Pricing Corporate Debt

Joel Reneby

AKADEMISK AVHANDLING
FÖR AVLÄGGANDE AV
EKONOMIE DOKTORSEXAMEN

Avhandlingen framläggs till offentlig granskning
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ERRATA
for
Pricing Corporate Debt
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Significant

- Page 29, LEMMA 5. It reads

$$C_L\{\omega_t, t; F, T\} = \omega_t \cdot Q^{\omega}\{\tau \nleq T, \omega_T > F\} - e^{-r(T-t)}F \cdot Q^{B}\{\tau \nleq T, \omega_T > F\}$$

It should read

$$C_L\{\omega_t, t; F, T\} = \omega_t \cdot \mathbf{e}^{-\beta(\mathbf{T}-\mathbf{t})} \cdot Q^{\omega}\{\tau \nleq T, \omega_T > F\} - e^{-r(T-t)}F \cdot Q^{B}\{\tau \nleq T, \omega_T > F\}$$

Less significant

- Page 29, LEMMA 4. The factor $\left(\frac{\omega_0}{L}\right)^{-\frac{2}{\sigma}h^{X\to m}}$ should read $\left(\frac{\omega_0}{L_0}\right)^{-\frac{2}{\sigma}h^{X\to m}}$.
- Page 72. Bibliography entry Ericsson (1997) should read "Ericsson, J. (1997) *Credit Risk in Corporate Securities and Derivatives – Valuation and Optimal Capital Structure Choice* PhD Thesis, Stockholm School of Economics, 1997".
- Page 77. The eighth row (bulleted item) should read "To change measure one uses $dW^X = dW^B + h^{B\to X}\mathbf{dt}$".

Pricing Corporate Debt

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A large part of this thesis has been produced in collaboration with Jan Ericsson. Section 3 of Chapter 2, Chapters 4, 5 and parts of Chapter 7 are more or less excerpts from joint work previously published in his 1997 thesis. In particular the following items are, however, specific to this paper: the approach to valuing equity, the introduction of a non-constant reorganisation barrier, and the consideration of the shape of the yield curve and the associated default probabilities.

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Contents

The majority of the derivations and a large number of convoluted formulae have been relegated to a substantial Appendix. The Appendix also contains a list of notation on page 75.

CHAPTER 1

Introduction

The application of contingent claims analysis to the pricing of corporate securities has a history of about 25 years. The emphasis has been on corporate debt and, after a period of inactivity during the 80's, the early 90's witnessed the evolving of a multitude of models. However, the focus usually has been on the development of new pricing formulae, rather than on the implementation of the models in practice.

The aim of this thesis is to provide a unified and readily implementable framework for pricing corporate debt instruments – in particular non-investment grade debt. The thesis covers both theoretical and empirical aspects. I first use contingent claims analysis to provide simple, yet flexible, closed form pricing formulae. Thereafter, I show how to apply them in practice, and examine the properties of price estimators.

The theoretical part shows how to use combinations of simple barrier contracts to obtain closed form solutions for relatively complex scenarios with considerable ease. The framework extends to compound claims, such as embedded options on corporate bonds. It is flexible enough to accomodate such exigencies as bankruptcy costs, corporate taxes and deviations from the absolute priority rule. The variable underlying the contingent claims is the value of the firm's assets, and the model presented consequently can be termed a firm value based pricing model. Such models try to capture the inherent *credit* risk in a security, and the presented framework can be expected to perform best when that particular risk component is large; i.e., for non-investment grade debt.

Due to the notoriously low liquidity and correspondingly low information content in price series of corporate securities other than equity, it is vital to be able to utilise stock prices for estimation purposes. There is a documented strong correlation between bond prices and corresponding stock prices for non-investment grade firms (Shane (1994)). Exploiting stock price information is a basic feature of the proposed model and the framework provides the equations necessary to do so.

Many previous firm value based models have valued equity as a residual claim to a single debt instrument. This model acknowledges that equity is the residual claim to a multitude of debt claims. There are two benefits with this approach. First, it is a more accurate description of the overall debt structure of the firm and therefore also of the therewith associated risk of running into financial distress. Second, the pricing formula for equity can de derived in closed form without restricting the modelling of the remaining corporate securities; something that considerably facilitates estimation.

The theoretical part also contains notes on some conceptions that are widespread in the literature. I show why the assertion that firm value based models in general cannot generate yield spreads or yield curves in line with actual ones is dubious, and demonstrate that the suggested framework features yield curves as well as default probabilities that agree with actual ones. Also, I suggest a way to avoid the common yet unsatisfactory assumption that the firm's assets are continuously traded.

Empirical applications of firm value based models are rare. The first test of pricing performance is by Jones, Mason and Rosenfeld (1984), who apply a model with no bankruptcy costs, no taxes, constant interest rates, continuous coupons and default possible only at maturity, but featuring callability.[1] Estimating prices of 27 American bonds, they find that their model overprices debt.

Since then, the models have been extended and refined (Kim et al. (1993), Nielsen et al. (1993), Mella-Barral & Perraudin (1997), Leland & Toft (1996) among others.) Also, a different estimation technique has been suggested by Duan (1994). Still, however, the test of Jones et al. (1984) is popularly referred to as having shown that firm value based models, in general, overprice debt.[2] Evidently, there is good reason to conduct a new test. This thesis provides the basis for such a test.

A brief outline of the suggested implementation method is the following. It can be viewed as consisting of two phases: *estimation* and *pricing*. During the first phase, maximum likelihood methods are used to find the (unobserved) asset value time series, most likely to have generated the (observed) stock price series. During the second phase, the estimates of asset value and its volatility are used, together with security-specific characteristics, to price corporate debt instruments.

The presented framework can price all sorts of debt claims including non-standardised or non-traded ones such as bank loans and private placements. Corporate bank loans are an important source of capital also in the United States. They supply more than half as much as the bond market (Sundaresan (1997), p. 320). Moreover, the private placement of debt is about 75% the size of the public issues of debt (Sundaresan (1997), p. 336). Intergrating these debt classes thus extends the usefulness of the model in a not negligible way.

The contribution of this thesis is that it provides a fully implementable framework for pricing corporate debt instruments. The model does not, as many other pricing models do, rely on the existence of traded bonds to be applicable – traded equity is sufficient. It is exemplified using real data and evaluated using simulated data where the small sample characteristics of the price estimators are investigated. The suggested approach appears to work well and the new estimation technique clearly outperforms the traditional one.

The composition of this thesis is the following. In the next chapter I present the economic setting. The setting is utilised in the three subsequent chapters to derive pricing formulae for various securities. First equity is valued. The chapter after that derives the pricing formulae for the basic securities, such as corporate bonds, and the following chapter examines compound securities, such as derivatives written on the basic securities. The sixth chapter features a discussion about the magnitude and shape of corporate yield spread curves. The seventh chapter describes how to perform an empirical investigation and the eighth chapter evaluates the model using simulated data. The final chapter holds a summary and some conclusions.

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[1] The paper is unclear in a number of respects - in particular, there seems to be no distinction between pricing equations for total debt and bonds - and the above list of features is my own interpretation.

[2] Zhou (1997) stretches the interpretation even further when he claims that "Jones, Mason and Rosenfeld (1984) find that credit spreads on corporate bonds are too high to be matched by *the diffusion approach.*" (my italics).

CHAPTER 2

Economic Setting

In this section, I discuss the basic economic setting on which I base my model. I make the standard assumptions of Black & Scholes (1973) and Merton (1974) about the economy – with the exception of the tradeability of the state variable, as I will discuss below. Arbitrage opportunities are ruled out and investors are price takers. Furthermore, for at least some large investors, there are no restrictions on short selling stocks or *riskfree* bonds and these can be traded costlessly and continuously in time. There are no assumptions about the tradeability of *corporate* bonds.

The constant riskfree interest rate is r. Many of the recently developed models incorporate stochastic interest rates.[1] This is important to the degree that credit risk is correlated with interest rate risk. Although from an economic perspective it appears natural that high interest rates are correlated with harsh conditions for firms and therewith increased default risks, it is not obvious that this effect is significant. Unique credit risk might be by far the most important risk factor for non-investment grade debt, which is in focus in this thesis. Indeed, to quote Fridson et al. (1997), "empirical investigations have not identified interest rates as an important determinant of default rates on high-yield bonds"; examples are Fridson & Kenney (1994) and Reilly & Wright (1994). On the other hand, Fridson et al. (1997) find that *lagged* real interest rates help to explain default rates. It therefore seems that determining the importance of allowing for stochastic interest rates is an empirical issue, and as a first step and as a benchmark, I choose to remain with the simplifying assumption of a constant interest rate.

The state variable used in the firm value based class of models is linked to the value of the firm's assets. The prices of the firm's securities depend on the share of the firm value that each security holder is entitled to, in solvency as well as in financial distress. In this setting, corporate securities and their derivatives are valued as claims contingent on the underlying asset state variable.

Throughout this thesis, I assume that the state variable determining the value of the firm's assets at a future point in time, v, follows a geometric Brownian motion.

$$(1) \qquad \begin{cases} dv_t = \mu\, v_t\, dt + \sigma\, v_t\, dW_t \\ v_0 = \underline{v} \end{cases}$$

Several other alternatives are conceivable, such as jump-processes and processes with non-constant parameters. A geometric process has been the most common specification in the past. The reason is, no doubt, that it is convenient to work with from a mathematical viewpoint, but also that it has acceptable economic features.

[1]E.g. Longstaff & Schwartz (1995), Kim et al. (1993), Saá-Requejo & Santa-Clara (1997) and Nielsen et al. (1993).

Observe that I do not assume that v_t equals the value of assets prior to that future date at which it settles their value – it is only an indicator of what the price will be and is, indeed, not a price at all until then. This implies that I do not assume that the assets are *continuously* traded. As common as this assumption may be, it is superfluous in the present setup. This issue is discussed in detail in Section 3 of this chapter.

It is acknowledged that the assets generate revenue that is not reinvested. This "free cash flow", or liquid assets, can be used to service debt or be paid out as dividends to shareholders. I assume that a constant fraction, β, of the return from assets is not reinvested.

The value of assets is denoted ω_t. It is shown in Section 3 that the process for the value of assets is given by the following equation

$$\left\{ \begin{array}{l} d\omega_t = (r + \lambda\sigma - \beta)\, \omega_t\, dt + \sigma\, \omega_t\, dW_t \\ \omega_0 = \underline{\omega} \end{array} \right. \tag{2}$$

The term $(r + \lambda\sigma)$ is the expected return from holding the firm's assets – including accumulating the free cash flow $\beta\omega$. The *growth rate* of assets is $(r + \lambda\sigma - \beta)$. The parameter σ is the volatility of the asset value and λ can be interpreted as the market price of risk associated with the operations of the firm. Next consider the firm's liability side.

1. The firm's liabilities

It is important to make a distinction between securities that, in nominal terms, constitute a small part of the firm's debt – such as a bond issue – and securities that appear on an aggregate level – such as equity. In many papers it is assumed, at least indirectly, that total debt is made up of a single bond issue. Consider for example the model of Geske (1977), which values a coupon payment as a compound option on later payments. Bankruptcy occurs when the value of assets is so low that equity holders no longer finds it profitable to honour a coupon payment. Although the idea of stockholders taking into account future debt obligations when deciding on debt service doubtless provides valuable insights, the model is not tractable for bond pricing. The reason is that in practice there will be a multitude of *other* payments, in between payments to the holder of the bond you want to value, that might be the cause of bankruptcy. To conclude: equity should generally not be valued as the residual of a single bond (or bond issue), but as a residual claim of a throng of liabilities. That is the approach used here.

The liability side of the firm's balance sheet is divided into equity and debt. Using that perspective, debt is composed of bank loans, bonds, accounts payable, salaries due, accrued taxes *etc.* Money due to suppliers, employees and the government are substitutes for other forms of debt. Part of the price of a supplied good and part of salary paid can be viewed as corresponding to compensation for the debt that, in substance, it constitutes. The cost of debt consequently includes not only regular interest payments to lenders and coupons to bondholders, but also fractions of most other payments made by a company. Debt service therefore can be argued to take place more or less every day: to price equity strictly, one would have to account for all these individual payments.

To obtain a simple, closed form pricing formula for equity, I make the assumption that debt service takes place continuously. Considering the frequency of actual payments, this is a fair hypothesis, at least if no individual payment is very large.

Moreover, since most companies do not have a maturity, fixed or otherwise, the firm is assumed, conditional on no default, to continue its operations forever.[2] Equity is then priced in the spirit of Black & Cox (1976) and Leland (1994) in Chapter 3.

Unlike in those models, however, the firm is allowed to increase its total debt over time. As time passes and the value of its assets increases, it is reasonable to expect debt obligations to increase as well – otherwise, the debt to equity ratio would tend towards zero as time goes by. As financial distress one way or the other is related to total debt, this implies that the risk of default disappears with time.[3] The importance for pricing of allowing total debt to grow is demonstrated in the following section on financial distress, and again in Chapter 6.

I denote total nominal debt with N_t. Although the increase in total debt is the result of many small debt issues, it is assumed that it can be approximated, on an aggregate level, by a continuous increase with rate α:

$$\begin{cases} dN_t = \alpha\, N_t\, dt \\ N_0 = \underline{N} \end{cases}$$

In other words: $\alpha\, N_t\, dt$ is the extra principal to which the firm must commit itself each instant in order to let total nominal debt grow at a rate α. For future reference, let $d\,\{\omega_t, t; \cdot\}\, dt$ denote the market value of a loan with principal $\alpha\, N_t\, dt$. Since all issues are assumed to be floated at a fair price, $d\,\{\omega_t, t; \cdot\}\, dt$ is also the amount borrowed each instant.

Total debt service at time t is denoted with C_t which increases at rate α as well. Note that the coupon increases because new loans are taken up and need to be serviced, and not because the coupon to a single loan increases. Coupons are tax-deductible and the corporate tax rate is ζ.

Assuming that aggregate debt has the simple structure outlined above implies that it will not be affected by the characteristics of the specific debt instrument one wants to price. Since equity is valued as the residual of aggregate debt, those characteristics will not affect the valuation formula for equity either. Intuitively, the singular debt issue is too small to have a significant affect on the overall debt level. In short, one can derive pricing formulae for equity and individual debt instruments "separately".

The full potential of this *separation assumption* appears when one is pricing instruments that potentially change the capital structure, such as convertibles or callable bonds. Normally, when solving for the value of equity in such a setting, one would have to account for the possibility that total debt changes stochastically. Among other things, one would have to account for a stochastic barrier. This would have an adverse effect on the tractability of the equity formula, and it would most certainly no longer be in closed form. That is avoided if one accepts the separation assumption laid down in the paragraph above. The idea applied to callable debt,

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[2] A specific debt contract (a coupon bond for example) is, of course, neither assumed to be serviced with continuous coupons, nor to have an infinite maturity.

[3] In a recent paper, Jesús Saá-Requejo and Pedro Santa-Clara (1997) claim that it is contrary to intuition that default intensity (or the probability of default conditional on no prior default) does *not* go to zero with the length of the time horizon. But their argument implies that old firms would be able to issue riskfree debt, something that is not observed.

It is possible that *some* firms, for example very new ones, or firms close to financial distress, have a decreasing default intensity over some time period, but I don't see why it should ever approach zero.

for example, is that when one bond issue is called, another loan is taken up, which leaves the overall debt structure approximately unaffected.

2. Financial distress

The firm is assumed to enter into financial distress, and start some form of reorganisation or possibly file for bankruptcy, if the value of its assets falls below L_t – the reorganisation barrier. The time of this unfortunate event is denoted τ. In the terminology of the model, the firm ceases to exist and the value of the assets, after some (stochastic) reorganisation cost has been incurred, is distributed to claimants. Due to violations of absolute priority, equity holders as well can be expected to acquire a fraction of the assets. Reorganisation encompasses such "mild" forms of financial distress as a writedown of the claims of some securities, supply of capital and changes in priority, but also downright bankruptcies. In the former case, it is apparent that reorganisation in the model does not correspond to an assumption that the firm, in effect, ceases to exist. The costs of reorganisation under such circumstances are made up of, for example, losses due to suspended deliveries by cautious suppliers and over- or under-investment. In the latter case, however, the costs can be expected to be higher, and also include direct bankruptcy costs such as fees to lawyers and accountants.

The payoff to claimants in reorganisation may be in the form of cash or new securities. I use ε_τ and δ_τ to denote the fractions of assets paid out to equity holders and the recovery rate for debtholders, respectively. Hence $\frac{\varepsilon_\tau \cdot L_\tau}{\varepsilon_\tau \cdot L_\tau + \delta_\tau \cdot N_\tau}$ quantifies the violations of the absolute priority rule. Note that the costs and payoffs in reorganisation are not assumed to be fixed or known – ε_τ and δ_τ can be thought of, and even modelled as, the outcome of a strategic game played between equity holders and various debtholders over the reorganising firm. I do assume that the payoffs are independent of time. Since the (relative) values of equity and total debt are independent of time, there is no reason that the outcome of the game should depend on time either. This hypothesis is supported by Altman & Kishore (1996), who find that the time to default from a bond's original date of issuance does not have any association with the recovery rate.

2.1. Choice of reorganisation barrier. There are several ways to interpret, and determine, the level of the reorganisation barrier. One is to choose it so as to equal the total amount of nominal debt, $L_t \equiv N_t$, or a fraction thereof. In many countries, corporate law states that financial distress occurs when the value of the firm's assets reaches some lower level, usually related to the total nominal value of outstanding debt. Apart from this judicial view, there are several economic justifications. One is to view the barrier as the level of asset value that is necessary for the firm to retain sufficient credibility to continue its operations or where, due to some covenant, it voluntarily files for bankruptcy. Another is to think of the barrier as the asset value at which it is no longer possible to honour the payments, be it by selling assets or issuing new securities.

A second interpretation of the reorganisation barrier is based on the supposition that equity holders are too small and scattered to contribute funds to satisfy creditors and avoid a reorganisation situation. Thus, assuming that internally generated funds are the only means to service debt, the firm is solvent as long as internally generated funds exceed the current coupon, i.e. as long as $\beta\omega_t \geq C_t$. Accordingly, in this case we obtain the reorganisation barrier from $L_t \equiv \frac{C_t}{\beta}$.

A third alternative for the reorganisation barrier is the level of asset value at which equity holders are no longer willing to contribute funds to stave off financial distress. This choice of barrier is the lowest possible since a lower equity value is not concordant with limited liability. Using this alternative, the level of barrier is endogenously determined within the model and Section 3.4 of the appendix derives a closed form expression. In Leland (1994) an endogenous barrier is interpreted as corresponding to a firm with unprotected debt, and the first alternative (setting the barrier equal to nominal debt) is interpreted as corresponding to debt with a safety covenant.

Finally, many authors have suggested that strategic considerations are an important determinant of financial distress.[4] Indubitably, this is true in many cases and the barrier L_t could be modelled as the outcome of a game played between debtors and creditors over the assets of the reorganised firm. However, in practice it is doubtful that considering strategic factors explicitly would enhance pricing performance – the ultimate goal of the current work. To accurately reflect the reorganisation decision, a model would become so complex that it would no longer be tractable, or even feasible, to use for pricing purposes.

As is apparent from the paragraphs above, the initial value of the reorganisation barrier will depend on its interpretation. Regardless of the choice of the reorganisation barrier in the current setup, however, it will grow exponentially at the same rate as total debt, α. This is due to the fact that the *change* in the "valuation environment" at the aggregate level of the firm is wholly captured by the change in total debt. The valuation environment should be understood as consisting of all the inputs necessary to price a security. The environment at the aggregate level can thus be said to be time homogeneous – a consequence of the infinite maturity of the aggregate securities. Consequently, the reorganisation decision cannot be altered by any other variable than total debt – which grows at a rate α. The fact that the reorganisation barrier grows exponentially (as opposed to linearly for example) is fundamental for the tractability of the pricing formulae to be derived in the following chapters. This is due to the existence of (well known) simple, closed form solutions for the probability of the first hitting time of two geometric Brownian motions.

2.2. Importance of increasing barrier. If a model does not account for an actual increase in total debt, and the therewith associated growth of the reorganisation barrier, the distribution of bankruptcy probabilities over time will be at odds with reality. Specifically, the default intensity will be heavily skewed to the early part of the company's life. This means that the probability of default, conditional on no previous default, very quickly approaches zero. In other words, if a firm does not go bankrupt in the immediate future, it probably never will.

This feature of a constant barrier, infinite horizon model severely biases the subsequent bond price estimates. Consider a long term corporate coupon bond floated by a company with a solid financial situation. In reality, even if the risk of default within the next five years is negligible, there is usually a risk that the company will encounter financial difficulties before the maturity of the bond. This

[4] Anderson & Sundaresan (1996) and Mella-Barral & Perraudin (1997), for example.

uncertainty is naturally reflected in real prices, but is not taken into account by a constant barrier model.[5]

The reason for the declining bankruptcy density is that the expected value of the asset value exponentially increases, and that therefore the expected ratio of asset value to the barrier (which determines the bankruptcy intensity) rapidly becomes very large. The remedy is, of course, to allow the barrier to grow as well. The following example illustrates this point.

EXAMPLE 1. *The value of a firm's assets has an expected return of 12% and a standard deviation of 15%. Let us choose the probability of reorganisation within the next 30 years to be 10%. Exhibit 1 depicts how the distribution of the default intensity depends on the growth rate of the reorganisation barrier (The default intensity can be thought of as the unconditional, instantaneous probability of default.).*

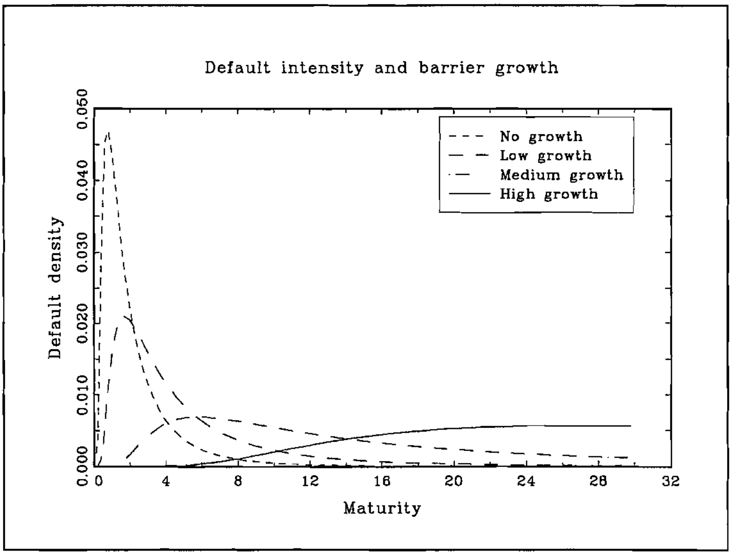

Exhibit 1: Distribution of the first hitting time.
The figure shows the distribution of the first hitting time density for four choices of barrier growth ($\alpha = \{0, 0.04, 0.07, 0.11\}$). To focus on the effect of the barrier growth, the cash flow, also affecting bankruptcy probabilities, is set to zero ($\beta = 0$).

The dashed line with the leftmost peek is the constant barrier case. It is apparent that it is unlikely that a default should occur more than 5 years hence – if the firm does not enter into bankruptcy within the next 5 years, it never will. Conversely, when the growth rate of the barrier is very high (the solid line), immediate default is unlikely, but as time goes by, the value of the assets may deteriorate and the default probability is spread evenly over time.

The ultimate concern is the effect on bond prices of neglecting to incorporate a growing barrier. This effect will be investigated in Chapter 8, where it is shown

[5] Note that this critique is only directly applicable when the barrier is the only trigger of default. If, for example, repayment of the principal also is critical, the skew of the default intensity will be mitigated.

that neglecting the increasing barrier introduces a pricing bias of as much as 18%, corresponding to several hundred basis points.

