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Financial Guarantees and Risky Debt

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Contents

A Summary of the Dissertation

This dissertation consists of six separate papers dealing with the valuation of financial guarantees and risky debt contracts. Each of these papers is independent and distinct, and can be read without necessarily consulting the others. The main theme is the valuation of financial securities by contingent claims analysis (CCA). It should be noted that the notation is not consistent throughout the dissertation. Furthermore, there are some appendices important for several papers but presented only in one paper. The dissertation is divided in two parts according to the following structure:

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Part I: Financial Guarantees

Paper 1: *Valuation of Financial Guarantees - A Presentation and a Critique*

The foundation by Merton (1977) that financial guarantees have similarities with put options has facilitated their valuation. By the use of option theory, preference-free pricing formulas of the guarantee value can be obtained. We begin the paper with a brief survey of the literature that followed the seminal work of Merton.

The first purpose of the paper is to derive a pricing formula for a deposit guarantee, when a bank's assets exhibit downward jumps due to extraordinary loan defaults. In this respect, we use the framework of Merton (1976), where a stock option is priced under the assumption of a jump-diffusion process for the underlying stock. In the valuation, we take into account an important difference between a put option and a guarantee, namely that the actual payment of the guarantee premium affects the underlying asset process, whereas the underlying stock process is independent of the value of the option. We show that the error of neglecting this issue, when pricing guarantees by option theory, can be significant.

We also show that, since the guaranteed part must make a payment of the premium from its assets, there is in existence a set of solvency ratios, where the premium payment will cause an immediate insolvency. The required solvency for making the payment increases in the standard deviation of the assets since the guarantee value, and thereby the premium, increases in the standard deviation. We present a critical relation between solvency ratios and asset standard deviations, which enables the guaranteed part to pay the fair premium without going insolvent. Since we use deposit insurance as an example of a financial guarantee, this critical border separates banks, healthy enough to be insured on an actuarial basis from banks that are in far too bad a shape to be included in the insurance system, unless they issue new equity or turn to less risky investments.

We also discuss other differences between option pricing and guarantee valuation. For example, the *meaning* of the exercise price is different between an option and a guarantee. Under the common CCA assumption of *perfect information*, it seems strange

that a guarantee contract can ever be in-the-money,[1] since this means that the guaranteed part is insolvent. A guarantor, perfectly observing this, would surely take some actions such as, for example, liquidating the guaranteed part and fulfilling its commitments. However, we do not consider this case in the valuation model, since it would, together with the assumption of a jump-diffusion process, complicate the derivation and the results. Instead, we focus on this issue in paper 2.

Paper 2: *Valuation of Deposit Insurance - An Alternative Approach*

This paper extends paper 1 in the respect that the guarantor, in this case a deposit insurance agency, will nullify the guarantee contract and liquidate the bank when it gets insolvent. The liquidation is assumed to involve some costs. Examples of such costs are legal, administrative and asset realization costs. In fact, since the guarantee contract will never get in-the-money, the guarantee will receive value only from these liquidation costs.

We value the deposit guarantee in two different models, for constant as well as for stochastic liquidation costs. The difference between these models is analyzed and we show that it is positive and non-linear in solvency.

A major advantage of the paper, compared to previous studies of deposit insurance valuation, is that it is consistent with the *perfect market* assumption in the sense that bankruptcies are possible in every point of time, without relying on audits revealing this. This is not the case in Merton (1978) and the numerous studies that have followed him.

Paper 3: *Financial Guarantees and Asymmetric Information*

Paper 1 and 2 consider CCA valuation of financial guarantees under the assumption of symmetric information between the guarantor and the guaranteed part. That is, the guarantor can fully observe the solvency process of the bank and nullify the guarantee contract when the bank becomes insolvent. In this paper we make the assumption that the guarantor cannot observe the solvency process of the bank, unless it carries out

[1] A put option on a stock is in-the-money when the stock price is below the exercise price.

audits. Since audits are often costly, and this cost burdens the guarantee value, the guarantor will search for an audit strategy, which minimizes the guarantee value.

We assume that the guarantor and the equity holders of the bank agree on a compensation contract, which induces the latter to signal insolvency voluntarily, when this event occurs. In this respect we are inspired by Merton (1993). The bank will signal insolvency if it receives a compensation from the guarantor equal to the equity value at the insolvency level. A crucial assumption for this to work, is that the bank actually believes that the guarantor will carry out audits that could reveal insolvency and thereby a liquidation of the bank. If they believe this, the equity value is a decreasing function of the audit probability.

After formulating this compensation contract, we carry out the pricing of the guarantee, which receives value partly from the audit costs and partly from the compensation. Since the compensation is decreasing and the audit cost is increasing in the audit intensity respectively, this setting makes it possible for the guarantor to choose an optimal audit strategy, which minimizes the guarantee value via the trade-off between audit costs and compensation.

The major advantage of this model over previous studies, à la Merton (1978), of deposit insurance valuation under asymmetric information, is that the bank will never operate under insolvency and that the guarantor can decide upon a realistic optimal audit strategy. The latter issue is also an advantage over Merton (1993), since the audit intensity does not appear in his guarantee valuation formula.

Part II: Risky Debt

Paper 4: *Valuation of Barrier Contracts - A Simplified Approach*

Many types of financial contracts can be classified as "barrier contracts". This description comes from their feature of allowing either contractual part to take some kind of action during the lifetime of the contract contingent on some prespecified event. In this sense, the deposit insurance contract in paper 2 can be regarded as a barrier contract.

The previous valuation models of barrier contracts are often considerably advanced and have tended to obscure the underlying economics. It is the path-dependence and stopping-time features that primarily make the derivation of these pricing formulas complicated. Our model simplifies this procedure by deriving the important first passage time distribution from a binomial model instead of using the reflection principle. It is then straightforward to use the convergence of the binomial distribution to normal distribution in order to calculate values of many types of barrier contracts in continuous time. For example, we value down-and-out and capped call options by this approach. We also model a financial guarantee contract with a reduced commitment as a capped put option. Here we observe that the inclusion of a barrier in a contract does not always decrease the value of the contract, which may contradict the intuition. We also find that two completely different barrier levels may generate the same contract value.

Paper 5: ***Valuation of Risky Debt in the Presence of Jumps, Safety Barriers and Collaterals***

This paper deals with different aspects of risky debt valuation with the CCA approach. The term, "risky", refers to the probability of default on the promised payment by the borrower.

In the first model of the paper we value a risky debt contract when there may be extraordinary losses in the borrower's asset portfolio. The model can be viewed as an application of Merton (1976) on Merton (1974). We have not found any previous studies of risky debt valuation where the borrower´s asset value is described by a jump-diffusion process. The main difference between our study and option valuation models following Merton (1976) is that we consider a complete market. This is the neccessary condition for a unique preference-free valuation of the debt.

The second model of the paper deals with risky debt contracts including a safety barrier. Thus, we are in need of the stopping-time framework from paper 4. Although there are several contributions on this issue, we investigate in more detail the impact on the debt value from changes of the barrier level. One interesting observation we have made is that the debt value is actually decreasing in underlying asset value for a certain range of barrier levels close to the promised payment. Moreover, there exists a set of barrier levels implying debt values exceeding the value of a risk-free bond.

The third model extends the basic risky debt model by allowing for a collateral or a guarantor in the contract. Thus, there will be a second underlying process, which together with the borrower´s assets, determines the repayment ability and thereby the value of the debt.

In the final model of the paper, we extend the second and third models by including both a collateral and a safety barrier in the debt contract. This contract is considerably complex and the derivation of the pricing formula makes use of several conditional probability distributions.

Paper 6: *Portfolio Selection and the Pricing of Personal Loan Contracts*

The CCA literature that follows Black and Scholes (1973), has mainly taken the underlying asset dynamics for given. Although it may be appropriate for stock options, we consider this assumption too simplifying with regards to personal loan contracts. It is obvious that the borrower's consumption-investment decisions affect his wealth process, on which the loan contract is contingent. Moreover, we believe that individuals actually have preferences to repay loans for different reasons such as the existence of reputational costs or legal penalties that affect the borrower in case of loan default.

In order to incorporate these features of the borrower into a valuation model of a risky loan, we derive the borrower's optimal consumption-investment rule in the presence of a "bequest function", reflecting the borrower's utility from having wealth at the maturity of the loan. When we have established this, we know the optimal dynamics of the borrower´s wealth process. This is used in the valuation model as the underlying process. We derive the pricing formula along the lines of Merton (1974) and investigate in detail the impact from the borrower's type on the value of the loan. The borrower´s type is characterized by his degree of relative risk aversion and preferences for repaying the loan.

One interesting observation is that the loan value is decreasing in relative risk aversion if the borrower has sufficiently high preferences for fulfilling the promised repayment. These investigations and the fact that the dynamics of the underlying process are consistent with utility maximization constitute the main contributions of the paper.

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Part I: Financial Guarantees

Paper 1: Valuation of Financial Guarantees - A Presentation and a Critique

1. Introduction

Merton (1977) was the first to show that financial guarantees can be valued by contingent claims analysis (CCA). This foundation has simplified the pricing of guarantees, since CCA formulas often end up with preference free closed form solutions. Alternative methods to value guarantees in an actuarial manner could be found in insurance mathematics.[1] However, these methods have some drawbacks compared to the CCA with respect to applications on *financial* insurances, since they do not necessarily rely on economic fundamentals such as equilibrium, agency preferences and arbitrage.

The first purpose of this paper is to derive a pricing formula for a deposit guarantee by the use of option theory. The valuation is undertaken under the assumption that the underlying asset in the guarantee contract follows a jump-diffusion process. This representation of the asset process was first formulated in Merton (1976) for the valuation of stock options. In the pricing formula we take the actual payment of the insurance premium into account. This has not been addressed in an adequate manner in the previous literature. This issue is one of several differences between options and guarantees.

The second purpose of the paper is to put forward such differences often ignored in the literature, and their implication for guarantee valuation by CCA. Therefore we briefly survey the literature of valuing financial guarantees by CCA and present our critique.

[1] See e.g. Gerber (1979), Beard et al.(1984) and Borch (1990).

The paper will be organized as follows.

Section 2 gives a brief discussion of the concept of financial guarantees. The major part of the academic interest in financial guarantees has been devoted to deposit insurance. This has motivated us to include a short presentation of this contract type and its practical implementation problems. This is done in Section 3. A brief survey of the contingent claims analysis literature of financial guarantees is presented in Section 4. In Section 5, we present our observations of some problems in the previous literature of financial guarantees and non-neglectable differences between guarantees and put options.

Section 6 presents a model for valuation of a deposit guarantee, where the value of the bank assets follows a jump-diffusion process. The value of a fair deposit insurance is given in numerical examples. We also show the numerical size of the bias that will occur if the valuation does not take into consideration the initial payment of the premium.

In Section 7 we continue the discussion of the problem that a payment of the initial guarantee premium may cause an immediate insolvency. We plot a critical relation between solvency ratios and standard deviations. This relation gives the minimum solvency ratio, for a given standard deviation, which enables the bank to pay the fair premium without going insolvent. A summary and some concluding remarks are given in Section 8.

2. The Concept of Financial Guarantees

Consider a financial transaction between a borrower and a lender. The lender may want to limit his risk and therefore engages a third part guarantor. In this context a financial guarantee is an instrument which transfers credit risk from the lender to the guarantor. A somewhat different, but analogous, situation is when the borrower buys a guarantee in order to be able to borrow at the riskfree rate. The guarantor commits himself to fulfilling the financial contract if the debtor fails to do so and charges the holder of the guarantee a premium as a compensation for this commitment. For this transfer of risk to be mutually beneficial, the guarantor must have a comparative advantage in risk bearing over the guaranteed part.

The need for a financial guarantee can also be illustrated by the following example.[2] Consider an agent who demands a default free insurance contract. The payout to this agent is not only contingent upon the event on which the agent is insured but also on the solvency of the insurance company. If the insured agent cannot costlessly hedge the insolvency of the insurance company on the capital market or diversify among several insurers there may be a need for a guarantee.[3] Thus, guarantees will be demanded if the costs of eliminating risks through alternative methods of obtaining the same assurance against default risk, exceed the costs of the guarantee.

The insurance literature on pricing of insurance contracts, see for example Beard et al. (1984) and Gerber (1979), is largely based on the law of large numbers, which is appropriate for e.g. life insurances. The use of this methodology when pricing financial guarantees may suffer from some drawbacks since it does not necessarily rely on economic fundamentals.

Instead, the use of CCA, which is based on arbitrage arguments and is consistent with economic equilibrium, may be a more appropriate technique. The advantage of this method, if the market is complete, is that the value of the guarantee will be invariant to the agents' preferences.

Merton (1977) was the first to use CCA in valuing financial guarantees on an individual basis and without dependence on historical loss data. He showed the analogy between a financial guarantee and a European put option. The value of a European put option, P, at the time for expiration, T, is:

$$P_T(A,D) = \max[D-A_T, 0]$$

where D and A represent strike price and stock price respectively. The expression could as well be valid for the value of a guarantee of a financial contract, where D and A represents the promised payment, which is to be guaranteed, and the collateral

[2] See also the motivation for guarantees in Merton and Bodie (1992) emanating from the distinction between customers and investors of financial intermediaries. A customer expects his payoff to be independent of the health of the intermediary, which is not the case for investors.

[3] Another way for the company to satisfy the customer's demand for a default free insurance policy is to raise an additional risk reserve, often in the form of equity. This alternative, however, may be inappropriate due to agency costs. See e.g. Jensen and Meckling (1976).

respectively.[4] At any time t<T the value of the guarantee is the discounted risk neutral expectation of this expression. The premium paid by the guaranteed part is actuarial if it is equal to the value of the guarantee.

3. Deposit Insurance

One of the most important types of financial guarantee, with respect to the size of the guaranteed balances as well as the academic interest, is deposit insurance. The academic interest for deposit guarantees has increased after the numerous failures in the U.S banking industry during the 1980s, where the U.S federal deposit insurance system exhibited a number of serious weaknesses. In this section we discuss the purposes and some aspects of the implementation of a deposit insurance system.

In principal, the main purpose of deposit insurance is to make deposits free of default risk. There are large information- and monitoring costs to be saved if depositors can be confident on the fulfilment of banks obligations.[5] As Merton (1977) points out, the advantages to depositors of using a bank instead of making direct market purchases of fixed-income securities are at hand only if deposits are riskless.[6] Also, if this purpose is achieved then the risk of bank runs, caused by public mistrust of the banking industry, is heavily reduced. In order to be fully credible in this sense, the insurance system ought to be managed by a governmental guarantee fund.[7]

The positive relation between the perceived risk and the expected return on individual investments makes it important to design the deposit insurance system in an incentive compatible way. In absence of a credible deposit insurance the interest rate offered to depositors of the bank would have to rise with the institution's leverage and with the

[4] In the case of deposit insurance, D and A represent the value of deposits and assets of the bank respectively.

[5] Gorton and Pennacchi (1990) present a rationale for deposit insurance emanating from the governmental goal of protecting small investors with respect to their information disadvantage relative to the informed larger investors.

[6] These advantages are e.g. smaller transactions costs, liquidity, and convenience.

[7] Benston et al.(1985), Sinkey (1992) and Merton and Bodie (1993) argue that there would in principle be no threat to the stability of the financial system from a run on banks if the run is not accompanied by a flight from deposits to currency.

riskiness of its portfolio.[8] These higher funding costs limit the incentives of the bank to increase its risk.

The existence of a credible deposit insurance system implies riskfree deposits and therefore no incentives for the depositors to monitor the investment policy of the bank and demand risk related deposit interest rates. Thus, a bank with completely guaranteed deposits can borrow at the riskless interest rate independent of its solvency level and the riskiness of its asset portfolio. Therefore, if the insurance premium paid by the bank is less than its actuarial value, the bank will gain from the implementation of a deposit insurance system. Thus, it is reasonable that the bank should be charged an actuarial insurance premium.

The experience from the U.S deposit insurance systems,[9] is that the purpose of preventing bank runs has been fulfilled. However, the system has exhibited a number of problems. The "flat premium"-construction[10] implies that relatively risky banks are subsidized on the account of relatively safe banks. This has probably encouraged higher risk taking by banks.[11]

Another problem of the U.S. deposit insurance system is the limitation of the incentives by the non-insured creditors to demand risk related interest rates.[12] This behavior comes from that fact that the most usual way of handling bank failures in the U.S. is the

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[8] This sensitivity of the bank's funding interest rate, to leverage, asset quality and duration gap, exists because rational depositors must demand compensation for the increasing risk that the bank will not be able to fulfil its obligations.

[9] The experiences from the U.S systems are covered in e.g. Kane (1985), Benston et al. (1986), Dotsey och Kuprianov (1990) and Pierce (1991).

[10] "Flat premium" means that all banks pay the same proportion of their deposits as insurance premium, independent of leverage and portfolio risk. In 1991 the premium was 19.5 cents per 100 dollar of deposits. See Sinkey (1992).

[11] Many academics argue that the incentive of banks to invest in riskier projects is the central problem of deposit insurance, see e.g. Kane (1985, 1990), Benston et al (1986) and Boot and Greenbaum (1992). However, some empirical studies, e.g., Marcus and Shaked (1984) and Duan et al.(1992) have shown that the problem may not be so widespread.

[12] E.g. the holders of certificates of deposits have not been formally insured by the U.S. deposit insurance system.

purchase and assumption strategy.[13] Further, the inclination of the guarantor to delay terminal sanctions on insolvent banks and actually bail out very large banks in trouble has probably produced a moral hazard behaviour. This forbearance by the guarantor has been a major reason for the deficit in the U.S. deposit insurance system.[14]

Thus, there have been a number of problems in the U.S. banking industry due to the existence of a deposit insurance. According to the discussion above of risk related pricing of guarantees, actuarial insurance premia would limit the risk taking behavior by banks. However, for this method to work it is important that the guarantor is able to measure the value of the deposit collateral. This may be difficult to achieve in practice due to the opaqueness of a major proportion of a bank's asset portfolio.[15] Another factor that normally could limit excessive risk taking is the charter value of the bank,[16] which will be jeopardized at too risky activities. However, a bank close to insolvency may have incentives to "take a last chance" and increase its risk in order to get into a safe solvent position and keep its charter, see Merton (1978).

Because of the numerous problems with the U.S. deposit insurance system there is some disagreement among economists whether deposit insurance is the best way of achieving the objective of default free deposits. A large part of the alternative proposals in the academic literature for achieving this goal suggest collateralization of the deposits by high quality and liquid assets, like short-term Treasury securities and highly rated commercial papers. These ideas could be grouped under the label "safe bank" or "narrow bank" proposals.[17] This means that other financial activities, including risky lending,

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[13] This means that the federal deposit insurance agency, FDIC, often ends up protecting even the non-insured creditors when resolving bank failures, see e.g. Pierce (1991) for a discussion of this matter.

[14] See e.g. Benston et al. (1986) and Boot and Greenbaum (1992) for a discussion of this time consistency problem.

[15] See e.g. Merton and Bodie (1993) and Pierce (1991) for a more detailed discussion of the problems of risk-based capital requirements and deposit insurance premia.

[16] The licence for running bank business and issuing deposits.

[17] See e.g. Kareken (1986), Pierce (1991), Boot and Greenbaum (1992) and Merton and Bodie (1993).

must be financed on the capital market.[18]

Since the theoretical rationale for banks is that it is efficient to finance the creation of illiquid loans with demand deposits, the safe bank proposals actually suggest the breaking up of the traditional institutional structure of banks. The technological progress and the associated explosive development of new markets and institutions, like for example commercial papers and money market mutual funds (MMFs), have implied that this rationale for banks is less clear. Gorton and Pennacchi (1992) provide empirical evidence that non-bank producers of liquidity like MMFs can successfully compete with banks.[19]

The criticism of the safe bank proposal mainly argues that the deposit insurance system would do fine if it moves towards risk adjusted insurance premia.[20] It has been argued that the costs of synergy losses and restructuring the banking industry according to the reform proposal would be very high. Further, the earnings on the riskfree collateral may be insufficient to pay the cost of providing deposit services. Another problem is that the volume of riskless instruments may not be sufficiently large for collateralization of all deposits. Since it is not the purpose of the present paper to discuss the arguments for and against the alternatives to deposit insurance in more detail we refer to the cited papers above for a thorough argumentation.

4. A Survey of the Literature

In this section we present some important studies in the CCA approach to value financial guarantees. A majority of the papers are concerned with deposit insurance. The seminal study by Merton (1977), briefly presented in Section 2, was followed by numerous

[18] According to Merton and Bodie (1993) the recent explosive development of nonbank credit sources are enough to satisfy the credit demand from households and smaller firms that do not have direct access to the capital market. Examples of such sources are "junk" bonds, securitized loans and nondepository finance companies.

[19] Merton and Bodie (1993) also support this view. However, Benston and Kaufman (1993) argue that there still exist significant synergies from deposit-taking, loan-making and other financial services.

[20] Examples of critical points on the safe bank reform could be found in Diamond and Dybvig (1986), Kaufman (1986) and Benston and Kaufman (1993).

studies, also using the CCA approach, of which a few are briefly presented below.[21]

The opaqueness of a large part of a bank's asset portfolio is an important issue in deposit insurance valuation. In Dahlfors and Jansson (1994c) we assume asymmetric information between the bank and the guarantor. That paper draws heavily on Merton (1978), which was the first study to take the surveillance problem into account explicitly. Merton (1978) derives a model for evaluating a fair one-time payment by banks for deposit insurance. In this perpetual model the guarantor carries out costly audits of the banks at random time intervals. In this setting the time to maturity is stochastic since the bank is liquidated when an audit reveals insolvency, which can occur at any point of time. The value of the guarantee comes partly from the expected discounted audit costs and partly from the expected discounted payout by the guarantor at the liquidation of the bank. The consideration of the surveillance problem in deposit insurance and inclusion of audits in the valuation model was followed by many studies of which Ronn and Verma (1986), Thomson (1987), Pennacchi (1987a,b), Acharya and Dreyfus (1989) and Allen and Saunders (1993) are surveyed below.[22]

Ronn and Verma (1986) present an empirical valuation methodology for arriving at estimates of deposit insurance premia using only market data.[23] The bank asset value and volatility is determined implicitly through market valuation of the equity as a call option on the assets. They argue that it is the future stochastic behavior of assets that should matter for the price of the guarantee. A somewhat different approach for using market information is presented in Thomson (1987). He shows that if uninsured depositors exist, the market's ex ante estimate of the fair value of deposit insurance can be observed from using market data on for example the risk premium on certificates of deposits.

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[21] Examples of other studies not presented here, are Sosin (1980), Flannery and James (1984), Pyle (1986), Selby, Franks and Karki (1988) and Crouhy and Galai (1991).

[22] Flannery (1991) also analyzes the problems of opaque bank assets but uses a different framework. He specifies his model with an uncertain volatility and concludes that uncertainty about the true volatility causes the guarantor to misprice the deposit insurance.

[23] Their analysis can be viewed as an extension of Marcus and Shaked (1984), who focus the attention on whether the flat premium construction of the U.S. deposit insurance system is fair. They estimate from bank-stock market data fair values of deposit insurance for 1979 and 1980 and find a majority of larger banks overcharged for the insurance.

Pennacchi (1987a) generalizes the model of Merton (1978) and considers alternative policy assumptions concerning the guarantor's pricing of insurance and methods for handling bankruptcy of banks. If a bank goes bankrupt, the choice of mergers allows the guarantor to recover charter value of the failed bank, which is not the case if direct payment to the depositors is chosen. However, a merger policy may, according to Pennachi´s model, give rise to a greater risk-taking incentive by banks. Another paper in this division of studies by the same author, Pennacchi (1987b), evaluates the extent of over or underpricing of deposit insurance by examining how the degree of government regulatory control over a bank can affect the valuation of the guarantor's liability. He studies the cases of full control, that is, following an audit regulators can force a capital deficient bank to add more capital, and no control, which means that regulators can close a bank only when it has negative net worth and cannot influence the bank's capital structure before that. His empirical results indicate that for a large sample of banks, the deposit insuring agency overcharges the banks if full control is at hand and undercharges if actual control is closer to no control.

Acharya and Dreyfus (1989) also analyze optimal policies for closing a bank. The policy decision is a function of the deposit interest rate, the rate of flow on bank deposits, the economy's riskfree interest rate and the regulator's administration/audit costs. Allen and Saunders (1993) model deposit insurance as a callable put in order to incorporate the ability of the guarantor to control the timing of the exercise of the guarantee via bank closure decisions.

