Credit Risk in Corporate Securities and Derivatives

Valuation and Optimal Capital Structure Choice

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Jan Ericsson

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Preface

In 1973 Fisher Black and Myron Scholes published their famous option pricing formula[1]. Assuming perfect markets and the absence of arbitrage, they were able to show how an option should be priced *relative* to the underlying stock.

They also pointed out that many claims on a firm's balance sheet were in fact analogous to options and that their newly developed theory could be applied to for example corporate debt. This particular extension was first provided by Robert Merton and was published a year later[2].

The basic idea of such models is to assume that the value of the firm's assets follows a given stochastic process. Then, invoking the same arbitrage argument as when pricing stock options, corporate liabilities can be priced relative to the value of the firm, or relative to each other. This idea has since been applied to the pricing of more complex capital structures and more complex financial instruments.

To date, there have been few empirical tests of this type of models. The most cited paper is one by Philip Jones[3], Scott Mason and Eric Rosenfeld. Their results were discouraging in that their model tended to systematically overprice the corporate bonds in their sample. They argued that perhaps the assumption of non random risk-free interest rates was too restrictive and that relaxing it would be a step in the right direction. This conjec-

[1] Black & Scholes (1973)
[2] Merton (1974)
[3] Jones et al. (1984)

ture spawned a number of papers which attempted to combine the risk associated with the firm's assets and that of the interest rate environment.

In Joon Kim[4], Krishna Ramaswamy and Suresh Sundaresan introduce interest rate risk into a model similar to the one used by Jones, Mason and Rosenfeld. They find that the spreads between the yields on corporate debt and identical government bonds are relatively insensitive to interest rate risk. Furthermore, and perhaps more importantly, they show that modelling default as triggered by a cash flow shortage and introducing debt writedowns explicitly result in model yield spreads that are much more in line with market levels than earlier models.

The motivation for the first two chapters of this dissertation relates to the latter observation. We believe that it is necessary to have a pricing framework which, while simple (and ideally yielding closed form price formulae), allows us to incorporate important real world features. The third paper is based on the belief that it is now necessary to put the recently developed models to the test on market data. It addresses the econometric issues of implementing models such as those developed in the first two chapters.

Another important area of applicability of the contingent claims approach is to analyse strategic problems relating to capital structure of the firm. It was realized at an early stage that given that the equity of a firm was analogous to a call option, it would lie in shareholders' interest (once debt has been issued) to select a higher business risk in order to transfer wealth from bondholders to themselves. However, the study of agency problems in corporate finance has, to a large extent, been studied in settings quite different from the typical Black & Scholes economy. As a result, although the mechanisms of the analysed problems have been understood, measures of their economic cost have not been available. Recently, a number of papers have tried to link the corporate finance and contingent claims literatures[5].

The final chapter of this volume is another attempt to incorporate ideas from both literatures. While introducing strategic interaction between shareholders and bondholders a theory of optimal leverage and maturity is proposed. The model also allows us to quantify the costs of the conflicting interests and their impact on a firm's capital structure decision.

This dissertation thus consists of four chapters. In principle, these are separate self-contained papers. However, papers 1, 2 and 3 follow naturally as a sequence. The first addresses the use of contingent claims models to price corporate liabilities. The second takes the analysis a step further and provides a model with which to price derivatives written on corporate

[4] Kim et al. (1993)

[5] A list of examples includes Anderson & Sundaresan (1996), Mella-Barral & Perraudin (1993), Leland & Toft (1996).

securities; and finally, paper 3 is concerned with a statistical estimation methodology that may be used to implement the models of the two preceding chapters in a practical situation. The final chapter is the "odd one out" although it is based on the same theoretical foundations. It establishes a link between the valuation theory employed in the first three chapters and the theory of corporate finance which has developed quite independently.

The four chapters are summarised as follows:

CHAPTER 1: A FRAMEWORK FOR VALUING CORPORATE SECURITIES (JOINT WITH JOEL RENEBY)

Early applications of contingent claims analysis to the pricing of corporate liabilities tend to restrict themselves to situations where debt is perpetual or where financial distress can only occur at debt maturity.

This paper relaxes these restrictions and provides an exposition of how most corporate liabilities can be valued as packages of two fundamental barrier contingent claims: a down-and-out call and a binary option. A down-and-out call is a standard call option with the added restriction that the underlying security price may not hit a predetermined barrier prior the call's expiration. The price of a down-and-out call was first derived by Robert Merton[6]. A binary option is one which at expiration pays out a fixed amount conditional on the underlying security price exceeding a predetermined exercise price.

A simple methodology to derive closed form solutions for quite complex securities is thus presented. Furthermore, it is shown how the comparative statics of the resulting pricing formulae can be derived.

CHAPTER 2: A NEW COMPOUND OPTION PRICING MODEL (JOINT WITH JOEL RENEBY)

This paper follows in the same direction as the previous chapter in that it provides a general framework for valuing securities. The most obvious application of the derived model is to value derivatives on the corporate securities discussed in the previous chapter.

In 1979 Robert Geske developed an option pricing formula which rested on the assumption that the underlying stock was in effect a call option on the firm's assets. Thus, a stock option would be an option on an option and hence, a *compound* claim. This allowed consistent pricing of both corporate claims as options on the firm and of derivatives on these claims within the same framework.

This paper extends the Geske (1979) model to the case where the security on which the option is written is a down-and-out call as opposed to a standard Black and Scholes call. The main economic rationale for carrying

[6] Merton (1973)

out this extension is that we believe that it is unreasonable to assume that financial distress only could occur at one *a priori* known point in time.

Furthermore, we develop a general and flexible framework for valuing options on more complex packages of contingent claims - any claim that can be valued using the ideas in chapter 1. This allows us to study the interaction between the detailed characteristics of a firm's capital structure and the prices of for example stock options. We provide a number of numerical examples and document the biases that would be induced by making simplifying assumptions about the maturity structure of debt and the size and timing of coupon payments.

CHAPTER 3: IMPLEMENTING FIRM VALUE BASED MODELS (JOINT WITH JOEL RENEBY)

The motivation for this chapter is that econometric implementations of firm value based models should be research priorities. A maximum likelihood approach has recently been suggested by Duan (1994). Although it appears to be an appealing method at first glance, its small sample properties are unknown. It is also unknown whether it will outperform methods used in the past.

Duan's idea is to use time series data of traded securities such as shares of common stock in order to estimate the dynamics of the firm's asset value. In this paper, we implement the maximum likelihood approach on simulated data and compare it to the method used by Jones et al. (1984).

We find that the estimators are efficient and unbiased for reasonable sample sizes. However, the performance of the estimators is related to debt maturity, leverage and business risk. This result, together with a conjecture that the method is likely to be sensitive to the choice of a particular model, leads us to suggest that a similar simulation study should be carried out before each empirical implementation.

Furthermore, we provide an argument which allows us to relax the (common) assumption that the firm's assets may be continuously traded. It is sufficient to assume that the firm's assets are traded at one particular point in time.

CHAPTER 4: ASSET SUBSTITUTION, DEBT PRICING, OPTIMAL LEVERAGE AND MATURITY

Chapters 1-3 have focused on the problem of pricing corporate securities. They have thus abstracted strategic aspects of corporate finance theory. This paper is an attempt to combine the contingent claims literature with the non-dynamic corporate finance literature.

I allow the management of the firm to alter their investment policies strategically. In doing so they are able to effectuate a wealth transfer from creditors. Since the shareholders will bear the cost of such policies ex ante when debt is issued, it will be in their interest to find a way to commit themselves not to transfer wealth. I argue that debt maturity functions as an instrument for debtors to align (at least partly) their incentives with creditors.

This yields a model which allows us to examine the relationship between bond prices, agency costs, optimal leverage *and* maturity. Furthermore, the relationship between deviations from the absolute priority rule in financial distress and the chosen capital structure of the firm is studied. The results contradict those of Eberhart et al. (1990*b*).

1
A Framework for Valuing Corporate Securities

1.1 Introduction

With the Black & Scholes (1973) model, a versatile methodology for the valuation of corporate securities was made available. Their insight was that the payoffs to many instruments on the firm's balance sheet were analogous to those of options. Thus the stock option pricing formula they derived could also be used to price corporate liabilities.

The aim of this paper is to show that some simple ideas related to barrier contracts can be applied to most corporate securities and that closed form expressions for relatively complex scenarios can be obtained with considerable ease. For example, while allowing the timing of default to be random we derive closed form solutions for the value of finite maturity corporate debt with discrete or continuous coupons. The framework we suggest is flexible enough to accommodate such exigencies as bankruptcy costs, corporate taxes and deviations from the absolute priority rule.

The idea of our approach is that corporate securities can be viewed as portfolios of two basic claims: a down-and-out call option and a down-and-out binary option, the Heaviside. A down-and-out claim is one that expires worthless if the value of the underlying asset hits a given barrier prior to the expiration date. A Heaviside is a contract that yields a unit payoff at the expiration date conditional on the value of the underlying asset exceeding the exercise price. Both the down-and-out call and the down-and-out Heaviside can be expressed as sums of standard (non-barrier) claims and hence have simple valuation formulae.

Another important issue is how the securities should be hedged, that is, to understand how the prices interact with the inputs of the model. The comparative statics of the corporate securities can easily be expressed as combinations of those of the standard basic claims - a list of these is provided.

The Black & Scholes (1973) paper spawned numerous applications to the valuation of various corporate debt instruments as well as equity instruments such as warrants. However, in the most common extensions of their framework, financial distress was only allowed to occur at one point in time, typically at the maturity of the outstanding debt, see for example Merton (1974). Black & Cox (1976) address this weakness by modelling a safety covenant as an absorbing barrier for the firm's asset value. They obtain closed form solutions for finite maturity discount debt and perpetual debt with continuous coupon payments. Leland (1994*b*) introduces taxes and bankruptcy costs into the latter model. Neither of these papers is able to combine finite maturity with coupons.

Geske (1977) suggests a compound option approach to overcome this problem. To value finite maturity discrete coupon debt, he argues that payments to bonds can be viewed recursively as compound options on later payments. This approach, however, does not allow for simple closed form expressions and is computationally cumbersome due to the necessity of evaluating multidimensional integrals.

Using our approach it is straightforward to value finite maturity coupon debt. Under certain restrictions on the default barrier, we can easily model discrete coupon payments as portfolios of binary options.

Our paper is organised as follows: in section 1.2 we discuss the general contingent claims approach to valuing corporate securities and our basic assumptions. Section 1.3 reviews results from the barrier option pricing literature. Section 1.4 exemplifies with some applications to the pricing of capital structures with common stock and debt and subsequently touches on the analysis of other equity-like securities. Section 1.5 provides a concluding discussion. We provide a list of the notation and the detailed comparative statics of the model building blocks in appendices.

1.2 Preliminaries

In the following subsections, we discuss the links between various possible economic assumptions and the technical assumptions that can be used to model these. In doing so, we believe that the limitations and potential of the framework we suggest will become clear.

The first assumption we make is the standard one about "perfect" capital markets.

Assumption 1 *Capital markets are "perfect" for at least some large investors*[1] *- no transaction costs, assets perfectly divisible, no arbitrage, constant risk-free rate, unlimited short sales, borrowing and lending at risk-free rate. At least one security on the firm's balance sheet is traded (for example equity).*

1.2.1 The Asset Value Process

Throughout this paper, we assume that the state variable determining the value of the firm's assets at time T follows an exogenous stochastic process.

Assumption 2 *The state variable determining the liquidation value of the firm's assets at T follows a geometric Brownian motion*

$$\begin{cases} dv = \mu v dt + \sigma v dW^P \\ v(0) = v_0 \end{cases}$$

where $W^P(t)$ with $W^P(0) = 0$ is a Wiener process under the objective probability measure.

Note that we do not need to assume that the assets are continuously traded - we only need to assume that they are traded at some prespecified date in the future when the firm is liquidated. With this assumption, *Lemma 1* in Ericsson & Reneby (1997) (this volume, Chapter 3) gives us the process for the *value* of assets

$$\begin{cases} d\omega = (r + \lambda\sigma)\,\omega dt + \sigma\omega dW^P \\ \omega(0) = \omega_0 \end{cases} \tag{1.1}$$

The term $(r + \lambda\sigma)$ is the expected return from holding the firm's assets and σ is their volatility. The parameter λ can be interpreted as the market price of risk associated with the operations of the firm.

Under the equivalent probability measure Q, where the discounted price processes are martingales, the asset value[2] has the following dynamics

$$\begin{cases} d\omega = r\omega dt + \sigma\omega dW^Q \\ \omega(0) = \omega_0 \end{cases} \tag{1.2}$$

---

[1] Merton (1990) (chapter 14) suggests a model in which many investors are unable to trade without transaction costs. Financial intermediaries, on the other hand, may do so by definition. He shows that in such a setting, if intermediation is efficient the products offered by intermediaries will be priced as in an economy without transaction costs.

[2] From standard arguments it follows that

$$\begin{aligned} \omega(t) &= e^{-r(T-t)} E^Q\left[v(T)\right] \\ &= v(t)\, e^{-(\mu-r)(T-t)} \end{aligned}$$

Under this measure[3] $W^Q(t) \equiv W^P(t) + \lambda t$ is a Wiener process.

Note again the fundamental difference between on the one hand, assuming that the firm's assets are continuously traded and hence the existence of (1.1) and on the other hand, assuming that the firm's assets are traded at a single future point in time (Assumption 2) and then *deriving* the process (1.1). The former assumption is in practice unnecessary and is not made here.

One could think of several other state variables that would affect the price of corporate securities. For example, when valuing bonds, a natural extension would be to model a stochastic risk-free interest rate[4]. In this paper, however, we assume a constant interest rate.

1.2.2 Corporate Securities

Having laid down the nature of the asset value process, one needs to specify the corporate securities of the firm: equity, debt, warrants, convertibles etc. In the setting of this paper, a corporate security is completely specified by its contracted payouts (coupons, dividends, principal) - for example, equity holders are in essence no longer in control of the firm but are reduced to residual claimholders to the firm's payouts. Hence, all securities are European, as opposed to American, in nature.

Assumption 3 *A corporate security is completely specified by its contracted payouts. Their holders can make no strategic decisions.*

There is one important distinction that needs to be made concerning contracted payouts, namely the one between soft and hard payments. A hard payment will force some form of reorganization as soon as it is not honoured. A soft payment will not. Securities with only hard payments are related to as hard claims; an example would be a coupon bond with an indenture that triggers financial distress when one of the coupon payments is not serviced. Securities with only soft payments are analogously termed soft claims; an example here is equity. There are of certainly corporate securities that will fall somewhere in between these two classes - for example a convertible bond.

The sum of the values of the corporate securities is termed the value of the firm, V. Note that this value does not in general equal the value of assets, ω. V will depend on the tax deductibility of interest payments and the costliness and likelihood of financial distress; tax deductibility adds to, and financial distress subtracts from, the firm value so that we may have both $\omega < V$ and $\omega > V$. In the literature, V is sometimes termed the value

[3] Popularly termed the risk neutral measure.

[4] See e.g. Nielsen et al. (1993) and Longstaff & Schwartz (1995).

of the levered firm and ω the value of the unlevered firm (since an all-equity firm has no deductible interest payments and cannot default).

A critical aspect of securities is their maturity. To keep the framework simple and obtain closed form solutions, we use a setting where all securities mature at the same date[5] - see section 1.2.4.

1.2.3 Financing of Payments

Another issue that has to be addressed is how contracted payments to security holders are financed. At maturity it is straightforward since the firm is liquidated, but prior to maturity, there is room for various solutions.

We may assume, as an example, that coupons and dividends are financed internally, from generated funds. Intuitively, part of the firm's assets are liquid cash assets and are used as a means of payment and the fraction retained is tied up in physical assets that are central to the firm's operations and cannot be liquidated without disrupting them. However, technically this assumption may create difficulties since the stochastic process of the asset value would be affected by each individual payment. A way to circumvent this problem while still allowing payments to be financed from the firm's activities, would be to assume that a *constant fraction* β of the return from assets is in liquid cash assets. To focus on the effects of using internally generated funds, suppose this is, in fact, the only way to raise money for security holders; if funds are not sufficient to service a hard payment, the firm defaults[6].

Assumption 4a *Free cash flow is the only means to service debt payments. The growth rate of asset value is $r + \lambda\sigma - \beta$ where β is the rate at which free cash flow is generated*[7].

With this assumption, the dynamics for the asset value under the Q-measure are

$$\begin{cases} d\omega = (r-\beta)\,\omega dt + \sigma\omega dW^Q \\ \omega(0) = \omega_0 \end{cases}$$

The parameter β is used to denote that part of the asset's return that cannot be reinvested. The expected *growth* of the assets is thus the expected return less the generated "free cash flow".

[5] This assumption can be relaxed without great effort if e.g. debt issues of shorter maturities have principals less than the default triggering barrier. Otherwise, a compound option model such as the one developed in chapter 2 could be used.

[6] A criticism to this approach is that, in some cases, it would be profitable for equity holders to inject capital.

[7] The drift of the state variable v can be written as

$$\mu(\omega,t) = \overline{\mu} - \beta - \frac{(\overline{\mu} - \beta - \lambda\sigma - r)\, e^{(\overline{\mu}-\beta-\lambda\sigma-r)(T-t)}}{\omega}$$

To finance payments internally is in essence equivalent to selling off assets. Many bond indentures, however, limit the extent of asset sales. Take the extreme case; assume that no asset sales at all are allowed. In this case, shareholders will finance all intermediate payments as long as this is consistent with limited liability. Since asset sales are not allowed, the process for ω is once more unaffected by financing decisions.

Assumption 4b *Equityholders service debt payments - no asset sales allowed.*

Another related issue is that we will throughout assume that coupons (but not the principal) are tax deductible. Thus the government becomes part of the analysis in that it partially finances such payments, namely, a fraction $\varkappa$ of a coupon c.

Assumption 5 *The corporate tax rate is* $\varkappa$.

The value of this tax shield will henceforth be denoted TS.

1.2.4 Reorganization Trigger

In many countries, corporate law states that financial distress occurs when the value of the firm's assets reaches some lower level, usually related to the total nominal value of outstanding debt. Technically, this is modelled as follows: the firm enters financial distress and some form of reorganization occurs if the value of its assets reaches an exogenous constant[8] barrier L.

Apart from this judicial view of the barrier, there are several economic interpretations. One is to view the barrier as a level of asset value which is necessary for the firm to retain sufficient credibility to continue its operations or where due to some covenant, it voluntarily files for bankruptcy. Another interpretation, with regard to hard intermediate payments, is to think of the barrier as the asset value for which it is no longer possible to honour the payments, be it by selling assets or issuing new securities. In the extreme case where payments to claimants have to be met by selling a restricted fraction of the assets, the barrier will be uniquely determined by the fraction β in (1.2) and the amount due.

As an alternative to using a barrier, one can let default on a hard payment directly trigger reorganization. Due to the dependence of the payoffs (if the firm defaults on one coupon, a claim to future coupons becomes worthless) risky coupon debt cannot be valued as a portfolio of discount bonds, however. To value corporate securities in such a setting, one could employ the compound option approach of Geske (1977). The drawback of

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[8] Note that dealing with a barrier that is an *exponential* function of time would not be difficult.

this approach is that it does not yield closed form solutions and is computationally cumbersome, as mentioned in the introduction.

Conversely, letting a barrier trigger financial distress allows the modelling of discrete coupons as a portfolio of discount bonds and will yield closed form solutions. This will demand a barrier which is sufficiently high for the limited liability of equity to hold at all times. Such an approach would also allow us to model corporate securities with different maturities. The technically critical point is that only the barrier may trigger financial distress prior to the last maturity date of the firm's securities.

Assumption 6 *Default occurs if*

$$\omega(s) \leq L, \textit{for some } s \leqslant T$$
$$\textit{or } \omega(T) < F$$

where F is the total face value of debt (of the longest maturity T)[9].

We will abstract from strategic behaviour regarding payments to claims. In the models of Anderson & Sundaresan (1996) and Mella-Barral & Perraudin (1993) it may be rational for creditors to accept a lower than contracted coupon to avoid bankruptcy.

1.2.5 Reorganization payoff

It is not critical to the suggested framework what assumptions one wishes to make about what happens when financial distress occurs. All that is required is that the payoffs in such an event are specified. Note in particular that this allows for deviations from absolute priority. The payoffs in the event of reorganization can be modelled in different ways. They can be taken to be a constant (or a constant fraction of a risk-free bond[10]) or a function of the firm's asset value when financial distress occurs[11]. We will concentrate on the latter.

Assumption 7 *When financial distress occurs, we assume that a cost k of reorganization is incurred.*

---

[9] Note that this yields a discontinuous reorganization trigger when $p > L$: at $T - \epsilon$ it is the barrier L, at T it is the principal p. This could be avoided with an exponential barrier if one considered such a discontinuity implausible. When $L > p$ the barrier and not the principal will trigger bankruptcy at maturity. Technically, in this case a discontinuity that should be present is not. For practical purposes this should be negligible.

[10] Longstaff & Schwartz (1995) and Nielsen et al. (1993) model risky discount debt in a world with stochastic interest rates. The payoff to bondholders in financial distress is a fraction of the value of an otherwise identical (credit) risk-free bond. In doing so, they rule out any link between the value of the firm when it defaults and the payoff to debt.

[11] In principle there is nothing to stop us from interpreting the payoffs we specify as expected outcomes of some bargaining process, conditional on the value of assets. This is feasible as long as these expectations are independent of time.

This cost may include legal fees, losses due to disrupted operations etc. The amount is deducted from the asset value of the firm before it can be shared by the firm's claimants. We will denote the market value of a fictive claim that receives realized reorganization costs with K.

The problem of determining payoffs in case of financial distress before maturity is simplified by our assumption of a deterministic barrier, since we know *a priori* what the firm will be worth if it enters reorganization (L). We may easily determine the payoffs to claimants as constants that must sum to $\max(L-k,0)$.

1.3 Valuation

The valuation method exploits the fact that most of the company's securities can be expressed as combinations of two basic claims: the down-and-out call and the down-and-out Heaviside.. Assuming the absence of arbitrage, two claims with identical payoff structures must have the same price. Hence to value a corporate security, one simply mimics the payoffs of that security with those of calls and Heavisides..

Moreover, the barrier claims used can be priced in terms of standard (non-barrier) claims. The reason for this result is that the reflection principle allows us to separate the density function of a Wiener process absorbed at a barrier into the density functions of two unabsorbed Wiener processes. An extension of this argument to a generalized Wiener process gives us the following expression for the density function (under the Q-measure) of $\ln \omega$ absorbed at $\ln L$

$$\begin{aligned} \varphi_{\ln L}(y;\cdot) &= \varphi\left(y; \ln \omega_0 + \left(r - \tfrac{1}{2}\sigma^2\right)T, \sigma\sqrt{T}\right) \\ &\quad - \left(\frac{L}{\omega_0}\right)^{\alpha} \varphi\left(y; \ln \tfrac{L^2}{\omega_0} + \left(r - \tfrac{1}{2}\sigma^2\right)T, \sigma\sqrt{T}\right) \\ &\quad \text{where} \\ \alpha &= \tfrac{2\left(r-\frac{1}{2}\sigma^2\right)}{\sigma^2} \end{aligned}$$

where φ (y; mean,standard deviation) is the density function for $y = \ln \omega(T)$ at $t = 0$. It follows that the price of a contingent claim on such a process can be written as the sum of prices of two separate non-barrier claims. A thorough treatment of these issues can be found in Björk (1994).

Below we will describe, for the basic claims, the payoff functions ($\Phi\{H\}$ for the Heaviside and $\Phi\{C\}$ the call[12]) and the valuation formulae (H_L for the down-and-out Heaviside and C_L for the down-and-out call). Generally we let subscript L denote a down-and-out claim with barrier L, and $\Phi\{R\}$ the payoff to a claim R.

The point is that since we are able to write the payoff functions of a corporate security G as $\Phi\{G\} = \sum \Phi\{H\} + \sum \Phi\{C\}$, we can write the value of the corporate security as $G = \sum H_L + \sum C_L$. Note that the relevant calls and Heavisides need not have the same maturity and that we readily can take infinite sums as will be the case when we analyse debt with continuous coupons. The tractability of the analysis thus stems from the ease with which one can obtain closed form expressions for the value of a (European) claim on the firm's assets, such as different classes of debt and equity, tax shields, convertibles and warrants by simply matching payoffs of standard claims.

1.3.1 The Basic Claims

1.3.1.1 The Heaviside

A standard Heaviside pays off unity at expiration T if the value of the underlying exceeds its exercise price X and zero otherwise

$$\Phi\{H(\cdot;X,T)\} = \begin{cases} 1 \text{ if } \omega(T) > X \\ 0 \text{ if } \omega(T) \leq X \end{cases}$$

Its price, when the value of the underlying is y at time t, is

$$\begin{aligned} H(y,t;X,T) &= e^{-r(T-t)}N(d_2) \\ \text{where } d_2 &= \frac{\ln\frac{y}{X} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

Intuitively, the price of the Heaviside is the discounted payoff of unity which occurs with Q-probability $N(d_2)$. The price of the down-and-out Heaviside is

$$H_L(\omega(t),t;X,T)$$

$$=$$

[12]The payoff function for any *down-and-out* claim R with barrier L and maturity T is

$$\Phi\{R_L\} = \begin{cases} \Phi\{R\} \text{ if } \omega(t) > L, \forall\, t \leq T \\ 0 \text{ if } \omega(t) \leq L, \text{ for some } t \leq T \end{cases}$$

Of course, this is the payoff function subsequently used to derive the value function of the down-and-out claim, R_L. To save space, however, only the standard payoff function, $\Phi\{R\}$ will be given throughout the paper.

$$\begin{cases} H(\omega(t),t;X,T) - \left(\frac{L}{\omega(t)}\right)^{\alpha} H\left(\frac{L^2}{\omega(t)},t;X,T\right) & \text{if } X \geq L \\ H_L(\omega(t),t;L,T) & \text{if } X < L \end{cases}$$

The down-and-out formula can be interpreted as the price of the standard claim less a discount reflecting the risk that it expires worthless. One can also see that when the value of the firm's assets is much higher than the barrier, and the risk of default therefore is small, $\frac{L}{\omega(t)}$ and $\frac{L^2}{\omega(t)}$ are small, and the value of the discount approaches zero. Thus, not surprisingly, the price of the down-and-out claim approaches the price of the standard claim in these cases.

Further, it is evident that deltas and gammas and other differentials of down-and-out claims can be computed with relative ease as combinations of the respective differentials of standard non-barrier claims. The comparative statics of the barrier claims may differ substantially from those of standard claims.

1.3.1.2 The Call

The payoff function of the standard call is

$$\Phi\left\{C\left(\cdot;X,T\right)\right\} = \begin{cases} \omega(T) - X \text{ if } \omega(T) > X \\ 0 \text{ if } \omega(T) \leq X \end{cases}$$

and its price is of course given by the Black-Scholes formula

$$\begin{aligned} C(y,t;X,T) &= yN(d_1) - Xe^{-r(T-t)}N(d_2) \\ \text{where } d_1 &= d_2 + \sigma\sqrt{T-t} \end{aligned}$$

The value of its down-and-out counterpart is:

$$C_L(\omega(t),t;X,T) \tag{1.3}$$

$$=$$

$$\begin{cases} C(\omega(t),t;X,T) - \left(\frac{L}{\omega(t)}\right)^{\alpha} C\left(\frac{L^2}{\omega(t)},t;X,T\right) & \text{if } X \geq L \\ C_L(\omega(t),t;L,T) + (L-X)\cdot H_L(\omega(t),t;L,T) & \text{if } X < L \end{cases}$$

Again the interpretation is that the price is that of the standard claim less a discount.

Below we describe two other claims that will prove useful. As will be shown, the values of both these claims are portfolios of down-and-out Heavisides and call options.

1.3.1.3 Down-and-out and -in Claims on the Assets

Denote with $\Omega(\omega(t), t; s)$ the value at t of a claim with payoff $\omega(s)$ at s

$$\Phi\{\Omega(\cdot; s)\} = \omega(s)$$

When no free cash flow is generated it obviously holds that $\Omega(\omega(t), t; T) = \omega(t)$, but when $\beta > 0$ the value of the asset claim is

$$\begin{aligned}\Omega(\omega(t), t; T) &= e^{-r(T-t)} E^Q[\omega(T)] \\ &= \omega(t) e^{-\beta(T-t)}\end{aligned}$$

The factor $e^{-\beta(T-t)}$ is the discount reflecting the value of payouts from the company that the holder of the asset claim Ω is not entitled to. Analogously, denote with $\Omega_L(\omega(t), t; s)$ a claim that gives the holder the right to $\omega(s)$ at time s given that the underlying process has not hit the barrier prior to that date. This is equivalent to a down-and-out call with exercise price zero. Hence, by equation (1.3)

$$\begin{aligned}\Omega_L(\omega(t), t; T) &= C_L(\omega(t), t; 0, T) \\ &= C_L(\omega(t), t; L, T) + L \cdot H_L(\omega(t), t; L, T)\end{aligned} \tag{1.4}$$

A down-and-in contract, henceforth denoted with *super*script L, is in essence the mirror image of the corresponding down-and-out contract. It gives the holder the right to the prespecified payoff only if the underlying process *has* hit the barrier during the security's lifetime. Since holding both contracts will guarantee the contracted payoff whether the underlying process has hit the barrier or not, we have the following value for the down-and-in claim

$$\Omega^L(\omega(t), t; T) = \Omega(\omega(t), t; T) - \Omega_L(\omega(t), t; T) \tag{1.5}$$

For ease of exposition we will sometimes suppress the input variables $(\omega(t), t)$ and specify claims by their parameters $(X, T, \ldots$ etc.) only.

1.3.2 Corporate Securities

The following proposition states the general pricing formula the assumptions of which have been discussed in the section 1.2.

Proposition 1.1 *If assumptions 1 through 7 hold a corporate security G with contracted payments*

$$\Phi\{G\} = \left\{ \begin{array}{c} \alpha\,\Phi\{\Omega\} \\ + \\ \sum_i \beta^{(i)}\,\Phi\left(C^{(i)}\right) \\ + \\ \sum_i \gamma^{(i)}\,\Phi\left(H^{(i)}\right) \end{array} \right\} \quad \textit{for } t \le \tau$$

can be valued as

$$G = \left\{ \begin{array}{c} \alpha\,\Omega_L \\ + \\ \sum_i \beta^{(i)} C_L^{(i)} \\ + \\ \sum_i \gamma^{(i)} H_L^{(i)} \end{array} \right\} \quad \text{for } t \leq \tau$$

where i are used to index Heavisides and options of different exercise price and maturity, $\beta^{(i)}$ and $\gamma^{(i)}$ are constants and τ is the time of default. The summation operator $\sum$ should be understood to encompass integrals when applicable.

1.4 Examples

We will now exemplify the proposed framework studying some different capital structures. The aim is to demonstrate its flexibility and manageability.

First we examine the case of firms financed by debt and equity (in section 1.4.1 we consider a single class of debt, introducing junior debt in section 1.4.2), and then briefly turn to simple forms of warrants (section 1.4.3) and convertibles (section 1.4.4).

1.4.1 Common Stock and One Class of Bonds

Denote the value of equity with E and the value of debt with B. The market value of the securities must add up to the market value of the company: $V = B + E$. Below, we will first consider the case where debt consists of discount bonds only and derive the valuation formulae in some detail. Later, we introduce coupons, considering both the case where asset sales are not allowed and the case where they are (i.e. internally generated funds).

1.4.1.1 Equity and Discount Debt

Apart from equity and debt, we will analyse the fictive claim relating to reorganization costs, K. Since there are no coupon payments in this case, there is no tax shield. Therefore it must hold that $V = \omega - K$; or $\omega = B + E + K$.

The values of each of these three claims will be the sums of the value of their payoff at maturity (denoted with ^{M}) and the value of their payoff in case of reorganization prior to maturity (denoted with $^\tau$). This can be written

$$\begin{aligned} B &= B^M + B^\tau \\ E &= E^M + E^\tau \\ K &= K^M + K^\tau \end{aligned}$$

Even though the two events {payoff at maturity} and {payoff prior to maturity} are mutually exclusive, claims to the payoffs naturally both have a value.

Receiving all payouts at maturity is equivalent to acquiring the company at maturity provided no prior default; that is having a down-and-out claim on the assets and hence $\Omega_L(\cdot;T) = B^M + E^M + K^M$. Similarly, it follows $\Omega^L(\cdot;T) = B^\tau + E^\tau + K^\tau$.

We start by deriving the value of the individual maturity payments and then turn to the value of payoffs in the event of reorganization prior to maturity.

Value of Payments at Maturity

The payoff-structure to the two securities depend on whether costs of reorganization (k) are larger or smaller than the face value of debt (F), and the Heavisides and calls necessary to mimic their payoffs therefore differ as well. But since k and F are assumed to be known ex-ante, this poses no problem.

Below we illustrate the payoffs at maturity to debt and equity for $F > k$ (figure 1.1) and for $F \leq k$ (figure 1.2).

Consider first figure 1.1. Formally, the payoff to debt at maturity is

$$\begin{aligned} \Phi\left(B^M\right) &= \begin{cases} F & \text{if } \omega(T) \geq F \\ \omega(T) - k & \text{if } F > \omega(T) \geq k \\ 0 & \text{if } k > \omega(T) \end{cases} \\ &= \Phi\left\{C\left(\cdot;k,T\right)\right\} - \Phi\left\{C\left(\cdot;F,T\right)\right\} + k \cdot \Phi\left\{H(\cdot;F,T)\right\} \end{aligned}$$

Or, verbally, to mimic the payoff to debt you need one long call with exercise price k, one short call with exercise price F and k long Heavisides with exercise price F. The value of debt is therefore the sum of these components. Since we have a barrier L we use the down-and-out claims to calculate the value of debt's maturity payments as

$$B^M(\cdot;\cdot) = C_L(\cdot;k,T) - C_L(\cdot;F,T) + k \cdot H_L(\cdot;F,T) \tag{1.6}$$

The dashed line in figure 1.1 represents the payoff to equity at maturity. Formally,

$$\Phi\left\{E^M\right\} = \left\{ \begin{array}{cc} \omega(T) - F & \text{if } \omega(T) > F \\ 0 & \text{if } \omega(T) \leq F \end{array} \right\} = \Phi\left\{C\left(\cdot;F,T\right)\right\}$$

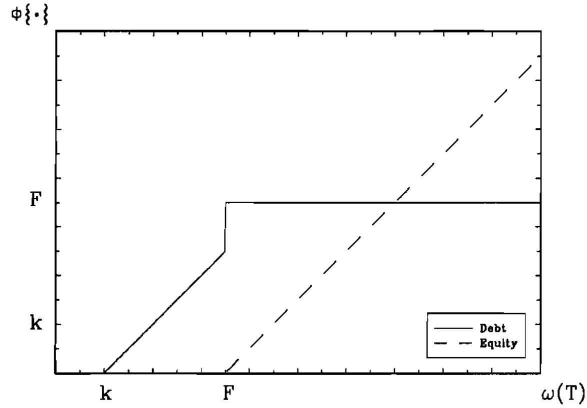

FIGURE 1.1. EQUITY AND DISCOUNT DEBT. Payoffs to debt and equity at maturity when $F > k$

The payoff to equity in this case is simply equal to the payoff from a call with exercise price F. Therefore, the value of equity can be expressed as

$$E^M(\cdot; F, T) = C_L(\cdot; F, T) \tag{1.7}$$

Now consider the case in figure 1.2 where reorganization costs exceed the principal:

Here, neither equity- nor debtholders receive any payoff if the firm cannot repay the principal at maturity - all value is lost to lawyers and in mismanagement. Therefore, the payoff to debtholders will be

$$\Phi\left\{B^M\right\} = \begin{cases} F \text{ if } \omega(T) > F \\ 0 \text{ if } \omega(T) \le F \end{cases}$$

Thus, the value of debt in this case is simply equal to F Heavisides with exercise price F

$$B^M(\cdot; \cdot) = F \cdot H_L(\cdot; F, T) \tag{1.8}$$

The payoff to equity is the same as when $k < F$; stockholders do not receive anything in default in either case.

From (1.6) and (1.8) it is apparent that the formulae for the value of debt differ when $k < F$ and when $k \ge F$. To avoid writing down two formulae for each security, we will define $k^F = \min(k, F)$. The adjusted reorganization cost k^F simply denotes the maximum possible realized costs at maturity -

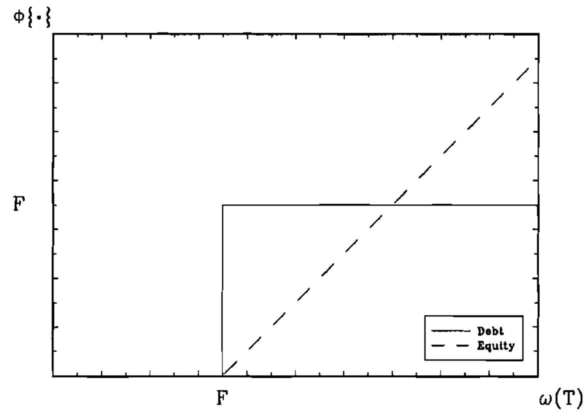

FIGURE 1.2. EQUITY AND DISCOUNT DEBT. Payoffs to debt and equity at maturity when $F \leq k$

reorganization costs can never exceed the value of assets in default[13]. Then we can summarize the values of maturity payoffs to debt and equity in (1.6) - (1.8) as

$$B^M(\cdot; F, T) = C_L(\cdot; k^F, T) - C_L(\cdot; F, T) + k^F \cdot H_L(\cdot; F, T) \tag{1.9}$$

$$E^M(\cdot; F, T) = C_L(\cdot; F, T) \tag{1.10}$$

We now derive the value of a claim to potential reorganization costs at maturity, K^M. When $k < F$ its payout function is

$$\begin{aligned} \Phi\left\{K^M\right\} &= \begin{cases} 0 & \text{if } \omega(T) > F \\ k & \text{if } F \geq \omega(T) \geq k \\ \omega(T) & \text{if } \omega(T) < k \end{cases} \\ &= \omega(T) - \Phi\left\{C\left(\cdot; k, T\right)\right\} - k \cdot \Phi\left\{H(\cdot; F, T)\right\} \end{aligned}$$

and when $k \geq F$ it is

$$\begin{aligned} \Phi\left\{K^M\right\} &= \begin{cases} 0 & \text{if } \omega(T) \geq F \\ \omega(T) & \text{if } \omega(T) < F \end{cases} \\ &= \omega(T) - \Phi\left\{C(\cdot; F, T)\right\} - F \cdot \Phi\left\{H(\cdot; F, T)\right\} \end{aligned}$$

[13]Of course, if $L > F$, realized bankruptcy costs in default *prior to* maturity may be higher than F.

Consequently, its value is

$$K^M(\cdot;\cdot) = \Omega_L(\cdot;T) - C_L(\cdot;k^F,T) - k^F \cdot H_L(\cdot;F,T) \tag{1.11}$$

Value of Payments in Case of Reorganization Prior to Maturity

The payoffs to the two securities and the fictive claim to reorganization costs are, for future reference, summarized in the following lemma.

Lemma 1.1 *Payouts to claimholders in financial distress prior to maturity are*

$$\Phi\{B^\tau\} = \min\left((L-k)^+, F\right) \tag{1.12}$$

$$\Phi\{E^\tau\} = (L-F-k)^+ \tag{1.13}$$

$$\Phi\{K^\tau\} = \min(k,L) \tag{1.14}$$

Even though the payoff to equityholders often is zero, for a barrier above the principal, it may very well be positive. Note also that in this setting it would be easy to accommodate violations of absolute priority by simply adjusting payout functions (1.12) and (1.13).

It is not necessary to assume that the payouts above are known or fixed - it is sufficient that the expected payout is independent of the time of default. To model stochastic bankruptcy costs and violations of absolute priority one simply replaces the constant payouts with their expected values.

Since reorganization by definition occurs when $\omega = L = \Phi\{B^\tau\} + \Phi\{E^\tau\} + \Phi\{K^\tau\}$, the payoffs to claimants (including reorganization costs) are known[14]. For purposes of valuation, consider the fixed *proportion* $\frac{\Phi\{R\}}{L}$ of L paid out to each claimant $R \in (B^\tau, E^\tau, K^\tau)$. The *value* of each claim will consequently be a fixed proportion $\frac{\Phi\{R\}}{L}$ of Ω^L. Thus

$$B^\tau = \frac{\Phi\{B^\tau\}}{L}\Omega^L(\cdot;T) \tag{1.15}$$

$$E^\tau = \frac{\Phi\{E^\tau\}}{L}\Omega^L(\cdot;T) \tag{1.16}$$

$$K^\tau = \frac{\Phi\{K^\tau\}}{L}\Omega^L(\cdot;T) \tag{1.17}$$

Combining the results in (1.9) - (1.11) and (1.15) - (1.17) above we have the following proposition.

[14]Note the difference from e.g. payoff at maturity, dependent on $\omega(T)$, which is stochastic. In the case of reorganization prior to maturity, we *know* what the value of the assets is *if* it occurs.