3. On the tradeability of the firm's assets

An almost ubiquitous assumption made in papers that deal with firm value based models is that the state variable follows a geometric Brownian motion under the objective probability measure (equation (1)). Then it is assumed (more or less explicitly) that v may be traded on frictionless markets. This implies that under a (unique) probability measure Q^B under which prices deflated by a unit of the money market account, B, are martingales[6], the state variable process will take on the following appearance (for ease of exposition I assume $\beta = 0$ in this subsection)

$$\begin{cases} dv_t = r\, v_t\, dt + \sigma\, v_t\, dW_t^B \\ v_0 = \underline{v} \end{cases}$$

where W^B is a Wiener-process under the new measure. It is often recognized that the assumption of traded assets is a strong one in many situations, but as a rule, few papers address this problem directly. If we were to relax the assumption of traded assets, we would be in an incomplete market setting in that we cannot replicate the state variable v. The process for v under a measure Q^B (now no longer uniquely determined[7]) would then be

$$\begin{cases} dv_t = (\mu - \lambda\sigma)\ v_t\, dt + \sigma\, v_t\, dW_t^B \\ v_0 = \underline{v} \end{cases}$$

Note that v now no longer describes the dynamics of a *price* variable (and hence can no longer be termed asset *value*). Furthermore, three asset-specific parameters (μ, λ, σ) as opposed to one (σ) determine the process.

Suppose, however, that at some point in time T, the firm's assets v will be traded. Let $E^B[A]$ denote the expected value of an event A under the probability measure Q^B. Standard theory then tells us that, given the existence of (1) and a *traded* derivative (e.g. equity) written on v, any derivative written on v can be valued by arbitrage methods. Specifically, a derivative paying off the value of assets at T can be priced. Formally,

$$\omega_t = e^{-r(T-t)} E^B[v_T] = v_t\, e^{(\mu-\lambda\sigma-r)(T-t)} \tag{3}$$

This contract may be interpreted as the value of a corresponding all-equity firm or simply the *value of assets.*[8] The dynamics for ω under the objective measure (by

[6]The probability measure Q^B is commonly referred to as the "risk neutral" probability measure. The Girsanov kernel for the transformation from the objective measure to Q^B is, when assets are traded, $\lambda \equiv \frac{\mu - r}{\sigma}$. This Girsanov kernel is frequently interpreted as "the market price of W-risk".

The money market account is defined through

$$\begin{cases} dB_t = r\, B_t\, dt \\ B_0 = \underline{B} \end{cases}$$

[7]That is, the Girsanov kernel λ is no longer uniquely determined (cf. footnote 6).

[8]Note that if the firm's assets are continuously traded, the Girsanov kernel is given by footnote (6). Then, from eq. (3), $\omega = v$, which illustrates that v can be interpreted as the value of assets only when they are *continuously* traded.

applying Itô's lemma on equation (1) and using (3)) are

$$(4)\qquad \begin{cases} d\omega_t = (r + \lambda\sigma)\,\omega_t\,dt + \sigma\,\omega_t\,dW_t \\ \omega_0 = \underline{\omega} \equiv v_0\,e^{(\mu-\lambda\sigma-r)T} \end{cases}$$

and under the measure Q^B

$$(5)\qquad \begin{cases} d\omega_t = r\,\omega_t\,dt + \sigma\,\omega_t\,dW_t^B \\ \omega_0 = \underline{\omega} \end{cases}$$

Given that the ultimate aim is to price a security and not to analyse the process for v itself, knowledge about the process for ω is sufficient. The reason is that the parameters (v_t, μ, λ) appear in the pricing formulae for corporate securities and derivatives as one entity only (i.e. as ω as defined in (3)). Therefore, for pricing purposes, the information contained in (ω_t, σ) is equivalent to the information contained in $(v_t, \mu, \lambda, \sigma)$.

This is of major importance when implementing a firm value based model. Since the asset process is most likely to be unobserved, estimation of its parameters must be based on some observed variables, for example the prices of traded securities. Since, as noted above, these only contain information about (v_t, μ, λ) as one entity, it will be impossible to estimate the process for the state variable (1) (for example it will not be possible to distinguish a situation with high μ and low v from a situation with low μ and high v). It will, however, be possible to estimate the process for the asset value, (4).

I can thus conclude the following:

- for pricing purposes, the pair (ω_t, σ) is a sufficient statistic
- in most cases it is therefore sufficient to estimate the process (4)
- the existence of the process (4) does not require continuously traded assets – sufficient conditions are that (i) equity is continuously traded and that (ii) assets are traded at a *discrete* (unspecified) future point in time

It is important to acknowledge the difference between *assuming* that the firm's assets are continuously traded and *deriving* that it is possible to trade continuously in a portfolio mimicking the value of the firm's assets.[9] It appears that, in many cases, unduly restrictive assumptions have been imposed – for pricing purposes, the assumption of contionuously traded assets is unnecessary.[10] However, the argument requires that the state variable is revealed *as a value* at some point in time when it can also be traded. Although this is a relatively mild assumption within the framework of a firm value based model, the argument is *not* valid for models with general non-traded state variables such as interest rate models.

Finally, note that even though the asset value process ω is fictitious in that it is not traded directly, it can be replicated by trading continuously in a traded security, such as equity, and the money market account. It is not possible to replicate the state variable v.

[9] An argument sometimes heard is: "Assets are not traded but we can always replicate them by trading in equity and the money market account – and thus pricing of debt and options can be carried out by standard arbitrage methods". That argument is not valid, however, since if the assets are *never* traded, the pricing formula for equity cannot be derived in the first place. Hence, its pricing formula and "hedge ratio" will be unknown.

[10] It is even more unsatisfactory to maintain this hypothesis in a situation where the asset process is being estimated – assets are in this case assumed to be traded while their value is not observable.

4. Summary of setting

Summing up, the proposed valuation setting is characterised by the following features:

- Assets
  - Future value of assets is the state variable.
  - Assets need not be continuously traded.
- Equity and debt
  - Total debt increases over time.
  - Equity and the isolated debt instument can be priced separately which simplifies formulae.
- Stock prices used as the main source of information – prices from bonds issued by the same firm are not needed.
- Reorganisation is triggered by a fall in the asset value to a barrier related to total debt. The reorganisation barrier thus increases over time.
- Constant interest rate.
- First and foremost applicable to price debt instruments that are non- investment grade.

The last section of this chapter discusses some alternative approaches to valuing corporate debt.

5. Alternative settings

Models applying contingent claims analysis to the pricing of corporate securities, in particular debt, are generally divided into two, more or less competing, strands: firm value based (or structural) models and intensity based (or reduced form) models. In the reduced form models, a default process is specified directly – the value of the firm's assets does not enter the model. The default process is usually estimated using the prices of bonds issued by the same firm. I argue that rather than being one of two separate approaches, the models can be classified on a scale ranging from purely firm value based such as Merton (1974), Geske (1979) and Kim et al. (1993), via Longstaff & Schwartz (1995) and Saá-Requejo & Santa-Clara (1997), to purely intensity based such as Jarrow et al. (1997) and Duffie & Singleton (1995).

Consider, for instance, the models of Longstaff & Schwartz (1995) or Saá-Requejo & Santa-Clara (1997), generally viewed as being firm value based models. Even though the default process is derived in a firm value based setting, the value of the firm is not needed for pricing purposes and can not, in fact, be estimated at all. The reason is that a default process, defined as the ratio of asset value to a variable related to debt, is a sufficient statistic for the pricing of risky bonds. Since only bonds (the value of equity is not derived) are used to estimate the parameters of the default process, it is not possibile to gauge the value of the underlying assets (or the firm). For example, one can not differentiate between a situation with high asset value and high nominal debt, from a situation with low asset value and low nominal debt.

The most important feature for classification is instead one of estimation. Neither the semi-firm value based models just discussed, nor the purely intensity based, can (as least as they stand) incorporate equity information, but rely solely on bond prices. When a company has several debt issues with good liquidity outstanding, this fact might be of minor importance, but it is naturally critical when a company has no traded public debt, and the model thereby discards the only source of information. Since bonds usually are liquid only for a short period of time following issuance, the latter situation may be the rule rather than the exception.

The proposed setting can potentially incorporate equity as well as bond instruments as sources of information, and can estimate the value of assets as a separate variable. It could consequently be classified somewhere between Kim et al. (1993) and Longstaff & Schwartz (1995).

So far, however, contingent claims models have not been acclaimed by practitioners. Instead, "empirical bond pricing models typically regress a cross-section of spreads over U.S. Treasuries against a set of variables that include the credit ratings of Moody's or Standard & Poor's" (Cantor et al. (1997)). One drawback of these "ad-hoc" models is this dependency on ratings. To value non-rated debt, it is vital to be able to abstract the information contained in equity prices using a formal model. But even if rating is available, stock prices can be useful. A price is more information sensitive than is a rating and reacts quicker to news about credit quality. Also, ratings may be unreliable. More than 50% of the bond issues have split ratings (Cantor et al. (1997)), i.e., different ratings by Moody's and Standard & Poor's.

CHAPTER 3

Equity

The purpose of this chapter is to derive the pricing equation for the firm's equity. Equity is valued as a residual claim to the firm's assets less the obligation to service debt, contingent on no default. To denote default formally, I use an indicator function. Generally $I_{\{A\}}$ denotes the indicator function for an event A. Specifically the indicator function $I_{\{\tau \nless s\}}$ takes on the value of unity if default (time of default is denoted τ) has not yet occurred at time s (i.e. τ is *not less* than s), and the value of zero if it has. Formally, the value of equity is calculated as the discounted expected value, under the pricing measure Q^B, of all payments made and received by equity holders in the future.

$$\mathcal{E}\{\omega_t, t; \cdot\} \equiv \lim_{T\to\infty} \left\{ \begin{array}{c} E^B\left[e^{-r(T-t)}\left(\omega_T - N_T\right) I_{\{\tau \nless T\}}\right] \\ +E^B\left[\int_t^T e^{-r(s-t)} \beta\omega_s I_{\{\tau \nless s\}} ds\right] \\ -E^B\left[\int_t^T e^{-r(s-t)}\left(1-\zeta\right) C_s I_{\{\tau \nless s\}} ds\right] \\ +E^B\left[\int_t^T e^{-r(s-t)} d\left\{\omega_s, s; \cdot\right\} I_{\{\tau \nless s\}} ds\right] \\ +E^B\left[e^{-r\tau}\varepsilon_\tau L_\tau I_{\{\tau \leq T\}}\right] \end{array} \right. \tag{6}$$

Thus the value of the firm's equity is composed of five parts. The first term

- a call option on the firm's assets,

is similar to how equity was valued in the first models of Black & Scholes and Merton in the early 70's. In addition, this model features

- generated cash flow,
- coupon payments and taxes (ζ denoted the tax rate),
- additional borrowing and costs of reorganisation[1] and,
- violations of the absolute priority rule

For future reference, observe that the dividend accruing to equity holders at time s is equal to generated cash flow *plus* borrowed money *less* (after tax) debt service:

$$\beta\omega_s + d\{\omega_s, s; \cdot\} - (1-\zeta)\, C_s \tag{7}$$

A negative dividend corresponds to a situation where equity holders contribute funds to stave off financial distress. This is, of course, done in their own self interest since the value of expected gains in the future exceeds those payments.

[1]The costs of reorganisation and the therewith associated recovery rate for debt enters the value of equity by affecting the market value of newly issued debt, $d\{\omega_s, s; \cdot\}$.

The first two lines of equation (6) correspond to two ways to capitalise the value of the firm's assets – selling them and receiving dividends. It is therefore covenient to summarise the value of the call and the value of the cash flow stream into one claim – the "asset claim" Ω. Formally

DEFINITION 1. *Define Ω as the value of a claim on the firm's assets:*

$$\Omega\{\omega_t, t; \cdot\} \equiv \lim_{T\to\infty} \begin{cases} E^B\left[e^{-r(T-t)}(\omega_T - N_T) I_{\{\tau \nless T\}}\right] \\ +E^B\left[\int_t^T e^{-r(s-t)}\beta\omega_s I_{\{\tau\nless s\}}ds\right] \end{cases}.$$

If I do not allow free cash flow to be generated ($\beta = 0$), do not allow debt to increase ($\alpha = 0$), and consider a finite maturity, the value of this claim would be the same as the value of equity in Merton (1974). In the present case, the value of the claim is the following:

LEMMA 1. *The value of a claim on the firm's assets, defined in Definition 1, is*

$$\Omega\{\omega_t, t; \cdot\} = \begin{cases} \omega_t \cdot \left(1 - \left(\frac{\omega_t}{L_t}\right)^{-\theta^\omega}\right) & \text{when} \quad \beta > 0 \text{ or } \beta = 0 \text{ and } \alpha \le r + \frac{1}{2}\sigma^2 \\ 0 & \text{when} \quad \beta = 0 \text{ and } \alpha \ge r + \frac{1}{2}\sigma^2 \end{cases}$$

where

$$\theta^\omega = \frac{\sqrt{\left(\frac{r-\beta-\alpha+0.5\sigma^2}{\sigma}\right)^2 + 2\beta} + \frac{r-\beta-\alpha+0.5\sigma^2}{\sigma}}{\sigma}$$

The result is derived in the Appendix on page 78.

To hold the asset claim is equivalent to having the right to the assets conditional on no default. Because of the default condition, the asset claim always has a smaller value than do the assets themselves. The value of the claim approaches the asset value as the latter increases and default becomes unlikely, and approaches zero as the asset value approaches the barrier and default seems imminent. To see this in the formula, note that θ^ω is always positive, and that therefore $0 \le \left(\frac{\omega_t}{L_t}\right)^{-\theta^\omega} \le 1$. The claim on the assets is always worthless if they generate no cash and, at the same time, borrowing is so excessive (α so high) that financial distress is expected in a not too distant future.

Before providing the solution to equation (6), I will define two additional claims, which will be useful building blocks in the subsequent formula. The claims are defined through their payoff functions, denoted Φ.

DEFINITION 2. *Define G as the value of a claim paying off unity in financial distress:*

$$\Phi\{G\{\omega_\tau, \tau; \cdot\}\} \equiv 1$$

DEFINITION 3. *Define G^α as the value of a claim paying off $e^{\alpha(\tau-t)}$ in financial distress:*

$$\Phi\{G^\alpha\{\omega_\tau, \tau; t, \cdot\}\} \equiv e^{\alpha(\tau-t)}$$

Time t denotes the time at which the the contract was entered.

The former claim can be said to quantify investors' appreciation of the risk of default. Its value is related to the probability of default, but adjusted for the time value of money. It will henceforth be referred to as the "dollar-in-default" claim. The latter claim, G^{α}, pays off a dollar in default which has increased with "interest rate" α. The values of these two claims are given in the following lemmas.

LEMMA 2. *The value of the dollar-in-default claim G, defined in Definition 2, is*

$$G\{\omega_t, t; \cdot\} = \left(\frac{\omega_t}{L_t}\right)^{-\theta\{r\}}$$

with the constant given by

$$\theta\{\rho\} = \frac{\sqrt{\left(\frac{r-\beta-\alpha-0.5\sigma^2}{\sigma}\right)^2 + 2\rho} + \frac{r-\beta-\alpha-0.5\sigma^2}{\sigma}}{\sigma}$$

The result is derived in the Appendix on page 79.

LEMMA 3. *The value of the claim G^{α}, defined in Definition 3, is*

$$G^{\alpha}\{\omega_t, t; t\} = \left(\frac{\omega_t}{L_t}\right)^{-\theta\{r-\alpha\}}$$

with the constant given by

$$\theta\{\rho\} = \frac{\sqrt{\left(\frac{r-\beta-\alpha-0.5\sigma^2}{\sigma}\right)^2 + 2\rho} + \frac{r-\beta-\alpha-0.5\sigma^2}{\sigma}}{\sigma}$$

The result is derived in the Appendix on page 79.

To understand these results, consider the following scenarios. As the value of assets approaches the reorganisation barrier, the value of a dollar-in-default, G, approaches unity. Conversely, as the value of assets become very large and default becomes extremely unlikely, the value of a dollar-in-default approaches zero. Thus we see that $0 \leq G \leq 1$. Now consider the formula for G^{α}, the "dollar-with-interest-in-default" claim. Since the payoff to the G^{α} always exceeds that of G, it must hold that $G \leq G^{\alpha}$. Finally, note that G^{α} can be worth more than unity only when $\alpha > r$ – ie., when the "interest rate" on the claim exceeds the riskfree interest rate. In the formula, this is due to the fact that in these cases $\theta\{r-\alpha\}$, in contrast to $\theta\{r\}$, can be negative.

With the help of the claims in Lemmas 1, 2 and 3, the solution to equation (6) is rather simple and intuitive. It is given in the following proposition. To denote the expected fraction paid out to equity holders, I use $\varepsilon \equiv E[\varepsilon_T]$ and the expected recovery rate for debt is $\delta \equiv E[\delta_T]$. These are the expected values under the ordinary (objective) probability measure.

PROPOSITION 1. *The price of equity is given by*

$$\begin{aligned}\mathcal{E}\{\omega_t, t; \cdot\} &= \Omega \\ & -N_t \cdot (1 - G) \\ & +\zeta C_t \cdot \begin{cases} \frac{1}{r-\alpha} \cdot (1 - G^\alpha) & \text{when } r \neq \alpha \\ \ln\left(\frac{\omega_t}{L_t}\right)^{\frac{1}{\beta + 0.5\sigma^2}} & \text{when } r = \alpha \end{cases} \\ & +\delta N_t \cdot (G^\alpha - G) \\ & +\varepsilon L_t \cdot G^\alpha\end{aligned}$$

where Ω, G and G^α are given by Lemmas 1, 2 and 3. The result is derived in the Appendix on page 80.

To interpret the formula in the proposition, consider the following correspondences between the cash flows of equation (6) and the building blocks of the proposition.

- The two first lines of equation (6) correspond (by definition) to term 1 of the proposition – the value to equity holders of the firm's assets.
- The third line of equation (6), the after-tax coupons, can be separated into three parts: debt service to current debtholders, the tax-shield and debt service to future debtholders. The value of coupons to current debtholders is captured by term 2 of the proposition and the value of the tax-shield[2] corresponds to term 3. Next consider the third part: the value of debt service to future debtholders. This value *plus* the value of debt recovered by future debtholders in a reorganisation, must, assuming debt is floated at a fair price, sum to the value of future borrowed money (line four in eq. (6)). The net of the value of debt service to future debtholders and line four in eq. (6), corresponds to term 4 in the proposition: $\delta N_t \cdot (G^\alpha - G)$. This term is the value, for equity holders, of borrowing using the assets as collateral. Since δ is the expected recovery rate for debt, δN_t can be interpreted as the current value of collateral. To summarise this paragraph: lines three and four of eq. (6) corresponds to terms 2, 3 and 4 of the proposition.
- Finally, the fifth line of equation (6), the value of expected payouts to equity holders in reorganisation, corresponds to term 5 of the proposition.

The correspondences between equation (6) and Proposition 1 are summarised in Exhibit 2. To obtain some additional intuition for the equity formula, consider the same scenarios as when interpreting the lemmas above. As the value of assets approaches the barrier, equity takes on a value equal to the expected payoff in a reorganisation: εL_t. Conversely, as the of assets increases very much and the risk of default becomes negligible, the value of equity approaches the value of assets *less* the value of the now riskfree debt *plus* the tax-shield: $\omega_t - N_t + \frac{\zeta C_t}{r-\alpha}$ (assuming $r \neq \alpha$).

As was mentioned in the introduction, implementing the model can be considered to consist of two phases: estimation and pricing. This chapter has derived the valuation formula for equity, which is necessary for estimation purposes. The next chapter derives the valuation formulae for debt instruments that constitute a small

[2]Note that $\ln\left(\frac{\omega_t}{L_t}\right)^{\frac{1}{\beta+0.5\sigma^2}}$ simply is the limit of $\frac{1}{r-\alpha} \cdot (1 - G^\alpha)$ as $\alpha \to r$.

Equation (6)		Proposition 1
Call on assets + *Generated cash*	→	*Assets*
− *After-tax coupons*	→	− *Current debt* + *Tax-shield* + *Future borrowing (net)*
+ *Future borrowing (gross)*	↗	
+ *Reorganisation payoff*	→	+ *Reorganisation payoff*

Exhibit 2: Correspondences between equation (6) and Proposition 1.

part of the firm. Due to the assumption that equity and debt instruments can be valued separately, characteristics of the latter securities will not affect the result in Proposition 1.

CHAPTER 4

Basic Securities

In this chapter, I explain how to utilise the framework to price the firm's basic securities, by which I mean claims whose underlying variable is the value of the assets. Examples are bonds and bank loans. The next section is devoted to compound securities, which are claims having a *basic security* as the underlying variable. An example of such a claim is the put option embedded in a putable bond.

The valuation method exploits the fact that contracted payoffs to most of the company's securities can be expressed as combinations of two building blocks: the down-and-out call option and the down-and-out heaviside. A down-and-out claim is one that expires worthless if the value of the underlying asset hits a given barrier prior to the expiration date – a heaviside is a binary option that yields a unit payoff at the expiration date conditional on the underlying asset exceeding the exercise price. Assuming the absence of arbitrage, two claims with identical payoff structures must have the same price. Hence, to value a corporate security, one simply mimics the contracted payoffs of that security with those of calls and heavisides.

To capture "non-contracted" payoffs, that is the payoffs in case of a reorganisation, one additional claim is required: the dollar-in-default claim. To value non-contracted payoffs, one simply scales this claim with the expected payoff in a reorganisation. The dollar-in-default claim of this chapter will differ from the one in Definition 2 of the previous chapter, in that it is equipped with an expiration date (equal to the maturity of the corporate security). Consequently, the dollar-in-default claim of this chapter will be worth less than its perpetual counterpart.

Before providing the exact definitions and formulae for these securities, I illustrate the technique of matching payoffs in an example.

EXAMPLE 2. *To show how payoffs are matched, it more bencicial to use a slightly different setting than the one in the main section. Let's therefore assume the a firm's only outstanding debt is a bond of finite maturity T and that there remains three coupons to be serviced. Default is triggered prior to maturity by a fall in asset value to a barrier and at maturity by the inability to repay the principal P. Should the firm default, the bondholders recover a fraction δ. The value of the bond is valued as the sum of three parts: the value of the coupons, the value of maturity payments and the value of debt recovered in a reorganisation prior to maturity.*

1. *In this setting, each coupon payment is paid in full conditional on no previous default. If the firm defaults prior to the scheduled coupon payment, it is not paid. A coupon is thus a binary payment and can be valued as a heaviside with maturity determined by the coupon date.*
2. *At maturity the story is different. If the value of the firm's assets exceeds the principal, equity holders repay the principal and keep the residual value. If, however, the assets do not suffice to repay the principal, the firm defaults and the assets are taken over by the debtholders. The payoff to debtholders is*

consequently $\max\{P, \omega_T\}$, *which can be written as the payoff from two call options:* $\max\{\omega_T - 0, 0\} - \max\{\omega_T - P, 0\}$. *This implies that one would get an identical payoff by holding a long call with exercise price zero and a short call with exercise price P – both options conditional on no default . Thus the value of payoffs to bondholders at maturity is equal to a down-and-out call option with exercise price zero minus a down-and-out call option with exercise price P. Both these calls expire at T.*

3. *If the firm defaults prior to maturity, bondholders receive δP. The value of this must be δP times the value of receiving one dollar-in-default. Hence the value of payoffs in default prior to maturity is equal to δP dollar-in-default claims of maturity T.*

The total value of the bond is therefore equal to three heavisides plus one call (with exercise price zero) less another call (with exercise price equal to P) plus δP dollar-in-default claims.

To formalise the idea of pricing by replicating payoffs, I first need to define the payoff functions, Φ, of the building blocks.

DEFINITION 4. *The payoff of a call, down-and-out at a barrier L_t, written on the asset value, with exercise price F and expiration at T is*

$$\Phi\{C_L\{\omega_T, T; F, T\}\} \equiv \begin{cases} \omega_T - F \text{ if } \omega_T > F \text{ and } \tau \nleq T \\ 0 \text{ if } \omega_T \leq F \text{ or } \tau \leq T \end{cases}$$

Generally, a subscript $_L$ denotes that a claim is down-and-out – i.e. that the payoff is contingent on the barrier not being hit ($\tau \nleq T$). A corresponding ordinary call would thus be denoted $C\{\omega_t, t; F, T\}$.

DEFINITION 5. *The payoff of a heaviside, down-and-out at a barrier L_t, written on the asset value, with exercise price F and expiration at T is*

$$\Phi\{H_L\{\omega_t, t; F, T\}\} \equiv \begin{cases} 1 \text{ if } \omega_T > F \text{ and } \tau \nleq T \\ 0 \text{ if } \omega_T \leq F \text{ or } \tau \leq T \end{cases}$$

DEFINITION 6. *The payoff of a dollar-in-default claim with expiration at T, is*

$$\Phi\{G\{\omega_T, \tau | \tau \leq T\}\} \equiv \begin{cases} 1 \text{ if } \tau \leq T \\ 0 \text{ if } \tau \nleq T \end{cases}$$

The price formulae for these three claims are given in Lemmas 5-7 below. They will all contain a term that expresses the probabilities (under different measures) of the asset value (ω_T) exceeding the exercise price (F) at maturity, without having hit the barrier prior to that date ($\tau \nleq T$) – in other words, the *in-the-money probability.* To clarify this common structure, I first state those probabilities in the following lemma. The probabilities are previously known, as are the formulae for the down-and-out call and heaviside in Lemmas 5 and 6 (see for example Björk (1998)).