Fries and Perraudin (1991) differs from the above mentioned "guarantee control" literature in one important way - they exclude audits in their analysis of deposit guarantees. They assume symmetric information between the guarantor and the bank and thereby avoid possible problems with audit strategies. The argument for doing this is that the guarantor should have the same possibility as investors to infer the true value of bank assets from stock market data. The techniques used permit a unified analysis of the shareholders' and guarantor's preferred closure policies. The paper shows that the value of the deposit insurance is the difference between the stock market value of the bank under unlimited liability and the value of the bank´s equity in the presence of a guarantee.

Since we deal with effects on debt values due to involvement of third party loan guarantors in Dahlfors and Jansson (1994b), Jones and Mason (1980) is an important paper form our point of view. They value different types of loan guarantees, applied on

callable and non-callable coupon debt, with respect to whether the guarantee is full or partial. They also present numerical solutions of the resulting guarantee values.

One study that deals with discontinuous sample paths and is thereby important for the present paper and Dahlfors and Jansson (1994b) is Cummins (1988), who values commitments of insurance guarantee funds. The analysis is carried out both for ongoing insurance with or without jumps in underlying asset values and in policy cohorts, where liabilities eventually run off to zero as claims are paid. His results indicate how the use of risk-related insurance premia can lessen the adverse risk incentives of firms arising from the use of non-risk-related premia.

5. Critique and Problems

This section will discuss some important differences between a put option and a financial guarantee and consequences of using option theory in order to value a financial guarantee.

(i) The price of a stock option is independent of the option value. This is not the case for the value of a guarantee, which will in many cases influence the underlying asset value process through the initial payment of the insurance premium.[24] This comes from the fact that the payment of the guarantee premium reduces the assets of the guaranteed business *from the guarantor's point of view.* Of course, from an objective point of view, the value of the assets of the guaranteed part is unchanged, when a guarantee is aquired. However, the guarantor cannot include the value of his own commitment when pricing the guarantee. Notice, that in the case of deposit insurance, the value of the guarantee cannot even objectively be included in the asset value of the bank, since the bank functions only as an intermediary between the depositors and the guarantor in this respect. The only actual value of the guarantee for the bank is the opportunity to get riskfree funding.

If this difference between a put option and a financial guarantee is not taken into consideration the value of the financial guarantee will, as we show in Section 6, be

[24] This is not at hand in the case where the claim holder acquires - and pays for - the guarantee. That case would not affect the initial value of the guaranteed business.

underestimated. Furthermore, there exists a set of combinations of solvency levels and solvency standard deviations implying too large premia in relation to the solvency to be charged. This feature of financial guarantees, that a guaranteed part with a combination of low solvency and high portfolio risk gets insolvent when paying a fair premium, is important and is discussed in detail in Section 7. These two problems has to our knowledge not been considered, or treated in an adequate manner, in the previous literature on financial guarantees based on option theory. In the deposit insurance literature, for example Merton (1977), Marcus and Shaked (1984) and Cummins (1988) the guarantee premium is expressed as a pure European put option. However, Pennacchi (1987a,b) treats the guarantee premium as a continuous payment from the bank to the governmental guarantor. The model in Section 6 assumes the premium to be paid up front, that is, at the initiation of the deposit insurance contract. This means a reduction of the initial value of the bank assets with the amount of the premium. One advantage of the assumption of an initial premium payment over a continuous payment, besides being more realistic, is that it captures the feature of whether the bank can actually afford to pay the guarantee premium.

(ii) A second difference between a put option and a guarantee, closely related to the first, is that the dynamics of the stock are independent of the expiration of the option whereas guarantees, like deposit insurance, has the property that when it expires "in the money" the bank will be closed. This would probably affect the risk taking behaviour of the bank and thereby the dynamics of the underlying assets.

(iii) Since a bank may become insolvent at any time during the life-span of the guarantee contract the pricing formula should take path-dependence into consideration. Therefore, it is seldom adequate to value a financial guarantee as a European put option, which is path-independent and does not permit insolvency except at the time for renegotiation of the contract.

(iv) To value the guarantee as an American put option would assuredly admit insolvency at all points of time. However, there is a difference between an American put option and a deposit guarantee regarding the right to exercise. A holder of an American put option can exercise the option whenever he finds it profitable. But the holder of a financial guarantee is normally not in the position of exercising the contract.[25] This is the privilege

[25] Note that there may exist special situations where the equity holders of the debtor firm, i.e. the holders of a loan guarantee, have preferences for self liquidation. This can occur if the opportunity

of the issuer of the contract, that is, the guarantor. The guarantor has no incentive to let the solvency process cross the insolvency barrier if it has the right to prevent this. Therefore, the guarantor will nullify the guarantee and close the bank exactly when the bank goes bankrupt.[26] Thus, the model leading to formula [5] in Section 6 would suffer from inconsistencies *if* the guarantor had the right to nullify the contract prior to maturity since the part of the probability distribution where the guarantee receives a value can never be reached.[27]

A practical problem with deposit insurance models is the estimation of the market value of the bank assets and the standard deviation of the returns of the assets. These two variables are difficult to observe directly because the information problems associated with a main part of the assets may not admit market pricing. However, these variables can be estimated by using the fact that equity can be valued as a call option written on the assets of the company. Since the equity of a bank in most cases is traded on a stock market it is possible to get to the value and return of the assets.[28] It is hard to see why a governmental guarantor should not have the same possibilities to infer from stock market prices the value of the bank asset portfolio as investors in general. The same can be said about the information mirrored by the market pricing of non-insured bank liabilities, like for example certificates of deposits. Further, the development of secondary markets for "bank asset backed securities" makes it easier to value bank assets on market conditions.

In the deposit insurance literature it is assumed that assets are traded continuously. This assumption is essentially motivated in order to value the guarantee by option theory. But a common feature in the literature of deposit insurance models is that audits are required to get knowledge of the market value of the bank assets. Thus, the assumption of market priced bank assets and audits are incompatible. This problem is present in for example Merton (1978).

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costs, plus reputation costs, of operating an insolvent firm is sufficiently high. See Allen and Saunders (1993) for a discussion of this issue. Self liquidation for opaque intermediaries at sufficiently high solvency levels due to costs associated with regulation and agency problems is briefly discussed in Merton (1993).

[26] In absence of reorganization costs and forbearance.

[27] This is valid in a pure diffusion model. In a jump-diffusion model the guarantee can receive a positive value due to "jumps" into insolvency.

[28] Marcus and Shaked (1984) as well as Ronn and Verma (1986) use this technique to study if the U.S deposit insurance has been over or under priced relative to its fair value.

In two other papers, Dahlfors and Jansson (1994a,c), we consider these problems and formulate alternative approaches to deposit insurance valuation.

Dahlfors and Jansson (1994a) deal with an economy where the guarantor and the bank are equally informed about the financial health of the bank. When the solvency process hits the insolvency level the guarantor closes the bank, realizes the assets and pays off the depositors. The guarantee gets a positive value due to the costs of closing or reorganizing the bank.

In Dahlfors and Jansson (1994c) the guarantor is imperfectly informed about the solvency of the bank and needs information-revealing devices such as audits. A contract is designed to induce the bank to reveal relevant information voluntarily. The guarantee value will then consist of two components, the audit costs and the compensation to the equity holders.

6. A Pricing Model of Deposit Insurance

In this section we derive a valuation formula for deposit insurance when the asset value of the bank may suffer from extraordinary reductions. Our study exploits the martingale connection of an arbitrage-free price system first observed by Cox and Ross (1976) and formalized by Harrison and Kreps (1979). Due to the possibility of jumps, we show that the economy in the model is not complete and we therefore conclude that there exists infinitely many probability measures that admit arbitrage free pricing. In order to get a unique guarantee value we must make assumptions about the "market price" for jump risk.[29]

Consider a deposit insurance contract formulated at time t=0 between the guarantor and the bank, which expires at time t=T.[30] The meaning of the contract is that the guarantor is commited, at time T, to pay the difference between the value of the guaranteed deposit value, D, and the collateral, A, the assets of the borrower if this is positive. Thus the

[29] For a detailed discussion of these issues see Dahlfors and Jansson (1994b).

[30] In a guarantee contract in general, the guaranteed party could be the debtor as well as the creditor. In our deposit insurance model the debtor, i.e. the bank, is assumed to be the guaranteed party, which is mirrored by the reduction in the collateral value due to payment of the insurance premium.

guarantor is committed to pay: $\max[0,D_T - A_T]$, where D and A is non-negative. This contract is analogous to a European put option written on the security A with strike price D.

At the time of the initiation of the contract, $t<T$, the guarantor charges the bank an insurance premium, β. In order to be an actuarial premium, β must equal the derived value of the guarantee, denoted $P(A,D,t,\beta)$. Also notice that the guarantor has no right to nullify the contract prior to time T no matter how far the underlying asset may decrease. This is an important feature of the contract in order to be valued by the model below.

The following assumptions are made:

Assumption 1: The assets of the bank are traded continuously in time on frictionless markets.

This assumption is important for the construction of a portfolio, which replicates the guarantee. However, if bank assets are not traded, and therefore unobserved by the guarantor, the use of the CCA approach depends on the existence of another traded security of the bank. If such an asset does not exist other securities that move closely together with the bank assets could be used. For a further discussion of this issue, see Merton and Mason (1985) and the appendix in Dahlfors and Jansson (1994c).

Assumption 2: There exists a riskfree,[31] both in terms of market and default risk, bond which has the following dynamics:[32]

$$\begin{aligned} dB_t &= rB_t dt \\ B_0 &= 1 \end{aligned} \tag{1}$$

where r = riskfree rate of return, known and constant through time.

[31] Short-term government securities can be considered as close to being riskfree assets since they experience relatively small price fluctuations and holders are guaranteed interest payments and repayment of the principal at maturity.

[32] It should be noted that the combination of assumptions 1 and 2 may threaten the rationale for banks and deposit insurance according to the discussion of the safe bank proposal in Section 3.

Assumption 3: The dynamics of the assets of the bank, which serve as collateral, is described by the following jump-diffusion process:

$$\begin{aligned} dA_t &= \alpha A_t dt + \sigma A_t dW_t + kA_{t-}dN \\ A_0 &> D_0 + \beta \end{aligned} \tag{2}$$

where A_t = the asset value at time t.

α = the instantaneous expected return on the assets between the jumps, $\alpha > r$.

σ = the instantaneous standard deviation of the return of the assets.

W_t = a standard Gauss-Wiener process that is a martingale with respect to a given filtered probability space $(\Omega, \Im, \{\Im_t\}, P)$ for the time interval $[0,\infty)$.

N = a Poisson process with $\{\Im_t\}$-intensity λ, which means we can interpret λ as the expected number of jumps over a small time interval, dt, conditioned on the filtration $\{\Im_t\}$, i.e., $E[dq_t|\Im_t] = \lambda dt$.[33]

k = a constant, $-1<k<0$, representing a proportional reduction of the assets in the event of a jump.

ß = guarantee premium, in dollars, that is charged at the initiation of the guarantee contract at time zero.

D_0 = initial value of deposits at time zero.

The deterministic part of the asset dynamics represents the expected rate of return between the jumps. The $\sigma A_t dW_t$ part describes the instantaneous part of the unanticipated return due to normal price vibrations, and $kA_{t-}dq$ describes the part due to abnormal value vibrations.[34] The proportional feature of k means that the size of the abnormal value reductions increases with business volume, which seems realistic. Proportionality of the dynamics in asset value captures the limited liability property by assigning a zero probability to negative asset values.

[33] Hence, the filtration $\{\Im_t\}$ is generated by the Wiener process, W and the Poisson process N. W and N are assumed to be independent.

[34] See Merton (1976) and Dahlfors and Jansson (1994b) for a more detailed analysis of contingent claims pricing with jump-diffusion processes.

One motivation for the inclusion of a jump component in the asset process is the existence of default on large loans in the bank's asset portfolio.[35] Another argument is the empirical observation that the tail of the return process distribution is fatter than would be the case if generated by pure lognormality, see for example Ball and Tourus (1983).[36] Further, we assume the insurance premium to be paid in advance, which means that the initial value of the assets is reduced by this premium, β.

The asset process is assumed to be described by a Right-Continuous Left-Limits sample path. When a Poisson event occurs in time t, that is, some extraordinary information on the assets arrives, the asset value is influenced as follows:

$$\Delta A_t \triangleq A_t - A_{t-} = (A_{t-} + kA_{t-}) - A_{t-} = A_{t-}k \tag{3}$$

where A_{t-} = the left limit of A at time t, that is, the asset value just before a jump.

Assumption 4: The guaranteed deposits are determined by the following process:

$$\begin{aligned} dD &= \mu D_t dt \\ D_0 &> 0 \end{aligned} \tag{4}$$

Where D_t = the value of deposits in time t

μ = the return on deposits, which is the riskfree rate minus the rate paid in the form of liquidity service, $\mu<r$.

Thus we assume that the guaranteed deposits are deterministic. Further, we assume no net withdrawals of deposits, which otherwise would have been reflected by a corresponding change in the asset dynamics [2].

Let $P(A,D,t,\beta) \varepsilon C^{2,2,1}$ denote the value of the deposit guarantee at time t. This function has similar properties to a European put option written on the asset A with a strike price D. The value of this guarantee at the time of expiration T, P(A,D,T), is $\max[D_T-A_T,0]$. In

[35] This motivates: $-1<k<0$.

[36] An alternative specification would be the use of so called Auto Regressive Conditional Heteroscedasticity models (ARCH) and its Generalizations (GARCH) see Engle (1982) and Bollerslev (1986). However we do not consider this approach in this paper.

order to value the guarantee at an arbitrary point of time we use a framework formalized by Harrison and Kreps (1979). With application of a general martingale pricing model and Girsanov's theorem we ensure an arbitrage free price system. To achieve this we must show that the discounted price processes, A and B, are martingales.[37] This is done by normalization by the riskless interest rate and the change from the original probability measure, P, to a risk neutral probability measure, Q. Given the existence of such a martingale measure, Q, the measure transformation, by application of the Girsanov theorem, ensures a preference-free valuation of the guarantee.

However, our model is not complete since there are too many risk sources in relation to existing securities. Therefore, there exist infinite numbers of martingale measures Q which martingalize the asset process, A. An arbitrage pricing of the guarantee is achieved if we can estimate the "market price" for jump risk. This price can be inferred from another traded security of the bank, for example, traded equity.

We use the linear homogenity of $P(A,D,t,\beta)$ in (A,D) and formulate P as a function in two arguments. This is done according to the following definition:

Definition 1: Let $X_t \equiv A_t/D_t$, be the asset to liability ratio, that is, the solvency of the bank, with $(X_0-\pi) = (A_0-\beta)/D_0$. Thus, $\pi = \beta D_0$ is the paid insurance premium per dollar of insured liabilities. The function $P(X,\pi,t)$ represents the value of the financial guarantee per dollar of insured liabilities at with T-t units of time to expiration of the guarantee.

The value at time t=0 of a deposit guarantee with T periods to expiration, which is a function of π, is given by:[38]

$$P(t=0,X_0,\pi) = \\ = \sum_{N=0}^{\infty} Po(\cdot)\left[e^{-(r-\mu)T}\Phi(d_N) - (X_0-\pi)e^{-\lambda(1+g)kT}(1+k)^N\Phi(d_N-\sigma\sqrt{T})\right] \tag{5}$$

where $\Phi(\cdot)$ denotes the standard cumulative normal distribution, Po() denotes the Poisson frequency function with intensity $\lambda(1+g)T$ and

[37] The process D will serve as a deterministic exercise price.

[38] The derivation of this expression is given in appendix.

$$d_N = \frac{-\ln(X_0 - \pi) - (r - \mu - \lambda(1+g)k - \frac{1}{2}\sigma^2)T - N\ln(1+k)}{\sigma\sqrt{T}} \qquad (6)$$

Thus, [5] expresses the value of the deposit guarantee as a function of the de facto paid premium π. To get an insurance premium that is actuarially fair, the up-front deduction from the assets of the guaranteed part must be equal to the net liability of the guarantor associated with the insurance obligation, that is, $P(t=0,X_0,\pi) - \pi = 0$. The numerical solution to this equality, that is, the actuarial value of the financial guarantee, is received through iterative estimation according to $|P(\cdot) - \pi| < \varepsilon$, where $\varepsilon > 0$ and is arbitrarily small. The existence and uniqueness of an actuarially fair premium is verified by treating it as a fix point of the value of the insurance. Assume a positive time to maturity, $T-t > 0$. If $\pi = 0$ then $P(\cdot) > \pi$ must follow. Further, we know that $0 \leq$ òP/òπ $= \Phi(\ ) < 1$. Therefore, there exists a point, π^*, such that $P(\pi^*) = \pi^*$.

This premium will always be greater than a premium derived under ignorance of initial deduction from the underlying assets. It is crucial to take into consideration this important difference between valuation of stock options, where the underlying procss is independent of the price of the option, and valuation of financial guarantees, where the underlying process in many cases is initially reduced by the guarantee premium.[39] Otherwise, the value of the financial guarantee will be underestimated. Our numerical examples below indicate a serious bias, particularly in cases of low solvency and/or high variance in the assets of the guaranteed part.

Notice, that in order to be feasible, the premium per dollar of insured deposits must be less than the current value of solvency of the bank minus the insolvency level, that is, $\pi < X_0 - 1$. Otherwise the initial charge of the premium will cause an immediate insolvency of the guaranteed part. This situation must, of course, be considered and ruled out. This is an interesting problem since it means that agents close to insolvency may not afford to be insured. We show how severe this issue may be in Section 7.

[39] Note that this is the case if the debtor is the guaranteed party. An alternative way of modelling the payment of the premium, though not so realistic, is the continuous reduction of the underlying asset.

6.1 Numerical Examples

The following tables show numerical examples of fair insurance premia per dollar of deposits calculated according to formula [5] for different parameter values. The tables also express premia calculated without considering the initial payment of the premium from the asset process. These values, called non-actuarial premia in the discussion below, are given in parantheses and the "bias" denotes their deviation from the fair premium. In all the examples below, $r=0.1$, $\mu=0.08$, $k=-0.1$ and $T=1$. All parameters are expressed on an annual basis. For example $\sigma=0.2$ means that the mean deviation from the expected value of the solvency is 20 percent per year and $\lambda' = \lambda(1+g)T = 1$ implies one expected number of jumps per year.

Table 1

$\sigma=0.1$	No jumps	$\lambda'=1$	$\lambda'=2$	$\lambda'=3$
X=1.5	$2.72\cdot10^{-7}$ $(2.72\cdot10^{-7})$ Bias: ≈0	0.00036451 (0.00036316) Bias:$1.4\cdot10^{-6}$	0.00153583 (0.0015179) Bias:$1.8\cdot10^{-5}$	0.0034188 (0.0033460) Bias:$7.2\cdot10^{-5}$
X=1.2	0.0008812 (0.0008643) Bias:$1.7\cdot10^{-5}$	0.0082113 (0.0075770) Bias:$6.3\cdot10^{-4}$	0.0167437 (0.0147974) Bias:0.00195	0.0256644 (0.0219344) Bias:0.00373
X=1.1	0.0072851 (0.00640316) Bias:$8.5\cdot10^{-4}$	0.0246573 (0.0196851) Bias:0.00497	0.0405332 (0.0305525) Bias:0.0148	0.0554174 (0.0401236) Bias:0.0153

Table 2

σ=0.2	No jumps	λ'=1	λ'=2	λ'=3
X=1.5	0.00146751 (0.00144837) Bias:$1.9\cdot10^{-5}$	0.0037759 (0.003682) Bias:$9.3\cdot10^{-5}$	0.0066205 (0.0063822) Bias:$2.4\cdot10^{-4}$	0.0098534 (0.0093957) Bias:$4.6\cdot10^{-4}$
X=1.2	0.0205529 (0.0176197) Bias:0.00293	0.0303809 (0.0252912) Bias:0.00510	0.039937 (0.03246748) Bias:0.00747	0.049244 (0.0392527) Bias:0.010
X=1.1	0.051008 (0.0362871) Bias:0.01472	0.0659348 (0.0456708) Bias:0.02026	0.0798844 (0.0541476) Bias:0.02574	0.09304 (0.061950) Bias:0.03109

Table 3

σ=0.3	No jumps	λ'=1	λ'=2	λ'=3
X=1.5	0.0135247 (0.0127105) Bias:$8.1\cdot10^{-4}$	0.0175799 (0.0163581) Bias:0.00122	0.0217567 (0.0200609) Bias:0.00170	0.0260247 (0.02379548) Bias:0.00223
X=1.2	0.0627416 (0.0482324) Bias:0.01451	0.071758 (0.054493) Bias:0.01726	0.080500 (0.0604767) Bias:0.020	0.0889925 (0.0662198) Bias:0.02277
X=1.1	0.114603* (0.0730858) Bias:0.04152	0.126247* (0.0798096) Bias:0.04644	0.13739* (0.0861865) Bias:0.05120	0.148083* (0.09226539) Bias:0.05582

We can observe that the premium is increasing in σ and λ' and decreasing in solvency, X. Further, the difference between actuarial premia and non-actuarial premia is significant, especially in the jump cases. Naturally, the bias decreases in X and increases in λ', which is also illustrated in Figures 1 and 2 below, since a low X and a high λ' means a large premium payment.

Note that the pair σ=0.3 and X=1.1 generate premia which is not feasable, since the payment of the premium will reduce the initial solvency from 1.1 to less than one, which implies insolvency. This problem is treated in more detail in the next section.

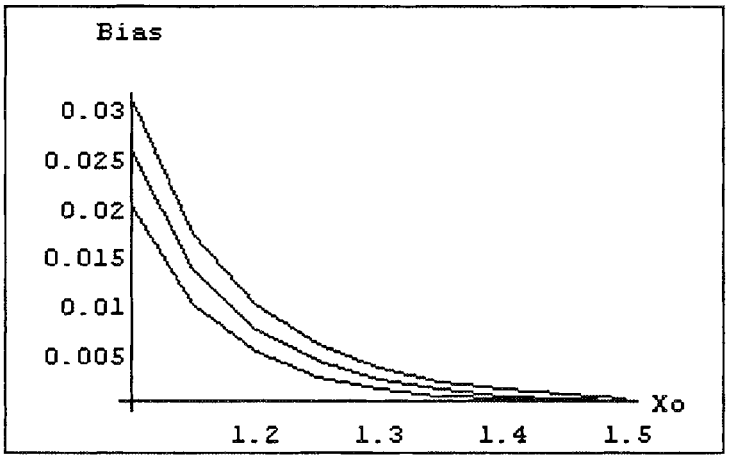

Figure 1

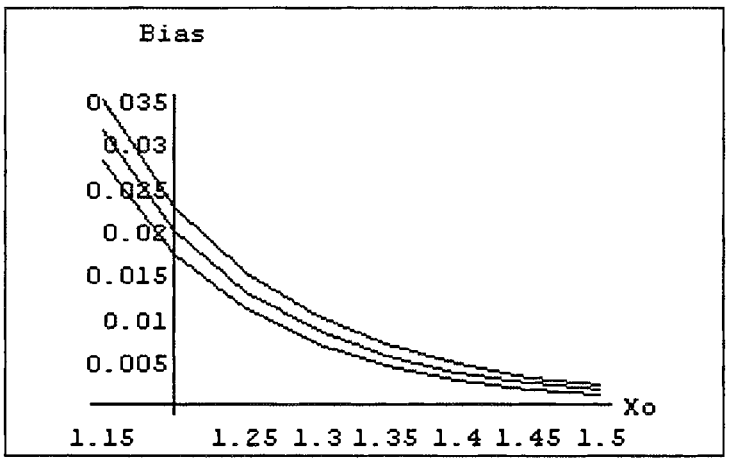

Figure 2

In Figures 1 and 2 we visualize the effect on the bias from changes in standard deviation and jump-intensity. The bias curves in the two figures are drawn for three different jump-

intensities, namely λ'=1,2,3. In Figure 1 we have that σ=0.2 and in Figure 2 that σ=0.3. The other parameter values are the same as in the tables above.

As we pointed out above, the bias is increasing in λ' and σ, which is verified by the figures since the lowest bias curves are for λ'=1 and the highest are for λ'=3.