Corollary 1.1 *Assumption 1-4b-7 hold. Consider a firm with a capital structure consisting of equity and discount debt with maturity T and principal F. The values of debt, equity and the fictive claim to reorganization costs k are then given by*

$$\begin{aligned} B(\cdot;F,T) &= -C_L(\cdot;F,T) + C_L(\cdot;k^F,T) + k^F \cdot H_L(\cdot;F,T) \\ &\quad + \frac{\Phi\{B^\tau\}}{L}\Omega^L(\cdot;T) \\ E(\cdot;F,T) &= C_L(\cdot;F,T) + \frac{\Phi\{E^\tau\}}{L}\Omega^L(\cdot;T) \\ K(\cdot;F,T) &= \Omega_L(\cdot;T) - C_L(\cdot;k^F,T) - k^F \cdot H_L(\cdot;F,T) \\ &\quad + \frac{\Phi\{K^\tau\}}{L}\Omega^L(\cdot;T) \end{aligned}$$

where $\Phi\{B^\tau\}$, $\Phi\{E^\tau\}$ and $\Phi\{K^\tau\}$ are given by Lemma 1.1.

1.4.1.2 Equity and Coupon Debt: Asset Sales Not Allowed

In this section we value corporate coupon debt when assumption 4b hold - that is when asset sales are not allowed and coupon payments are assumed to be financed by equity[15]. These payments are made conditional on no reorganization and will be termed solvency payments. Denote the constant percentage coupon due at time t_i $(t_i < T, i = 1...N)$ by c. The principal, F, is due at time T[16]. The assumption of constant coupon payments can be relaxed at the cost of notational inconvenience. For example we could easily accommodate sinking fund features into the valuation problem.

As explained in section *1.2.4 Reorganization trigger* we throughout the paper assume that default is driven by a barrier and the principal repayment. However, we must guarantee that equity (which finances coupons) is always positive. This places a restriction on possible debt contracts (on the barrier in relation to coupon payments), since limited liability does not allow equity to finance a coupon that would create a negative equity value. Therefore we require that when creditors and shareholders agree on the contractual terms of the debt contract they will make sure equity value is positive for all asset values above the barrier. Formally we require that

$$E(\omega, t; c, F, L, \cdot) > 0 \ , \forall\ t, \ \omega > L$$

[15] We could model an intermediate case where payments would be financed by equity should internally generated funds not be sufficient. For simplicity we abstract from this possibility although it would be technically similar.

[16] To avoid cumbersome notation we assume that no coupon is due at maturity ($t_i < T$). The coupon dates may, however, be arbitrarily close to maturity.

Should this assumption be violated the valuation formulae are no longer applicable to a debt and an equity contract since the latter is no longer subject to limited liability.

When we add coupons to the analysis, we must consider the issue of tax deductibility - we let TS denote the value of the tax shield. Hence the value of the firm now equals $V = \omega + TS - K$, so that the value of the firm may exceed the value of assets.

The value of a corporate security will be a sum of three parts: value of maturity payments (^{M}), value of payments in case of reorganization prior to maturity ($^\tau$) and value of intermediate payments in solvency (^{S}):

$$\begin{aligned} B &= B^M + B^\tau + B^S \\ E &= E^M + E^\tau + E^S \\ K &= K^M + K^\tau \\ TS &= TS^S \end{aligned}$$

The payoffs at maturity and in case of reorganization prior to maturity equal those of the previous analysis, and their value is hence the same as in Corollary 1.1. We therefore turn to analyse the value of coupons to debt.

A coupon cF is paid out to bondholders given no reorganization up to and including t_i. Hence, each coupon payment is equivalent to cF down-and-out Heavisides of maturity t_i. The exercise price equals the barrier L. The value of the coupons consequently is

$$B^S(\cdot;\cdot) = cF \sum_{i=1}^{N} H_L(\cdot; L, t_i)$$

The coupon payments decrease the value of equity. But since coupons are assumed to be tax deductible, only a fraction $1 - \varkappa$ of the payments is borne by equity holders. Hence,

$$E^S(\cdot;\cdot) = -(1-\varkappa) \cdot cF \sum_{i=1}^{N} H_L(\cdot; L, t_i)$$

The value of the tax shield is

$$TS^S(\cdot;\cdot) = \varkappa \cdot cF \sum_{i=1}^{N} H_L(\cdot; L, t_i)$$

Combining these three equations with Corollary 1.1, we get

Corollary 1.2 *Assumptions 1-4b-7 hold. Consider a firm with a capital structure consisting of equity and debt with coupon payment cF at dates t_i*

for $i = 1...N$ and principal repayment of F at $T > t_N$. The values of debt, equity, reorganization costs and tax shield are then given by

$$
\begin{aligned}
B(\omega,t;\cdot) &= \begin{cases} -C_L(\omega,t;F,T) + C_L(\omega,t;k^F,T) + k^F \cdot H_L(\omega,t;F,T) \\ \qquad +\frac{\Phi\{B^\tau\}}{L}\Omega^L(\omega,t;T) \\ \qquad +cF\sum_{i=1}^N H_L(\omega,t;L,t_i) \end{cases} \\
E(\omega,t;\cdot) &= C_L(\omega,t;F,T) + \frac{\Phi\{E^\tau\}}{L}\Omega^L(\omega,t;T) \\
&\quad -(1-\varkappa)\cdot cF\sum_{i=1}^N H_L(\omega,t;L,t_i) \\
K(\omega,t;\cdot) &= \Omega_L(\omega,t;T) - C_L(\omega,t;k^F,T) - k^F \cdot H_L(\omega,t;F,T) \\
&\quad +\frac{\Phi\{K^\tau\}}{L}\Omega^L(\omega,t;T) \\
TS(\omega,t;\cdot) &= \varkappa \cdot cF\sum_{i=1}^N H_L(\omega,t;L,t_i)
\end{aligned}
$$

where $\Phi\{B^\tau\}$, $\Phi\{E^\tau\}$ and $\Phi\{K^\tau\}$ are given by Lemma 1.1.

1.4.1.3 Equity and Coupon Debt: Internally Financed Coupon Payments

We now analyse a situation where coupons are financed solely by internally generated funds - assumption 4a as opposed to 4b now hold. As discussed in section 1.2.3 we assume funds are generated continuously as a fraction β of the asset value. In order to obtain closed form solutions, we assume that coupons, as well, are continuous[17]. If the generated free cash flow each moment is sufficient to pay the after-tax coupon the company will do so and distribute the remaining $\beta\omega - (1-\varkappa)cF$ as dividends. Reorganization occurs when $\beta\omega < (1-\varkappa)cF$. Hence we have a barrier in terms of asset value $L \equiv \omega(\tau) = (1-\varkappa)cF/\beta$. Note that for a given cash flow generation potential and tax rate the barrier is uniquely determined by the scheduled payments of the debt contract (c, F).

The value of payouts at maturity and in case of reorganization is not affected by the choice between external or internal financing of coupons (except, of course, that we now have $\beta > 0$ instead of $\beta = 0$). We therefore turn to the valuation of the coupon payments.

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[17] Otherwise we would have to consider the effect of allowing for accumulation of cash flow which would produce a path-dependent problem.

Valuation of Intermediate Payments in Solvency

Consider first the debt contract. When the firm is solvent it simply receives the coupon payment - a continuous stream $cFdt$. Since the barrier is equal for all coupon payments and they all cease simultaneously we can value the coupon stream as a continuous portfolio of cF down-and-out Heavisides

$$B^S(\omega;T) = cF \int_t^T H_L(\omega; L, s)ds \tag{1.18}$$

By a similar argument, the value of the tax shield is

$$TS^S(\omega;T) = \varkappa cF \int_t^T H_L(\omega; L, s)ds \tag{1.19}$$

Equity receives the residual, i.e. each moment receiving

$$\max(\beta\omega - (1-\varkappa)cF, 0)\, ds$$

or $\beta \max(\omega - L, 0)$ which is equivalent to the payoff from β down-and-out call options with exercise price L. Hence

$$E^S(\omega;T) = \beta \int_t^T C_L(\omega; L, s)ds \tag{1.20}$$

The total value of intermediate payments in solvency is

$$B^S + E^S - TS^S = \beta \int_t^T \Omega_L(\omega(t), t; s)ds$$

This is the value of accumulating the stream $\beta\omega dt$ conditional on solvency between today and maturity T.

Corollary 1.3 *Assumptions 1-4a-7 hold. Consider a firm with a capital structure consisting of debt with principal F, percentage coupon c and of maturity T and equity that receives the generated cash flow $\beta\omega$ net of coupon payments. The values of debt, equity, reorganization costs and tax shield are then given by*

$$B(\omega,t;\cdot) = \begin{cases} -C_L(\omega,t;F,T) + C_L(\omega,t;k^F,T) + k^F \cdot H_L(\omega,t;F,T) \\ \qquad + \frac{\Phi\{B^\tau\}}{L}\Omega^L(\omega,t;T) \\ \qquad + cF\int_t^T H_L(\omega,t;L,s)ds \end{cases}$$

$$E(\omega;\cdot) = C_L(\omega,t;F,T) + \frac{\Phi\{E^\tau\}}{L}\Omega^L(\omega,t;T) + \beta\int_t^T C_L(\omega,t;L,s)ds$$

$$
\begin{aligned}
K(\omega;\cdot) &= \Omega_L(\omega,t;T) - C_L(\omega,t;k^F,T) - k^F \cdot H_L(\omega,t;F,T) \\
&\quad + \frac{\Phi\{K^\tau\}}{L}\Omega^L(\omega,t;T) \\
TS(\omega;\cdot) &= \varkappa cF \int_t^T H_L(\omega,t;L,s)ds
\end{aligned}
$$

where Φ^{B^τ}, Φ^{E^τ} *and* Φ^{K^τ} *are given by Lemma 1.1.*

1.4.1.4 A Remark: Continuous versus Discrete Coupon Payments

In the literature it is common to approximate a series of actual discrete coupons with a continuous coupon stream. In this subsection we briefly analyse the consequences of this. Modelling the situation where assets sales are not allowed, we can use *Proposition 1.1* to value debt. Debt with continuous coupons is found as a limiting case when $N \to \infty$.

Figure 1.3 below depicts the percentage amount by which the value of continuous coupon debt exceeds the value of discrete coupon debt[18] for different firm values. When the value of the firm approaches the barrier we see that the continuous coupon bond increases in value relative to the discrete one.

To understand this note that as ω decreases, the probability of entering into financial distress prior to the next coupon date increases. Suppose that we expect default to occur with high probability within 6 months and the coupon is due in a year's time. The value of the coupon is very low for the discrete coupon bond whereas the continuous coupon bond may continue to receive coupon payments for some time yet. Hence if we choose to approximate the value of a coupon bond with discrete payments by using a model with continuous coupons, we are likely to overestimate the value of the bond if the likelihood of default is high (ω close to L).

1.4.2 Common Stock and Two Classes of Bonds

Extending the analysis to the case of a capital structure consisting of equity and debt with different classes of seniority is easily done as long as the debt contracts have the same maturity. This is what we will consider here. Moreover, in the interests of clarity, we consider only two classes of *discount* debt. More debt classes, and coupon payments, could be introduced at the expense of notational simplicity.

Denote the principals of senior and junior debt by F_S and F_J respectively. If we let $F = F_S + F_J$ be the total nominal amount due at maturity, we can continue to use the definition $k^F = \min(F,k)$ to denote maximum reorganization costs that can be realized at maturity. The value of senior

[18] The two debt contracts have the same principal and coupon series with equal (risk free) present values.

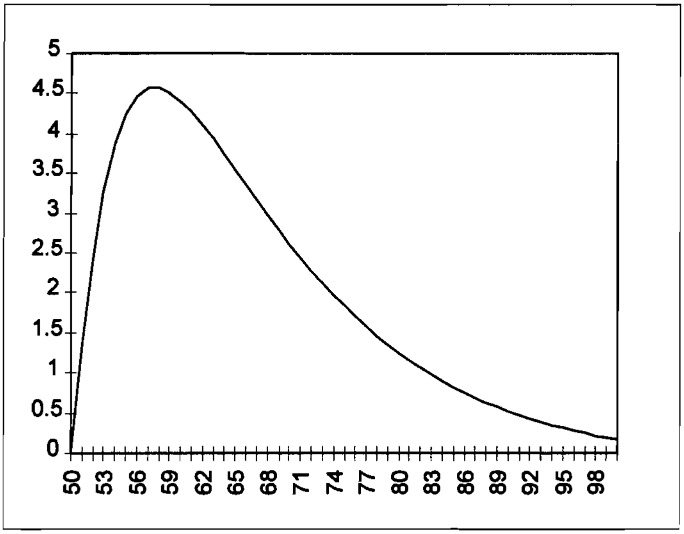

FIGURE 1.3. APPROXIMATING DISCRETE WITH CONTINUOUS COUPON PAYMENTS. Percentage difference between the value of discrete and continuous coupon debt. $L = 50$, $\sigma = 0.30$, $T = 5$, $k = 20$, $c = 0.1$ (corresponding discrete coupon $c = 0.126$), $r = 0.1$.

and junior debt may be expressed as

$$\begin{aligned} B_S &= B_S^M + B_S^\tau \\ B_J &= B_J^M + B_J^\tau \end{aligned}$$

We now turn to the valuation of these blocks.

1.4.2.1 Value of Payments at Maturity

When we have two classes of debt the payoff structures at maturity differ depending on how the principals and reorganization costs are interrelated. In figure 1.4 we depict the payoffs to the different securities in the situation when $k < F_J < F_S$.

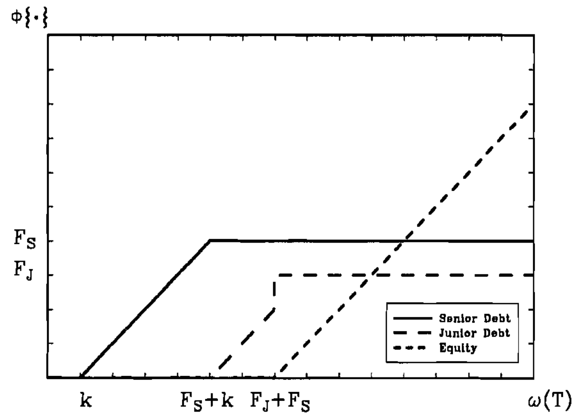

FIGURE 1.4. SENIOR AND JUNIOR DEBT. Payoffs at maturity to equity, senior and junior debt when $k < F_J < F_S$.

Formally, in this particular case, the payoff functions of debt, equity and reorganization costs are mimicked by the following portfolios of basic claims payoffs

$$\begin{aligned}
\Phi\left\{B_J^M\right\} &= \Phi\left\{C(\cdot;F_S+k,T)\right\} - \Phi\left\{C(\cdot;F,T)\right\} + k\cdot\Phi\left\{H(\cdot;F,T)\right\} \\
\Phi\left\{B_S^M\right\} &= \Phi\left\{C(\cdot;k,T)\right\} - \Phi\left\{C(\cdot;F_S+k,T)\right\} \\
\Phi\left\{E^M\right\} &= \Phi\left\{C(\cdot;F,T)\right\} \\
\Phi\left\{K^M\right\} &= \Phi\left\{\Omega(\cdot;T)\right\} - \Phi\left\{C(\cdot;k,T)\right\} - k\cdot\Phi\left\{H(\cdot;F,T)\right\}
\end{aligned}$$

The mimicking claims in the other cases, when the ordering of principals and bankruptcy costs is different, will be stated in the corollary.

1.4.2.2 Value of Payments in Case of Reorganization Prior to Maturity

The payoffs in intermediate financial distress are given by the following expressions:

$$\Phi\left\{B_S^\tau\right\} = \min\left((L-k)^+, F_S\right) \tag{1.21}$$

$$\Phi\left\{B_J^\tau\right\} = \min\left((L-k-F_S)^+, F_J\right) \tag{1.22}$$

$$\Phi\left\{E^\tau\right\} = (L-F-k)^+ \tag{1.23}$$

$$\Phi\left\{K^\tau\right\} = \min(L,k) \tag{1.24}$$

In analogy with the analysis above (when the capital structure of the firm consisted only of equity and a single class of debt), the value of the payments must sum to a down-and-in contract on the asset value of the firm. Hence the values of payments in financial distress must be fixed fractions of the corresponding down-and-in contract.

Corollary 1.4 *Assumptions 1-4b-7 hold. Consider a firm with a capital structure consisting of equity and two discount debt issues, one senior with principal F_S and the other junior with principal F_J, both of maturity T. The values of senior debt, junior debt, equity and reorganization costs are then given by*

$$B_S(\cdot;\cdot) = \begin{cases} C_L(\cdot;k^F,T) - C_L(\cdot;F_S+k^F,T) & \text{if } k^F < F_J \\ C_L(\cdot;k^F,T) - C_L(\cdot;F,T) + \left(k^F - F_J\right) H_L(\cdot;F,T) & \text{if } k^F \geq F_J \end{cases} + \frac{\Phi\left\{B_S^\tau\right\}}{L}\Omega^L(\cdot;T)$$

$$B_J(\cdot;\cdot) = \begin{cases} C_L(\cdot;F_S+k^F,T) - C_L(\cdot;F,T) + k^F H_L(\cdot;F,T) & \text{if } k^F < F_J \\ F_J H_L(\cdot;F,T) & \text{if } k^F \geq F_J \end{cases} + \frac{\Phi\left\{B_J^\tau\right\}}{L}\Omega^L(\cdot;T)$$

$$E(\cdot;\cdot) = C_L(\cdot;F,T) + \frac{\Phi\left\{E^\tau\right\}}{L}\Omega^L(\cdot;T)$$

$$K(\cdot;\cdot) = \Omega_L(\cdot;T) - C_L(\cdot;k^F,T) - k^F H_L(\cdot;F,T) + \frac{\Phi\left\{K^\tau\right\}}{L}\Omega^L(\cdot;T)$$

where $\Phi\left\{B_S^\tau\right\}$ and $\Phi\left\{B_J^\tau\right\}$ are given by (1.21) and (1.22). $\Phi\left\{E^\tau\right\}$ and $\Phi\left\{K^\tau\right\}$ are given in Lemma 1.1.

1.4.3 Common Stock, Warrants and Debt

The purpose of this and the following section on convertible debt is to illustrate how the suggested technique can be applied to other than standard debt and equity securities. We will make a number of simplifying assumptions in order to make the exposition transparent and thus we do not suggest that the derived pricing formulae are well suited for actual pricing purposes without further extension. In particular, as we have already pointed out, we restrict ourselves to the case where these securities are European, an assumption which may be less appropriate here than for debt.

Consider a firm whose capital structure consists of n shares of common stock, m warrants with value $\mathfrak{W}$ and an issue of discount debt with principal F. The holder of a warrant has the right to obtain a newly issued share of common stock at an exercise price X at time T. As in the previous sections, we assume an equal maturity for the corporate securities. When the maturity of debt exceeds the time to expiration of the warrant and if the proceeds from the exercise are reinvested in the firm, Crouhy & Galai (1994) show that a transfer of wealth from shareholders to bondholders may occur. This is the result of a decreased probability of reorganization due to the increase in the firm's value relative to its debt obligations.

We can graphically represent the payments at maturity to the holders of the respective securities as in figure 1.5.

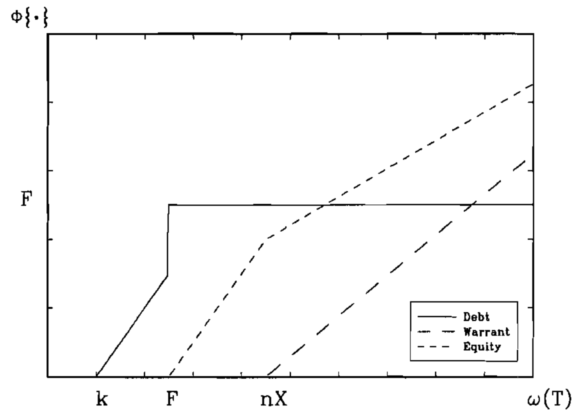

FIGURE 1.5. WARRANTS. Payoffs at maturity to equity, debt and warrants.

At maturity, if it is worthwhile exercising the warrants, their value will be

$$\begin{aligned} m\mathfrak{W}(T) &= \left(\frac{m}{n+m}\right)(\omega(T)+mX-F)-mX \\ &= \left(\frac{m}{n+m}\right)(\omega(T)-F-nX) \end{aligned}$$

so we require that $\omega(T) - F > nX$ in order for the holder of the warrant to exercise. Note that the value of debt is not affected by the exercise of the warrants since this will only happen when the firm is solvent at maturity.

Given the payoff functions implicit in figure 1.5 and the payoffs to the respective securities in financial distress we have the following result.

Corollary 1.5 *Assumptions 1-4b-7 hold. Consider a firm with a capital structure consisting of n shares of common stock, m warrants with exercise price X and an issue of discount debt with principal F. The values of debt and reorganization costs remain as in Corollary 1.1. The values of equity and the warrants are given by*

$$\begin{aligned} E(\cdot;F,T) &= C_L(\cdot;F,T) - \frac{n}{n+m}C_L(\cdot;X+F,T) + \frac{\Phi\{E^\tau\}}{L}\Omega^L(\cdot;T) \\ \mathfrak{W}(\cdot;T) &= \frac{n}{n+m}C_L(\cdot;X+F,T) \end{aligned}$$

where $\Phi\{E^\tau\}$ is given by Lemma 1.1.

1.4.4 Common Stock and Convertible Debt

In this section we suggest a pricing formula for the price of a discount European convertible bond (CB) when the capital structure in addition to this instrument contains only common stock. Hence we have

$$\begin{aligned} CB &= CB^\tau + CB^M \\ E &= E^\tau + E^M \end{aligned}$$

Since the convertible is European the payoffs in intermediate financial distress are the same as for standard discount bonds, and CB^τ and E^τ are therefore given by equations (1.15) and (1.16). To value maturity payments we need to study the conversion behaviour.

The conversion behaviour will differ substantially from the standard case discussed in Cox & Rubinstein (1985) due to the presence of reorganization costs and the fact that all debt is convertible. There will be multiple regions of conversion, both in the lower and the higher end of the range of asset values at maturity.

Suppose the firm has outstanding n shares of common stock and z convertible bonds. A convertible can be converted at maturity to ε shares of newly issued stock at no cost (as opposed to the case of warrants above when the holder has to pay an exercise price up front on conversion). So if holders of a convertible decide to convert they will own a fraction

$$\delta = \frac{z\varepsilon}{n + z\varepsilon}$$

of the firm's value. Hence in order for conversion to be optimal we require that $\delta\omega(T)$ be higher than the payoff he would obtain if he did not convert.

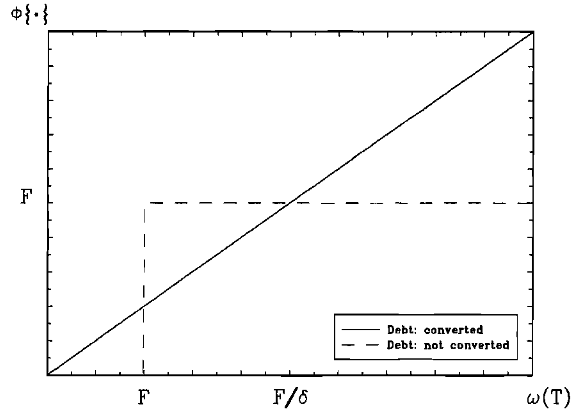

FIGURE 1.6. CONVERTIBLE DEBT. Payoffs to a convertible: conversion vs no conversion when $k > F$.

The conversion behaviour will depend on whether convertible holders receive anything in reorganization; i.e. on whether k is larger or smaller than F. Consider first the extreme case where reorganization costs exceed the principal, $k > F$. This situation is depicted in diagram 1.6.

First in the higher range of maturity asset values, when $\omega(T) > \frac{F}{\delta}$, conversion will take place as in the standard case without costs to financial distress, reflecting the normal embedded call options in the debt contract. When asset value falls in the range $\left[F...\frac{F}{\delta}\right]$, bondholders will be better of by accepting repayment of the principal. Now consider what happens when the asset value is insufficient to repay the principal. If they force the firm into reorganization, they will receive nothing since $k > F$ but if they convert they will receive $\delta\omega(T)$. Hence it will be optimal to convert in order to avoid the costs of reorganization. So no matter what happens the payoff to debt will be positive and the firm will never default at maturity. Graphically we can see from figure 1.6 that conversion will take place when the payoff schedule from not converting lies below the linear payoff schedule of conversion.

Second, consider the case where reorganization costs are less than the principal, $k < F$. This situation is depicted in figure 1.7.

As in the previous case, conversion will take place when the payoff schedule from not converting lies below the payoff schedule of conversion. First, as above, conversion will take place when $\delta\omega(T) > F$. Initially when $\omega(T) < \frac{F}{\delta}$ conversion will not take place. When the firm cannot pay back the principal, as opposed to the case above, it will not pay to convert until payoff in

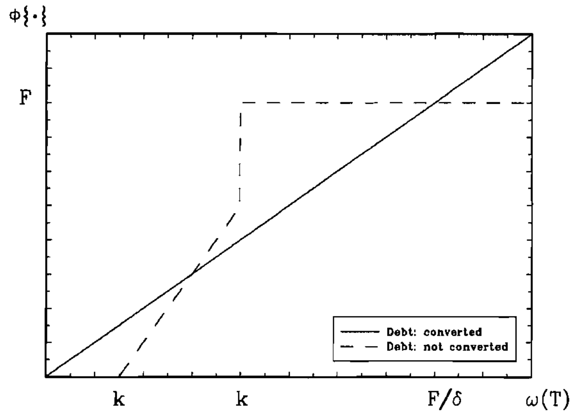

FIGURE 1.7. PAYOFFS TO A CONVERTIBLE: conversion vs no conversion when $k < F$.

reorganization is less then the conversion value, i.e. when $\omega(T) - k < \delta\omega(T)$ or $\omega(T) < \frac{k}{1-\delta}$. Hence we see that at the lower end of the range of asset values reorganization will not occur, while it *may* in an intermediate range. If $\frac{k}{1-\delta} > F$ it will always be optimal for bondholders to convert to avoid reorganization.

With the previous discussion and figures 1.6 and 1.7, we can plot the payoff functions at maturity:

The calls and Heavisides necessary to mimic these payoffs are given in the corollary below.

Corollary 1.6 *Assumptions 1-4b-7 hold. Consider a firm with a capital structure consisting of n shares of common stock and z convertible bonds, each which can be converted into ω shares of the stock at maturity. The values of debt, equity and reorganization costs are then given by*

$$\begin{aligned}
when\ k &> F \\
CB(\cdot;T) &= \delta\left(\Omega_L(\cdot;T) - C_L(\cdot;F,T) + C_L\left(\cdot;\frac{F}{\delta},T\right)\right) \\
&\quad +(1-\delta)F\cdot H_L(\cdot;F,T) \\
&\quad +\frac{\Phi\{CB^{\tau}\}}{L}\Omega^L(\cdot;T) \\
E(\cdot;T) &= (1-\delta)\left(\Omega_L(\cdot;T) - F\cdot H_L(\cdot;F,T)\right)
\end{aligned}$$

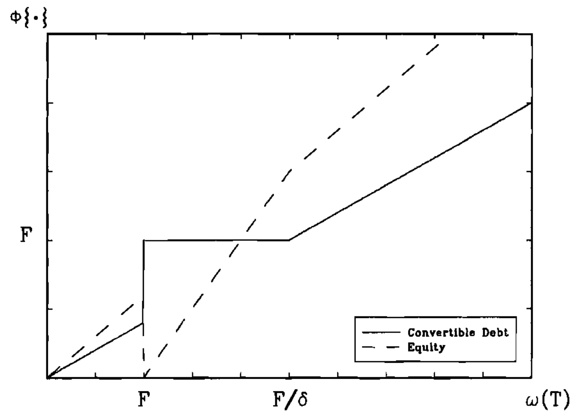

FIGURE 1.8. PAYOFFS TO DEBT AND EQUITY at maturity when $k > F$.

$$
\begin{aligned}
& +\delta\left(C_L(\cdot;F,T) - C_L\left(\cdot;\frac{F}{\delta},T\right)\right) \\
& +\frac{\Phi\{E^\tau\}}{L}\Omega^L(\cdot;T) \\
K(\omega;T) &= \frac{\Phi\{K^\tau\}}{L}\Omega^L(\cdot;T)
\end{aligned}
$$

$$
\begin{aligned}
\textit{when } k &< F \\
CB(\cdot;T) &= \delta\Omega_L(\cdot;T) + (1-\delta)C_L\left(\cdot;\frac{k}{1-\delta},T\right) - C_L(\cdot;F,T) \\
&\quad +kH_L(\cdot;F,T) + \delta C_L\left(\cdot;\frac{F}{\delta},T\right) + \frac{\Phi\{CB^\tau\}}{L}\Omega^L(\cdot;T) \\
E(\cdot;T) &= (1-\delta)\left(\Omega_L(\cdot;T) - C_L\left(\cdot;\frac{k}{1-\delta};T\right)\right) - kH_L\left(\cdot;\frac{k}{1-\delta},T\right) \\
&\quad +C_L(\cdot;F,T) - \delta C_L\left(\cdot;\frac{F}{\delta},T\right) + \frac{\Phi\{E^\tau\}}{L}\Omega^L(\cdot;T) \\
K(\omega;T) &= k\cdot\left(H_L\left(\cdot;\frac{k}{1-\delta},T\right) - H_L(\cdot;F,T)\right) + \frac{\Phi\{K^\tau\}}{L}\Omega^L(\cdot;T)
\end{aligned}
$$

where $\Phi\{B^\tau\}$, $\Phi\{E^\tau\}$ *and* $\Phi\{K^\tau\}$ *are given by Lemma 1.1.*

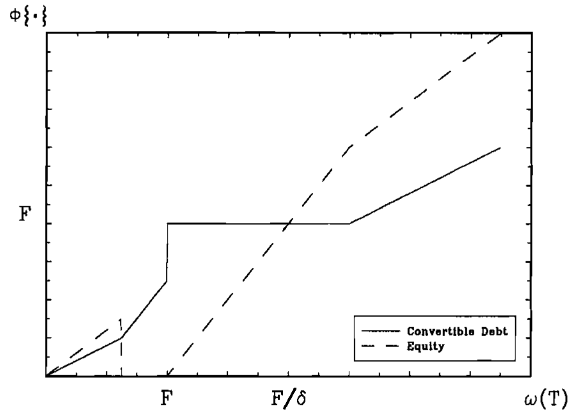

FIGURE 1.9. PAYOFFS TO DEBT AND EQUITY at maturity when $k < F$.

1.5 Concluding Remarks

This paper suggests a framework for valuing corporate securities. The framework can accommodate quite general securities while allowing for financial distress prior to maturity through the assumption that the firm will default once the value of its assets reaches some exogenously given barrier[19]. Various economic modelling assumptions present in the literature are discussed and put in relation with the technical assumptions that are required to obtain closed form solutions.

[19]There is nothing in our setup to preclude the modelling of an endogenous default barrier. We will as an illustration derive the expression for equity in Leland (1994*b*) extension of Black & Cox (1976) using our approach. The value of equity is equal to a perpetual down-and-out call option minus a perpetual down and out coupon stream $E(\cdot;\infty) = C_L(\cdot;L,\infty) - (1-\varkappa)cF\int_t^\infty H_L(\cdot;0,s)ds$. From eq. (1.3) the call is worth $\omega - \left(\frac{L}{\omega}\right)^\alpha \frac{L^2}{\omega}$. Using $H = H_L + H^L$, the fact that $\int_t^\infty H^L(\cdot;0,s)ds = \frac{\Omega_L(\cdot;\infty)}{L}\frac{1}{r}$ together with equations (1.5) and (1.4) the coupon stream can be found to be worth $\frac{1}{r}\left\{1-\left(\frac{L}{\alpha}\right)^{\alpha+1}\right\}$. Thus the value of equity is

$$E(\cdot;\infty) = \omega - (1-\varkappa)\,cF\frac{1}{r} + \left\{(1-\varkappa)\,cF\frac{1}{r} - L\right\}\left(\frac{L}{\omega}\right)^\alpha$$

Maximizing this expression with respect to L and plugging in the optimal barrier we obtain Leland's formula. The expression for debt can be derived along the same lines. Setting bankruptcy costs and taxes to zero, we obtain the Black and Cox model.

We apply the suggested approach to a number of different capital structures and show how the relevant corporate securities can be valued in terms of two down-and-out contracts, a call option and a Heaviside contract (a binary option). Since these claims in turn can be represented in terms of standard non-barrier claims, we are also able to indirectly value the securities as portfolios of these. We obtain closed form solutions for the analysed corporate securities - for example we provide closed form formulae for discrete coupon debt with finite maturity.

For practical purposes it is important to know how to hedge a corporate security. In our framework it is straightforward to obtain hedge parameters in terms of the hedge parameters of the two "building blocks" - the call and the Heaviside. *Appendix 1.6* provides a list of these.

In short, we provide an easy-to-apply yet flexible tool for the valuation of corporate securities.

1.6 Appendix: Comparative Statics of the Building Blocks

We have noted above that we can in general separate the price of a barrier claim into the price of a portfolio of standard non-barrier claims. This allows us to use the hedge statistics of standard contracts in order to obtain the comparative statics of the basic barrier claims that we will be using to value corporate securities in what follows.

We first need to define some notation. Superscripts will denote the security that we are taking the derivative of. A subscript L will be denote a down-and-out feature. The symbols for the different partial derivatives are given in the following table.

Security	w.r.t. ω	2nd w.r.t. ω	w.r.t. r	w.r.t. t	w.r.t. σ
Call	Δ^C	Γ^C	R^C	Θ^C	Λ^C
Heaviside	Δ^H	Γ^H	R^H	Θ^H	Λ^H
DAO Call	Δ^C_L	Γ^C_L	R^C_L	Θ^C_L	Λ^C_L
DAO Heaviside	Δ^H_L	Γ^H_L	R^H_L	Θ^H_L	Λ^H_L

First we list the comparative statics of the two standard (non-barrier) claims, the Heaviside and the call.

$$
\begin{aligned}
\Delta^H(\omega, X) &= \frac{e^{-r(T-t)} N'(d_2)}{\sigma\omega\sqrt{T-t}} \\
\Delta^C(\omega, X) &= N(d_1) \\
\Gamma^H(\omega, X) &= \frac{e^{-r(T-t)} N'(d_2)}{\sigma^2\omega^2 (T-t)} d_2 \\
\Gamma^C(\omega, X) &= \frac{N'(d_1)}{\sigma\omega\sqrt{T-t}} \\
R^H(\omega, X) &= -(T-t)H(\omega, X) + e^{-r(T-t)} \frac{\sqrt{T-t}N'(d_2)}{\sigma} \\
R^C(\omega, X) &= (T-t)Xe^{-r(T-t)} N(d_2) \\
\Lambda^H(\omega, X) &= -e^{-r(T-t)} N'(d_2) \frac{d_1}{\sigma} \\
\Lambda^C(\omega, X) &= \omega\sqrt{T-t}N'(d_1) \\
\Theta^H(\omega, X) &= rH(\omega, X) + e^{-r(T-t)} N'(d_2) \left(\frac{\ln\frac{\omega}{X} - (\tilde{r} - \frac{1}{2}\sigma^2)(T-t)}{2\sigma(T-t)^{\frac{3}{2}}} \right) \\
\Theta^C(\omega, X) &= -\frac{\omega N'(d_1)\sigma}{2\sqrt{T-t}} - rXe^{-r(T-t)} N(d_2)
\end{aligned}
$$

The formulae for the comparative statics of the building blocks, the down-and-out Heaviside and call, are then given by:

$$\Delta_L^H(\omega,X) = \begin{cases} \Delta^H(\omega,X) - \left(\frac{L}{\omega}\right)^\alpha \Delta^H\left(\frac{L^2}{\omega},X\right) + \frac{\alpha}{\omega}\left(\frac{L}{\omega}\right)^\alpha H\left(\frac{L^2}{\omega},X\right) & \text{if } X \geq L \\ \Delta^H(\omega,L) - \left(\frac{L}{\omega}\right)^\alpha \Delta^H\left(\frac{L^2}{\omega},X\right) + \frac{\alpha}{\omega}\left(\frac{L}{\omega}\right)^\alpha H\left(\frac{L^2}{\omega},L\right) & \text{if } X < L \end{cases}$$

$$\Delta_L^C(\omega,X) = \begin{cases} \Delta^C(\omega,L) - \left(\frac{L}{\omega}\right)^\alpha \Delta^C\left(\frac{L^2}{\omega},X\right) + \frac{\alpha}{\omega}\left(\frac{L}{\omega}\right)^\alpha C\left(\frac{L^2}{\omega},X\right) & \text{if } X \geq L \\ \Delta^C(\omega,L) - \left(\frac{L}{\omega}\right)^\alpha \Delta^C\left(\frac{L^2}{\omega},L\right) + \frac{\alpha}{\omega}\left(\frac{L}{\omega}\right)^\alpha C\left(\frac{L^2}{\omega},L\right) \\ \quad +(L-X)\Delta_L^H(\omega,L) & \text{if } X < L \end{cases}$$

$$\Gamma_L^H(\omega,X) = \begin{cases} \Gamma^H(\omega,X) - \left(\frac{L}{\omega}\right)^\alpha \Gamma^H\left(\frac{L^2}{\omega},X\right) + \frac{2\alpha}{\omega}\left(\frac{L}{\omega}\right)^\alpha \Delta^H\left(\frac{L^2}{\omega},X\right) \\ \quad -\frac{(1+\alpha)\alpha}{\omega^2}\left(\frac{L}{\omega}\right)^\alpha H\left(\frac{L^2}{\omega},X\right) & \text{if } X \geq L \\ \Gamma^H(\omega,L) - \left(\frac{L}{\omega}\right)^\alpha \Gamma^H\left(\frac{L^2}{\omega},L\right) + \frac{2\alpha}{\omega}\left(\frac{L}{\omega}\right)^\alpha \Delta^H\left(\frac{L^2}{\omega},L\right) \\ \quad -\frac{(1+\alpha)\alpha}{\omega^2}\left(\frac{L}{\omega}\right)^\alpha H\left(\frac{L^2}{\omega},L\right) & \text{if } X < L \end{cases}$$

$$\Gamma_L^C(\omega,X) = \begin{cases} \Gamma^C(\omega,X) - \left(\frac{L}{\omega}\right)^\alpha \Gamma^C\left(\frac{L^2}{\omega},X\right) + \frac{2\alpha}{\omega}\left(\frac{L}{\omega}\right)^\alpha \Delta^C\left(\frac{L^2}{\omega},X\right) \\ \quad -\frac{(1+\alpha)\alpha}{\omega^2}\left(\frac{L}{\omega}\right)^\alpha C\left(\frac{L^2}{\omega},X\right) & \text{if } X \geq L \\ \Gamma^C(\omega,L) - \left(\frac{L}{\omega}\right)^\alpha \Gamma^C\left(\frac{L^2}{\omega},L\right) + \frac{2\alpha}{\omega}\left(\frac{L}{\omega}\right)^\alpha \Delta^C\left(\frac{L^2}{\omega},L\right) \\ \quad -\frac{(1+\alpha)\alpha}{\omega^2}\left(\frac{L}{\omega}\right)^\alpha C\left(\frac{L^2}{\omega},L\right) + (L-X)\Gamma_L^H(\omega,L) & \text{if } X < L \end{cases}$$

$$\Theta_L^H(\omega,X) = \begin{cases} \Theta^H(\omega,X) - \left(\frac{L}{\omega}\right)^\alpha \Theta^H\left(\frac{L^2}{\omega},X\right) & \text{if } X \geq L \\ \Theta^H(\omega,L) - \left(\frac{L}{\omega}\right)^\alpha \Theta^H\left(\frac{L^2}{\omega},L\right) & \text{if } X < L \end{cases}$$

$$\Theta_L^C(\omega,X) = \begin{cases} \Theta^C(\omega,X) - \left(\frac{L}{\omega}\right)^\alpha \Theta^C\left(\frac{L^2}{\omega},X\right) & \text{if } X \geq L \\ \Theta^C(\omega,L) - \left(\frac{L}{\omega}\right)^\alpha \Theta^C\left(\frac{L^2}{\omega},L\right) \\ \quad +(L-X)\Theta_L^H(\omega,L) & \text{if } X < L \end{cases}$$

$$R_L^H(\omega, X) = \begin{cases} R^H(\omega, X) - \left(\frac{L}{\omega}\right)^\alpha R^H\left(\frac{L^2}{\omega}, X\right) \\ -\frac{2}{\sigma^2}\ln\frac{L}{\omega}\left(\frac{L}{\omega}\right)^\alpha H\left(\frac{L^2}{\omega}, X\right) \text{ if } X \geq L \\ \\ R^H(\omega, L) - \left(\frac{L}{\omega}\right)^\alpha R^H\left(\frac{L^2}{\omega}, L\right) \\ -\frac{2}{\sigma^2}\ln\frac{L}{\omega}\left(\frac{L}{\omega}\right)^\alpha H\left(\frac{L^2}{\omega}, L\right) \text{ if } X < L \end{cases}$$

$$R_L^C(\omega, X) = \begin{cases} R^C(\omega, X) - \left(\frac{L}{\omega}\right)^\alpha R^C\left(\frac{L^2}{\omega}, X\right) \\ -\frac{1}{\sigma^2}\left(\frac{L}{\omega}\right)^{\alpha+2} C\left(\frac{L^2}{\omega}, X\right) \text{ if } X \geq L \\ \\ R^C(\omega, L) - \left(\frac{L}{\omega}\right)^\alpha R^C\left(\frac{L^2}{\omega}, L\right) \\ -\frac{1}{\sigma^2}\left(\frac{L}{\omega}\right)^{\alpha+2} C\left(\frac{L^2}{\omega}, L\right) + (L - X) R_L^H(\omega, L) \text{ if } X < L \end{cases}$$

$$\Lambda_L^H(\omega, X) = \begin{cases} \Lambda_L^H(\omega, X) - \left(\frac{L}{\omega}\right)^\alpha \Lambda^H\left(\frac{L^2}{\omega}, X\right) \\ +\frac{4\tilde{r}}{\sigma^3}\ln\frac{L}{\omega}\left(\frac{L}{\omega}\right)^\alpha H\left(\frac{L^2}{\omega}, X\right) \text{ if } X \geq L \\ \\ \Lambda_L^H(\omega, L) - \left(\frac{L}{\omega}\right)^\alpha \Lambda^H\left(\frac{L^2}{\omega}, L\right) \\ +\frac{4\tilde{r}}{\sigma^3}\ln\frac{L}{\omega}\left(\frac{L}{\omega}\right)^\alpha H\left(\frac{L^2}{\omega}, L\right) \text{ if } X < L \end{cases}$$

$$\Lambda_L^C(\omega, X) = \begin{cases} \Lambda_L^C(\omega, X) - \left(\frac{L}{\omega}\right)^\alpha \Lambda^C\left(\frac{L^2}{\omega}, X\right) \\ +\frac{4\tilde{r}}{\sigma^3}\ln\frac{L}{\omega}\left(\frac{L}{\omega}\right)^\alpha C\left(\frac{L^2}{\omega}, X\right) \text{ if } X \geq L \\ \\ \Lambda_L^C(\omega, L) - \left(\frac{L}{\omega}\right)^\alpha \Lambda^C\left(\frac{L^2}{\omega}, L\right) \\ +\frac{4\tilde{r}}{\sigma^3}\ln\frac{L}{\omega}\left(\frac{L}{\omega}\right)^\alpha C\left(\frac{L^2}{\omega}, L\right) + (L - X)\Lambda_L^H(\omega, L) \\ \text{if } X < L \end{cases}$$

2

A New Compound Option Pricing Model

2.1 Introduction

Contingent claims analysis has frequently been used to price corporate securities and related derivatives. The idea is that securities can be valued as options and that consequently ordinary (stock) options are valued as compound options. Only when the models extend the basic Black-Scholes framework, however, do the pricing formulae for the corporate securities generate prices in line with actual ones. Yet no such model also incorporating derivatives in a general way has been developed.