I use $Q^m\{A\}$ to denote the probability under the measure Q^m of event A occuring: $Q^m\{A\} \equiv E^m\left[I_{\{A\}}\right]$. Specifically, I will consider the three probability measures Q^B, Q^ω and Q^G. They are characterised by having the money market account, the value of assets and the dollar-in-default claim, respectively, as numeraires. Readers unfamiliar with probability measures or numeraires can simply think of Q^m with $m = \{\omega, B, G\}$ as referring to different formulae signified by 'm'. In the lemma below, $\phi\{d\}$ denotes the cumulative standard normal distribution function with limit d.

LEMMA 4. *The probabilities of the event*

$$A = \{\tau \nleq T, \omega_T > F\}$$

(the in-the-money event) under the probability measures $Q^m : m = \{\omega, B, G\}$ *are*

$$Q^m\{A\} = \phi\left\{d_T^m\left\{\frac{\omega_0}{Fe^{-\alpha(T-t)}}\right\}\right\} - \left(\frac{\omega_0}{L}\right)^{-\frac{2}{\sigma}h^{X\to m}}\phi\left\{d_S^m\left\{\frac{L_0^2}{\omega_0 \cdot Fe^{-\alpha(T-t)}}\right\}\right\}$$

where[1]

$$d_t^m\{x\} = \frac{\ln x}{\sigma\sqrt{t}} + h^{X\to m}\sqrt{t}$$

$$\begin{cases} h^{X\to B} = \frac{r-\beta-\alpha-\frac{1}{2}\sigma^2}{\sigma} \\ h^{X\to\omega} = h^{X\to B} + \sigma \\ h^{X\to G} = h^{X\to B} - \theta\{r\}\sigma \end{cases}$$

Using the lemma, the pricing equations for the building blocks C_L, H_L and $G\{\cdot\,|\tau \leq T\}$ can be written down in a convenient form (for a derivation see e.g. Björk (1998)).

LEMMA 5. *The price of a down-and-out call (with payoff given by Definition 4) is*

$$C_L\{\omega_t, t; F, T\} = \omega_t \cdot Q^\omega\{\tau \nleq T, \omega_T > F\} - e^{-r(T-t)}F \cdot Q^B\{\tau \nleq T, \omega_T > F\}$$

with probability given by Lemma 4.

The formula is similar in structure to the ordinary Black-Scholes call option formula. When there is a *constant* barrier ($\alpha = 0$) and no generated cash flow ($\beta = 0$), the formula simplifies to the ordinary down-and-out call option formula. A price for that claim was first derived by Merton (1973). When there is *no* barrier (implying that the condition $\tau \nleq T$ becomes redundant), the formula simplifies to the Black-Scholes formula. The normal cumulative distribution functions appearing in all these formulae simply are in-the-money probabilities under two probability measures.

LEMMA 6. *The price of a down-and-out heaviside (with payoff given by Definition 5) is*

$$H_L\{\omega_t, t; F, T\} = e^{-r(T-t)} \cdot Q^B\{\tau \nleq T, \omega_T > F\}$$

with probability given by Lemma 4.

Hence, a heaviside is merely the "latter half" of a call option.

LEMMA 7. *The price of a dollar-in-default claim (with payoff given by Definition 6) is*

$$G\{\omega_t, t\,|\tau < T\} = G\{\omega_t, t\} \cdot \left(1 - Q^G\{\tau \nleq T, \omega_T > L_T\}\right)$$

[1]The term $h^{X\to m}$ is equal to the Girsanov kernel used to go from probability measure Q^X, where the process $X \equiv \frac{1}{\sigma}\ln\frac{\omega_t}{L_t}$ is a Wiener process, to the measure Q^m.

with probability given by Lemma 4. The result is derived in the Appendix on page 81.

Note the accordance of Lemma 2 on page 22 and Lemma 7; it holds that $\lim_{T\to\infty} G\{\omega_t, t\,|\tau < T\} = G\{\omega_t, t\}$.

The claim in Lemma 7 gives us the value of receiving one dollar in case of a reorganisation. Denote with ψ_τ the recovery rate for the debt contract and with P the principal. The payout $\psi_\tau P$ can be a partial repayment, or come in the form of new securities. As has been argued previously (in Section 2 of Chapter 2), I will assume that the expected recovery rate, $\psi \equiv E[\psi_\tau]$, is independent of time. Consequently, the value of payouts in reorganisation to debtholders is $\psi P \cdot G\{\omega_t, t\,|\tau < T\}$.

Having defined the payoff functions and pricing formulae for the building blocks, I can formalise the idea described in the beginning of the chapter. I let $D\{\omega_t, t; \cdot\}$ denote the value of a general basic security as a function of asset value and time.

PROPOSITION 2. *A corporate security D with contracted payments*

$$\Phi\{D\{\omega_t, t; \cdot\}\} = \begin{cases} \sum_i a^{(i)}\,\Phi\left(C_L\{\omega_t, t; F_i, t_i\}\right) \\ + \\ \sum_i b^{(i)}\,\Phi\left(H_L\{\omega_t, t; F_i, t_i\}\right) \end{cases}$$

and an expected recovery rate ψ of a principal P, can be valued as

$$D\{\omega_t, t; \cdot\} = \begin{cases} \sum_i a^{(i)} C_L\{\omega_t, t; F_i, t_i\} \\ + \\ \sum_i b^{(i)} H_L\{\omega_t, t; F_i, t_i\} \\ + \\ \psi P \cdot G\{\omega_t, t\,|\tau < T\} \end{cases}$$

where i are used to index heavisides and options of different exercise prices and maturities and $a^{(i)}$ and $b^{(i)}$ are constants. The summation operator $\sum$ should be understood to encompass integrals when applicable.

The tractability of the analysis thus stems from the ease with which one can obtain closed form expressions for the values of (European) claims on the firm's assets, such as different classes of debt, by simply matching payoffs of standard claims. Next I turn to some examples to demonstrate its flexibility.

EXAMPLE 3. ***A straight coupon bond.*** *This is the corporate security to be implemented in Chapters 7 and 8. The M contracted payments of the straight bond are simply the coupons $\{c\}$ and the repayment of the principal, P. Thus*

$$\begin{aligned} D\{\omega_t, t; \cdot\} &= \sum_{i=1}^{M-1} c \cdot H_L\{\omega_t, t; L_{t_i}, t_i\} \\ &\quad + (c+P) \cdot H_L\{\omega_t, t; L_T, T\} \\ &\quad + \psi P \cdot G\{\omega_t, t\,|\tau < T\} \end{aligned}$$

If the bond is secured, ψ is probably close to unity, but if it is a subordinated bond, ψ might be close to zero. On the other hand, coupons are often higher in the latter case.

EXAMPLE 4. ***A guarantee.*** *A guarantee to reimburse the bondholder for the loss of principal would simply be worth*

$$D\{\omega_t, t; \cdot\} = (1-\psi)\, P \cdot G\{\omega_t, t\,|\tau < T\}$$

EXAMPLE 5. ***Preference shares.*** *As a final example, consider a preference share. A preference share is a hybrid security that has characteristics of both debt and equity. Like debt, it has a contracted fixed coupon (denoted c). However, legally it is an equity security in that failure to pay the coupon does not give holders of such shares the right to take over the firm.*
Assume that when the value of the assets approaches the reorganisation barrier, the firm will first decide to curtail the dividend and finally cut it altogether. Assume that the level of asset value at which the dividend is first reduced can be predicted with some certainty and assume further that the reduction is linear in asset value at a rate 'b'. This means that as asset value decreases by 1*, the dividend decreases by 'b'. Denote the upper level of the asset value, at which the dividend is first reduced, with* $\omega_t^{'}$ *(the time indicator is necessary since the level of asset value depends on the level of the barrier). Then the lower level (under which dividends are zero) is equal to* $\omega_t^{''} = \omega_t^{'} - \frac{c}{b}$. *It thus holds that* $L_t < \omega_t^{''} < \omega_t^{'}$. *Noting that most preference shares are perpetuals, we can write down their value as*

$$D\{\omega_t, t; \cdot\} = \sum_{i=1}^{\infty} b \cdot \left(C_L\left\{\omega_t, t; \omega_t^{''}, t_i\right\} - C_L\left\{\omega_t, t; \omega_t^{'}, t_i\right\}\right) + \psi P \cdot G\{\omega_t, t\}$$

For preference shares, ψ *is usually very low, since holders of preference shares rank below all debt securities. As reported by Crabbe (1996), the recovery rate is only 6% on average.*

CHAPTER 5

Compound Securities

This section is concerned with derivatives written on the basic securities. Examples of these compound securities include embedded options like call and put option features of bonds. I have not studied derivatives written on equity, although pricing formulae, similar to those of Toft & Prucyk (1997), can be derived in closed form.

In the previous chapter, it was shown how basic corporate securities could be valued as combinations of three simple barrier contingent claims. Hence, in order to value derivatives written on a firm's securities we will only need to consider claims on these three contracts. The basic idea is the following. Since corporate securities can be valued as a combination of down-and-out calls, heavisides and a dollar-in-default claim, an option on a corporate security can be valued as an option on a portfolio of these three contracts. Furthermore, an option on a portfolio can be treated as a portfolio of options on the parts. The reason this result holds in the present case is that the exercise price of the compound claim can be expressed in terms of asset value, as will be explained below. To conclude, an option on a corporate security can be valued as a portfolio of compound options. That is the essence of the following proposition.

Consider for example a call option written on a straight coupon bond. In Example 3 it was demonstrated that the straight coupon bond is valued as a portfolio of M heavisides plus ψP dollar-in-default claims. Hence the call can be valued as a portfolio of M calls on heavisides plus ψP calls on dollar-in-default claims.

An issue that has to be addressed, when pricing derivatives, is what happens to the value of an option when the firm defaults. If the option is embedded, such as in a putable bond, it is reasonable to assume that the option becomes worthless as well. If, on the other hand, the option is written by a third party, the option will not default along with the underlying asset. In this case, the value of the underlying corporate security after default becomes crucial. Since the securities are sometimes no longer traded, or do not even exist, their values might be difficult to determine. I will assume that for purposes of determining the payoff to derivative holders there is a provision stating that corporate securities have a post-default value of zero.[1] Thus, at expiration, a call option expires worthless whereas a put option takes on a value equal to the exercise price.

The chapter is composed as follows. The first section derives the price of a call option written on a corporate security. The second section considers the special case of the corporate security being equal to a single down-and-out call option, and thus extending the result of Geske (1979) to the case where the underlying call is

[1] This assumption is not critical. Formulae would not change much if one instead assumes, for example, that it is the distribution in reorganisation that determines the price of the corporate security.

down-and-out. The third section values a put option. The chapter ends with an example.

1. A call on a corporate security

Standard theory tells us that a call with expiration date S and exercise price K on a corporate security D can be calculated as its discounted, expected payoff at expiration under the pricing measure.

$$\begin{aligned} C\{D\{\omega_t,t;\cdot\},t;K,S\} &= e^{-r(S-t)}\cdot E^B\left[(D\{\omega_S,S;\cdot\}-K)\cdot I_{\{D_S>K\}}\right] \\ &= e^{-r(S-t)}\cdot E^B\left[D\{\omega_S,S;\cdot\}\cdot I_{\{D_S>K\}}\right] \\ &\quad -e^{-r(S-t)}K\cdot E^B\left[I_{\{D_S>K\}}\right] \end{aligned}$$

The indicator function takes on a value of unity if the call is in-the-money at expiration. Next, insert the value of the underlying security at S, as given by Proposition 2 (p. 30), with $\{t_i\}>S$.

$$\begin{aligned} &C\{D\{\omega_t,t;\cdot\},t;K,S\} \\ &= e^{-r(S-t)}\cdot\left\{\begin{array}{c} \sum_i a^{(i)}\cdot E^B\left[C_L\{\omega_S,S;F_i,t_i\}\cdot I_{\{D_S>K\}}\right] \\ + \\ \sum_i b^{(i)}\cdot E^B\left[H_L\{\omega_S,S;F_i,t_i\}\cdot I_{\{D_S>K\}}\right] \\ + \\ \psi P\cdot E^B\left[G\{\omega_S,S\,|\tau<T\}\cdot I_{\{D_S>K\}}\right] \end{array}\right\} \\ &\quad -e^{-r(S-t)}K\cdot E^B\left[I_{\{D_S>K\}}\right] \end{aligned} \tag{8}$$

The terms within braces contain expected values of the building blocks of the underlying corporate security – conditional on the call being in-the-money at S. The formula for a call option on a corporate security can consequently be regarded as being made up of *conditional* building blocks. It will prove useful to formalise this idea.

DEFINITION 7. *For a general building block Y maturing at $T>S$, its conditional counterpart, $\mathcal{Y}$, is defined using the pay-off functions*

$$\Phi\{\mathcal{Y}\{\omega_T,T;\cdot\,|D_S>K\}\}\equiv\Phi\{Y\{\omega_T,T;\cdot\}\}\cdot I_{\{D_S>K\}}$$

The value, at t, of a conditional claim can be calculated as

$$\mathcal{Y}\{\omega_t,t;\cdot\,|D_S>K\}=e^{-r(S-t)}\cdot E^B\left[Y\{\omega_S,S;\cdot\,|D_S>K\}\cdot I_{\{D_S>K\}}\right]$$

From the structure of this formula it can be deduced that the expected values in equation (8) can be expressed as various conditional claims. More formally, in analogy with Definition 7, the following correspondences hold:

$$(9)\quad \begin{cases} \mathcal{C}_L\{\omega_t, t; F_i, t_i\,|D_S > K\} = e^{-r(S-t)} \cdot E^B\left[C_L\{\omega_S, S; F_i, t_i\} \cdot I_{\{D_S>K\}}\right] \\ \mathcal{H}_L\{\omega_t, t; F_i, t_i\,|D_S > K\} = e^{-r(S-t)} \cdot E^B\left[H_L\{\omega_S, S; F_i, t_i\} \cdot I_{\{D_S>K\}}\right] \\ \mathcal{G}\{\omega_t, t; |\tau \nless t_i, D_S > K\} = e^{-r(S-t)} \cdot E^B\left[G\{\omega_S, S; |\tau \nless t_i\} \cdot I_{\{D_S>K\}}\right] \end{cases}$$

In order to solve for the value of a call on a corporate security, I thus have to find the values of three conditional claims. As in the previous chapter, the formulae will contain a mutual probability, but under different measures. That is the probability that the call is in-the-money at S and that also the underlying claim is in-the-money at T – for ease of exposition I use T to denote a general maturity of an underlying security instead of t_i. It is natural to express the compound in-the-money event in terms of asset value. Therefore I first define the *exercise price in terms of asset value*, $\overline{\omega}$, as the value of assets at S at which the call is exercised:

$$D\{\overline{\omega}, S; \cdot\} \equiv K$$

The monotonicity of the bond value with respect to the asset value implies that the in-the-money event $\{D_S > K\}$ is equivalent to $\{\omega_S > \overline{\omega}, \tau \nleq S\}$. The two relevant in-the-money events are as follows:

- $A_T = \{\omega_S > \overline{\omega}, \omega_T > F, \tau \nleq T\}$
- $A_S = \{\omega_S > \overline{\omega}, \tau \nleq S\} = \{D_S > K\}$

i.e. the "all is well" events at times T and S, respectively. The probabilities of the latter event were given in Lemma 4 (p. 29), and the probabilities of the former are given in the following proposition.

PROPOSITION 3. *The probabilities of the event A_T occurring under the probability measures Q^B, Q^ω and Q^G are given by*

$$Q^m\{A_T\} =$$

$$\begin{aligned} & \phi\left\{d_S^m\left\{\frac{\omega_t}{\overline{\omega}e^{-\alpha(S-t)}}\right\}, d_T^m\left\{\frac{\omega_t}{Fe^{-\alpha(T-t)}}\right\}, \sqrt{\tfrac{S-t}{T-t}}\right\} \\ -\left(\frac{\omega_t}{L_t}\right)^{-\frac{2}{\sigma}h^{X\to m}} & \phi\left\{d_S^m\left\{\frac{L_t^2}{\omega_t \cdot \overline{\omega}e^{-\alpha(S-t)}}\right\}, d_T^m\left\{\frac{L_t^2}{\omega_t \cdot Fe^{-\alpha(T-t)}}\right\}, \sqrt{\tfrac{S-t}{T-t}}\right\} \\ -\left(\frac{\omega_t}{L_t}\right)^{-\frac{2}{\sigma}h^{X\to m}} & \phi\left\{-d_S^m\left\{\frac{\overline{\omega}e^{-\alpha(S-t)}}{\omega_t}\right\}, d_T^m\left\{\frac{L_t^2}{\omega_t \cdot Fe^{-\alpha(T-t)}}\right\}, -\sqrt{\tfrac{S-t}{T-t}}\right\} \\ + \quad & \phi\left\{-d_S^m\left\{\frac{\omega_t \cdot \overline{\omega}e^{-\alpha(S-t)}}{L_t^2}\right\}, d_T^m\left\{\frac{\omega_t}{Fe^{-\alpha(T-t)}}\right\}, -\sqrt{\tfrac{S-t}{T-t}}\right\} \end{aligned}$$

where, for $m = \{\omega, B, G\}$

$$d_v^m\{x\} = \frac{\ln x}{\sigma\sqrt{v-t}} + h^{X\to m}\sqrt{v-t}$$

$$\begin{cases} h^{X\to B} = \frac{r-\beta-\alpha-\frac{1}{2}\sigma^2}{\sigma} \\ h^{X\to\omega} = h^{X\to B} + \sigma \\ h^{X\to G} = h^{X\to B} - \theta\{r\}\sigma \end{cases}$$

The result is derived in the Appendix on page 81.

To understand the structure of the in-the-money probability in the proposition, consider the probability of an event A conditional on not hitting a barrier. By decomposing A into complementary events we may write[2]

$$
\begin{aligned}
Q(A,\tau \nleq T) = \quad & Q(A) \\
- \quad & Q(A,\tau_S \leq S) \\
- \quad & Q(A,S < \tau_T \leq T) \\
+ \quad & Q(A,\tau_S \leq S, S < \tau_T \leq T)
\end{aligned} \tag{10}
$$

The total probability is the unconditional (as regards the barrier) probability (line 1) *less* the probability conditional on the barrier being hit before S (line 2) *less* the probability conditional on it being hit between S and T (line 3) *plus* the probability of hitting the barrier both before S *and* between S and T (line 4). The structure of this partition is precisely that of the expression for the compound probabilities in the proposition (for the measures Q^B and Q^ω).[3]

Using the result of the proposition, the solutions for the values of the conditional claims in equation (9) can be written down in a convenient way.

COROLLARY 1. *The price of a conditional call is*

$$\mathcal{C}_L\left\{\omega_t, t; F, T \,|D_S > K\right\} = \omega_t e^{-\beta(T-t)} \cdot Q^\omega\left\{A_T\right\} - e^{-r(T-t)} F \cdot Q^B\left\{A_T\right\}$$

with the probabilities given by Proposition 3. The corollary is derived in the Appendix on page 84.

Notice that the structure of the call formula is similar to that of the standard Black-Scholes formula. Setting $\alpha = \beta = K = L = 0$ the expression collapses to the latter formula.

COROLLARY 2. *The price of a conditional heaviside is*

$$\mathcal{H}_L\left\{\omega_t, t; F, T \,|D_S > K\right\} = e^{-r(T-t)} Q^B\left\{A_T\right\}$$

with the probability given by Proposition 3.

Just as a heaviside constitutes the "latter half" of a call option, so is a conditional heaviside the latter half of a conditional call.

COROLLARY 3. *The price of a conditional dollar-in-default claim is*

$$\mathcal{G}\left\{\omega_t, t \,|\tau \leq T, D_S > K\right\} = G\left\{\omega_t, t\right\} \cdot \left(Q^G\left\{A_S\right\} - Q^G\left\{A_T\right\}\right)$$

where the simple probability is given by Lemma 4 and the compound probability is given by in Proposition 3 with

$$F = L_T$$

The corollary is derived in the Appendix on page 85.

---

[2]I use τ_S and τ_T to denote the first hitting times in the intervals $[t, S]$ and $(S, T]$, respectively.

[3]Some readers may think it peculiar that the first line, expressing the unconditional (as regards the barrier) probability of being in-the-money at T, appears to contain the parameter α (the growth rate of the barrier). However, it does not - the α appearing together with $\overline{\omega}$ is netted out against the α in the Girsanov kernel $h^{X \to m}$ for $m = \{B, \omega\}$.

The reason that the interpretation does not hold for the Q^G-probability, is that this measure has G, a *barrier* contingent claim, as numeraire. Consequently, the measure itself depends on the barrier, and we can not express a probability, under this measure, that is unconditional with respect to the barrier. Conversely, Q^B and Q^Q have non-barrier claims as numeraires, and these measures are therefore independent of the barrier.

These corollaries constitute the penultimate step in solving for the value of a call on a corporate security. The final step is to note that the last term of equation (8) is equal to K down-and-out heavisides (cf. Lemma 6 on page 29). The following proposition sums up the result.

PROPOSITION 4. *A call option on the corporate security of Proposition 2, can be valued as*

$$\begin{aligned} C\{D\{\omega_t,t;T,\cdot\},t;K,S\} &= \sum_{i=1}^{M} a^{(i)}\cdot\mathcal{H}_L\{\omega_t,t;N_{t_i},t_i\,|D_S>K\} \\ &\quad +\sum_{i=1}^{M} b^{(i)}\cdot\mathcal{C}_L\{\omega_t,t;N_{t_i},t_i\,|D_S>K\} \\ &\quad +\psi\cdot\mathcal{G}\{\omega_t,t;\cdot\,|\tau<T,D_S>K,\} \\ &\quad -K\cdot H_L\{\omega_t,t;\overline{\omega},S\} \end{aligned}$$

where $D\{\overline{\omega},S;\cdot\}\equiv K$. The values of the conditional claims are found in Corollaries 1-3 and the value of the heaviside is obtained from in Lemma 6 on page 29.

As can be seen by comparing this result with Proposition 2, the price of a call is analogous to the price of the underlying security – with conditional rather than ordinary down-and-out claims, and an adjustment for the requirement to pay the exercise price (the K down-and-out heavisides). The intuition behind this structure is that a call option, although formally a claim on the underlying security at S, is ultimately a claim on the assets of the firm at T. Technically, a call that expires at S written on a security of maturity T can be viewed as a call of maturity T on ω that pays off only if $\{\omega_S>\overline{\omega}\}$, *less* the value of the requirement to pay the exercise price at S.

2. A special case: the compound call

Consider a call with a particularly simple choice of underlying security – a down-and-out call option. The underlying call matures at T and has exercise price F and the compound call matures at $S<T$ with exercise price K. This can be thought of as a call with exercise price K written on equity in a firm that has issued discount debt with face value F only.

We know from standard theory that we can write the price of the compound call as its discounted expected payoff under the pricing measure Q^B at expiration.

$$C\{C_L\{\omega_t,t;F,T\},t;K,S\}=e^{-r(S-t)}\cdot E^B\left[(C_L\{\omega_S,S;F,T\}-K)\cdot I_{\{C_L>K\}}\right]$$

Utilising Corollary 1 and Lemma 6 (p. 29), this expression is equivalent to

$$C\{C_L\{\cdot\}\cdot\}=\omega_t e^{-\beta(T-t)}\cdot Q^\omega\{A_T\}-e^{-r(T-t)}F\cdot Q^B\{A_T\}-e^{-r(S-t)}K\cdot Q^B\{A_S\}$$

The structure of the formula is not tied to the assumptions of constant asset volatility and interest rate – see Geman, El Karoui & Rochet (1995). However, they guarantee the existence of closed form solutions for the probabilities.

Moreover, the composition of the formula is identical to that of the formula for a compound call in Geske (1979). Setting $\alpha = \beta = 0$ in the expression above extends his result to the case where the underlying call is down-and-out. Setting $L = 0$ yields Geske's formula.