In option pricing the standard deviation and, if jumps are likely, the jump-intensity are often non-observable variables. This problem is certainly present also in deposit insurance valuation due to the opaque nature of a large part of a bank's assets. This implies estimation problems. For example, if a standard deviation of 0.1 is used in the valuation formula when the true standard deviation is 0.3 then it can be observed from the tables that, for a solvency of 1.2, the under-estimation of the guarantee value will be 0.06274-0.00088=0.06186, for the no jump case. The estimation error will of course be even worse if the solvency process in reality takes unanticipated jumps.

7. Critical Border of (σ,X_0)

As we briefly mentioned in Section 6 there is a set of pairs (σ,X_0) generating fair premia causing immediate bankruptcy, that is, $X_0-\pi < 1$. In this section we introduce the concept "critical border", which is the set of pairs (σ,X_0) generating premia, when charged make the bank exactly insolvent. Here follows a more formal definition of the critical border.

Definition 2: The *critical border* connects solvency ratios and standard deviations, such that:

(i) $P(X_0,\pi,\sigma) = \pi$

and

(ii) $X_0-\pi = 1$

Condition (i) implies that the premium is fair, that is, the premium is equal to the value of the guarantee. Condition (ii) gives the minimum solvency level required to pay the premium without beeing insolvent.

Thus, the critical border separates feasible pairs of standard deviation and initial solvency ratio from infeasible pairs. In the figures below the feasible pairs are located above the border and the infeasible pairs on and below the border.

In the following figure, which plots the critical border for $\sigma \in [5\%,30\%]$, we can observe a positive slope of the curve, which means that if the standard deviation of the bank assets increases, the required solvency must also increase. Furthermore, this slope is increasing in time to maturity, τ, and decreasing in the riskfree rate of return, r. In the figures below $\tau=1$ and $r=0.1$. The other parameter values are $\lambda'=1$, $\mu=0.08$ and $k= -0.1$.

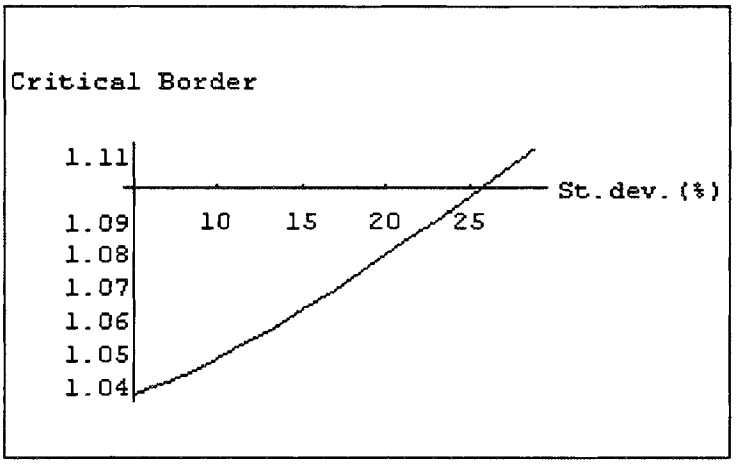

Figure 3

In Figure 4 below, the critical border is shown for different jump parameter values. The lower curve is the no-jump case. The other curves are valid for $\lambda'=1$, $\lambda'=2$ and $\lambda'=3$ (the upper curve).

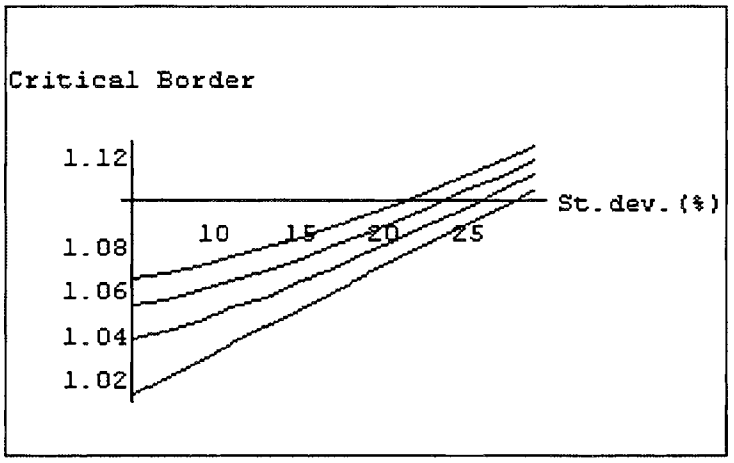

Figure 4

From the figure we observe that if the assets of the bank have a σ of, for example, 0.25 the guaranteed part must have an initial solvency of at least 1.089 (in the no-jump case), 1.097 (in the jump case for $\lambda'=1$), 1.105 (in the jump case for $\lambda'=2$) and 1.112 (in the jump case for $\lambda'=3$) in order to be able to pay the fair premium and be insured. Notice that the intercept of the critical border is increasing in the jump-intensity. The intercept for the no-jump case is 1.0, because when $\sigma \rightarrow 0$ the guarantee value becomes zero and no initial payment of the premium has to be done. For the jump cases the intercept is located above 1.0, because the jump risk implies a positive guarantee value, even when $\sigma \rightarrow 0$, and therefore an initial payment of the premium. Furthermore, the slope of the critical border is increasing in time-to-maturity, τ, and decreasing in the interest rate, r.

Note that the critical border is derived for a given time-to-maturity of the deposit guarantee contract, $\tau=1$. A smaller τ means that the slope of the critical border becomes flatter, which in turn means a lower required initial solvency for a given standard deviation and vice versa. Thus, a shorter life-span of the guarantee contract makes it easier for banks to pay the actuarial premium. If $\tau \rightarrow 0$, then the critical border would have an intercept of 1.0 and a zero slope, in the limit. Then all solvent banks would be able to able to pay the premium, regardless of their solvency and standard deviation.

A feature of the critical border, not observable in the figures, is that it converges to 2.0 for $\sigma \rightarrow \infty$ since 1.0 is the highest possible payout for the guarantor. A similar upper limit for the critical border is present in the alternative approach to deposit insurance pricing, presented in Dahlfors and Jansson (1994a), however determined by liquidation costs.

Since the critical border shows the minimum solvency for a bank to be able to issue insured deposits it could serve as a tool for determinining capital requirement at the initiation of the guarantee contract. If the premium can be charged, then the depositors are fully insured during the life-span of the contract and no further solvency requirement is needed until the premium payment of the next guarantee contract. If a bank does not have a sufficient solvency at the time for the premium payment and is located below the border, it has to issue new equity. If it is not possible to attract new equity the government withdraws the charter, that is, closes the bank.

In the case of the banking industry, regulatory requirements[40] generally focus on the solvency level of the bank rather than the variance. However, in our model, in order to be insured, that is, to be able to pay the fair premia, it is not sufficient to require a bank to have a certain solvency ratio. The associated variance must also be sufficiently low in order to avoid an immediate bankruptcy. Thus, a more appropriate governmental safety rule would be to regard the pair of solvency and variance according to some critical values associated with the cost of deposit insurance.

8. Concluding Remarks

The main contribution of this paper is the critique of valuing deposit insurance as a pure European put option. First, an important difference between a deposit insurance and a put option is that the former has impact on the underlying process, which is not the case in the latter. It is a fact that the value of the fair deposit guarantee must be withdrawn from the bank assets. The under-estimation of the premia, between considering this fact and neglecting it, is significant and is expressed in numerical examples.

Second, we have given attention to another major difference between a deposit insurance and a put option. The payment of the insurance may cause the immediate bankruptcy of the bank. We have shown a critical relation, betweeen the solvency of the bank and the standard deviation of the bank's assets, which separates feasible from non-feasible premia in this respect.

Third, the valuation of the insurance as a European put option does not permit insolvency in other points of time than at the maturity date. Instead, the frameworks of Dahlfors and Jansson (1994 a,c) work out better.

[40] For example; the BIS capital adequacy standards applied in Sweden and 11 other OECD countries demand preferred stock and subordinated debt to be at least 8% of the assets weighted by four different risk categories. The numerous shortcomings of these standards are discussed in Pierce (1991) and criticism of capital requirements as regulatory tools in general is presented in Merton and Bodie (1993).

Appendix

The dynamics of the assets B_t, A_t and the liabilities, D_t, are given by [1], [2] and [4] respectively.

Now assume that the value of a financial guarantee, P(t,A,D), is a function of time, the asset process, A_t and the liability process, D_t. The boundary condition, at the maturity date T, is $P(T,A_T,D_T)=\max[D_T-A_T,0]$. Hence, D_T is the deterministic exercise price. In order to achieve a risk-neutral valuation, the discounted price processes, A_t/B_t and B_t/B_t must be martingales. Therefore, in order to find a probability measure Q when this is fullfilled, we must change the drift of [2] by transforming the objective probability measure P to the equivalent martingale measure Q.

The probability measure Q is defined by $dQ = L_t dP$, where $dL = hLdW + gL(dN - \lambda dt)$ is the differential of the Radon-Nikodym derivative for Q with respect to P. It is assumed that the Wiener process, W and the Poisson process, N are independent under the probability P.

Now we can use the Girsanov theorem, which states that the Wienerprocess W under P, becomes a Wienerprocess V, plus a drift term under Q according to:[41]

$$dW = hdt + dV \tag{7}$$

The Poisson process, N, will have the intensity $\lambda(1+g)$ under measure Q and will generate a martingale, MG, according to:

$$dN^Q = \lambda(1+g)dt + dMG \tag{8}$$

The discounted price process is defined by $Z^A = A/B$, where the differential of B is given by [1]. Its differential, under measure P, is given by:

[41] See for example Duffie (1988) p.229.

$$dZ_t = (\alpha - r)Z_t dt + \sigma Z_t dW + kZ_{t-}dN \tag{9}$$

If we substitute the Wiener- and Poisson processes under Q into [9] we get the dynamics of the discounted price process under the equivalent probability measure Q:

$$dZ_t = \left(\alpha - r + h\sigma + k\lambda(1+g)\right)Z_t dt + \sigma Z_t dV + kZ_{t-}dMG \tag{10}$$

Now we want to choose h and g so that [10] becomes a martingale, that is, $E^Q[dZ] = 0$. Therefore the following equation must be satisfied:

$$h = \frac{r - \alpha - \lambda(1+g)k}{\sigma} \tag{11}$$

We see that [11] cannot be uniquely determined, because for every choice of g there is a certain h. Therefore the probability measure Q is not unique. This is due to incompleteness in the model, that is, there are two risk sources but only one plus one traded assets. However, the probability measure Q can be uniquely determined if the "market price" for, for example, jump risk can be determined, either by assumption or estimation from the market. Once this is determined, g is determined and thereby h. For a further discussion of this issue, see Dahlfors and Jansson (1994 b,c).

Now substitute [11] into [7] and then [7] into [2] and we have the dynamics of the asset process under probability measure Q:

$$dA_t = \left(r - \lambda(1+g)k\right) A dt + \sigma A dV + kA dN^Q \tag{12}$$

We see that [12] has the local drift equal to the risk less rate, since dN^Q is given by [8] under probability measure Q.

The value of the guarantee at time t=0 is:

$$P(t=0,A_0,D_0) = e^{-rT}E^Q\max[D_T - A_T, 0] =$$

$$= e^{-rT}E^Q\max\left[D_0e^{\mu T} - A_0e^{\left(r-\lambda(1+g)k-\frac{1}{2}\sigma^2\right)T+\sigma V(T)+N(T)\ln(1+k)}, 0\right] \tag{13}$$

Use the linear homogenity of P(t,A,D) in (A,D) and use definition 1, that is, X≡A/D. Divide through by D_0, then [13] can be expressed as:

$$P(t=0,X_0) =$$

$$= e^{-(r-\mu)T}E^Q\max\left[1 - X_0e^{(r-\mu-\lambda(1+g)k-\frac{1}{2}\sigma^2)T+\sigma V(T)+N(T)\ln(1+k)}, 0\right] \tag{14}$$

The solution, that is, the price of the financial guarantee at time zero, is:

$$P(t=0,X_0) = e^{-(r-\mu)T}\sum_{N=0}^{\infty}Po(\cdot)\int_{-\infty}^{\infty}(2\pi\sigma^2T)^{-\frac{1}{2}}\exp\left[-\frac{\varepsilon^2}{2\sigma^2T}\right]\times$$

$$\times\max\left[1 - X_0\exp\left[\left(r-\mu-\lambda(1+g)k-\frac{1}{2}\sigma^2\right)T+\varepsilon+N\ln(1+k)\right], 0\right]d\varepsilon \tag{15}$$

where Po(°) is the Poisson frequency function with intensity λ(1+g)T.

The max expression can be dropped if we integrate up to ε^*_N, instead of infinity, where

$$\varepsilon_N^* = -\ln X_0 + \left(\mu - r + \lambda(1+g)k + \frac{1}{2}\sigma^2\right)T - N\ln(1+k) \tag{16}$$

The solution is then:

$$P(t=0, X_0) =$$

$$= \sum_{N=0}^{\infty} Po(\cdot)\left[e^{-(r-\mu)T}\Phi(d_N) - X_0 e^{-\lambda(1+g)kT}(1+k)^N \Phi(d_N - \sigma\sqrt{T})\right] \tag{17}$$

where

$$d_N = \frac{-\ln X_0 - \left(r - \mu - (1+g)k - \frac{1}{2}\sigma^2\right)T - N\ln(1+k)}{\sigma\sqrt{T}} \tag{18}$$

As discussed in Section 6 the initially paid premium reduces the assets and this will affect the value of [17]. Therefore, if we substitute X_0-π for X_0 in [17], we arrive at formula [5].

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Paper 2 Valuation of Deposit Insurance An Alternative Approach

1. Introduction

The contingent claims approach to the valuation of financial guarantees, that has generally been followed since Merton (1977), uses the analogy between a guarantee and a put option.[1] However, this approach has several drawbacks. The following three important differences between the valuation of a put option and a financial guarantee are discussed in Dahlfors and Jansson (1994a).[2]

(i) The value of the guarantee will in many cases influence the underlying process through the initial payment of the insurance premium. This is not the case in option valuation, where the stock process is independent of the option value. If this difference is not considered, the value of the guarantee will be under-estimated.

(ii) It is seldom realistic to value a guarantee as a European put option, since it implies that insolvency can only take place on the expiration date. One approach to handling this problem in the previous literature is to view the expiration date as stochastic, or rather; the time of the governmental audits of the bank, which would reveal insolvency, as random.[3] However, this approach has one drawback. The assumption of perfect information is inconsistent with the need of audits for the guarantor and the fact that the bank can operate under insolvency.

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[1] For a brief discussion of the contributions in the literature, see Dahlfors and Jansson (1994a)

[2] A fourth difference between a stock option and a guarantee that was discussed in Dahlfors and Jansson (1994a), but not explicitly considered in the present paper, is that the dynamics of the stock are independent of the expiration of the option. Guarantees like the deposit insurance, on the other hand, has the property that when it expires "in the money" the bank will be closed. This would probably affect the risk taking behaviour of the bank and thereby the dynamics of the bank assets.

[3] This approach has been used by many authors, e.g. Merton (1978) and Pennachi (1987).

(iii) To value the guarantee as an American put option is neither a proper solution to the premature expiration problem, although this type of contract allows for early exercise. The reason for this has to do with another difference between an option and a guarantee, namely the fact that it is the guarantor, that is, the issuer of the contract, that is entitled to nullify the contract. An option contract, on the other hand, gives the holder of the contract the right to exercise. This feature of a guarantee implies that, together with an assumption of perfect information, the guarantee will never be nullified below the exercise price. This, in turn, gives the guarantee a value of zero, which seems inconvenient.

From the aspects (i)-(iii) above, we will formulate an alternative approach to value a deposit insurance. This approach will take into consideration the fact that the initial payment of the insurance premium will affect the solvency process of the bank. Furthermore, we allow for insolvency at every instant.[4] The value of the guarantee comes from the presence of liquidation costs, assumed both as constant and stochastic. To our knowledge, this treatment of deposit insurance valuation has not been dealt with in the literature.

Since the main source, in the present paper, for the guarantee to receive a value is the presence of liquidation costs, a short discussion of these are motivated. In previous studies of deposit insurance pricing, it has been assumed that when a bank is found insolvent the guarantor can realize the full value of the bank's assets and pay off the depositors. However, in reality, this possibility is seldom at hand, since there exist liquidation costs of different kinds.[5] For example, there are reasons to believe that the liquidation of a bank will involve legal and administrative costs.

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[4] In other words, we assume that the guarantor, with perfect observability of the market valued bank solvency, is able to nullify the contract and close the bank exactly at the insolvency level. However, in reality, even with the best in monitoring, it is not always possible to catch a bank just when its net worth reaches zero. Further, the guarantor may act in a discretionary manner and delay the liquidation in the hope of a recovery of the bank. See e.g. Allen and Saunders (1993) for a discussion of this forbearance behavior of the U.S. deposit guarantor.

[5] According to Sinkey (1992) the average life of a bank liquidation was almost 12 years in the U.S in the 80's. Therefore, the costs of a liquidation process can be quite substantial. Sinkey discusses the nature of these costs and gives some references to empirical studies of their magnitude.

There are also other kinds of costs associated with liquidation of banks that are connected to information problems.[6] Such costs often arise when the guarantor realizes the assets of a failed bank in order to pay off the insured depositors. The sale of a large volume of illiquid assets, that to a large extent suffer from information problems, often into a thin market of buyers who are aware of the guarantor's predicament, will probably induce a reduction in asset value.[7]

The paper will be structured as follows:

In the model presented in Section 2 we assume the bank to be closed exactly when its solvency process hits the insolvency barrier. Furthermore, we assume the liquidation cost of the bank to be constant. The value of the deposit insurance is given in numerical examples. We also analyze the consequences for the valuation of neglecting the up-front payment of the guarantee premium.

In Section 3 we extend the model by letting the liquidation cost be stochastic. It is reasonable to assume this uncertainty to increase in time. This feature can be captured by modelling the cost as a Brownian motion. We present a closed form solution for a fair guarantee value. We also table some numerical values and compare the results with the constant liquidation cost model of Section 2. In Section 4 we discuss some critical model assumptions and concluding remarks are given in Section 5.

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[6] See e.g. Barnea et.al.(1985) and Milgrom and Roberts (1992) for a presentation of the agency problems connected to bankruptcy proceedings in general.

[7] The most common way of handling bank failures in the U.S. deposit insurance system is the Purchase and Assumption Transaction (P&A), which means that the guarantor arranges for an acquisition of the failed bank by another institution. The acquiring bank purchases the assets and assumes the liabilities of the failed bank. An advantage of the P&A option over the simple deposit pay off strategy is that it is easier to preserve the going concern value of the failed bank. However, it is not always possible for the guarantor to get favourable bids on the failed bank from other banks. So, the remaining option for the insurer may be to pay off the depositors and liquidate the bank's assets, and be prepared to suffer a loss due to the realization reduction in asset value. The total resolution costs for bank failures 1985 and 1986 were 30 percent of the assets, according to Sinkey (1992). If the P&A method was used the costs were only 20 percent since the assets are worth more in a going concern than in a liquidation. Although it is, on average, costly to liquidate a bank there may be cases where an aquiring firm in fact is willing to pay a premium for goodwill and intangible assets like charter value. See e.g. Benston et.al. (1986) and Pierce (1991) for a further discussion of this issue.

2. A Model with Constant Liquidation Cost

2.1 The Model

Consider a governmental insurance agency, which guarantees all the liabilities of a bank.[8] Since we assume all the liabilities to be covered by the guarantee, we will use the terms liabilities and deposits interchangeable. Let X_t be the solvency process, that is, asset to liability ratio. This means that the value of the liabilities is normalized to be 1. The traditional framework in the literature, following Merton (1977), stipulates the value of the guarantee on the expiration date, T, to be $\max[1-X_T,0]$. Thus, if $X_T > 1$, the bank is solvent and the guarantee is worth zero. On the other hand, if $X_T \leq 1$ the bank is declared insolvent and the guarantor has to make a payout of $1 - X_T$ in order to fulfil his obligations to the claimholders of the bank. Thus, the deposit insurance is equivalent to a guarantee of $1 - X_t$. In order to make this obligation, the guarantor charges the bank an insurance premium, which can be calculated by option pricing theory.

However, the Merton (1977) framework has some drawbacks, as we mentioned in the introduction. One problem with his model is that the bank can operate under insolvency for long times. In our model, which is an extension, the bank is declared insolvent when the solvency process hits the level of $X_t = 1$, that is, the nominal value of the liabilities.[9,10] When the bank is declared insolvent and is liquidated by the guarantor the closing procedure involves a liquidation cost for the guarantor.

In order to value the deposit insurance, under the circumstances mentioned above, we make the following assumptions and definitions:

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[8] Thus, we assume homogenous debts. In reality, in most existing deposit insurance systems there are types of deposits not included in the system. Further, there may be upper limits of protection of insured deposits, e.g., up to $100.000 in the U.S. deposit insurance system, see Benston et al (1986).

[9] An alternative insolvency definition could be flow-based. In such case the bank is declared insolvent when it fails to obtain financing for its due payments. This criteria is used in Nielsen et al. (1993)

[10] This is a simplifying assumption since the correct insolvency barrier is the discounted value of the libabilities. Since liabilities are guaranteed, the discount factor must be the risk-free interest rate.

Assumption 1: All securities are continuously traded on a perfect market.[11]

Assumption 2: The solvency process X_t, which is assumed to be a continuously traded asset, is given by:[12]

$$dX_t = \mu X_t dt + \sigma X_t dZ_t$$
$$X_0 > 1 \tag{1}$$

where μ = the instantaneous expected value of the rate of return of the solvency process.

σ = the instantaneuos standard deviation of the rate of return of the solvency process.

Z_t = a standard Brownian motion.

Assumption 3: There is also a risk-free bond, which has the following dynamics, where r is the constant risk free rate of return:

$$dB_t = rB_t dt$$
$$B_0 = 1 \tag{2}$$

Assumption 4: The guarantor will nullify the guarantee and close the bank when it is declared insolvent. This is so because the guarantor, who continuously observe the solvency process, will never gain in closing the bank at $X_t < 1$ and is assumed legally prohibited to do so if $X_t > 1$.

[11] See Section 4 for a comment on this assumption.

[12] This is a simplification of the probably more realistic assumption of modelling bank asset dynamics as a jump-diffusion process due to the existence of large loan defaults. This assumption is made in Dahlfors and Jansson (1994a).

Assumption 5: The liquidation procedure involves a constant liquidation cost for the guarantor of C dollar per dollar of deposits.[13] These costs represent legal and administrative expenses associated with the liquidation of the bank. It may be realistic to view these as constant and institutionally determined.

Definition 1: Let $P(t,x)\varepsilon C^{1,2}$ be the value of a contingent claim, called the deposit insurance, which pays C dollars per dollar of deposits if and only if the process X_t, defined by [1], hits the barrier X=1.

Definition 2: Let $\tau \equiv \inf\{t\varepsilon[0,T]: X_t=1\}$, $X_s > 1$, $\forall s: 0 \leq s \leq t$. Thus, τ is the first time X_t hits the insolvency barrier $X_t=1$. That is, τ is a stopping time.

Since the economy is complete,[14] we can make a risk neutral valuation of the deposit insurance by constructing a self-financing portfolio, which mimics the dynamics of the insurance contract. After carrying out the standard portfolio construction procedure we end up in the following differential equation:[15]

$$\frac{1}{2}\sigma^2 X^2 P_{xx} + rXP_x + P_t - rP = 0 \tag{3}$$

A necessary condition for the economy to be free from arbitrage, is that [3] is valid for all $X_t>1$ on the time-interval [0,T].

The boundary condition to this differential equation is:

$$P[\tau,1] = C, \quad X_s > 1, \quad \forall s < \tau \tag{4}$$

The solution to [3] and [4] is:

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[13] This proportionality of liquidation costs may not be realistic since there may exist economics of scale in resolving bankruptcies so that the costs of liquidating a large bank are smaller as a proportion of its liabilities.

[14] We have only one risk source in the economy and two traded securities. This implies that the state space is spanned by these two securities.

[15] See Black and Scholes (1973).

$$P[t=0,\ X_0\ ] = C\int_0^T e^{-rs}G_s^{'}ds \tag{5}$$

Where G' is the risk neutral first passage time density of X_t passing through the absorbing barrier X=1.[16]

The value of the deposit insurance per dollar of deposits, at time t=0, with a constant liquidation cost C, is therefore:[17]

$$P(t,X_0,\pi) = C\left[(X_0-\pi)^{-\frac{2r}{\sigma^2}}\left(\frac{-\ln(X_0-\pi)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)+\right.$$
$$\left.+\ (X_0-\pi)\Phi\left(\frac{-\ln(X_0-\pi)-(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right] \tag{6}$$

where $\phi(\cdot)$ = the cumulative standard normal distribution.