We suggest a comprehensive model which allows us to incorporate common contractual features and stylized facts. More specifically, we derive a closed form solution for the price of a call option on a down-and-out call. We then show how the obtained result can be generalized in order to price options on complex corporate securities, allowing among other things for corporate taxation, costly financial distress and deviations from the absolute priority rule. The characteristics of the model are illustrated with numerical examples.

Black & Scholes (1973) suggest for the first time that corporate securities can be viewed as options on the underlying firm value and Merton (1974) provides an application of these ideas to the pricing of corporate debt. The same insight is exploited by Geske (1979) who takes a step further in pricing stock options as compound options on the firm's asset value. In doing, so the stock volatility becomes an endogenous stochastic process dependent on firm leverage, relaxing the empirically refuted assumption of constant stock return volatility.

A shortcoming of early models is that financial distress only can occur at maturity of debt. Kim et al. (1993) show that Merton's 1974 model is unable to generate credit spreads in line with levels observed in practice. Hence, the corresponding equity prices and thus also option prices are likely to be biased. Black & Cox (1976) allow for financial distress prior to the maturity of debt by modelling default as taking place when the value of the firm's assets hits a lower boundary[1]. However, they value only corporate securities and do not address the pricing of options. Toft & Prucyk (1996), on the other hand, do value stock options in a firm value based model but restrict their analysis to a class of capital structures with perpetual debt.

Ericsson & Reneby (1995) suggest a simple framework to apply the option approach to the valuation of corporate securities when default is triggered by a barrier in addition to the inability to repay debt at maturity. Claims to the firm's assets with general payoff structures can be priced as portfolios of standard and binary barrier options. In this paper we build on these ideas in order to develop a framework for valuing European options on corporate securities as compound contingent claims when default can occur at a random point in time. It is our view that the framework should, above all, be applied to non-standardized securities such as OTC-derivatives, new debt issues with embedded options and financial guarantees. A new debt issue, for instance, alters the capital structure. Other models, that do not take this into account, will fail to capture the ensuing changes in the volatilities of the firm's securities. Options, being inherently sensitive to volatility changes, thus require a structural model such as ours to be accurately priced in those circumstances.

We view the main contribution of this paper as being twofold. First, we extend Geske's compound option pricing model to encompass an underlying down-and-out call. Second, we provide a general framework in which both underlying corporate securities and derivatives can be valued consistently in an environment where their volatility is driven by changes in leverage.

The paper is organized as follows. Section 2.2 lays out the modelling environment and the basic assumptions. Section 2.3 derives the price of a call written on a down-and-out call option. Section 2.4 generalizes to options on general corporate securities. The section which follows contains some numerical results and section 2.6 concludes.

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[1] Other models that have since employed this approach of modelling financial distress are Nielsen et al. (1993), Kim et al. (1993), Leland (1994*b*), Leland & Toft (1996) and Longstaff & Schwartz (1995).

The pricing of barrier options was pioneered by Merton (1973) and has since been treated in a multitude of papers including Björk (1994), Rich (1994) and Carr (1995).

2.2 Setup

We make the standard Black and Scholes assumptions about the economy. The interest rate is constant, trading takes place continuously without transaction costs, unlimited short sales are permitted and arbitrage opportunities are ruled out.

We assume that the state variable determining the value of the firm's assets at time T follows an exogenous stochastic process

$$\begin{cases} dv = \mu v dt + \sigma v dW^P \\ v(0) = v_0 \end{cases}$$

where $W^P(t)$ with $W^P(0) = 0$ is a Wiener process under the objective probability measure.

Note that we do *not* assume that the assets are continuously traded - we only need to assume that they are traded at some prespecified date in the future when the firm is liquidated. With this assumption, Lemma 1 in Ericsson & Reneby (1997), chapter 3 in this volume, gives us the process for the *value* of assets[2]

$$\begin{cases} d\omega = (r + \lambda\sigma)\,\omega dt + \sigma\omega dW^P \\ \omega(0) = \omega_0 \end{cases} \tag{2.1}$$

where $W^P(t)$ is a Wiener-process. The term $(r + \lambda\sigma)$ is the expected return from holding the firm's assets and σ is their volatility. The parameter λ can be interpreted as the market price of risk associated with the operations of the firm.

Under the equivalent probability measure Q, where discounted price processes are martingales, the asset value has the following dynamics

$$\begin{cases} d\omega = r\omega dt + \sigma\omega dW^Q \\ \omega(0) = \omega_0 \end{cases} \tag{2.2}$$

Under this measure[3] $W^Q(t) \equiv W^P(t) + \lambda t$ is a Wiener process.

Ericsson & Reneby (1997) discuss the state variable assumptions for firm value based pricing models, in some detail. They show that one does not have to assume that the firm's assets are continuously traded. The existence of (2.1) follows as a result, as long as one is willing to assume that at one future date the value of the firm's assets will be traded.

[2] From standard arguments it follows that

$$\begin{aligned} \omega(t) &= e^{-r(T-t)}E^Q\left[v(T)\right] \\ &= v(t)\,e^{-(\mu-r)(T-t)} \end{aligned}$$

[3] Popularly termed the risk neutral measure.

Default prior to maturity occurs when the value of the firm's assets reaches a lower barrier L, an exogenous constant[4]. To develop a comprehensive model of the bankruptcy decision would complicate the valuation procedure considerably if it resulted in a non constant barrier. We therefore use a constant barrier to approximate the outcome of an endogenized bankruptcy mechanism. For a more detailed discussion of these issues see Ericsson & Reneby (1995), chapter 1 of this volume.

We take the firm's investment policy to be independent of its financial policy which in this setting implies that the parameters of the asset value process are exogenous constants. Note that this rules out any strategic considerations such as, for example, opportunistic investment policy changes by the management on behalf of shareholders[5].

The assumption of a constant risk-free interest rate may seem restrictive, in particular if one wishes to value corporate debt and associated options. However, Kim et al. (1993) show that although yields on Treasury and noncallable corporate bonds are sensitive to the modelling of interest rate risk, the spread between the yields on these securities is relatively insensitive to interest rate uncertainty. Thus, in situations where assuming a constant interest rate is clearly inappropriate, our model may be used to price credit risk discounts.

As already mentioned, corporate securities can be valued as combinations of barrier contingent claims, given that financial distress prior to maturity of debt is caused by the value of the firm's assets hitting an exogenous barrier. Expressly, it is sufficient with two kinds of claims, down-and-out call options and down-and-out Heavisides.

Remark 2.1 *We define a Heaviside contract as a binary option with unit payoff if, at its expiry date, the underlying state variable exceeds the strike price, and zero otherwise. A down-and-out contract becomes worthless when the value of the underlying state variable hits a predetermined barrier. The pricing functions of down-and-out calls and Heavisides can be found in Appendix 2.8.*

Hence, in order to value derivatives on a firm's securities we will only need to consider claims on these two down-and-out contracts. In what follows we focus on a call option on a down-and-out call. In section 2.4 the pricing formula is generalised in order to value options on complex corporate securities in a straightforward manner.

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[4] We could easily extend the analysis to allow for a barrier that evolves exponentially over time.

[5] See Ericsson (1997), chapter 4 in this volume, for a model in which this is not the case. In this paper the volatility parameter σ is left to the discretion of the firm's management.

2.3 The Compound Barrier Call

The objective of this section is to price a call option on a down-and-out call. The underlying call matures at T and has exercise price F and the compound call matures at $S < T$ with exercise price K. This can be thought of as a call with exercise price K written on equity in a firm that has issued discount debt with face value F.

Denote with $C(f(t); X, S)$ the price of a call at time t with exercise prices X and maturity at S written on an asset f. Also, let subscript $_L$ denote the down-and-out feature with barrier L. For notational convenience we let all pricing take place at time 0. Then we can write the price of the compound call as

$$C\left(C_L(\omega_0; F, T); K, S\right)$$

NOTATION

- Let $I_{\{A,B,C,...\}}$ be an indicator function taking the value one if all the events $A, B, C, ...$ have occurred.
- Denote with τ_S the first passage time of ω to L in the interval $\theta_S = [0, S]$ and with τ_T the first passage time in the interval $\theta_T = (S, T]$.

With this notation the down-and-out feature is equivalent to multiplying the claim's payoff by $I_{\{\tau_S \notin \theta_S, \tau_T \notin \theta_T\}}$. Thus $C_L(\omega(T); X, T) = C(\omega(T); X, T) \times I_{\{\tau_S \notin \theta_S, \tau_T \notin \theta_T\}}$.

- Superscripts will generally refer to probability measures: Q^i denotes a probability measure, W^i denotes a Wiener process under Q^i and $R^{j \to i}$ denotes the Radon-Nikodym derivative $R^{j \to i} = \frac{dQ^i}{dQ^j}$ with associated Girsanov-kernel $h^{j \to i}$. Thus superscripts "$j \to i$" should be read *from probability measure 'j' to probability measure 'i'*.
- $\Im_t$ denotes the information structure at time t.
- Define $Q^i(A) \equiv E^{Q^i}\left[I_{\{A\}} | \Im_0\right]$, that is the Q^i-probability of event A.
- Denote with $N(a, b, \rho)$ and $N(a)$ the standardized normal cumulative distribution functions for the bi- and univariate cases, respectively.

We take the probability measure Q^1 to be the one under which all price processes normalized by a unit of the money market account are martingales (the so called risk neutral probability measure). We then know from standard theory that we can write the price of the compound call as

$$C(C_L(\omega_0; F, T); K, S) = e^{-rS} \cdot E^{Q^1}\left[\left(C_L\left(\omega\left(S\right); F, T\right) - K\right) \cdot I_{\{C_L > K\}} | \Im_0\right]$$

The value of the underlying call at time S will be

$$C_L\left(\omega\left(S\right);F,T\right)$$

$$=$$

$$e^{-r(T-S)}\cdot E^{Q^1}\left[\left(\omega(T)-F\right)\cdot I_{\{\omega(T)>F,\tau_T\notin\theta_T\}}\left|\Im_S\right.\right]\cdot I_{\{\tau_S\notin\theta_S\}}$$

Inserting, we obtain

$$\begin{aligned} & C\left(C_L(\omega_0;F,T);K,S\right) \\ = \;& e^{-rT}\cdot E^{Q^1}\left[\omega(T)\cdot I_{\{\omega(T)>F,\tau_T\notin\theta_T,\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}}\left|\Im_0\right.\right] \\ & -e^{-rT}F\cdot E^{Q^1}\left[I_{\{\omega(T)>F,\tau_T\notin\theta_T,\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}}\left|\Im_0\right.\right] \\ & -e^{-rS}K\cdot E^{Q^1}\left[I_{\{\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}}\left|\Im_0\right.\right] \end{aligned} \tag{2.3}$$

where we denote with $\overline{\omega}$ the value of $\omega\left(S\right)$ that solves $C_L\left(\omega\left(S\right);F,T\right)=K$; that is the exercise price in terms of the state variable[6]. The monotonicity of C_L with respect to ω then implies that the event $\{C_L>K\}$ is equivalent to $\{\omega(S)>\overline{\omega}\}\cap\{\tau_S\notin\theta_S\}$.

We can separate the two variables within the expectation brackets of the first term on the right hand side of (2.3) by a change of probability measure. Define a new measure Q^2 through

$$dQ^2=:R^{1\to2}dQ^1 \quad \text{with} \quad R^{1\to2}=\frac{\omega(T)}{E^{Q^1}\left[\omega(T)\left|\Im_0\right.\right]}$$

where prices normalized with the asset value process ω are martingales. Then[7] we can rewrite the first term on the RHS of (2.3) as

$$e^{-rT}\cdot E^{Q^1}\left[\omega(T)\left|\Im_0\right.\right]\cdot E^{Q^2}\left[I_{\{\omega(T)>F,\tau_T\notin\theta_T,\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}}\left|\Im_0\right.\right]$$

Moreover, if we let

- A_S be the event $\{\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}$
- A_T be the event$\{\omega(T)>F,\tau_T\notin\theta_T,\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}$

(that is the events that "all is well" at times S and T, respectively) we are able to rewrite our pricing formula (2.3) as

$$C(C_L(\omega_0;F,T);K,S)$$

[6]Throughout the analysis, to make it non trivial, we assume $\overline{\omega},F,\omega_0>L$.
[7]See Geman et al. (1995) or Björk (1994).

$$= \omega_0 \cdot Q^2 (A_T) - e^{-rT} F \cdot Q^1 (A_T) - e^{-rS} K \cdot Q^1 (A_S) \tag{2.4}$$

The structure of this formula is not tied to the assumptions of constant asset volatility and interest rate - see Geman et al. (1995). However, they guarantee closed form solutions for the probabilities.

What remains is to find analytic expressions for these probabilities and hence for the price of the compound call. This is done explicitly in *Appendix 2.7* for the $\{A_T\}$-event and the result is given in the following lemma:

Lemma 2.1 *Define the event*

$$A_T = \{\omega(T) > F, \tau_T \notin \theta_T, \omega(S) > \overline{\omega}, \tau_S \notin \theta_S\} \quad : \; S < T$$

Then the associated probabilities under the probability measures Q^2 and Q^1 are given by

$$\begin{aligned} Q^m (A_T) = & \quad N\left(d_S^{3\to m}\left(\frac{\omega_0}{\overline{\omega}}\right), d_T^{3\to m}\left(\frac{\omega_0}{F}\right), \rho\right) \\ & - \left(\frac{L}{\omega_0}\right)^{\frac{2}{\sigma}h^{3\to m}} N\left(d_S^{3\to m}\left(\frac{L^2}{\omega_0 \cdot \overline{\omega}}\right), d_T^{3\to m}\left(\frac{L^2}{\omega_0 \cdot F}\right), \rho\right) \\ & - \left(\frac{L}{\omega_0}\right)^{\frac{2}{\sigma}h^{3\to m}} N\left(-d_S^{3\to m}\left(\frac{\overline{\omega}}{\omega_0}\right), d_T^{3\to m}\left(\frac{L^2}{\omega_0 \cdot F}\right), -\rho\right) \\ & + \quad N\left(-d_S^{3\to m}\left(\frac{\omega_0 \cdot \overline{\omega}}{L^2}\right), d_T^{3\to m}\left(\frac{\omega_0}{F}\right), -\rho\right) \end{aligned}$$

where $m = \{1, 2\}$,

$$d_s^{3\to m}(x) = \frac{\ln x}{\sigma\sqrt{s}} + h^{3\to m}\sqrt{s}$$

$$h^{3\to 2} = \frac{r + \frac{1}{2}\sigma^2}{\sigma}, \qquad h^{3\to 1} = h^{3\to 2} - \sigma$$

and

$$\rho = \sqrt{\frac{S}{T}}$$

To understand the structure of the derived formula consider the probability of an event A conditional on not hitting a barrier. By decomposing A into complementary events we may write

$$\begin{aligned} Q(A, \tau_T \notin \theta_T, \tau_S \notin \theta_S) = & \quad Q(A) \\ & - \; Q(A, \tau_S \in \theta_S) \\ & - \; Q(A, \tau_T \in \theta_T) \\ & + \; Q(A, \tau_T \in \theta_T, \tau_S \in \theta_S) \end{aligned} \tag{2.5}$$

The total probability is the unconditional (as regards the barrier) probability (line 1) *less* the probability conditional on the barrier being hit before S (line 2) *less* the probability conditional on it being hit between S and T (line 3) *plus* the probability of hitting the barrier both before S *and* between S and T (line 4). This structure of this partition (with $A = \{\omega(T) > F, \omega(S) > \overline{\omega}\}$) is precisely that of the expression in *Lemma 2.1.*

The probability for the $\{A_S\}$-event is well known[8] and is

Lemma 2.2 *Define the event*

$$A_S = \{\omega(S) > \overline{\omega}, \tau_S \notin \theta_S\}$$

Then the associated probabilities under the probability measures Q^2 and Q^1 are given by

$$Q^m(A_S) = N\left(d_S^{3\to m}\left(\frac{\omega_0}{\overline{\omega}}\right)\right) - \left(\frac{L}{\omega_0}\right)^{\frac{2}{\sigma}h^{3\to 1}} N\left(d_S^{3\to m}\left(\frac{L^2}{\omega_0 \cdot \overline{\omega}}\right)\right)$$

where $m = 1,2$ and $d_S^{3\to m}(x)$ is given in Lemma 2.1.

Applying *Lemma 2.1* and *Lemma 2.2* to our pricing equation (2.4) we state the price of the compound call:

Proposition 2.1 *The price of a call with exercise price K and maturity S on a down-and-out call with exercise price F of maturity T is given by*

$$C(C_L(\omega_0; F, T); K, S) = \omega_0 \cdot Q^2(A_T) - e^{-rT}F \cdot Q^1(A_T) - e^{-rS}K \cdot Q^1(A_S)$$

where the probabilities are given in Lemma 2.1 and Lemma 2.2.

To interpret the proposition note that a compound call, although formally a claim on the underlying call at S, is ultimately a claim on the assets of the firm at T. Technically, a compound call of maturity S on a call of maturity of T can therefore be viewed as a call of maturity T on v that pays off only if $\{\omega(S) > \overline{\omega}\}$ *less* the value of the requirement to pay the exercise price at S. To help formalize this idea we make the following, slightly more general, definition.

[8] The event A_S is the in-the-money event for a down-and-out call. The price of a down-and-out call was first derived by Merton (1973)

Definition 2.1 *Define a* ***conditional call*** $\mathcal{C}(\omega_0; F, T\,|A)$ *as a derivative that pays off as an ordinary call option conditional on the event* A :

$$\mathcal{C}(\omega(T); F, T\,|A) = C(\omega(T); F, T) \cdot I_{\{A\}}$$

Using earlier notation, with subscript L denoting the barrier feature, we can write its down-and-out counterpart as

$$\mathcal{C}(\omega_0; F, T\,|A \cap \{\tau_S \notin \theta_S\} \cap \{\tau_T \notin \theta_T\}) \equiv \mathcal{C}_L(\omega_0; F, T\,|A)$$

In particular

$$\mathcal{C}_L(\omega_0; F, T\,|\omega(S) > \overline{\omega}) = \omega_0 \cdot Q^2(A_T) - e^{-rT} F \cdot Q^1(A_T) \qquad (2.6)$$

Accordingly, we can rewrite the price of the compound down-and-out call in *Proposition 1* as

$$C(C_L(\omega_0; F, T); K, S) = \mathcal{C}_L(\omega_0; F, T\,|\omega(S) > \overline{\omega}) - K \cdot H_L(\omega_0; \overline{\omega}, S)$$

where we also use that $e^{-rS}Q^1(A_S)$ is equivalent to a down-and-out Heaviside (*Appendix 2.8*). This expression formalizes the interpretation above of a compound call ultimately being a claim on the firm's assets.

Consider now in more detail the conditional call option of *Definition 2.1.* Remembering the probability decomposition in equation (2.5) we can write the call price as

$$\begin{aligned}\mathcal{C}_L(\cdot\,|\omega(S) > \overline{\omega}) = \quad & \mathcal{C}(\cdot\,|\{\omega(S) > \overline{\omega}\}) \\ - \quad & \mathcal{C}(\cdot\,|\{\omega(S) > \overline{\omega}\} \cap \{\tau_S \in \theta_S\}) \\ - \quad & \mathcal{C}(\cdot\,|\{\omega(S) > \overline{\omega}\} \cap \{\tau_T \in \theta_T\}) \\ + \quad & \mathcal{C}(\cdot\,|\{\omega(S) > \overline{\omega}\} \cap \{\tau_S \in \theta_S\} \cap \{\tau_T \in \theta_T\})\end{aligned}$$

The three last claims are thus down-and-*in* calls with different partial barrier arrangements. Similar options with barriers that only partially cover the options' lives are studied in Heynen & Kat (1994), Carr (1995) and Bermin (1995).

The conditional call will be useful for pricing more complex compound derivatives. For future reference we also make the following analogous definition:

Definition 2.2 *Let a* ***conditional Heaviside*** $\mathcal{H}(\omega_0; F, T\,|A)$ *be a Heaviside of maturity* T *with exercise price* F *conditional on the event* A :

$$\mathcal{H}(\omega(T); F, T\,|A) = H(\omega(T); F, T) \cdot I_{\{A\}}$$

Using earlier notation, we can write its down-and-out counterpart as

$$\mathcal{H}(\omega_0; F, T\,|A \cap \{\tau_S \notin \theta_S\} \cap \{\tau_T \notin \theta_T\}) \equiv \mathcal{H}_L(\omega_0; F, T\,|A)$$

In particular, when $A = \{\omega(S) > \overline{\omega}\}$ (cf.. (2.6))

$$\mathcal{H}_L(\omega_0; F, T\,|\omega(S) > \overline{\omega}) = e^{-rT}Q^1(A_T)$$

2.4 Options on Corporate Securities

In this section we consider the pricing of options on corporate securities, extending the idea that the compound call could be interpreted as a option on equity in a firm with discount debt only. We begin by reviewing the valuation of the underlying securities, and then proceed to the options themselves. We conclude by briefly discussing the applicability of the results to other areas.

2.4.1 Pricing Corporate Securities

In the framework of Ericsson & Reneby (1995) securities can be valued as portfolios of down-and-out call calls and down-and-out Heavisides. A full exposition of this idea would take up too much space and the reader is referred to that paper for details, but a general description of the method is as follows.

The valuation method exploits the fact that payoffs to the company's securities can be replicated with payoffs from two basic claims: a call and a Heaviside. Assuming the absence of arbitrage, two claims with identical payoff structures must have the same price. Hence to value a corporate security, one simply mimics the payoffs of that security with those of (down-and-out) calls and Heavisides.

An important condition for this result is that default prior to maturity of debt is driven by a barrier only. Specifically, when equity finances coupon payments, the barrier must be high enough to ensure equity a non negative value. When coupons are financed internally, from the firm's assets, a condition is instead that the asset value process (2.1) is unaffected by any payments to outside parties, in particular coupon payments[9].

In this setting, *Proposition 1.1* in (Ericsson & Reneby (1995) this volume Chapter 1) gives us the value of a corporate security. For convenience, we restate it here as a lemma. The letter $\Phi\{f\}$ denotes the contracted payoff to a claim f and Ω denotes a claim which pays off $\omega(T)$ at T.

Lemma 2.3 *A corporate security* $G(\omega(t);\cdot)$ *with contracted payments*

$$\Phi\{G\} = \left\{ \begin{array}{c} \alpha\,\Phi\{\Omega\} \\ + \\ \sum_i \beta^{(i)}\,\Phi\{C^{(i)}\} \\ + \\ \sum_i \gamma^{(i)}\,\Phi\{H^{(i)}\} \end{array} \right\} \quad \textit{for } t \leq \tau$$

[9]One would in that case be restricted to assuming that assets are sold continuously at a constant rate to leave the asset value process unaffected. In the interest of clarity, however, we refrain from that possibility here.

can be valued as

$$G(\omega(t);\cdot) = \left\{ \begin{array}{c} \alpha\Omega \\ + \\ \sum_i \beta^{(i)} C_L^{(i)} \\ + \\ \sum_i \gamma^{(i)} H_L^{(i)} \end{array} \right\} \quad \text{for } t \leq \tau$$

where i are used to index Heavisides and options of different exercise price and maturity, $\beta^{(i)}$ and $\gamma^{(i)}$ are constants and τ is the time of default. The summation operator $\sum$ should be understood to encompass integrals when applicable.

To get some intuition for this result, consider the following example.

Example 2.1 *For pricing purposes, the value of a security is split into three parts: (i) the payments at maturity, (ii) intermediate cash flows such as coupons or dividends and (iii) payments in the event of a reorganization prior to maturity. First consider maturity payments. They are valued as down-and-out calls and Heavisides - maturity payments to debt are in the most simple case without bankruptcy costs valued as a risk-free down-and-out bond less a down-and-out put option. In a case with bankruptcy costs and violations of the absolute priority rule the valuation becomes slightly more complicated and we obtain the following payoffs to equity and debt at maturity*[10]

Denote by γ the percentage deviation from the absolute priority rule and by k the bankruptcy costs. Debt payoffs, for example, are then replicated as follows:

- $(1-\gamma)$ *long calls with exercise price k*
- $(1-\gamma)$ *short calls with exercise price F*
- $(\gamma(F-k)+k)$ *long Heaviside contracts with exercise price F*

Now consider payments prior to maturity. Coupons, whether discrete or continuous, can be valued as sums (integrals) of down-and-out Heavisides. Payoffs to security holders in the event of a reorganization prior to maturity of debt are valued with a down-and-in contract (which in turn may be valued as a combination of down-and-out calls and Heavisides).

2.4.2 Pricing Options

We now turn to the valuation of options. An issue that has to be addressed is what happens with the value of an option when the firm defaults. In

[10]Note that the non-monotonicity of the payoff function equity may in extreme situations yield prices that are non-monotonic as well. Deviations from absolute priority and equity pricing in a similar setting have been studied by Eberhart & Senbet (1993).

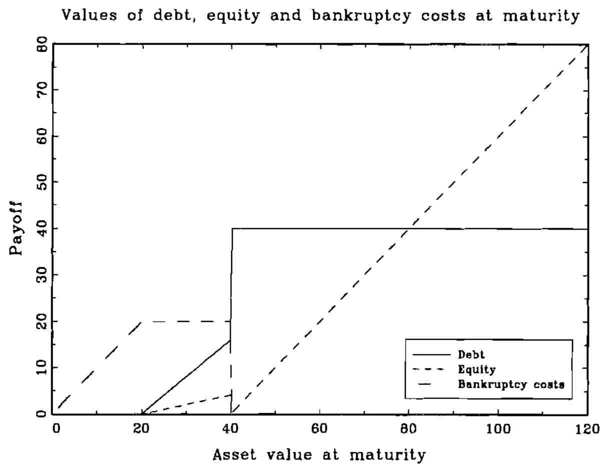

FIGURE 2.1. THE DIVISION OF PAYOFFS TO CLAIMHOLDERS AT MATURITY. The facevalue of debt (F) is 40, bankruptcy (K) costs are 20. There are deviations from the absolute priority in default states in the sense that shareholders receive 20% of the net proceeds.

other words, what is the value of the underlying corporate security after default? Since the securities are often no longer traded, their values are difficult to determine. We will assume that for purposes of determining the payoff to derivative holders there is a provision stating that corporate securities have a post-default value of zero. Thus, at expiration, a call option expires worthless whereas a put option takes on a value equal to the exercise price. We formalize this idea by defining the process determining derivative payoffs

$$\widehat{G}\left(\omega(0);\cdot\right) \equiv G\left(\omega(0);\cdot\right) \cdot I_{\{\tau_S \notin \theta_S\}}$$

Since corporate securities can be valued as down-and-out calls and Heavisides, an option on a corporate security can be valued as an option on a portfolio of these two contracts. Furthermore, since an option on a portfolio can be treated as a portfolio of options on the parts, the option can be valued as a sum of conditional calls and Heavisides.

Proposition 2.2 *The price (at time 0) of a call option of maturity S and exercise price K on the corporate security of Lemma 2.3 is given by*

$$C\left(\widehat{G}\left(\omega(0);\cdot\right);K,S\right)=\left\{\begin{array}{c}\alpha\cdot\mathcal{C}_L\left(\omega_0;0,T\,|\omega(S)>\overline{\omega}\right)\\+\\\sum_{i\left(t^{(i)}\geq S\right)}\beta^{(i)}\cdot\mathcal{C}_L\left(\omega_0;F^{(i)},t^{(i)}\,|\omega(S)>\overline{\omega}\right)\\+\\\sum_{i\left(t^{(i)}\geq S\right)}\gamma^{(i)}\cdot\mathcal{H}_L\left(\omega_0;F^{(i)},t^{(i)}\,|\omega(S)>\overline{\omega}\right)\\-\\K\cdot H_L\left(\omega_0;\overline{\omega},S\right)\end{array}\right.$$

where $\overline{\omega}$ *solves* $G(\omega(S);\cdot)=K$. *For a proof see Appendix 2.9 for a derivation.*

As can be seen by comparing this result with *Lemma 2.3*, the price of a call is analogous to the price of the underlying security - with conditional rather than ordinary down-and-out claims and an adjustment for the requirement to pay the exercise price (K down-and-out Heavisides).

Remark 2.2 *The price of a put option on the corporate security of the above proposition may be calculated using put call parity:*

$$P\left(G(\omega_0;T,\cdot);K,S\right)=e^{-rS}K-G(\omega_0;T,\cdot)+C(G(\omega_0;T,\cdot);K,S)$$

2.4.3 Pricing Other Claims

The proposed approach can readily be extended to value claims other than options on corporate securities. Basically, any compound claim can be valued while allowing for a barrier.

One obvious application would be to debt and equity while allowing debt of two different maturities in the capital structure. Equity and short term debt would in such an environment be valued as compound barrier claims. A related area is financial guarantees, which could be valued as compound barrier claims on debt of different maturities. A similar approach could be used for warrants.

Moreover, as already noted in section 2.3, we have implicitly valued some compound down-and-in claims. Using the method of derivation in *Appendix 2.7*, the analysis could easily be extended to a case with a single barrier of any type (up- or down, in- or out, partial or non partial), if one should find an interesting application for such a compound claim.

Even though the previous analysis has dealt with options as the compound claim, other derivatives can readily be valued as well as long as their payoffs can be replicated by calls and Heavisides. One then needs two additional compound claims: the Heaviside on a down-and-out call and the

Heaviside on a down-and-out Heaviside. These are just simpler versions of the compound claims in this paper.

2.5 Some Numerical Results

This section presents some numerical results for the suggested option pricing model. Our choice of comparisons is partially dictated by existing models. We are able to nest the Geske (1979) model by setting the default barrier to zero and a version of the Toft & Prucyk (1996) model by using perpetual continuous coupon debt.

We focus on the effect of the barrier and the finite debt maturity, since the combination of those features are the essence of the paper. This is the subject of the following two subsections. We assume no bankruptcy costs or taxes and also assume that the assets of the firm are traded to be able to conduct the comparative analysis in a straightforward manner. In this setting the value of the firm equals the value of assets, and hence two firms with equal asset value and leverage will also have the same debt and equity values. This is convenient for comparing different pricing models consistently. Moreover, to isolate the effect of the barrier and the maturity of debt we refrain from introducing coupons until later. Throughout we limit ourselves to considering a European call option only, which in this setting simply will be equal to the compound option of *Proposition 2.1.*

We investigate the effects of coupons in the third subsection (2.5.3).

2.5.1 The Effect of The Barrier

It is the future evolution of the value of equity that governs the price of the option. Therefore, it will be useful to begin the analysis by looking at the effect of a barrier on the distribution of the stock price. The following figure plots the distribution for three different levels of the default barrier. Leverage is held constant by varying the principal of debt. Hence we have the same current stock price for all three scenarios although the time S expectations may differ.

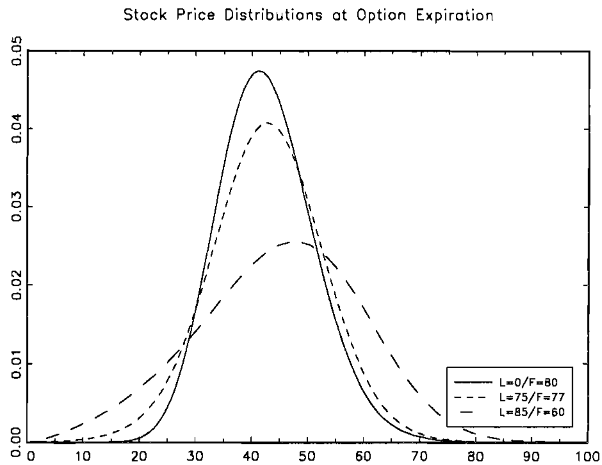

FIGURE 2.2. THE EFFECT OF THE DEFAULT BARRIER ON STOCK PRICE DISTRIBUTIONS AT EXPIRATION. $\sigma = 0.15, r = 0.06$, $T = 5$, $S = 5/12$, current equity price 41.4.

Most notably an increased barrier will yield an increase in stock price volatility. The higher barrier increases the probability that the stock becomes worthless. Furthermore since the principal is lower for the high barrier cases the probability of high stock prices will also increase; the intuition being that when default is unlikely shareholders are better off with a lower principal. This effect on stock price volatility will drive most of the results we present below.

We start by analysing the barrier's influence on prices, then study the influence on hedge ratios and conclude by looking at the effects of using the Black-Scholes' model to calculate implied volatilities.

2.5.1.1 The Effect on Prices

A hypothetical trader observes market values of equity and debt but values options using a non barrier model and we can thus calculate the effect of omitting the barrier. He will select the principal of debt so that his model matches observed market prices of the corporate securities, and thereafter compute the option price. An *option price ratio* is obtained by dividing this option price by the true option price.

The following figure plots the option price ratio against leverage. The firm's leverage ratio is increased by decreasing the asset value of the firm. For each leverage, the exercise price of the option is set so as to place it at-the-money. The figure thus shows how the pricing error, resulting from using a non-barrier model for an at-the-money option, is affected by the leverage of the firm.

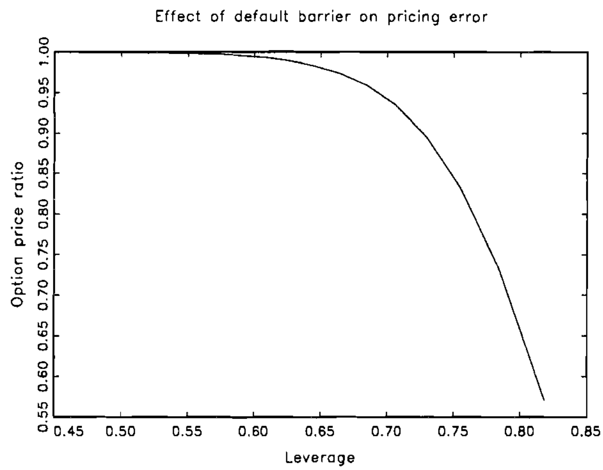

FIGURE 2.3. DEFAULT BARRIER AND OPTION PRICES. The ratio between option prices calculates using models without and with barrier respectively. $L = 50, \omega_0 = 51...100, \sigma = 0.15, r = 0.06, T = 5, F = 65, S = 5/12$, K =at-the-money.

We note that the non-barrier model will underprice the option in general and that this error becomes more significant the higher the leverage. The most important effect at work here is that the non-barrier model will underestimate stock price volatility and hence option prices. The higher the leverage (the closer the asset value is to the barrier) the higher the equity volatility, and the more crucial the effect of neglecting the barrier becomes.

2.5.1.2 The Effect on Deltas

In this section we study the option delta (Δ) - its sensitivity to changes in the underlying stock price. Below we plot both barrier and non barrier deltas (hedge ratios) as a function of the stock price.

Consider a call option. As the stock price increases as a result of an increase in the underlying asset value the likelihood of default decreases. As this happens the deltas converge for the two models - that of the barrier model approaches from below.

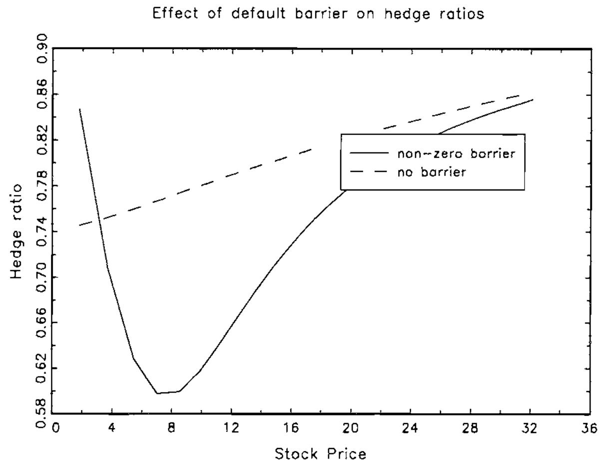

FIGURE 2.4. DEFAULT BARRIER AND HEDGE RATIOS. The deltas for barrier and non-barrier models as functions of the underlying stock price. $L = 50$, $\omega_0 = 51, ..., 100$, $\sigma = 0.15$, $r = 0.06$, $T = 5$, $F = 65$. Option is at-the-money.

When one approaches financial distress the deltas may differ considerably. Initially the delta for the option in the barrier model decreases relative to that of the standard compound option model. Then as default becomes more imminent the relationship is reversed and the delta of the barrier option pricing model may exceed the non barrier one considerably. This indicates that the higher sensitivity of standard barrier contingent claims prices near the barrier carries over to compound barrier options.

In practice, these results would translate into the following situation. Suppose that a trader uses the non-barrier model to replicate a long position in the call in a situation where a barrier is an important determinant of option prices. If financial distress is imminent he will underinvest in the underlying stock. On the other hand as the firm becomes less likely to default his hedge portfolio may consist of too much stock.

Investigating in- and out-of-the-money options shows that deltas are familiarly increasing the deeper in-the-money they are.

2.5.1.3 The Effect on Implied Volatilities

Equity volatilities are in practice often estimated as the volatilities implied by the Black-Scholes model from traded option prices. If it were a correct description of reality, all options written on the same stock would of course imply the same equity volatility. However, one usually finds that implied volatilities increase the deeper in-the-money the options are (the volatility

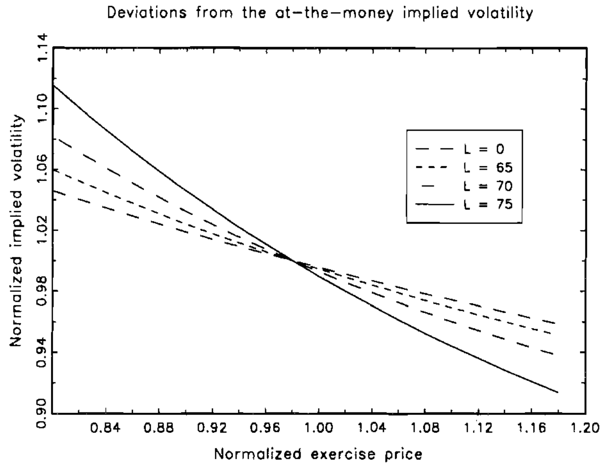

FIGURE 2.5. DEFAULT BARRIER AND IMPLIED VOLATILITIES. Implied volatilities normalized by the at-the-money implied volatility as a function of the normalized exercise price. $\omega_0 = 100$, $\sigma = 0.15$, $r = 0.06$, $T = 5$, $F = 82.05$ / 81.77 / 81.10 / 79.31. Leverage is 60% and $S = 4/12$

"smirk") - this is equivalent to overpricing in-the-money call options, see for example Rubinstein (1994).

Below, we plot implied volatilities (normalized by the at-the-money implied volatility) as a function of the exercise to stock price ratio. Again, we let our model represent reality, and hence yield correct prices. Implied volatilities are calculated using the Black-Scholes' model. Moreover, this is done for several levels of the barrier, L. For each choice of L in the figure below we adjust the principal of debt so that the market value of equity (and hence leverage) is held constant. Thus, the figure indicates how the volatility "smirk" is influenced by a barrier.

As can be seen, the volatility smirk becomes more pronounced the higher the barrier. The explanation is as follows. Equity volatility decreases (increases) with increases (decreases) in the asset value due to a lower (higher) leverage. This effect obtains even with $L = 0$ and leads to relatively higher values (and thus implied volatilities) for in-the-money calls and relatively lower for out-of the money calls. This explains the general downward sloping curve shape of the volatility skew. Furthermore this leverage effect is amplified by a higher default barrier. The model of Toft & Prucyk (1996) yields the same pattern. Furthermore they find empirical support for such leverage driven pricing biases.