3. A put on a corporate security

To value a put option, denoted Þ, some care is required. If the put is issued by a third party, put-call-parity holds and the value of the put is straightforward to derive.

REMARK 1. *The price of a put option, issued by a third party, on a corporate security may be calculated using put-call-parity:*

$$Þ\{D\{\omega_t, t; T, \cdot\}, t; K, S\} = e^{-r(S-t)}K - D\{\omega_t, t; T, \cdot\} + C\{D\{\omega_t, t; T, \cdot\}, t; K, S\}$$

If the put option is embedded, however, the put-call-parity relation does not hold with the call in Proposition 4. The embedded put option has a lesser value than the ordinary put since it runs the risk of becoming worthless if the value of the firm deteriorates *too* much (i.e. enters reorganisation) – and this is when the ordinary put is worth the most. Instead, the following relationship may be used.

REMARK 2. *The price of an embedded put option on a corporate security may be calculated using the following formula:*

$$\begin{aligned} Þ\{D\{\omega_t, t; T, \cdot\}, t; K, S\} &= K \cdot H_L\{\omega_t, t; L_S, S\} - D\{\omega_t, t; T, \cdot\} \\ &\quad + C\{D\{\omega_t, t; T, \cdot\}, t; K, S\} \end{aligned}$$

As an example of how to utilise this framework for options on corporate securities, consider a putable bond. A putable bond gives the holder the right to sell the bond back to the issuer prior to maturity. Typically, the put is "European-style" in that it can be exercised for a very short time only (Hardy (1995)). About 900 bonds had some kind of put feature by 1995 (Hardy (1995)).

EXAMPLE 6. ***A putable bond.*** *Assume that the holders of the straight coupon bond of Example 3 on page 30 have an option to redeem it at par at time S. The value of this putable bond is the value of the original bond plus the put or, using the relation in Remark 2 above with $K = P$, the value of a number of P down-and-out heavisides plus the corresponding call. Formally*

$$\begin{aligned} D\{\omega_t, t; T, \cdot \,|\mathit{putable}{:}P, S\} &= D\{\omega_t, t; T, \cdot\} + Þ\{D\{\omega_t, t; T, \cdot\}, t; P, S\} \\ &= P \cdot H_L\{\omega_t, t; L_S, S\} + C\{D\{\omega_t, t; T, \cdot\}, t; P, S\} \end{aligned}$$

Inserting the value of the call yields

$$D\{\omega_t,t;T,\cdot\,|putable{:}P,S\} = P\cdot H_L\{\omega_t,t;L_S,S\} + \begin{cases} \sum_{i=1}^{M-1} c\cdot \mathcal{H}_L\{\omega_t,t;L_{t_i},t_i\,|D_S>K\} \\ +(c+P)\cdot \mathcal{H}_L\{\omega_t,t;L_T,T\,|D_S>K\} \\ +\psi P\cdot \mathcal{G}\{\omega_t,t\,|\tau<T,D_S>K\} \\ -P\cdot H_L\{\omega_t,t;\varpi,,S\} \end{cases}$$

To better understand the formula, think about the following two outcomes. If, on the expiration day of the put, the bond is worth more than par, the put is not utilised and the holder retains the bond. In the formula, this means that, at time S*,* $D\{\omega_S,S;T,\cdot\}>P$ *and the conditional claims are "transformed" to ordinary (un-conditional) claims which, in aggregate, equal the value of the bond. The first and last terms are both equal to* P *(since both heavisides are in-the-money) and consequently add to zero. The total sum thus is equal to the value of the bond,* $D\{\omega_S,S;T,\cdot\}=\sum cH_L+(c+P)H_L+\psi P\cdot G\{\cdot\,|\tau<T\}$*.*
Contrarily, if on the expiration day of the put, the bond is worth less than par, the put is utilised and the holder obtains the principal. In the formula, the same result is obtained since $D\{\omega_S,S;T,\cdot\}<P$ *and the conditional claims expire worthless. The first term is equal to* P*, since this heaviside (having exercise price* L_S*) is in-the-money, but in this case the last term is equal to zero, since this heaviside (having exercise price* ϖ*) expires out-of-the-money. The total sum in this case is therefore equal to* P*.*

Perhaps it would seem natural to also include callable bonds among the examples. However, as has been shown in theory (Kim et al. (1993)) as well as in practice (Duffee (1996)), the value of the call feature stems mainly from the interest rate risk. Since such risk is excluded from the setup of this thesis by assumption, the relevance of callability is diminished as well. Basically, credit risk is more relevant for putable than for callable bonds – with slight exaggeration, it is true that a callable bond is called when interest rates fall, and a putable bond is sold back to the firm when credit quality falls.

CHAPTER 6

Yields

There has been a focus on yield spreads, or default premiums, in recent bond pricing literature. The intention of this chapter is to discuss some of the popular topics. The first section considers the size of yields in general. The second section briefly discusses liquidity problems and their effect on yields in contingent claims models. The third section compares yield curves implied by the model with those found by Sarig & Warga (1989).

1. The size of yields

Firm value based models are often criticized on the grounds of them yielding too small credit spreads. In this section, I aim to show that such critique often is irrelevant, or based on misconceptions.

Many authors seem to forget that what ultimately matters is prices rather than yields. Thus the correct measure of credit risk in a bond is the credit discount, the difference in price between the risky bond and a comparable riskfree bond, and not the yield spread. Different bonds, with the same credit discount but with different coupons and maturity, will not have the same yield. Moreover, even if a model produces the correct price it will not necessarily produce the correct yield. An example is given here:

EXAMPLE 7. *We want to value a 5 year $100 corporate bond with a semi-annual 8% coupon. With a riskfree interest rate of 6%, the equivalent riskfree bond is worth $108. We apply a model in the line with Black & Cox (1976) and Leland (1994) to the corporate bond. In these models, debt service to a given bond is approximated by an infinite, constant coupon stream (with no repayment of the principal). Assume the model produces a yield spread of 18 basis points for the corporate bond. Assume further that the bond is traded with a spread of 70 basis points. The model obviously runs the risk of being accused of producing too small spreads. The disparity is, however, a delusion – the* ***price*** *produced by the model and the* ***price*** *quoted by traders is the same, namely $105. The reason is the different relationships between yields and prices in the model and in reality.*[1]

[1]In a model like Black & Cox (1976), the yield is obtained as

$$y = \frac{rP}{D}$$

whereas for an actual bond, it is obtained by numerically solving the following expression

$$D = \sum_{i=1}^{M-1} e^{-y \cdot t_i} c + e^{-y \cdot t_M} (c + P)$$

Another area where, in my view, misunderstandings are prevalent, concerns the size of costs of financial distress. The critique, in this case, is that unrealistically high bankruptcy costs are required to produce the desired yield spreads. Estimation of these costs are usually based on measurable costs relating to the bankruptcy procedure (direct costs), or the value of the firm's assets before and after bankruptcy. In this way, various estimates have been produced. Warner (1977) finds that direct costs on average constitute 5% of the value of the firm, Altman (1983) reports figures in the range 5-15%, whereas Alderson & Betker (1995) detect total costs of reorganisation of between 13% and 62%. Whatever the estimated size, I argue that for the purposes of pricing bonds, these are the wrong figures to focus on.

To come up with estimates relevant for bond pricing, one should start at the other end – that is, at the recovery ratios for bonds. For example, from Altman & Kishore (1996), we know that the recovery rate for bonds in the United States during the last 25 years or so is around 40% on average. The costs of financial distress (plus violations of absolute priority) must be set, in the valuation model, in relation to the level of the barrier to reflect this fact. If, for example, companies are assumed to enter financial distress whenever the value of their assets falls below the nominal value of debt, the expected costs of distress should be around 60%, since this figure generates expected recovery rates in line with actual ones (ignoring violations of absolute priority). If, on the other hand, companies are assumed to enter financial distress when the value of their assets falls below 70% of the nominal value of debt, the expected costs of distress should be around 43%. And, if the barrier is equal to 40% of the total debt, expected costs should be zero!

The examples in the previous paragraph highlight one important point when discussing whether a given model produces "reasonable outputs with reasonable inputs" and that is the importance of focusing on readily observable parameters (if such are available). In this case, the conclusion is: to arrive at reasonable yield spreads, the point of departure should be reasonable recovery rates (which are readily observable), and not reasonable bankruptcy costs and barriers (which are difficult to observe and hence can be set more arbitrarily).

2. Liquidity

Besides the credit risk component, actual yield spreads usually have a liquidity (marketability) premium as well. The latter premium reflects the expected costs involved in selling the bond, i.e., the risk of selling it below its intrinsic value. This premium can not be captured by standard contingent claims models, since pricing relies on the possibility of costlessly constructing replicating portfolios – something that breaks down in the presence of illiquidity. Thus, paradoxically, even if a model perfectly captures the credit risk of a security, it would generate too small yield spreads.

Empirical investigations of the size of the liquidity premium are sparse. Amihud & Mendelson (1991) and Daves & Erhardt (1993) investigated the yield on treasury instruments that have identical features but for dissimilar perceived liquidities. They found that yields differ in the range 20-40 basis points. Those figures, however, constitute but a lower limit for the liquidity premium for corporate bonds. First, the relatively more liquid treasury instrument may also carry some liquidity premium. Second, corporate bonds are generally less liquid than government bonds – especially in the secondary market (Sundaresan (1997) page 24). Empirical

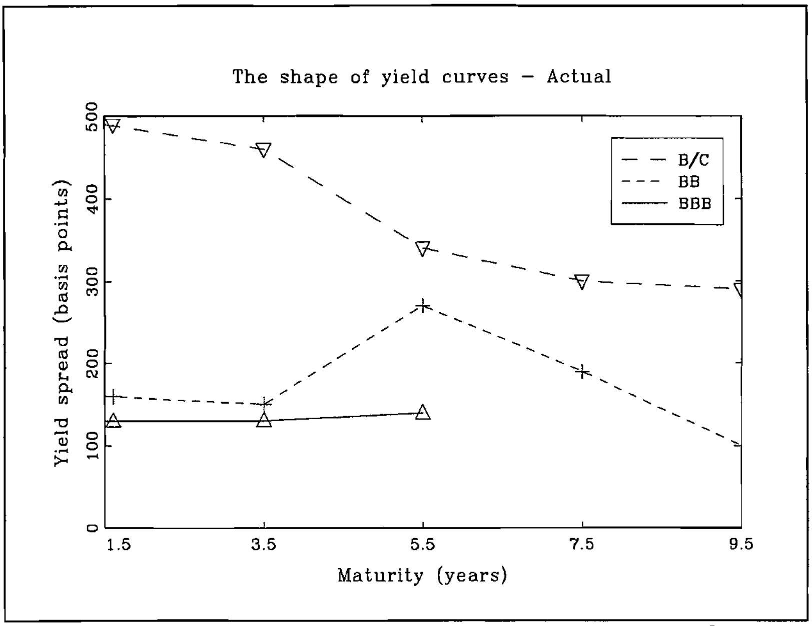

Exhibit 3: Yield spreads for corporate zero coupon bonds.
February 1985 through September 1987.
Source: Table 1 in Sarig & Warga (1989).

studies of the corporate bond market are even more rare, unfortunately. In an attempt, Fridson & Bersh (1993) measure liquidity in a bond issue as a function of the amount outstanding. They find that yields in the primary market *ceteris paribus* increase by 40 basis points when the size of the issue decreases by $100 million.

The fact that liquidity problems are aggravated in the secondary market can potentially introduce a pricing bias in intensity based models. Typically, existing (vintage) bonds issued by the same firm are used to calibrate the model. These bonds are usually burdened with a liquidity premium, which translates into a higher estimate of the default intensity. In turn, this translates into a higher estimated yield on the bond one wants to price. If the bond-to-price is vintage too, no bias is likely to arise since that bond, as well, is burdened with a liquidity premium. But if it is a new issue, one will get a pricing bias since the predicted yield will be too high, in this case erroneously incorporating a liquidity premium.

To conclude, the framework in this thesis should be expected to perform best when the credit risk component is large – and not expected to perform well when pricing illiquid bonds issued by AAA-rated companies. I will, however, not consider liquidity further in this thesis.

3. The shape of yield curves

In this section, I leave the general discussion and focus instead on the characteristics of the model. The most important output is the price of a debt instrument. However, it is also a one-dimensional output that conveys very little about how it is determined – and hence also very little about how well other instruments will be priced. Therefore, while not forgetting the critique levelled against a singular focus on yields, I will focus on the zero coupon yield curve, which gives the price of all fixed income securities issued by the firm.

In a widely quoted paper, Sarig & Warga (1989) use corporate zero coupon bond prices to investigate what the zero coupon yield spread curve looks like for different classes of credit rating. Figure 1 in their paper depicts the average yield spreads as a function of maturity. The yield spread averages for three groups of credit rating – BBB, BB and B/C – are reprinted here as Exhibit 3. The paper by Sarig & Warga is excellent, but care is required so as not to interpret the results as the final truth about yield spread curves. Indeed, as the authors stress themselves "the results presented here must be considered only a preliminary investigation awaiting a richer and longer time series that will enable more formal tests". As they only cover 42 companies, the average yields, especially at the longer end for the three depicted credit groups, are uncertain. A general point is also that curves that are calculated as cross-sectional averages, do not necessarily conform to the shape of the yield curve for each firm in the class. As regards the *size* of the yield spreads, in retrospect it has become apparent that the period their data stems from, February 1985 through September 1987, was one with comparatively low yield spreads (Cheung et al. (1992)). Still, a model ought to be able to produce yield curves in line with those in their paper.

Exhibit 4 depicts three zero coupon yield spread curves. They are meant to correspond to the three rating groups in Exhibit 3 and are produced by changing the leverage in the model.[2] The curves for the high-yield bonds coincide fairly well, with regard to size as well as shape. The investment-grade spread curve, however, has a larger spread in the Sarig & Warga set – at least for short maturities. A possible explanation is the following. Short maturity bonds are generally those of the oldest vintage and consequently of the lowest liquidity. Since the model does not capture the liquidity premium, these are the circumstances when the model can be expected to perform the worst.

[2]The recovery rates for zeros are related to the price when first floated, instead of to the principal amount. Consequently, a zero coupon bond issued at t will recover $\psi\tau \cdot D\{\omega_t, t; \cdot\}$ in default. The initial price of the zero is $D\{\omega_t, t; \cdot\} = H_L\{\omega_t, t; T\} + \psi \cdot D\{\omega_t, t; \cdot\} \cdot G\{\omega_t, t; \cdot \,|\tau \leq T\}$ or $D\{\omega_t, t; \cdot\} = H_L\{\omega_t, t; \cdot; T\} \,/\, (1 - \psi \cdot G\{\omega_t, t; \cdot\,|\tau \leq T\})$.

The following parameters were used for all classes of debt: $N = L = 1000$, $\beta = 2\%$, $r = 7\%$, $\sigma = 15\%$, $\delta = \psi = 40\%$. The class specific parameters were for the B/C-class $\omega = 1250$, $\alpha = 0.5\%$, for the BB-class $\omega = 1450$, $\alpha = 2\%$, and for the BBB-class $\omega = 1900$, $\alpha = 4\%$ (corresponding to leverages of 78% , 69% and 53%, respectively).

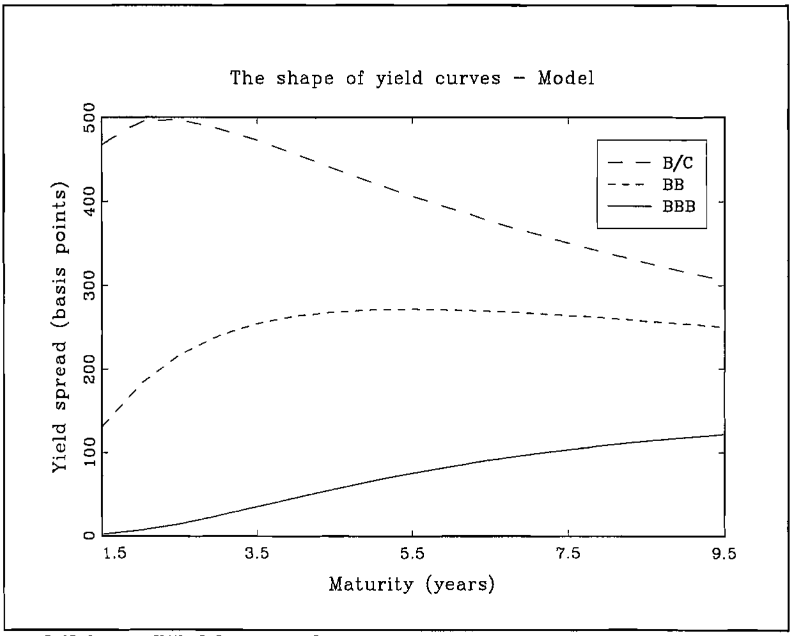

Exhibit 4: Yield spreads for corporate zero coupon bonds.
Source: model of this paper.

CHAPTER 7

Implementation

The ultimate aim of this thesis is to present an easily implementable framework for the pricing of corporate securities. The previous chapters have dealt with the necessary prerequisites – the pricing formulae to be used. This chapter shows how to implement them in practice.

Imagine that we want to price a straight coupon bond, i.e., the security of Example 3 on page 30 The formula for the bond is restated here in full for convenience.

$$
\begin{aligned}
D\{\omega_t, t; \sigma, P, c, \{t_i\}, r, L, \alpha, \psi, \beta\} &= \sum_{i=1}^{M-1} c \cdot H_L\{\omega_t, t; L_{t_i}, t_i, \cdot\} \\
&\quad + (c+P) \cdot H_L\{\omega_t, t; L_T, T, \cdot\} \qquad (11) \\
&\quad + \psi P \cdot G\{\omega_t, t; \cdot\, |\tau < T\}
\end{aligned}
$$

The following inputs are needed, according to the formula: current value and volatility of assets, the principal amount, the coupon rate and the coupon dates, the riskfree interest rate, the reorganisation barrier and its growth rate, the expected recovery rate in case of a reorganisation and, finally, the rate at which cash is generated by the assets. One way of quantifying these is as follows. The first three are acquired from the bond indenture. The riskfree interest rate is chosen to price the equivalent riskfree bond correctly. The level of the barrier is determined in the chosen manner – equal to total debt, as a cash flow constraint, *etc* – and the growth rate can be estimated from past balance sheets. The next parameter, the expected recovery rate, will depend on seniority and whether the bond is secured. Thus one needs the balance sheet as well as the indenture to find this parameter. It also depends on the industry the company is in. Should one be so fortunate as to have access to a price quote on another bond of equal seniority issued by the same company, the expected recovery rate could be estimated (see below). The cash flow parameter can be set to yield the correct, current, stock dividend (equation (7) on page 21). Thus, of the necessary parameters, all are "known" save the ones describing the asset value process (ω_t, σ). I now turn to the task of estimating these.

The idea, first proposed by Duan (1994), is to use price data from one or several derivatives written on ω to infer, using maximum likelihood (ML) techniques, the characteristics of the underlying, unobserved, process. In principle the "derivatives" can be any of the firm's securities that are traded in liquid markets (only prices reported from liquid markets can be expected to carry relevant information). In practice, only equity is likely to be able to present an undisrupted price series since corporate bonds tend to be liquid only immediately following issuance. Therefore, the price data used will probably be a series of stock prices and perhaps a few discretely observed bond prices. It is not necessary that the bond prices are from the pricing date t; it suffices that they are not older than the first stock price used. If one or more of the bonds are of the same seniority class as the bond one is interested in, the recovery rate parameter ψ for that class can be found by calibrating the model. In practice, this can correspond to a situation when the company floated another issue a year before. This issue is now illiquid and the current prices can not be relied upon. However, assume it was liquid when it was first issued. In that case ψ can be estimated as the value that gave the correct price of the old bond at the time of issuance.

By inspecting the equity formula in Proposition 1, one finds that three additional parameters are needed to price equity – besides those needed to price the bond. Expressly, one needs the total nominal amount of debt, the expected payouts in a reorganisation to equity holders and the recovery rate for total debt – that is $(N_t, \varepsilon, \delta)$. The first of these is easily obtained from the balance sheet but like the recovery rate for the bond, there is no categorical way of finding the two latter. The most reasonable approach perhaps is to use historical industry averages – an excellent source for procuring relevant figures is Altman & Kishore (1996).

Applying the model can, as stated previously, be regarded as a two-phased process – estimation and pricing. Each phase makes use of its particular group of securities. It should be pointed out that for the first group of securities, closed form pricing expressions are vital, whereas for the second group, it is much less so. The reason is that the formulae for the first group are inverted and solved for numerically many times during the estimation process – if the formulae have to be solved numerically to start with, inversion may prove a very cumbersome and time consuming task. The pricing formulae in the second group of securities, on the other hand, naturally are never inverted and need only be utilized once. The drawback of not having closed form solutions for this group is that the distributions of the price estimators may be difficult to calculate.

If one ignores stock price data, and relies solely on bond prices, estimation will resemble calibration of an intensity based model. In the remainder of this thesis, I will concentrate on the other extreme and *only* make use of a stock price series. Intensity based models cannot be applied in this case since they presume the existence of traded bonds issued by the same company. As claimed in the introduction, prices on corporate securities other than equity tend to be less liquid and therefore contain less information than stock prices. In many cases, equity may be the only price information available. Such a situation not only arises when the old bond issues are no longer traded, but also when a bank considers lending money to smaller companies without any bonds outstanding whatsoever. In many countries in Europe, bank loans are indeed the most common way also for large companies to borrow money.

Furthermore, I will only analyse the pricing of a non-callable straight coupon bond. The reasons for choosing this debt instrument are threefold: firstly, it is usually useful to start with a basic case; secondly, the straight coupon bond is tractable from a data availability perspective; and thirdly, the straight coupon bond is becoming increasingly popular on the market. For example, in 1982, only 20% of the issues were non-callable, whereas in 1990, only 20% were *callable* (Crabbe (1991)). I can sum up the last two paragraphs as follows: I rely on equity only in the estimation phase, and consider a straight coupon bond only in the pricing phase.

This rest of this chapter is organised as follows. Section 1 is concerned with the first implementation phase, estimation, and contains the explicit equations to use when estimating the asset value process using a series of equity prices. Section 2 is concerned with the second implementation phase, pricing, and in particular with the asymptotic distributions of the estimators. In Section 3, I exemplify the method with an application to real data. The purpose of the last section is merely to illustrate the method and to point to its potential: an empirical test is beyond the scope of this thesis. Before the model is brought to bear on market data, however, it is necessary to gauge the small sample properties of the price estimators. This is the purpose of Chapter 8.

1. Estimation

The problem at hand is thus maximum likelihood estimation of the relevant parameters of the asset value process: (ω_t, σ). This will be accomplished using a time series of stock prices, $\mathcal{E}^{obs} = \left\{\mathcal{E}_i^{obs} : i = 1...n\right\}$. Subscript '$i$' is used to index observations, in contrast to subscript 't', which refers to a point in time in years. Typically, $\mathcal{E}_n^{obs}$ is the current stock price and $\mathcal{E}_1^{obs}$ is the stock price $(n-1)$ days ago.

We require the likelihood function of the observed price variable. I start by defining $f(\cdot)$ as the conditional density for $\mathcal{E}_i^{obs}$ which gives us the following log-likelihood function for equity

$$L_{\mathcal{E}}\left\{\mathcal{E}^{obs}; \sigma, \lambda\right\} = \sum_{i=2}^{n} \ln f\left\{\mathcal{E}_i^{obs} \left| \mathcal{E}_{i-1}^{obs} \right. ; \sigma, \lambda\right\} \tag{12}$$

To derive the density function for equity, I make a change of variables as suggested in Duan (1994). Note that $\mathcal{E}_i = \mathcal{E}_i\{\omega_i, t_i; \cdot\}$ denotes the equity *formula* in Proposition 1 on page 23, whereas $\mathcal{E}_i^{obs}$ denotes an observed stock price, a *constant.* The change of variables results in[1]

$$\begin{aligned} f\left\{\mathcal{E}_i^{obs} \left| \mathcal{E}_{i-1}^{obs} \right. ; \sigma, \lambda\right\} &= g\left\{\ln \omega_i \left| \ln \omega_{i-1} \right. ; \sigma, \lambda\right\}\Big|_{\omega_i = \vartheta\left\{\mathcal{E}_i^{obs}, t_i; \sigma\right\}} \\ &\quad \times \left[\left. \frac{\partial \mathcal{E}_i}{\partial \ln \omega_i} \right|_{\omega_i = \vartheta\left\{\mathcal{E}_i^{obs}, t_i; \sigma\right\}} \right]^{-1} \end{aligned} \tag{13}$$

The function transforming equity to asset value, ϑ, is defined as follows

$$\omega_i \equiv \vartheta\left\{\mathcal{E}_i^{obs}, t_i; \sigma\right\} = \mathcal{E}^{-1}\left\{\mathcal{E}_i^{obs}, t_i; \sigma\right\} \tag{14}$$

This is the inverse of the equity pricing function in Proposition 1. There is a one-to-one correspondence between the stock price $\mathcal{E}_i^{obs}$ and the asset value ω_i (given σ). Note that we do not require λ to invert this function.

