End value of option premium: $1158e^{0.055 \cdot 4}$ | 1442 |
| Hedging error | $ –91 |

The percentage hedging error is given by the present value of the hedging error divided by the option premium.

| | |
|---|---|
| Percentage hedging error | |
| Stochastic volatility process: $-91e^{-0.055 \cdot 4} / 1158$ | –6.3% |

Table B.3 Hedging error for the stochastic volatility process.

| Date | Magnesium 99.9% CIS [U$/TN] | Interest rate Cont. comp. [%] | Time to maturity [years] | 5-year put option Price | 5-year put option Delta | 5-year put option "Vega" | 10-year put option Price | 10-year put option Delta | 10-year put option "Vega" | No of assets | No of 10-year puts | Additional payments to hold asset | Overnight account | Interest payments (next day) | Payment of net convenience yield (next day) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3-Apr-95 | 3875 | 0.061 | 5.00 | 1158.0 | -0.401 | 58.254 | 1381.6 | -0.207 | 1.158 | 10.035 | 50.313 | -108399 | -108399 | -25.6 | 22.3 |
| 4-Apr-95 | 3875 | 0.061 | 4.99 | 1157.7 | -0.401 | 59.516 | 1381.6 | -0.208 | 1.174 | 10.115 | 50.677 | -812 | -109215 | -25.8 | 22.5 |
| 5-Apr-95 | 3875 | 0.061 | 4.99 | 1157.4 | -0.401 | 60.769 | 1381.6 | -0.208 | 1.191 | 10.190 | 51.015 | -755 | -109973 | -26.0 | 22.7 |
| 6-Apr-95 | 3875 | 0.061 | 4.99 | 1157.1 | -0.401 | 62.012 | 1381.6 | -0.208 | 1.208 | 10.259 | 51.328 | -700 | -110676 | -26.2 | 22.8 |
| 7-Apr-95 | 3875 | 0.061 | 4.99 | 1156.7 | -0.401 | 63.243 | 1381.5 | -0.208 | 1.225 | 10.323 | 51.616 | -646 | -111325 | -26.1 | 23.0 |
| 10-Apr-95 | 3875 | 0.061 | 4.98 | 1155.9 | -0.402 | 64.650 | 1381.5 | -0.208 | 1.257 | 10.296 | 51.427 | 366 | -110962 | -26.1 | 22.9 |
| 11-Apr-95 | 3875 | 0.061 | 4.98 | 1155.6 | -0.402 | 65.859 | 1381.5 | -0.208 | 1.275 | 10.349 | 51.664 | -536 | -111502 | -26.2 | 23.0 |
| 12-Apr-95 | 3875 | 0.061 | 4.97 | 1155.3 | -0.402 | 67.055 | 1381.5 | -0.208 | 1.292 | 10.399 | 51.880 | -489 | -111994 | -26.2 | 23.1 |
| : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : |
| : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : |
| : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : |
| 25-Mar-99 | 2300 | 0.049 | 1.02 | 1688.9 | -0.859 | 1.214 | 1850.0 | -0.403 | -9.974 | -0.908 | -0.122 | -3 | 2470 | 0.5 | -1.2 |
| 26-Mar-99 | 2300 | 0.049 | 1.01 | 1688.6 | -0.859 | 1.201 | 1850.0 | -0.403 | -9.989 | -0.907 | -0.120 | -3 | 2466 | 0.5 | -1.2 |
| 29-Mar-99 | 2300 | 0.049 | 1.01 | 1687.9 | -0.860 | 1.160 | 1850.2 | -0.403 | -10.010 | -0.907 | -0.116 | -9 | 2456 | 0.5 | -1.2 |
| 30-Mar-99 | 2300 | 0.049 | 1.01 | 1687.6 | -0.860 | 1.147 | 1850.2 | -0.403 | -10.024 | -0.907 | -0.114 | -3 | 2452 | 0.5 | -1.2 |
| 31-Mar-99 | 2300 | 0.049 | 1.00 | 1687.3 | -0.861 | 1.134 | 1850.2 | -0.404 | -10.037 | -0.906 | -0.113 | -3 | 2448 | 0.5 | -1.2 |
| 1-Apr-99 | 2300 | | | 1687.1 | | | 1850.3 | | | | | | 2448 | | |

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