TABLE 2.1. CHANGE IN OPTION PRICE AS A RESULT OF A CHANGE IN DEBT MATURITY

	$T=1$ (F=62) (price change)	$T=5$ (F=80) (price)	$T=10$ (F=118) (price change)
In-the-money option (K=33)	+3.2%	9.85	−1.2%
At-the-money option (E=K=41)	+6.4%	4.41	−5.0%
Out-of-the-money option (K=50)	+11.2%	1.26	−12.6%

L=50, $\omega_0 = 100, \sigma = 0.15, r = 0.06,\ S = 5/12$

2.5.2 The Effect of Debt Maturity

In this section we briefly analyse how the maturity of debt affects the value of the option. It is done in the following way. First option prices in-, at-, and out-of-the-money are calculated for a benchmark maturity of $T = 5$ years. Then the price changes resulting from a change of debt maturity are calculated. The stock price is held constant at 41 throughout by varying the principal. The described procedure can thus be interpreted as the effect on the option price from a change in the capital structure that would leave both debt and equity holders indifferent.

Table 2.1 indicates that the maturity of debt influences option prices significantly. Again, the reason is volatility - the longer the maturity, the lower the volatility of equity during the lifetime of the option since the impact of the principal is weakened.

The influence of volatility increases with out-of-the-moneyness. The reason is the well known observation that out-of-the-money calls "have nothing to loose" from increased risk - only the right tail of the stock price distribution affects the call price (in contrast to the price of an *in*-the-money option).

In section 2.5.3.3 we look at the effects of approximating debt of finite maturity with infinite maturity when coupons are paid continuously.

2.5.3 Some Additional Numerical Results

In this section we introduce coupons, bankruptcy costs, (corporate) taxes and violation of the absolute priority rule (APR). We do not claim that the analysis is complete, but it points to some interesting features of our and alternative models.

The notion of bankruptcy costs should be interpreted as the total decrease in asset value (at time τ) that occurs before distribution to debt- and equityholders take place. Thus this includes not only direct costs such as legal fees, deferred taxes and wages which have priority but also indirect costs such as production disturbances and damaged reputation. A violation of the APR is modelled as the percentage of the firms assets less bankruptcy costs that are distributed to shareholders following financial distress[11] - even though they are formally not entitled to it.

2.5.3.1 Time Passes

The intention of this section is to exemplify pricing of options in a more realistic setting and to illustrate the quantitative price change of a European call option as time passes. We will also see how the pricing biases of alternative models change.

We will use the following set of parameters:

External factors

ω_0	r	σ	Tax rate	Violation of APR	Bankruptcy costs
100	0.06	0.15	0.35	0.08	10

Capital structure

F	c	T	L
60	0.08	5	50

Option contract details

K	S
at-the-money	$\frac{5}{12}$

The coupon is paid semi-annually. The setup above gives rise to the following capital structure:

Value of the firm	106
Equity ($\sigma_E = 34\%$)	43
Debt	63
Leverage	59%

The tax shield is worth 6.5. The price of the call option is 4.37. The Black-Scholes' and the non-barrier models produce a modest underpricing of 1% and 2%, respectively.

Now consider what happens to the value of the option as time passes. The following table contains future option prices for different scenarios. For five "reasonable" (given an asset volatility of 15%) outcomes of ω in three months it gives the corresponding equity and option prices. It also shows how the pricing biases of two alternative models change (NB abbreviates non barrier[12] and BS abbreviates Black-Scholes) .

[11] That is $[violation\,of\,APR] = \frac{[to\,equityholders]}{L-[bankruptcy\,costs]}$.

[12] With coupons, it is no longer obvious what maturity of debt the non barrier user should choose when applying his model. As suggested by Geske (1979), maturity of debt is chosen to match the (riskfree) duration of the true debt contract.

TABLE 2.2. 3 MONTHS LATER - 2 MONTHS TO EXPIRATION (T=0.25).

$\omega(0.25)$	Equity value	Option price	NB price bias	BS price bias
104	47 (+7.1%)	4.64	−1%	−1%
102	45 (+2.6%)	3.30	−1%	−1%
100	43 (−1.9%)	2.12	−2%	±0%
98	41 (−6.5%)	1.38	−4%	+3%
96	39 (−11%)	0.80	−6%	+7%

$L = 50, \sigma = 0.15$, 8% deviations from APR, $r = 0.06$, $T = 4.75$, $c = 0.08$, $F = 60$, $S = 2/12$, $K = 43$, $BC = 10$, taxes 30%.

The next table contains five possible scenarios yet 7 weeks later (1 week prior to expiration) for the same option.

TABLE 2.3. 4.75 MONTHS LATER - 1 WEEK TO EXPIRATION (T=0.4)

$\omega(0.4)$	equity change	option price	NB price bias	BS price bias
106	48 (+11%)	4.65	±0%	±0%
103	45 (+3.7%)	1.94	±0%	±0%
100	42 (−3.1%)	0.36	−4%	+3%
97	39 (−10%)	0.02	−14%	+21%
94	36 (−17%)	0.00	−34%	+209%

L=50, $\sigma = 0.15$, 8% deviations from APR, $r = 0.06$, $T = 4.6$, $c = 0.08$, $F = 60$, $S = 0.25/12$, $K = 43$, $BC = 10$, taxes 35%.

The pricing biases increase as expiration approaches and as equity value falls, driving the options out-of-the-money. This is the same story that was told before. As equity value falls, the downward volatility bias increases for the non barrier model. And the more out-of-the-money the option is, the more it would stand to gain from a higher volatility.

For the Black–Scholes' model, however, the opposite is true - the model uses the correct (instantaneous) equity volatility. But since it is assumed that it is constant, it fails to take into account that volatility actually decreases when equity increases (due to lowered leverage). This overestimation of volatility at higher stock prices is particularly noticeable for out-of-the-money options.

TABLE 2.4. EFFECT OF FIRST COUPON PAYMENT

First coupon payment in...		option price	NB price bias
...3 months (T=4.75)	Expiration before (S=2/12)	2.63	-2%
	Expiration after (S=4/12)	4.70	-20%
...6 months (T=5))	Expiration before (S=5/12)	4.37	-2%
	Expiration after (S=7/12)	6.12	-15%

$L = 50$, $\omega_0 = 100$, $\sigma = 0.15$, 8% deviations from APR=0.08, $r = 0.06$, $T = 4.75/5$, $c = 0.08$, $F = 60$, $S = 2/12$, K =at-the-money, $BC = 10$, taxes 35%.

Bankruptcy Costs and Taxes

The presence of bankruptcy costs does, *ceteris paribus*, mainly affect debt value. Only for very high leverages does it effect the value of equity and equity options[13] (when the asset value is close to the barrier). When there are deviations from absolute priority, equity will loose value from an increase in bankruptcy costs - and hence the option price will decrease.

The effect of taxes are difficult to analyse since a change in τ affects ω - the *after-tax* value of assets - in an exogenous way[14]. It is not possible in the model to determine the increase in the value of assets following a decrease in the tax rate - a comparison of option prices when taxes are high and low would therefore be dubious. We can, however, compare the mispricing of the alternative models in a world with and without taxes. Doing so, we obtain results equivalent to those with low and high coupons - taxes merely serve to lower the cost of coupon payments.

Concluding, taxes and bankruptcy costs are an important determinant of equity option prices only indirectly. They have to be modelled for purposes of matching equity and debt values implied by the model with true values.

2.5.3.2 Option Maturity and the First Coupon Payment

The option price is sensitive to whether the option matures before or after a coupon payment. Table 2.4 gives the true option price and the pricing bias using a non barrier (and thus non coupon) model for different combinations of option and coupon maturities.As can be seen, the mispricing of the non barrier model dramatically increases when there is a coupon payment during the lifetime of the option. Moreover, the effect increases when the

[13]For debt and debt derivatives, bankruptcy costs are of course bound to have a larger effect.

[14]Potentially, a changed tax rate may also affect (μ, λ, σ).

coupon payment is more imminent, that is when its impact on equity prices and its volatility is larger. The explanation is the following.

Immediately after a coupon payment equity increases by the amount of the coupon. This explains why the option increases from 2.63 (4.37) to 4.70 (6.12) when its maturity is increased with 2 months. This effect is of course neglected in a model without coupons. Only the "usual" time-to-maturity effect increases option prices in that case.

2.5.3.3 Continuous Coupon Payments

An alternative to modelling discrete coupons, is to use an approximation with continuous payments. This section briefly looks at the potential effects of such an approximation.

For the option in the section termed *Time Passes*, the continuous coupon model overestimates the price with 15%. This magnitude of overpricing extends to a wide range of underlying capital structures. On expiration of the option the equity price will be heavily influenced by the imminent coupon payment, which lowers the expected value of equity and hence the value of the option. In the continuous coupon model this effect will be much less pronounced, since the burden of the coupon is spread over time.

The closer the coupon payment follows the expiration of the option, the more the continuous coupon model overprices. If, on the other hand, the expiration of the option follows immediately *after* the coupon payment, the pricing bias is negligible.

Infinite Maturity Approximation with Continuous Coupons. This section takes a brief look at what happens when finite maturity debt is approximated by infinite maturity debt. To focus on the maturity effect (and not the discrete/continuous coupon effect) we use continuous coupons throughout. First we calculate call option prices for different maturities of debt keeping leverage constant (at 60%) by varying the coupon. Then the coupon necessary to obtain the same leverage in an infinite maturity environment and the corresponding option price are computed. This corresponds to a trader implementing an erroneous (infinite maturity) model. Table 2.5 shows the resulting price biases (when the true price is given by the finite maturity model).

As can be seen, overestimation of option prices when using an infinite maturity approximation may be far from negligible. The reason using an infinite maturity overestimates option prices is that a higher coupon rate means that a larger fraction of debt value will be repaid during the lifetime of the option. The expected value of equity at maturity of the option is therefore higher in the infinite maturity (high coupon) case. Therefore, the value of the option will be higher as well. Moreover (not shown in the table), overestimation increases with out-of-the-moneyness and maturity of the option.

TABLE 2.5. PRICE BIAS AS A RESULT OF APPROXIMATING FINITE WITH INFINITE MATURITY DEBT

Capital structure	Option price	Option price bias ($T=\infty, c=5.2\%$)
$T=2.5$, $c=0\%$	4.28	+19%
$T=5$, $c=2.7\%$	4.67	+8%
$T=10$, $c=4.2\%$	4.90	+3.5%

$L=50$, $\omega_0=100$, $\sigma=0.15$, $r=0.06$, $F=70$, $S=5/12$, K=at-the-money.

Note that to apply a continuous coupon, infinite maturity approximation to a discrete coupon, finite maturity reality results in two sources of bias - both of which are positive.

2.6 Concluding Remarks

We have done two things in this paper. First we have presented an extension of Geske's compound option pricing model to the case of an option on a down-and-out call. Second we provide a general and unified method for pricing (analytically) both credit risky corporate securities and related options in an environment where their volatility is driven by changes in leverage.

Numerical results show that using alternative models that do not account for intermediate financial distress may result in considerable pricing errors when leverage is high and especially when financial distress is likely. The pricing biases also carry over to the hedge parameters which may be both over- and underestimated. The suggested model is consistent with the volatility "smirk" effect observed in practice. The driving force of these results is that models that do not incorporate a default barrier may implicitly underestimate equity volatility and thus option prices.

Furthermore our results suggest the importance of accommodating a finite maturity. Detailed capital structure information - in particular the size and timing of coupon payments - has an important impact on option prices.

In summary, the comparative advantage of our model is precisely that it is capable of incorporating comprehensive balance sheet data. Suppose we want to price an option. Often good estimates of the underlying volatility can be extracted from market data. However if the derivative we want to price is part of, say, a debt issue which will affect the capital structure as a whole, it will be far more difficult to obtain a reliable estimate. In such a situation the model we suggest appears an appealing alternative. Since the model can directly incorporate the changes in the capital structure the volatilities of the underlying securities will reflect the impact of the new issue.

2.7 Appendix: In-the-money Probabilities

This appendix derives explicitly the formulae for the probabilities in *Lemma 2.1*. First we consider the probability under the Q^2−measure and consider, by a simple analogy, the Q^1−measure in section 2.7.2.

2.7.1 The Q^2−probability of being In-the-money at T

Consider the expression

$$Q^2(A_2) = E^{Q^2}\left[I_{\{\omega(T)>F,\tau_T\notin\theta_T,\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}}|\Im_0\right] \quad (2.7)$$

The object now is to transform ω into a Wiener process through a normalization (to set the standard deviation to unity) followed by yet another Girsanov-transformation (to remove the drift) (section 2.7.1.1). The purpose of this exercise is to allow us to invoke the reflection principle[15] and derive the density functions (section 2.7.1.2). Finally, in section 2.7.1.3, we integrate to end up with bivariate cumulative normal distribution functions.

2.7.1.1 Transformation into a Wiener process

The Normalized Process $\eta(t)$

Note that the Girsanov kernel associated with the change of measure from Q^1 to Q^2 is $h^{1\to2} = \sigma$ so[16] that the process for ω under Q^2 will be

$$\begin{cases} d\omega = \left(r+\sigma^2\right)\omega dt + \sigma\omega\, dW^2 \\ \omega(0) = \omega_0 \end{cases}$$

Define

$$\eta(t) \equiv \frac{\ln\frac{\omega(t)}{\omega_0}}{\sigma}$$

Under the Q^2−measure, the dynamics of η are

$$\begin{cases} d\eta = \left(\frac{r+\frac{1}{2}\sigma^2}{\sigma}\right)dt + dW^2 \\ \eta(0) = 0 \end{cases} \quad (2.8)$$

---

[15] To express a probability conditional on a barrier not being hit as an unconditional probability, i.e. to find an x' such that

$$P(W(T) > x, \tau \notin \theta) = P\left(W(T) > x'\right)$$

See Harrison (1985) for details.

[16] Remember that Q^2 is the measure under which prices normalized with the asset value process ω are martingales (in particular $Z(t) \equiv \frac{B(t)}{\omega(t)}$ with $B(t)$ the money market account $(dB(t) = rB(t)dt)$, is a martingale).

Define also the normalized barrier and the normalized exercise prices at time S and T, respectively

$$\Lambda = \frac{\ln \frac{L}{\omega_0}}{\sigma}, \qquad X_S = \frac{\ln \frac{\overline{\omega}}{\omega_0}}{\sigma} \quad \text{and} \quad X_T = \frac{\ln \frac{F}{\omega_0}}{\sigma}$$

Then we can rewrite (2.7) as

$$Q^2(A_2) = E^{Q^2}\left[I_{\{\eta(T)>X_T,\tau_T\notin\theta_T,\eta(S)>X_S,\tau_S\notin\theta_S\}}\,|\Im_0\right] \tag{2.9}$$

The Q^3-measure

Define (implicitly) a probability measure Q^3 through

$$dQ^2 = R^{3\to2}dQ^3 \quad \text{with} \quad R^{3\to2} = e^{h^{3\to2}W^3(T)-\frac{1}{2}\left(h^{3\to2}\right)^2T}$$

so that η is a Wiener process under this measure $\left(d\eta = dW^3\right)$. By inspecting (2.8) we see that we need the following Girsanov kernel

$$h^{3\to2} = \left(\frac{r+\frac{1}{2}\sigma^2}{\sigma}\right)$$

With the help of this new probability measure we can rewrite (2.9) as

$$Q^2(A_2) = E^{Q^3}\left[R^{3\to2}\cdot I_{\{W^3(T)>X_T,\tau_T\notin\theta_T,W^3(S)>X_S,\tau_S\notin\theta_S\}}\,|\Im_0\right]$$

On integral form it becomes

$$Q^2(A_2) = \int_{X_T}^{\infty}\int_{X_S}^{\infty} R^{3\to2}Q^3\left(W^3(T)\in dw_T,\tau_T\notin\theta_T,W^3(S)\in dw_S,\tau_S\notin\theta_S\right) \tag{2.10}$$

where w_t denotes a specific realization of $W^3(t)$.

The term $Q^3\left(W^3(T)\in dw_T,\tau_T\notin\theta_T,W^3(S)\in dw_S,\tau_S\notin\theta_S\right)$ can safely be thought of as the probability of the Wiener process passing through the infinitesimal intervals dw_S at S and dw_T at T without hitting or having hit the barrier Λ.

The object of the next section is to compute the density function for $Q^3\left(W^3(T)\in dw_T,\tau_T\notin\theta_T,W^3(S)\in dw_S,\tau_S\notin\theta_S\right)$ so that we in section 2.7.1.3 can solve the integral in expression (2.10).

2.7.1.2 Q^3–densities

The aim is now to decompose

$$Q^3\left(W^3(T)\in dw_T,\tau_T\notin\theta_T,W^3(S)\in dw_S,\tau_S\notin\theta_S\right),$$

a barrier contingent probability, into a sum of non-barrier probabilities using the reflection principle. Note that the probability is defined through

$$\begin{aligned}
& Q^3\left(W^3(T) > X_T, \tau_T \notin \theta_T, W^3(S) > X_S, \tau_S \notin \theta_S\right) \qquad (2.11) \\
\equiv & \int_{X_T}^{\infty} \int_{X_S}^{\infty} Q^3\left(W^3(T) \in dw_T, \tau_T \notin \theta_T, W^3(S) \in dw_S, \tau_S \notin \theta_S\right)
\end{aligned}$$

By complementarity

$$\begin{aligned}
& Q^3\left(W^3(T) > X_T, \tau_T \notin \theta_T, W^3(S) > X_S, \tau_S \notin \theta_S\right) \qquad (2.12) \\
= & Q^3\left(W^3(T) > X_T, W^3(S) > X_S\right) \\
& -Q^3\left(W^3(T) > X_T, W^3(S) > X_S, \tau_S \in \theta_S\right) \\
& -Q^3\left(W^3(T) > X_T, \tau_T \in \theta_T, W^3(S) > X_S\right) \\
& +Q^3\left(W^3(T) > X_T, \tau_T \in \theta_T, W^3(S) > X_S, \tau_S \in \theta_S\right)
\end{aligned}$$

Having transformed ω into a Wiener-processes, we are ready to employ the reflection principle.

The Reflection Principle

- The first term of (2.12).

$$\begin{aligned}
& Q^3\left(W^3(T) > X_T, W^3(S) > X_S\right) \\
= & Q^3\left(-W^3(T) < -X_T, -W^3(S) < -X_S\right)
\end{aligned}$$

- The second term of (2.12).

$$\begin{aligned}
& -Q^3\left(W^3(T) > X_T, W^3(S) > X_S, \tau_S \in \theta_S\right) \\
= & -Q^3\left(W^3(T) < 2\Lambda - X_T, W^3(S) < 2\Lambda - X_S, \tau_S \in \theta_S\right) \\
= & -Q^3\left(W^3(T) < 2\Lambda - X_T, W^3(S) < 2\Lambda - X_S\right)
\end{aligned}$$

- The third term of (2.12).

$$\begin{aligned}
& -Q^3\left(W^3(T) > X_T, \tau_T \in \theta_T, W^3(S) > X_S\right) \\
= & -Q^3\left(W^3(T) < 2\Lambda - X_T, \tau_T \in \theta_T, W^3(S) > X_S\right) \\
= & -Q^3\left(W^3(T) < 2\Lambda - X_T, W^3(S) > X_S\right) \\
= & -Q^3\left(W^3(T) < 2\Lambda - X_T, -W^3(S) < -X_S\right)
\end{aligned}$$

- The fourth term of (2.12).

$$\begin{aligned}
& Q^3\left(W^3(T) > X_T, \tau_T \in \theta_T, W^3(S) > X_S, \tau_S \in \theta_S\right) \\
= & Q^3\left(W^3(T) < 2\Lambda - X_T, \tau_T \in \theta_T, W^3(S) < 2\Lambda - X_S, \tau_S \in \theta_S\right) \\
= & Q^3\left(W^3(T) < 2\Lambda - X_T, \tau_T \in \theta_T, W^3(S) < 2\Lambda - X_S\right) \\
= & Q^3\left(W^3(T) > X_T, \tau_T \in \theta_T, W^3(S) < 2\Lambda - X_S\right) \\
= & Q^3\left(-W^3(T) < -X_T, W^3(S) < 2\Lambda - X_S\right)
\end{aligned}$$

Note that we use that $Q^3\left(-W^3(T) < X\right) = Q^3\left(W^3(T) < X\right)$. Summing up

$$\begin{aligned}
& Q^3\left(W^3(T) > X_T, \tau_T \notin \theta_T, W^3(S) > X_S, \tau_S \notin \theta_S\right) \\
= \; & Q^3\left(-W^3(T) < -X_T, -W^3(S) < -X_S\right) \\
& -Q^3\left(W^3(T) < 2\Lambda - X_T, W^3(S) < 2\Lambda - X_S\right) \\
& -Q^3\left(W^3(T) < 2\Lambda - X_T, -W^3(S) < -X_S\right) \\
& +Q^3\left(-W^3(T) < -X_T, W^3(S) < 2\Lambda - X_S\right)
\end{aligned} \tag{2.13}$$

Deriving the Density Functions

The next step is to derive the bivariate standardized density functions corresponding to the probabilities in (2.13). We immediately see that

$$\begin{aligned}
& Q^3\left(W^3(T) > X_T, \tau_T \notin \theta_T, W^3(S) > X_S, \tau_S \notin \theta_S\right) \\
= \; & N\left(\tfrac{-X_T}{\sqrt{T}}, \tfrac{-X_S}{\sqrt{S}}, \sqrt{\tfrac{S}{T}}\right) - N\left(\tfrac{2\Lambda - X_T}{\sqrt{T}}, \tfrac{2\Lambda - X_S}{\sqrt{S}}, \sqrt{\tfrac{S}{T}}\right) \\
& -N\left(\tfrac{2\Lambda - X_T}{\sqrt{T}}, \tfrac{-X_S}{\sqrt{S}}, -\sqrt{\tfrac{S}{T}}\right) + N\left(\tfrac{-X_T}{\sqrt{T}}, \tfrac{2\Lambda - X_S}{\sqrt{S}}, -\sqrt{\tfrac{S}{T}}\right)
\end{aligned}$$

where $N(\cdot)$ is the standardized bivariate cumulative distribution function. On integral form the expressions reads

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \left\{ \begin{array}{c}
\int_{\frac{X_T}{\sqrt{T}}}^{\infty} \int_{\frac{X_S}{\sqrt{S}}}^{\infty} \exp\left\{\frac{z_T^2 - 2\rho z_S z_T + z_S^2}{-2(1-\rho^2)}\right\} dz_S dz_T \\
-\int_{\frac{X_T - 2\Lambda}{\sqrt{T}}}^{\infty} \int_{\frac{X_S - 2\Lambda}{\sqrt{S}}}^{\infty} \exp\left\{\frac{z_T^2 - 2\rho z_S z_T + z_S^2}{-2(1-\rho^2)}\right\} dz_S dz_T \\
-\int_{\frac{X_T - 2\Lambda}{\sqrt{T}}}^{\infty} \int_{\frac{X_S}{\sqrt{S}}}^{\infty} \exp\left\{\frac{z_T^2 + 2\rho z_S z_T + z_S^2}{-2(1-\rho^2)}\right\} dz_S dz_T \\
+\int_{\frac{X_T}{\sqrt{T}}}^{\infty} \int_{\frac{X_S - 2\Lambda}{\sqrt{S}}}^{\infty} \exp\left\{\frac{z_T^2 + 2\rho z_S z_T + z_S^2}{-2(1-\rho^2)}\right\} dz_S dz_T
\end{array} \right.$$

Changing integration variables from standard normal to Wiener process we standardize by dividing with the respective standard deviations $\sqrt{S}$ and $\sqrt{T}$ and so obtain

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \times$$

$$\left\{\begin{array}{l} +\frac{1}{\sqrt{S}}\frac{1}{\sqrt{T}}\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty}\exp\left\{\frac{\left(\frac{w_T}{\sqrt{T}}\right)^2-2\rho\frac{w_S}{\sqrt{S}}\frac{w_T}{\sqrt{T}}+\left(\frac{w_S}{\sqrt{S}}\right)^2}{-2(1-\rho^2)}\right\}dw_S dw_T \\ -\frac{1}{\sqrt{S}}\frac{1}{\sqrt{T}}\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty}\exp\left\{\frac{\left(\frac{w_T-2\Lambda}{\sqrt{T}}\right)^2-2\rho\frac{w_T-2\Lambda}{\sqrt{T}}\frac{w_S-2\Lambda}{\sqrt{S}}+\left(\frac{w_S-2\Lambda}{\sqrt{S}}\right)^2}{-2(1-\rho^2)}\right\}dw_S dw_T \\ -\frac{1}{\sqrt{S}}\frac{1}{\sqrt{T}}\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty}\exp\left\{\frac{\left(\frac{w_T-2\Lambda}{\sqrt{T}}\right)^2+2\rho\frac{w_T-2\Lambda}{\sqrt{T}}\frac{w_S}{\sqrt{S}}+\left(\frac{w_S}{\sqrt{S}}\right)^2}{-2(1-\rho^2)}\right\}dw_S dw_T \\ +\frac{1}{\sqrt{S}}\frac{1}{\sqrt{T}}\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty}\exp\left\{\frac{\left(\frac{w_T}{\sqrt{T}}\right)^2+2\rho\frac{w_T}{\sqrt{T}}\frac{w_S-2\Lambda}{\sqrt{S}}+\left(\frac{w_S-2\Lambda}{\sqrt{S}}\right)^2}{-2(1-\rho^2)}\right\}dw_S dw_T \end{array}\right.$$

Then we can write[17]

$$=\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty}\left(\begin{array}{c} f\left(0,\sqrt{T};0,\sqrt{S};\rho\right)-f\left(2\Lambda,\sqrt{T};2\Lambda,\sqrt{S};\rho\right) \\ -f\left(2\Lambda,\sqrt{T};0,\sqrt{S};-\rho\right)+f\left(0,\sqrt{T};2\Lambda,\sqrt{S};-\rho\right) \end{array}\right)dw_S dw_T$$

Finally, comparing with (2.11) we obtain bivariate normal density functions:

$$\begin{aligned} & Q^3\left(W^3(T)\in dw_T,\tau_T\notin\theta_T,W^3(S)\in dw_S,\tau_S\notin\theta_S\right) \quad (2.14) \\ = \; & f\left(0,\sqrt{T};0,\sqrt{S};\rho\right)dw_S dw_T \\ & -f\left(2\Lambda,\sqrt{T};2\Lambda,\sqrt{S};\rho\right)dw_S dw_T \\ & -f\left(2\Lambda,\sqrt{T};0,\sqrt{S};-\rho\right)dw_S dw_T \\ & +f\left(0,\sqrt{T};2\Lambda,\sqrt{S};-\rho\right)dw_S dw_T \end{aligned}$$

Thus we are ready to integrate expression (2.10).

2.7.1.3 Integration with Square Completion

Inserting (2.14) into (2.10) and using[18] $R^{3\to m}=e^{h^{3\to m}W^3(T)-\frac{1}{2}\left(h^{3\to 2m}\right)^2 T}$

$$\begin{aligned} & Q^2(A_2) \\ = \; & \int_{X_T}^{\infty}\int_{X_S}^{\infty} e^{h^{3\to m}w_T-\frac{1}{2}\left(h^{3\to m}\right)^2 T} f\left(0,\sqrt{T};0,\sqrt{S};\rho\right)dw_S dw_T \end{aligned}$$

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[17] Denote with $f(\mu_X,\sigma_X;\mu_Y,\sigma_Y;\rho)$ the bivariate normal density function for $\{X=x;Y=y\}$ where $X\sim N(\mu_X,\sigma_X)$ and $Y\sim N(\mu_Y,\sigma_Y)$ when the correlation between them is ρ.

[18] We use the more general notation with m anticipating the solution for the Q^1-probabilities in section 2.7.2.

$$-\int_{X_T}^{\infty}\int_{X_S}^{\infty} e^{h^{3\to m}w_T-\frac{1}{2}\left(h^{3\to m}\right)^2T} f\left(2\Lambda,\sqrt{T};2\Lambda,\sqrt{S};\rho\right) dw_S dw_T$$
$$-\int_{X_T}^{\infty}\int_{X_S}^{\infty} e^{h^{3\to m}w_T-\frac{1}{2}\left(h^{3\to m}\right)^2T} f\left(2\Lambda,\sqrt{T};0,\sqrt{S};-\rho\right) dw_S dw_T$$
$$+\int_{X_T}^{\infty}\int_{X_S}^{\infty} e^{h^{3\to m}w_T-\frac{1}{2}\left(h^{3\to m}\right)^2T} f\left(0,\sqrt{T};2\Lambda,\sqrt{S};-\rho\right) dw_S dw_T$$

or

$$Q^2(A_2) = e^{-\frac{1}{2}\left(h^{3\to m}\right)^2T}\int_{X_T}^{\infty}\int_{X_S}^{\infty} e^{h^{3\to m}w_T} f\left(0,\sqrt{T};0,\sqrt{S};\rho\right) dw_S dw_T$$
$$-e^{-\frac{1}{2}\left(h^{3\to m}\right)^2T}\int_{X_T}^{\infty}\int_{X_S}^{\infty} e^{h^{3\to m}w_T} f\left(2\Lambda,\sqrt{T};2\Lambda,\sqrt{S};\rho\right) dw_S dw_T$$
$$-e^{-\frac{1}{2}\left(h^{3\to m}\right)^2T}\int_{X_T}^{\infty}\int_{X_S}^{\infty} e^{h^{3\to m}w_T} f\left(2\Lambda,\sqrt{T};0,\sqrt{S};-\rho\right) dw_S dw_T$$
$$+e^{-\frac{1}{2}\left(h^{3\to m}\right)^2T}\int_{X_T}^{\infty}\int_{X_S}^{\infty} e^{h^{3\to m}w_T} f\left(0,\sqrt{T};2\Lambda,\sqrt{S};-\rho\right) dw_S dw_T \quad (2.15)$$

As a remark, we show how a square completion is performed in the general case before proceeding.

Remark 2.3 ***Completing the square - general bivariate case.*** *Consider the following expression*

$$\left\{\begin{array}{c} exp\left\{aw_2+bw_1\right\} \\ \times f\left(\mu_{w_T},\sigma_{w_T};\mu_{w_S},\sigma_{w_S};\rho\right) \end{array}\right.$$

that is

$$\left\{\begin{array}{c} \exp\left\{aw_T+bw_S\right\} \\ \times\frac{1}{2\pi\sigma_{w_T}\sigma_{w_S}\sqrt{1-\rho^2}} \\ \times\exp\left\{-\frac{\left(\frac{w_T-\mu_{w_T}}{\sigma_{w_T}}\right)^2-2\rho\frac{w_T-\mu_{w_T}}{\sigma_{w_T}}\frac{w_S-\mu_{w_S}}{\sigma_{w_S}}+\left(\frac{w_S-\mu_{w_S}}{\sigma_{w_S}}\right)^2}{\sqrt{1-\rho^2}}\right\} \end{array}\right.$$

Then this is equal to

$$\left\{\begin{array}{c} \exp\left\{a\mu_{w_T}+b\mu_{w_S}+\frac{1}{2}\left(a^2\sigma_{w_T}^2+2ab\sigma_{w_Tw_S}+b^2\sigma_{w_S}^2\right)\right\} \\ \times\frac{1}{2\pi\sigma_{w_T}\sigma_{w_S}\sqrt{1-\rho^2}} \\ \times\exp\left\{-\frac{\left(\frac{w_T-\overline{\mu}_{w_T}}{\sigma_{w_T}}\right)^2-2\rho\left(\frac{w_T-\overline{\mu}_{w_T}}{\sigma_{w_T}}\right)\left(\frac{w_S-\overline{\mu}_{w_S}}{\sigma_{w_S}}\right)+\left(\frac{w_S-\overline{\mu}_{w_S}}{\sigma_{w_S}}\right)^2}{\sqrt{1-\rho^2}}\right\} \end{array}\right.$$

$$\begin{aligned}\overline{\mu}_{w_S} &= \mu_{w_S} + b\sigma_{w_S}^2 + a\sigma_{w_T w_S}\\ \overline{\mu}_{w_T} &= \mu_{w_T} + a\sigma_{w_T}^2 + b\sigma_{w_T w_S}\end{aligned}$$

where $\sigma_{w_T w_S} = \rho\sigma_{w_T}\sigma_{w_S}$. *Rewritten*

$$\left\{\begin{array}{c}\exp\left\{a\mu_{w_T} + b\mu_{w_S} + \frac{1}{2}\left(a^2\sigma_{w_T}^2 + 2ab\sigma_{w_T w_S} + b^2\sigma_{w_S}^2\right)\right\}\\ \times f\left(\mu_{w_T} + a\sigma_{w_T}^2 + b\sigma_{w_T w_S}, \sigma_{w_T}; \mu_{w_S} + b\sigma_{w_S}^2 + a\sigma_{w_T w_S}, \sigma_{w_S}; \rho\right)\end{array}\right.$$

Now we return to expression (2.15). Completing squares with, using the above notation, $a = h^{3\to m}$ and $b = 0$, we obtain

$$Q^2(A_2) = e^{-\frac{1}{2}\left(h^{3\to m}\right)^2 T} \times$$

$$\left[\begin{array}{c}\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty}\left\{\begin{array}{c}e^{\frac{1}{2}\left(h^{3\to m}\right)^2 T}\times\\ f\left(h^{3\to m}T, \sqrt{T}; h^{3\to m}S, \sqrt{S}; \rho\right)\end{array}\right\}dw_S dw_T\\ -\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty}\left\{\begin{array}{c}e^{h^{3\to m}2\Lambda + \frac{1}{2}\left(h^{3\to m}\right)^2 T}\times\\ f\left(2\Lambda + h^{3\to m}T, \sqrt{T}; 2\Lambda + h^{3\to m}S, \sqrt{S}; \rho\right)\end{array}\right\}dw_S dw_T\\ -\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty}\left\{\begin{array}{c}e^{h^{3\to m}2\Lambda + \frac{1}{2}\left(h^{3\to m}\right)^2 T}\times\\ f\left(2\Lambda + h^{3\to m}T, \sqrt{T}; -h^{3\to m}S, \sqrt{S}; -\rho\right)\end{array}\right\}dw_S dw_T\\ +\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty}\left\{\begin{array}{c}e^{\frac{1}{2}\left(h^{3\to m}\right)^2 T}\times\\ f\left(h^{3\to m}T, \sqrt{T}; 2\Lambda - h^{3\to m}S, \sqrt{S}; -\rho\right)\end{array}\right\}dw_S dw_T\end{array}\right]$$

Cancelling terms we get

$$\begin{aligned}&Q^2(A_2)\\ =\ &\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty} f\left(h^{3\to m}T, \sqrt{T}; h^{3\to m}S, \sqrt{S}; \rho\right)dw_S dw_T\\ &-e^{h^{3\to m}2\Lambda}\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty} f\left(2\Lambda + h^{3\to m}T, \sqrt{T}; 2\Lambda + h^{3\to m}S, \sqrt{S}; \rho\right)dw_S dw_T\\ &-e^{h^{3\to m}2\Lambda}\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty} f\left(2\Lambda + h^{3\to m}T, \sqrt{T}; -h^{3\to m}S, \sqrt{S}; -\rho\right)dw_S dw_T\\ &+\int\limits_{X_T}^{\infty}\int\limits_{X_S}^{\infty} f\left(h^{3\to m}T, \sqrt{T}; 2\Lambda - h^{3\to m}S, \sqrt{S}; -\rho\right)dw_S dw_T\end{aligned}$$

Changing integration variables to standard normal again yields

$$Q^2(A_2)$$

$$= \int_{\frac{X_T - h^{3\to m}T}{\sqrt{T}}}^{\infty} \int_{\frac{X_S - h^{3\to m}S}{\sqrt{S}}}^{\infty} f(0,1;0,1;\rho)\, dz_S dz_T$$
$$-e^{h^{3\to m}2\Lambda} \int_{\frac{X_T - h^{3\to m}T - 2\Lambda}{\sqrt{T}}}^{\infty} \int_{\frac{X_S - h^{3\to m}S - 2\Lambda}{\sqrt{S}}}^{\infty} f(0,1;0,1;\rho)\, dz_S dz_T$$
$$-e^{h^{3\to m}2\Lambda} \int_{\frac{X_T - h^{3\to m}T - 2\Lambda}{\sqrt{T}}}^{\infty} \int_{\frac{X_S + h^{3\to m}S}{\sqrt{S}}}^{\infty} f(0,1;0,1;-\rho)\, dz_S dz_T$$
$$+ \int_{\frac{X_T - h^{3\to m}T}{\sqrt{T}}}^{\infty} \int_{\frac{X_S + h^{3\to m}S - 2\Lambda}{\sqrt{S}}}^{\infty} f(0,1;0,1;-\rho)\, dz_S dz_T$$

Finally, integrating, we obtain

$$\begin{aligned} Q^2(A_2) &= N\left(\tfrac{h^{3\to m}T - X_T}{\sqrt{T}}, \tfrac{h^{3\to m}S - X_S}{\sqrt{S}}, \rho\right) \\ &\quad -e^{h^{3\to m}2\Lambda} N\left(\tfrac{2\Lambda + h^{3\to m}T - X_T}{\sqrt{T}}, \tfrac{2\Lambda + h^{3\to m}S - X_S}{\sqrt{S}}, \rho\right) \\ &\quad -e^{h^{3\to m}2\Lambda} N\left(\tfrac{2\Lambda + h^{3\to m}T - X_T}{\sqrt{T}}, \tfrac{-h^{3\to m}S - X_S}{\sqrt{S}}, -\rho\right) \\ &\quad +N\left(\tfrac{h^{3\to m}T - X_T}{\sqrt{T}}, \tfrac{2\Lambda - h^{3\to m}S - X_S}{\sqrt{S}}, -\rho\right) \end{aligned}$$

Noting $e^{h^{3\to m}2\Lambda} = \left(\frac{L}{\omega_0}\right)^{\frac{2}{\sigma}h^{3\to m}}$ we obtain the probability in Lemma 1.

2.7.2 The Q^1−probability of being In-the-money at T

Now consider the *second* probability of (2.4):

$$Q^1(A_T) = E^{Q^1}\left[I_{\{\omega(T)>F, \tau_T \notin \theta_T, \omega(S)>\overline{\omega}, \tau_S \notin \theta_S\}} \,|\Im_0\right]$$

Comparing with expression (2.7) we see that the only difference is that we take expectations under the Q^1-probability measure, where the dynamics for $\eta(t)$ are given by

$$\begin{cases} d\eta = \left(\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right) dt + dW^1 \\ \eta(0) = 0 \end{cases}$$

The Girsanov transformation $dQ^1 = R^{3\to 1} dQ^3$ with kernel

$$h^{3\to 1} = \left(\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right)$$

is therefore necessary to transform the second term into the analogue of (2.10):

$$\int_{X_T}^{\infty} \int_{X_S}^{\infty} R^{3\to 1} Q^3\left(W^3(T) \in dw_T, \tau_T \notin \theta_T, W^3(S) \in dw_S, \tau_S \notin \theta_S\right) \tag{2.16}$$

If we insert (2.14) into (2.16), perform the square completions (with $m = 1$) and solve the integrals we obtain, as before, a sum of four bivariate normal cumulative distribution functions.

2.8 Appendix: Prices and Probabilities

Below we list some previous results with our notation.