Since the maximum pay-out of the guarantee is C, the value of the guarantee is bounded by $0\leq P(t,X_0,\pi) < C$, for all t and $X_0>1$.

In Figure 1 below we have plotted the guarantee value, P, for different liquidation costs, $C\in[0,0.25]$, and different initial solvency levels, $X_0\in[1.01,1.2]$.

As expected, the value of the guarantee is close to the liquidation costs for initial solvencies, X_0, close to the insolvency level, X=1. The reason for this is that insolvency, and thereby the cost of C, will occur almost certain. For a given value of C, the value of the guarantee is convex in X. This is the usual observation for ordinary put options

[16] The expression for G' is given in the appendix.

[17] The derivation of [6] is given in the appendix.

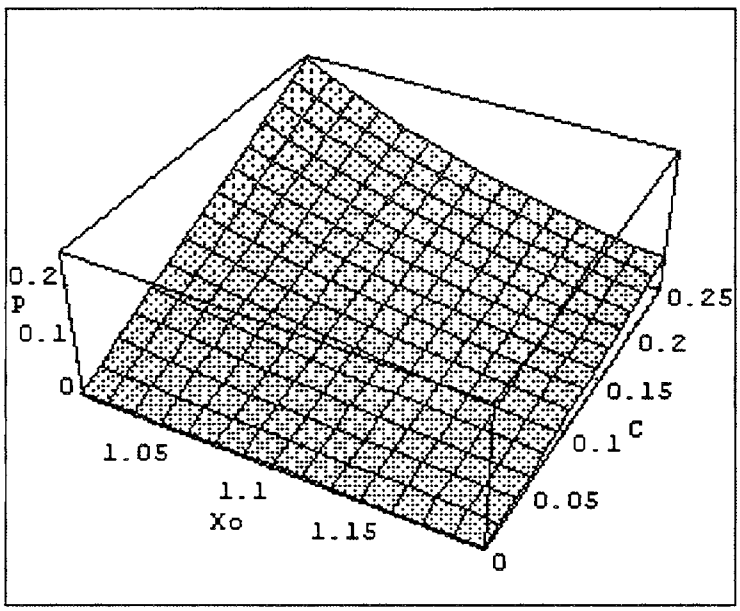

Figure 1

As we discuss in another, Dahlfors and Jansson (1994a), deposit insurance is a type of guarantee arrangement where the bank makes the actual payment of the guarantee premium. This is mirrored by the reduction of the initial solvency value in [1].[18] Thus, the value of the guarantee is also a function of the initially paid premium, π. Therefore, in order to get a fair premium we have to solve the following equality: $P(C,\pi) - \pi = 0$. The premium, π, which solves this equality is defined as a fair premium, since it equals the value of the guarantee. This issue is discussed in general in Dahlfors and Jansson (1994a). The impact from neglecting this issue in the present model is analyzed in connection to Figures 2 and 3 below.

Tables 1 and 2 below exhibit numerical values of formula [6] for different initial solvencies, liquidation costs and standard deviations. The other parameter values are $r=0.1$ and $T=1$. For some triplets (X_0,C,σ) the premium is too high to be charged, compared to the solvency level. That is, if $\pi > (X_0-1)$ the guaranteed part cannot afford to pay the premium and still be solvent. These triplets are not available, and are denoted n.a. in the tables. The importance of this phenomena is discussed in connection to Figures 5 and 6 below, which exhibits what we call the critical border.

[18] The value of the guarantee will of course not appear as an asset, in the balance sheet of the bank, in the valuation of the deposit guarantee. However, for a non-bank in position of acquiring the bank, the guarantee may be considered as an intangible asset since it makes it possible to issue deposits at the risk-free rate.

Table 1

σ=0.2	C=0.01	C=0.1	C=0.2
X_0=2.0	$1.136 \cdot 10^{-6}$	$1.136 \cdot 10^{-5}$	$2.270 \cdot 10^{-5}$
X_0=1.5	0.000166	0.001684	0.003423
X_0=1.2	0.002345	0.028926	0.097032
X_0=1.1	0.005149	0.099983	n.a.

Table 2

σ=0.3	C=0.01	C=0.1	C=0.2
X_0=2.0	0.0001248	0.001255	0.002525
X_0=1.5	0.0012888	0.013620	0.029221
X_0=1.2	0.004746	0.058881	0.199978
X_0=1.1	0.007095	0.099997	n.a.

As expected from option pricing theory, the insurance premium increases in σ and C and decreases in initial solvency, X_0. We also notice from the tables that the premia are almost proportional in C for high X_0. This is due to the fact that a change in C affects the premium, which is payed up-front and thereby affects the initial X_0-value. This, in turn, influences the probability of hitting the insolvency level at a certain point of time. For relatively high values of X_0 the changes in the probability distribution is almost neglectable, which thereby explains the linearity in C. For lower initial solvencies, changes in C, have greater impact on the insolvency probability. This explaines why the effect on the premium is more than proportional for lower X_0-values, when C is changed.

2.2 The Valuation Bias

As discussed above and in the introduction, one important feature of a financial guarantee is that the guarantee value affects the underlying process via the up-front payment of the premium. If the valuation does not take account for this the value of the guarantee will be underestimated. This bias, defined as the difference between the fair premium and the premium neglecting the up-front payment, is sometimes quite

substantial, which is seen in Figures 2 and 3 below. The figures are plotted for two different liquidation costs and for r=0.1, T=1 and σ=0.2.

Figure 2 below is plotted for C=0.01.

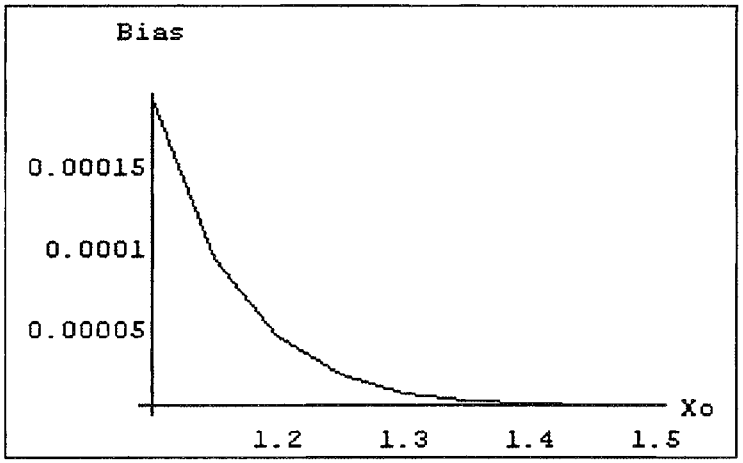

Figure 2

Figure 3 below is plotted for C=0.1.

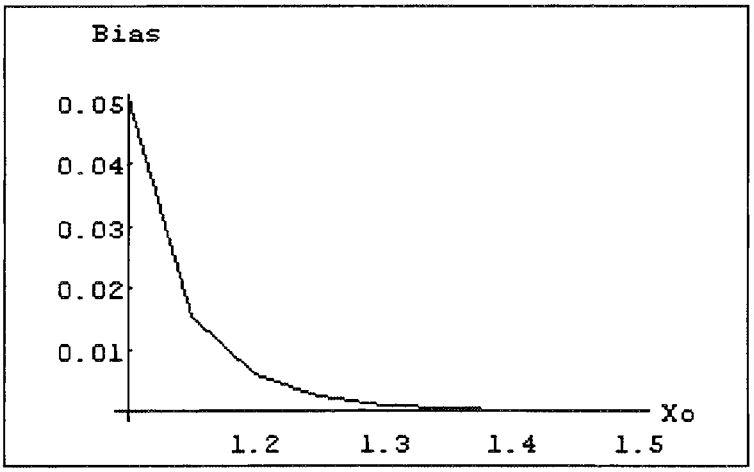

Figure 3

Comparing Figures 2 and 3, we observe the valuation bias to increase in C. This is because the premium itself increases in C. Further, for relatively low initial solvencies,

X_0, the bias is quite substantial. In Dahlfors and Jansson (1994a) we discuss this valuation bias in more detail and derive it for different standard deviations and jump-intensities.

2.3 The Critical Border

In another paper, Dahlfors and Jansson (1994a), we discussed a critical relation between the initial solvency and the standard deviation of the solvency process. This was defined as follows:

Definition 3: The critical border connects solvency ratios and standard deviations, such that:

(i) $P(X_0,\pi,\sigma)=\pi$

and

(ii) $X_0-\pi=1$

Condition (i) implies that the premium, π, is fair. That is, the premium is equal to the value of the guarantee, P.

Condition (ii) gives the minimum solvency level required to pay the premium without becomming insolvent. That is, the solvency is above the insolvency level, X=1, after the premium has been paid.

In Figures 4 and 5, the critical border is computed for liquidation costs of 10 and 20 percent of deposits respectively. We observe that the border converges to 1.1 and 1.2 when the standard deviation goes to infinity. The convergence of the required initial solvency to $X_0=1+C$ comes from the fact that the probability of hitting X=1 converges to one in standard deviation and that the supremum of P is C.

Figure 4 is plotted for C=0.1.

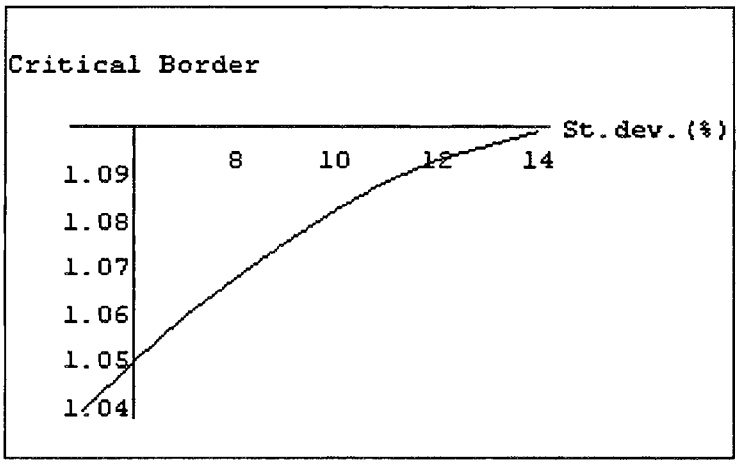

Figure 4

Figure 5 is plotted for C=0.2.

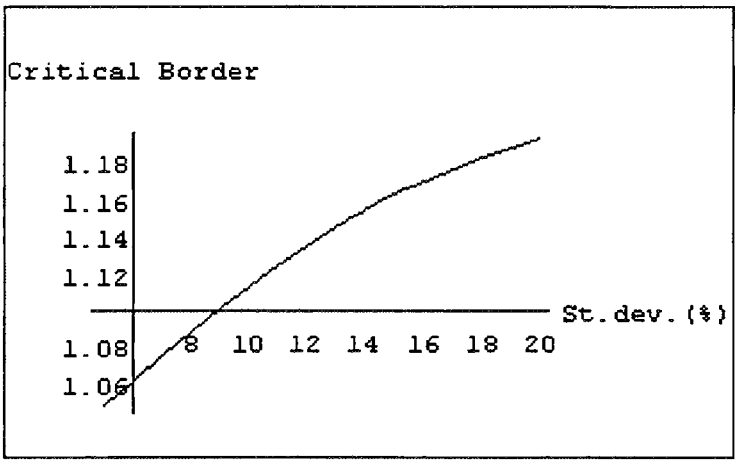

Figure 5

Combinations of solvency ratios and standard deviations located below the graph implies guarantee premia, which are not feasible in the sense that they can not be charged without forcing the bank into insolvency. Thus, the critical border separates feasible from non-feasible premia with respect to different pairs of X_0 and σ. Further, the level of the critical border increases in the liquidation cost. For example, if C = 0.2 and σ = 0.1 an initial solvency of X_0 = 1.11 is required to be able to pay the premium. If, on the other hand, C = 0.1 and σ = 0.1 an initial solvency of X_0 = 1.08 is required.

The critical border is computed for a given time-to-maturity. If we shorten the time-to-maturity of the guarantee, the border will shift downwards. Hence, a shorter time to maturity of the guarantee implies a lower required initial solvency for a given standard deviation and vice versa. In the limit, when time to maturity goes to zero, all solvent banks would be able to pay the fair premium.

In reality, a bank with a solvency below the border may succeed in issuing new equity, which would increase its solvency and thereby help the bank to climb over the critical border and continue its business. However, this opportunity is not considered in our model.

In the next section we extend the analysis by allowing the liquidation cost to be stochastic.

3. A Model with Stochastic Liquidation Costs

The assumption of a constant liquidation cost may in many cases be too strong. Sometimes it can be more realistic to model these costs as random over time. This could for example be motivated by a stochastic nature of the reduction in asset value associated with the realization of the assets when closing the bank and paying off the depositors.

In the introduction we mentioned that these types of liquidation costs may be connected to information problems.[19] However, it should be noted that this does not imply any imperfections concerning observability of the asset value dynamics *prior* to liquidation. It is only during the realization of the failed bank's assets these costs arise. However, we argue that liquidation costs, are not necessarily connected to information problems and are thereby in accordance with the used assumptions. This is so because it is possible that the failed bank's assets are worth less when separated and realized in pieces compared to remaining in a going concern.

Thus, *if* liquidation costs are stochastic, the model below should be appropriate. We consider it realistic to let the uncertainty of these liquidation costs be an increasing function of time. To model this uncertainty we assume the existence of a continuously

[19] Such information imperfections could be in conflict with the standard option pricing assumptions of perfect and frictionless markets.

traded security which captures these properties.[20] Consider the following process which determines the stochastic cost of liquidating the bank at time t:

$$dC_t = \delta\ C_t dW_t$$
$$C_0 > 0 \tag{7}$$

where δ = the instantaneous standard deviation.
W_t = a standard Brownian Motion.

Assumption 6: The liquidation cost process, [7], is independent of the solvency process, [1].[21]

Thus, the guarantee value is not only a function of time and solvency, but also of the liquidation cost, that is, P(t,x,c). We now proceed in the same manner as in Section 2 by constructing a self financing portfolio, which also includes [7]. According to this, the arbitrage free argument leads us to the following differential equation, which must hold for $(t,x,c) \in [0,T] \times (1,\infty) \times (0,\infty)$:

$$\frac{1}{2}\sigma^2 X^2 P_{xx} + \frac{1}{2}\delta^2 C^2 P_{cc} + rXP_x + rCP_c + P_t - rP = 0 \tag{8}$$

The boundary condition to this differential equation is:

$$P[\tau,\ 1,\ C] = C_t, \qquad X_s > 1, \quad \forall s < \tau \tag{9}$$

The solution to [8] and [9] is:

$$P[t=0, X_0, C_0] = \int_0^T e^{-rs} E^Q[C_S] G_S' ds \tag{10}$$

Where E^Q is the risk-neutral expectation operator.

[20] See Section 4 for a remark on this assumption.

[21] This assumption is discussed in Section 4.

The value of the deposit insurance per dollar of deposits, at time t=0, with stochastic liquidation cost, which is solution to [10] is:[22] $P(t, X_0, C_0; \pi)$

$$P(t, X_0, C_0; \pi) = C_0 \left[\Phi\left(-\frac{-\ln(X_0 - \pi) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) + \right.$$

$$\left. + (X_0 - \pi)^{-\frac{2r}{\sigma^2}+1} \Phi\left(-\frac{-\ln(X_0 - \pi) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right] \tag{11}$$

The reason for [11] to be rather simple is that C_t and X_t are assumed independent. A somewhat surprising feature of [11] is that the premium does not depend on the uncertainty of the liquidation cost process, δ. This is because it is only the expected value of C_t that matters, which is seen in [10].

In Tables 3 and 4 below, we give some numerical values of [11] for different parameter values, where r=0.1 and T=1.

Table 3

σ=0.2	C_0=0.01	C_0=0.1	C_0=0.2
X_0=2.0	$1.240 \cdot 10^{-6}$	$1.240 \cdot 10^{-5}$	$2.480 \cdot 10^{-5}$
X_0=1.5	0.000179	0.001816	0.003696
X_0=1.2	002464	0.030631	0.108700
X_0=1.1	0.005306	0.100000	n.a.

Table 4

σ=0.3	C_0=0.01	C_0=0.1	C_0=0.2
X_0=2.0	0.000135	0.001358	0.002734
X_0=1.5	0.001372	0.014542	0.031322
X_0=1.2	0.004925	0.061088	0.200000
X_0=1.1	0.007245	0.100000	n.a.

[22] The derivation of [11] is analogous to the derivation of [6].

These values are higher than the premia for the constant liquidation case. The primary source for the difference is that in [11], the discount factor is eliminated by the local drift of C_t, which is **rdt** under risk neutral valuation.

In Figure 6 below we have plotted the guarantee value, P, for different initial liquidation costs, $C_0 \in [0,0.25]$ and initial solvencies, $X_0 \in [1.01,1.2]$. We have used the same parameter values as in the tables and a standard deviation, $\sigma = 0.2$.

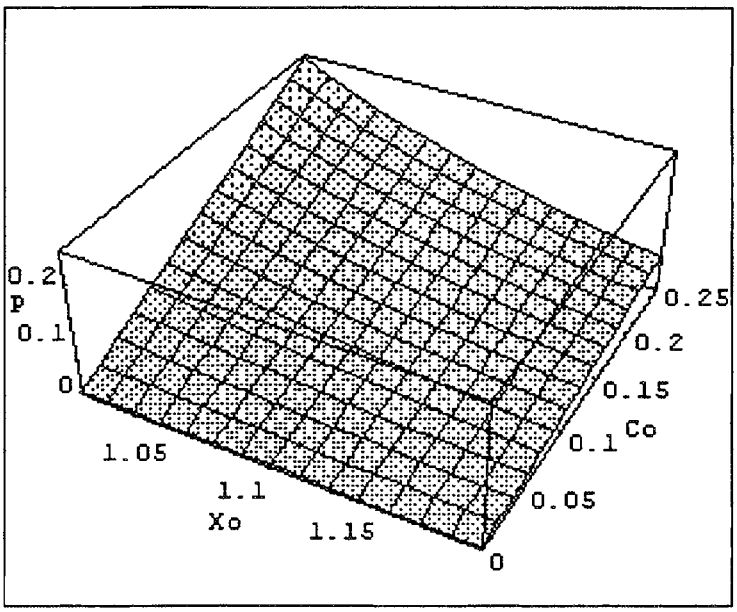

Figure 6

Refering to the tables, the values are higher than for the constant liquidation cost case. Regarding the shape of the plane we refer to the comments in connection to Figure 1 in Section 2 and Figures 7 and 8 below.

In order to see the effect from the introduction of randomness in the liquidation cost more clearly we have plotted the differences in guarantee values between stochastic and constant liquidation costs respectively for different initial solvencies. To compare the two cases C_0, in the present section, is equal to the constant C in Section 2. This is done in Figure 7 and 8 below. The other parameter values are $r = 0.1$ and $T = 1$.

In Figure 7 below this difference in guarantee value, for different initial solvencies, is plotted for C_0=C=0.2 (the upper curve) and $C_0 = C = 0.1$ (the lower curve) respectively. The standard deviation of the solvency process is $\sigma = 0.2$.

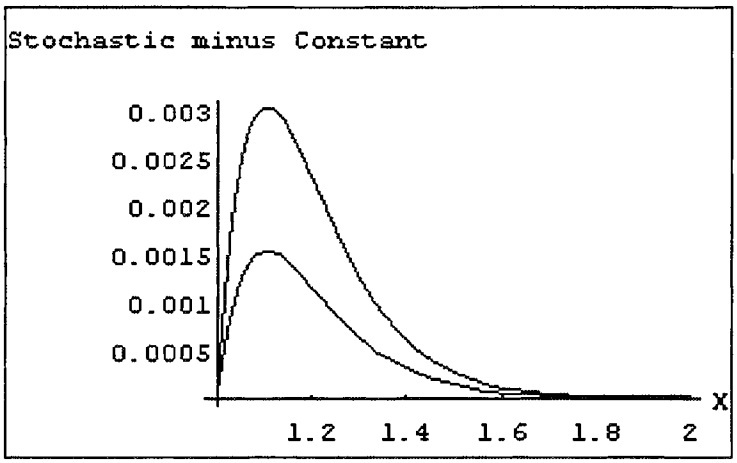

Figure 7

In Figure 8 below, the corresponding difference is plotted for different standard eviations of the solvency process, namely $\sigma = 0.1$ (the lower curve) and $\sigma = 0.2$ (the upper curve). The liquidation cost is $C_0 = C = 0.1$.

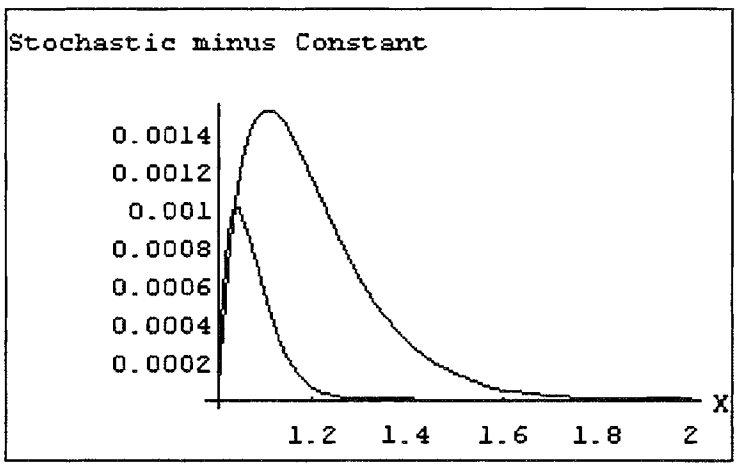

Figure 8

From the discussion in connection to formula [11] we know that the primaly difference between the two values is the discounting effect, which accordingly explains the shapes of the graphs in Figures 7 and 8 above.

As $X_0 \rightarrow 1$, the expected point of time of insolvency goes to zero. The importance of the discount factor, and thereby the difference between the two curves, goes to zero, which is mirrored by the equality of the two cases for $X = 1$.

As $X_0 \rightarrow \infty$, the probability of insolvency goes to zero, which means that the guarantee value goes to zero. This occurs irrespectively of the stochastic feature of the liquidation costs. Thus, the difference between the guarantee values becomes zero for these extreme values of X, which is seen in the figures.

We also know, from the formulas, that the guarantee value with stochastic liquidation cost is never less than the corresponding guarantee with constant liquidation cost due to the discount effect. Accordingly, in Figure 7 this effect has its largest impact at approximately $X = 1.1$, for the chosen parameter values. Further, the difference between the two cases increases in C.

In Figure 8 we also observe that the difference increases in the standard deviation. The reason for this is that the guarantee value increases in the standard deviation and the difference in the discount effect, of this increased value, is intensified.

4. A Short Discussion of the Model Assumptions

First of all, how relevant is it to assume a continuously traded solvency process? To be strict, it is not realistic. Bank assets are often too opaque to be priced on a perfect market. However, if there is at least one security issued by the bank traded on a perfect market, for example certificates of deposits, then it is possible to estimate the market price for risk. Further, the equity of major banks is traded on the stock market. Therefore it may be reasonable to believe that this market valuation indirectly gives a good measure of the solvency process, since there is a unique relation between equity and solvency.[23] Thus, our assumption does not imply too serious practical problems with respect to the valuation.

[23] This is confirmed by the following relation between equity, f and solvency, $X \equiv$ (asset value/promised payment (D)):

$$\frac{f}{D} = X\Phi\left(\frac{\ln X + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - e^{-rT}\Phi\left(\frac{\ln X + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

Furthermore, it could be desireable if the insolvency level corresponded to a zero valued equity. If this would be the case, then the equity of the bank could be valued as a down-and-out call option, which is a contract that is nullified if the underlying asset process hits a lower barrier. However, it is not resonable to believe that bank equity is independent of the distance to such a liquidation barrier, which is the case for a down-and-out stock option. In order to model the solvency process as state dependent the standard deviation could be a function of the solvency itself. However, this setting will not give rise to a closed form solution and we disregard it in this study.