The point of the change of variables is that $g\{\cdot\}$ is the well known density function for a normally distributed variable – the log of the asset value. Its dynamics are

$$\left\{ \begin{array}{c} d\ln\omega = \left(r - \beta + \lambda\sigma - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW \\ \ln \omega_{(t=0)} = \ln \omega_{(i=n)} \end{array} \right.$$

[1]Equation (13) corrects an error in equation (4.6) in Duan (1994) (where the derivative erroneously is taken with respect to ω_i instead of $\ln \omega_i$).

The first two (one-period) conditional moments of the distribution are given by

$$\left.\begin{aligned} m_i &= E\left[\ln\omega_i \,|\ln\omega_{i-1}\right] = \ln\omega_{i-1} + \left(r - \beta + \lambda\sigma - \tfrac{1}{2}\sigma^2\right)\Delta t \\ s_i^2 &= E\left[\left(\ln\omega_i - m_i\right)^2 |\ln\omega_{i-1}\right] = \sigma^2\Delta t \end{aligned}\right\} \text{for } i = 2...n$$

and the conditional normal density for $\ln\omega_i$ is thus

$$g\left\{\ln\omega_i \,|\ln\omega_{i-1}\,;\sigma,\lambda\right\} = \frac{1}{\sqrt{2\pi s_i}}\exp\left\{-\frac{\left(\ln\omega_i - m_i\right)^2}{2s_i^2}\right\}$$

Inserting (13) into (12) we obtain the log-likelihood of the vector $\mathcal{E}^{obs}$ for a given choice of σ and λ as

$$\begin{aligned} L_{\mathcal{E}}\left\{\mathcal{E}^{obs};\sigma,\lambda\right\} &= \sum_{i=2}^{n}\ln g\left\{\ln\omega_i \,|\ln\omega_{i-1}\,;\sigma,\lambda\right\}\Big|_{\omega_i=\vartheta\left\{\mathcal{E}_i^{obs},t_i;\sigma\right\}} \\ &\quad -\sum_{i=2}^{n}\ln\left.\frac{d\,\mathcal{E}\left\{\omega_i,t_i;\sigma\right\}}{d\ln\omega_i}\right|_{\omega_i=\vartheta\left\{\mathcal{E}_i^{obs},t_i;\sigma\right\}} \end{aligned}$$

or, noting that the first sum is simply the log-likelihood for $\ln\boldsymbol{\omega}$, as

$$\begin{aligned} L_{\mathcal{E}}\left\{\mathcal{E}^{obs};\sigma,\lambda\right\} &= L_{\ln\omega}\left\{\ln\vartheta\left\{\mathcal{E}_i^{obs},t_i;\sigma\right\} : i = 2...n;\sigma,\lambda\right\} \\ &\quad -\sum_{i=2}^{n}\omega_i\ln\left.\frac{d\,\mathcal{E}\left\{\omega_i,t_i;\sigma\right\}}{d\,\omega_i}\right|_{\omega_i=\vartheta\left\{\mathcal{E}_i^{obs},t_i;\sigma\right\}} \end{aligned} \tag{15}$$

What remains is to find the explicit formula for the derivative of equity with respect to asset value. It is straightforward to derive by differentiating the equity formula in Proposition 1, but the result is a long and not very intuitive expression and is therefore relegated to the Appendix (p. 87).

Having derived the likelihood function, the estimated parameter vector $\left(\widehat{\sigma},\widehat{\lambda}\right)$ is obtained by maximising equation (15) with respect to (σ,λ). The market price of risk is estimated as a consequence of the chosen estimation method, even though it is not relevant for pricing purposes. Finally, an estimate of the value of assets is simply obtained using the inverse equity function:

$$\widehat{\omega}_n = \vartheta\left\{\mathcal{E}_n^{obs},t_n;\widehat{\sigma}\right\}$$

Having obtained the couplet $(\widehat{\omega}_t,\widehat{\sigma})$, I now turn to pricing and the associated asymptotic distributions of the estimators.

2. Pricing

I denote with $D\{\sigma\}$ the estimator for the price of an asset as a function of the volatility and the estimated price is thus $\widehat{D} = D\{\widehat{\sigma}\}$. Following Lo (1986), I derive the asymptotic distributions of the parameter and price estimators. The asymptotic distributions are used to approximate the distribution in small samples (the next chapter investigates how well this turns out).

For any function of a variable, it holds that the ML estimator of the function is the function of the ML estimator of the variable. Furthermore, in this case it holds that the estimator is asymptotically normally distributed

$$\sqrt{n}\left(D\{\widehat{\sigma}\} - D\{\sigma\}\right) \xrightarrow{L} N\left\{0,\Gamma_{\hat{\sigma}}\frac{dD}{d\sigma}\right\} \tag{16}$$

where $\Gamma_{\hat{\sigma}}$ is the asymptotic standard deviation of $\widehat{\sigma}$. (I denote the standard deviation of an estimate k with γ_k, the *estimated* standard deviation with $\widehat{\gamma}_k$ and the *asymptotic* standard deviation with $\Gamma_k = \lim_{n\to\infty} \gamma_k$)

$$\sqrt{n}\,(\hat{\sigma} - \sigma) \xrightarrow{L} N\{0, \Gamma_{\hat{\sigma}}\}$$

An estimate of the standard deviation of a price estimate is obtained as

$$\widehat{\gamma}_{\widehat{D}} = \widehat{\gamma}_{\hat{\sigma}} \frac{dD}{d\sigma}\,|_{\sigma=\hat{\sigma}}$$

The parameter $\gamma_{\hat{\sigma}}$ is the small sample standard deviation of the estimated asset volatility. It is estimated using a Taylor-series expansion of the likelihood function (see Appendix p. 91). This estimate is generally provided by the computer package.

Since a pricing formula is evaluated on the form $D_i = D\left\{\vartheta\left\{\sigma, \mathcal{E}_i^{obs}\right\}, t_i; \sigma\right\}$, the full differential is

$$\frac{dD}{d\sigma} = \frac{\partial D}{\partial \omega}\frac{\partial \vartheta}{\partial \sigma} + \frac{\partial D}{\partial \sigma}$$

On the right hand side of this equation, we recognize two "hedge parameters": $\frac{\partial D}{\partial \omega}$ is the "delta" of the security and $\frac{\partial D}{\partial \sigma}$ is the "vega" – measuring the sensitivity of the price to changes in volatility.

To sum it up, we obtain two figures for each security we are interested in: the price estimate, $\widehat{D}$, and its estimated standard deviation, $\widehat{\gamma}_{\widehat{D}}$. The exact expression for the estimated standard deviation of the straight coupon bond is given in the Appendix (p. 91).

3. An empirical example

To illustrate the proposed method, consider a bond issued by *Ann Taylor Stores*, a clothing company.[2] The stock of this firm started trading in mid-1991 and the bond was floated on the 28th of June 1993 (the first price quote is from the 13th of July). Assume you are an investor who considers investing in this bond, and consequently needs an estimate of its price. The following information is then procured. (All data were kindly provided by DATASTREAM.)

The bond carries a 8.75% coupon paid semi-annually on the 15:th of June and December, and thus $c = 4.375$. On the 15th of June 2000 the total amount outstanding ($110 millions) is redeemed, and so $T = 6.91$. An estimate of the expected recovery rate is obtained from Altman & Kishore (1996), where one can find that the average recovery rate for a senior, unsecured bond in the "Textile and apparel products"-industry is 34%. Assuming that the average is representative of the actual payoffs in the clothing industry, I set $\psi = 34\%$.

No other bonds were traded at this time, and so the only price information available are stock prices. A price series of market values of equity, starting on the 28th of January 1992 ($\mathcal{E}_1 = 328.29M$), the day of the first published balance sheet, and ending on the 13th of July 1994 ($\mathcal{E}_n = 471.53M$), was constructed ($n = 381$). The cash rate was set rather low, $\beta = 1\%$, to reflect the fact that the firm pays no dividends.

[2]This company was chosen in the following arbitrary way. In exhibit 8 in Fridson & Bersh (1993) containing a sample of firms that had issued bonds rated B- or lower, Ann Taylor Stores was the only company available in DATASTREAM and at the same time being neither convertible nor callable.

From the two available balance sheets, total nominal debt was found to be \$262 millions on the 28th of January 1992 and \$242 millions one year later.[3] To obtain estimates of the amount of nominal debt over the rest of the sample period, and in particular on the 13th of July 1993, linear interpolation was used. In this way, I estimated the nominal amount on the day of the pricing to be $N_n = 247.62M$. Moreover, I chose to set the reorganisation barrier equal to this amount, $L_n = N_n = 247.62M$.

Judging by the reported figures in the previous paragraph, the growth rate of debt is negative. However, two figures do not give a very reliable estimate for the long run growth, and from a practical standpoint, it does not seem reasonable to have an ever decreasing amount of debt, especially considering that Ann Taylor has a rather low leverage (34%). I applied $\alpha = 2.5\%$ initially, but checked for the robustness of this parameter.

The constant riskfree interest rate should optimally be chosen to price the equivalent riskfree bond correct. For simplicitly, I chose the interest rate to equal the yield on a treasury benchmark bond with 7 years to maturity – more or less the same maturity as the bond. On July 13th 1993, the yield on the treasury bond was $r = 5.25\%$.

The recovery rate for debt was taken from Altman & Kishore (1996), who reports that the average recovery rate for the textile industry is $\delta = 37\%$. Equity was expected to violate the absolute priority rule and obtain 20% of the assets in reorganisation. This corresponds to procuring 9.25% of the pre-reorganisation value of assets: $\varepsilon = 9.25\%$.

Maximising the likelihood function with respect to σ and λ given the chosen parameters, produced the following estimates:

$$\begin{cases} \widehat{\sigma} = 31.0\%, \ \widehat{\gamma}_{\widehat{\sigma}} = 1.7\% \\ \widehat{\lambda} = 51.3\%, \ \widehat{\gamma}_{\widehat{\lambda}} = 82.9\% \end{cases}$$

The estimator of the volatility seems to be pretty efficient, whereas the estimator of the market price of risk is extremely inefficient. Of course, the latter result is closely related to the well known fact that it is very difficult to estimate the drift of lognormal diffusion processes (see Merton (1980)). I conclude that one cannot use estimates of the market price of risk for any practical purposes. However, since we price by arbitrage, it is not required, either.

Inserting the estimated volatility into the inverse equity function yields the estimate for asset value:

$$\widehat{\omega}_n = \vartheta\{471.53M, t_n; 0.31\} = 703.37M, \ \ \widehat{\gamma}_{\widehat{\omega}} = 1.34$$

The final step is to price the bond. The estimated price is

$$\widehat{D}_n = D\{703.37M, t_n; 0.31\} = 103.70, \ \ \widehat{\gamma}_{\widehat{D}} = 1.66$$

which corresponds to a 95% confidence interval of

$$[100.45, 106.95]$$

As can be seen, the model produced an estimate that was approximately \$3 too low, but that the confidence interval did contain the actual value of the bond, \$100.76.

[3]This figures for total debt were found by adding Total Current Liabilities (Datastream no. 389), Total Long Term Provisions (Datastream no. 314) and Total Loan Capital (Datastream no. 321).

This exercise being merely meant as an example, I will not, however, digest on whether the result is good or bad.

To see how robust the result is for changes in α, I priced the bond with two alternative values of that, in this case, elusive parameter. Setting $\alpha = 1\%$ produced a bond price estimate of \$105.17, and setting $\alpha = 4\%$ yielded the estimate \$102.33. It seems that an accurate estimation of the growth rate of debt is important to obtain reliable price estimates.

CHAPTER 8

Performance

The main purpose of this chapter is to gauge the small sample properties of the price estimators. For a sufficiently long time series, the asset value process could be estimated without error, and, given that the model is correct, securities could be priced exactly right. In a practical situation however, the time series used will not be long enough to do so.[1] This study will investigate the "not long enough" aspect, i.e. the small sample characteristics, of the parameter estimates and how these carry over to the prices of the debt contracts. A secondary purpose is to compare the suggested estimation method with the traditional one, as will be explained below. I will also illustrate the importance of allowing for an increasing reorganisation barrier.

The first section describes the design of the simulation study, and the second section explains the different output parameters produced to judge the model. The third section, finally, holds the results.

1. Experiment design

The tool used to carry out the study was Monte Carlo simulation. For a chosen scenario for the firm, I generated 1000 sample paths for the asset value variable ω which all may have resulted in the price of equity observed today.[2] For each path, I first computed the resulting stock price path using Proposition 1 and then used it to estimate current asset value and the parameters of its process (i.e., compute the maximum likelihood estimates $\widehat{\lambda}, \widehat{\sigma}, \widehat{\omega}$). Finally, I used the parameter estimates $(\widehat{\sigma}, \widehat{\omega})$ to price the bond (using the formula in Example 3), compute the standard error of the price estimates (using equation (11) on page 47) and construct the corresponding confidence intervals. These steps were repeated for each sample path in order to ultimately assess the sampling distribution.

The whole exercise is repeated for four different scenarios. These are referred to as four different firms, even though they just as well could be thought of as corresponding to the same firm in different situations. The constructed firms all correspond to non-investment grade issuers, since these are the target of the pricing model, and also because their issues are the most difficult to price, in an absolute sense (carrying the highest risk premium). The high-yield bond market is a material

[1]Even if a time series of stock prices exists several decades back, it is not reasonable, of course, that the parameters of the asset value process should stay constant that long – even if it is an assumption of the model that this should be the case.

[2]I first generated a realisation of the underlying Wiener-process over the chosen sample period. Thereafter, the state variable path was generated *backwards* starting from today. The idea was to construct a *ceteris paribus* situation in the sense that except for the past, all things (e.g. prices, volatilities, leverage) were equal irrespective of the generated path. Paths that at some point hit the barrier were discarded and a new path was generated.

			Asset risk	
			Low (σ=20%)	High (σ=30%)
Financial risk	Low (ω=1538)	Expected return	17%	21%
		Volatility	54%	81%
		Leverage	65%	65%
		Dividend yield	2%	1%
		1-year organisation probability	3%	14%
		10-year organisation probability	42%	63%
	High (ω=1176)	Expected return	25%	32%
		Volatility	109%	159%
		Leverage	85%	85%
		Dividend yield	-4%	-5%
		1-year organisation probability	39%	58%
		10-year organisation probability	75%	86%

Exhibit 5: Summary of firm scenarios. The following parameter values were used: $N_t = L_t = 1000$, $\alpha=5\%$, $r=9\%$, $\delta=40\%$, $\varepsilon=5\%$, $\beta=3.5\%$, $\zeta=20\%$

supplier of corporate capital. After some turbulence during the early 90's, the high-yield bond market appears to have stabilised and constitutes roughly 25% of the corporate bond market (Fabozzi & Fabozzi (1995), p. 311).

For each firm scenario, I considered four different debt issues. I will generally use the term "bond", even though the contract could equally well be interpreted as a bank loan. The firm scenarios and the bond issues are discussed in the following two subsections.

1.1. Firm scenarios. I first defined a base scenario which is supposed to correspond to a firm whose bonds have a credit rating of B. According to Standard and Poor's *Great Comments* (1993), cited in Fabozzi & Fabozzi (1995) as Exhibit 18-1, firms issuing debt of that credit quality have a (nominal) leverage of around 65% (1989-1992, industrial firms). I chose the riskfree interest rate to be 9% and assumed that the firm had a stock beta of 1; with a market risk premium of 8% the expected return of the firm's stock therefore was 17% (this was accomplished by setting the market price of risk in the model to 15%). The dividend yield was 2%. The volatility of the firm's asset value was 20% which translated into an (instantaneous) stock volatility of 54%.

A figure in Jónsson & Fridson (1996) depicts yearly default probabilities, based on the 1971-1994 experience, for B-bonds as a function of time outstanding (from Altman & Kishore (1995)). The first year default probability is 2%, a peak is reached after 3 years with 7% and then probabilities decline – after seven years they are 3-4%. These figures are matched by the base scenario. The total probability for the firm to enter financial distress within the next 10 years is 42%.

I chose to let the reorganisation barrier equal total debt and the growth rate for debt be equal to 5%. With that increase in total debt, the firm would not expect to change the debt-to-equity ratio of the firm in the future. In the case of a reorganisation, the expected recovery rate for debt was 40%. Moreover, equity holders would expect to receive 5% of the assets which corresponds to a violation of absolute priority of 12.5%.

To construct three additional scenarios, I changed base parameters along two dimensions: financial risk and asset risk. The base scenario was defined as having low financial risk and low asset risk. I represented high financial risk as a nominal leverage of 85% – that is, the average leverage of firms issuing debt with a credit rating of CCC (*Great Comments* (1993)). High asset risk was arbitrarily quantified as being equal to 30%. Exhibit 5 summarises the characteristics of the four scenarios.

When leverage is high, equity holders need to contribute funds to the firm to service debt which is reflected by the negative dividend yield. Firms with high leverage and low asset risk, or low leverage and high asset risk, can be thought of as firms whose senior debt is CCC-rated. Firms with high risk along both dimensions are on the brink of default and issue debt of the very lowest credit quality.

1.2. Bond issues. For each firm scenario, four bond issues were considered. These were defined by having either long or short maturity, and either being senior or junior. A long maturity was defined as 30 years and a short as 3 years. Senior bonds had an expected recovery rate of 58% and and the corresponding figure for junior (subordinated) bonds was 31% (Altman & Kishore (1996)). The coupon, paid semi-annually, was set to be 12%. Exhibit 6 reports the resulting prices of the corporate bonds.

As can be seen, the risk premiums increase with maturity and reduced seniority. For the very risky bonds – such as a long, senior bond issued by a firm with both high asset risk and high financial risk – the value is made up almost entirely of the expected payoff in a reorganisation. This is not surprising considering the bondholders only have a 9 % chance of ever being reimbursed in full.

From Cheung et al. (1992), quoted in Fabozzi & Fabozzi (1995), it can be deduced that yield spreads on corporate B-rated bonds have varied between 250 and 550 basis points during the 1977-1989 period.[3] The additional yield spread on CCC-rated bonds varied between 11 and 280 basis points – in total 400 to 750.

2. Output

The aim of this section is to delineate the outputs produced and the tests performed.[4] Since the ultimate intended use of the implemented model is to price, the first question to address is whether price estimates $\left(\widehat{D}\right)$ are biased, $\left(E\left[\widehat{D}\right] \neq D\right)$, in small samples. Second, we need to know how reliable, or efficient, they are. I use the standard deviation of the estimator $\left(\gamma_{\widehat{D}}\right)$ as one measure of this. If the estimates were normally distributed, this would suffice as a measure of efficiency. As this is sometimes not the case, I also provide the interval within which 95% of the estimates can be found. Of course, the wider the interval, the less efficient the estimator.

[3]Using Exhibits 15-12 and 15-15 in Fabozzi & Fabozzi (1995).

[4]When using the term "estimate", I refer to the estimate for a particular sample path. The expected value of an estimate is calculated as the mean of estimates across generated sample paths.

			Low financial risk			
			Low asset risk		High asset risk	
			Maturity		Maturity	
			Short (T=3)	Long (T=30)	Short (T=3)	Long (T=30)
Riskfree			107.18	128.32	107.18	128.32
Priority	Senior (ψ=58%)	Price	96.89	95.12	85.98	82.10
		Discount	9.6%	26%	20%	36%
		Spread (basis p.)	386	325	848	515
	Junior (ψ=31%)	Price	91.13	82.64	74.35	64.87
		Discount	15%	36%	31%	49%
		Spread (basis p.)	623	506	1418	874
			High financial risk			
Priority	Senior (ψ=58%)	Price	75.73	73.84	68.72	66.96
		Discount	29%	43%	36%	48%
		Spread (basis p.)	1346	667	1731	821
	Junior (ψ=31%)	Price	58.83	53.63	48.13	44.16
		Discount	45%	58%	55%	66%
		Spread (basis p.)	2355	1224	3180	1650

Exhibit 6: Bonds' credit risk premiums

A secondary issue is to examine if the asymptotic distributions, discussed in Section 2 of the previous chapter, carry over to small samples. I measure the skewness (Sk.) and kurtosis (Ku.) of the sampled distributions and perform Bowman-Shelton (BS) tests of normality.[5] A superindexed asterisk (*) on a Bowman Shelton (BS) statistic in the tables holding the results, indicates that the null hypothesis of normality could not be rejected with 5% significance (the cut-off point is 5.99). Even if a particular estimator is found not to be normally distributed, the estimated standard deviation $\widehat{\gamma}_{\widehat{D}}$ might be used for hypothesis testing and to calculate confidence intervals in small samples. Therefore, I investigated whether it was biased $\left(E\left[\widehat{\gamma}_{\widehat{D}}\right] \neq \gamma_{\widehat{D}}\right)$ and as a measure of its efficiency I used *its* standard deviation, $\gamma_{\widehat{\gamma}_{\widehat{D}}}$.

To further pursue this issue I carried out a *size test* – i.e., I looked at how often the true value of an estimated price parameter fell outside the confidence interval

[5] The BS-statistic is calculated as follows

$$1000 \times \left[\frac{skewness^2}{6} + \frac{(kurtosis - 3)^2}{24}\right]$$

calculated using the estimated standard deviation.[6] This is termed the *size*. If the size is close to the chosen *nominal size* (I use 1%, 5% and 10%), one may conclude that the asymptotic distribution is useful for purposes of calculating confidence intervals. A superindexed asterisk on the population size denotes that the null of the size being equal to the nominal size could not be rejected with 5% significance (the cut-off points are 0.7-1.3% at nominal size 1%, 4.3-5.7% at nominal size 5% and 9.1-10.9% at nominal size 10%). Thus, loosely speaking, the more superindexed asterisks a line in a table contains, the more reasonable is the normal distribution assumption in small samples for that particular estimator.

The price estimates ultimately depend on the estimates of the parameter(s) of the asset process. To help understand the results, I therefore report the same output as above for the parameter estimates $(\widehat{\sigma}, \widehat{\omega})$ as well.

Finally, as a comparison, I included results of using the most common method to date of estimating the asset value process and thereafter pricing bonds (see for example Jones et al. (1984) and Ronn & Verma (1986)). As I wanted to investigate the estimation performance, the same economic setting was used and consequently also the same pricing formulae. The difference was in the estimation of (ω_t, σ) (the market price of risk was not estimated), which required the following steps to be carried out.

1. The stock price volatility $\sigma_{\mathcal{E}}$ is estimated using historical data. Denote this estimate $\widehat{\sigma}_{\mathcal{E}}^{VR}$.
2. The parameters σ and ω are obtained by solving the following system of equations

$$\left.\begin{array}{r} \sigma \cdot \frac{\partial \mathcal{E}}{\partial \omega} \cdot \omega_n = \widehat{\sigma}_{\mathcal{E}}^{VR} \cdot \mathcal{E}_n^{obs} \\ \mathcal{E}\left(\omega_n, t_n; \sigma\right) = \mathcal{E}_t^{obs} \end{array}\right\} \longrightarrow \left(\widehat{\omega}_n^{VR}, \widehat{\sigma}^{VR}\right)$$

   The first equality is implied by the application of Itô's lemma to equity as a function of ω and t and the second from matching the theoretical equity price with the observed market price.

There are two problems with this method. The instantaneous stock price volatility $\sigma_{\mathcal{E}}$ is most often estimated, in step 1, under the assumption of it being constant – even though it is a known function of ω and t, which in fact iss used to derive the first equation in step 2. Furthermore, that equation is redundant since it was used to derive the equity price formula in the *second* equation.[7] Another disadvantage of this approach is that it does not allow the straightforward calculation of the distributions of the estimators for ω and σ.