2.8.1 Probabilities

The in-the-money probability under Q^m for the standard call and Heaviside is

$$Q^m\left(\omega(S) > \overline{\omega}\right) = N\left(d_S^{3\to m}\left(\tfrac{\omega_0}{\overline{\omega}}\right)\right)$$

and for the down-and-out call and Heaviside it is

$$Q^m\left(A_S\right) = N\left(d_S^{3\to m}\left(\tfrac{\omega_0}{\overline{\omega}}\right)\right) - \left(\frac{L}{\omega_0}\right)^{\frac{2}{\sigma}h^{3\to 1}} N\left(d_S^{3\to m}\left(\frac{L^2}{\omega_0\cdot\overline{\omega}}\right)\right)$$

For the standard compound call it is

$$Q^m\left(\omega(T) > F, \omega(S) > \overline{\omega}\right) = N\left(d_S^{3\to m}\left(\tfrac{\omega_0}{\overline{\omega}}\right), d_T^{3\to m}\left(\tfrac{\omega_0}{F}\right), \rho\right)$$

The formulae for $d_s^{3\to m}(x)$ and ρ are given in *Lemma 1.*

2.8.2 Heavisides

A Heaviside has payoff function

$$H\left(\omega(S); X, S\right) = \begin{cases} 1 \text{ if } \omega(S) \geq X \\ 0 \text{ otherwise} \end{cases}$$

The price at time zero of a Heaviside is

$$H\left(\omega_0; X, S\right) = e^{-rS} Q^1\left(\omega(S) > X\right)$$

and the price of a down-and-out Heaviside is

$$H_L\left(\omega_0; X, S\right) = e^{-rS} Q^1\left(\omega(S) > X, \tau_S \notin \theta_S\right)$$

2.8.3 Calls

The price of a call is given by Black & Scholes (1973)

$$C\left(\omega_0; X, S\right) = \omega_0 \cdot Q^2\left(\omega(S) > X\right) - e^{-rS} X \cdot Q^1\left(\omega(S) > X\right)$$

The price of a down-and-out call is

$$\begin{aligned} & C_L(\omega_0; X, S) \\ = \;& \omega_0 \cdot Q^2(\omega(S) > X, \tau_S \notin \theta_S) - e^{-rS} X \cdot Q^1(\omega(S) > X, \tau_S \notin \theta_S) \end{aligned}$$

and the price of a compound call Geske (1979) is

$$\begin{aligned} C(C(\omega_0; F, T); X, S) \;=\;& \omega_0 \cdot Q^2(\omega(S) > X, \omega(T) > F) \\ & -e^{-rT} F \cdot Q^1(\omega(S) > X, \omega(T) > F) \\ & -e^{-rS} X \cdot Q^1(\omega(S) > X) \end{aligned}$$

2.9 Appendix: Derivation of Proposition 2

Standard arguments give us the value of a call on a security G as the discounted value of Q^1-expected payoffs.

$$\begin{aligned} & C\left(\widehat{G}(\omega_0;\cdot);K,S\right) \\ = \; & e^{-rS}E^{Q^1}\left[\left(\widehat{G}(\omega(S);\cdot)-K\right)\cdot I_{\{G>K\}}\right] \\ = \; & e^{-rS}E^{Q^1}\left[\left(G(\omega(S);\cdot)\cdot I_{\{\tau_S\notin\theta_S\}}-K\right)\cdot I_{\{\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}}\right] \end{aligned}$$

where $\overline{\omega}$ solves $G(\omega(S);\cdot)=K$. Recognizing $E^{Q^1}\left[I_{\{\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}}\right]$ as the Q^1-expected payoff to a Heaviside we can write

$$\begin{aligned} C\left(\widehat{G}(\omega_0;\cdot);K,S\right) \; = \; & e^{-rS}E^{Q^1}\left[G(\omega(S);\cdot)\cdot I_{\{\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}}\right] \\ & -K\cdot H_L\left(\omega_0;\overline{\omega},S\right) \end{aligned}$$

Inserting the price of a from *Lemma 2.3*

$$\begin{aligned} C\left(\widehat{G}(\omega_0;\cdot);K,S\right) \; = \; & e^{-rS}E^{Q^1}\left[\left\{\begin{array}{c}\alpha\Omega \\ + \\ \sum_i\beta^{(i)}C_L^{(i)} \\ + \\ \sum_i\gamma^{(i)}H_L^{(i)}\end{array}\right\}\cdot I_{\{\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}}\right] \\ & -K\cdot H_L\left(\omega_0;\overline{\omega},S\right) \end{aligned}$$

Since the event $\{\tau_S\notin\theta_S\}$ is included in the down-and-out event

$$\begin{aligned} C\left(\widehat{G}(\omega_0;\cdot);K,S\right) \; = \; & e^{-rS}E^{Q^1}\left[\left\{\begin{array}{c}\alpha\Omega\cdot I_{\{\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}} \\ + \\ \sum_i\beta^{(i)}C_L^{(i)}\cdot I_{\{\omega(S)>\overline{\omega}\}} \\ + \\ \sum_i\gamma^{(i)}H_L^{(i)}\cdot I_{\{\omega(S)>\overline{\omega}\}}\end{array}\right\}\right] \\ & -K\cdot H_L\left(\omega_0;\overline{\omega},S\right) \end{aligned}$$

Moving the expectations operator and the discount factor inside the summation operator

$$C\left(\widehat{G}(\omega_0;\cdot);K,S\right) = \left\{\begin{array}{c}\alpha e^{-rS}E^{Q^1}\left[\Omega\cdot I_{\{\omega(S)>\overline{\omega},\tau_S\notin\theta_S\}}\right] \\ + \\ \sum_i\beta^{(i)}e^{-rS}E^{Q^1}\left[C_L^{(i)}\cdot I_{\{\omega(S)>\overline{\omega}\}}\right] \\ + \\ \sum_i\gamma^{(i)}e^{-rS}E^{Q^1}\left[H_L^{(i)}\cdot I_{\{\omega(S)>\overline{\omega}\}}\right]\end{array}\right\}$$

$$-K \cdot H_L(\omega_0; \overline{\omega}, S)$$

The first row is the expected payoff to a claim giving its holder the value of assets at S conditional on no previous default and $\omega(S) > \overline{\omega}$; i.e. a conditional (on $\{\omega(S) > \overline{\omega}\}$) down-and-out call with an exercise price equal to zero. The second and third rows contain expected payoffs to conditional down-and-out calls and Heavisides. We denote the exercise price and maturity of a call or Heaviside $^{(i)}$ with $F^{(i)}$ and $t^{(i)}$, respectively. Thus we obtain *Proposition 2.*

$$C\left(\widehat{G}(\omega(0);\cdot);K,S\right) = \begin{cases} \alpha \cdot \mathcal{C}_L(\omega_0; 0, T \,|\omega(S) > \overline{\omega}) \\ + \\ \sum_i \beta^{(i)} \cdot \mathcal{C}_L\left(\omega_0; F^{(i)}, t^{(i)} \,|\omega(S) > \overline{\omega}\right) \\ + \\ \sum_i \gamma^{(i)} \cdot \mathcal{H}_L\left(\omega_0; F^{(i)}, t^{(i)} \,|\omega(S) > \overline{\omega}\right) \\ - \\ K \cdot H_L(\omega_0; \overline{\omega}, S) \end{cases}$$

3

Implementing Firm Value Based Pricing Models

3.1 Introduction

An important application of contingent claims analysis is to the pricing of corporate liabilities and non standardized options. The idea is to use price information from traded securities to estimate the dynamics for the value of the firm and subsequently price other securities. Although the theoretical literature in this field has grown in recent years, the models, termed structural form or firm value based (FVB) models, have not been satisfactorily tested. Before the models are brought to market data, however, it is necessary to gauge the properties of the estimators.

The aim of this paper is to evaluate maximum likelihood estimation of and pricing with a simple firm value based model. We perform Monte Carlo experiments to assess the small sample properties of price estimators for debt and options. We rely on the basic framework of Black & Scholes (1973) and Geske (1979) to price debt and options respectively, but suggest a way to avoid the common yet unsatisfactory assumption that the firm's assets are continuously traded.

It was not until option pricing theory was developed by Black & Scholes (1973) and Merton (1973) that a systematic theory for the pricing of corporate liabilities became available. Their insight was that one can view the firm's value as opposed to the stock price as the underlying state variable. Geske (1979) priced stock options as compound options in this framework.

However, the initial enthusiasm about the practical applicability was frustrated by the results of attempts to empirically implement the models[1].

In the early 1990s, interest in FVB models was renewed. A number of stylized facts were incorporated into the models - such as violations of the absolute priority rule, taxes, costly financial distress and cash flow triggered default[2]. It was felt that these features would enhance model performance (see John (1993)). While these models, unlike their predecessors, are able to generate prices in line with market quotes with reasonable parameters, this alone does not guarantee that they will actually perform any better on market data. A research priority should therefore be to test the pricing performance of the recent models empirically.

Most earlier empirical tests have used a theoretically inconsistent estimation procedure. Recently, Duan (1994) has suggested a maximum likelihood technique. It is probable that using this consistent estimation procedure would enhance the FVB models performance. The small sample properties of the implied estimators, however, have not been examined until now.

We make two contributions in this paper. First, we show how to relax the vexing assumption of traded assets when implementing FVB models. Second, using simulated data, we evaluate a maximum likelihood (ML) technique for implementing a simple case of such a model.

We find that the ML technique works well and clearly outperforms the traditional estimation method. This appears promising for future empirical work.

The paper is organized as follows. Section 3.2 briefly reviews the economic framework and the pricing results we base our simulations on. The first issue addressed in this section is the tradeability of assets. Section 3.3 describes the design of the study and section 3.4 presents the results. A concluding discussion is presented in section 3.5.

3.2 The Economic Setting

In this section we discuss the basic economic setup on which we will base our subsequent study. In particular we discuss assumptions relating to the state

[1] For example, Jones et al. (1984) apply an extension of the Merton (1974) model to a sample of American firms and find that their model systematically underestimates yield spreads.

[2] Recent models of corporate debt (and related securities) along these lines include Longstaff & Schwartz (1995), Nielsen et al. (1993), Kim et al. (1993), Anderson & Sundaresan (1996), Leland (1994*b*), Leland (1994*a*), Leland & Toft (1996), Toft & Prucyk (1996), Ericsson & Reneby (1995) and (1996). The implications of some of these models have been tested. For example Longstaff & Schwartz (1995) and Toft & Prucyk (1996) find empirical support for restrictions implied by their models. However, whether these models are better qualified than their predecessors to match observed price data is still an open question.

variable that underlies our pricing expressions. We show that in many cases the assumption of continuously traded assets can be relaxed at a negligible cost in terms of tractability. We then briefly review the pricing results of Black & Scholes (1973) and Geske (1979).

The state variable used in the FVB class of models is linked to the value of a firm's assets. The prices of the firm's securities depend on the share of the firm value that each security holder is entitled to when the firm value is divided among claimants at some future point in time. We make the standard Black & Scholes (1973), Merton (1974) assumptions about the economy with the exception of the tradeability of the state variable. We thus assume that there are no transaction costs or taxes. Furthermore, arbitrage opportunities are ruled out and investors are price takers who can borrow and lend freely at the constant risk-free rate r. Furthermore there are no restrictions on short sales, assets are perfectly divisible and trading takes place continuously in time.

In this setting, corporate securities and their derivatives are valued as claims contingent on the underlying asset state variable. The assumption regarding the tradeability of the state variable is important for the appearance of the obtained pricing expressions. When assets are not continuously traded, those expressions will require the knowledge of more parameters than otherwise. For example, investors' attitude towards risk will enter directly into the formulae. Most papers in this field, perhaps for this reason, maintain the assumption of continuously traded assets. In the following subsection, we will show how to relax this assumption without loss of tractability. This is achieved through a simple transformation of variables.

3.2.1 The State Variable Assumption

An almost ubiquitous assumption made in papers that deal with FVB models is that the state variable follows a geometric Brownian motion under the objective probability measure P.

$$\begin{cases} dv = \mu v dt + \sigma v dW^P \\ v(0) = v_n \end{cases} \tag{3.1}$$

Then it is assumed (more or less explicitly) that v may be traded on frictionless markets[3]. This implies that under a (unique) probability measure Q under which prices deflated by a unit of the money market account are martingales[4], the state variable process will take on the following appear-

[3] A time index in subscript denotes an ordinal (daily) position and a time index within brackets refers to the time in years. E.g. it holds that $v\left(\frac{7}{365}\right) \equiv v_{n+7}$ since $v(0) = v_n$.

[4] The Girsanov kernel for the transformation from P to Q is, when assets are traded, $\lambda \equiv \frac{\mu - r}{\sigma}$. The probability measure Q is commonly referred to as the "risk neutral" probability measure.

ance

$$\begin{cases} dv = rvdt + \sigma v dW^Q \\ v(0) = v_n \end{cases} \tag{3.2}$$

It is often recognized that the assumption of traded assets is a strong one in many situations, but as a rule, few papers address this problem directly. If we were to relax the assumption of traded assets, we would be in an incomplete market setting in that we cannot replicate the state variable v. The process for v under a measure Q (now no longer uniquely determined[5]) would then be

$$\begin{cases} dv = (\mu - \lambda\sigma)\, vdt + \sigma v dW^Q \\ v(0) = v_n \end{cases} \tag{3.3}$$

where λ may be interpreted as the market price of risk for the operations of the firm. Note that v now no longer describes the dynamics of a *price* variable (and hence can no longer be termed asset *value*). Furthermore, three firm-specific parameters (μ, λ, σ) as opposed to one (σ) determine the process.

Suppose, however, that at some point in time T, the firm's assets v will be traded. We can then calculate the value of a contract denoted $\omega(t)$ that entitles the holder to $v(T)$ at time T as

$$\omega(t) = e^{-r(T-t)} E^Q [v(T)] = v(t)\, e^{(\mu - \lambda\sigma - r)(T-t)} \tag{3.4}$$

This contract may be interpreted as the value of a corresponding all-equity firm or simply the *value of assets*[6]. The dynamics for ω under the objective measure P (by applying Itô's lemma on equations (3.4) and (3.1)) are

$$\begin{cases} d\omega = (r + \lambda\sigma)\, \omega dt + \sigma\omega dW^P \\ \omega(0) = \omega_n \end{cases} \tag{3.5}$$

and under the measure Q

$$\begin{cases} d\omega = r\omega dt + \sigma\omega dW^Q \\ \omega(0) = \omega_n \end{cases} \tag{3.6}$$

Given that the ultimate aim is to price a security and not to analyse the process for v itself, knowledge about the process for ω is sufficient. The reason is that the parameters $(v(0), \mu, \lambda)$ only appear in the pricing formulae for corporate securities and derivatives (see section (3.2.2)) as one entity (that is as ω as defined in (3.4)). Therefore, for pricing purposes, the information contained in $(\omega(0), \sigma)$ is equivalent to the information contained in $(v(0), \mu, \lambda, \sigma)$.

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[5] That is, the Girsanov kernel λ is no longer uniquely determined (cf. footnote 4).

[6] Note that if the firm's assets are continuously traded, the Girsanov kernel is given by footnote (4). Then, from eq. (3.4), $\omega = v$, which illustrates that v can be interpreted as the value of assets only when they are *continuously* traded.

This is of major importance when implementing an FVB model. Since the asset process is most likely to be unobserved, estimation of its parameters must be based on some observed variables, for example the prices of traded securities. Since, as noted above, these only contain information about $(v(0), \mu, \lambda)$ as one entity, it will be impossible to estimate the process for the state variable (3.1) (for example it will not be possible to distinguish a situation with high μ and low v from a situation with low μ and high v); it will, however, be possible to estimate the process for the asset value, (3.5).

We can thus conclude the following:

- for pricing purposes the pair $(\omega(0), \sigma)$ is a sufficient statistic
- it is therefore in most cases sufficient to estimate the (fictive) process (3.5)
- the existence of the process (3.5) does not require continuously traded assets - a sufficient condition is that assets are traded at a discrete future point in time (after or at the maturity of the security to be priced)

It is important to note the fundamental difference between *assuming* that the firm's assets are continuously traded and *deriving* that it is possible to trade continuously in a portfolio mimicking the value of the firm's assets[7]. It appears that in many cases unduly restrictive assumptions have been imposed - for pricing purposes, the continuous tradeability assumption is in practice unnecessary[8]. However, our argument requires that the state variable is revealed *as a value* at some known point in time when it can also be traded. We feel that this is a relatively mild assumption within the framework of an FVB model. The argument is *not* valid for models with general non-traded state variables such as interest rate models.

Finally, note that even though the asset value process ω is fictive in that it is not traded directly, it can be replicated by trading continuously in a traded security, such as equity, and the money market account. It is not possible to replicate the state variable v.

[7] An argument sometimes heard is: "Assets are not traded but we can always replicate them by trading in equity and the money market account - and thus pricing of debt and options can be carried out by standard arbitrage methods". That argument is not valid, however, since if the assets are never traded, the price of equity cannot be derived with arbitrage methods in the first place. Hence, its pricing formula and "hedge ratio" will be unknown.

[8] It is even more unsatisfactory to maintain this hypothesis in a situation where the asset process is being estimated - assets are in this case assumed to be traded while their value is not observable.

3.2.2 Pricing

We consider a hypothetical firm financed with an issue of discount debt of maturity T and principal F. Financial distress occurs at maturity of debt if the value of the assets at that point in time does not suffice to pay off the creditor. Given these assumptions and the state variable assumptions discussed above, the values of debt and equity are well known and the formulae for debt and equity are[9]

$$D(\omega(t), t; \sigma) = \omega(t) \cdot N(-d_1(T, F)) + Fe^{-r(T-t)} N(d_2(T, F)) \quad (3.7)$$

and

$$E(\omega(t), t; \sigma) = \omega(t) \cdot N(d_1(T, F)) - Fe^{-r(T-t)} N(d_2(T, F)) \quad (3.8)$$

respectively, where

$$\begin{aligned} d_1(s, p) &= \frac{\ln \frac{\omega(t)}{p} + \left(r + \frac{1}{2}\sigma^2\right)(s - t)}{\sigma\sqrt{s - t}} \\ d_2(s, p) &= d_1(s, p) - \sigma\sqrt{s - t} \end{aligned} \quad (3.9)$$

Denote the price of a call option with time to expiration S and exercise price X written on the stock by $C(\omega, t; S, X)$. Since the stock is viewed as a call option on the firm's assets the stock options will be compound options. They will accordingly be priced with the Geske (1979) model.

$$\begin{aligned} C(\omega(t), t; \sigma, S, X) &= \omega(t) \cdot N(d_1(S, \overline{\omega}), d_1(T, F), \rho) \\ &\quad -Fe^{-r(T-t)} N(d_2(S, \overline{\omega}), d_2(T, F), \rho) \\ &\quad -Xe^{-r(S-t)} N(d_1(S, \overline{\omega})) \end{aligned} \quad (3.10)$$

where $N(\cdot)$ and $N(\cdot, \cdot)$ denote the univariate and bivariate normal cumulative distribution functions respectively, and $\rho = \sqrt{\frac{S-t}{T-t}}$. We denote by $\overline{\omega}$

[9] See Black & Scholes (1973). We have chosen to write the formulae in terms of the asset value ω instead of the state variable v. Thus equation (3.7) is shorthand for

$$D(v(t), t; \mu, \lambda, \sigma) = v(t) e^{(\mu - \lambda\sigma - r)(T-t)} \cdot N(-d_1(T, F)) + Fe^{-r(T-t)} N(d_2(T, F))$$

where

$$\begin{aligned} d_1(s, p) &= \frac{\ln \frac{v(t)}{p} + \left(\mu - \lambda\sigma + \frac{1}{2}\sigma^2\right)(s - t)}{\sigma\sqrt{s - t}} \\ d_2(s, p) &= d_1(s, p) - \sigma\sqrt{s - t} \end{aligned}$$

the solution to the following equation

$$E(\overline{\omega}, S) = X$$

that is the level of the asset value ω for which the option expires precisely at-the-money (the exercise price translated into asset value terms).

Note that the pricing expressions for debt, equity and the options implicitly contain μ, λ and $v(t)$ although they need not be entered other than in the form of $\omega(t)$. In this sense they are not "preference free" as the traditional Black & Scholes (1973) formulae would be (cf.. footnote 9).

3.3 Experiment Design

The idea of the study we will conduct is the following. Imagine that at $t = 0$ a new debt contract is floated or a number of call options on the firm's stock are introduced. We are thus in need of price estimates for these instruments. The face value of debt, F, its maturity T, the exercise prices X_j, the option expiration dates S_j and the risk-free interest rate r can be observed. Thus, of the parameters necessary to price the calls and debt all are known save the ones describing the asset value process, that is $(\sigma, \omega(0))$. To estimate these (along with the market price of risk, λ, although it is not needed for pricing purposes) we use a time series of price quotations for the firm's stock. The price estimates of calls and debt are thereafter obtained by plugging these estimates into the respective formulae.

For a sufficiently long time series, the asset value process could be estimated without error. In a practical situation however, the time series used will generally not be long enough[10] to do so. This study will investigate the "not long enough" aspect, that is small sample characteristics, of the parameter estimates and how these carry over to the prices of call options and debt contracts. The tool will be Monte Carlo simulation.

More specifically, for a chosen scenario $(T, F, \sigma, \mu, r, \lambda, v(0), n)$, we generate 1000 sample paths for the state variable v which all may have resulted in the price of equity observed today[11]. For each path we compute the resulting stock price path using equations (3.4) and (3.8) and use it to estimate the current asset value and the parameter of its process (that

[10] Even if a time series of stock prices exists several decades back, it is of course not reasonable that the parameters of the asset value process and the amount of debt should stay constant that long - even if it is an assumption of the model that this should be the case.

[11] For each path, we first generate a realization of the underlying Wiener-process over the chosen sample period. Thereafter, the state variable path is generated *backwards* starting with the value at $t = 0$ (today). The idea is to construct a "ceteris paribus" situation in the sense that except for the past, all things (e.g. prices, volatilities, leverage) are equal irrespective of the generated path.

is compute the ML estimates $\widehat{\lambda}, \widehat{\sigma}, \widehat{\omega}(0))$. With the parameter estimates $(\widehat{\sigma}, \widehat{\omega}(0))$ it is straightforward to calculate option and debt prices estimates (using equations (3.10) and (3.7)), standard errors of price estimates (using equations (3.18) and (3.17)) and corresponding confidence intervals. This exercise is repeated for each sample path in order to ultimately gauge the sampling distribution. Appendix 3.7 presents a flow chart of the successive steps carried out in the experiment and provides an example.

The whole exercise is repeated for eight different scenarios. The scenarios are defined by choosing long or short debt ($T = 10/1$), high or low asset risk ($\sigma = 25\%/10\%$) and high or low leverage[12] (80%/40%). The values are chosen to give visible indications about the impact of these inputs on estimation and pricing, rather than to correspond to the situations most often encountered in practice. We also vary n to investigate the importance of the estimation period. The parameters (μ, λ) are kept constant throughout and $v(0)$ is chosen to yield a constant current asset value - for reasons explained in section 3.2.1 the individual value of these parameters do not affect the analysis. The interest rate r is also kept constant.

Below we first describe the output that will be generated and define some terminology. Thereafter we briefly describe the Maximum Likelihood estimation technique used. The last subsection discussed the derivation of the (asymptotic) distributions of the price estimators.

3.3.1 Output

The aim of this section is to delineate the outputs produced and the tests performed. When using the term "estimate" we refer to the estimate for a particular sample path. The expected value of an estimate is calculated as the mean of estimates across generated sample paths.

Since the ultimate intended use of the implemented model is to price, the first question to address is whether price estimates $(\widehat{\pi})$ are biased, $(E[\widehat{\pi}] \neq \pi)$, in small samples. Second, we need to know how reliable, or efficient, they are. We use the standard deviation of the estimator $(\gamma_{\widehat{\pi}})$ as one measure of this (We denote the standard deviation of an estimator k with γ_k, the *estimated* standard deviation with $\widehat{\gamma}_k$ and the *asymptotic* standard deviation with Γ_k). Were the estimator normally distributed, this would suffice as a measure of efficiency. As this is often not the case, we also provide the interval within which 95% of the estimates can be found. Of course, the wider the interval, the less efficient the estimate.

A secondary issue is to examine if the asymptotic distributions of estimates carry over to small samples. In large samples the estimator distributions are normal

$$\sqrt{n}\,(\widehat{\pi} - \pi) \xrightarrow{L} N\,(0, \Gamma_{\widehat{\pi}})$$

[12] Leverage is set by adjusting the face value of debt, F.

where the asymptotic standard deviations $\Gamma_{\hat{\pi}}$ for the different pricing formulae are given below in section (3.3.3). We measure the skewness and kurtosis of the sampled distributions and perform Bowman-Shelton tests[13] of normality. Even if a particular estimator is found not to be normally distributed, it may be the case that the estimated standard deviation $\widehat{\gamma}_{\hat{\pi}}$ can be used for hypothesis testing and to calculate confidence intervals in small samples. The standard deviation is estimated using the asymptotic distribution as explained below in section 3.3.3. To see if this estimate is unbiased we compare the expected standard deviation estimate $E\left[\widehat{\gamma}_{\hat{\pi}}\right]$ with the true small sample standard deviation $\gamma_{\hat{\pi}}$. As a measure of the efficiency of this estimate we use *its* standard deviation $\gamma_{\hat{\gamma}_{\hat{\pi}}}$.

To further pursue this issue we carry out a *size test* - that is we see how often the true value of an estimated price parameter falls outside the confidence interval calculated using the estimated standard deviation. This is termed the *simulated population size.* If the *simulated size* is close to the *nominal size* (we use 1%, 5% and 10%), one may conclude that the asymptotic distribution is useful for purposes of calculating confidence intervals.

The price estimates ultimately depend on the estimates of the parameter(s) of the asset process. To help understand the results we therefore report the same output as above for the parameter estimates as well.

Finally, as a comparison, we include results of using the to date most common method of estimating the asset value process and thereafter pricing options and debt (see for example Jones et al. (1984) and Ronn & Verma (1986)). The same economic setting is used and consequently also the same pricing formulae. The difference lies in the estimation of $(\omega(0), \sigma)$ (the market price of risk is not estimated), which requires the following steps to be carried out.

- The instantaneous stock price volatility $\sigma_E = \sigma_E\left(\omega(0), 0; \sigma\right)$ is estimated using historical data.
- The diffusion parameter σ and ω are obtained by solving the following system of equations

$$\left.\begin{aligned} \sigma \tfrac{\partial E}{\partial \omega}\omega(0) &= \widehat{\sigma}_E E^{obs} \\ E\left(\omega(0), 0; \sigma\right) &= E^{obs} \end{aligned}\right\} \rightarrow \left(\widehat{\omega(0)}^{RV}, \widehat{\sigma}^{RV}\right)$$

The first equality is implied by the application of Itô's lemma to equity as a function of $\omega(t)$ and t and the second from matching the theoretical equity price with the observed market price.

[13] The BS-statistic is calculated as follows

$$1000 \times \left[\frac{skewness^2}{6} + \frac{(kurtosis - 3)^2}{24}\right]$$

There are two problems with this method. The instantaneous stock price volatility σ_E is most often estimated under the assumption of it being constant - even though it is a known function of ω and t - which is in fact used to derive the first equation above. Furthermore, the first equation is redundant since it was used to derive the equity price formula in the *second* equation[14]. Another disadvantage of this approach is that it does not allow the straightforward calculation of the distributions of the estimators for ω and σ.

It is, however, a practical and frequently used approach that needs to be evaluated. Furthermore it serves as a benchmark for the performance of the ML approach evaluated in this paper. We will throughout the remainder of this paper refer to this estimation approach as the volatility restriction (VR) approach.

3.3.2 Estimation

The problem at hand is thus maximum likelihood estimation of the parameters of the asset value process. This will be accomplished using a time series of market prices of equity, $\mathbf{E}^{obs} = \{E_i^{obs} : i = 1...n\}$ but in principle any security that can be valued as a derivative on the asset value would serve our purpose. We thus require the likelihood function of the observed price variable. In this section we review how this is achieved.

We start by defining $f(\cdot)$ as the conditional density for E_i^{obs} which gives us the following log-likelihood function

$$L_E\left(\mathbf{E}^{obs};\sigma,\lambda\right) = \sum_{i=2}^{n} \ln f\left(E_i^{obs}\left|E_{i-1}^{obs}\right.;\sigma,\lambda\right) \tag{3.11}$$

To derive the density function for equity we make a change of variables as suggested in Duan (1994)[15]

$$\begin{aligned} f\left(E_i^{obs}\left|E_{i-1}^{obs}\right.;\sigma,\lambda\right) &= g\left(\ln\omega_i\left|\ln\omega_{i-1}\right.;\sigma,\lambda\right)\Big|_{\omega_i=\vartheta\left(E_i^{obs},t_i;\sigma\right)} \\ &\quad\times\left[\left.\frac{\partial E_i}{\partial\ln\omega_i}\right|_{\omega_i=\vartheta\left(E_i^{obs},t_i;\sigma\right)}\right]^{-1} \end{aligned} \tag{3.12}$$

The function transforming equity to asset value[16], ϑ, is defined as follows

[14] It is interesting to note this implies that if in fact the estimation of σ_E would produce the correct estimate $(\widehat{\sigma}_E = \sigma_E(\omega(0), 0; \cdot))$, one of the equations would be redundant. Thus, the first theoretical inconsistency (assuming constant stock price volatility) is necessary to find a unique solution to a system of equations which, theoretically, has an infinite number of solutions.

[15] This expression corrects an error in equation 4.6 in Duan (1994) (where the derivative erroneously is taken with respect to ω_i instead of $\ln\omega_i$).

[16] The inverse of the equity pricing function in equation (3.8). Note that we do not require λ to invert this function.

$$\omega_i = \vartheta\left(E_i^{obs}, t_i; \sigma\right) = E^{-1}\left(E_i^{obs}, t_i; \sigma\right) \tag{3.13}$$

The point of the change of variables is that $g(\cdot)$ is the well known density function for a normally distributed variable - the log of the asset value. Its dynamics are

$$\begin{cases} d\ln\omega = \left(r + \lambda\sigma - \frac{1}{2}\sigma^2\right)dt + \sigma\, dW^P \\ \ln\omega(0) = \ln\omega_n \end{cases}$$

The (one-period) conditional first two moments of the distribution are given by

$$\left.\begin{aligned} m_i &= E\left[\ln\omega_i \,|\ln\omega_{i-1}\right] = \ln\omega_{i-1} + \left(r + \lambda\sigma - \tfrac{1}{2}\sigma^2\right)\Delta t \\ s_i^2 &= E\left[\left(\ln\omega_i - m_i\right)^2 |\ln\omega_{i-1}\right] = \sigma^2\Delta t \end{aligned}\right\} \text{for } i = 2...n$$

and the conditional normal density for $\ln\omega_i$ is thus

$$g\left(\ln\omega_i \,|\ln\omega_{i-1}\,; \sigma, \lambda\right) = \frac{1}{\sqrt{2\pi}s_i}\exp\left\{-\frac{\left(\ln\omega_i - m_i\right)^2}{2s_i^2}\right\}$$

Inserting (3.12) into (3.11) we obtain the log-likelihood of $\mathbf{E}^{obs}$ for a given choice of σ and λ as[17]

$$\begin{aligned} L_E\left(\mathbf{E}^{obs}; \sigma, \lambda\right) &= \sum_{i=2}^{n} \ln g\left(\ln\omega_i \,|\ln\omega_{i-1}\,; \sigma, \lambda\right)\Big|_{\omega_i = \vartheta\left(E_i^{obs}, t_i; \sigma\right)} \\ &\quad - \sum_{i=2}^{n} \ln \left.\frac{dE\left(\omega_i, t_i; \sigma\right)}{d\ln\omega_i}\right|_{\omega_i = \vartheta\left(E_i^{obs}, t_i; \sigma\right)} \end{aligned} \tag{3.14}$$

or, noting that the first sum is simply the log-likelihood for $\ln\omega$ ($L_{\ln\omega}$), as

$$\begin{aligned} L_E\left(\mathbf{E}^{obs}; \sigma, \lambda\right) &= L_{\ln\omega}\left(\ln\vartheta\left(E_i^{obs}, t_i; \sigma\right) : i = 2...n; \sigma, \lambda\right) \\ &\quad - \sum_{i=2}^{n} \ln \left.\frac{dE\left(\omega_i, t_i; \sigma\right)}{d\ln\omega_i}\right|_{\omega_i = \vartheta\left(E_i^{obs}, t_i; \sigma\right)} \end{aligned}$$

We now turn to the asymptotic distributions of the parameter and price estimators.

3.3.3 *Asymptotic Distributions of Price Estimators*

Following Lo (1986) we derive the asymptotic distributions of the parameter and price estimators. For any function of a variable it holds that the ML

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17 $\frac{dE(\omega_i, t_i; \sigma)}{d\ln\omega_i} = \omega_i \cdot N\left(d_1\left(T, F\right)\right)$

estimator of the function is the function of the ML estimator of the variable. Furthermore, it holds that

$$\sqrt{n}\left(\Pi\left(\widehat{\sigma}\right)-\Pi\left(\sigma\right)\right)\xrightarrow{L}N\left(0,\Gamma_{\hat{\sigma}}\frac{\partial\Pi}{\partial\sigma}\right) \tag{3.15}$$

where $\widehat{\sigma}$ is the maximum likelihood estimate of σ and $\Gamma_{\hat{\sigma}}$ is its asymptotic standard deviation[18]. We denote with $\Pi(\sigma)$ the estimator for the price of an asset as a function of the volatility and the estimated price is thus $\widehat{\pi}=\Pi(\widehat{\sigma})$.

The asymptotic distributions are used to approximate the distribution in small samples. An approximation of the standard deviation of a price estimate is obtained as[19]

$$\widehat{\gamma}_{\hat{\pi}}=\widehat{\gamma}_{\hat{\sigma}}\frac{\partial\Pi}{\partial\sigma}\Big|_{\sigma=\hat{\sigma}}$$

In the subsections below we derive the asymptotic distributions for the price estimators for debt and options, but we first need the asymptotic distribution of the asset value estimator. It is derived in detail in appendix 3.6 and can be written as

$$\sqrt{n}\left(\vartheta\left(\widehat{\sigma}\right)-\vartheta\left(\sigma\right)\right)\xrightarrow{L}N\left(0,\Gamma_{\hat{\sigma}}\frac{\partial\vartheta}{\partial\sigma}\right)$$

[18] That is

$$\sqrt{n}\left(\hat{\sigma}-\sigma\right)\xrightarrow{L}N\left(0,\Gamma_{\hat{\sigma}}\right)$$

[19] We use the GAUSS Constrained Maximum Likelihood Application. The estimates this application provides are based on a Taylor-series approximation to the likelihood function (see e.g. Amemiya (1985), page 111) which yields that (letting $\theta=(\sigma,\lambda)$)

$$\sqrt{n}\left(\hat{\theta}-\theta\right)\xrightarrow{L}N\left(0,A^{-1}BA^{-1}\right)$$

where

$$\begin{aligned}A&=E\left[\frac{\partial^2L}{\partial\theta\,\partial\theta'}\right]\\B&=E\left[\left(\frac{\partial L}{\partial\theta}\right)'\left(\frac{\partial L}{\partial\theta}\right)\right]\end{aligned}$$

which can be estimated as

$$\begin{aligned}\widehat{A}&=\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2L_i}{\partial\theta\,\partial\theta'}\\\widehat{B}&=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\partial L_i}{\partial\theta}\right)'\left(\frac{\partial L_i}{\partial\theta}\right)\end{aligned}$$

From Inc. (1995).

Thus $\hat{\gamma}_{\hat{\sigma}}=\begin{pmatrix}1&0\end{pmatrix}\left(\widehat{A}^{-1}\widehat{B}\widehat{A}^{-1}\right)\begin{pmatrix}1\\0\end{pmatrix}$

where

$$\frac{\partial \vartheta}{\partial \sigma} = -\frac{Fe^{-r(T-t_i)}N'\left(d_2^{\vartheta}\right)\sqrt{T-t_i}}{N\left(d_1^{\vartheta}\right)} \tag{3.16}$$

3.3.3.1 *Debt*

The asymptotic distribution of debt may be written as

$$\sqrt{T}\left(D\left(\widehat{\sigma}\right) - D\left(\sigma\right)\right) \xrightarrow{L} N\left(0, \Gamma_{\widehat{\sigma}}\left(\frac{\partial D}{\partial \omega}\frac{\partial \vartheta}{\partial \sigma} + \frac{\partial D}{\partial \sigma}\right)\right) \tag{3.17}$$

The derivative of debt with respect to asset value is given by

$$\frac{\partial D}{\partial \omega} = N\left(-d_1\right)$$

The derivative of the asset value with respect to volatility was derived in (3.16) and that of debt with respect to the volatility is

$$\frac{\partial D}{\partial \sigma} = -\omega\left(0\right)\sqrt{T-t_i}N'\left(d_1\right)$$

3.3.3.2 *Options*

The asymptotic distribution of the call price estimate is

$$\sqrt{n}\left(C\left(\widehat{\sigma}\right) - C\left(\sigma\right)\right) \xrightarrow{L} N\left(0, \Gamma_{\widehat{\sigma}}\left(\frac{\partial C}{\partial \omega}\frac{\partial \vartheta}{\partial \sigma} + \frac{\partial C}{\partial \sigma}\right)\right) \tag{3.18}$$

The partial derivative of the call option price with respect to the asset value is given by

$$\frac{\partial C}{\partial \omega} = N\left(d_1\left(S, \overline{\omega}_i\right), d_1\left(T, F\right), \sqrt{\frac{S}{T}}\right)$$

and $\frac{\partial \vartheta}{\partial \sigma}$ is again given by (3.16). The derivative with respect to σ is[20]

$$\begin{aligned}&\frac{\partial C}{\partial \sigma}\\ = \ &\omega\left(0\right)\left\{\begin{array}{c}\int_{-\infty}^{d_1(T,F)} f\left(x, d_1\left(S,\overline{\omega}\right), \sqrt{\frac{S}{T}}\right)dx\left(\sqrt{S} - \frac{1}{\sigma}d_1\left(S,\overline{\omega}\right)\right)\\ +\\ \int_{-\infty}^{d_1(S,\overline{\omega})} f\left(x, d_1\left(T,F\right), \sqrt{\frac{S}{T}}\right)dx\left(\sqrt{T} - \frac{1}{\sigma}d_1\left(T,F\right)\right)\end{array}\right\}\end{aligned}$$

[20] The formula given here (when translated into a derivative with respect to σ^2 instead of σ) corrects formula (12) in Geske (1979) p 72.

$$-e^{-rT}F\left\{\begin{array}{c}\int_{-\infty}^{d_2(T,F)} f\left(x,d_2(S,\overline{\omega}),\sqrt{\frac{S}{T}}\right)dx\left(-\sqrt{S}-\frac{1}{\sigma}d_2(S,\overline{\omega})\right)\\+\\\int_{-\infty}^{d_1(S,\overline{\omega})} f\left(x,d_2(T,F),\sqrt{\frac{S}{T}}\right)dx\left(-\sqrt{T}-\frac{1}{\sigma}d_2(T,F)\right)\end{array}\right\}$$

$$+e^{-rS}X\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\cdot(d_1(S,\overline{\omega}))^2}\left(\sqrt{S}+\frac{1}{\sigma}d_2(S,\overline{\omega})\right)$$

3.4 Monte Carlo Results

Below we present the results of the Monte Carlo simulation. First we discuss the importance of the estimation period. The subsequent subsections report the results for the estimation of asset risk, market price of risk, asset value, debt prices and option prices. The bulk of the results are presented in tables in the appendix 3.8. After that we compare our findings with the VR method.

We also include two pages, 87 and 88, with density plots of estimates. The dotted line depicts simulated density for the VR-approach and the solid line depicts the simulated density for the ML-approach. Each page contains 12 density plots according to the following scheme:

Sigma	Long ITM option	Long ATM option	Long OTM option
Asset value	Medium ITM option	Medium ATM option	Medium OTM option
Debt	Short ITM option	Short ATM option	Short OTM option

3.4.1 Estimation Period

In this section we examine how the distributions of the estimators depend on the length of the equity price sample (n) chosen to estimate the parameters in (3.5). In the same scenario we use estimation periods of 31, 93, 365 and 1095 days. Figure 3.3 illustrates the convergence rate of the ML estimator of σ by plotting its sampled distribution:

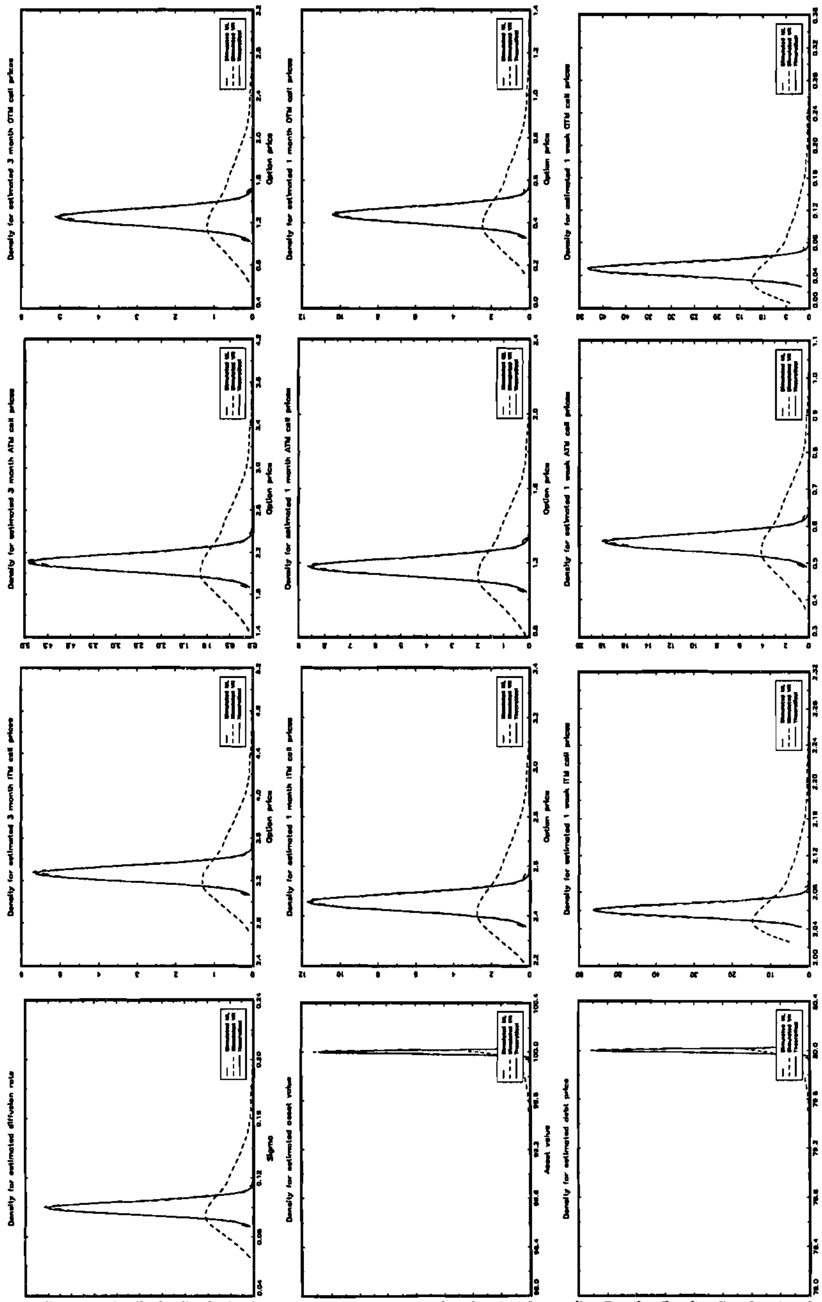


FIGURE 3.1. DENSITY PLOTS - Leverage 80%, asset volatility 10 and 1 year to maturity.