Another assumption is the exogenously determined intervention barrier. In the case of deposit insurance under symmetric information, which is the assumption in the present paper, this may be reasonable due to bank legislation. Nevertheless, in other types of guarantee arrangements, under asymmetric information, it is not always clear where the level of intervention should be located. In these cases, it seems tractable to determine this barrier endogenously by contract design.[24]

We end this section with a remark on the liquidation cost. We have assumed the existence of a continuously traded security, which captures the features of the stochastic liquidation cost. The realism of this assumption may be questionable. However, if bank bankruptcies are common, it would be possible to estimate these costs for a given bank liquidation from the outcomes of previous bankruptcy proceedings. Therefore we could think of traded claims contingent on an index of liquidation costs, which consists of the outcomes from perfect auctions of failing banks' assets. Furhtermore, we assumed the liquidation costs to be independent of the solvency process. At first sight, this may seem as a strong assumption. But on the other hand, why should there be a dependence between them if symmetric information is at hand? A dependence could be justified if liquidation costs emanate from asymmetric information between sellers and buyers of a failed bank's assets, according to the discussion in the introduction. The assumption of independence is also motivated in order to keep the model simple.

[24] See e.g. Acharya and Dreyfus (1989), Fries and Perraudin (1991) and Allen and Saunders (1993) for a discussion of this matter with respect to deposit insurance. Anderson and Sundaresan (1992) analyze endogenous liquidation decisions within a debt contract design framework.

5. Concluding Remarks

Our observations of some drawbacks of earlier studies of deposit insurance have motivated us to provide an alternative framework for analyzing financial guarantees. In doing so we have focused on the costs of bankruptcy procedures. The value of a deposit guarantee depends on the magnitude of these costs, which we have considered both constant and stochastic, and the point of time when the bankruptcy occurs. A major advantage of this approach over the previous models is that it is consistent with the assumption of perfect markets and permits bankruptcies in every possible point of time during the life of the guarantee.

Appendix

The objective of this appendix is to derive expression [6] from [5]. Thus, we want to calculate the expected discounted risk neutral value of the liquidation cost C, which arises if and only if the solvency process hits the barrier y = 1. The probability of this event in time t, conditioned on that it has never hit the barrier before t, is dG_t/dt, where $G_t(X,y)$ is the one-sided first passage time distribution.

Referring to, for example, Dahlfors and Jansson (1994b) and Karlin and Taylor (1975) we know that this probability distribution, for a geometric Brownian motion with the drift rdt, with a lower barrier, y, is:

$$G_t(X,y) = \Phi\left(\frac{\ln\frac{y}{X_0} - (r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) + \left(\frac{y}{X_0}\right)^{\frac{2r}{\sigma^2}-1} \Phi\left(\frac{\ln\frac{y}{X_0} + (r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \qquad (12)$$

The derivative of [12] with respect to t is:

$$G' \equiv \frac{dG_t(X,y)}{dt} = \ln\frac{X_0}{y}\left[\sigma^2 t^3 2\pi\right]^{-\frac{1}{2}} \exp - \left[\frac{\left[-\ln\frac{y}{X_0} + (r-\frac{1}{2}\sigma^2)t\right]^2}{2\sigma^2 t}\right] \qquad (13)$$

Thus, we multiply C by this probability [13] over all dt-intervals and discount it by the corresponding discount factor exp(-rt). After integrating over the interval, [0,T], we have that:

$$P(t=0, X_0) = C\int_0^T e^{-rt}G'dt \qquad (14)$$

If we substitute G' from [13], with y=1, into the integral in [14] we get:

$$C \int_0^T e^{-rt} \ln X_0 \left[\sigma^2 t^3 2\pi\right]^{-\frac{1}{2}} \exp - \left[\frac{\left[\ln X_0 + (r - \frac{1}{2}\sigma^2\right]^2}{2\sigma^2 t}\right] dt =$$

$$= CX_0 \int_0^T \ln X_0 \left[\sigma^2 t^3 2\pi\right]^{-\frac{1}{2}} \exp - \left[\frac{\left[\ln X_0 + (r + \frac{1}{2}\sigma^2)t\right]^2}{2\sigma^2 t}\right] dt = \qquad (15)$$

$$= C\left[X_0 \Phi\left(\frac{-\ln X_0 - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + X_0^{-\frac{2r}{\sigma^2}} \Phi\left(\frac{-\ln X_0 + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right]$$

When X_0 is reduced by the initial payment of the premium π, we arrive at expression [6].

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Paper 3 Financial Guarantees and Asymmetric Information

1. Introduction

Since Merton (1977) it has been known that financial guarantees can be valued by contingent claims analysis (CCA). This result was based on the isomorphic relation between a put option and a guarantee. However, the CCA approach may involve some problems if the underlying asset, on which the guarantee is contingent, is not marketable. This is because the CCA technique relies on dynamic hedging strategies, which require trading in the underlying asset, or equilibrium valuation.

An important kind of financial guarantees is the deposit insurance. A problem of using CCA when valuing this contract has to do with a major function of a bank, which is to solve information asymmetries between depositors and issuers of opaque financial assets.

The previous literature of deposit insurance pricing mainly assumes that the guarantor is imperfectly informed of the financial health of the bank.[1] Therefore, in order to use the CCA approach, it has been common to assume that the guarantor needs some information revealing device, for example audits. This is the case in Merton (1978).[2] Since audits are often costly, it is reasonable to assume that the guarantor will search for a cost minimizing audit strategy.[3]

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[1] Two exception are Dahlfors and Jansson (1994b) and Fries and Perraudin (1991) where perfect information is assumed. See Dahlfors and Jansson (1994a) for a review and discussion of the deposit insurance literature in general.

[2] Studies that have followed this seminal paper in this respect are e.g. Ronn and Verma (1987), Pennachi (1987 a,b) and Acharya and Dreyfus (1989).

[3] This would be reasonable for e.g. a governmental guarantor acting in the interest of the public.

Another kind of information device is to induce the manangers of the bank to voluntarily reveal the financial health. This may be achieved by formulating a contract along the lines of Merton (1993a), which makes it optimal for the bank to signal insolvency voluntarily. This issue will be analyzed in the present paper, which can be viewed as a synthesis of Merton (1978) and Merton (1993a).

The assumption in Merton (1978), that audits are the only information revealing device, has the following two implications for the valuation of the bank equity and the deposit guarantee:

(i) If the probability of an audit goes to zero, the equity of the bank can be valued, in the limit, as a pure perpetual call option. This is so because the equity will never be nullified, even if the bank is insolvent. In this case the guarantee value is zero since asset values of the bank will never be observed and therefore the bank will never be declared insolvent, irrespectively of its financial health. This, however, would contradict the very existence of the guarantee.

(ii) If the probability of an audit goes to one, we have the case of perfect information. Then the equity can be valued, in the limit, as a down-and-out call option with exercise price and knock-out price both equal to one. This is so because the equity will be nullified at the insolvency level with probability one. Therefore, the insolvency level will correspond to a knock-out level in a down-and-out call option contract. This implies a value of the guarantee consisting only of audit costs.

From these implications we can conclude that a guarantee value minimizing audit strategy would be never to conduct any audits. This is a problem with the Merton (1978)-model.

In Merton (1993a) a payment function is introduced in order to induce the equity holders to signal insolvency voluntarily. This is done to minimize the need of audits. However, it is unclear whether the equity holders will take this payment function seriously since the probability of audits is not included in their equity pricing formula in Merton's model. In fact, this probability is not specified in the model at all. This corresponds to the inappropriate case (i) above. Therefore, it is important that the probability for audits *is* included in the equity valuation formula.

In the present paper, we value equity in the presence of stochastic audits to find the smallest payment to the equity holders, inducing them to signal insolvency voluntarily. In the equity valuation we include the probability of audits and therefore avoid the problems in Merton (1993a). The deposit insurance is then valued with respect to the compensation. It is shown that the insurance gets value partly from the compensation and partly from the costly audits.

The paper also presents policy implications concerning optimal audit strategies. It is shown that, for every initial solvency, there exists an audit strategy, which minimizes the value of the deposit insurance. This has significance for the *choice* of audit intensity. In models without compensation to the equity holders, it is optimal never to audit if a major concern is to minimize the value of the guarantee. However, this anomaly is not at hand in the presence of a compensation and is thus not present in the model of this paper.

The paper is organized as follows: The model for deposit insurance valuation under asymmetric information is introduced in Section 2. In Section 3 a pricing formula for the bank equity is derived. This result is used in Section 4 in order to formulate a compensation function for the equity holders to signal insolvency voluntarily. In Section 5 we finally derive the pricing formula for the deposit insurance. Section 6 is devoted to a discussion of the policy implications of the model concerning optimal audit strategies. A comparison between our model and the Merton (1978) model is done in Section 7. In Section 8 we discuss some central assumptions concerning choice of audit strategies and liquidation levels. Concluding remarks are given in Section 9.

2. The Model

2.1 Introducing the model

We assume that a bank must issue default free deposits in order to stay in business. The primary motivation for this can be found in the discussion of the distinction between customers and investors to financial intermediaries in Merton and Bodie (1992). In this context, a depositor is a customer who has preferences for pay-offs invariant to the financial health of the bank. On the contrary, an investor in stocks or bonds issued by the bank, expects his pay-off to be affected by the bank's profits and losses. One way to

meet the depositors' demands is to supply a deposit guarantee.[4]

The bank finances its purchase of financial assets by equity and deposits. It is assumed that all deposits are fully insured for the indefinite future.[5] The fully credible guarantor is compensated for this commitment by an initially payed insurance premium.[6]

Assume the ratio of assets to default-free value of deposits of the bank, denoted X, to have the following dynamics:[7]

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$
$$X_0 > 1 \tag{1}$$

where

μ = instantaneous expected value of the rate of return of the solvency process, assumed to be constant.

σ = instantaneous standard deviation of the rate of return of the solvency process, assumed to be constant.[8]

W_t = a standard Brownian motion.

Thus, since deposits are assumed constant over time the stochastic feature of the solvency process comes from the assets.

There is also a risk-free asset, both in terms of default risk and market risk, which has the following dynamics, where r is the constant risk-free rate:

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4 One alternative to deposit insurance, according to this feature, is to invest the deposits in default free assets only. This is proposed by the supporters of the "narrow bank" idea. See e.g. Pierce (1991) for a review .

5 The level of deposits is assumed to be constant.

6 In Dahlfors and Jansson (1994a) we valued a deposit guarantee taking into account the initial reduction of the bank's asset due to the payment of the premium. As a simplification, we do not consider this feature in the present model.

7 For a motivation and description of this solvency process, see Dahlfors and Jansson (1994a).

8 This may be a strong assumption since experiences from the U.S. indicate behavioral changes towards higher risk-taking, i.e. higher σ, by banks close to insolvency in presence of deposit insurance. See e.g. Kane (1985).

$$dB_t = rB_t dt$$
$$B_0 = 1 \tag{2}$$

The following assumptions are crucial for the analys is:

Assumption 1: The guarantor has preferences for a liquidation of the bank at $X_t = 1$.[9]

Assumption 2: The guarantor cannot perfectly observe the X_t-process, whereas the managers of the bank have perfect information and act in the interest of the equity holders.[10]

Assumption 3: The only information about X available to the guarantor is the initial value of X and the knowledge of μ and σ. The guarantor receives information of X through audits at stochastic points of time. The audits are Poisson distributed with an exogenously given intensity λ. Between the audits, the guarantor does not know the location of X.

Assumption 4: The guarantor and the managers of the bank have the same opinion about the audit probability. Thus, the bank really believes that audits will be carried out and that an audit revealing insolvency means immediate liquidation.[11]

Assumption 5: It is assumed that the equity of the bank is nullified if the bank is liquidated.

2.2 The Information Structure

This section will clarify the information aspect of the economy defined above.

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[9] This assumption is ambiguous. We will discuss this matter in Section 8.

[10] Thus, we disregard possible agency problems between managers and equity holders discussed in e.g. Jensen and Meckling (1976). The information structure is discussed further in Section 2.2.

[11] Closure of a bank is assumed to be free from costs of any kind.

It is supposed that *all* relevant information of the bank is generated by the solvency process [1]. At every point of time managers of the bank have knowledge of the solvency, X_t. This information is not available to the guarantor, unless an audit is carried out. Therefore, an audit equalizes the sets of information of managers and guarantor.[12]

If X_t were traded continuously on a perfect market, the information of X_t would be contained in the prices and there would not be a need for audits.[13]

In this paper we assume that X_t is not traded in a perfect market. The reason for this has to do with a major function of a bank, namely to solve information asymmetries between investors and issuers of opaque financial assets. The agency costs associated with these assets may preclude market valuation.[14]

3. Equity Valuation

The purpose of this section is to derive a valuation formula for the equity of the bank. This will give us the value of the equity at the insolvency level and thereby the smallest compensation the guarantor has to pay to the bank in order to signal insolvency.[15]

Recall that the value of the equity of a firm can be viewed as a call option on the firm, denoted F(t,X) below.[16] Therefore we can express the ratio of equity through default-free deposits by a call option, F(t,X), on X_t with strike price equal to one. It may be realistic to assume the equity to be long-lived. Therefore, a perpetual call option F(X),

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[12] If the solvency process were deterministic, audits would not be necessary.

[13] See e.g. Merton (1974) for a description of the perfect market concept.

[14] See Merton and Bodie (1992) for a liquidity criteria of whether an instrument will be traded directly on a securities market or handled by an intermediary. The intermediary increases the liquidity of an instrument if the bid-asked spread charged, is smaller than the one charged by market-makers, if traded in an organized market.

[15] This is developed in Section 4.

[16] See Black and Scholes (1973).

where we have dropped the variable t, may be a good approximation.[17] Since X_t is not priced continuously on a perfect market, it is not possible to create a mimic hedge portfolio and thereby achieve a preference free valuation. However, assume the existence of a traded security, whose price is only dependent on X_t and t. Then it is possible to find the risk premium per unit of volatility of X_t.[18] Despite the existence of the traded security, audits are assumed to be necessary for the guarantor to get the correct value of X_t. Furthermore, the probability of an audit is assumed to be uncorrelated with any non-diversifiable market risk.

In equilibrium, the dynamics of the equity satisfy the following equations, dependent on whether $X_t \geq 1$ or $X_t \leq 1$:

$$\frac{1}{2}\sigma^2 X^2 F_{xx} + rXF_x - (r+\lambda)F = 0, \qquad X \leq 1 \tag{3}$$

$$\frac{1}{2}\sigma^2 X^2 F_{xx} + rXF_x - rF = 0, \qquad X \geq 1 \tag{4}$$

The motivation behind [3] is that if an audit is carried out by the guarantor when $X_t \leq 1$, that is, when the bank is insolvent, then the bank is liquidated and the equity is nullified. This is reflected by the $-\lambda F$ term.

Expression [4] means that the equity is independent of audits if the bank is solvent.

The general solution to [3] is:

$$F^1(X) = A_1 X^{\xi_1} + A_2 X^{\xi_2}, \qquad X \leq 1 \tag{5}$$

where

[17] The rationale behind this type of contract is that the boundary conditions become time independent which in turn reduce the Black-Scholes partial differential equation to an ordinary differential equation, which is easier to handle. For a description, see Ingersoll (1987).

[18] See appendix for a discussion.

$$\xi_1 = \left(\frac{1}{2} - \frac{r}{\sigma^2}\right) + \left[\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}\right]^{\frac{1}{2}} > 0$$

$$\xi_2 = \left(\frac{1}{2} - \frac{r}{\sigma^2}\right) - \left[\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}\right]^{\frac{1}{2}} < 0 \qquad (6)$$

Since $F^1(0) = 0$ and $\xi_2 < 0$ it follows that $A_2=0$.

The general solution to [4] is:

$$F^2(X) = A_3X + A_4X^{-\gamma}, \quad X \geq 1$$

$$\text{where } \gamma = \frac{2r}{\sigma^2} \qquad (7)$$

Since $F^2(X) = X$ when $X \rightarrow \infty$, it follows that $A_3=1$.

Recall that if the probability of an audit is zero, the equity can be valued as a perpetual call option, that is, F(X)=X. Further, we would expect F(X) to be continuous and differentiable for all audit probabilities. Thus, since F(X) is a solution for all $X \in [0,\infty)$ and we have one solution for $X \leq 1$ and $X \geq 1$ respectively, we need conditions on F(X) at X=1 in order to preserve the properties of continuity and differentiability. The continuity condition means that the function values of [5] and [7] are equal at X=1. Differentiability requires that the derivatives of [5] and [7] must be equal at X=1. Thus, the following conditions must hold:

$$F^1(1) = F^2(1)$$

$$\frac{\delta F^1(1)}{\delta x} = \frac{\delta F^1(1)}{\delta x} \qquad (8)$$

Combining [5] and [7] with [8] gives:

$$F^1(X) = \frac{1+\gamma}{\xi_1+\gamma} X^{\xi_1}, \quad X \leq 1$$

$$F^2(X) = X - \left[\frac{\xi_1 - 1}{\xi_1 + \gamma}\right] X^{-\gamma}, \quad X \geq 1 \tag{9}$$

In Figure 1 below, the equity value acording to [9], is plotted for parameter values, r=0.1, σ=0.2 and three different cases of jump-intensities, λ:

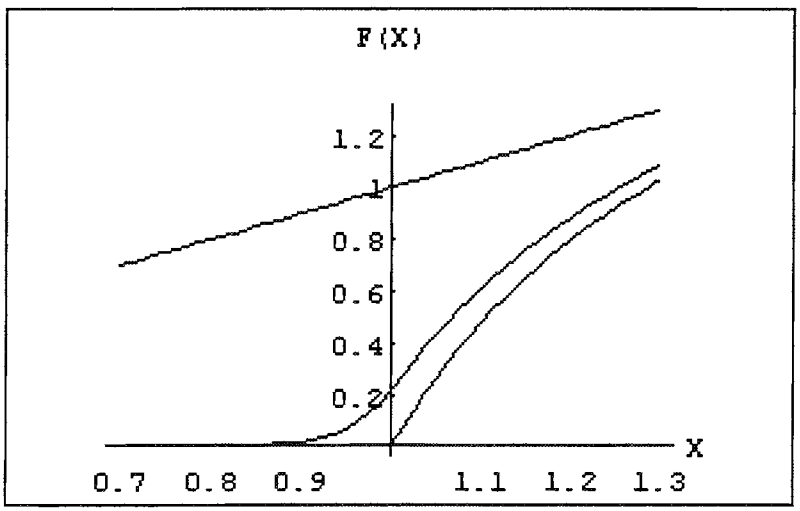

Figure 1

The intuition behind the shape of the graphs can be summarized as follows:

(i) In the absence of audits, that is λ=0, F(X) is a perpetual call option, that is, F(X)=X. This is represented by the *upper curve* in the figure.

(ii) The *middle curve* illustrates a case where it is a positive probability of nullifiation of F(X) when X≤1, in this case we have choosen λ=12. Then, F(X) is non-linear and worth less than X. Thus, the non-linearity comes from the audits. But notice that when X goes to infinity F(X) converges with X. We can also observe that, since F is concave in X for X≥1 the derivative of F with respect to X is larger than one. This differs from an ordinary call option, where the corresponding derivative is between zero and one.

(iii) The *lower curve* represents the case of perfect information, that is, the probability of an audit in the next instant is one, that is $\lambda \to \infty$. Hence, this curve can be interpreted as a down-and-out call option, with a knock-out barrier equal to one.

The rest of this section is devoted to some comparative statics. We investigate the sensitivity of the equity with respect to the standard deviation of the solvency process, denoted sigma below. In Figure 2 we have plotted the equity, F as a function of the standard deviation (sigma) for X=1.01 (the upper curve), X=0.99 (the middle curve) and X=0.95 (the lower curve). The rest of the parameter values are r=0.1 and λ=12.

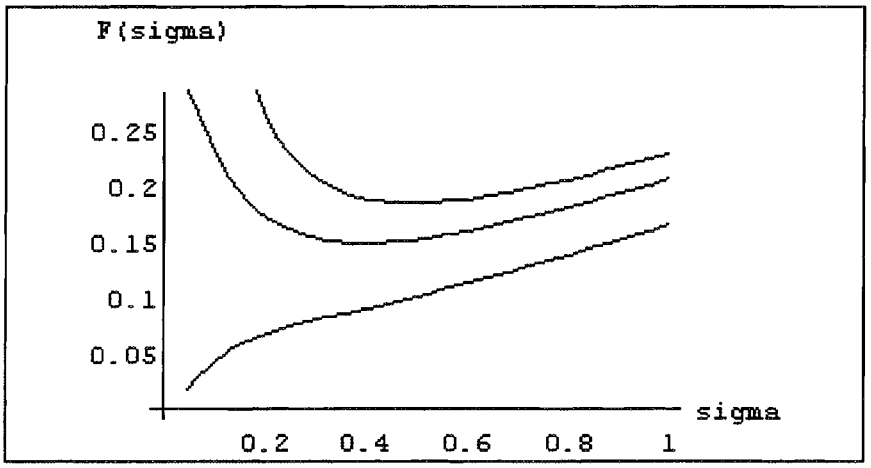

Figure 2

We observe that F is decreasing in sigma for the two upper curves for reasonable values of sigma. This may contradict the intuituion concerning the usual results for call options, where the call option value is always increasing in sigma. The explanation is that a solvent bank, which increases its sigma increases the likelihood of becomming insolvent. This effect is strong enough to obtain even when the bank is moderately insolvent, which is seen by the middle curve, where X=0.99.[19] Furthermore, we also observe that two completely different sigma-values may give rise to the same equity value, for the two upper curves. The lower curve, where X=0.95, exhibits a positive slope for all sigma-values. The explanation is that a substantially insolvent bank increases its value by increasing sigma, since the likelihood of becomming solvent by the time of the next audit increases.

[19] However, if λ is increased substantially, then F will increase in sigma even when X=0.99.

4. The Signalling Compensation

In Section 3 it was shown how the value of the equity was dependent on the solvency and the audit intensity. We know that if an audit occurs when the bank is insolvent, its equity value is nullified. However, even if the bank is insolvent, its equity value is positive due to the positive probability of becoming solvent before the next audit. Thereby it will not be optimal for the bank to reveal its insolvency voluntarily without a compensation.

In this section we will give the condition for the equity holders to voluntarily signal insolvency at X=1. This is done by formulating a compensation, which serves as a lower boundary condition in the equity valuation and induces the equity holders to call for liquidation at X=1.

In order to find the compensation we use that [9] is the value of the bank equity, F(X). Hence, at X=1, the value of the equity is equal to:

$$F(1) = \frac{1+\gamma}{\xi_1+\gamma} \tag{10}$$

Thus, at X=1 the equity holders must receive at least [10] in order to signal insolvency, that is, in terms of option theory, give up their call option on the value of the bank.[20] Therefore, since the equity holders are assumed to prefer this payment instead of not signal insolvency at X=1, the value of equity, for $1 \leq X \leq \infty$, is:

$$F(X) = X - \left[\frac{\xi_1 - 1}{\xi_1+\gamma}\right] X^{-\gamma} \tag{11}$$

Notice that [11] is analogous to a perpetual down-and-out call option with a rebate of [10] and a knock-out barrier equal to 1.0. The major difference is the interpretation of the barrier. The knock-out barrier for a down-and-out call option is exogenously given

[20] Note that we assume that the equity holders actually prefer the compensation given by [10] instead of being indifferent to keeping their call option. This assumption is not crucial for the model since we could add an arbitrarily small amount to [10] and thereby have an expression the equity holders would prefer.

in the contract. Furthermore, the information concerning the underlying process is perfectly available, therefore the down-and-out call option will always be nullified at X=1. The equity valuation in [11] can be said to have a endogenous knock-out barrier, since we know that the behaviour of equity holders is to signal insolvency at X=1, which thereby nullifies the equity.

We end this section with a final remark on the compensation to the equity holders. In Figure 3 below, we have plotted the compensation, according to [10], in σ, for r=0.1 and λ=12.