It is, however, a practical and frequently used approach that needs to be evaluated. Furthermore it serves as a benchmark for the performance of the maximum likelihood approach evaluated in this paper. Throughout the remainder of this

[6]Note the difference between the interval within which 95% of the estimates can be found, mentioned above, and the 95% confidence interval. The former is a *population* feature, and 95% quantifies the *proportion* of estimates contained in a given interval; whereas the latter is a *single sample* feature, where 95% quantifies the *probability* that the constructed interval contains the true value.

[7]It is interesting to note that this implies that if the estimation of $\sigma_{\mathcal{E}}$ would produce the *correct* estimate $(\widehat{\sigma}_{\mathcal{E}}^{VR} = \sigma_{\mathcal{E}}(\omega_t, t; \cdot))$, one of the equations would be redundant. Thus, the first theoretical inconsistency (assuming constant stock price volatility) is necessary to find a unique solution to a system of equations which, theoretically, has an infinite number of solutions.

paper I will refer to this estimation approach as the volatility restriction (VR) approach.

3. Monte Carlo results

Below I present the results of the Monte Carlo simulations. The initial subsection investigates the importance of the chosen estimation period, or sample size, in order to determine which sample size to use for the main simulations. The results of these are contained in the following subsection. Then I investigate the consequences of neglecting to incorporate an increasing barrier. The next subsection compares the maximum likelihood method with the volatility restriction method. The last subsection briefly addresses the effect of applying an endogenous barrier. Finally, there is a summary.

3.1. Estimation period. In this section, I examine how the distributions of the asset value and asset risk estimators depend on the length of the equity price sample (n). Using the base scenario, with low risk along both dimensions, I investigate four sample sizes : 30, 90, 250 and 500 days. The results of this investigation are summarised in Exhibit 10 on page 65

The estimator for asset risk is somewhat biased for the smallest sample size, and also inefficient – to contain 95% of the estimates the interval has to be as wide as 13.8-25.5%. The 30-day-distribution at best can be termed "fairly normal". The estimator for asset value is much less biased, and could probably even be referred to as being "unbiased" for practical purposes, and is also much more efficient. On the other hand, the distribution is not even close to normal. Still, however, as indicated by the size tests, the estimated standard deviations seem useful for constructing confidence intervals and therefore also for hypothesis testing.

As the sample size increases, the efficiency increases dramatically and the distributions of estimators tend towards the normal. When the sample size is 250, the estimators seem to be normally distributed. Extending the sample size to 500 does not improve on the results much (oddly enough, the normality of the the asset value estimator decreases when you increase the sample size to 500). Based on these results I chose to go ahead with a sample size of 250.

3.2. The bond price estimator. The results for the bonds are presented in Exhibits 11-14 on pages 66-69. Each table holds the results for a particular firm scenario. The overall impression is that, for practical purposes estimators are unbiased as well as efficient.

The relative bond price bias never exceeds one half percent. The highest standard deviation is 2.11, which should be put in relation to the bond price in that case: 82.64. The intervals containing 95% of the estimates appear to be reasonably tight – the intervals are less than 4 units of currency wide. Of course, it is a delicate matter to quantify what constitutes a "good" estimator in this case. When judging the estimator, one has to keep in mind, however, what kind of securities the model aims to price: i.e., non-investment grade debt instruments, including loans, issued by firms with no traded debt. This means the estimation procedure has no price information but stock prices to rely on. Adding one bond price would probably enhance pricing performance considerably.

	Bond		Estimation method	
			Maximum likelihood	Volatility restriction
	Junior debt	Short maturity	0.22%	7.40%
Bias		Long maturity	0.27%	7.69%
	Senior debt	Short maturity	0.12%	3.75%
		Long maturity	0.62%	4.07%
	Junior debt	Short maturity	1.50	8.31
Efficiency		Long maturity	1.53	8.34
	Senior debt	Short maturity	0.98	16.28
		Long maturity	1.12	6.08

Exhibit 7: Bias and efficiency of maximum likelihood and volatility restriction estimators. The table reports the average bias and standard deviation of estimators for the four bonds. Averages are calculated across firm scenarios.

The asymptotic distributions carry over to small samples for the base scenario, but when asset risk and, in particular, financial risk increase, the normality attribute vanishes. The Bowman-Shelton test always rejects the null hypothesis of the estimators being normally distributed when leverage is high, and the estimated standard deviations can only be used exceptionally to construct reliable confidence intervals. Specifically, the intervals are too wide, which is also reflected by the fact that, in these cases, the estimate of the standard deviation exceeds the true standard deviation ($E\left[\widehat{\gamma}_{\widehat{D}}\right] > \gamma_{\widehat{D}}$).

In general, the estimators of (σ, ω_t) perform better. They are both unbiased and very efficient, and this holds true for the estimated standard deviations as well. Normality for the asset risk estimator can never be rejected, but for the asset value estimator, it can be very firmly rejected, except for the base scenario. Still, judging from the size test, the estimated standard deviation appears to be useful for hypothesis testing and constructing confidence intervals.

3.3. Volatility restriction. Exhibit 7 presents the results from the comparison of the suggested approach with the volatility restriction approach. For each bond, the table gives the average absolute bias and efficiency across firm scenarios.[8] First of all, the average biases with the VR-estimators are considerably higher than with the corresponding ML-estimators. Also, the average efficiency of the VR-estimators are markedly lower. This is not just an average result – the same difference holds for each scenario (not displayed). A visual interpretation of Exhibit

[8]For example, the average absolute bias for the volatility restriction method is calculated as $\frac{1}{4}\sum_{s=1}^{4}\left|\frac{E[\widehat{D}_s^{VR}]-D_s}{D_s}\right|$, where s indexes scenarios.

7 is in Exhibit 8. It plots the densities obtained from estimating with the maximum likelihood and volatility restriction methods, respectively.

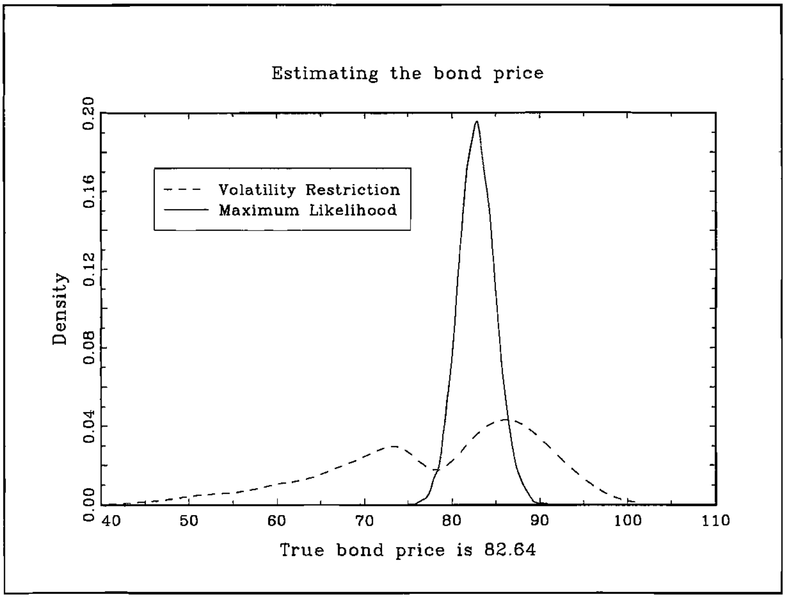

Exhibit 8: Density plots for maximum likelihood and volatility restriction estimators. The figure depicts density plots for estimates of a long, junior bond in the base case firm scenario with the maximum likelihood and the volatility restriction methods, respectively.

As can be seen, the figure reinforces the numeric results that the maximum likelihood approach is superior.

3.4. Ignoring debt growth. To ignore the fact that the firm increases its debt over time severely biases the bond price estimates. Exhibit 9 depicts the results of estimating the asset value and its volatility, and subsequently pricing bonds issued by the base scenario firm, under the assumption that the current amount of debt remains unchanged forever, even though it in fact increases at a rate of 5%. As can be seen, the estimated bond prices are too high. The average bias is 2-3% for the short bonds but as much as 11-18% for the long bonds. As has been discussed earlier, a model with a constant barrier underestimates the bankruptcy probabilities in the future, and quite naturally the effect on the prices is more severe the longer the maturity of the bond.

3.5. An endogenous barrier. To examine how the choice of reorganisation barrier influences pricing performance, I reconsidered the base firm scenario with an endogenous barrier: $L_t = 579.43$. The largest effect of an endogenous barrier is on the price of the bond, of course. A lower reorganisation barrier decreases the probability of running into financial distress during the first 10 years from 42% to 8%. Also, the expected return of stock has decreased to 15% and the volatility to 37% (cf. Exhibit 5). The short bond becomes virtually riskfree, but the long bond still carries a risk premium: the spread is 96 and 139 basis points for the long senior and junior bonds, respectively (cf. Exhibit 6)

Parameter/Bond		Bias			Efficiency		Distribution
		True value	Exp. est.	Rel. bias	Std.	95% interval	Exp. est. std.
			(%)	(%)			
		π	$E[\widehat{\pi}]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$		$E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$
Asset risk: σ		20%	20.7%	3.3%	1.2%	18.1-22.8%	0.9% (0.1%)
Asset value: ω		1538	1479	-3.8%	5.37	1467-1488	4.02 (0.20)
Junior debt	Short maturity	91.13	94.23	3.4%	1.94	90.65-98.35	1.52 (0.09)
	Long maturity	82.64	97.38	18%	3.44	91.31-104.94	2.59 (0.13)
Senior debt	Short maturity	96.89	98.86	2.0%	1.25	96.54-101.52	1.04 (0.06)
	Long maturity	95.12	105.81	11%	2.51	101.38-111.32	1.91 (0.09)

Exhibit 9: Effect of ignoring debt growth. Firm scenario with low asset risk and low financial risk. Estimation under the assumption that $\alpha = 0$. True $\alpha = 5\%$.

Exhibit 15 on page 70 presents the results of using an endogenous barrier (all parameter values are the same as in the base scenario, with the exception of the level of the barrier). The price estimates for the prices of the long bonds are still, for practical purposes, unbiased and efficient. The estimators are "fairly normally" distributed, but to a slightly lesser degree than in the original scenario. The estimator of the price of the short bond is extremely efficient, but also definitely not normally distributed. The former characteristic depends, of course, on that there is hardly any risk to price, and the latter is related to the fact that the price estimates are bounded from above by the value of the riskfree bond (107.18), which introduces a marked skew to the distribution.

It seems as endogenous barrier has its primary effect on the absolute level of prices, rather than on the performance of price estimators.

4. Summary of simulation study

Only the asymptotic distributions are known for the estimators suggested by this framework. This chapter presents the result of a simulation study to find the small sample properties of estimators. The bond price estimators seem unbiased and efficient, also for very risky debt. The asymptotic (normal) distribution carry over to estimators when credit risk is modest, but is lost for the highly risky scenarios. Even so, the estimated standard deviation is often useful for calculating confidence intervals and conducting hypothesis tests. Furthermore, the characteristics of the estimators do not seem to be affected by the choice of the reorganisation barrier, except when the bond issue becomes virtually riskfree. In the latter cases, the estimates are no longer normally distributed.

A secondary purpose of this chapter was to compare the proposed estimation method with an alternative method. The results are strongly in favour of the proposed method.

This chapter also examined the consequences of neglecting to incorporate an increasing barrier in the current setting. Since this underestimates the risk of encountering financial distress, especially for long maturities, bond prices will be underestimated.

Estimation period	Parameter		Bias		Efficiency			Distribution					
			Exp. est. (%)	Rel. bias (%)	Std.	95% interval	Exp. est. std.	Sk.	Ku.	B-S	Simulated size 1%	5%	10%
n		π	$E\left[\widehat{\pi}\right]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$		$E\left[\widehat{\gamma}_{\widehat{\pi}}\right]$ $\left(\gamma_{\widehat{\gamma}_{\widehat{\pi}}}\right)$						
30	Asset risk: σ	20%	19.4%	-3.2%	2.9%	13.8-25.5%	2.9% (0.4%)	0.2	2.9	6.6	3.1%	7.6%	12.9%
	Asset value: ω	1538	1531	-0.5%	20.37	1482-1564	19.79 (3.26)	-0.7	3.6	88	1.5%*	5.5%*	9.9%*
90	Asset risk: σ	20%	19.8%	-1.0%	1.7%	16.3-23.1%	1.7% (0.1%)	-0.1	2.9	2.4*	2.5%	6.1%	11.3%
	Asset value: ω	1538	1536	-0.1%	11.54	1509-1555	11.15 (1.21)	-0.6	3.4	66	1.1%*	5.4%*	9.9%*
250	Asset risk: σ	20%	19.9%	-0.4%	1.1%	17.9-22.0%	1.1% (0.1%)	0.2	3.0	4.3*	1.3%*	5.4%*	10.2%*
	Asset value: ω	1538	1537	-0.1%	6.78	1523-1550	6.65 (0.52)	-0.2	2.9	4.4*	0.9%*	5.7%*	10.0%*
500	Asset risk: σ	20%	19.9%	-0.3%	0.7%	18.5-21.3%	0.7% (0.0%)	-0.1	3.2	3.1*	1.4%	5.1%*	10.7%*
	Asset value: ω	1538	1538	-0.0%	4.65	1528-1546	4.66 (0.32)	-0.4	3.3	24	0.9%*	4.6%*	9.9%*

Exhibit 10: Importance of sample size.

Parameter/Bond		Bias			Efficiency			Distribution					
		True value	Exp. est. (%)	Rel. bias (%)	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Simulated size 1%	5%	10%
		π	$E\left[\widehat{\pi}\right]$	$\frac{E\left[\widehat{\pi}\right]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$		$E\left[\widehat{\gamma}_{\widehat{\pi}}\right]$ $\left(\gamma_{\widehat{\gamma}_{\widehat{\pi}}}\right)$						
Asset risk: σ		20%	19.9%	-0.5%	1.1%	17.9-22.0%	1.0% (0.1%)	0.2	3.0	4.3*	1.3%*	5.4%*	10.2%*
Asset value: ω		1538	1537	-0.1%	6.78	1523-1550	6.65 (0.52)	-0.2	2.9	4.4*	0.9%*	5.7%*	10.0%*
Junior debt	Short maturity	91.13	91.27	0.2%	1.66	87.89-94.44	1.71 (0.11)	-0.2	2.9	4.3*	1.0%*	4.7%*	8.6%
	Long maturity	82.64	82.87	0.3%	2.11	78.72-87.04	2.11 (0.09)	-0.0	2.9	0.6*	0.9%*	4.9%*	9.6%*
Senior debt	Short maturity	96.89	96.98	0.1%	1.07	94.80-99.03	1.20 (0.08)	-0.2	2.9	4.5*	0.6%	3.2%	6.4%
	Long maturity	95.12	95.29	0.2%	1.54	92.25-98.34	1.59 (0.07)	-0.0	2.9	0.6*	0.6%	4.4%*	8.8%

Exhibit 11: Pricing the bonds – firm scenario with low financial risk and low asset risk

Parameter/Bond			Bias		Efficiency		Distribution						
			Exp. est. (%)	Rel. bias (%)	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Simulated size 1%	5%	10%
		π	$E\left[\widehat{\pi}\right]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$		$E\left[\widehat{\gamma}_{\widehat{\pi}}\right]$ $\left(\gamma_{\widehat{\gamma}_{\widehat{\pi}}}\right)$						
Asset risk: σ		20%	20.0%	-0.1%	1.1%	18.0-22.0%	1.1% (0.0%)	-0.0	2.6	7.7	0.1%	3.5%	8.8%
Asset value: ω		1176	1176	-0.0%	2.0	1172-1179	2.1 (0.22)	-0.3	2.7	21.3	0.2%	3.6%	7.8%
Junior debt	Short maturity	58.83	58.90	0.1%	1.37	56.35-61.60	1.59 (0.06)	0.1	2.6	10.7	0.0%	1.1%	5.3%
	Long maturity	53.63	53.71	0.1%	1.27	51.25-56.22	1.38 (0.06)	0.2	2.6	11.6	0.2%	3.2%	7.4%
Senior debt	Short maturity	75.73	75.78	0.1%	0.90	74.10-77.55	1.23 (0.05)	0.1	2.6	10.7	0.0%	0.2%	1.8%
	Long maturity	73.84	73.90	0.1%	0.93	72.18-75.73	1.06 (0.05)	0.2	2.6	11.7	0.0%	2.0%	5.7%

Exhibit 12: Pricing the bonds – firm scenario with high financial risk and low asset risk

Parameter/Bond			Bias		Efficiency		Distribution						
			Exp. est. (%)	Rel. bias (%)	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Simulated size 1%	5%	10%
		π	$E[\widehat{\pi}]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$		$E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$						
Asset risk: σ		30%	29.8%	-0.6%	1.4%	27.0-32.6%	1.5% (0.1%)	-0.1	2.8	1.9*	1.0%*	5.7%*	11.2%
Asset value: ω		1538	1537	-0.0%	3.13	1531-1543	3.14 (0.35)	-0.5	3.1	35.7	0.8%*	4.4%*	10.5%*
Junior debt	Short maturity	74.35	74.63	0.4%	1.89	71.05-78.46	2.00 (0.05)	0.2	2.8	6.0	0.5%	4.5%*	9.6%*
	Long maturity	64.87	65.16	0.5%	1.82	61.77-68.89	1.86 (0.06)	0.2	2.9	9.4	0.5%	4.9%*	10.9%*
Senior debt	Short maturity	85.98	86.17	0.2%	1.24	83.82-88.67	1.40 (0.03)	0.2	2.8	5.8*	0.2%	3.0%	7.1%
	Long maturity	82.10	82.31	0.3%	1.33	79.83-85.05	1.39 (0.04)	0.2	2.9	9.4	0.4%	4.1%	9.8%*

Exhibit 13: Pricing the bonds – firm scenario with low financial risk and high asset risk

Parameter/Bond		π	Bias: Exp. est. (%) $E[\widehat{\pi}]$	Bias: Rel. bias (%) $\frac{E[\widehat{\pi}]-\pi}{\pi}$	Efficiency: Std. $\gamma_{\widehat{\pi}}$	Efficiency: 95% interval	Efficiency: Exp. est. std. $E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$	Distribution: Sk.	Distribution: Ku.	Distribution: BS	Simulated size 1%	Simulated size 5%	Simulated size 10%
Asset risk: σ		30%	29.9%	-0.25%	1.5%	26.9-33.0%	1.5% (0.1%)	-0.0	2.9	0.7*	1.0%*	6.0%	11.3%
Asset value: ω		1176	1176	-0.01%	0.8	1174-1177	0.73 (0.12)	-0.7	3.7	92	1.6%	5.4%*	11.7%
Junior debt	Short maturity	48.12	48.22	0.2%	1.07	46.23-50.47	1.28 (0.05)	0.3	3.1	14.6	0.6%	3.7%	8.3%
	Long maturity	44.16	44.24	0.2%	0.93	42.53-46.20	0.92 (0.05)	0.3	3.1	17.4	0.9%*	5.6%*	10.1%*
Senior debt	Short maturity	68.72	68.78	0.1%	0.70	67.48-70.25	0.85 (0.04)	0.3	3.1	15.1	0.3%	1.2%	4.3%
	Long maturity	66.96	67.03	0.1%	0.67	65.79-68.44	0.68 (0.04)	0.3	3.1	18.2	0.8%*	4.4%*	8.9%

Exhibit 14: Pricing the bonds – firm scenario with high financial risk and high asset risk

Parameter/Bond			Bias		Efficiency		Distribution						
			Exp. est. (%)	Rel. bias (%)	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Simulated size 1%	5%	10%
		π	$E\left[\widehat{\pi}\right]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$		$E\left[\widehat{\gamma}_{\widehat{\pi}}\right]$ $\left(\gamma_{\widehat{\gamma}_{\widehat{\pi}}}\right)$						
Asset risk: σ		20%	19.9%	-0.4%	1.0%	18.0-21.9%	1.0% (1.0%)	0.1	3.0	2.0	1.8%	6.3%	11.3%
Asset value: ω		1538	1537	-0.0%	5.09	1526-1546	4.90 (0.51)	-0.3	3.0	18	1.9%	5.7%*	11.9%
Junior debt	Short maturity	106.78	106.77	-0.0%	0.18	106.36-107.03	0.17 (0.06)	-1.0	4.5	264	5.3%	9.6%	13.0%
	Long maturity	112.01	112.14	0.1%	1.85	108.43-115.62	1.83 (0.16)	-0.1	3.0	3.5	1.7%	6.2%	10.7%*
Senior debt	Short maturity	106.93	106.92	-0.0%	0.11	106.66-107.09	0.11 (0.04)	-1.0	4.5	265	4.7%	8.7%	12.0%
	Long maturity	116.59	116.68	0.1%	1.34	113.99-119.20	1.37 (0.12)	-0.1	3.0	3.7	1.5%	5.3%*	9.4%*

Exhibit 15: An endogenous barrier.

CHAPTER 9

Conclusion

This thesis suggests an integrated and easily implementable framework for pricing the credit risk in corporate debt. It concentrates on non-investment grade debt, where credit risk is likely to be the major determinant of the price discount. The presented model is characterised by the following features.

First, it can utilise information from stock as well as bond prices. This is important because many firms do not have any actively traded bonds outstanding.

Second, equity is valued separately from the debt instrument of interest. This not only facilitates estimation but also provides a more realistic setup for equity, reflecting the multitude of debt payments made regularly by firms.

Third, it is acknowledged that as the value of assets increases over time, the firm may also take up more debt. This feature prevents the bankruptcy probability from being amassed in the immediate future. In turn, this is crucial for the pricing of long maturity debt instruments.

The framework is easy to apply and all the necessary equations are provided in this paper. A simulation study is performed to investigate its properties. Although it is difficult to bring out clearly what constitutes "good" performance, the results indicate that the model works well. The employed estimation method clearly outperforms the most common method to date.

A straight coupon bond is the focus of this thesis, but the framework extends to several other securities: e.g., guarantees, preference shares and putable bonds. It can also readily be used to price non-standardised and non-traded debt such as loans and private placements.

The next step is to evaluate the model with an empirical investigation. Jan Ericsson and myself are currently conducting such a study on American bonds. The results from that investigation will suggest directions for future research, but preliminary work lead us to consider the following extension from the outset. It is possible that some parameters that are not directly observable, such as the growth rate of debt (α), the expected payoffs in financial distress ($\delta, \psi, \varepsilon$), or the reorganisation barrier (L_t), should be estimated structurally together with σ and λ.

Furthermore, the following theoretical extensions are among those that should be considered. The first is to include stochastic interest rates, something that is also current work. Variable interest rates would render a meaningful analysis of callable bonds possible. The second important extension is to take liquidity problems into account. Most bonds and definitely all loans are afflicted with a liquidity premium, and the premium depends on the vintage of the bond. To improve on the pricing of vintage bonds, some consideration has to be given to liquidity.

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CHAPTER 10

Appendix

The appendix consists of three sections. The first is a list of notation, the second contains the derivations and the last section holds some formulae.

1. List of notation

Parameter	Meaning
B_t	Money market account
C	Total debt service/Call option
C_L	Down-and-out call option
$\mathcal{C}_L$	Conditional down-and-out call option
c	Coupon for debt contract
D	General debt contract
d	Market value of (small) loan at aggregate level
E	Expected value under objective probability measure
E^m	Expected value under probability measure Q^m
$\mathcal{E}$	Equity
F	Exercise price of underlying call
f	Conditional density for equity
G	Dollar-in-default claim
$\mathcal{G}$	Conditional dollar-in-default
G^α	Dollar-with-interest-in-default claim
g	Conditional density for $\ln \omega$
H	Heaviside
H_L	Down-and-out heaviside
$\mathcal{H}_L$	Conditional down-and-out heaviside
I	Indicator function
K	Exercise price of compound call
L_t	Reorganisation barrier/Likelihood function (without time-index)
M	Number of contracted payments
N_t	Total nominal debt
n	Sample size
P	Principal of debt contract
Q^m	Probability measure/Probability of event under that measure
r	Riskfree interest rate
S	Maturity of underlying call
T	Maturity of compound call/General maturity
W^m	Wiener process under the Q^m-measure
v_t	State variable (v_T is value of assets at T)
X_t	Reorganisation process

Parameter	Meaning
α	Growth rate of debt (and barrier)
β	Cash flow rate
$\gamma_{\widehat{\pi}}$	Standard deviation of estimate of parameter/price π
$(\delta), \delta_\tau$	(Expected) Recovery rate for total debt
$(\varepsilon), \varepsilon_\tau$	(Expected) Relative payout to equity in reorg.
ζ	Corporate tax rate
ϑ	Inverse equity function
λ	Market price of W-risk
μ	Drift of state variable (v)
π	Parameter/Price
ρ	Correlation coefficient $\left(\frac{\sqrt{S-t}}{\sqrt{T-t}}\right)$
σ	Volatility of state variable (v)
τ	Time of reorganisation
Φ	Payoff function
ϕ	Cumulative normal distribution function
$(\psi), \psi_\tau$	(Expected) Recovery rate for debt contract
ω	Value of assets
Ω	Asset claim
Þ	Put option

2. Derivations

I first go through some notation and previous results, and thereafter derives the relevant lemmas and propositions. However, the mathematics are not very formal.