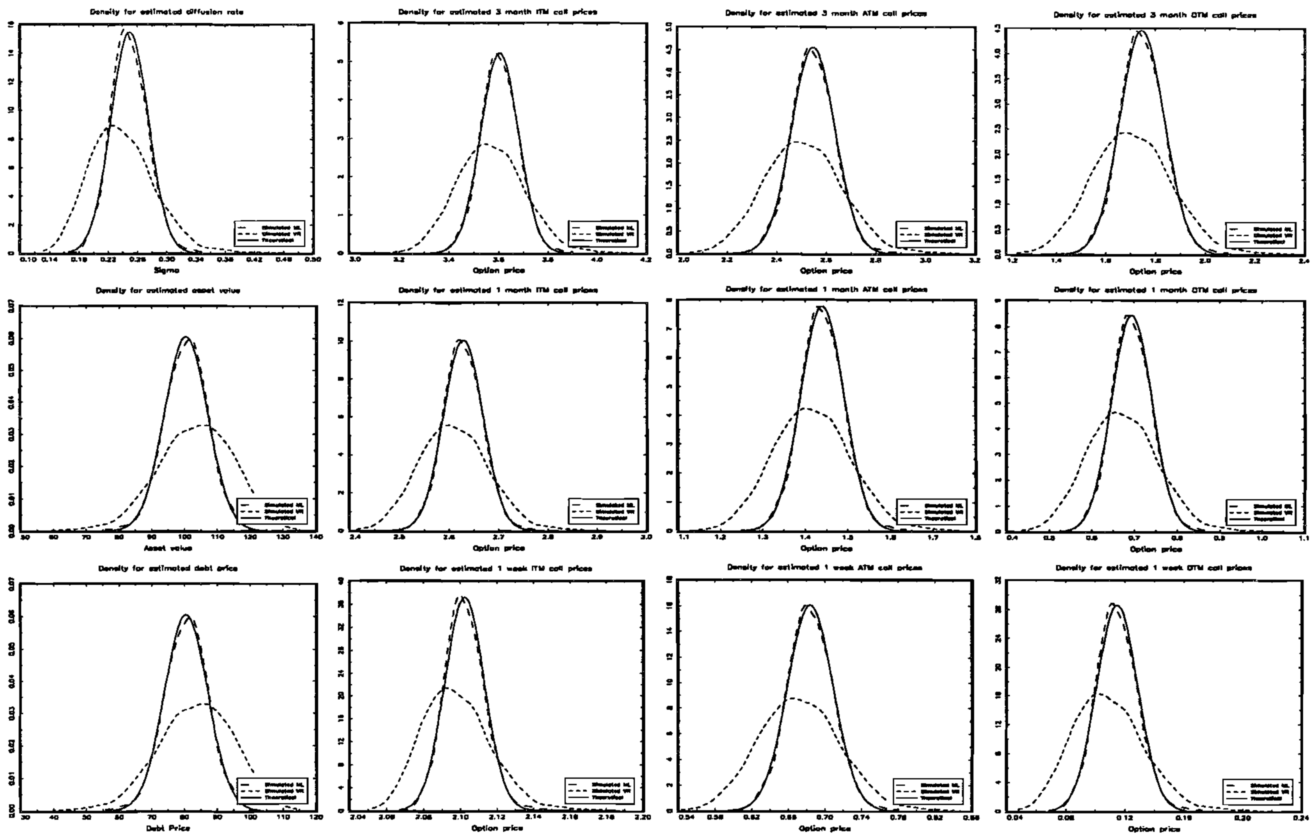


FIGURE 3.2. DENSITY PLOTS - Leverage 80%, asset volatility 25%, 10 years to maturity

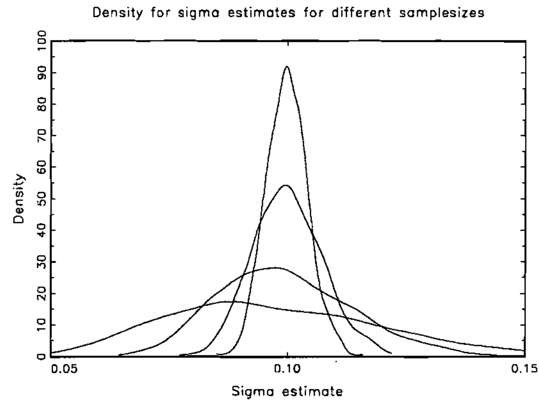


FIGURE 3.3. SIMULATED DENSITIES FOR THE ASSET RISK ESTIMATOR. The flattest distribution corresponds to a sample size of 31 days. The other densities correspond to 93, 365 and 1095 days respectively. The true asset risk is 10%.

The relative estimator biases and standard deviations are reported in table 3.1.

For a 31 day sample the bias is about -2.5% . The relative standard deviations of the price estimates lie in the range 0.3% (for short ITM) and 76% (for short OTM).

We find that the option price estimators are all more or less unbiased when we have a one year sample. Relative standard deviations are in the range 0.1% to 22%. Simulated sizes become reasonable for most calls and normality can only be rejected for long OTM, short ATM and OTM calls.

The relative standard errors of debt fall from 4% to 1% when the estimation period increases from a month to a year. The distribution of the debt price estimator is negatively skewed (as a result of the upper bound dictated by the price of the risk-free bond). This leaves us with a distribution for which normality can be distinctly rejected even when we use three years of data. On the other hand, it is sufficient with a year's worth of data to obtain reasonable simulated sizes.

A year of data is required in order for the ML estimator of σ to be unbiased. With an estimation period of one year normality can still be rejected but the population sizes are very similar to the nominal ones.

Based on these results we choose to proceed with a sample of daily stock prices over a year. Furthermore this seems a reasonable figure from a practical perspective in the sense that such a period could pass without major changes in the financial structure of the firm and its business risk.

TABLE 3.1. IMPORTANCE OF THE ESTIMATION PERIOD LENGTH. Leverage is 80%, debt has 10 years to maturity and the risk parameter is set to 10%. This yields a situation at $t = 0$ where the instantaneous equity volatility is about 37% p.a. and the credit spread on debt amounts to 72 basis points.

Number repeated samples	Instrument or parameter	Relative bias (%)	Relative efficiency (%)	BS	Distribution simulated size 1%	5%	10%
n	x	$\frac{E[\hat{x}]-x}{x}$	$\frac{\gamma_{\hat{x}}}{x}$				
31	Debt	+0.1	4	626	10.8	16.6	20.1
	Short ATM	-2.6	13	7.5	3.5	9.9	14.2
	Long ATM	-2.5	12	7.4	3.5	9.9	14.2
	σ	-2.5	25	205	7.5	12.3	17.4
93	Debt	+0.1	2	60	5.3	9.4	13.3
	Short ATM	-1.1	7	0.8*	2.2	6.5	11.2
	Long ATM	-1.0	7	0.8*	2.2	6.5	11.2
	σ	-1.3	14	15	3.6	7.8	12.3
365	Debt	-0.1	1	28	1.6	6.1	10.6*
	Short ATM	-0.0	4	1.1*	0.8*	5.2*	10.2*
	Long ATM	-0.0	3	1.1*	0.8*	5.2*	10.2*
	σ	+0.2	7	11	1.0*	5.1*	10.5*
1095	Debt	0.0	1	10	0.7*	5.8	10.4*
	Short ATM	0.0	2	0.8*	0.6	5.0*	9.9*
	Long ATM	0.0	2	0.8*	0.6	5.0*	9.9*
	σ	0.0	4	4.0*	0.6	5.3*	10.1*

3.4.2 Asset Risk

The estimator of asset risk is unbiased and efficient. The relative standard deviation of the estimate is in the range 4-10%. A typical 95% interval for $\sigma = 10\%$ is 9-11%.

Estimated standard deviations are also fairly unbiased and efficient. Normality is rejected when leverage is high, but on the other hand, the size is much better. About half the size tests yield a rejection of the null of being equal to the nominal sizes. However the sizes appear reasonable for practical purposes. Most population sizes exceed their nominal counterparts and when testing the model this would thus at worst lead to unnecessarily frequent cases of model rejection.

We find that the standard deviations of the estimate (both in absolute and relative terms) are increasing in the true level of risk, in leverage and (for a given leverage) in maturity.

3.4.3 Market Price of Risk

The estimator of the market price of risk is biased and extremely inefficient. The 95% intervals are in the range -2 to +2. We conclude that one cannot use estimates of the market price of risk for any practical purposes. This result is of course closely related to the well known fact that it is very difficult to estimate the drift of lognormal diffusion processes (see Merton (1980)).

3.4.4 Value of Assets

An estimate of the value of assets is in a sense the mirror image of the estimate of asset risk. Since σ and ω are chosen to match implied with observed equity prices, a good estimate of σ will yield a good estimate of $\omega(0)$ - and a too high estimate of σ corresponds with a too low estimate of $\omega(0)$.

The estimator of the value of assets has a one-to-one correspondence with the debt price estimator through $\widehat{\omega}(0) = \widehat{\pi}_{Debt} + E_n^{obs}$. For the results on the asset value estimator, see the next subsection.

3.4.5 Debt

The debt price estimator is generally both unbiased and very efficient. When leverage is low and debt is virtually risk-free the estimates are almost non-stochastic. The interesting cases are when leverage is high so that debt becomes risky. The estimation of debt still performs well. The percentage bias does not exceed 0.35% and the highest relative standard error is 8%. This occurs when leverage is 80%, debt matures in 10 years and the asset risk is 25%. In that particular situation the credit spread amounts to 559 basis points. Quite naturally it appears that in general longer debt combined with high asset risk is harder to price.

Normality of the price estimators can be rejected in all cases but one. The population sizes are poor when leverage is low; only for highly levered firms with long debt do the sizes become reasonable. It is not surprising that normality is rejected when the estimates are non-stochastic. But also when debt is not riskless price estimates are bounded from above by the value of riskless debt, rendering the distribution non-normal. Only for very risky debt, when the bound from above is too far away to have an effect, do the distribution tend to the normal.

3.4.6 Options

The option estimates appear to be unbiased. Generally the estimator is also efficient with relative standard deviations below 5%. When the option is out-of-the money, however, standard deviations go up and if in addition time to expiration is short they become huge. For short out-of-the money options relative standard errors reach 100%, rendering estimates useless.

The distribution of the price estimates is fairly normal in most cases and corresponding simulated sizes are in line with their nominal sizes. Notable exceptions are shorter in-the-money options. The estimates of these are very efficient (relative standard deviations below 0.1%) and the distribution is more or less a spike which has the effect of firmly rejecting normality and producing poor sizes. This is illustrated by the density plots in appendix on pages 86 and 86.

For in- and at-the-money options, efficiency increases with maturity of debt and decreases with asset risk and leverage. The opposite generally holds for out-of-the-money options.

3.4.7 The Volatility Restriction Approach

Examining the results from using the VR approach reviewed on page 81 to estimate the asset value process yields significantly different conclusions from those of the previous subsections. Tables 3.2 and 3.3 below show the average (over scenarios) bias and efficiency (relative standard deviation of estimators) for the maximum likelihood and restricted volatility methods respectively.

First of all the average biases with the VR-estimators are higher than with the ML-estimators by a factor 10. Also, the average efficiency of the VR-estimators are markedly lower by about a factor 3. But not only is the ML-estimator better on average - studying the different scenarios we find that the ML-estimator outperforms the VR-estimator on both unbiasedness and efficiency in *all* cases. We conclude that the ML method is clearly superior.

TABLE 3.2. BIAS AND EFFICIENCY OF ML AND VR ESTIMATORS OF ASSET RISK, ASSET VALUE AND DEBT PRICE. The first row for each variable represents the average (over the 8 scenarios analyzed) absolute value of the percentage bias. The number beneath in parentheses is the corresponding standard deviation (in %).

	ML Ave. bias (%) (Ave. efficiency (%))	*VR* Ave. bias (%) (Ave.efficiency (%))
σ	0,3	2,6
	(1,1)	(15,1)
$\omega(0)$	0,1	0,5
	(1,1)	(2,2)
D	0,1	0,6
	(1,5)	(3,0)

TABLE 3.3. BIAS AND EFFICIENCY OF ML AND VR ESTIMATORS OF OPTION PRICES. The first row for each variable represents the average absolute value of the percentage bias (over the 8 scenarios analyzed). The numbers in parentheses are the corresponding standard deviations (in %).

	3 months		1 month		1week	
	ML	*VR*	*ML*	*VR*	*ML*	*VR*
	Ave. bias (%) (Ave. eff. (%))		Ave. bias (%) (Ave. eff. (%))		Ave. bias (%) (Ave. eff. (%))	
ITM	0.1	1.1	0.1	0.8	0.0	0.3
	(1.6)	(5.3)	(1.0)	(3.5)	(0.3)	(1.3)
ATM	0.2	2.0	0.2	2.0	0.2	2.1
	(3.5)	(9.9)	(3.6)	(10.2)	(3.8)	(10.5)
OTM	0.3	3.6	0.5	6.4	0.5	34.7
	(7.2)	(17.4)	(12.0)	(29.3)	(36.7)	(105.9)

3.4.8 Summary of Results

Our overall impression is that the estimation procedure works well on the simulated data. Estimates of both debt and option prices are most often biased by less than 0.5% and, except in some extreme cases, reasonably efficient. They are also in many cases more or less normally distributed; at least, it is usually reasonable to apply the normality assumption when constructing confidence intervals or performing hypotheses tests. Throughout, the suggested approach outperforms the VR method. This is seen clearly in the density plot diagrams on pages 87 and 88.

Estimates of the asset risk parameter σ follow the same pattern as the price estimates, whereas the estimator of the market price of risk λ performs very poorly. The reason is that this parameter does not affect prices (only price *changes* and thus expected returns) and thus very little information about λ is reflected in them.

3.5 Concluding Remarks

We have evaluated the performance of a maximum likelihood procedure for implementing firm value based contingent claims models. We find that the proposed procedure is superior to methods traditionally used since it yields lower pricing biases and substantially lower standard errors. Performance depends on variables such as financial leverage, the risk of the firm's operations and the maturity structure of debt. Furthermore, it is likely that the particular model employed will have an impact on the performance of the estimation methodology[21]. Thus, we suggest that the simulation study outlined in this paper be carried out before a particular model is brought to market data.

We also present a way of relaxing the assumption of continuously traded assets without increasing the difficulty of estimation. We show that it is sufficient to assume that the assets are traded at *one* particular future point in time. In the context of firm value based models, this is a reasonable assumption.

For the pricing of options written on actively traded stocks, well developed models exist. When the underlying security is not traded, however, these cannot be applied. If another security on the balance sheet is traded, firm value based models constitute an alternative. Examples here include the corporate debt with embedded options and financial guarantees when the stock is traded. Furthermore firm value based models are able to cap-

[21] For example, models with barriers tend to produce prices which are more sensitive to changes in the underlying variable and standard deviation. This may clearly have an impact on both the likelihood function of the observed equity series and the asymptotic distribution.

ture the effect of changes in a firm's capital structure. This is valuable when pricing new issues of corporate securities. It is in these areas that we believe firm value based models have their greatest potential.

3.6 Appendix: The Asymptotic Distribution of the Asset Value Estimator

In this case $\Pi(\widehat{\sigma}) = \vartheta\left(E_i^{obs}, t_i; \widehat{\sigma}\right)$ and we therefore need an expression for $\frac{\partial \vartheta}{\partial \sigma}$. We know that

$$\begin{aligned} E_i^{obs} &\equiv E\left(\vartheta\left(E_i^{obs}, t_i; \sigma\right), t_i; \sigma\right) \\ &\quad \text{or} \\ E_i^{obs} &\equiv \vartheta\left(E_i^{obs}, t_i; \sigma\right) N\left(d_1^{\vartheta}\right) - F e^{-r(T-t_i)} N\left(d_2^{\vartheta}\right) \end{aligned}$$

where superscript ϑ indicates that we are using (3.9) with ϑ instead of ω. Since $\frac{\partial E_i^{obs}}{\partial \sigma} = 0$ differentiating both sides of this expression with respect to σ yields

$$\begin{aligned} 0 &= \frac{\partial \vartheta}{\partial \sigma} N\left(d_1^{\vartheta}\right) + \vartheta\left(E_i^{obs}, \sigma\right) N'\left(d_1^{\vartheta}\right) \frac{\partial d_1^{\vartheta}}{\partial \sigma} \\ &\quad - F e^{-r(T-t_i)} N'\left(d_2^{\vartheta}\right) \frac{\partial d_2^{\vartheta}}{\partial \sigma} \end{aligned}$$

where

$$\frac{\partial d_1^{\vartheta}}{\partial \sigma} = \frac{\partial d_2^{\vartheta}}{\partial \sigma} - \sqrt{T - t_i}$$

and thus

$$\begin{aligned} 0 &= \frac{\partial \vartheta}{\partial \sigma} N\left(d_1^{\vartheta}\right) + \left\{\vartheta\left(E_i^{obs}, t_i; \sigma\right) N'\left(d_1^{\vartheta}\right) - F e^{-r(T-t_i)} N'\left(d_2^{\vartheta}\right)\right\} \frac{\partial d_1^{\vartheta}}{\partial \sigma} \\ &\quad + F e^{-r(T-t_i)} N'\left(d_2^{\vartheta}\right) \sqrt{T - t_i} \end{aligned}$$

Using that $\vartheta\left(E_i^{obs}, t_i; \sigma\right) N'\left(d_1^{\vartheta}\right) - F e^{-r(T-t_i)} N'\left(d_2^{\vartheta}\right) = 0$ we finally arrive at the following expression

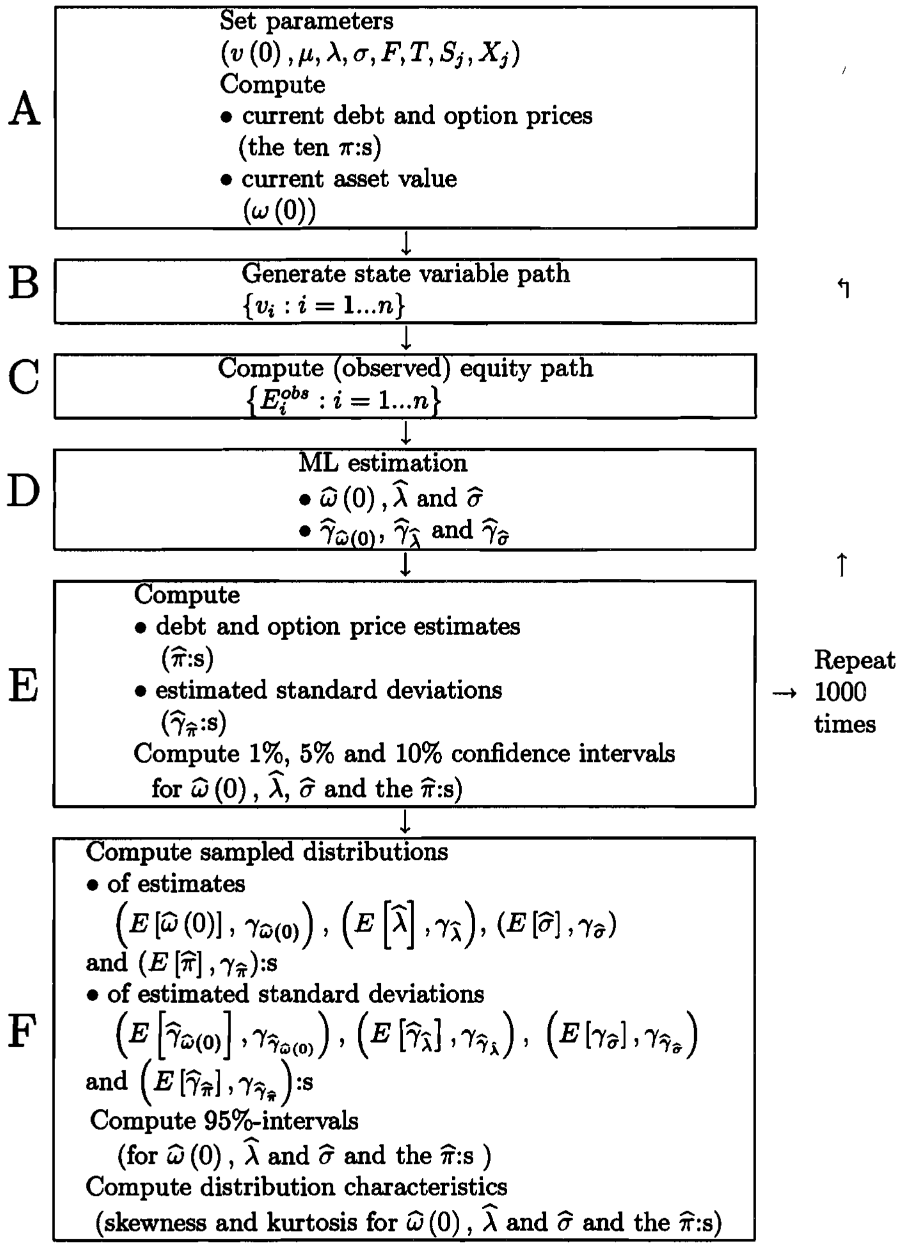
$$\frac{\partial \vartheta}{\partial \sigma} = -\frac{F e^{-r(T-t_i)} N'\left(d_2^{\vartheta}\right) \sqrt{T - t_i}}{N\left(d_1^{\vartheta}\right)} \tag{3.19}$$

The asymptotic distribution of the estimate of the state variable is therefore

$$\sqrt{n}\left(\vartheta\left(E_i^{obs}, t_i; \widehat{\sigma}\right) - \vartheta\left(E_i^{obs}, t_i; \sigma\right)\right) \xrightarrow{L} N\left(0, \Gamma_{\widehat{\sigma}} \frac{F e^{-r(T-t_i)} N'\left(d_2^{\vartheta}\right) \sqrt{T - t_i}}{N\left(d_1^{\vartheta}\right)}\right)$$

3.7 Appendix: The Monte Carlo Exercise

This appendix contains a brief description of how the Monte Carlo simulation was carried out. A flowchart illustrating the different steps is depicted below.



Step A

Consider a situation where the current leverage of the firm is 80%. The risk of the firm's assets (σ) is 10% p.a. The debt has a term to maturity (T) of 10 years and a face value (F) amounting to 157.63. The risk-free interest rate is $r = 0.06$. The market price of risk (λ) is 0.25 and the drift of the state variable (μ) is 10% p.a. The state variable, $v(0)$, is chosen to be 86.07 which, through equation (3.4), yields a current asset value ($\omega(0)$) of 100.

Finally to provide a basis for comparing obtained price estimates we compute the true debt and option prices. We illustrate with the debt issue and an option with time to expiration equal to 3 months ($S = \frac{3}{12}$) and an exercise price of $X = 18$).

- $D(100, 0; 0.1) = 80$
- $C\left(100, 0; 0.1, \frac{3}{12}, 18\right) = 0.811$

Step B

Drawing an n-vector of Wiener-increments from the following distribution

$$\Delta \mathbf{W} \sim \mathbf{N}\left(\mathbf{0}, \sqrt{\Delta t} \cdot \mathbf{1}\right)$$

we generate (backwards, as commented on in footnote 11) a sample path for the state variable v using (3.1).

Step C

Using equation (3.8) we translate this path into a path for equity $E_i^{obs} = E(\omega_i, t_i; \cdot)$ for the sample period $\{i : 1...n\}$.

During the sample period, the expected return from holding the stock fluctuated between 32% and 38%, and the volatility between 38% and 44% - the average volatility is 41.3% whereas the current one is 36.9%. This reflects a systematic decrease in leverage during the sample path.

Step D

The aim is now to price the debt issue and the call option given the information contained in a one year sample ($n = 365$) of stock prices. To do this we first compute the ML estimates of σ and λ by maximizing the likelihood in equation (3.14) with respect to these parameters. Given these estimates ($\widehat{\sigma}$, $\widehat{\lambda}$) and the current stock price E_n^{obs} we obtain an estimate of the current asset value $\omega(0) = \omega_n$ by equation (3.13). In this particular case we obtain the following estimates (with associated standard deviations)

- $\widehat{\sigma} = 11.4\%$, $\widehat{\gamma}_{\widehat{\sigma}} = 0.8\%$

- $\widehat{\lambda} = 1.48$, $\widehat{\gamma}_{\widehat{\lambda}} = 1.0$
- $\widehat{\omega}(0) = 100.3$, $\widehat{\gamma}_{\widehat{\omega}(0)} = 1.1$

Step E

Using the parameter estimates from Step D we can easily compute the maximum likelihood estimates for debt and option prices. The estimated standard deviations are obtained by setting $\Gamma_{\widehat{\vartheta}} = \widehat{\gamma}_{\widehat{\vartheta}}$ and $\omega(0) = \widehat{\omega}(0)$ in equations (3.17) and (3.18) respectively. For our example

- $\widehat{\pi} = \widehat{D} = D(100.3, 0; 0.114) = 80.30$, $\widehat{\gamma}_{\widehat{D}} = 1.13$
- $\widehat{\pi} = \widehat{C} = C\left(100.3, 0; 0.114, \frac{3}{12}, 18\right) = 0.795$, $\widehat{\gamma}_{\widehat{C}} = 0.06$,

We calculate the confidence intervals with significance levels 1%, 5% and 10%. In this case, the confidence intervals with 5% significance for the debt and option estimates are

- $[78.09 : 82.26]$ and $[0.677 : 0.913]$.

Hence in this particular case the true values, computed in Step A, fell within the confidence intervals.

Steps B to E are then repeated a thousand times.

Step F

With thousand estimates of each parameter and price (for example $\widehat{\pi}^j$: $j = 1...1000$) we can calculate the sampled distributions - their means and standard deviations (for example $(E[\widehat{\pi}], \gamma_{\widehat{\pi}})$). With the thousand estimates of the standard deviation of estimates (for example $\widehat{\gamma}^j_{\widehat{\pi}} : j = 1...1000$) we can calculate *their* sampled distributions (for example $\left(E[\widehat{\gamma}_{\widehat{\pi}}], \gamma_{\widehat{\gamma}_{\widehat{\pi}}}\right)$).

The 95%-intervals are obtained the mid-950 estimates from a vector containing estimates sorted in ascending order (for example $\left[\widehat{\pi}_s^{26} : \widehat{\pi}_s^{976}\right]$, where subindex s is used to denote that the superindex refers to the position in the sorted vector).

The results of the size test are calculated by adding up the times the true estimate fell outside its respective confidence interval in Step E.

Finally, the skewness and kurtosis of estimates are computed. The skewness and kurtosis are also used for a Bowman-Shelton test of normality.

3.8 Appendix: Tables

The first five columns describe the scenario of a particular Monte Carlo simulation: the first column reports the leverage of the firm $\left(\frac{\text{value of debt}}{\text{value of firm (assets)}}\right)$, the second column the volatility of the state variable, the third the maturity and the fourth the facevalue of debt. Finally, the fifth column reports the mean (over samples) of the average volatility during the sample periods.

The following two columns report the results on the unbiasedness of the estimator - both the expected estimate and the relative bias of the estimator. The option tables also include a column giving the true price.

The next two columns describe the efficiency of the estimator: the first of these reports the standard deviation of the estimate (and for prices its relative standard deviation as well) and the second gives the interval within which 95% of the estimates will be found.

The last six columns investigates the usefulness of the asymptotic distribution: first the expected estimated standard deviation of the estimate (and *its* standard deviation within brackets below) is reported (to be compared with the true standard deviation) and then follow the skewness and kurtosis of the distribution. These are used to compute the Bowman-Shelton test statistic reported in the following column. The last three columns contain the (simulated) population sizes.

A superindex* on a BS-statistic indicates that the null of normality could not be rejected with 5% significance (the cut-off point is 5.99). A similar superindex on a simulated size denotes that the null hypothesis of the true population size being equal to the nominal size could not be rejected with 5% significance (cut-off intervals are $0.7-1.3$, $4.3-5.7$ and $9.1-10.7$ respectively). Thus, loosely speaking, the more superindexed stars a row contain, the more reasonable the normal distribution assumption in small samples.

Scenario					Bias		Efficiency		Distribution						
Lev.	Asset vol. (%)	Mat.	Face value	Ave. eq. vol. (%)	Exp. est. (%)	Rel. bias (%)	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
	σ	T	F	$\overline{\sigma_E}$	$E[\widehat{\sigma}]$	$\frac{E[\widehat{\sigma}]-\sigma}{\sigma}$	$\gamma_{\widehat{\sigma}}$		$E[\widehat{\gamma}_{\widehat{\sigma}}]$ ($\gamma_{\widehat{\gamma}_{\widehat{\sigma}}}$)						
40	10	1	42	17	10.0	-0.2	0.4	9.2—10.7	0.4 (0.000)	0.0	2.8	1.5*	1.2* (67/33)	6.0 (70/30)	12.8 (69/31)
		10	73	17	10.0	-0.1	0.4	9.2—10.7	0.4 (0.000)	-0.1	3.0	0.6*	1.6 (88/12)	5.9 (76/24)	10.6* (67/23)
	25	1	42	43	24.9	-0.4	1.0	23.0—26.8	0.9 (0.001)	0.1	3.2	1.9*	1.8 (83/17)	5.5* (67/23)	11.0 (65/35)
		10	80	39	24.9	-0.2	1.3	22.4—27.4	1.2 (0.001)	0.1	2.9	2.6*	1.5 (93/7)	6.5 (77/23)	11.5 (71/29)
80	10	1	85	51	10.0	-0.3	0.4	9.1—10.8	0.4 (0.001)	0.4	5.5	279	1.1* (91/9)	4.3* (81/19)	10.1* (75/25)
		10	157	37	10.0	0.2	0.7	8.7—11.5	0.7 (0.001)	0.3	3.1	11	1.0* (100/0)	5.1* (82/18)	10.5* (76/24)
	25	1	88	95	24.9	-0.4	1.7	21.6—28.4	1.7 (0.004)	0.3	3.7	34	1.5 (80/20)	5.0* (78/22)	9.1* (70/30)
		10	255	60	25.0	-0.4	2.6	20.1—29.5	2.6 (0.004)	0.2	3.0	8	1.3* (100/0)	6.8 (81/19)	11.8 (75/25)

TABLE 3.4. ASSET RISK - σ

Scenario					Bias		Efficiency		Distribution						
Lev.	Asset vol.	Mat.	Face value	Ave. eq.vol.	Exp. est.	Bias	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
(%)	(%)			(%)		(%)									
	σ	T	F	$\overline{\sigma_E}$	$E\left[\widehat{\lambda}\right]$	$\frac{E\left[\widehat{\lambda}\right]-\lambda}{\lambda}$	$\gamma_{\widehat{\lambda}}$		$E\left[\widehat{\gamma}_{\widehat{\lambda}}\right]$ $\left(\gamma_{\widehat{\gamma}_{\widehat{\lambda}}}\right)$						
40	10	1	42	17	0.23	-8	1.01	-1.71—2.12	1.00 (0.07)	-0.0	3.0	0.0*	1.2* (58/42)	4.3* (51/49)	9.4* (52/48)
		10	73	17	0.23	-8	1.00	-1.82—2.14	1.00 (0.05)	-0.0	3.3	4.0*	1.5 (60/40)	4.6* (57/43)	9.5* (55/45)
	25	1	42	43	0.28	+12	1.04	-1.77—2.34	1.00 (0.11)	0.0	2.8	1.9*	1.1* (45/55)	6.2 (44/56)	11.7 (48/52)
		10	80	39	0.30	+20	1.04	-1.71—2.45	1.00 (0.07)	0.0	3.1	0.5*	1.9 (47/53)	6.3 (36/64)	10.2* (42/58)
80	10	1	85	51	0.20	-20	1.03	-1.84—2.15	1.00 (0.07)	-0.1	3.1	3.2*	1.6 (75/25)	5.6* (59/41)	10.2* (57/43)
		10	157	37	0.28	-8	1.01	-1.73—2.16	1.00 (0.08)	0.3	3.1	11	1.0* (53/47)	5.1* (56/44)	10.5* (48/52)
	25	1	88	95	0.23	-8	1.03	-1.75—2.33	1.01 (0.24)	0.1	3.1	1.3*	1.3* (38/62)	5.9 (44/56)	10.2* (58/42)
		10	255	60	0.23	-8	0.99	-1.77—2.28	1.00 (0.06)	0.2	3.0	4.3*	0.5 (20/80)	5.7* (47/53)	10.3* (50/50)

TABLE 3.5. MARKET PRICE OF RISK - λ

Scenario					Bias		Efficiency		Distribution						
Lev.	Asset vol.	Mat.	Face value	Ave. eq. vol.	Exp. est.	Bias	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
(%)	(%)			(%)		(%)									
	σ	T	F	$\overline{\sigma_E}$	$E[\widehat{\omega}]$	$\frac{E[\widehat{\omega}]-\omega}{\omega}$	$\gamma_{\widehat{\omega}}$		$E[\widehat{\gamma}_{\widehat{\omega}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\omega}}})$						
40	10	1	42	17	100.00	-0.0	0.0	100.00-100.00	0.0 (0.0)	-18	335	∞	n.a.	n.a.	n.a.
		10	73	17	99.99	-0.0	0.0	99.98-100.01	0.0 (0.0)	-1.1	4.8	338	4 0/100	7.3 1/99	11.1 9/91
	25	1	42	43	100.00	-0.0	0.0	100.00-100.00	0.0 (0.0)	-1.6	6.2	839	8.7 0/100	13.2 0/100	16.4 0/100
		10	80	39	100.01	+0.0	0.6	98.81-101.15	0.6 (0.1)	-0.2	2.9	10	2.2 0/100	6.2 13/87	9.7* 24/76
80	10	1	85	51	99.99	-0.0	0.0	99.97-100.02	0.0 (0.0)	-0.9	5.1	335	3.4 0/100	7.9 3/97	11.9 9/91
		10	157	37	99.96	-0.0	1.0	97.73-101.70	1.0 (0.2)	-0.4	3.3	29	1.1* 0/100	5.6* 16/48	9.3* 28/82
	25	1	88	95	100.01	0.0	0.6	98.76-101.15	0.6 (0.2)	-0.3	3.4	22	3.3 0/100	7.0 4/96	11.5 19/81
		10	255	60	100.45	0.5	6.6	87.44-113.41	6.6 (0.7)	-0.1	3.3	6	1.4 36/64	6.4 31/69	10.5* 35/65

TABLE 3.6. ASSET VALUE - $\omega(0)$

Scenario						Bias		Efficiency			Distribution					
Lev.	Asset vol.	Mat.	Face value	Ave. eq.vol.	True value	Exp. est.	Bias	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
(%)	(%)			(%)			(%)									
	σ	T	F	$\overline{\sigma_E}$	π	$E[\widehat{\pi}]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$ $\left[\frac{\gamma_{\widehat{\pi}}}{\pi}\right]$ (%)		$E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$						
40	10	1	42	17	6.96	6.96	-0.0	0.027 [0.4]	6.91—7.02	0.026 (0.003)	0.3	2.9	12	2.1 (95/5)	7.0 (84/11)	13.1 (75/25)
		10	73	17	6.96	6.96	+0.0	0.026 [0.4]	6.91—7.01	0.026 (0.003)	0.2	3.0	3.9*	2.2 (95/5)	6.6 (85/15)	11.2 (78/22)
	25	1	42	43	8.92	8.91		0.155 [1.7]	8.60—9.22	0.154 (0.010)	0.1	3.2	2.9*	1.8 (83/17)	5.6* (70/30)	11.4 (67/33)
		10	80	39	8.59	8.58	-0.1	0.137 [1.6]	8.30—8.84	0.133 (0.011)	0.1	2.9	1.5*	1.4 (93/7)	6.6 (74/26)	11.1 (70/30)
80	10	1	85	51	3.27	3.27	-0.1	0.072 [2.2]	3.13—3.40	0.072 (0.010)	0.3	5.1	190	1.1* (91/9)	4.2 (9/21)	10.4* (73/27)
		10	157	37	2.82	2.82	-0.0	0.042 [1.5]	2.74—2.90	0.042 (0.005)	0.1	2.9	2.3*	0.8* (100/0)	5.3* (74/26)	10.5* (71/29)
	25	1	88	95	5.16	5.15	-0.3	0.151 [2.9]	4.84—5.44	0.151 (0.033)	0.0	3.5	11	1.0* (50/50)	4.8* (71/29)	8.8 (61/39)
		10	255	60	3.61	3.61	-0.1	0.078 [2.2]	3.45—3.76	0.077 (0.009)	0.0	2.9	0.7*	1.0* (80/20)	5.8 (72/28)	11.9 (67/33)

TABLE 3.7. LONG ITM OPTION PRICES

Scenario						Bias		Efficiency			Distribution					
Lev.	Asset vol	Mat.	Face value	Ave. eq.vol.	True value	Exp. est.	Bias	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
(%)	(%)			(%)			(%)									
	σ	T	F	$\overline{\sigma_E}$	π	$E[\widehat{\pi}]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$ $\left[\frac{\gamma_{\widehat{\pi}}}{\pi}\right]$ (%)		$E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$						
40%	10	1	42	17	2.44	2.44	-0.1	0.075 [3.1]	2.29—2.58	0.072 (0.003)	0.1	2.8	1.6*	1.1* (73/27)	6.0 (70/30)	12.8 (69/31)
		10	73	17	2.44	2.44	-0.1	0.073 [3.0]	2.29—2.58	0.072 (0.003)	-0.1	3.0	0.8*	1.6 (88/12)	6.1 (74/26)	10.7* (66/34)
	25	1	42	43	5.38	5.37		0.187 [3.5]	4.99—5.74	0.186 (0.010)	0.1	3.2	1.8*	1.8 (83/17)	5.5* (67/33)	11.0 (65/35)
		10	80	39	5.01	4.99	-0.2	0.175 [3.5]	4.64—5.34	0.171 (0.013)	0.0	2.9	0.8*	1.5 (87/13)	6.4 (72/28)	11.4 (67/33)
80%	10	1	85	51	2.11	2.11	-0.3	0.082 [3.9]	1.95—2.26	0.082 (0.011)	0.3	4.9	165	1.2* (83/17)	4.0 (78/22)	10.2* (72/28)
		10	157	37	1.61	1.61	-0.0	0.054 [3.4]	1.50—1.71	0.055 (0.006)	0.1	2.9	1.1*	0.8* (100/0)	5.2* (73/27)	10.2* (70/30)
	25	1	88	95	4.19	4.18	-0.4	0.164 [3.9]	3.84—4.50	0.164 (0.039)	0.0	3.5	11	1.0* (50/50)	4.8* (71/29)	8.8 (61/39)
		10	255	60	2.55	2.55	-0.2	0.089 [3.5]	2.38—2.73	0.088 (0.010)	0.0	2.9	0.6*	0.9* (78/22)	5.6* (71/29)	11.9 (66/34)

TABLE 3.8. LONG ATM OPTION PRICES

Scenario						Bias		Efficiency			Distribution					
Lev.	Asset vol.	Mat.	Face value	Ave. eq.vol.	True value	Exp. est.	Bias	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
(%)	(%)			(%)			(%)									
	σ	T	F	$\overline{\sigma_E}$	π	$E[\widehat{\pi}]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$ $[\frac{\gamma_{\widehat{\pi}}}{\pi}]$ (%)		$E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$						
40%	0.10	1	42	17	0.43	0.43	-0.3	0.048 [11.0]	0.34—0.53	0.046 (0.003)	0.2	2.9	4.6*	1.5 (87/13)	6.3 (81/19)	12.8 (71/29)
		10	73	17	0.43	0.43	-0.2	0.047 [10.8]	0.34—0.52	0.046 (0.004)	0.0	3.0	0.4*	1.9 (95/5)	6.2 (82/18)	11.2 (71/29)
	0.25	1	42	43	2.95	2.94		0.179 [6.1]	2.58—3.30	0.177 (0.010)	0.1	3.2	2.3*	1.8 (83/17)	5.6* (64/36)	11.2 (66/34)
		10	80	39	2.63	2.62	-0.4	0.169 [6.4]	2.28—2.95	0.156 (0.013)	0.1	2.9	1.1*	1.5 (87/13)	6.5 (72/28)	11.2 (70/30)
80%	0.10	1	85	51	1.26	1.25	-0.4	0.079 [6.3]	1.10—1.14	0.080 (0.011)	0.3	5.1	194	1.1* (91/9)	4.2 (79/21)	10.4* (73/27)
		10	157	37	0.81	0.81	-0.0	0.052 [6.4]	0.71—0.91	0.053 (0.006)	0.1	2.9	1.7*	0.8* (100/0)	5.3* (74/26)	10.3* (71/29)
	0.25	1	88	95	3.37	3.36	-0.5	0.170 [5.1]	3.00—3.69	0.170 (0.037)	0.0	3.5	11	1.0* (50/50)	4.8* (71/29)	8.8 (61/39)
		10	255	60	1.75	1.75	-0.3	0.091 [5.2]	1.57—1.93	0.089 (0.010)	0.0	2.9	0.6*	1.0* (80/20)	5.7* (72/28)	11.8 (67/33)