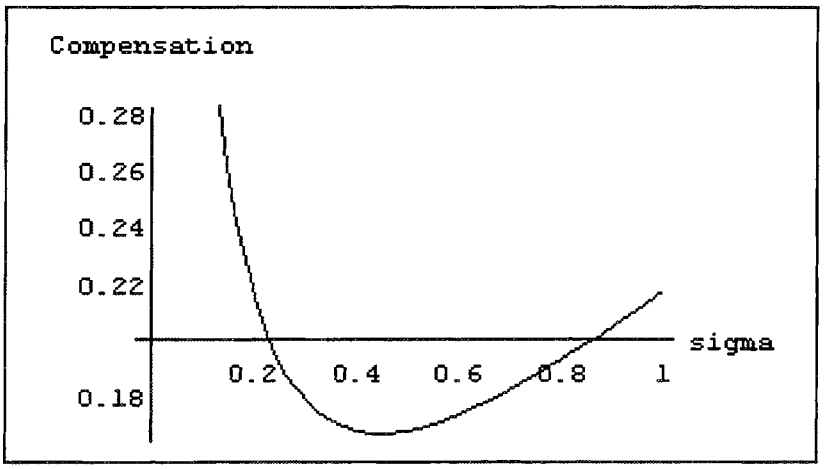

Figure 3

The shape of the figure is explained as follows: For sigma-values close to zero the required compensation will be close to one. This is so because the solvency process will be almost deterministic and the probability of becomming insolvent is almost neglectable. When sigma is increased the probability of becomming insolvent increases. This effect dominates the usually positive effect on equity from an increased sigma. Hence, the net effect will be a reduced value of the equity and thereby the compensation. This explaines the downward sloping segment in the figure.[21] However, that effect is reversed for sigma-values higher than approximately 0.4. In that region the positive impact on the equity value, from an increased sigma, dominates the negative impact from an increased insolvency probability. This explaines the upward sloping segment in the figure. The implication of this discussion is that two different values of sigma may give rise to the same compensation. This also follows from the discussion in

[21] The downward slope will also become steeper for higher audit intensities, λ. Further, for λ=0 the compensation will be equal to one, independent of sigma, since the probability of being caught insolvent by an audit is zero.

connection to Figure 2.

5. Valuation of the Guarantee

The objective in this section is to use the results from Sections 3 and 4 to value a perpetual guarantee of the deposits in a bank. That is, the bank pays a one time premium and its deposits are insured for an infinite time-period.

Let G(X) be the value per dollar of deposits of an insurance with a guaranteed value of 1, that is, depositors are fully insured.[22] The dynamics of G(X), expressed by [12] below, on a small time interval are dependent on whether the guarantor carries out an audit or not. The audits, which are Poisson distributed according to assumption 3, are assumed costly according to the following assumption:

Assumption 6: The audit cost per dollar of deposits is a constant k.

The dynamics of the guarantee are then given by:

$$dG = \left[\left(\frac{1}{2}\sigma^2X^2G_{xx} + \mu XG_x + G_t + k\right)dt + \sigma XG_x dW\right]\lambda$$

$$dG = \left[\left(\frac{1}{2}\sigma^2X^2G_{xx} + \mu XG_x + G_t\right)dt + \sigma XG_x dW\right](1-\lambda) \tag{12}$$

The first row in [12] is the change in G if there is an audit and the second is the corresponding change without an audit.

Taking expectation over audits in [12], we end up with the following equation, which must hold in equilibrium.[23]

$$\frac{1}{2}\sigma^2X^2G_{xx} + rXG_x - rG + \lambda k = 0 \tag{13}$$

[22] Since G(X) is expressed in dollar per deposits, "1" means that 100% of the deposits are guaranteed.

[23] The derivation is analogous to the equity valuation that was carried out in Section 3 and appendix.

Since we know that the equity holders will signal insolvency when X=1, if the compensation is sufficient, the guarantee is exercised at this level and is therefore valid for $X \in [1,\infty)$.

The general solution to [13] is:

$$G(X) = AX + BX^{-\gamma} + \frac{\lambda k}{r} \tag{14}$$

Recall from Section 4 that the guarantor pays an amount given by [10] to the equity holders when they signal insolvency at X=1. Therefore the constants A and B can be determined by the following boundary conditions:

$$\begin{aligned} &\text{(i)} \quad G(\infty) = \frac{\lambda k}{r} \\ &\text{(ii)} \quad G(1) = \frac{1+\gamma}{\xi_1 + \gamma} = A + B + \frac{\lambda k}{r} \end{aligned} \tag{15}$$

The meaning of [15] (i) is that when X goes to infinity the actuarial value of the guarantee goes to zero and the guarantee receives value only from the discounted expected audit costs.

Hence, from [14] we see that A must equal zero. B is then determined by the second boundary condition, [15] (ii).

The solution to G(X), for $1 \leq X_t \leq \infty$, is then:

$$G(X) \equiv G_1 + G_2 = \left[\frac{1+\gamma}{\xi_1 + \gamma}\right] X^{-\gamma} + \frac{\lambda k}{r}(1 - X^{-\gamma}) \tag{16}$$

Expression [16] is the value of the deposit guarantee when a pay-out to the bank is made at X=1, according to [10], with the probability of λdt for an audit, whose cost per dollar of deposits is k. Therefore, the value of the guarantee can be decomposed into two parts, G_1 and G_2. G_1 represents the expected value of the compensation and G_2 is the expected value of the audit costs. One issue worth mentioning concerning G_2, is that a solvency close to X=1, implies a relatively low expected audit cost, since the probability of

insolvency, that is termination of the contract, is high. Furthermore, it is seen that G_1 and G_2 work in the opposite direction for different values of X. Hence, a high (low) value of X implies a relatively low (high) value of G_1 and a relatively high (low) value of G_2. A further discussion of these decomposition results will take place in connection to Figures 5 to 10 below.

In Figure 4 below we graph the solution of G(X), that is expression [16], for different initial values of X and for r=0.1, σ=0.2, λ=12 and k=0.0001.[24]

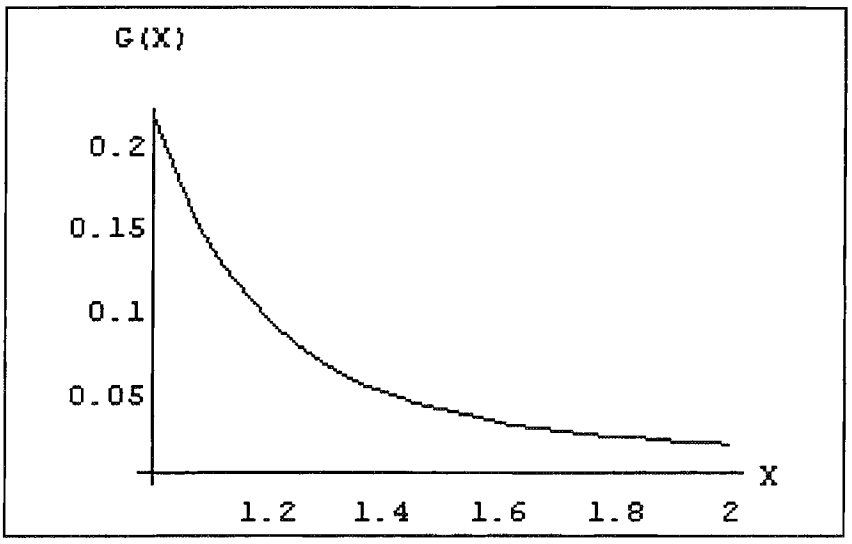

Figure 4

The graph illustrates the value of the guarantee, modelled as a perpetual put option. For the given parameter values, the guarantee value ranges between $G(X) \in [0.2048, 0.012)$ for $X \in [1,\infty)$. The explanation of the extreme values is as follows: When X=1, G(X) equals G_1, since the probability of insolvency, and thereby paying out the compensation of 0.2048 to the equity holders, is one. In this case G_2 equals zero, since the contract will be terminated and no further audits will be carried out. When $X \to \infty$, $G(X) \to G_2$, that is, the guarantee consists only of audit costs, since the probability of paying out the compensation goes to zero. Thus, the expected discounted audit costs are 0.012.

[24] Thus, the annual risk-free rate of return is 10%, the annual standard deviation is 20%, the expected number of audits in one year is 12 and the cost for one audit is 0.0001 dollars per dollar of deposits.

In order to see how sensitive the guarantee value is to changes in the standard deviation (sigma) of the solvency process we present this in Figures 5 to 10 below. We have decomposed the guarantee value in its two components G_1 and G_2 so that the effects from changes in sigma can be investigated in more detail. All the figures are plotted for r=0.1, k=0.0001 and X=1.2. Figures 5 to 7 are plotted for λ=12 and Figures 8 to 10 are plotted for λ=70.

In Figure 5 the guarantee value is plotted for λ=12. The decomposition of G(X) into G_1 and G_2 is shown in Figures 6 and 7.

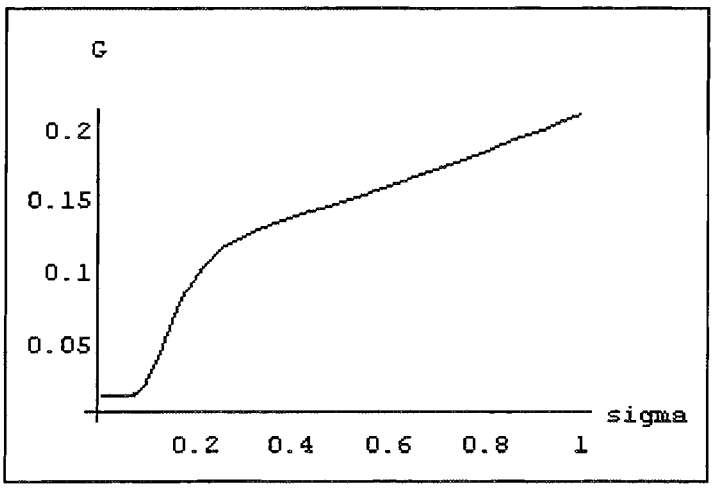

Figure 5

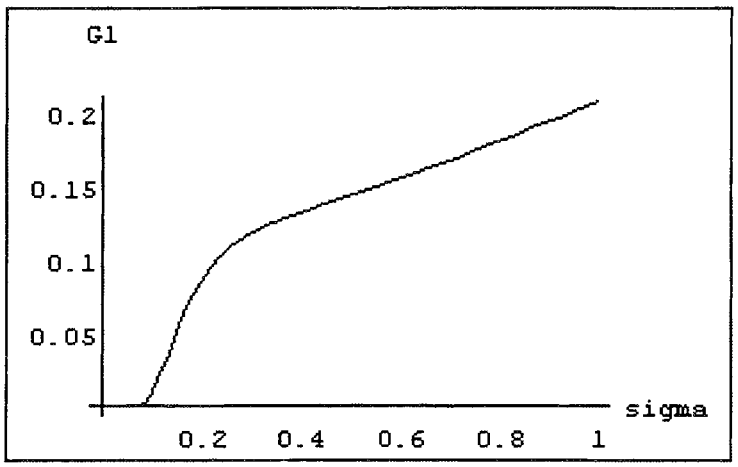

Figure 6

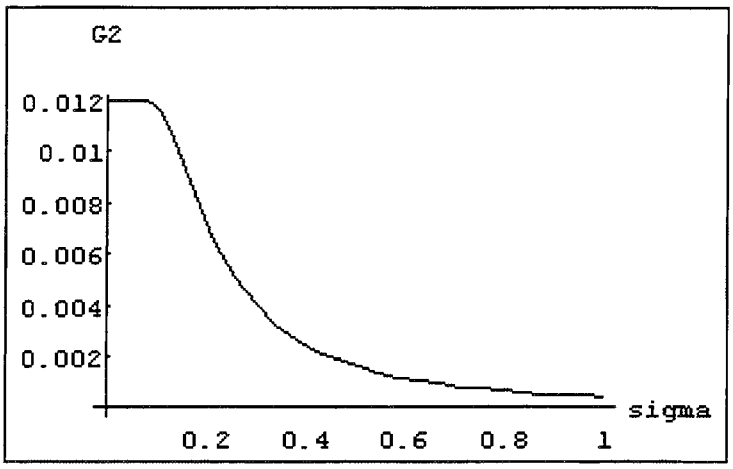

Figure 7

In Figure 5 we can see that the guarantee value responds to changes in the standard deviation as we expected, namely that there is a positive relation. We can also observe that the shape and position of G is almost identical to G_1 since the value of the compensation strongly dominates the expected audit costs, that is G_2, for the given audit intensity of λ=12. It is only for low sigmas the impact from the audit costs is visible. Since we have plotted the guarantee value for X=1.2, a sigma close to zero means that it is a low probability for a termination of the bank. Notice that the demanded compensation is close to one, which was discussed in connection to Figure 3, *but* the probability of paying out the compensation is almost zero. Therefore, G_1 is close to zero for low sigma-values. Thus, the value of the guarantee consists only of expected audit costs, which are high, since the expected life of the contract is long.

The only difference between the following three figures and the figures above is that we have increased the audit intensity. In Figure 8 the guarantee value is plotted for λ=70. The decomposition of G(X) into G_1 and G_2 is shown in Figures 9 and 10.

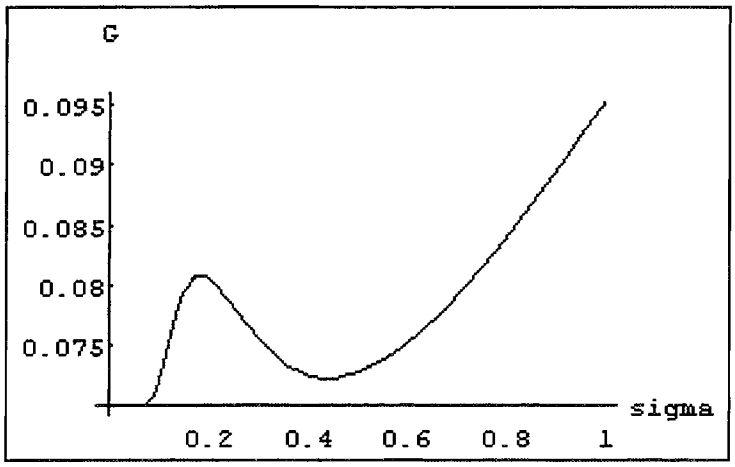

Figure 8

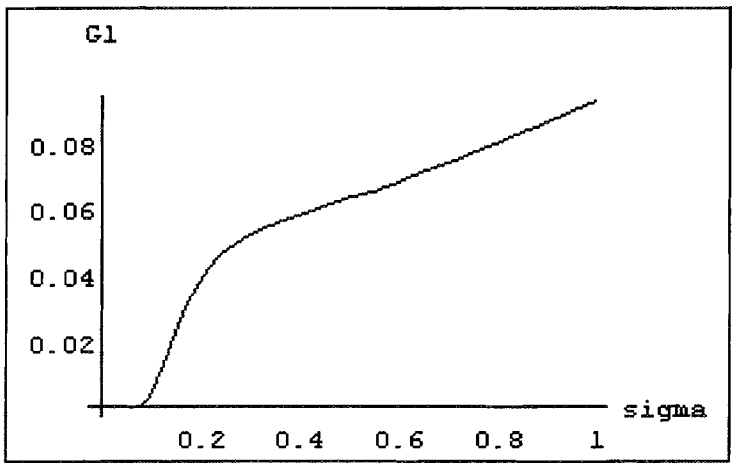

Figure 9.

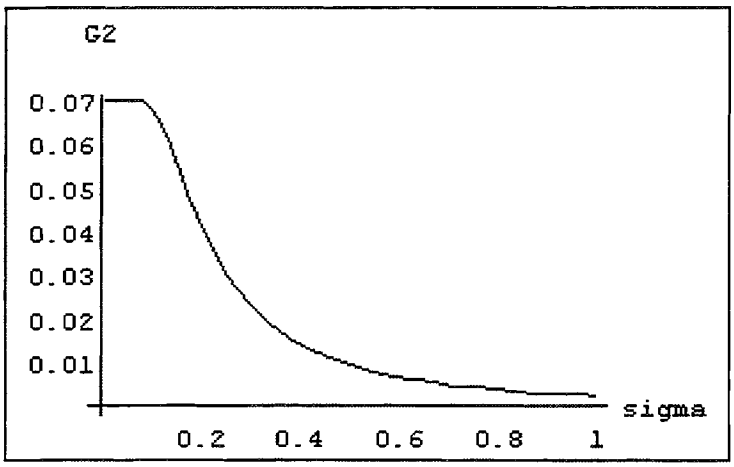

Figure 10

We observe from Figure 8 that the shape of the guarantee value is significantly changed when the audit intensity is substantially increased. However, the shapes of G_1 and G_2 are not changed. It is the *levels* of G_1 and G_2 that have changed. G_1 exhibits lower values for all sigmas when the audit intensity is increased due to the higher probability of beeing caught insolvent. G_2 increases since more audits are expected. Compared with the case of $\lambda = 12$, where G_1 exhibits much higher values than G_2, the two guarantee components are almost equal in strength. The total relative impact from G_1 and G_2 on G is that for sigmas up to 0.2, G_1 increases more than G_2 decreases. In the sigma interval between 0.2 and 0.4, G_2 decreases more than G_1 increases. For the higher sigma-values, the shape of G_1 again dominates the shape of G_2.

As was the case for $\lambda = 12$, the compensation component is very low for low sigmas and the expected audit costs completely determine the guarantee value.

We end this section with a remark on the payment of the deposit insurance premium by the bank. In Dahlfors and Jansson (1994a) we pointed out that a guarantee premium is paid from the assets, up front or continuously, which in turn decreases the solvency process and thereby increases the guarantee value. However, since this is not a central issue in the present paper and we have analyzed this feature in the paper mentioned, we have made a simplifying assumption concerning this and disregard it.

6. Optimal Audit Strategies

In the model above we have assumed the audit intensity, λ, to be constant. In practice this is a policy variable and should therefore be regarded as endogenous, in some sense. The guarantor's choice of this variable affects the value of the guarantee through the compensation contract and the audit cost. Since G(X) is convex in λ, a cost minimizing guarantor would choose a λ which minimizes the guarantee value. The rationale for doing this is obvious if the guarantor faces private competition and rational for a governmental guarantor acting in the interest of society. We think this is a reasonable criteria for choosing an audit strategy.

The relation between λ and the guarantee value, G(X), is shown in the figure below for $X = 1.2$. The parameter values are: $r = 0.1$, $\sigma = 0.15$ and $k = 0.0001$.

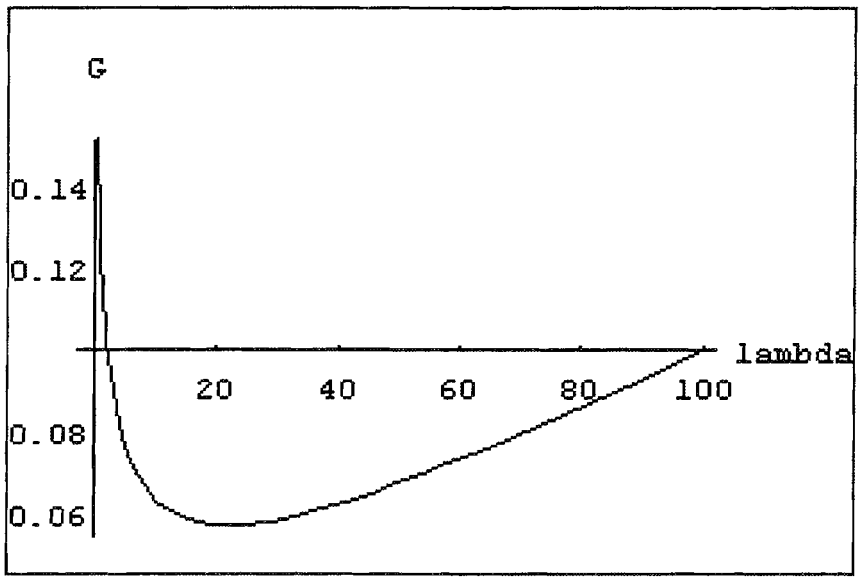

Figure 11

From the figure above we observe that there is a λ, which minimizes the guarantee value, for a given initial value of X. The shape of the graph can be explained as follows: For a given initial X and parameter values, a very small λ gives rise to a relatively high guarantee value since the compensation for the equity holders must be relatively large in order to induce them to signal insolvency.[25] On the other hand, when λ is relatively large the needed compensation is small and the audit costs constitute the major part of the guarantee value. Furthermore, the optimal choice of audit intensity increases in σ and decreases in X and k.

Thus, the policy implication for the guarantor is to choose an audit intensity which minimizes the insurance premium. This means, for example, that if the initial solvency is 1.2 as in Figure 11, the cost minimizing guarantor will choose an audit intensity of about 20 audits a year which corresponds to an insurance premium of about 6 cent per dollar of deposits.

Notice that in audit models without any compensation, like for example, Merton (1978), we cannot find these kinds of policy implications, since in those models the guarantee value minimizing audit intensity is zero. This is further discussed in Section 7.

[25] Recall that when λ goes to zero, $F(X) \rightarrow X$ and the compensation contract receives a value of one.

7. A Comparison with the Merton (1978) Model

In this section we compare the valuation formula of our model with the corresponding formula in Merton (1978).

In Figure 12 below we have plotted the guarantee value from Section 5 with an audit intensity of $\lambda = 40$. The other parameter values are $\sigma = 0.2$, $r = 0.1$, $k = 0.0001$. In the same figure we have graphed the guarantee value according to Merton (1978), for same parameter values, represented by the lower curve.

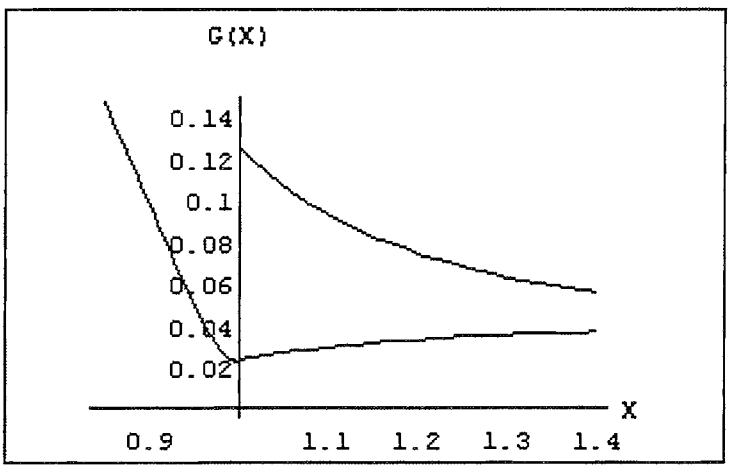

Figure 12

We can see from the figure that both curves converge to the discounted expected audit costs, 0.04, as $X \to \infty$. So, the mechanism is the same in both models for large solvency levels. For $X>1$ the guarantee value in the Merton model actually increases in X, which may be surprising. This is because expected audit costs completely dominate the reduction in "the pure guarantee part". In our model this effect works in the other direction. For levels of X equal or just above 1 our model produces much higher values than Merton's. It comes from the fact that the probability of paying out the compensation goes to one as $X \to 1$.

A feature of our model is that the guarantee payout is limited to the compensation to equity holders, whereas in the Merton model this limit is the sum of the total guaranteed value and expected audit costs.

An advantage of our model is that running an insolvent bank for long times is ruled out. On the other hand, in the Merton model this unsatisfactory behaviour is likely.

A direct comparison between our model and the Merton (1993a)-model is not fruitful, since that model is driven by cash-flow dynamics as the underlying state variable, rather than the solvency process as in our model. His model is also based on a compensation contract, which is introduced in order to limit the need of costly audits. However, the audit cost is not visable in the pricing formulas. Therefore, it is dubious if the equity holders will believe that audits actually will take place.

8. A Short Discussion on Some Central Assumptions

In this section we address two issues which may give rise to some model anomalies.[26] They concern the liquidation level and the audit strategy. First, regarding assumption 1, why should a guarantor choose to close the bank at the insolvency level, X=1?[27] In deposit insurance models without compensation to the equity holders, for example Merton (1978), a guarantee value minimizing choice of liquidation level would be at zero-solvency. This comes from the perpetual capped put feature of the guarantee. In spite of this, we assume that the guarantor chooses to liquidate the bank at the insolvency level, $X = 1$. The motivation for this is that institutional legislations, such as bankruptcy rules, can be viewed as exogenously given liquidation levels, like $X = 1$.[28]

The second issue closely related to the first deals with the audit strategy. If it is possible for the bank to operate insolvent, why should the guarantor audit at all? It would be costless, that is, the guarantee will have a zero value, if the guarantor does nothing. This problem is present in models à la Merton (1978) and ***not*** in the present paper. The reason why our model does not suffer from this problem is the existence of an audit strategy, which minimizes the guarantee value. That is, the guarantee value is convex in

[26] The main reason for these anomalies is the perpetual feature of the contracts.

[27] In presence of liquidation costs as in e.g. Fries and Perraudin (1991) and Acharya and Dreyfus (1989), the optimal closure level may deviate from the insolvency level. See also Allen and Saunders (1993) for a discussion on guarantor forbearance behaviour.

[28] Further, it is in practice impossible for the bank to operate deeply insolvent during a long period of time. For example, in such situations the bank may have difficulties in fulfilling its obligations due to liquidity problems. Note that this feature is not captured by the model.

the audit intensity. Hence, in this context, the inclusion of the compensation in fact solves the audit puzzle.