2.1. Preliminaries.

- $R^{j\to m}$ denotes the Radon-Nikodym derivative $R^{j\to m} = \frac{dQ^i}{dQ^m}$ with associated Girsanov-kernel $h^{j\to m}$. Thus superscripts "$j \to m$" can be read *from probability measure 'j' to probability measure 'i'.*
- To change measure one uses $dW^X = dW^B + h^{B\to X}$
- $\Im_t$ denotes the information structure at time t.

2.1.1. *The default process.* Define the default process

$$\begin{cases} X_t \equiv \frac{1}{\sigma}\ln\frac{\omega_t}{L_t} \\ X_0 \equiv \frac{1}{\sigma}\ln\frac{\omega_0}{L_0} \end{cases} \tag{17}$$

Default is defined through

$$X_\tau \equiv 0$$

The dynamics of the default process are

$$dX = \mu_X^m dt + dW^m$$

where W^m is a Wiener-process under probability measure Q^m and μ_X^m is its appurtenant drift.

- Q^B is the measure under which price processes normalised with the money market account are martingales (the pricing measure)
- Q^ω is the measure under which price processes normalised with

  $$\omega_t' \equiv e^{-rT}E^B\left[\omega_T\right] = e^{-\beta(T-t)}\omega_t$$

  are martingales. The Girsanov kernel used to go from the pricing measure is $h^{B\to\omega} = -\sigma$.
- Q^G is the measure under which price processes normalised with $G\left\{\omega_t, t; \cdot\right\}$ are martingales. The Girsanov kernel used to go from the pricing measure is $h^{B\to G} = \theta\left\{r\right\}\sigma$.
- Q^X is the measure under which X is a martingale. The Girsanov kernel used to go from the pricing measure is obviously $h^{B\to X} = \mu_X^B$.

The relevant drifts are

$$\begin{aligned} \mu_X &= \frac{r + \lambda\sigma - \beta - \alpha - 0.5\sigma^2}{\sigma} \\ \mu_X^B &= \frac{r - \beta - \alpha - 0.5\sigma^2}{\sigma} \\ \mu_X^\omega &= \frac{r - \beta - \alpha + 0.5\sigma^2}{\sigma} \\ \mu_X^G &= -\sqrt{\left(\mu_X^B\right)^2 + 2r} \\ \mu_X^X &\equiv 0 \end{aligned}$$

2.1.2. *Numeraires and change of probability measures.* Loosely speaking it holds that, for a general measure Q^j,

$$E^j[m \cdot Y] = E^j[m] \cdot E^m[Y]$$

where E^m is the expected value under the probability measure Q^m defined through

$$\begin{cases} dQ^m = R^{j\to m} dQ^j \\ \\ R^{j\to m} = \frac{m}{E^j[m]} \end{cases}$$

The corresponding Girsanov-kernel is denoted $h^{j\to m}$. See e.g. Geman et al. (1995).

2.1.3. *First passage time.* First note that the first passage time density at 0 for process the X_t under probability measure Q^m

$$f^m\{X_t; s\} = \frac{X_t}{\sqrt{2\pi(s-t)^3}} e^{-\frac{1}{2}\left(\frac{X_t - \mu_X^m (s-t)}{\sqrt{s-t}}\right)^2}$$

Remark 3. *It holds that for a density function for a process like (17) with drift μ_X^m*

$$\int_t^\infty e^{-\rho(s-t)} f^m\{X_t; s\}\, ds = e^{-X_t\left(\sqrt{(\mu_X^m)^2 + 2\rho} + \mu_X^m\right)}$$

if $(\mu_X^m)^2 + 2\rho \geq 0$. See e.g. Ericsson (1997) pp. 139-140.

2.2. Deriving Lemma 1. I solve for the terms of Definition 1 one at a time and denote them $C_L\{\infty\}$ (a perpetual down-and-out call on the assets) and \$ (a claim on generated cash), respectively. Thus $\Omega = C_L + \$$. Consider the first line:

$$\begin{aligned} C_L\{\infty\} &= \lim_{T\to\infty} E^B\left[(\omega_T - N_T) e^{-r(T-t)} I_{\{\tau \nless T\}}\right] \\ &= \lim_{T\to\infty} e^{-r(T-t)} E^B\left[\omega_T I_{\{\tau \nless T\}}\right] - N_t \lim_{T\to\infty} \frac{Q^B(\tau \nless T)}{e^{(r-\alpha)(T-t)}} \end{aligned}$$

The last term obviously equals zero whenever $r \geq \alpha$. Applying the rule of Hôpital it can be shown that this is true also when $r < \alpha$. To solve for the value of the first term utilise the probability measure Q^ω to separate the two variables within the expectation brackets in the first term

$$\begin{aligned} C_L\{\infty\} &= \lim_{T\to\infty} e^{-r(T-t)} E^B[\omega_T] E^\omega\left[I_{\{\tau \nless T\}}\right] - 0 \\ &= \omega_t \lim_{T\to\infty} e^{-\beta(T-t)} (1 - Q^\omega\{\tau \leq T\}) \end{aligned}$$

(This result holds in this case even though the measures Q^ω and Q^B are no longer equivalent as $T \to \infty$.) Only when $\beta = 0$ does the call have a positive value. When the drift is non-positive ($\mu_X^m \leq 0$) the probability of default is unity since we have an infinite horizon:

$$Q^m\{\tau \leq \infty\} = 1$$

Hence the call is worthless if $\beta = 0$ but $\mu_X^m \leq 0$. From Remark 3 above it follows that when the drift is positive ($\mu_X^m > 0$) it holds that

$$Q^m\{\tau \leq \infty\} = e^{-2\cdot\mu_X^m \cdot X_t}$$

Thus the value of the perpetual call is

$$C_L\{\infty\} = \begin{cases} \omega_t \cdot \left(1 - \left(\frac{\omega_t}{L_t}\right)^{-2\frac{r-\alpha+\frac{1}{2}\sigma^2}{\sigma^2}}\right) & \text{when} \quad \alpha \leq r + \frac{1}{2}\sigma^2 \text{ and } \beta = 0 \\ 0 & \text{when} \quad \alpha \geq r + \frac{1}{2}\sigma^2 \text{ or } \beta > 0 \end{cases}$$

Now consider second term of Definition 1:

$$\begin{aligned} \$ &= \lim_{T\to\infty} E^B\left[\int_t^T e^{-r(s-t)}\beta\omega_s I_{\{\tau \nless s\}} ds\right] \\ &= \beta \lim_{T\to\infty} \int_t^T e^{-r(s-t)} E^B\left[\omega_s\right] E^\omega \left[I_{\{\tau \nless s\}}\right] ds \\ &= \beta\omega_t \int_t^\infty e^{-\beta(s-t)}\left(1 - Q^\omega\left\{\tau \leq s\right\}\right) ds \end{aligned}$$

Using that

$$Q^\omega\left\{\tau \leq s\right\} = \int_t^s f^\omega\left\{X_t; z\right\} dz$$

and applying integration by parts I arrive at

$$\$ = \beta\omega_t\left(\left[-\frac{e^{-\beta(s-t)}}{\beta}\left(1 - Q^\omega\left\{\tau \leq s\right\}\right)\right]_t^\infty - \int_t^\infty \frac{e^{-\beta(s-t)}}{\beta} f\omega\left\{X_t; s\right\} ds\right)$$

(Since by definition $\$ = 0$ when $\beta = 0$, I only need consider $\beta > 0$.) Finally, using Remark 3 again I obtain

$$\$ = \begin{cases} \omega_t \cdot \left(1 - \left(\frac{\omega}{N}\right)^{-\frac{\sqrt{\left(\mu_X{}^\omega\right)^2 + 2\beta} + \mu_X{}^\omega}{\sigma}}\right) & \text{when} \quad \beta > 0 \\ 0 & \text{when} \quad \beta = 0 \end{cases}$$

In a sense $C_L\{\infty\}$ and $\$$ are "mutually exclusive". Note also that $\lim_{\beta\to 0} \$ = C_L\{\infty\}$.

Lemma 1 follows directly from $\Omega = C_L\{\infty\} + \$$.

2.3. Deriving Lemmas 2 and 3. Standard theory tells us that

$$\begin{aligned} G\left\{\omega_t, t; \cdot\right\} &= E^B\left[e^{-r(\tau-t)}\right] \\ G^\alpha\left\{\omega_t, t; t, \cdot\right\} &= E^B\left[e^{-r(\tau-t)}e^{\alpha(\tau-t)}\right] \end{aligned}$$

For a general parameter ρ it holds that

$$E^B\left[e^{-\rho(\tau-t)}\right] = \int_t^\infty e^{-\rho(s-t)} f^B\left\{X_t; s\right\} ds$$

Applying Remark 3 this is equal to

$$\int_t^\infty e^{-\rho(s-t)} f^B\left\{X_t; s\right\} ds = e^{-X_t\left(\sqrt{\left(\mu_X^B\right)^2 + 2\rho} + \mu_X^B\right)}$$

and the lemmas follow immediately.

2.4. Deriving Proposition 1. First, rewrite equation (6) as

$$\mathcal{E}\{\omega_t, t; \cdot\} \equiv \begin{cases} \lim_{T\to\infty} E^B\left[e^{-r(T-t)}(\omega_T - N_T)\, I_{\{\tau \not< T\}}\right] \\ +E^B\left[\int_t^\infty e^{-r(s-t)}\beta\omega_s I_{\{\tau\not< s\}}ds\right] \\ -E^B\left[\int_t^\infty e^{-r(s-t)}(1-\zeta)\, C_s I_{\{\tau\not< s\}}ds\right] \\ +E^B\left[\int_t^\infty e^{-r(s-t)}d\{\omega_s, s; \cdot\}\, I_{\{\tau\not< s\}}ds\right] \\ +E^B\left[e^{-r\tau}\varepsilon_\tau L_\tau\right] \end{cases} \tag{18}$$

Now split the coupon stream (row three) into two parts-debt service to current debtholders and debt service to future debtholders:

$$\begin{aligned} & \int_t^\infty e^{-r(s-t)}(1-\zeta)\, C_s I_{\{\tau\not< s\}}ds \\ = & \int_t^\infty e^{-r(s-t)}(1-\zeta)\, C_t I_{\{\tau\not< s\}}ds \\ & + \int_t^\infty e^{-r(s-t)}(1-\zeta)\, C_t \left(e^{\alpha(s-t)} - 1\right) I_{\{\tau\not< s\}}ds \end{aligned} \tag{19}$$

(The coupon accruing to debt issued after time t, at time s, is $C_s - C_t = C_t e^{\alpha(s-t)} - C_t$.) The value of future borrowed money must be equal to the value of total debt service to future borrowed money plus the value of payouts in default accruing to future borrowers. Formally

$$\begin{aligned} & E^B\left[\int_t^\infty e^{-r(s-t)}d\{\omega_s, s; \cdot\}\, I_{\{\tau\not< s\}}ds\right] \\ = & E^B\left[\int_t^\infty e^{-r(s-t)}C_t\left(e^{\alpha(s-t)} - 1\right) I_{\{\tau\not< s\}}ds\right] \\ & +\delta\cdot E^B\left[e^{-r(\tau-t)}(L_\tau - L_t)\right] \end{aligned} \tag{20}$$

Applying (19) and (20) to (18) and cancelling terms, we arrive at

$$\begin{aligned} E\{\omega_t, t; \cdot\} = & \lim_{T\to\infty} E^B\left[(\omega_T - N_T)\, e^{-r(T-t)} I_{\{\tau\not< T\}}\right] \\ & +E^B\left[\int_t^T e^{-r(s-t)}\beta\omega_s I_{\{\tau\not< s\}}ds\right] \\ & -C_t\cdot E^B\left[\int_t^\infty e^{-r(s-t)} I_{\{\tau\not< s\}}ds\right] \\ & +\zeta C_t\cdot E^B\left[\int_t^\infty e^{-(r-\alpha)(s-t)} I_{\{\tau\not< s\}}ds\right] \\ & +\delta L_t\cdot E^B\left[e^{-(r-\alpha)(\tau-t)} - e^{-r(\tau-t)}\right] \\ & +\varepsilon L_t\cdot E^B\left[e^{-(r-\alpha)(\tau-t)}\right] \end{aligned}$$

Proposition 1, for $r \neq \alpha$, follows from the results in subsections 2.2 and 2.3. The result for $r = \alpha$ is obtained as the limit $\alpha \to r$.

2.5. Deriving Lemma 7. The value of receiving a dollar if default occurs prior to T must be equal to receiving a dollar-in-default claim with no maturity, less a claim where you receive a dollar in default conditional on it *not* occurring prior to T. Formally

$$\begin{aligned} G\{\omega_t;\sigma\,|\tau < T\} &= E^B\left[e^{-r(\tau-s)}\cdot I_{\{\tau<T\}}\right] \\ &= G\{\omega_t;\sigma\} - E^B\left[e^{-r(T-t)}G\{\omega_T;\sigma\}\cdot I_{\{\tau\nless T\}}\right] \end{aligned}$$

Using the Q^G-measure, we can separate the variables (cf. Subsection 2.2 above) within the expectation brackets.

$$\begin{aligned} G\{\omega_t;\sigma\,|\tau < T\} &= G\{\omega_t;\sigma\} - e^{-r(T-t)}E^B\left[G\{\omega_T;\sigma\}\right]\cdot E^G\left[I_{\{\tau\nless T\}}\right] \\ &= G\{\omega_t;\sigma\}\cdot\left(1 - Q^G\{\tau \nleq T\}\right) \end{aligned}$$

The probability $Q^G\{\tau \nless T\}$ is known from Lemma 4.

2.6. Deriving Proposition 3. Consider the expression (for notational convenience, set the time of pricing equal to zero: $t = 0$)

$$Q^m(A_T) = E^m\left[I_{\{\omega_T>F,\omega_S>\overline{\omega},\tau\nleq T\}}\right] \tag{21}$$

for $m = \omega, B, G$. Defining the normalised exercise prices at time S and T

$$\overline{\omega}_X \equiv \frac{1}{\sigma}\ln\frac{\overline{\omega}}{L_S} \qquad\qquad F_X \equiv \frac{1}{\sigma}\ln\frac{F}{L_T}$$

we can write

$$Q^m(A_T) = E^m\left[I_{\{X_T>F_X,X_S>\overline{\omega}_X,\tau\nleq T\}}\right] \tag{22}$$

To remove the drift of the X-process, define (implicitly) the measure Q^X (under which X is a Wiener process) with associated Radon-Nikodym derivative:

$$dQ^m = R^{X\to m}dQ^X \quad \text{with} \quad R^{X\to m} = e^{h^{X\to m}W_T^X - \frac{1}{2}\left(h^{X\to m}\right)^2 T}$$

With the help of this new probability measure I can rewrite (22) as

$$\begin{aligned} Q^m(A_T) &= E^X\left[R^{X\to m}\cdot I_{\{X_T>F_X,X_S>\overline{\omega}_X,\tau\nleq T\}}\right] \\ &= \int_{F_X}^{\infty}\int_{\overline{\omega}_X}^{\infty} R^{X\to m}Q^X\left(X_T\in dx_t, X_S\in dx_t,\tau\nleq T\right) \qquad (23) \end{aligned}$$

where x_t denotes a specific realisation of the default process X_t.

The term $Q^X(X_T\in dx_T, X_S\in dx_S,\tau\nleq T)$ can safely be thought of as the probability of the Wiener process X_t passing through the infinitesimal intervals dx_S at S and dx_T at T without hitting or having hit zero. The next step is to derive this density.

2.6.1. *The Q^X-density function.* The corresponding cumulative probability can, by complementarity, be rewritten as

$$
\begin{aligned}
& Q^X\left(X_T > F_X, X_S > \overline{\omega}_X, \tau \nleq T\right) \\
= \; & Q^X\left(X_T > F_X, X_S > \overline{\omega}_X\right) \\
& - Q^X\left(X_T > F_X, X_S > \overline{\omega}_X, \tau_S < S\right) \\
& - Q^X\left(X_T > F_X, X_S > \overline{\omega}_X, S < \tau_T < T\right) \\
& + Q^X\left(X_T > F_X, X_S > \overline{\omega}_X, \tau_S < S, S < \tau_T < T\right)
\end{aligned}
$$

(Here, I use τ_S to denote the first hitting time in the interval $[0, S]$ and τ_T to denote the first hitting time in the interval $(S, T]$.) Applying the reflection principle on the three latter terms yields

$$
\begin{cases}
Q^X\left(X_T > F_X, X_S > \overline{\omega}_X, \tau_S < S\right) = Q^X\left(X_T < -F_X, X_S < -\overline{\omega}_X\right) \\
Q^X\left(X_T > F_X, X_S > \overline{\omega}_X, S < \tau_T < T\right) = Q^X\left(X_T < -F_X, X_S > \overline{\omega}_X\right) \\
Q^X\left(X_T > F_X, X_S > \overline{\omega}_X, \tau_S < S, S < \tau_T < T\right) = Q^X\left(X_T > F_X, X_S < -\overline{\omega}_X\right)
\end{cases}
$$

Summing up, and expressing the conditions as the probability of X being *less than* a constant produces the following formula.

$$
\begin{aligned}
& Q^X\left(X_T > F_X, X_S > \overline{\omega}_X, \tau \nleq T\right) \\
= \; & Q^X\left(X_T < 2X_0 - F_X, X_S < 2X_0 - \overline{\omega}_X\right) \\
& - Q^X\left(X_T < -F_X, X_S < -\overline{\omega}_X\right) \\
& - Q^X\left(X_T < -F_X, -X_S < 2X_0 - \overline{\omega}_X\right) \\
& + Q^X\left(-X_T < 2X_0 - F_X, X_S < -\overline{\omega}_X\right)
\end{aligned}
$$

The next step is to derive the corresponding density functions. We immediately see that the bivariate standardised cumulative functions are

$$
\begin{aligned}
& Q^X\left(X_T > F_X, X_S > \overline{\omega}_X, \tau \nleq T\right) \\
= \; & \phi\left(\frac{X_0 - F_X}{\sqrt{T}}, \frac{X_0 - \overline{\omega}_X}{\sqrt{S}}, \sqrt{\frac{S}{T}}\right) - \phi\left(\frac{-F_X - X_0}{\sqrt{T}}, \frac{-\overline{\omega}_X - X_0}{\sqrt{S}}, \sqrt{\frac{S}{T}}\right) \\
& - \phi\left(\frac{-F_X - X_0}{\sqrt{T}}, \frac{X_0 - \overline{\omega}_X}{\sqrt{S}}, -\sqrt{\frac{S}{T}}\right) + \phi\left(\frac{X_0 - F_X}{\sqrt{T}}, \frac{-\overline{\omega}_X - X_0}{\sqrt{S}}, -\sqrt{\frac{S}{T}}\right)
\end{aligned}
$$

Keeping in mind that integration starts at F_X and $\overline{\omega}_X$, respectively, and that the start value for the bankruptcy process is X_0, the density functions are

$$\begin{aligned}
Q^X\left(X_T \in dx_T, X_S \in dx_S, \tau \nleq T\right) &= f\left\{0, \sqrt{T}; 0, \sqrt{S}; \sqrt{\frac{S}{T}}\right\} dx_T\, dx_S \\
&\quad - f\left\{-2X_0, \sqrt{T}; -2X_0, \sqrt{S}; \sqrt{\frac{S}{T}}\right\} dx_T\, dx_S \\
&\quad - f\left\{-2X_0, \sqrt{T}; 0, \sqrt{S}; -\sqrt{\frac{S}{T}}\right\} dx_T\, dx_S \\
&\quad + f\left\{0, \sqrt{T}; -2X_0, \sqrt{S}; -\sqrt{\frac{S}{T}}\right\} dx_T\, dx_S
\end{aligned}$$

Having derived the Q^X-density function we can return to equation (23) and integrate.

2.6.2. *Integration.* Inserting the Radon-Nikodym derivative and the derived density function into equation (23) and changing integration variables to w_t (identical to X_t but starting at 0) we obtain:

$$\begin{aligned}
Q^m\left(A_T\right) &= e^{-\frac{1}{2}\left(h^{X\to m}\right)^2 T} \int_{F_X - X_0}^{\infty} \int_{\overline{\omega}_X - X_0}^{\infty} \\
&\left\{ \begin{array}{c}
e^{h^{X\to m} w_T} \times f\left\{0, \sqrt{T}; 0, \sqrt{S}; \sqrt{\frac{S}{T}}\right\} \\
-e^{h^{X\to m} w_T} \times f\left\{-2X_0, \sqrt{T}; -2X_0, \sqrt{S}; \sqrt{\frac{S}{T}}\right\} \\
-e^{h^{X\to m} w_T} \times f\left\{-2X_0, \sqrt{T}; 0, \sqrt{S}; -\sqrt{\frac{S}{T}}\right\} \\
+e^{h^{X\to m} w_T} \times f\left\{0, \sqrt{T}; -2X_0, \sqrt{S}; -\sqrt{\frac{S}{T}}\right\}
\end{array} \right. \quad dw_T\, dw_S
\end{aligned}$$

Completing the square yields

$$Q^m\left(A_T\right) = e^{-\frac{1}{2}\left(h^{X\to m}\right)^2 T} \int_{F_X - X_0}^{\infty} \int_{\overline{\omega}_X - X_0}^{\infty} ff\left\{w_t, w_t\right\} dw_T\, dw_S$$

with the bivariate density function $ff\{w_t, w_t\}$ given by

$$ff\{w_t, w_t\} \equiv \begin{cases} \begin{array}{c} e^{\frac{1}{2}\left(h^{X\to m}\right)^2 T} \\ \times f\left\{h^{X\to m}T, \sqrt{T}; h^{X\to m}S, \sqrt{S}; \sqrt{\frac{S}{T}}\right\} \end{array} \\ - \begin{array}{c} e^{-2X_0 h^{X\to m}+\frac{1}{2}\left(h^{X\to m}\right)^2 T} \\ \times f\left\{-2X_0 + h^{X\to m}T, \sqrt{T}; -2X_0 + h^{X\to m}S, \sqrt{S}; \sqrt{\frac{S}{T}}\right\} \end{array} \\ - \begin{array}{c} e^{-2X_0 h^{X\to m}+\frac{1}{2}\left(h^{X\to m}\right)^2 T} \\ \times f\left\{-2X_0 + h^{X\to m}T, \sqrt{T}; -h^{X\to m}S, \sqrt{S}; -\sqrt{\frac{S}{T}}\right\} \end{array} \\ + \begin{array}{c} e^{\frac{1}{2}\left(h^{X\to m}\right)^2 T} \\ \times f\left\{h^{X\to m}T, \sqrt{T}; -2X_0 - h^{X\to m}S, \sqrt{S}; -\sqrt{\frac{S}{T}}\right\} \end{array} \end{cases}$$

Cancelling terms ($e^{\frac{1}{2}\left(h^{X\to m}\right)^2 T}$ *etc.*) and integrating gives the result.