TABLE 3.9. LONG OTM OPTION PRICES

Scenario						Bias		Efficiency			Distribution					
Lev.	Asset vol.	Mat.	Face value	Ave. eq.vol.	True value	Exp. est.	Bias	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
(%)	(%)			(%)			(%)									
	σ	T	F	$\overline{\sigma_E}$	π	$E[\widehat{\pi}]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$ $[\frac{\gamma_{\widehat{\pi}}}{\pi}]$ (%)		$E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$						
40%	0.10	1	42	17	6.28	6.28	+0.0	0.003 [0.1]	6.27—6.29	0.003 (0.001)	0.6	3.4	65	3.5 (100/0)	9.3 (94/6)	13.1 (85/15)
		10	73	17	6.28	6.28	+0.0	0.003 [0.1]	6.27—6.28	0.003 (0.001)	0.5	3.4	49	3.6 (100/0)	7.4 (96/4)	11.6 (86/14)
	0.25	1	42	43	6.96	6.95		0.070 [1.0]	6.82—7.10	0.070 (0.006)	0.2	3.2	5.8*	1.8 (89/11)	5.4* (72/28)	11.3 (68/32)
		10	80	39	6.81	6.81	-0.0	0.059 [0.9]	6.69—6.92	0.058 (0.006)	0.1	2.9	3.7*	1.8 (94/6)	6.6 (80/20)	11.7 (72/28)
80%	0.10	1	85	51	2.46	2.46	-0.1	0.035 [1.4]	2.39—2.52	0.035 (0.005)	0.4	5.4	252	1.2* (92/8)	4.7* (83/17)	10.1* (75/25)
		10	157	37	2.25	2.25	+0.0	0.018 [0.8]	2.22—2.29	0.018 (0.002)	0.2	3.0	5.6*	1.0* (100/0)	5.3* (79/21)	10.5* (72/28)
	0.25	1	88	95	3.45	3.44	-0.2	0.083 [2.4]	3.27—3.60	0.083 (0.018)	0.0	3.5	12	1.0* (50/50)	4.8* (71/29)	9.0 (62/38)
		10	255	60	2.63	2.63	-0.1	0.040 [1.5]	2.55—2.71	0.040 (0.005)	0.1	2.9	1.2*	0.9* (89/11)	6.1 (75/25)	12.1 (68/32)

TABLE 3.10. MEDIUM ITM OPTION PRICES

Lev. (%)	Asset vol .(%) σ	Mat. T	Face value F	Ave. eq.vol. (%) $\overline{\sigma_E}$	True value π	Exp. est. $E[\widehat{\pi}]$	Bias (%) $\frac{E[\widehat{\pi}]-\pi}{\pi}$	Std. $\gamma_{\widehat{\pi}}$ $\left[\frac{\gamma_{\widehat{\pi}}}{\pi}\right]$ (%)	95% interval	Exp. est. std. $E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
	Scenario					**Bias**		**Efficiency**			**Distribution**					
40%	10	1	42	17	1.30	1.29	-0.1	0.044 [3.4]	1.21—1.38	0.042 (0.002)	0.1	2.8	1.5*	1.2* (67/33)	6.0 (70/30)	12.8 (69/31)
		10	73	17	1.29	1.29	-0.1	0.043 [3.3]	1.21—1.37	0.042 (0.002)	-0.1	3.0	0.8*	1.6 (88/12)	6.1 (74/26)	10.7* (66/34)
	25	1	42	43	3.00	3.00		0.109 [3.6]	2.78—.321	0.108 (0.006)	0.1	3.2	1.8*	1.8 (83/17)	5.5* (67/33)	11.0 (65/35)
		10	80	39	2.79	2.78	-0.2	0.102 [3.7]	2.57—2.98	0.108 (0.006)	0.0	2.9	0.8*	1.5 (87/13)	6.4 (72/28)	11.4 (67/33)
80%	10	1	85	51	1.18	1.18	-0.3	0.047 [4.0]	1.09—1.27	0.047 (0.006)	0.3	4.9	161	1.3* (77/23)	4.0 (78/22)	10.2* (71/29)
		10	157	37	0.89	0.89	-0.0	0.031 [3.5]	0.83—0.95	0.032 (0.003)	0.1	2.9	1.1*	0.8* (100/0)	5.2* (73/27)	10.2* (70/30)
	25	1	88	95	2.37	2.36	-0.4	0.094 [4.0]	2.17—2.54	0.094 (0.021)	0.0	3.5	11	1.0* (50/50)	4.8* (71/29)	8.8 (61/39)
		10	255	60	1.44	1.44	-0.2	0.052 [3.6]	1.34—1.55	0.051 (0.006)	0.0	2.9	0.6*	0.9* (78/22)	5.6* (71/29)	11.8 (67/33)

TABLE 3.11. MEDIUM ATM OPTION PRICES

Scenario						Bias		Efficiency			Distribution					
Lev.	Asset vol.	Mat.	Face value	Ave. eq.vol.	True value	Exp. est.	Bias	Std est.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
(%)	(%)			(%)			(%)									
	σ	T	F	$\overline{\sigma_E}$	π	$E[\widehat{\pi}]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$ $[\frac{\gamma_{\widehat{\pi}}}{\pi}]$ (%)		$E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$						
40%	0.10	1	42	17	0.030	0.031	+0.6	0.007 [23.3]	0.018—0.045	0.007 (0.001)	0.4	3.2	34	2.7 (100/0)	8.6 (88/12)	12.8 (81/19)
		10	73	17	0.030	0.031	+0.7	0.007 [22.8]	0.018—0.045	0.007 (0.001)	0.3	3.2	22	3.1 (100/0)	7.0 (93/7)	11.5 (83/17)
	0.25	1	42	43	0.933	0.927		0.083 [8.9]	0.762—1.096	0.083 (0.006)	0.1	3.2	4.3*	1.8 (89/11)	5.5* (71/29)	11.3 (68/32)
		10	80	39	0.79	0.78	-0.4	0.076 [9.7]	0.63—0.93	0.074 (0.007)	0.1	2.9	2.7*	1.5 (93/7)	6.8 (78/22)	11.5 (71/29)
80%	0.10	1	85	51	0.439	0.437	-0.5	0.039 [8.9]	0.364—0.510	0.039 (0.006)	0.4	5.4	254	1.2* (92/8)	4.7* (83/17)	10.1* (75/25)
		10	157	37	0.232	0.231	+0.1	0.023 [9.8]	0.188—0.277	0.023 (0.003)	0.2	3.0	4.2*	0.9* (100/0)	5.2* (79/21)	10.6* (72/28)
	0.25	1	88	95	1.555	1.549	-0.5	0.094 [6.0]	1.355—1.731	0.091 (0.021)	0.0	3.5	12	1.0* (50/50)	4.8* (71/29)	9.0 (62/38)
		10	255	60	0.697	0.695	-0.4	0.048 [6.9]	0.601—0.792	0.048 (0.005)	0.1	2.9	1.0*	1.0* (80/20)	5.9 (75/25)	12.0 (67/33)

TABLE 3.12. MEDIUM OTM OPTION PRICES

Scenario					Bias			Efficiency			Distribution					
Lev. (%)	Asset vol .(%)	Mat.	Face value	Ave. eq.vol. (%)	True value	Exp. est.	Bias (%)	Std.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
	σ	T	F	$\overline{\sigma_E}$	π	$E[\widehat{\pi}]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$ $\left[\frac{\gamma_{\widehat{\pi}}}{\pi}\right]$ (%)		$E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$						
40%	0.10	1	42	17	6.062	6.062	+0.0	0.000 [0.0]	6.062—6.062	0.000 (0.000)	2.5	13.0	4879	11.4 (100/0)	14.8 (100/0)	17.2 (100/0)
		10	73	17	6.062	6.062	+0.0	0.000 [0.0]	6.062—6.062	0.000 (0.000)	2.6	15.0	7124	10.1 (100/0)	14.6 (100/0)	17.4 (100/0)
	0.25	1	42	43	6.109	6.109		0.010 [0.2]	6.091—6.131	0.010 (0.002)	0.5	3.5	46	2.6 (100/0)	6.5 (86/14)	12.1 (80/20)
		10	80	39	6.090	6.090	+0.0	0.007 [0.1]	6.078—6.104	0.006 (0.001)	0.5	3.2	37	3.3 (100/0)	7.5 (92/8)	12.9 (84/16)
80%	0.10	1	85	51	2.061	2.060	-0.0	0.007 [0.4]	2.048—2.074	0.007 (0.001)	0.8	8.0	1131	1.8 (94/6)	5.9 (88/12)	10.1* (82/18)
		10	157	37	2.028	2.028	+0.0	0.002 [0.1]	2.025—2.032	0.002 (0.000)	0.5	3.4	50	2.5 (100/0)	6.8 (93/7)	10.9* (83/17)
	0.25	1	88	95	2.380	2.379	-0.1	0.032 [1.4]	2.313—2.442	0.032 (0.014)	0.1	3.6	14	1.3* (69/31)	5.1* (73/27)	9.2* (67/33)
		10	255	60	2.103	2.103	-0.0	0.011 [0.5]	2.082—2.125	0.011 (0.002)	0.2	3.0	6.3	1.2* (100/0)	6.6 (80/20)	11.8 (74/26)

TABLE 3.13. SHORT ITM OPTION PRICES

Scenario					Bias			Efficiency			Distribution					
Lev.	Asset vol.	Mat.	Face value	Ave. eq.vol.	True value	Exp. est.	Bias	Std. est.	95% interval	Exp. est. std.	Sk.	Kt.	BS	Sampled size (% above/below) 1%	5%	10%
(%)	(%)	(%)		(%)			(%)									
	σ	T	F	$\overline{\sigma_E}$	π	$E[\widehat{\pi}]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$ $\left[\frac{\gamma_{\widehat{\pi}}}{\pi}\right]$ (%)		$E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$						
40%	0.10	1	42	17	0.587	0.586	-0.2	0.021 [3.6]	0.544—0.627	0.020 (0.001)	0.0	2.8	1.5*	1.2* (67/33)	6.0 (70/30)	12.8 (67/33)
		10	73	17	0.587	0.586	-0.1	0.021 [3.5]	0.542—0.625	0.020 (0.001)	-0.1	3.0	0.8*	1.6 (88/12)	6.1 (74/26)	10.7* (66/34)
	0.25	1	42	43	1.415	1.411		0.053 [3.7]	1.305—1.517	0.052 (0.003)	0.1	3.2	1.8*	1.8 (83/27)	5.5* (67/33)	11.0 (65/35)
		10	80	39	1.312	1.308	-0.2	0.049 [3.8]	1.207—1.404	0.048 (0.004)	0.0	2.9	0.8*	1.5 (87/23)	6.4 (72/28)	11.4 (67/33)
80%	0.10	1	85	51	0.558	0.556	-0.3	0.023 [4.1]	0.513—0.598	0.023 (0.003)	0.3	4.9	159	1.3* (77/23)	4.0 (78/22)	10.2* (71/29)
		10	157	37	0.419	0.418	-0.0	0.015 [3.6]	0.389—0.448	0.015 (0.002)	0.1	2.9	1.1*	0.8* (100/0)	5.2* (73/27)	10.2* (70/30)
	0.25	1	88	95	1.128	1.126	-0.4	0.045 [4.0]	1.031—1.212	0.045 (0.010)	0.0	3.5	11	1.0* (50/50)	4.8* (71/29)	8.8 (61/39)
		10	255	60	0.686	0.685	-0.2	0.025 [3.7]	0.635—0.736	0.025 (0.003)	0.0	2.9	0.6*	0.9* (78/22)	5.7* (72/28)	11.8 (67/33)

TABLE 3.14. SHORT ATM OPTION PRICES

Scenario						Bias		Efficiency			Distribution					
Lev.	Asset vol.	Mat.	Face value	Ave. eq.vol.	True value	Exp. est.	Bias	Std. of est.	95% interval	Exp. est. std.	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
(%)	(%)			(%)			(%)									
	σ	T	F	$\overline{\sigma_E}$	π	$E[\widehat{\pi}]$	$\frac{E[\widehat{\pi}]-\pi}{\pi}$	$\gamma_{\widehat{\pi}}$ $\left[\frac{\gamma_{\widehat{\pi}}}{\pi}\right]$ (%)		$E[\widehat{\gamma}_{\widehat{\pi}}]$ $(\gamma_{\widehat{\gamma}_{\widehat{\pi}}})$						
40%	0.10	1	42	17	0.000	0.000	+0.0	0.000 [97.7]	0.000—0.000	0.000 (0.000)	2.1	9.5	2451	10.3 (100/0)	13.7 (100/0)	16.1 (100/0)
		10	73	17	0.000	0.000	+0.0	0.000 [94.1]	0.000—0.000	0.000 (0.000)	2.1	10.6	3142	9.5 (100/0)	12.4 (100/0)	16.0 (100/0)
	0.25	1	42	43	0.071	0.071		0.014 [19.0]	0.046—0.100	0.013 (0.002)	0.4	3.4	34	2.3 (100/0)	6.4 (86/14)	11.7 (79/21)
		10	80	39	0.049	0.049	+0.0	0.011 [21.4]	0.030—0.072	0.010 (0.002)	0.4	3.12	27	2.9 (100/0)	7.4 (91/9)	12.8 (80/20)
80%	0.10	1	85	51	0.048	0.048	-0.5	0.008 [17.6]	0.033—0.064	0.008 (0.002)	0.8	7.9	1100	1.8 (94/6)	5.9 (88/12)	10.0* (82/18)
		10	157	37	0.012	0.012	+1.2	0.003 [22.3]	0.008—0.018	0.003 (0.001)	0.5	3.3	39	2.4 (100/0)	6.7 (90/10)	10.9* (83/17)
	0.25	1	88	95	0.421	0.419	-0.7	0.037 [8.9]	0.342—0.492	0.037 (0.008)	0.1	3.5	14	1.3* (69/31)	5.1* (73/27)	9.1* (67/33)
		10	255	60	0.115	0.115	-0.5	0.014 [12.3]	0.088—0.145	0.014 (0.002)	0.2	3.0	4.7*	0.8* (100/0)	6.3 (78/22)	12.0 (73/27)

TABLE 3.15. SHORT OTM OPTION PRICES

Scenario						Bias		Efficiency			Distribution					
Lev. (%)	Asset vol. (%) σ	Mat. T	Face value F	Ave. eq.vol. (%) $\overline{\sigma_E}$	Spread (basis pts)	Mean est. $E[\hat{\pi}]$	Bias (%) $\frac{E[\hat{\pi}]-\pi}{\pi}$	Std. $\gamma_{\hat{\pi}}$ $\left[\frac{\gamma_{\hat{\pi}}}{\pi}\right]$ (%)	95% interval	Exp. est. std. $E[\hat{\gamma}_{\hat{\pi}}]$ $(\gamma_{\hat{\gamma}_{\hat{\pi}}})$	Sk.	Ku.	BS	Sampled size (% above/below) 1%	5%	10%
40	0.10	1	42	17	0.0	40.00	0	0.00 [0.0]	40.00—40.00	0.00 (0.00)	∞	∞	∞	n.a. n.a.	n.a. n.a.	n.a. n.a.
		10	73	17	0.3	40.00	-0.0	0.01 [0.0]	39.99—40.01	0.01 (0.00)	-0.9	4.4	223	5.3 (0/100)	9.3 (1/99)	12.2 (4/96)
	0.25	1	42	43	0.1	40.00		0.00 [0.0]	40.00—40.00	0.00 (0.00)	-1.9	8.6	1874	7.7 (0/100)	7.7 (0/100)	7.7 (0/100)
		10	80	39	91	40.03	+0.0	0.62 [2.0]	38.75—41.18	0.60 (0.09)	-0.3	3.0	14	7.7 (0/100)	12.0 (14/86)	13.9 (22/78)
80	0.10	1	85	51	5.1	80.00	-0.0	0.02 [0.0]	79.97—80.02	0.01 (0.01)	-2.7	32.2	36663	3.2 (0/100)	8.0 (1/99)	11.2 (7/93)
		10	157	37	72	80.01	-0.1	1.00 [1.0]	77.85—81.78	1.01 (0.18)	-0.4	3.2	28	1.6 (0/100)	6.1 (13/87)	10.6* 18/82
	0.25	1	88	95	368	80.00	+0.0	0.58 [1.0]	78.78—81.07	0.58 (0.16)	-0.5	4.3	117	2.1 (5/95)	5.6* (14/86)	9.8 (20/80)
		10	255	60	559	80.28	+0.6	6.72 [8.0]	66.77—93.21	6.60 (0.72)	-0.0	2.8	1.2*	1.0* (20/80)	6.6 (30/70)	12.8 (36/64)

TABLE 3.16. Debt prices

4

Asset Substitution, Debt Pricing, Optimal Leverage and Maturity

4.1 Introduction

A considerable part of the literature in corporate finance has focused on a firm's capital structure decision, notably on the nature and relative amount of debt and equity financing. Although this has led to an understanding of some of the determinants, few attempts have been made to provide quantitative guidelines. Recently Leland (1994*b*), Leland (1994*a*), Mella-Barral & Perraudin (1993) and Leland & Toft (1996) have developed models capable of both pricing corporate securities and quantifying optimal mixes thereof.

This paper will follow in this direction by allowing corporate debt and equity to be priced while both leverage *and* maturity are chosen optimally. This simultaneous choice will result from the presence of risk shifting opportunities.

The management's option to shift asset return risk when the incentives to do so become sufficiently strong is modelled explicitly. Given the resulting endogenous risk policy, the *ex ante* effect of this behaviour on the chosen capital structure (leverage and maturity) may be assessed. It is shown explicitly how debt maturity serves as an instrument to curb risk shifting incentives and it is thus possible to quantify the associated agency costs and the impact on the capital structure decision.

Asset substitution is modelled as follows. The firm currently operates at a given level of risk, represented by the standard deviation of a diffusion process. The management (acting in the interests of shareholders) may at its discretion change this risk to another (higher) level. Creditors realize the extent of this problem and will *ex ante* demand a higher return on debt as compensation for *ex post* losses due to changes in the firm's operations. Hence, shareholders will bear the full cost of this agency problem and it may be in their interest to find a way of committing themselves to not indulging in this form of opportunistic behaviour or at least limit the scope for it. It is suggested that they do this by a capital structure decision which involves a simultaneous choice of leverage and debt maturity. Shorter term debt is less vulnerable to higher asset risk and consequent increases in the likelihood of default. A shorter debt issue leaves less wealth to be expropriated by shareholders and hence maturity may be used as an instrument to curtail asset substitution incentives.

The paper is organized as follows. In section 4.2 the paper is related to the existing literature on asset substitution, debt maturity structure and debt pricing. Section 4.3 presents the setup and derives the values of the corporate securities. In section 4.4, the time consistent risk policy of the shareholders is derived; section 4.5 discusses the simultaneous choice of leverage and maturity. Section 4.6 examines the impact of deviations from the absolute priority rule on optimal capital structures. The conclusions are presented in section 4.7.

4.2 Related Research

Jensen & Meckling (1976) suggest a capital structure theory based on agency costs stemming from conflicts between different groups of agents related to a firm. They discuss the adverse incentive effects of an owner-manager who has the opportunity to issue debt and at a later stage decide on the details of the investment policy. They find that he may very well transfer wealth to himself from bondholders by taking on excessive risk. Such behaviour will henceforth be referred to as asset substitution or risk shifting. The capital structure decision in the Jensen & Meckling setting involves trading off the agency costs related to the firm's different claimholders.

Gavish & Kalay (1983) formalize asset substitution in a one-period model and investigate the effect of leverage on the incentives of an owner-manager. They find that shareholders gains from an unexpected increase in asset return variance do not increase monotonically in the amount of debt in the capital structure and thus cast some doubt on the proposition of Jensen & Meckling (1976). Green & Talmor (1986) take the analysis in Gavish & Kalay (1983) a step further and derive the optimal risk policy of a firm in a similar setting. They find that the "optimal amount" of asset substitution, measured by the asset return volatility, increases monotonically in leverage although shareholder gains do not.

Although these one period models have yielded important insights into the problem, they are unable to measure the quantitative effects of the studied agency conflicts. Furthermore, they may fail to capture some qualitative results that arise in dynamic models. For example, it is not clear that the "optimal risk" from a shareholder perspective in a multiperiod setting increases monotonically in leverage[1].

Recently, a number of papers have attempted to link the well developed literatures on contingent claims pricing in the wake of Black & Scholes (1973) and Merton (1973) and the study of agency problems in corporate finance. For example, Mello & Parsons (1992) extend the Brennan & Schwartz (1986) model for the valuation of a mine in order to allow for debt financing. Having done so, they examine the changes induced in the operating policy by external finance and are able to provide a measure of the agency costs of debt. Anderson & Sundaresan (1996) and Mella-Barral & Perraudin (1993) allow a game of strategic debt service[2] to enter into a model of corporate debt valuation and are thus able to analyse effects on pricing.

The incentives to indulge in asset substitution have recently been examined in a continuous time environment by Leland (1994*b*), Leland (1994*a*) and Leland & Toft (1996). These papers develop pricing models for corporate debt and equity which yield interior capital structures where bankruptcy costs and the tax advantage of debt are traded off. They examine the comparative statics of debt and equity values with respect to the asset return standard deviation as a measure of the potential for asset substitution. However, they do not let this costly behaviour enter into their pricing equations and thus do not allow it to affect the optimal capital structures they derive. By modelling risk shifting opportunities explicitly, I provide an extension of Leland (1994*a*) analogous to that provided by Green & Talmor (1986) for Gavish & Kalay (1983). The leverage and maturity decision is based on a trade-off between the benefits and agency costs of debt and thus adds a new dimension to Leland's model.

Different rationales for debt maturity choice have been suggested in the corporate finance literature. Myers (1977) argues that short term debt reduces the debt overhang problem[3] and Barnea et al. (1980) argue that short term debt reduces incentives for asset substitution. Diamond (1991) shows

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[1] This is due to the failure of the often invoked analogy between equity and a European call option when intermediate financial distress is introduced. A more appropriate comparison would be between equity and a down-and-out call option whose value is not a monotonic funtion of the asset value volatility.

[2] Shareholders may choose to pay less than the contractual debt service due to the non-credibility of a liquidation threat.

[3] If debt matures before growth options are exercised investment incentives will not be distorted.

that although firms will want to borrow long term they may be screened out of that market segment due to a lack of credit quality.

Recent empirical work on debt maturity includes Barclay & Smith (1995) who find that maturity rises with firm size as well as credit quality and falls with growth opportunities. Stohs & Mauer (1994) find that long term debt is used by large firms with low risk and growth opportunities. This is consistent with the view that high contracting costs induce shorter maturities[4]. Similarly, the model suggested in this paper assumes the impossibility of writing contracts that prevent shareholders from expropriating wealth from creditors *ex post* and suggests maturity as an alternative contracting device.

A model conceptually similar to the one in this paper has been developed by Boyle & Lee (1994) in the context of deposit insurance[5]. They extend the Merton (1977) to allow for the bank asset risk to take on two distinct values depending on whether or not the asset value has crossed a barrier. Their model is different in that financial distress can occur only at maturity (audit time) and that the switching barriers are assumed to be constant and exogenously given[6]. The model in this paper derives the barrier endogenously and allows financial distress to occur at any point in time.

4.3 The Model

We will now turn to a more formal description of the setup. The asset value of the firm is assumed to follow a geometric Brownian motion

$$\begin{cases} dv = (\mu - \beta)\, vdt + \sigma(v,t)\, vdW^P \\ v(0) = v_0 \end{cases} \tag{4.1}$$

[4]These studies are concerned with overall maturity whereas Guedes & Opler (1995) tackle the issue of debt maturity by examining new debt issues as opposed to the aggregate balance sheet measure. In their sample, firms with a Standard & Poor's rating less than BBB rarely issue short term notes and are normally found in the 5 to 29 year brackets. Very short and very long term debt (with a term exceeding 29 years) is issued by investment grade firms.

[5]I am grateful to Kenneth Vetzal for bringing this reference to my attention.

[6]Given that their barrier option pricing model has finite maturity, the optimal switching barrier cannot be constant.

which under the risk neutral probability measure Q will have the following appearance[7]

$$\begin{cases} dv = (r-\beta)\,vdt + \sigma(v,t)\,vdW^Q \\ v(0) = v_0 \end{cases} \tag{4.2}$$

The firm's assets generate a proportional payout rate[8] of β and are expected to grow[9] at a rate $\mu - \beta$. These parameters are assumed to be exogenous and constant. The diffusion parameter $\sigma(v,t)$ may take on either of two values depending on whether a change in asset return risk has taken place or not. This parameter is discussed in detail below. The firm is subject to the following debt policy (suggested by Leland (1994*a*)):

- In each instant the firm has debt outstanding with a total principal of P. This debt pays a constant total coupon of C.
- A fraction m of debt is continuously rolled over → the firm withdraws *and* issues $mPdt$ new debt. The newly issued debt is identical in all respects (seniority, coupon, principal, amortization rate).
- At $t = 0$ debt is issued at par.

A debt issue is fully characterized by the triplet (c, p, m) and thus the capital structure of the firm by the triplet (C, P, m). The cash flow at time t to a particular debt issue floated at time $s < t$ is thus $e^{-m(t-s)}(c + mp)$, where the first factor reflects the amortization that has taken place between t and s. The second factor represents the sum of the coupon and the amortization payments.

This debt policy may be likened to debt with a sinking fund provision[10], a feature often observed in practice. The appeal of this particular debt

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[7] I assume for convenience that the firm's assets may be traded continuously. If we were not willing to make this (admittedly restrictive) assumption, the process under the risk neutral measure would take on the following appearance

$$\begin{cases} dv = (\mu - \beta - \lambda\sigma(v,t))\,vdt + \sigma(v,t)\,vdW' \\ v(0) = v_0 \end{cases}$$

where λ is the market price of risk associated with the firm's operations. Since we would no longer be in a complete market environment, a choice of the market price of risk associated with the firm' operations would be required to uniquely define the martingale measure Q.

See Ericsson & Reneby (1997) for a discussion about the relaxation of the tradeability assumption.

[8] We assume that asset sales are precluded. This is often the case in practice where bond covenants restrict the disposition of assets, see Smith & Warner (1979).

[9] $(\mu - \beta)$ is the expected appreciation rate of the state variable v whereas μ is the expected return of the gain process, that is of benefitting from the growth *and* accumulating the payout stream βvdt.

[10] The debt repayment schedule chosen here implies that debt amortization is an exponential function of time. This assumption may be relaxed at the expense of more complex pricing expressions than those obtained in this paper (see Leland & Toft (1996)).

structure is that the overall debt in the firm's capital structure will be time independent and that the obtained default and risk shifting barriers will be constants and may be solved for analytically.

The parameter m, the repayment speed, may be interpreted as the inverse of debt maturity T. To see this consider the average maturity of debt

$$T = \int_0^\infty tme^{-mt}dt = \frac{1}{m}$$

where me^{-mt} is the fraction of the principal of current $(t = 0)$ debt retired at time t.

4.3.1 Financial Distress

Assume that financial distress[11] occurs when the net operating cash flow βvdt and the proceeds from newly issued debt $d(v)\,dt$ are not sufficient to cover after-tax payments to current debtholders[12]. When the firm defaults a fraction α of the firm's assets is lost in reorganization costs or in inefficient running of the firm. At this point let us assume that the absolute priority rule is enforced, that is that shareholders recover nothing and creditors $(1-\alpha)\,L$, where L denotes the value of the firm's assets in financial distress. Hence debt that is issued just at the time of default recovers $(1-\alpha)\,mL$. Recall that m is the fraction of outstanding debt which is rolled over.

Thus for a solvent firm the following relationship will hold

$$\beta vdt + d(v)\,dt \geq (1-\tau)\,Cdt + mPdt$$

Define L as the value of v such that this inequality binds, that is when the firm becomes insolvent:

$$\beta L + (1-\alpha)\,mL = (1-\tau)\,C + mP$$

The second term on the left hand side equals the proceeds from newly issued debt which as noted is equal to the recovery in default of such an issue - a necessary condition for a creditor to be willing to lend anything to an insolvent firm. Solving for L we obtain the following expression for the default barrier

$$L = \frac{(1-\tau)\,C + mP}{\beta + (1-\alpha)\,m} \tag{4.3}$$

[11] We do not model what happens once a firm is in default explicitly. We will use the terms financial distress and default interchangeably to describe anything from an informal workout process to formal bankrupcty proceedings.

[12] This financial distress triggering mechanism differs from that used in Leland (1994*a*) in that we do not allow new equity to be issued at $v = L$ to finance debt service. In his paper default is caused by the unwillingness of shareholders to finance debt payments. Kim et al. (1993) and Anderson & Sundaresan (1996) use a trigger similar to the one in this paper although new debt is not issued.

TABLE 4.1. THE COMPARATIVE STATICS OF THE DEFAULT BARRIER

with respect to	Sensitivity of the default barrier
Principal $\left(\frac{dL}{dP}\right)$	> 0
Coupon $\left(\frac{dL}{dC}\right)$	> 0
Maturity $\left(\frac{dL}{dT}\right)$	$\begin{cases} < 0 & \text{if } \frac{P}{1-\alpha} > \frac{(1-\tau)C}{\beta} \\ > 0 & \text{otherwise} \end{cases}$
Tax rate $\left(\frac{dL}{d\tau}\right)$	< 0
Default costs$\left(\frac{dL}{d\alpha}\right)$	> 0
Cash flow rate $\left(\frac{dL}{d\beta}\right)$	< 0

Table 4.1 reports the comparative statics of the default barrier. We observe that *ceteris paribus* increasing the amount of debt (be it by increasing the coupon or the principal) will raise the default trigger, a natural result since the cash flow constraint is tightened. Conversely a higher tax rate decreases the barrier since for a given coupon it will decrease the amount that has to be financed by internally generated funds. Increasing costs of reorganization raises L since it decreases the proceeds from issuing debt to finance debt service, in particular near default where the proceeds approach $(1-\alpha)\,mL$.

An increase in the cash flow proportion will lower the barrier since a lower asset value will suffice to generate the critical cash flow.

The partial derivative with respect to m, the inverse of debt maturity, merits some comment. Consider the limits of L as m goes to zero (maturity becomes infinite) and as m goes to infinity (debt is rolled over instantly), respectively:

$$\lim_{m\to 0} L = \frac{(1-\tau)\,C}{\beta}$$

$$\lim_{m\to\infty} L = \frac{P}{1-\alpha}$$

For perpetual debt the default trigger will depend solely on the coupon, tax rate and cash flow proportion. The principal repayment is no longer of any importance and L equals the ratio between the after tax coupon rate and the cash flow generation rate.

As maturity tends to zero the coupon loses its influence on L and the default condition becomes a requirement that the recovery in default equals the outstanding principal, that is $(1-\alpha)L = P$. Thus debt becomes essentially risk-free for very short maturities.

The values of the default barrier in these polar cases will determine whether L is a decreasing or increasing function of debt term to maturity. Default triggers for firms with low payout ratios will *ceteris paribus*[13] be likely to default at lower firm values the shorter the maturity. Similarly very costly financial distress is likely to yield lower default barriers for longer maturities.

4.3.2 Risk Shifting

The firm's management has two investment policies to choose from. These are characterized by the standard deviations (σ_1, σ_2) of the asset value process, where $\sigma_1 < \sigma_2$. It is assumed that when the firm is initiated it will be in the management's (shareholders') interests[14] to operate at σ_1. At any time in the future management can opt for the alternative policy. It is assumed that they can do this at no cost. A justification for this would be as follows: when they decide to switch the risk level they sell of the current project and are able to invest the proceeds in a (perfectly divisible) new project. Another way of interpreting this would be that they only shift some of their productive assets into different operations and that the overall effect is an increase in the standard deviation of the asset value process, without any injection of additional funds. The change in risk policy, once decided upon, is assumed to be irreversible.

Suppose for the time being that the management decides to switch at a barrier $K : v_0 \geq K \geq L$. Note that $K > v_0$ and $K < L$ imply that shifting has already taken place and will never occur, respectively. We will return to the endogenous determination of the risk shifting barrier K after deriving the values of debt, equity, reorganization costs and the tax shield.

4.3.3 Valuing Corporate Securities and the Levered Firm

To value debt, we need the present value of the debt service stream conditional on solvency and the present value of the recovery conditional on

[13] Maturity will influence the amount of debt the firm will optimally take on and thus affect this relationship.

[14] The focus in this paper is on the divergence of interests between creditors and the debtor. Agency conflicts between management and shareholders are not considered.

financial distress. Assume for the time being that the absolute priority rule is respected. This assumption will be relaxed later. Denote the value of debt at $t = 0$ for given default and risk shifting barriers by $d(v_0; L, K)$.

The cash flow accruing at time $t > s$ to debt issued at time s is

$$e^{-m(t-s)}(c + mp)\,dt$$

and in financial distress the associated recovery will be

$$d(L; L, K) = (1 - \alpha) L e^{-m(t-s)} \frac{p}{P}$$

The valuation is thus carried out by dividing the value of debt into the following two components:

- The value of the coupon and withdrawal stream given *no* financial distress. The value of this stream will depend on the (risk neutral) likelihood of default which in turn will depend on whether a risk switch has taken place. Having assumed that $v_0 \geq K \geq L$ we divide this stream into the following subcomponents
  - A debt service stream which terminates at the barrier K.
  - A debt service stream that begins after the first passage time of v to K and terminates at the first passage time of v to L.
- The value of the payment in the event of financial distress. Since financial distress cannot occur without the asset value first hitting K, we can value this payment as one which is "in" subject to K and "out" subject to L.

Carrying out the calculations (see appendix 4.9) we arrive at the following proposition.

Proposition 4.1 *The value of a debt issue (c, p, m) when financial distress occurs at L and management change risk at $K : v_0 \geq K \geq L$ is given by*

$$\begin{aligned} d(v_0; L, K) &= \left(\frac{c + mp}{r + m}\right)(1 - Q(v_0, L, K)) \\ &\quad + (1 - \alpha) mLQ(v_0, L, K) \end{aligned}$$

where

$$Q(v_0, L, K) \equiv \left(\frac{v_0}{K}\right)^{-\theta(\sigma_1)} \left(\frac{K}{L}\right)^{-\theta(\sigma_2)}$$

and

$$\theta(\sigma) = \frac{r - \beta - \frac{1}{2}\sigma^2 + \sqrt{\left(r - \beta - \frac{1}{2}\sigma^2\right)^2 + 2(r + m)\sigma^2}}{\sigma^2}$$

Proof: see Appendices 4.8 and 4.9.

The function $Q(\cdot)$ is related to the value of one dollar received upon the occurrence of financial distress. More precisely it is the fraction of a dollar received in that event that would accrue to the current debtholders.

Since $d(v_0, L, K)$ is the value of debt[15] with principal p and the total principal outstanding at each point in time is $P = \frac{p}{m}$ the total amount of debt $D(\cdot)$ in the capital structure is

$$\begin{aligned} D(v_0, L, K) &= \frac{d(v_0, L, K)}{m} \\ &= \left(\frac{C + mP}{r + m}\right)(1 - Q(v_0, L, K)) \\ &\quad + (1-\alpha) LQ(v_0, L, K) \end{aligned} \tag{4.4}$$

Next we value the levered firm. In order to do this we need values for bankruptcy costs, $B(v_0, L, K)$, and the tax shield, $T(v_0, L, K)$, generated by the tax deductibility of coupon payments. These can be derived straightforwardly using the techniques in appendix 4.8 and are given by (see appendix 4.9)

$$B(v_0, L, K) = \alpha L\widetilde{Q}(v_0, L, K)$$

$$T(v_0, L, K) = \frac{\tau C}{r}\left(1 - \widetilde{Q}(v_0, L, K)\right)$$

where

$$\widetilde{Q}(v_0, L, K) = \left(\frac{v_0}{K}\right)^{-\varphi(\sigma_1)}\left(\frac{K}{L}\right)^{-\varphi(\sigma_2)}$$

and

$$\varphi(\sigma) = \frac{r - \beta - \frac{1}{2}\sigma^2 + \sqrt{\left(r - \beta - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}}{\sigma^2}$$

The function $\widetilde{Q}(v_0, L, K)$ may be interpreted as the value of one dollar conditional on default occurring. Note that it may also be interpreted as the discounted unconditional risk neutral probability of default. We can thus see the product as the probability of default given that we have just switched from σ_1 to σ_2 times the probability of changing to σ_2.

The current value of the levered firm (V_0) is given by

$$V_0 = v_0 + T(v_0, L, K) - B(v_0, L, K)$$

and thus the value of equity must be

$$E_0 = V_0 - D(v_0, L, K)$$

[15] More accurately $d(v_0, L, K)\,dt$ is the amount of debt issued at $t = 0$.

$$= v_0 + \frac{\tau C}{r}\left(1 - \widetilde{Q}(v_0, L, K)\right) - \alpha L\widetilde{Q}(v_0, L, K) \quad (4.5)$$
$$- \left(\frac{C + mP}{r + m}\right)(1 - Q(v_0, L, K))$$
$$- (1 - \alpha) LQ(v_0, L, K)$$

4.3.4 Cost of Asset Substitution

A measure of the cost of asset substitution ($A_0 \geq 0$) for a given set of parameters (including the triplet (C, P, m)) is the difference between firm value if operated at the lower risk with a sustainable commitment and firm value when this is not possible (and risk shifting may take place).

$$\begin{aligned}
A_0 &= V_0|_{K \equiv L} - V_0 \\
&= \frac{\tau C}{r}\left(\widetilde{Q}(v_0, L, L) - \widetilde{Q}(v_0, L, K)\right) \\
&\quad -\alpha L\left(\widetilde{Q}(v_0, L, K) - \widetilde{Q}(v_0, L, L)\right) \\
&= \left(\frac{\tau C}{r} - \alpha L\right)\left(\widetilde{Q}(v_0, L, K) - \widetilde{Q}(v_0, L, L)\right)
\end{aligned}$$

We will term this quantity the *agency discount* as it measures how large a cost is incurred at the current capital structure (C, P, m) as a result of risk shifting incentives. We can see from this expression that the agency discount enters by altering the value of the tax shield and the costs of financial distress. More accurately it arises from an endogenous change in the (risk neutral) probability of default. Thus although the default cost proportion α and the tax rate are exogenously determined, their impact on the value of the levered firm is endogenous and tied to the incentive structure of equity.

Another way of expressing this measure of the agency problem is the difference between the credit spread on debt in the two alternative firms above. Note that the two measures discussed here do not reflect equilibrium agency costs since leverage and maturity are not chosen optimally. The optimal capital structure's influence on agency costs is discussed later.

Having valued the firm's capital structure for a given K, the problem at hand is to determine the time consistent level at which shareholders will want to increase asset return risk.

4.4 Time Consistent Risk Shifting Policies

I model a firm which is financed by a nominally stationary capital structure. Once the financing decisions are made there will be an opportunity for the management (representing the shareholders) to select a higher level of risk and thus effectuate a wealth expropriation from the firm's creditors. These are aware of these adverse incentives and will demand compensation when the terms and cost of the financing are decided upon. An implicit assumption made is that enforcing a contract which precludes this behaviour is impossible.

More precisely, suppose that at time 0 the firm is set up. A debt issue (C, P, m) is floated on the grounds that the firm will operate at σ_1 until the value of the firm's assets reach a level K at which the risk of the firm's activities becomes σ_2. We will now determine the endogenous level K that will be chosen by the firm's management.

The highest value the shareholders can obtain *ex ante* would be by committing not to switch, that is

$$L = \arg\max_{\{K\}} E(\cdot, K)$$

This condition simply states that shareholders are currently better off if management continues to operate at σ_1 and could commit to not altering this risk choice. It also defines the first best level of firm value. This level of K will however not in general be sustainable *ex post.* There will exist a level of firm asset value for which the post switching value exceeds the value of waiting until $K = L$ to switch (that is not to switch at all). In order to find the time consistent K we require the following smooth pasting condition (see Dixit (1993)) to be fulfilled:

$$\frac{\partial E(v, \sigma_1, \sigma_2)}{\partial v}\Big|_{v=K} = \frac{\partial E(v, \sigma_2, \sigma_2)}{\partial v}\Big|_{v=K}$$

The first term is simply the derivative of equity in equation (4.5). The second is the derivative of equity when a switch has taken place and the firm will operate at σ_2. This equation can then be solved for K, yielding the following proposition.

Proposition 4.2 *The time consistent risk shifting policy is such that the policy σ_2 will be selected at the first passage time of v to K, where*

$$K = L\left(\Lambda\frac{\frac{C+mP}{r+m} - (1-\alpha)L}{\frac{\tau C}{r} + \alpha L}\right)^{\Gamma} \tag{4.6}$$

and

$$\Lambda = \frac{\varphi(\sigma_1) - \varphi(\sigma_2)}{\theta(\sigma_1) - \theta(\sigma_2)}$$

$$\Gamma = \frac{1}{\theta(\sigma_2) - \varphi(\sigma_2)}$$

Proof. *See appendix 4.10 with* $\gamma = 0$

The denominator in (4.6) is the realized loss to the levered firm in the event of financial distress, that is the "risk-free" values of the tax shield plus the costs of default. Since shareholders hold the residual claim to the firm it may also be interpreted as the realized loss to equity. In the nominator we have the realized loss to current bondholders[16]. Hence we see that the level of firm value at which a shift will take place will be determined by a trade-off between equity losses and debt losses. The higher the loss to current creditors relative to equity, the stronger the incentives for asset substitution[17].

A measure of the strength of risk shifting incentives is the ratio between the switching barrier and the default barrier, $\frac{K}{L}$. If this ratio is unity then there will be no asset substitution problem. If it is greater than one there will be risk shifting at some level of asset value.

To analyse the effect of debt maturity on this ratio consider the limiting cases of perpetual debt and instantaneously rolled over debt and their associated $\frac{K}{L}$ ratios.