Another aspect of audit strategies is that it may be questionable to believe that the guarantor will keep the same audit strategy irrespectively of the revealed information from the audits. Instead, one realistic extension of the analysis is to let λ be a function of the observed X. That is, the probability of the next audit should increase if the audit reveals a decreasing solvency, and vice versa. This feature could assuredly be achieved in the framework of the present model if the guarantor, at every audit, considers the observed solvency as the new initial X-value and calculates a new minimizing λ according to the information from the audit. However, it is difficult to model this behavior rigorously. One attempt to approximate this could be to define the audit intensity as λ/X^{α}, where $\alpha>1$, for all $X\geq 1$ and λ for $X<1$. This approach has the obvious drawback that the guarantor is assumed to be able to formulate an audit strategy at every instant based on a variable, which he cannot actually observe.

Another critical assumption made in the paper is that agents are risk neutral. This assumption is made in order to determine the market price for risk to be zero. Nevertheless, as we have mentioned in the appendix, the existence of a traded security, whose price is dependent only of X and time, enables us to determine the market price for risk if agents actually were risk averse. However, the very existence of such security, denoted h(t,X) in the appendix, can conflict the necessity of audits, since the value of the solvency could be infered from the h(t,X) security. This is a general problem in studies involving audits and relying on CCA techniques. For example, in Merton (1978) continuous trading in all securities is assumed, also in the assets of the bank, and still, audits are assumed necessary to determine the solvency.

9. Concluding Remarks

In this paper we have focused on the valuation of deposit insurance when asymmetric information is at hand. The paper is very much a synthesis between Merton (1978) and Merton (1993a). It is shown how the presence of stochastic audits affects the compensation, which an equity holder of a bank requires in order to signal insolvency voluntarily. The value of the guarantee, which contains the compensation to equity holders and audit costs, could be interpreted as the price of imperfect information.

Furthermore, in contrast to the previous related literature, our model can be used to find optimal monitoring strategies of opaque financial assets.

APPENDIX

The objective of this appendix is to value the contingent claim F(t,X) when X is not a traded security.

Since the solvency process, X, is not a traded security, there is a problem of valuing F(t,X) by the standard procedures of hedge arguments or equilibrium pricing. Therefore we assume the existence of a traded security, whose price is dependent only on the value of t and X.

Definition 1: Let h(t,X) be a traded security with the following dynamics:

$$dh(t,X) = \alpha_h h dt + \sigma_h h dW \tag{17}$$

where

$$\alpha_h = \frac{\frac{1}{2}\alpha^2 X^2 h_{xx} + \mu x h_x + h_t}{h}$$

$$\sigma_h = \frac{\sigma x h_x}{h} \tag{18}$$

Definition 2: Let δ be the riskpremium per unit of volatility of security h, dependent only on time and asset X, i.e.

$$\delta = \frac{\alpha_h - r}{\sigma_h} \tag{19}$$

Since the security h is perfectly correlated to security F(t,X) in Section 3 it follows that:

$$\delta = \frac{\alpha_F - r}{\sigma_F} \tag{20}$$

where

$$\alpha_F = \frac{\frac{1}{2}\sigma^2 X^2 F_{xx} + \mu X F_x + F_t}{F}$$

$$\sigma_F = \frac{\sigma X F_x}{F} \tag{21}$$

From [19] we see that δ does not depend on F and from [20] we see that δ does not depend on h. Therefore, δ will not depend on either h or F. Given that we have estimated δ from the market or have made any assumption about it, we can express [21] according to:

$$\frac{1}{2}\sigma^2 X^2 F_{xx} + (\mu - \delta\sigma) X F_x + F_t - rF = 0 \tag{22}$$

Assume $\delta = 0$, i.e. the agents have risk neutral preferences, then $\mu = r$ and [22] reduces to equation [4] in Section 3, where F is a perpetual contingent claim.

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Part II: Risky Debt

Paper 4: Valuation of Barrier Contracts - A Simplified Approach

1. Introduction

A common feature of many types of financial contracts is the inclusion of clauses that entitle both contractual parts to take some kind of action during the lifetime of the contract, conditioned upon some prespecified event. These clauses are often referred to as barriers for the value process, which the contract is contingent on. The existing valuation models of these contracts are often considerably advanced and have tended to obscure the underlying economics. This paper provides a simplified approach to the valuation of financial contracts including barrier features.

Corporate financial contracts can be complex with respect to different provisions and barrier clauses that the contractual parties wish to include. One type of such provision that is commonly included in a debt contract is a bond covenant, which entitles the bondholders to obtain the ownership of the firm if the value of the assets falls to a prespecified "safety" barrier.[1] Other examples are different kinds of callable and puttable provisions.[2] The principle of a callable contract is that the bond can be bought back by the issuing firm at a predetermined call price, that is, an upper barrier where the underlying value process is absorbed. A puttable bond contains provisions that allow the holder of the contract to demand early redemption at a certain price.[3]

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[1] See Black and Cox (1976) and Mason and Bhattacharya (1981) for valuation models of these types of safety covenants. See Smith and Warner (1979) for a comprehensive investigation of various kinds of bond covenants that can be included in debt contracts.

[2] Examples of studies of bond contracts including such covenants are Merton (1974) and Brennan and Schwartz (1977).

[3] See e.g. Hull (1989) for a more complete description of these bond types.

If the objective is to value the equity of the borrowing firm, we know that option theory can be used.[4] However, the presence of a debt contract with a safety barrier implies that an ordinary call option is not the adequate valuation instrument. This is because the valuation formula must take the barrier into consideration. Instead, it is more appropriate to use the so called down-and-out call option formula. This option is identical to a European call with the additional feature that the contract is nullified if the underlying asset price hits a prespecified boundary *below* the exercise price. Thus, the payoff of this option is restricted by an absorbing lower barrier. An up-and-out put option has similar characteristics except that this option is nullified if the value of the underlying asset reaches a certain level *above* the exercise price.

Though not as common as these two instruments, up-and-out call options and down-and-out put options could as well be traded. These, so called capped options, also belong to the "out instrument"-family. These contracts are nullified when the underlying asset hits a barrier located in-the-money. When this occurs the holder is compensated by a rebate.

Another type of barrier option is the "in" option. The principle of this instrument is that if the underlying asset value drops/rises to a "knock-in" price, a call/put comes into existence. When activated, the payoffs of these options are the same as standard options with similar terms. If the "knock-in" level is never crossed the "in" option expires worthless.[5]

The present paper deals primarily with contracts including down-and-out or cap features. The existing literature in this field is far from vast. One of the first studies of barrier options in the preference-free contingent claims analysis literature can be found in Merton (1973), who briefly analyses down-and-out options. Cox and Rubinstein (1985) includes a section on up-and-out put options. Valuation formulas of capped options have been formalized by Boyle and Turnbull (1989) and Flesaker (1992).

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[4] Seminal studies of this subject are Black and Scholes (1973) and Merton (1973,1974). See also Dahlfors and Jansson (1994b) for a brief presentation of the contingent claims approach of corporate securities valuation.

[5] See Gastineau (1993) for an introduction to these barrier options. See also Kunitomo and Ikeda (1992) for exemplification of a number of barrier options that have recently been introduced in the Tokyo financial markets.

One important feature of barrier contracts is path-dependence. The general meaning of this feature is that the value of the contract does not only depend on the underlying asset's value at expiration but also on the path of the asset value process.[6] The path-dependence of barrier contracts comes from the reduction of "forbidden" paths, passing through the barrier, at the valuation of the contract. Therefore, one important component of many valuation models of barrier contracts is the first passage time distribution. This can be derived for Brownian motions by the reflection principle.[7] However, this methodology is rather complicated and requires knowledge of measure theory.

Therefore, the first purpose of this paper is to derive an expression for the first passage time distribution using a simplified approach. Since we use a binomial methodology when deriving this expression, it is straightforward to incorporate our results with the simplified approach for pricing standard European options discovered by Sharpe (1978) and formulated by Cox et al. (1979) in order to value barrier options. It is then possible, due to convergence of the binomial distribution to the normal distribution, to arrive at results in continuous time. The advantage of this, is that pricing formulas in continuous time are often easier to interpret and it is possible to carry out a comparative statics analysis of the results.

The second purpose is to apply the continuous correspondence of the binomial model on financial contracts including provisions that can be interpreted as barriers for the underlying stochastic process. This is done for both down-and-out and capped options. Our model of capped options differs from the studies of Boyle and Turnbull (1989) and Flesaker (1992) only in the derivation of the formulas. We take the conditional binomial model as a starting point, which is an approach never explicitly used in the earlier literature. Further, we believe that our approach is more pedagogical and therefore easier to interpret.

We also apply the valuation model for capped options on a financial guarantee contract, which permits a reduced commitment for the guarantor. This has some similarities with Jones and Mason (1980), who value a partially guaranteed issue of non-callable debt. However, since their contract does not include any "safety barriers" for the guarantor, our analysis is significantly different. In that aspect, Black and Cox (1976) are more

[6] Thus, all "in" and "out" options are path-dependent as well as "look-back" and "Asian" options.

[7] See Harrison (1985) or Karlin and Taylor (1975).

close to our study since they examine the effect of safety covenants on the value of a debt issue. We value the limited guarantee as a capped put option and show that this contract exhibits some interesting features. For example, two completely different barrier levels may imply the same contract value. Further, the value of the cap feature can be both negative and positive depending on how close the barrier is to the exercise price.

The paper is organized as follows:

In Section 2 we present the expression for the first passage time distribution, emanating from a binomial model. In Section 3 we apply the model on down-and-out call options and show an explicit formula with corrections on the condition that the barrier has never been hit. In Section 4 we extend the analysis by considering capped options. The put version of this contract type is analogous to a loan guarantee that enables the writer to limit his obligations to a certain amount. We perform the valuation of this type of limited guarantee in Section 5. Concluding remarks are given in Section 6.

2. A Binomial Derivation of the First Passage Time Distribution

In this section we develop an expression for the one-sided first passage time distribution for a stochastic process using a simple binomial model. This probability distribution is of crucial importance when calculating a value of a financial barrier contract, as we will see in the sections below. Another area of application of this expression is in the theory of collective risk, where the first passage time distribution is used in so called ruin problems.[8]

It is shown in the model below that in order to calculate the probability that the process has never hit the barrier up to time t, the binomial coefficients have to be revised in a certain way. The binomial tree has to be "cut" in order to prevent the paths of the stochastic process from crossing the barrier before the maturity time t. If this is done in a proper way, which gives us the first passage time distribution, then we can move on to the problems of valuing barrier contracts of different kinds.

[8] See, for example, Gerber (1979) and Beard et al. (1984).

2.1 The Model

Assume the stochastic process X_t to be given by:[9]

$$X_t = X_0 u^{t-k} d^k , \qquad 0 \leq k \leq t \tag{1}$$

where

u = positive growth term, $u > 1$.
d = negative growth term, $0 < d < 1$.

The stochastic variable k, the number of downward movements, is supposed to be binomial distributed with probability q. The evolution of X_t is determined by t independent Bernoulli trials.

We will now introduce some definitions in order to derive a probability distribution, which take path-dependence of X_t into consideration.

Definition 1: X_s is said to pass through the barrier, y, at time s, if $X_s < y$ and $X_{s-1} \geq y$, for all $s \in \{1,2,..,t\}$.

Since X_t is a geometric process, it can never reach zero. The lower barrier, y, which X_t is not allowed to pass through, must therefore be strictly greater than zero, in order to be meaningful.

Definition 2: Let $R_t(X,y) = \text{prob}[X_t \geq y \wedge \inf X_s > y \ ; \ 0 \leq s \leq t]$, that is, the joint probability that X_t is not below y at time t, *and* that it has never been below y before t. *This is the one-sided first passage time distribution.*

Definition 3: Let n be the integer of $[\ln(y/X_0)/\ln(d)]$, $0 \leq n \leq t$, representing the maximum *initial* consecutive downward movements which can occur without passing through the barrier, y, up to time t.

[9] Thus, the process has t stochastic movements, k down-moves and t-k up-moves.

Then $R_t(X,y)$is given by:[10]

$$R_t(X,y) = \sum_{k=0}^{n} p^{t-k} q^k \binom{t}{k} + \sum_{k=n+1}^{\left\lfloor \frac{t+n}{2} \right\rfloor} p^{t-k} q^k \left[\binom{t}{k} - \binom{t}{k-1-n} \right] \qquad (2)$$

where

p = the probability for an upward movement

q = the probability for a downward movement

$\lfloor (t+n)/2 \rfloor$ = the integer of (t+n)/2. This represents the maximum number of permitted downward movements, such that X_t is not below y at time t.

Definition 4: Let $\Phi(k;t,q) = \text{prob}[Z \leq k]$ represent the cumulative binomial distribution function. Thus, this is the probability that the number of downward movements are not greater than k, with t trials, where the probability of a downward movement is q.

Then [2] can be written as:

$$R_t(X,y) = \left[\Phi\left(\left\lfloor \frac{t+n}{2} \right\rfloor, t;q \right) - \left(\frac{q}{p} \right)^{n+1} \Phi\left(\left\lfloor \frac{t-n-2}{2} \right\rfloor, t;q \right) \right] \qquad (3)$$

The intuition of [2] and [3] can be found if we consider a binomial tree, which is cut in order to eliminate "forbidden" paths. Since many financial contracts have this feature, it is a straightforward analogy to value down-and-out options as well as many other types of barrier contracts by this model. This is done in the following sections in a continuous-time framework.

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[10] A heuristic derivation of expression [2] is given in appendix A0. The derivation of the first passage time distribution can also be done using the reflection principle, see for example Harrison (1985).

3. Down-and-out Options

In this section we use the model from Section 2 in order to develop a formula for the pricing of down-and-out options.[11] This option is identical to a European call option with the additional feature that the option contract is nullfied if the underlying asset crosses a lower barrier y, which is at a level where the call option is out-of-the-money. When this barrier is crossed a rebate, which can be time-dependent, may be paid to the holder of the option. In the model below we assume the rebate to be zero.

Consider the following asset, for example a stock, whose price, X_t, follows a geometric binomial process:

$$X_t = X_0 u^{t-k} d^k , \qquad 0 \leq k \leq t \tag{4}$$

where

u-1 = change in stock value if there is an upward movement over one period, u>1

d-1 = change in stock value if there is a downward movement over one period, 0<d<1.

In this economy, there is also a risk-free asset with the constant return R=1+r, where r is the interest rate over one period.

The no-arbitrage condition requires that u > R > d.[12]

A call option, C, on this stock with one period left to maturity, which occurs at time t, has the following risk neutral valuation:[13]

$$C_{t-1} = \left[(1-q)C_u + qC_d\right] / R$$

[11] See Gastineau (1993) for practical applications of the down-and-out contract.

[12] If u > d > R , then an investor can issue a riskless loan and invest it in the risky asset. The payoff in the worst state is strictly higher than the cost of the loan. Therefore, in this case we have an arbitrage opportunity, which is not consistent with market efficiency.

[13] See Cox et al.(1979) for the economics behind this.

where

q = (R-u)/(d-u), the martingale probability of a downward movement. Hence, 1-q is the martingale probability of an upward movement.

C_i = $\max[0\,,\, iX_{t-1} - E]$, $i = u,d$

E = exercise price.

Provided that the lower barrier y has never been crossed, the option value, with t periods left to maturity, is given by:

$$C = \frac{\left[\sum_{k=0}^{n} p^{t-k}q^{k}\binom{t}{k} + \sum_{k=n+1}^{\left\lfloor \frac{t+n}{2} \right\rfloor} p^{t-k}q^{k}\left[\binom{t}{k} - \binom{t}{k-1-n}\right]\right] \max\left[0, u^{t-k}d^{k}X_0 - E\right]}{R^t} \qquad (5)$$

where n is given by definition 2.

Notice that the first part of [5] is identical to expression [2], that is, the probability of not having crossed the barrier y, up to time t. This probability is then used to arrive at the discounted expected pay-off of the option.

In order to eliminate the max-expression in [5], we use the following definition:

***Definition* 5:** Let a be the integer of $[\ln(X_0u^t/E) \;/\; \ln(u/d)]$, representing the maximum downward movement such that the option will end up in-the-money at the maturity date, that is, $X_t > E$.

This means that for all numbers of downward movements, $k > a$, we have that $\max[0,u^{t-k}d^kX_0-E]=0$. For $k \leq a$, we have that $\max[0,u^{t-k}d^kX_0-E]=u^{t-k}d^kX_0-E$.

If we use definition 5, the max-expression can be eliminated and the pricing formula [5] can be written as:

$$C = \frac{\left[\sum_{k=0}^{\min[a,n]} p^{t-k}q^{k}\binom{t}{k} + \sum_{k=n+1}^{a} p^{t-k}q^{k}\left[\binom{t}{k} - \binom{t}{k-1-n}\right]\right]\left[u^{t-k}d^{k}X_0 - E\right]}{R^t} \qquad (6)$$

To be able to express [6] with the cumulative binomial distribution, we rewrite it as:[14]

$$C = X_0 \left[\sum_{k=0}^{\min[a,n]} (pu/R)^{t-k} (qd/R)^k \binom{t}{k} + \right]$$

$$+ \left[\sum_{k=n+1}^{a} (pu/R)^{t-k} (qd/R)^k \left(\binom{t}{k} - \binom{t}{k-1-n} \right) \right] - \tag{7}$$

$$- ER^{-t} \left[\sum_{k=0}^{\min[a,n]} p^{t-k} q^k \binom{t}{k} + \sum_{k=n+1}^{a} p^{t-k} q^k \left(\binom{t}{k} - \binom{t}{k-1-n} \right) \right]$$

Now we can express [7] with the cumulative binomial distribution.

$$C = X_0 \left[\Phi(a;t,q') - \left(\frac{q'}{p'}\right)^{n+1} \Phi(a-n-1;t,q') \right] -$$

$$- ER^{-t} \left[\Phi(a;t,q) - \left(\frac{q}{p}\right)^{n+1} \Phi(a-n-1;\, t,q) \right] \tag{8}$$

where

$q = (R-u)/(d-u)$

$q' = (qd/R)$

$p' = 1 - q'$

In previous formulas, time-to-maturity, t, and the number of trading opportunities have been the same. Now, in order to transform [8] to its continuous correspondence, we let t and T denote the "number of trading opportunities" and the "fixed time interval" respectively. Hence, if we let $t \to \infty$ we have accomplished instantaneous trading. For the results to be meaningful we have to adjust u, d, R and q. If the probability for ups and downs are equal, u and d must converge to 1 when $t \to \infty$. Otherwise the stock will move to zero or infinity. If we wish either u or d to deviate strictly from 1 in convergence, then the probability for it must converge to zero. Use that when $t \to \infty$ the

[14] Notice that if min[a,n]=a, then the second and fourth summation of formula [7] disappears.

binomial distribution converges to the normal distribution. Then [8] can be written as:[15]

$$C = X_0 \left[N\left(\frac{\ln \frac{X_0}{E} + \left(r + \frac{\sigma^2}{2} T \right)}{\sigma\sqrt{T}} \right) - \left(\frac{X_0}{y} \right)^{-\frac{2r}{\sigma^2} - 1} N\left(\frac{\ln \frac{y^2}{X_0 E} + \left(r + \frac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}} \right) \right] -$$

$$-Ee^{-rT} \left[N\left(\frac{\ln \frac{X_0}{E} + \left(r - \frac{\sigma^2}{2} T \right)}{\sigma\sqrt{T}} \right) - \left(\frac{X_0}{y} \right)^{-\frac{2r}{\sigma^2} + 1} N\left(\frac{\ln \frac{y^2}{X_0 E} + \left(r - \frac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}} \right) \right] \tag{9}$$

where

$N(\cdot)$	=	the standardised cumulative normal distribution.
r	=	the instantaneous rate of interest.
σ	=	the instantaneous standard deviation of the return of the X_t-process.
T	=	the time-to-maturity.

Thus, Formula [9] expresses the value of a down-and-out call option in continuous time.

In Figure 1 below we have plotted the value of a standard call option (the upper curve) and a down-and-out call option (the lower curve) with the knock-out barrier $y = 0.8$, for $r = 0.1$, $\sigma = 0.2$, $E = 1$ and $T = 1$.

[15] Derivation is given in appendix A1-A3.

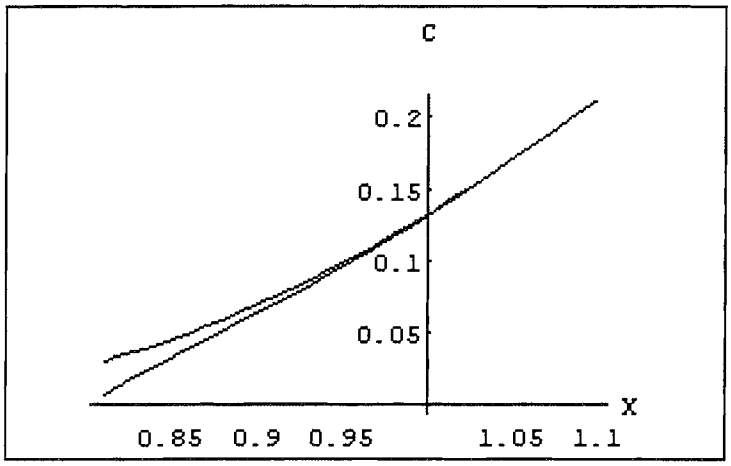

Figure 1

Even though it seems like the two curves coincide for high X-values, it should be noted that the down-and-out call option is always worth strictly less than the standard call option.[16] Since the down-and-out option is worthless at y but the standard call has always a positive value as long as there is a positive time-to-maturity, the divergence between the options increases when the possibility of early termination increases, that is, the closer X_t is to y. Thus, the reason why the down-and-out call option is always worth strictly less than a standard call option, C*, is because the barrier limits the possible X-paths that end up in-the-money. This is verified by the positive difference C* - C:

$$C^* - C = X_0 \left(\frac{X_0}{y}\right)^{-\frac{2r}{\sigma^2}-1} N\left(\frac{\ln\frac{y^2}{X_0E} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - \tag{10}$$

$$- Ee^{-rT}\left(\frac{X_0}{y}\right)^{-\frac{2r}{\sigma^2}+1} N\left(\frac{\ln\frac{y^2}{X_0E} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) > 0$$

This difference is decreasing in X and increasing in y. This is verified by the two figures below. The difference between the value of a standard call option and a down-and-out

[16] Except in the limit, $X \to \infty$, where the difference is zero.

option, which can be viewed as the value of the knock-out barrier, is plotted in Figure 2 and 3 below for the following parameter values: $E = 1$, $r = 0.1$, $\sigma = 0.2$ and $T = 1$.

In Figure 2 below, expression [10] is plotted for different X_0-values and a barrier $y = 0.9$.

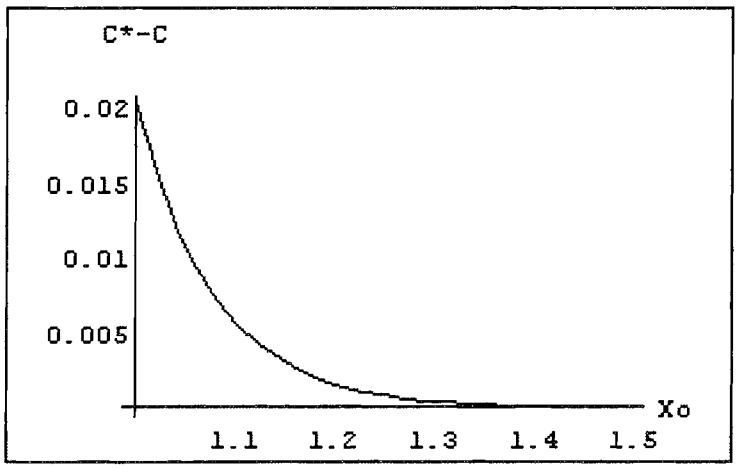

Figure 2

In Figure 3 below, expression [10] is plotted for different barrier levels and $X_0=1$.