2.7. Deriving Corollary 1. From equation (9)

$$\mathcal{C}_L\{\omega_t, t; F_i, t_i\,|D_S > K\} = e^{-r(S-t)} \cdot E^B\left[\mathcal{C}_L\{\omega_S, S; F_i, t_i\} \cdot I_{\{D_S>K\}}\,|\Im_t\right]$$

The value of underlying call at expiration of the compound option is

$$\mathcal{C}_L\{\omega_S, S; F_i, t_i\} = e^{-r(T-S)}E^B\left[(\omega_T - F)\cdot I_{\{\omega_T>F,\tau\nleq T\}}\,|\Im_S\right]$$

Inserting the latter into the former yields

$$\mathcal{C}_L\{\omega_t, t; F_i, t_i\,|D_S > K\} = e^{-r(T-t)} \cdot E^B\left[(\omega_T - F)\cdot I_{\{\omega_T>F,\tau\nleq T\}} \cdot I_{\{D_S>K\}}\,|\Im_t\right]$$

Noting that $\{\omega_T > F, \tau \nleq T\} \cap \{D_S > K\} = A_T$ (as defined on page 35) this expression can be rewritten

$$\begin{aligned} \mathcal{C}_L\{\omega_t, t; F_i, t_i\,|D_S > K\} &= e^{-r(T-t)} \cdot E^B\left[\omega_T \cdot I_{\{A_T\}}\right] \\ &\quad -e^{-r(T-t)} \cdot F \cdot E^B\left[I_{\{A_T\}}\right] \end{aligned}$$

To separate the two terms within the expectations brackets in the first term, I change probability measure as indicated in section 2.1.2 on page 78,

$$\begin{aligned} \mathcal{C}_L\{\omega_t, t; F_i, t_i\,|D_S > K\} &= e^{-r(T-t)} \cdot E^\omega\left[\omega_T\right] \cdot E^B\left[I_{\{A_T\}}\right] \\ &\quad -e^{-r(T-t)} \cdot F \cdot E^B\left[I_{\{A_T\}}\right] \end{aligned}$$

which yields the result.

2.8. Deriving Corollary 3. The value of receiving a dollar if default occurs in the interval $(S,T]$ conditional on $D_S > K$, i.e., $\omega_S > \overline{\omega}$, must be equal to receiving a long position in a (perpetual) dollar-in-default claim at S conditional $\omega_S > \overline{\omega}$ and no default prior to S, less receiving a short position in a (perpetual) dollar-in-default claim at T conditional $\omega_S > \overline{\omega}$ and no default prior to T. Formally

$$\begin{aligned}\mathcal{G}\left\{\omega_t, t\,|\tau \leq T, D_S > K\right\} &= e^{-r(S-t)}E^B\left[G\left\{\omega_S, S\right\}\cdot I_{\{\overline{\omega}<\omega_S,\tau\nleq S\}}\right] \\ &\quad -e^{-r(T-t)}E^B\left[G\left\{\omega_T, T\right\}\cdot I_{\{\overline{\omega}<\omega_S,\tau\nleq T\}}\right]\end{aligned}$$

Change measure as in subsection 2.5:

$$\begin{aligned}\mathcal{G}\left\{\omega_t, t\,|\tau \leq T, D_S > K\right\} &= e^{-r(S-t)}E^B\left[G\left\{\omega_S, S\right\}\right]E^G\left[I_{\{\overline{\omega}<\omega_S,\tau\nleq S\}}\right] \\ &\quad -e^{-r(T-t)}E^B\left[G\left\{\omega_T, T\right\}\right]E^G\left[I_{\{\overline{\omega}<\omega_S,\tau\nleq T\}}\right]\end{aligned}$$

The expected return of G under the pricing measure Q^B is r and so

$$\begin{aligned}\mathcal{G}\left\{\omega_t, t\,|\tau \leq T, D_S > K\right\} &= G\left\{\omega_t, t\right\}\cdot Q^G\left\{\overline{\omega} < \omega_S, \tau \nleq S\right\} \\ &\quad -G\left\{\omega_t, t\right\}\cdot Q^G\left\{\overline{\omega} < \omega_S, \tau \nleq T\right\}\end{aligned}$$

Noting that $Q^G\left\{\overline{\omega} < \omega_S, \tau \nleq T\right\} = Q^G\left\{\overline{\omega} < \omega_S, \tau \nleq T, L_T < \omega_T\right\}$ gives the corollary.

3. Formulae

The intention of this section is to present all the formulae necessary to use the framework to price a corporate coupon bond. Specifically, these are the derivative of equity with respect to asset value (which is used in the estimation process) and the standard deviations of estimators (which are used to measure the efficiency of a given price estimate. The standard deviations require the derivatives with respect to asset value and volatility (being calculated on the form $\gamma_{\widehat{\pi}} = \gamma_{\widehat{\sigma}}\left(\frac{\partial\pi}{\partial\omega}\frac{\partial\vartheta}{\partial\sigma} + \frac{\partial\pi}{\partial\sigma}\right)$). In sum, we consequently need the derivatives with respect to asset value and volatility for all claims we want to price.

The section is organised by claims. First the building blocks at the aggregate level (G, G^{α} and Ω), followed by the equity and asset value. Thereafter are the building blocks for the bond (H_L and $G\{\cdot|\tau \leq T\}$), followed by the derivatives of the corporate bond itself. For notational convenience, I assume all pricing takes place at $t = 0$ and drop exuberant parameters. E.g., I use L to denote L_0.

3.1. The dollar-in-default claim: G. The derivative with respect to asset value is

$$\frac{\partial G}{\partial\omega} = -\frac{\theta\{r\}}{\omega}G$$

The derivative with respect to volatility is

$$\frac{\partial G}{\partial\sigma} = -\frac{\partial\theta\{r\}}{\partial\sigma}G\cdot\ln\frac{\omega}{L}$$

where

$$\frac{\partial\theta\{\rho\}}{\partial\sigma} = -\frac{2}{\sigma^3}\left(\frac{\left(\frac{r-\beta-\alpha}{\sigma^2} - 0.5\right)(r-\beta-\alpha)+\rho}{\sqrt{\left(\frac{r-\beta-\alpha}{\sigma^2} - 0.5\right)^2 + \frac{2\rho}{\sigma^2}}} + r - \beta - \alpha\right)$$

3.2. The "α-increasing" dollar-in-default claim: G^{α}. The derivative with respect to asset value is

$$\frac{\partial G^{\alpha}}{\partial\omega} = -\frac{\theta\{r-\alpha\}}{\omega}G^{\alpha}$$

The derivative with respect to volatility is

$$\frac{\partial G^{\alpha}}{\partial\sigma} = -\frac{\partial\theta\{r-\alpha\}}{\partial\sigma}G^{\alpha}\cdot\ln\frac{\omega}{L}$$

The value of $\frac{\partial\theta\{r-\alpha\}}{\partial\sigma}$ is in the previous subsection.

3.3. The asset claim: Ω. The derivative with respect to asset value is

$$\frac{\partial\Omega}{\partial\omega} = \begin{cases} \begin{array}{l} 1-\left(\frac{\omega}{L}\right)^{-\theta^{\omega}} \\ +\theta^{\omega}\cdot\left(\frac{\omega}{L}\right)^{-\theta^{\omega}} \end{array} & \text{when} \quad \begin{array}{c} (\beta > 0) \\ \text{or} \\ (\beta = 0 \text{ and } \alpha \leq r + \frac{1}{2}\sigma^2) \end{array} \\ 0 & \text{when} \quad (\beta = 0 \textit{ and } \alpha \geq r + \frac{1}{2}\sigma^2) \end{cases}$$

The derivative with respect to volatility is

$$\frac{\partial\Omega}{\partial\sigma}=\begin{cases}\omega\frac{\partial\theta^{\omega}}{\partial\sigma}\left(\frac{\omega}{L}\right)^{-\theta^{\omega}}\ln\frac{\omega}{L} & \text{when} \quad \begin{array}{c}(\beta>0)\\ \text{or}\\ \left(\beta=0 \text{ and } \alpha\leq r+\frac{1}{2}\sigma^2\right)\end{array}\\ 0 & \text{when} \quad \left(\beta=0 \textit{ and } \alpha\geq r+\frac{1}{2}\sigma^2\right)\end{cases}$$

3.4. Equity. The derivative with respect to asset value (required for the estimation process) is

$$\begin{aligned}\frac{\partial\mathcal{E}}{\partial\omega} &= \frac{\partial\Omega}{\partial\omega}\\ &+N\cdot\frac{\partial G}{\partial\sigma}\\ &+\zeta C\cdot\begin{cases}\frac{1}{r-\alpha}\cdot\frac{\partial G^{\alpha}}{\partial\sigma} & \text{when } r\neq\alpha\\ \frac{2}{\sigma^2\omega} & \text{when } r=\alpha\end{cases}\\ &+\delta N\cdot\left(\frac{\partial G^{\alpha}}{\partial\sigma}-\frac{\partial G}{\partial\sigma}\right)\\ &+\varepsilon L\cdot\frac{\partial G^{\alpha}}{\partial\sigma}\end{aligned}$$

The expressions for the derivatives of the building blocks were given in the previous subsections.

The derivative with respect to volatility (needed to calculate the standard deviation of asset value in the next subsection) is

$$\begin{aligned}\frac{\partial\mathcal{E}}{\partial\sigma} &= \frac{\partial\Omega}{\partial\sigma}\\ &+N\cdot\frac{\partial G}{\partial\sigma}\\ &+\zeta C\begin{cases}\frac{1}{r-\alpha}\cdot\frac{\partial G^{\alpha}}{\partial\sigma} & \text{when } r\neq\alpha\\ \ln\left(\frac{\omega_t}{N_t}\right)^{\frac{4}{\sigma^3}} & \text{when } r=\alpha\end{cases}\\ &+\delta N\cdot\left(\frac{\partial G^{\alpha}}{\partial\sigma}-\frac{\partial G}{\partial\sigma}\right)\\ &+\varepsilon L\cdot\frac{\partial G^{\alpha}}{\partial\sigma}\end{aligned}$$

The expressions for the derivatives of the building blocks were given in the previous subsections.

The endogenous reorganisation barrier is the value of assets at which the value of equity in solvency equals the value of equity in reorganisation.

$$\mathcal{E}\left\{\omega_t,t;\cdot\right\}=\varepsilon\cdot L_t$$

Applying a smooth pasting condition then implies that

$$\frac{\partial\mathcal{E}\left\{\omega_t,t;\cdot\right\}}{\partial\omega_t}\bigg|_{\omega_t=L_t}=\varepsilon$$

The optimal barrier can be solved for in closed form. As an example, the following expression gives the value of the barrier when $\beta > 0$ and $r \neq \alpha$.

$$L_t \big|_{\beta>0,r\neq\alpha} = \frac{\zeta \frac{r}{r-\alpha} \theta \{r-\alpha\} - \theta \{r\}}{\varepsilon \cdot (1+\theta\{r-\alpha\}) - \theta^{\omega} + \delta \frac{N_0}{L_0} \cdot (\theta\{r\} - \theta\{r-\alpha\})} N_t$$

As can be seen, the endogenous barrier grows at a rate α.

3.5. Asset value. The estimated value of assets is

$$\widehat{\omega} = \vartheta \left\{ \mathcal{E}_n^{obs}, t_n; \widehat{\sigma} \right\}$$

An estimate of the standard deviation is obtained as

$$\widehat{\gamma}_{\widehat{\omega}} = \widehat{\gamma}_{\widehat{\sigma}} \cdot \frac{d\,\vartheta \left\{ \mathcal{E}_n^{obs}, t_n; \widehat{\sigma} \right\}}{d\,\sigma} \Big|_{\sigma=\widehat{\sigma}}$$

The derivative of asset value with respect to sigma is obtained by differentiating the following identity with respect to sigma

$$\mathcal{E}_n^{obs} \equiv \mathcal{E} \left\{ \vartheta \left\{ \mathcal{E}_n^{obs}, t_n; \widehat{\sigma} \right\}, t_n; \widehat{\sigma} \right\}$$

which results in

$$0 = \frac{\partial\,\mathcal{E}}{\partial\,\omega} \frac{\partial\,\vartheta}{d\,\sigma} + \frac{\partial\,\mathcal{E}}{\partial\,\sigma}$$

Hence

$$\frac{\partial\,\vartheta}{d\,\sigma} = -\frac{\frac{\partial\,\mathcal{E}}{\partial\,\sigma}}{\frac{\partial\,\mathcal{E}}{\partial\,\omega}} \tag{24}$$

The derivatives of equity were given in Section 3.4 above.

3.6. The down-and-out heaviside: H_L. The value of the heaviside is

$$H_L = e^{-rT} \cdot Q^B \{ \tau \not< T, \omega_T > F \}$$

where the probability is given by

$$\begin{aligned} Q^B \{ \tau \not< T, \omega_T > F \} &= \phi \{ d^B \} \\ &\quad - \left(\frac{\omega}{L} \right)^{-2 \frac{r-\beta-\alpha-0.5\sigma^2}{\sigma^2}} \cdot \phi \{ D^B \} \end{aligned}$$

$$\begin{aligned} d^B &= \frac{\ln \frac{\omega}{F e^{-\alpha(T-t)}} + \left(r - \beta - \alpha - 0.5\sigma^2 \right) T}{\sigma \sqrt{T}} \\ D^B &= \frac{\ln \frac{L^2}{\omega F e^{-\alpha(T-t)}} + \left(r - \beta - \alpha - 0.5\sigma^2 \right) T}{\sigma \sqrt{T}} \end{aligned}$$

The derivative with respect to asset value is

$$\frac{\partial\,H_N \{T\}}{\partial\,\omega} = e^{-rT} \cdot \frac{\partial\,Q^B \{ \tau \not< T, \omega_T > F \}}{\partial\,\omega}$$

where

$$\begin{aligned}\frac{\partial Q^B\{\tau \not< T, \omega_T > F\}}{\partial \omega} &= \phi'\{d^B\}\cdot\frac{1}{\omega\sigma\sqrt{T}} \\ &+2\frac{r-\beta-\alpha-0.5\sigma^2}{\sigma^2}\frac{1}{\omega_t}\left(\frac{\omega_t}{L_t}\right)^{-2\frac{r-\beta-\alpha-0.5\sigma^2}{\sigma^2}}\cdot\phi\{D^B\} \\ &+\frac{\left(\frac{\omega}{L}\right)^{-2\frac{r-\beta-\alpha-0.5\sigma^2}{\sigma^2}}}{\omega\sigma\sqrt{T}}\cdot\phi'\{D^B\}\end{aligned}$$

with

$$\phi'\{x\} = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

The derivative with respect to volatility is

$$\frac{\partial H_L}{\partial \sigma} = e^{-rT}\cdot\frac{\partial Q^B\{\tau \not< T, \omega_T > F\}}{\partial \sigma}$$

and using that $\frac{\partial d^B}{\partial \sigma} = -\frac{1}{\sigma}d^\omega$ and that $\frac{\partial D^B}{\partial \sigma} = -\frac{1}{\sigma}D^\omega$, the derivative of the probability is

$$\begin{aligned}\frac{\partial Q^B\{\tau \not< T, \omega_T > F\}}{\partial \sigma} &= -\phi'\{d^B\}\frac{1}{\sigma}d^\omega \\ &-4\frac{r-\beta-\alpha}{\sigma^3}\left(\frac{\omega}{L}\right)^{2\frac{r-\beta-\alpha-0.5\sigma^2}{\sigma^2}}\ln\frac{\omega}{L}\cdot\phi\{D^B\} \\ &+\left(\frac{\omega}{L}\right)^{-2\frac{r-\beta-\alpha-0.5\sigma^2}{\sigma^2}}\cdot\phi'\{D^B\}\frac{1}{\sigma}D^\omega\end{aligned}$$

with

$$\begin{aligned}d^\omega &= \frac{\ln\frac{\omega}{Fe^{-\alpha(T-t)}} + \left(r-\beta-\alpha+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \\ D^\omega &= \frac{\ln\frac{L^2}{\omega Fe^{-\alpha(T-t)}} + \left(r-\beta-\alpha+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\end{aligned}$$

3.7. The (finite maturity) dollar-in-default claim: $G\{\cdot|\tau < T\}$. The value of a dollar-in-default claim is

$$G\{\cdot|\tau < T\} = G\cdot\left(1 - Q^G\{\tau \not< T, \omega_T > L_T\}\right)$$

where the probability is given by

$$\begin{aligned}Q^G\{\tau \not< T, \omega_T > F\} &= \phi\{d^G\} \\ &-\left(\frac{\omega}{L}\right)^{-2\frac{r-\beta-\alpha-\left(\frac{1}{2}+\theta\{r\}\right)\sigma^2}{\sigma^2}}\phi\{D^G\}\end{aligned}$$

with

$$\begin{aligned}d^G &= \frac{\ln\frac{\omega}{Fe^{-\alpha(T-t)}} + \left(r-\beta-\alpha-\left(\frac{1}{2}+\theta\{r\}\right)\sigma^2\right)T}{\sigma\sqrt{T}} \\ D^G &= \frac{\ln\frac{L^2}{\omega Fe^{-\alpha(T-t)}} + \left(r-\beta-\alpha-\left(\frac{1}{2}+\theta\{r\}\right)\sigma^2\right)T}{\sigma\sqrt{T}}\end{aligned}$$

Note that

- $r-\beta-\alpha-\left(\frac{1}{2}+\theta\{r\}\right)\sigma^2=-\sqrt{\left(\frac{r-\beta-\alpha-\frac{1}{2}\sigma^2}{\sigma}\right)^2+2r}$

The derivative with respect to asset value is

$$\begin{aligned}\frac{\partial G\{\cdot|\tau<T\}}{\partial\omega} &= \frac{\partial G}{\partial\omega}\cdot\left(1-Q^G\{\tau\nless T,\omega_T>L_T\}\right)\\ &\quad -G\frac{\partial Q^G\{\tau\nless T,\omega_T>L_T\}}{\partial\omega}\end{aligned}$$

where

$$\begin{aligned}\frac{\partial Q^G\{\tau\nless T,\omega_T>F\}}{\partial\omega} &= \phi'\left\{\frac{\ln\frac{\omega}{Fe^{-\alpha(T-t)}}+\left(r-\beta-\alpha-0.5\sigma^2\right)T}{\sigma\sqrt{T}}\right\}\frac{1}{\omega\sigma\sqrt{T}}\\ &\quad -\left(2\frac{r-\beta-\alpha-\left(\frac{1}{2}+\theta\{r\}\right)\sigma^2}{\sigma^2}\frac{1}{\omega}-\frac{1}{\omega\sigma\sqrt{T}}\right)\\ &\quad \times\left(\frac{\omega}{L}\right)^{-2\frac{r-\beta-\alpha-\left(\frac{1}{2}+\theta\{r\}\right)\sigma^2}{\sigma^2}}\\ &\quad \times\phi\left\{\frac{\ln\frac{L^2}{Fe^{-\alpha(T-t)}\omega}+\left(r-\beta-\alpha-\left(\frac{1}{2}+\theta\{r\}\right)\sigma^2\right)T}{\sigma\sqrt{T}}\right\}\end{aligned}$$

and $\frac{\partial G}{\partial\omega}$ is given above.

The derivative with respect to volatility is

$$\begin{aligned}\frac{\partial G\{\cdot|\tau<T\}}{\partial\sigma} &= \frac{\partial G}{\partial\sigma}\cdot\left(1-Q^G\{\tau\nless T,\omega_T>F\}\right)\\ &\quad -G\frac{\partial Q^G\{\tau\nless T,\omega_T>F\}}{\partial\sigma}\end{aligned}$$

We know $\frac{\partial G}{\partial\sigma}$, $Q^G\{\tau\nless T,\omega_T>F\}$ and G from before. The derivative of the probability is

$$\begin{aligned}\frac{\partial Q^G\{\tau\nless T,\omega_T>F\}}{\partial\sigma} &= \phi'\left\{d^G\right\}\frac{\partial d^G}{\partial\sigma}\\ &\quad -\left(4\frac{r-\beta-\alpha}{\sigma^3}+\frac{\partial\theta\{r\}}{\partial\sigma}\right)\left(\frac{\omega}{L}\right)^{-2\frac{r-\beta-\alpha-\left(\frac{1}{2}+\theta\{r\}\right)\sigma^2}{\sigma^2}}\\ &\quad \times\ln\frac{\omega}{L}\cdot\phi\left\{D^G\right\}\\ &\quad -\left(\frac{\omega}{L}\right)^{-2\frac{r-\beta-\alpha-\left(\frac{1}{2}+\theta\{r\}\right)\sigma^2}{\sigma^2}}\phi'\left\{D^G\right\}\frac{\partial D^G}{\partial\sigma}\end{aligned}$$

where

$$\frac{\partial d^G}{\partial\sigma}=-\frac{\ln\frac{\omega}{Fe^{-\alpha(T-t)}}+(r-\beta-\alpha)T}{\sigma^2\sqrt{T}}-\frac{\partial\theta\{r\}}{\partial\sigma}\sqrt{T}\sigma-\left(\frac{1}{2}+\theta\{r\}\right)\sqrt{T}$$

and

$$\frac{\partial D^G}{\partial\sigma}=-\frac{\ln\frac{N^2}{\omega Fe^{-\alpha(T-t)}}+(r-\beta-\alpha)T}{\sigma^2\sqrt{T}}-\frac{\partial\theta\{r\}}{\partial\sigma}\sqrt{T}\sigma-\left(\frac{1}{2}+\theta\{r\}\right)\sqrt{T}$$

3.8. The straight coupon bond. An estimate for the value of the straight coupon bond is

$$\begin{aligned}\widehat{D} &= \sum_{i=1}^{M-1} c \cdot H_L\left\{\widehat{\omega}_n, t_n; L_{t_i}, t_i, \widehat{\sigma}\right\} \\ &\quad + (c+P)\cdot H_L\left\{\widehat{\omega}_n, t_n; L_T, T, \widehat{\sigma}\right\} \\ &\quad + \psi P \cdot G\left\{\widehat{\omega}_n, t_n; \widehat{\sigma}\,|\tau < T\right\}\end{aligned}$$

An estimate for the standard deviation of the price estimate is

$$\widehat{\gamma}_{\widehat{D}} = \widehat{\gamma}_{\widehat{\sigma}} \cdot \frac{dD}{d\sigma}\Big|_{\sigma=\widehat{\sigma}} \tag{25}$$

$$\begin{aligned}\frac{dD}{d\sigma} &= \sum_{i=1}^{M-1} c\cdot \frac{dH_L\left\{\vartheta\left\{\mathcal{E}_n^{obs}, t_n; \sigma\right\}, t_n; L_{t_i}, t_i, \sigma\right\}}{d\sigma} \\ &\quad + (c+P)\cdot \frac{dH_L\left\{\vartheta\left\{\mathcal{E}_n^{obs}, t_n; \sigma\right\}, t_n; L_T, T, \sigma\right\}}{d\sigma} \\ &\quad + \psi P\cdot \frac{dG\left\{\vartheta\left\{\mathcal{E}_n^{obs}, t_n; \sigma\right\}, t_n; \sigma\,|\tau < T\right\}}{d\sigma}\end{aligned}$$

The full differentials of the heaviside and the dollar-in-default claim are

$$\frac{dH_L}{d\sigma} = \frac{\partial H_L}{\partial \omega}\frac{\partial \vartheta}{\partial \sigma} + \frac{\partial H_L}{\partial \sigma}$$

and

$$\frac{dG\{\cdot\,|\tau<T\}}{d\sigma} = \frac{\partial G\{\cdot\,|\tau<T\}}{\partial \omega}\frac{\partial \vartheta}{\partial \sigma} + \frac{\partial G\{\cdot\,|\tau<T\}}{\partial \sigma}$$

I use the GAUSS Constrained Maximum Likelihood Application to estimate the standard deviation of the volatility, $\gamma_{\widehat{\sigma}}$. The estimates this application provides are based on a Taylor-series approximation to the likelihood function (see e.g. Amemiya (1985), page 111) which yields that (letting $\boldsymbol{\eta} = (\sigma, \lambda)$)

$$\sqrt{n}\left(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta}\right) \xrightarrow{L} N\left(0, A^{-1}BA^{-1}\right)$$

where

$$\begin{aligned}A &= E\left[\frac{\partial^2 L}{\partial \boldsymbol{\eta}\,\partial \boldsymbol{\eta}'}\right] \rightarrow \widehat{A} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2 L_i}{\partial \boldsymbol{\eta}\,\partial \boldsymbol{\eta}'} \\ B &= E\left[\left(\frac{\partial L}{\partial \boldsymbol{\eta}}\right)'\left(\frac{\partial L}{\partial \boldsymbol{\eta}}\right)\right] \rightarrow \widehat{B} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\partial L_i}{\partial \boldsymbol{\eta}}\right)'\left(\frac{\partial L_i}{\partial \boldsymbol{\eta}}\right)\end{aligned}$$

From *Gauss Applications - Constrained Maximum Likelihood* (1995).

Thus $\hat{\gamma}_{\widehat{\sigma}} = \begin{pmatrix} 1 & 0 \end{pmatrix}\left(\widehat{A}^{-1}\widehat{B}\widehat{A}^{-1}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

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