Remark 4.1 *For perpetual debt* ($m = 0$) *the firm will operate at* σ_2 *immediately if* $r < \beta$ *and never otherwise. The condition* $r < \beta$ *states that the realized loss to bondholders in financial distress is greater than that of equity. More formally*

$$\lim_{m \to 0} \left(\frac{K}{L} \right) = \begin{cases} \infty & \text{if } r < \beta \\ 0 & \text{otherwise} \end{cases}$$

Remark 4.2 *For very short term debt there will not be any asset substitution.*

$$\lim_{m \to \infty} \left(\frac{K}{L} \right) = 1$$

Consider now the effect of distress costs and tax deductibility of coupon payments on the incentives of shareholders to alter the riskiness of the

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[16]We will henceforth assume that the nominator is positive, implying that the debt contract is such that creditors do not actually gain in the event of financial distress.

[17]The parameter Γ is positive and Λ is a number between 0 and 1 (prove). Hence the ratio of debt losses to equity losses in financial distress has to be larger than one for K to be larger than L. Note however that this is only a necessary condition. Whether or not K exceeds L will also depend on the difference of the risk levels, the payout rate, the risk free rate and debt maturity through Λ and Γ.

firm's operations. When these are not present any change in the risk policy is irrelevant to the value of the firm and will only cause a redistribution of value between creditor and debtor.

Remark 4.3 *In the absence of costs of financial distress and tax deductibility ($\alpha = \tau = 0$) shareholders will always want the higher risk level* [18]

$$\lim_{\tau,\alpha \to 0} \left(\frac{K}{L}\right) = +\infty$$

The gains from asset substitution stem from an asymmetry in losses to bondholders and shareholders. So for asset substitution to be (*ex post*) optimal for shareholders in this model we require that the measure of this asymmetry be significant - that the ratio of debt losses to equity losses is greater than unity (which is always the case when $\tau = \alpha = 0$). Furthermore only current creditors can be expropriated by changing the risk policy and maturity serves to control the "amount" of current debt. If maturity is infinite current creditors are also the future creditors and the potential gains from risk shifting (if any) are at their maximum (*ceteris paribus*). For instantly rolled over debt there are in a sense no current creditors to expropriate and thus no gains to asset substitution. Recall also that

$$\lim_{m \to \infty} L = \frac{P}{1-\alpha}$$

which implies that instantly rolled over debt will enjoy full recovery in default.

Figure 4.1 plots the ratio between the risk shifting barrier K and the default barrier L as a function of the principal of debt and its maturity. Remember that the coupon is set so that debt is initially priced at par. The most important effect noticeable in the plot is that for very short maturities the ratio is equal to 1 regardless of leverage, confirming remark 4.2. The longer the maturity the higher the ratio. Long term debt thus increases the scope for asset substitution.

There is also a tendency in the plotted example for the ratio to decrease in leverage. Although in this case it is evident only for long maturities it does provide a counterexample to the results of Green & Talmor (1986) mentioned in section 4.2.

[18] Recall that we have assumed that $\frac{C+mP}{r+m} - (1-\alpha)L > 0$.

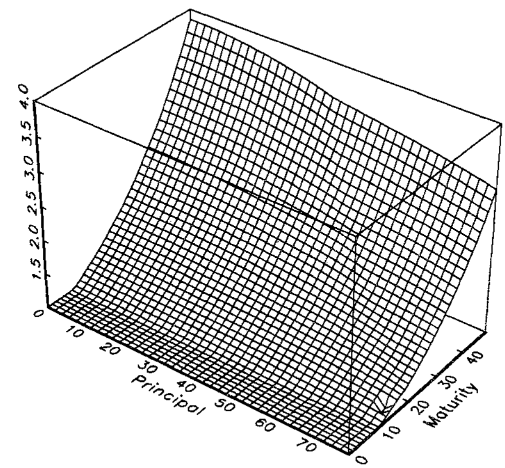

FIGURE 4.1. RISK SHIFTING INCENTIVES AS A FUNCTION OF LEVERAGE AND MATURITY. The ratio $\frac{K}{L}$ as a function of debt principal and maturity. The coupon is set so that debt sells at par. $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\alpha = 0.20$, $\tau = 0.35$, $r = 0.06$, $\beta = 0.075$.

Now that the firm's capital structure has been valued and the time consistent risk shifting policy determined, the problem of selecting an optimal capital structure will be addressed.

4.5 Optimal Capital Structure

In the first subsection below we will examine the initial choice of leverage for given maturities. Then we consider the simultaneous choice of these two variables. The examples in this section are constructed using the following set of base case parameters:

Asset value v_0	Short rate r	Reorg. costs α	C.F. rate β	Low risk σ_1	High risk σ_2	Tax rate τ
100	0.06	0.15	0.075	0.15	0.3	0.35

These are selected so as to generate outputs consistent with empirical observations. Direct costs of bankruptcy are estimated at 2.9 percent of the book value of assets by Weiss (1990), whereas Altman (1984) reports an estimate of the indirect costs in the range of 11 to 17 percent of the predistress firm value. We set α to 15 percent as an estimate of the total costs. The cash flow generation parameter β is chosen so as to allow for some asset substitution incentives while being consistent with observed payouts[19]. The selected levels of asset risk generate reasonable levels of equity volatility in the range of 20% to 30%.

4.5.1 *Optimal Leverage for a Given Maturity*

Table 4.2 reports the characteristics of the optimal capital structures obtained in the base case presented above for different maturities

TABLE 4.2. CHARACTERISTICS OF OPTIMAL CAPITAL STRUCTURES FOR DIFFERENT MATURITIES

Maturity	0.5	2.5	5	10	15	20	25	30
Leverage (%)	32.9	36.9	41.5	46.9	46.7	45.1	43.0	40.6
Firm value	107.2	108.1	109.1	110.6	110.8	110.7	110.3	109.9
Default barrier	40.5	42.2	44.4	46.3	43.5	40.4	37.2	34.3
Switching barrier	40.5	42.2	44.4	48.4	53.2	57.7	62.4	67.2
Principal	35.2	39.9	45.3	51.9	51.7	50.0	47.4	44.6
Coupon	2.1	2.4	2.8	3.4	3.5	3.5	3.3	3.2
Equity Vol. (%)	22.3	23.8	25.6	27.7	26.9	25.8	24.6	23.5
Credit Spr. (%)	0.0	0.0	0.2	0.6	0.8	0.9	1.0	1.1
Agency Discount	0.0	0.0	0.0	0.3	1.3	2.2	2.9	3.4
Agency Spread	0.0	0.0	0.0	0.0	0.1	0.2	0.3	0.4
Agency Costs	0.0	0.0	0.0	0.2	1.3	2.2	3.3	4.2
Lev. if no AS	32.9	36.9	41.5	47.4	50.6	52.6	53.9	54.9

For maturities less than or equal to 5 years there are no incentives for asset substitution. This is in line with the argument above that short term debt will alleviate these incentives. In the 0.5 to 15 year bracket we also note that the recovery in default for current creditors is relatively high and that optimal leverage increases in maturity. Given that debt is issued at

[19] Leland & Toft (1996) argue that a figure of approximately 7% is consistent by historical standards for larger firms.

par today a longer maturity will allow more debt to be taken on and hence yield a more valuable tax shield[20].

For longer term capital structures (in this example terms to maturity exceeding 15 years) risk shifting may well occur long before default and increasing debt maturity no longer increases optimal leverage and the associated firm value. The increased risk (in expected terms) that the potential for asset substitution entails leads to higher expected costs of financial distress and thus a lower optimal leverage and firm value. We see that a maturity of roughly 15 years would be chosen to balance the tax benefits of debt, default and agency costs of debt.

Equity volatilities lie in the range 20 to 30 percent annually and not surprisingly mirror the behaviour of the optimal leverage for different terms to maturity[21].

For maturities at which shareholders have the incentive to shift risk the levered firm value departs from its first best level. Letting debt maturity vary between 15 and 30 years yields agency discounts between 0.8 and 4.9 percent of the levered firm value. The associated agency spread may be a large component of the total spread over the risk-free rate paid to creditors. For 15 year debt the agency spread constitutes about 12 percent of the total credit spread of 80 basis points. Increasing debt maturity to 30 years implies that more than a third of the 110 point spread is due to asset substitution.

Consider figure 4.2 which plots the firm value at the optimal leverage as a function of debt maturity. Two different policies are depicted - the first best operating policy where management is able to commit to σ_1 and that in which they cannot and at some point will indulge in asset substitution. The area between the σ_1 commitment schedule and that for the no commitment case represents the agency discounts for a given maturity as reported in table 4.2. It is clear from this plot that firm value in the absence of adverse risk incentives is monotonically increasing in maturity and hence that perpetual debt would be selected by shareholders. Introducing these incentives makes the levered firm value a non-monotone function of maturity instead, yielding an interior leverage and maturity.

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[20] The shorter the maturity, the higher (ceteris paribus) the impact of an increase in the principal on the default barrier.

[21] The instantaneous equity volatility is related to the asset value standard deviation through the following equation

$$\sigma_E = \sigma \frac{\partial E}{\partial v} \frac{v}{E}$$

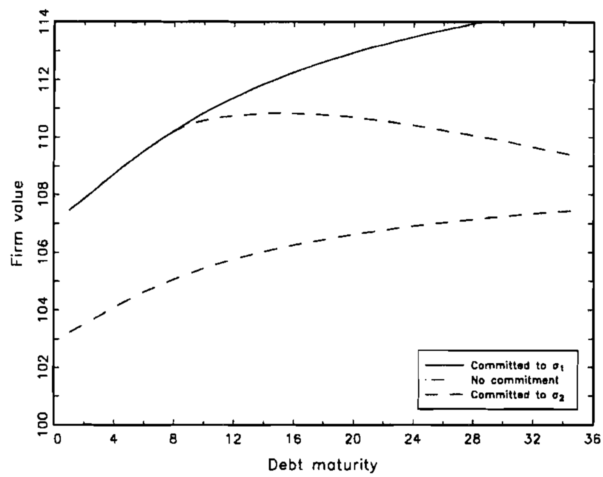

FIGURE 4.2. FIRM VALUE AND DEBT MATURITY. Levered firm value (V_0) as a function of debt maturity. Parameters as in the base case.

Recall that the agency discount only measures a *ceteris paribus* loss in firm value. The true agency costs for a given maturity is the difference at the optimal leverages. The optimal leverage that would have been chosen in the absence of risk shifting incentives is reported in the penultimate row of table 4.2. These costs are naturally larger than the agency discounts since the latter are firm value differences when one is at a suboptimal leverage.

We have thus seen that as opposed to the case where risk shifting is not modelled an additional dimension is added to the leverage decision. In the no risk shifting case the trade-off stands between an increase in firm value from the tax shield and a loss from expected costs of financial distress. When asset substitution is modelled explicitly the value of agency costs enter into the trade-off. Debt maturity and amount is chosen so as to equalize the marginal benefits of the tax shield and the marginal costs of financial distress and agency.

4.5.2 Optimal Leverage and Maturity

In this section we examine how the optimal leverage *and* maturity of a firm depend on the various parameters that define its environment. Table 4.3 presents values for different asset return standard deviations and risk shifting scenarios. For each σ_1 we calculate optimal leverage, maturity and other measures for two different high to low risk ratios $\left(\frac{\sigma_1}{\sigma_2}\right)$, that is when the standard deviation that shareholders may shift to is 50% and 100% higher respectively.

TABLE 4.3. OPTIMAL LEVERAGE AND MATURITY: DIFFERENT RISK CONSTELLATIONS

σ_1	0.1		0.15		0.2	
σ_2	0.15	0.2	0.225	0.30	0.3	0.4
Optimal leverage (%)	54.48	51.12	48.02	46.62	43.21	43.46
Optimal maturity	38.46	22.58	26.47	15.27	20.33	11.63
Firm value	115.89	114.10	111.90	110.83	109.15	108.57
Default barrier	45.76	45.39	41.80	43.33	38.67	41.64
Switching barrier	79.86	65.82	64.28	53.40	53.62	44.92
Principal	63.14	58.32	53.73	51.66	47.17	47.18
Coupon	4.31	3.89	3.77	3.50	3.38	3.25
Equity Vol. (%)	19.46	19.00	26.61	26.88	33.27	34.36
Credit Spread (%)	0.83	0.67	1.01	0.78	1.17	0.88
Agency Discount	2.55	2.52	1.89	1.40	1.29	0.47
Agency Spread	0.22	0.17	0.18	0.10	0.14	0.03
Equilib. agency Cost	7.54	9.33	6.50	7.57	5.40	5.89
Lev. without AS	68.97	68.97	61.40	61.40	54.82	54.82

We observe that the higher the initial asset return risk the lower the optimal leverage. This is natural since for a given triplet (C, P, m) higher standard deviations imply higher financial distress probabilities and expected costs of default and hence lower leverage. In general the behaviour of the optimal leverage as we increase the high to low risk ratio follows the same pattern except when the risk is so high (and the resulting leverage so low) that risk shifting incentives are mitigated. In such a situation optimal leverage may increase slightly when σ_2 increases as the scope for asset substitution is depleted.

The fact that the calculated firm values are decreasing in the standard deviation should not be over-interpreted since we are comparing firms with the same exogenous "all equity" value and different risk levels.

Credit spreads are increasing in the initial risk level as debt prices are decreasing in asset volatility. However, increasing the risk level that shareholders may switch to will lower the spread since the optimal leverage is likely to be lower and thus also the likelihood of default.

The equilibrium agency discounts are non-zero. They range between 5 and 8 percent of levered firm value. Maturity is only used up to a certain point to curtail asset substitution incentives. At some point the benefit of shortening the maturity is outweighed by the loss in tax benefits. So although shareholders bear the full cost risk shifting will occur in equilibrium.

Interestingly both the agency discounts and the equilibrium agency costs are decreasing both in the initial risk levels and the high to low risk ratios. This is due to the chosen leverages and maturities being lower and that incentives to shift risk will accordingly be weakened. The equilibrium agency costs are almost threefold larger and the associated leverages (reported in the last row) are considerably higher than when asset substitution is allowed for.

TABLE 4.4. OPTIMAL LEVERAGE AND MATURITY: THE IMPACT OF DEFAULT COSTS

α	0.05	0.10	0.15	0.20
Optimal leverage (%)	54.85	50.37	46.62	43.43
Optimal maturity	10.95	13.09	15.27	17.46
Firm value	112.75	111.70	110.83	110.09
Default barrier	51.82	47.17	43.33	40.10
Switching barrier	59.20	56.10	53.41	51.03
Principal	61.85	56.26	51.66	47.81
Coupon	4.21	3.82	3.50	3.24
Equity Vol. (%)	31.81	28.91	26.88	25.37
Credit Spread (%)	0.80	0.79	0.78	0.78
Agency Discount	1.13	1.30	1.40	1.47
Agency Spread	0.06	0.08	0.10	0.11
Equilibrium Agency Cost	7.41	7.53	7.57	7.56
Leverage without AS	65.78	63.50	61.39	59.45

Table 4.4 reports optimal capital structures for different levels of financial distress costs. Not surprisingly the amount of debt chosen increases the lower the cost parameter α. However, as the selected leverage increases asset substitution incentives become more pronounced for a given maturity and a shorter term to maturity will be chosen. When default costs increase from 5% to 20% the optimal maturity increases by about 60% and optimal leverage falls by 20 %.

The credit spreads decrease in α, although they do not change much in the situations depicted in table 4.4.

4.6 Deviations from Absolute Priority

The absolute priority rule (APR) stipulates that shareholders shall not receive anything in bankruptcy until creditors are fully compensated. There is strong empirical evidence that departures from this rule are commonplace[22]. Denote with γ the fraction of the firm value in default net of bankruptcy costs that accrues to shareholders. The price of debt is now given by

$$\begin{aligned} D(v_0, L, K) &= \left(\frac{C + mP}{r + m}\right)(1 - Q(v_0, L, K)) \\ &\quad + (1 - \alpha)(1 - \gamma) LQ(v_0, L, K) \end{aligned}$$

The value of equity is as before obtained as the residual and will be increased precisely by $\gamma(1 - \alpha) LQ(v_0, L, K)$. The default triggering firm value will also change, since the value at which debt can be issued is lowered. The financial distress condition now becomes

$$\beta L + (1 - \alpha)(1 - \gamma) mL = (1 - \tau) C + mP$$

and the default barrier thus becomes

$$L = \frac{(1 - \tau) C + mP}{\beta + (1 - \alpha)(1 - \gamma) m} \tag{4.7}$$

Furthermore extending proposition 4.2 to a non-zero γ yields the following expression for the risk shifting barrier

$$K = L\left(\Lambda \frac{\frac{C+mP}{r+m} - (1 - \alpha)(1 - \gamma) L}{\frac{\tau C}{r} + \alpha L}\right)^{\Gamma} \tag{4.8}$$

with Λ and Γ defined as before.

We begin by examining the change in incentives and security values that result from introducing APR deviations for the chosen nominal capital structures (C, P, m) in table 4.3. We find that leverage is lower for all risk constellations. Comparing default and risk shifting barriers we see that these are higher. The reason the default barrier is higher is clear from equation 4.7. The cash flow available to service debt is generated internally and by issuing new debt. The amount of internally generated debt is unaffected by APR deviations whereas the proceeds from newly issued debt are lower. Hence the cash flow constraint is tighter with APR deviations than without.

The reason the risk shifting barrier increases can be inferred from a comparison between equations 4.6 and 4.8. The occurrence of asset substitution is dictated by the ratio of creditor losses to debtor losses. This

[22] See Eberhart et al. (1990*a*), Weiss (1990) and Franks & Torous (1989).

TABLE 4.5. THE CETERIS PARIBUS EFFECT OF INTRODUCING DEVIATIONS FROM THE ABSOLUTE PRIORITY RULE ($\gamma = 0.08$).

σ_1	0.10		0.15		0.2	
σ_2	0.15	0.2	0.225	0.30	0.3	0.4
Leverage (%)	53.6	50.4	47.2	46.0	42.4	43.0
Maturity	38.5	22.6	26.5	15.3	20.3	11.6
Firm value	114.5	112.1	110.4	108.8	107.7	106.6
Default barrier	46.6	46.6	42.9	44.8	39.8	43.4
Switching barrier	97.2	79.2	80.4	66.2	68.8	57.4
Principal	63.1	58.3	53.7	51.7	47.2	47.2
Coupon	4.3	3.9	3.8	3.5	3.4	3.3
Equity Vol. (%)	19.3	19.2	26.6	27.3	33.3	35.0
Credit Spread (%)	1.03	0.88	1.24	0.99	1.40	1.09
Agency Discount	3.7	4.1	3.0	3.0	2.4	1.9

ratio increases with APR deviations since creditors receive less in default. The nominator is unaffected since it relates only to the tax shield and bankruptcy costs.

This result is at odds with the predictions of Eberhart & Senbet (1993) who argue that giving shareholders a part in the payoffs in default states would alleviate the asset substitution problem. The model in this paper would have us believe that sharing payoffs in these states would exacerbate the problem. Now shareholders not only have nothing to loose but will gain from unfavourable outcomes as a result of increased risk.

The reason for this difference between the models is twofold. First, Eberhart & Senbet (1993) use a standard Black & Scholes (1973) model and thus do not allow the timing of financial distress to be random. They assume that the firm is near default in the sense that the asset value is lower than the impending principal repayment. If on the other hand (as in this model) default can be triggered by a cash flow shortage, the value of the payment to equity in default will tend to increase with asset volatility. This effect contrasts with Eberhart & Senbet (1993) case since their equity payment is in the money and will decrease with asset volatility. The second related reason is that there is no clear maturity of equity in our model.

From this we can conclude that the result of Eberhart & Senbet (1993) is sensitive to the situation at hand. If one is in a situation where a large repayment is due shortly, asset substitution incentives may be alleviated by deviations from the absolute priority rule. Nonetheless the possibility of default before debt maturity is likely to counteract this result. If the situation is similar to the one in this paper where debt contracts are rolled over in small amounts, deviations are likely to increase risk shifting incentives.

TABLE 4.6. DEVIATIONS FROM ABSOLUTE PRIORITY AND CAPITAL STRUCTURE

σ_1	0.15
σ_2	0.30
Optimal leverage (%)	42.00
Optimal maturity	13.67
Firm value	109.55
Default barrier	40.88
Switching barrier	56.44
Principal	46.02
Coupon	3.14
Equity Vol. (%)	25.17
Credit Spread (%)	0.81
Agency Discount	1.89
Equilibrium Agency Cost	0.30
Leverage without AS	42.88

Table 4.6 reports the optimal leverage and maturity with APR deviations for the scenario of the fourth column in table 4.3. The chosen leverage is yet lower and the optimal maturity shorter as a result of the amplified risk shifting incentives.

4.7 Concluding Remarks

Kim et al. (1993) show that the traditional Merton (1974) debt pricing model is unable to generate realistic credit spreads without resorting to unreasonable inputs. John (1993) argues that basic contingent claims model of corporate debt are at odds with reality on four counts. First, default is in reality likely to be triggered by a cash flow criterion. Second, financial distress is costly. Third, relevant strategic aspects should enter into the model; and finally, the significant deviations from the absolute priority rule observed in practice should be accounted for. The model in this paper is an attempt to address aspects of all these issues.

To do so, I have developed a model for the valuation of corporate securities which yields interior optimal leverage *and* debt maturity. These quantities result from a trade-off between the value of the tax shield and expected bankruptcy costs. The value of costly default may be separated into the value which would result if equity could commit to not shifting risk and that which arises endogenously from the asset substitution incentives. Hence agency costs of debt enter explicitly into the capital structure decision.

The maturity of debt is used to abate incentives to increase risk; a shorter term to maturity reduces the scope for wealth expropriation. At the optimal capital structure, there may still be significant agency costs (more than 5% of firm value) because the gains from decreasing agency costs are outweighed by lower tax benefits.

It is shown that deviations from the absolute priority rule may aggravate the asset substitution problem, in contrast with results of other studies. Other things being equal, APR violations lead to lower leverages and shorter maturities due to stronger incentives to increase risk.

4.8 Appendix: Integrals of First Passage Time Densities

Consider the following diffusion process

$$\begin{aligned} dX &= \mu dt + dW \\ X(0) &= a \end{aligned}$$

let τ denote the first passage time of this process at 0. The density of the first passage time is given by

$$f(t) = \frac{|a|}{\sqrt{2\pi t^3}} exp\left\{-\frac{1}{2}\left(\frac{a+\mu t}{\sqrt{t}}\right)^2\right\}$$

We know that the integral of this first passage time between zero and infinity is

$$\int_0^\infty f(t)\,dt = e^{-2\mu a}$$

We will now derive an expression for the following integral

$$\int_0^\infty e^{\alpha t} f(t)\,dt$$

In more detail this integral may be written as

$$\frac{|a|}{\sqrt{2\pi t^3}} \int_0^\infty exp\left\{\alpha t - \frac{1}{2}\left(\frac{a+\mu t}{\sqrt{t}}\right)^2\right\} dt$$

$$\begin{aligned} & \alpha t - \frac{1}{2}\left(\frac{a+\mu t}{\sqrt{t}}\right)^2 \\ = \;& \alpha t - \frac{a^2 + 2a\mu t + \mu^2 t^2}{2t} \\ = \;& -\frac{a^2 + 2a\mu t + \mu^2 t^2 - 2\alpha t^2}{2t} \end{aligned}$$

$$\begin{aligned} & a^2 + 2a\mu t + \mu^2 t^2 - 2\alpha t^2 \\ = \;& a^2 + \left(\mu^2 - 2\alpha\right) t^2 + 2a\mu t \\ = \;& \left(a + \sqrt{\mu^2 - 2\alpha t}\right)^2 + X \end{aligned}$$

$$X = 2a\mu t - 2\sqrt{(\mu^2 - 2\alpha)}ta$$

$$
\begin{aligned}
& exp\left\{\alpha t - \frac{1}{2}\left(\frac{a+\mu t}{\sqrt{t}}\right)^2\right\} \\
= \; & exp\left\{-\frac{\left(a+\sqrt{\mu^2-2\alpha t}\right)^2 + X}{2t}\right\} \\
= \; & exp\left\{-\frac{X}{2t}\right\} exp\left\{-\frac{\left(a+\sqrt{\mu^2-2\alpha t}\right)^2}{2t}\right\} \\
= \; & exp\left\{a\left(\sqrt{(\mu^2-2\alpha)}-\mu\right)\right\} exp\left\{-\frac{\left(a+\sqrt{\mu^2-2\alpha t}\right)^2}{2t}\right\}
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \int_0^\infty e^{\alpha t} f(t)\, dt \\
= \; & \frac{|a|}{\sqrt{2\pi t^3}} e^{a\left(\sqrt{(\mu^2-2\alpha)}-\mu\right)} \int_0^\infty e^{-\frac{\left(a+\sqrt{\mu^2-2\alpha t}\right)^2}{2t}} dt
\end{aligned}
$$

Define

$$
g(t) = \frac{|a|}{\sqrt{2\pi t^3}} e^{-\frac{\left(a+\sqrt{\mu^2-2\alpha t}\right)^2}{2t}}
$$

so that

$$
\int_0^\infty e^{\alpha t} f(t)\, dt = e^{a\left(\sqrt{(\mu^2-2\alpha)}-\mu\right)} \int_0^\infty g(t)\, dt
$$

Note that $g(t)$ is the first passage time to zero of a unity variance process with drift $\sqrt{\mu^2-2\alpha}$. Hence

$$
\int_0^\infty g(t)\, dt = e^{-2a\sqrt{\mu^2-2\alpha}}
$$

so that

$$
\begin{aligned}
\int_0^\infty e^{\alpha t} f(t)\, dt &= \exp\left\{a\left(\sqrt{\mu^2-2\alpha}-\mu\right) - 2a\sqrt{\mu^2-2\alpha}\right\} \\
&= \exp\left\{a\left(\sqrt{\mu^2-2\alpha}-\mu-2\sqrt{\mu^2-2\alpha}\right)\right\} \\
&= \exp\left\{-a\left(\sqrt{\mu^2-2\alpha}+\mu\right)\right\}
\end{aligned}
$$

We are interested in the first passage time of the following process

$$
\begin{aligned}
dv &= (\mu-\beta-\lambda\sigma)\, v dt + \sigma v dW \\
v(0) &= v_0
\end{aligned}
$$

but we have derived the above result for a process of the form

$$\begin{aligned} dX &= \mu dt + dW \\ X(0) &= a \end{aligned}$$

If we define

$$X = \frac{\ln \frac{v}{L}}{\sigma}$$

then

$$\begin{aligned} dX &= \frac{\left(\mu-\beta-\lambda\sigma-\frac{1}{2}\sigma^2\right)}{\sigma} dt + dW \\ X(0) &= \frac{\ln \frac{v_0}{L}}{\sigma} \end{aligned}$$

so that

$$\begin{aligned} a &= \frac{\ln \frac{v_0}{L}}{\sigma} \\ \mu &= \frac{\left(\mu-\beta-\lambda\sigma-\frac{1}{2}\sigma^2\right)}{\sigma} \\ \alpha &= -r-m \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^\infty e^{-(r+m)t} f(t)\, dt = \exp\left\{-a\left(\sqrt{\mu^2-2\alpha}+\mu\right)\right\} \\ &= \exp\left\{-\frac{\ln \frac{v_0}{L}}{\sigma}\left(\frac{\left(\mu-\beta-\lambda\sigma-\frac{1}{2}\sigma^2\right)}{\sigma} + \sqrt{\frac{\left(\mu-\beta-\lambda\sigma-\frac{1}{2}\sigma^2\right)^2}{\sigma^2} + 2(r+m)}\right)\right\} \\ &= \exp\left\{-\ln \frac{v_0}{L}\left(\frac{\left(\mu-\beta-\lambda\sigma-\frac{1}{2}\sigma^2\right)+\sqrt{\left(\mu-\beta-\lambda\sigma-\frac{1}{2}\sigma^2\right)^2+2(r+m)\sigma^2}}{\sigma^2}\right)\right\} \\ &= \left(\frac{v_0}{L}\right)^{-\theta} \end{aligned}$$

where

$$\theta = \frac{\left(\mu-\beta-\lambda\sigma-\frac{1}{2}\sigma^2\right)+\sqrt{\left(\mu-\beta-\lambda\sigma-\frac{1}{2}\sigma^2\right)^2+2(r+m)\sigma^2}}{\sigma^2} \tag{4.9}$$

when $m = 0$

$$\theta \equiv \varphi = \frac{\left(\mu-\beta-\lambda\sigma-\frac{1}{2}\sigma^2\right)+\sqrt{\left(\mu-\beta-\lambda\sigma-\frac{1}{2}\sigma^2\right)^2+2r\sigma^2}}{\sigma^2} \tag{4.10}$$

4.9 Appendix: the Value of Debt with Asset Substitution

4.9.1 *Component 1: Debt Service Stream Terminated at K*

A debt service stream which terminates at the barrier K. The value of this stream is given by

$$d_1(v_0) = \int_0^\infty e^{-rt} e^{-mt} (c + mp)(1 - F(t, v_0, K))\, dt$$

The probability of having hit the barrier K at time t is given by

$$F(t, v_0, K) = \int_0^t f(t, v_0, K)\, dt$$

where f is the density of the first passage time of the geometric Brownian motion in 4.2 with $\sigma(t) = \sigma_1$ to the barrier K. We can then write

$$d_1(v_0) = (c + mp) \int_0^\infty e^{-(r+m)t} (1 - F(t, v_0, K))\, dt$$

Integrating by parts:

$$\int_0^\infty u(t)\, v'(t)\, dt = [u(t)\, v(t)]_0^\infty - \int_0^\infty u'(t)\, v(t)\, dt$$

let

$$\begin{aligned} v'(t) &= e^{-(r+m)t} \rightarrow v(t) = -\frac{1}{r+m} e^{-(r+m)t} \\ u(t) &= 1 - F(t, v_0, K) \rightarrow u'(t) = -f(t, v_0, K) \end{aligned}$$

Hence we can rewrite the integral as

$$\begin{aligned} \int_0^\infty e^{-(r+m)t} (1 - F(t, v_0, K))\, dt &= \left[-\frac{1}{r+m} e^{-(r+m)t} (1 - F(t, v_0, K)) \right]_0^\infty \\ &\quad - \frac{1}{r+m} \int_0^\infty e^{-(r+m)t} f(t, v_0, K)\, dt \\ &= \frac{1}{r+m} \left(1 - \int_0^\infty e^{-(r+m)t} f(t, v_0, K)\, dt \right) \end{aligned}$$

and hence

$$d_1(v_0) = \frac{c + mp}{r+m} \left(1 - \int_0^\infty e^{-(r+m)t} f(t, v_0, K)\, dt \right)$$

It is shown in appendix 4.8 that

$$\int_0^\infty e^{-(r+m)t} f(t, v_0, K)\, dt = \left(\frac{v_0}{K} \right)^{-\theta(\sigma_1)}$$

where θ is given by (4.9) and thus we have

$$d_1(v_0) = \left(\frac{c+mp}{r+m}\right)\left(1-\left(\frac{v_0}{K}\right)^{-\theta(\sigma_1)}\right)$$

4.9.2 Component 2: Stream Started at K and Terminated at L.

We now want to derive the value of a stream analogous to the one above save that it only starts once the value of the firm's assets hits the barrier K. When this happens we receive one unit of the following

$$\left(\frac{c+mp}{r+m}\right)\left(1-\left(\frac{K}{L}\right)^{-\theta(\sigma_2)}\right)$$

Since K is known *a priori*, this quantity is a constant. Multiplying this with the value of the fraction of one dollar paid out conditional on default to currently issued debt. The value of this dollar is given by

$$\left(\frac{v_0}{K}\right)^{-\theta(\sigma_1)}$$

where $\varphi(\sigma_1)$ is given by 4.10

Hence the value of this component is given by

$$d_2(v_0) = \left(\frac{v_0}{K}\right)^{-\theta(\sigma_1)}\left(\frac{c+mp}{r+m}\right)\left(1-\left(\frac{K}{L}\right)^{-\theta(\sigma_2)}\right)$$

4.9.3 Component 3: the Payout in Financial Distress

When the firm defaults creditors receive a fraction $(1-\alpha)$ of the value of the assets L. Let $d_3(\cdot)$ denote the value of the payout in default. Consider its value at $v_0 = K$, remembering that we now have $\sigma(t) = \sigma_2$.

$$\begin{aligned} & (1-\alpha) L \int_0^\infty e^{-rt}\left(e^{-mt}\frac{p}{P}\right) f(t, v_0, L)\, dt \\ = \; & (1-\alpha) L \frac{p}{P} \int_0^\infty e^{-(r+m)t} f(t, v_0, L)\, dt \\ = \; & (1-\alpha) mL \left(\frac{K}{L}\right)^{-\theta(\sigma_2)} \end{aligned}$$

where $e^{-mt}\frac{p}{P}$ is fraction of the outstanding principal that pertains to this particular issue.

Using the same reasoning as above, this expression is a constant given that we have reached K, hence the value of the third component is simply

$$d_3(v_0) = (1-\alpha)\, mL \left(\frac{K}{L}\right)^{-\theta(\sigma_2)} \left(\frac{v_0}{K}\right)^{-\theta(\sigma_1)}$$

Hence the value of a the currently floated issue is

$$\begin{aligned}
d(v_0, L, K) &= d_1(v_0) + d_2(v_0) + d_3(v_0) \\
&= \left(\frac{c+mp}{r+m}\right)\left(1 - \left(\frac{v_0}{K}\right)^{-\theta(\sigma_1)}\right) \\
&\quad + \left(\frac{v_0}{K}\right)^{-\theta(\sigma_1)} \left(\frac{c+mp}{r+m}\right)\left(1 - \left(\frac{K}{L}\right)^{-\theta(\sigma_2)}\right) \\
&\quad + (1-\alpha)\, mL \left(\frac{K}{L}\right)^{-\theta(\sigma_2)} \left(\frac{v_0}{K}\right)^{-\theta(\sigma_1)} \\
&= \left(\frac{c+mp}{r+m}\right)\left(1 - \left(\frac{v_0}{K}\right)^{-\theta(\sigma_1)} \left(\frac{K}{L}\right)^{-\theta(\sigma_2)}\right) \\
&\quad + (1-\alpha)\, mL \left(\frac{K}{L}\right)^{-\theta(\sigma_2)} \left(\frac{v_0}{K}\right)^{-\theta(\sigma_1)}
\end{aligned}$$

4.10 Appendix: Derivation of the Time Consistent Risk Shifting Barrier

The value of equity is given by

$$\begin{aligned}\mathcal{E}(v,\sigma_1,\sigma_2) &= v+\frac{\tau C}{r}\left(1-\widetilde{Q}(v,L,K)\right)-\alpha L\widetilde{Q}(v,L,K)\\ &-\left(\frac{C+mp}{r+m}\right)(1-Q(v,L,K))\\ &-(1-\alpha)(1-\gamma)LQ(v,L,K)\end{aligned}$$

where

$$\begin{aligned}\widetilde{Q}(v,L,K) &= \left(\frac{v}{K}\right)^{-\varphi(\sigma_1)}\left(\frac{K}{L}\right)^{-\varphi(\sigma_2)}\\ \text{for } v &> K \text{ and}\\ \widetilde{Q}(v,L,K) &= \left(\frac{v}{L}\right)^{-\varphi(\sigma_2)} \text{ for } v<K\end{aligned}$$

and

$$\begin{aligned}Q(v,L,K) &= \left(\frac{v}{K}\right)^{-\theta(\sigma_1)}\left(\frac{K}{L}\right)^{-\theta(\sigma_2)}\\ \text{for } v &> K \text{ and}\\ Q(v,L,K) &= \left(\frac{v}{L}\right)^{-\theta(\sigma_2)} \text{ for } v<K\end{aligned}$$

The following smooth pasting condition has to be fulfilled at $v=K$ (see (Dixit 1993, 1993))

$$\frac{\partial\mathcal{E}_1(v,\sigma_1,\sigma_2)}{\partial v}|v=K=\frac{\partial\mathcal{E}_2(v,\sigma_1,\sigma_2)}{\partial v}|v=K$$

where $\mathcal{E}_1$ denotes the pre-switching equity value and $\mathcal{E}_2$ the post switching value. Writing out this condition explicitly we obtain

$$\begin{aligned}&1+\frac{\varphi(\sigma_1)}{K}\left(\frac{K}{L}\right)^{-\varphi(\sigma_2)}\left[\frac{\tau C}{r}+\alpha L\right]\\ &+\frac{\theta(\sigma_1)}{K}\left(\frac{K}{L}\right)^{-\theta(\sigma_2)}\left[(1-\alpha)(1-\gamma)L-\frac{C+mp}{r+m}\right]\\ =\ &1+\frac{\varphi(\sigma_2)}{K}\left(\frac{K}{L}\right)^{-\varphi(\sigma_2)}\left[\frac{\tau C}{r}+\alpha L\right]\\ &+\frac{\theta(\sigma_2)}{K}\left(\frac{K}{L}\right)^{-\theta(\sigma_2)}\left[(1-\alpha)(1-\gamma)L-\left(\frac{C+mp}{r+m}\right)\right]\end{aligned}$$

Collecting terms and simplifying we have

$$[\varphi(\sigma_1) - \varphi(\sigma_2)] \left(\frac{K}{L}\right)^{-\varphi(\sigma_2)} \left[\frac{\tau C}{r} + \alpha L\right]$$
$$+ [\theta(\sigma_1) - \theta(\sigma_2)] \left(\frac{K}{L}\right)^{-\theta(\sigma_2)} \left[(1-\alpha)(1-\gamma) L - \frac{C+mp}{r+m}\right] = 0$$

or

$$AK^{-\varphi(\sigma_2)} + BK^{-\theta(\sigma_2)} = 0$$

where

$$A = \frac{[\varphi(\sigma_1) - \varphi(\sigma_2)]}{L^{-\varphi(\sigma_2)}} \left[\frac{\tau C}{r} + \alpha L\right]$$

$$B = \frac{[\theta(\sigma_1) - \theta(\sigma_2)]}{L^{-\theta(\sigma_2)}} \left[(1-\alpha)(1-\gamma) L - \frac{C+mp}{r+m}\right]$$

$$A + BK^{\varphi(\sigma_2)-\theta(\sigma_2)} = 0$$

$$K^{\varphi(\sigma_2)-\theta(\sigma_2)} = -\frac{A}{B}$$

$$\begin{aligned}
K &= \left(-\frac{A}{B}\right)^{\frac{1}{\varphi(\sigma_2)-\theta(\sigma_2)}} \\
&= -\left(\frac{\frac{[\varphi(\sigma_1)-\varphi(\sigma_2)]}{L^{-\varphi(\sigma_2)}}\left[\frac{\tau C}{r} + \alpha L\right]}{\frac{[\theta(\sigma_1)-\theta(\sigma_2)]}{L^{-\theta(\sigma_2)}}\left[(1-\alpha)(1-\gamma) L - \frac{C+mp}{r+m}\right]}\right)^{\frac{1}{\varphi(\sigma_2)-\theta(\sigma_2)}} \\
&= L\left(\frac{[\varphi(\sigma_1) - \varphi(\sigma_2)]\left[\frac{\tau C}{r} + \alpha L\right]}{[\theta(\sigma_1) - \theta(\sigma_2)]\left[\frac{C+mp}{r+m} - (1-\alpha)(1-\gamma) L\right]}\right)^{\frac{1}{\varphi(\sigma_2)-\theta(\sigma_2)}}
\end{aligned}$$

Notation

B	total value of debt
B_J	total value of junior debt
B_S	total value of senior debt
B^{τ}	value of payments in reorganization prior to maturity
B^M	value of maturity payments of debt
B^S	value of intermediate (solvency) payments to debt
c	percentage coupon
C	value of call option
C_L	value of down-and-out call option with barrier L
CB^{τ}	same as B^F but for a convertible bond
CB^M	same as B^M but for a convertible bond
CB	value of a convertible bond
E	total value of equity
E^{τ}	value of payments in reorganization prior to maturity of equity
E^M	value of maturity payments of equity
E^S	value of intermediate (solvency) payments of equity
F	value of (fictive) claim paying off 1 when ω hits the barrier
H	value of heaviside
H_L	value of down-and-out heaviside with barrier L
k	reorganisation costs
k^F	maximum possible realized reorganization costs at maturity
K	total value of a (fictive) claim to reorganization costs
K^{τ}	value of "intermediate" bankruptcy costs
K^M	value of maturity payments of the "reorganization claim"
L	the reorganization barrier (in terms of asset value)
m	number of warrants (Ch 1)/debt amortization rate(Ch 4)
n	number of shares
N	number of coupon payments
$N(\cdot)$	cumulative normal distribution function

F	total principal of debt
F_J	principal of junior debt
F_S	principal of senior debt
r	constant riskfree interest rate
t	current time
S	pre-maturity payments in solvency
T	maturity of debt, warrant, convertible, firm
TS	total value of tax shield ($= TS^S$)
V	value of firm
X	exercise price
v	state variable
W^P	a Wiener process under the objective probability measure
W^{Q^i}	a Wiener process under the probability measure Q^i
$\mathfrak{W}$	value of a warrant
$\varkappa$	tax rate
z	number of convertibles
α	$\frac{2(r-0.5\sigma^2)}{\sigma^2}$
β	rate at which free cash flow is generated
γ	standard deviation of estimate k
δ	fraction of equity held by convertible holders after conversion
ε	number of shares one convertible can be converted into
ϑ	inverse equity function
λ	market price of risk of assets (of W)
μ	expected return on assets
π	price
Π	pricing function
σ	standard deviation of return on assets
τ	time of default
φ	density function of normal distribution
$\Phi\{R\}$	payoff function for claim or security i
ω	value of assets
Ω	value of asset claim

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