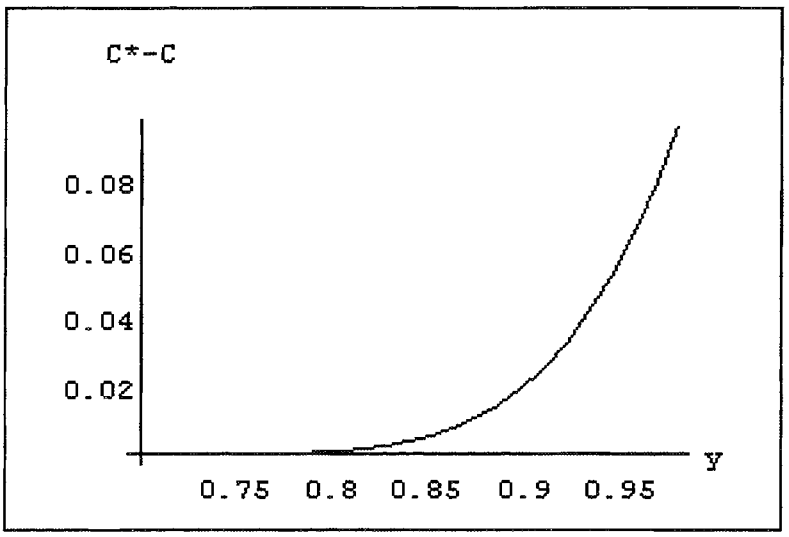

Figure 3

Thus, for the given parameter values, the value of a barrier of, for example, 0.9 is approximately 2.5 percent.

The analysis above concerns finite-time horizon contracts. It is possible to observe more interesting differences between the down-and-out contract and the standard option if we look at these contract's perpetual counterparts. This analysis is to some extent carried out in Dahlfors and Jansson (1994d), where it is shown, for example, that the delta of a perpetual down-and-out call option is greater than unity, which is never the case for standard call options.

4. Capped Options

A capped call option is identical to a European call option with the additional feature that the contract is nullified if the underlying asset, X_t, reaches a prespecified upper boundary, z, where $z > X_0$. When this barrier is hit by the underlying asset process a rebate of arbitrary size is payed to the owner of the contract. Here, we denote the rebate as z-E, where E is the exercise price. The main difference to a down-and-out option is that such a contract is cancelled when the underlying asset is out-of-the money, while a capped call option is nullified when the underlying asset is in-the-money. This type of contract reduces the number of pay-off states and will therefore limit the risk for a writer of the contract.

The existing literature on capped options is rather limited. Boyle and Turnbull (1989) analyze valuation and hedging of such options. Furthermore, they give some examples of applications of capped options such as collar loans and index currency option notes (ICON's).[17] Flesaker (1992) presents a theoretical analysis of capped stock index options and shed light upon some problems with the rules of the existing trading in the U.S. In Flesaker's derivation the option value is decomposed in three parts:

(i) the value of early exercise
(ii) the value of terminal payment
(iii) the reduction in terminal payment due to early exercise.

[17] Capped stock options are traded by the Chicago Board of Options Exchange on the S&P 100 and S&P 500. See Gastineau (1993) for a discussion of circumstances where these instruments are particularly attractive to investors.

We consider our valuation more straightforward and intuitive, in terms of the contract specification. There are two possible cash-flows generated by the contract and we arrive at a pricing formula, which is decomposed in the expected discounted value of these. Note that our formula can be transformed into Flesaker's expression.

The value of a capped call option, C_{cap}, is given by:[18]

$$C_{cap} = C_1 + C_2$$

where

C_1 = discounted expected value of the rebate z-E if the CAP barrier, z, is hit

C_2 = the value of the terminal payment without any early exercise.

The formulas for these components are given by:[19]

$$C_1 = (z-E)\left[\left(\frac{X_0}{z}\right)N\left(\frac{\ln\frac{X_0}{z} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + \right.$$

$$\left. + \left(\frac{z}{X_0}\right)^{\frac{2r}{\sigma^2}} N\left(\frac{\ln\frac{X_0}{z} - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right] \tag{11}$$

[18] The principles of the risk neutral valuation of this formula can be found in Dahlfors and Jansson (1994c, Section 2).

[19] The derivation of C_1 is given in appendix A4 and A5 and the derivation of C_2 is given in appendix A6.

$$C_2 = X_0 \left[N\left(\frac{\ln \frac{z}{X_0} - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln \frac{E}{X_0} - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) + \right.$$

$$\left. + \left(\frac{z}{X_0}\right)^{\frac{2r}{\sigma^2}+1} \left(N\left(\frac{\ln \frac{X_0 E}{z^2} - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln \frac{X_0}{z} - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) \right) \right] -$$

$$- Ee^{-rT} \left[N\left(\frac{\ln \frac{z}{X_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln \frac{E}{X_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) - \right. \tag{12}$$

$$\left. - \left(\frac{z}{X_0}\right)^{\frac{2r}{\sigma^2}-1} \left(N\left(\frac{\ln \frac{X_0}{z} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln \frac{X_0 E}{z^2} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) \right) \right]$$

The following figures are valid for σ=0.2, r=0.1, T=1, E=1 and z=1.2.

Figure 4 exhibits one of the two components of C_{cap}, namely the value of terminal payment conditioned on no early exercise (C_2). This component has a more interesting shape than C_1, which is convex and has the value of 0.2 at X=1.2 and converges to zero for lower X-values.

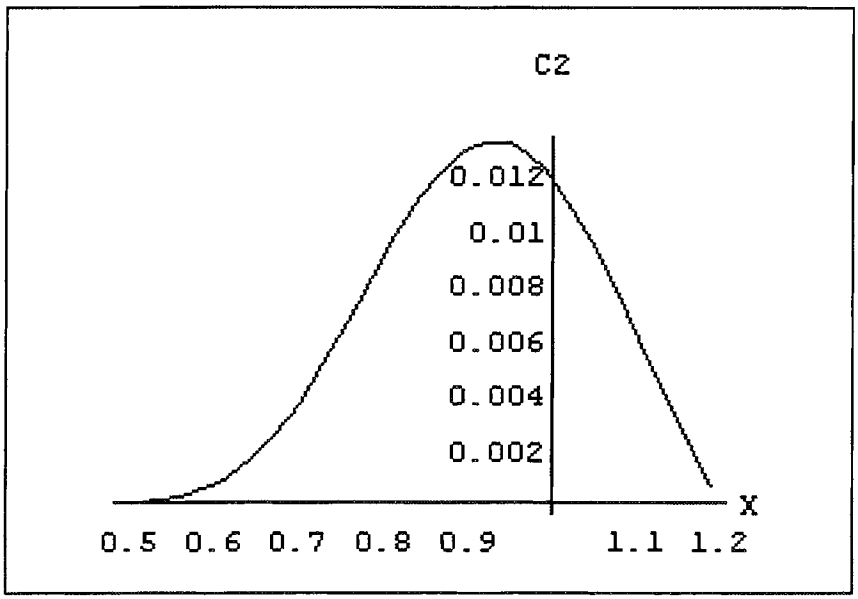

Figure 4

The intuition behind the shape of C_2 is as follows: For X-values close to the cap level, z=1.2, C_2 is close to zero. This is because the contract almost surely will be pre-exercised, which means that terminal payment is unlikely. For low X-values C_2 is also relatively low, since the probability of ending up in-the-money is increasing in X. Not surprisingly, C_2 loses value faster the closer it is to z. C_2 will have a maximum value since negative values are ruled out, which is seen in the figure. Furthermore, the peak moves to the right for shorter time-to-maturities.

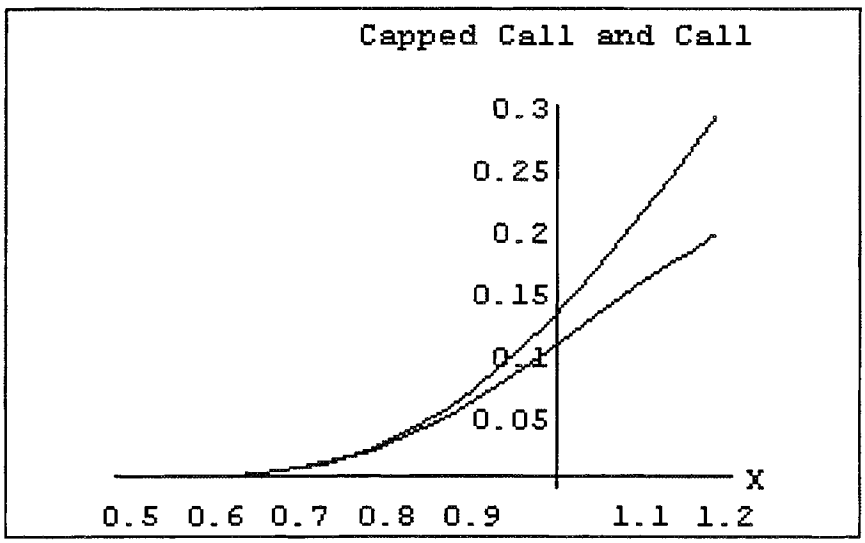

Figure 5

Figure 5 above presents the value of a standard call option (the upper curve) and a capped call option (the lower curve). The standard call option is always more valuable than the capped call contract, for given parameter values. This is so because the capped call loses value in relation to the standard call due to the elimination of paths of the underlying asset passing through the upper barrier z.[20]

Note that there is an important difference to the capped put option in this respect. As will be shown in the next section the difference between the capped put and the standard put contract is positive for most barrier levels, except for barriers close to the exercise price. The analysis of the capped put option will be carried out as an application on a certain type of financial guarantee.

5. Capped Put Option - An Application on Financial Guarantees

If we generalize the concept of barrier options to guarantees in general, a capped put option would correspond to a contract, in which a guarantor wishes to reduce his commitments. A hypothetical situation when such a contract would be preferable is as follows:

Suppose a private bank loan with a promised repayment of E dollars at time T, which is guaranteed by a guarantor that has preferences for limiting his commitment up to an amount of E-y dollars. If the borrower's wealth, X_t, hits the barrier y then the guarantor has the right, according to the contract, to take the borrower's wealth into possession and pay back the loan to the bank. Thus, in this case he must supply an amount of E-y.[21] If, however, the borrower's wealth, never hits the barrier y, then either $E-X_T$ or zero will be the commitment for the guarantor when the contract expires at the maturity date of the loan contract, T.

In order to make the obligation described above, the guarantor must receive a value corresponding to a capped put option:

[20] There are infinitely many paths eliminated. However, paths to very high X values have very low probabilities.

[21] This presupposes that prepayment is both possible and costless.

$P_{cap} = P_1 + P_2$

where

P_1 = value of the payment E-y at early exercise.

P_2 = value of terminal payment conditioned on the event of no early exercise.

The formulas for each component are given by:[22]

$$P_1 = (E-y)\left[\left(\frac{X_0}{y}\right)N\left(\frac{\ln\frac{y}{X_0} - \left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + \left(\frac{y}{X_0}\right)^{\frac{2r}{\sigma^2}} N\left(\frac{\ln\frac{y}{X_0} + \left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right] \tag{13}$$

$$P_2 = Ee^{-rT}\left[N\left(\frac{\ln\frac{E}{X_0} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\frac{y}{X_0} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) + \right.$$

$$\left. + \left(\frac{y}{X_0}\right)^{\frac{2r}{\sigma^2}-1}\left(N\left(\frac{\ln\frac{X_0}{y} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\frac{X_0 y}{y^2} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right)\right] - \tag{14}$$

$$- X_0\, N\left[\left(\frac{\ln\frac{E}{X_0} - \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\frac{y}{X_0} - \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - \right.$$

$$\left. - \left(\frac{y}{X_0}\right)^{\frac{2r}{\sigma^2}+1}\left(N\left(\frac{\ln\frac{X_0 E}{y^2} - \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\frac{X_0}{y} - \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right)\right]$$

[22] The derivation of these components are analogous to the derivation of [11] and [12], which is given in appendix A4-A6.

Before discussing the value of the financial guarantee, we will discuss the P_1 and P_2 components.[23]

P_1 has almost the same features as C_1 in Section 4. P_2 requires some extra attention for the same reasons as C_2. The following figure, which thus shows the value of the terminal payment conditioned on no early exercise, is plotted for y=0.8.

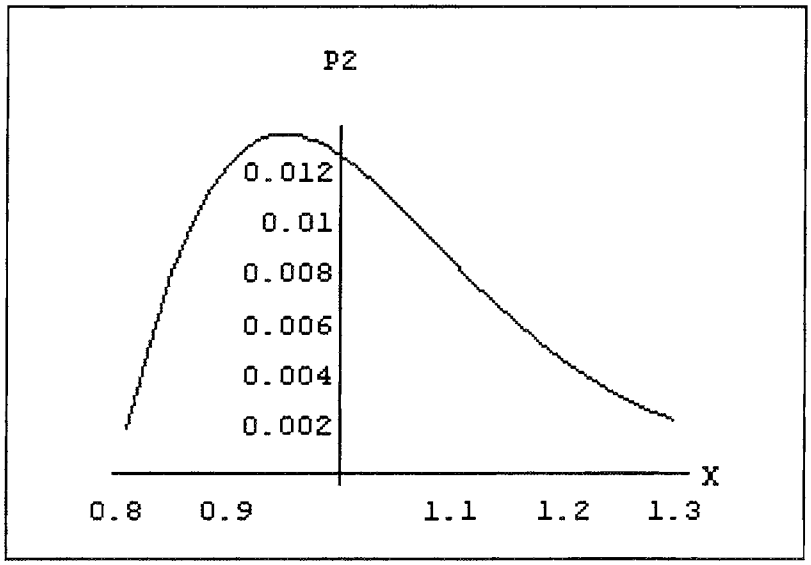

Figure 6

The intuition behind the shape of the graph is as follows: For X-values close to the cap level, y=0.8, P_2 is close to zero. This is because terminal payment is unlikely, since the contract almost surely will be pre-exercised. For large X-values, P_2 is also relatively low, since the probability of ending up in-the-money is decreasing in X. Since negative values are ruled out, P_2 will have a maximum, which is seen in the figure. The major difference to C_2 is the skewness, since the contract value goes to zero faster towards the barrier.

Figure 7 below, shows values of the guarantee, P_{cap}, for different barrier values, y and X=1.0. When y goes zero the capped put option value goes to the value of a standard put option which, for given parameter values, is 0.03787. This is because the importance of the barrier is increasing in y and for sufficiently low y-values the barrier feature does not have any value at all. When y goes to E, in this case E=1, the value of the capped put

[23] Notice, as a simplification, we disregard the effect from the initially paid premium. For a discussion of this feature of financial guarantees, see Dahlfors and Jansson (1994a).

option goes to zero. The reason for the upward slope segment is that P_1 increases in y, in this segment, because of two effects:

(i) the probability of pre-exercise increases in y.

(ii) the pay-out, E-y, decreases in y.

In the upward sloping segment the first effect dominates the second. After a certain barrier level, in this case $y \approx 0.88$, the probability of pre-exercise is certainly increasing but the value of pre-exercise decreases faster. Further, P_2 has a convex shape with the following asymptots: the Black-Scholes put option value for $y \rightarrow 0$ and zero for $y \rightarrow E$.

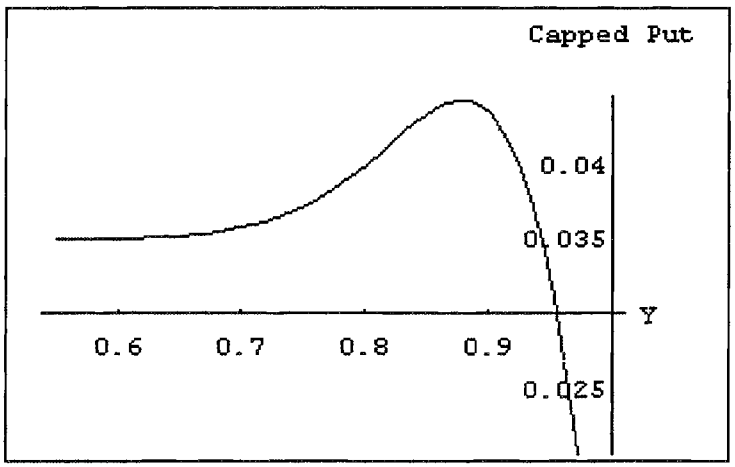

Figure 7

One interesting implication of the shape of the graph in Figure 7 is that in many cases the capped put option has the same value for two different barrier levels. The most extreme example of this is that the contract has the value of 0.03787 if y is close to zero but also for $y \approx 0.93$. This means that a guarantor would receive the same compensation for a nearly unlimited guarantee and a guarantee limited to E-y, that is, 1-0.93 = 0.07. One natural question is weather the guarantor would prefer one of these contracts to the other. The "advantage" of the unlimited guarantee is that no pre-exercise payment has to be done. The "disadvantage" is that this contract is open to extreme outcomes. The "advantage" of the limited guarantee is that extreme pay-outs are ruled out. The

"disadvantage" is that a pre-exercise payment of E-y can take place. The choice between the two contracts depends on the guarantor's preferences for these features.

The following figures exhibit the difference between a standard European put option and a capped put option. The parameter values are as presented in the introduction of this section.

Figure 8 plots the standard put option (the lower curve) and a capped put option (the upper curve) for y=0.8.

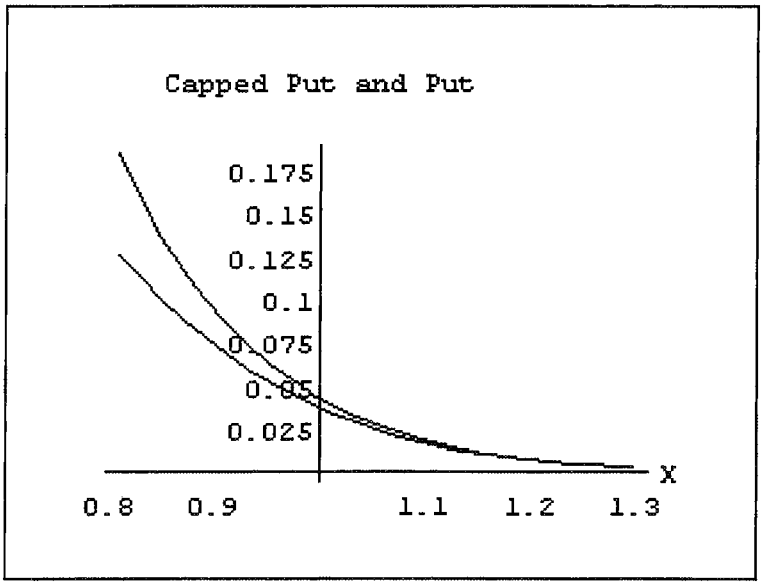

Figure 8

We can see that the capped put curve converges with the standard put, that is, the cap feature has a zero value, when X receives sufficiently high values. Needless to say, the value of the capped put equals E-y at X=y, that is, 0.2 in this case.

Contrary to the call option case of Section 4, where the standard option was always more valuable than the capped option, the value difference between the capped put and the standard put option can be both positive and negative, which is shown in Figure 9 below. This figure, which plots the difference between the capped put option and the standard put option for different barrier levels $Y \in [0.65, 0.999]$, is valid for X=1.0.

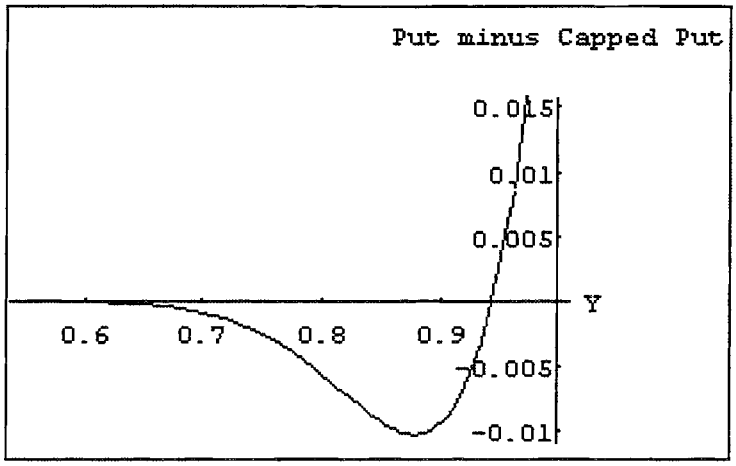

Figure 9

The shape of this figure is obvious if it is compared with Figure 7, which is almost its mirror. As a final remark, we conclude that barriers are often considered as a value-reducing device in a contract. However, as we have seen from the results in this section this is not always the case.

6. Concluding Remarks

In this paper we have developed a simple binomial expression of the first passage time distribution. The continuous correspondence of this distribution is an important tool for the continuous time valuation of financial contracts including barriers. We have shown how different types of such contracts can be valued by this model, for example, down-and-out options and capped options. We have also given an application of this on a financial guarantee with a limited commitment. This analysis has given some interesting results. For example, it is not obvious that the inclusion of a barrier in a contract, in order to rule out extreme pay-outs in case of default of the borrower, lowers the value of the contract.

Issues discussed in this essay are extended in Dahlfors and Jansson (1994 b,c), where the techniques developed here are used in order to value risky debt and other financial guarantee contracts.

Appendix

Section A0 below discusses the heuristic of [2]. Sections A1 - A3 show the derivation of the limit of [8], which is [9]. Sections A4 and A5 show the derivation of [11] and A6 shows the derivation of [12].

A0

Consider the binomial tree, in Figure 10 below, of order t=8, which is divided into two symmetric parts. The interpretation of this tree is as follows:

Node A is reached for eight upward moves and zero downward moves. Accordingly, node B requires seven upward moves and one downward move to be reached. The same pattern is valid for the rest of the end-nodes, C to I.

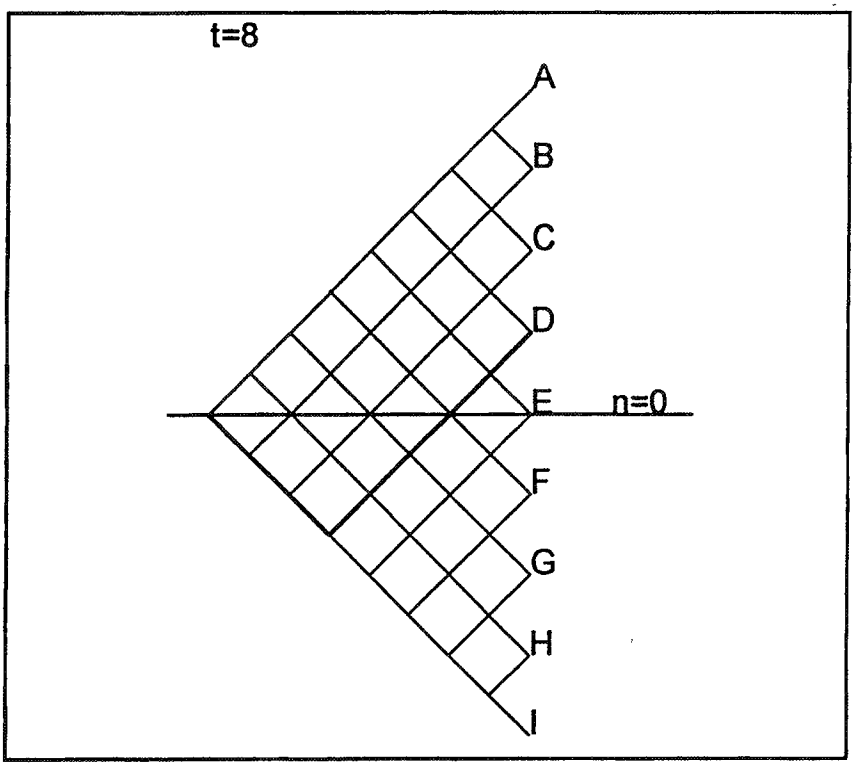

Figure 10

The division line, which is seen in the figure, is called the lower barrier and is denoted n. Symmetric division of the tree, as in Figure 10, means that n=0.

Now, suppose that it is forbidden to pass through the lower barrier, n, at any time, up to time t=8. We know that the probability of reaching a single end-node with k downward moves and 8-k upward moves is $p^{8-k}q^{k}$ multiplied with the number of possible ways of doing this. However, the case must not arise where some of these paths pass through the barrier on their way to the end-node. Therefore, the number of paths to each end-node that pass through the barrier must be eliminated. The path to the highest node A, that is with k=0, will never pass through the barrier. However, the number of paths to the next highest node B, with k=1 downs, must be reduced by one forbidden path. That is, the path that starts with one down and passes through the barrier and after that takes 8-1=7 ups. The nodes below this are corrected with 8 over 1 forbidden ways, 8 over 2 and so on.

This is the result if we have symmetric division of the tree. If the tree is divided below its horizontal midpoint, that is, n>0, the barrier is moved downwards. This in turn means the existence of less "forbidden" paths for a given end-node.

In Figure 11 below we have an example for t=8 and n=2.

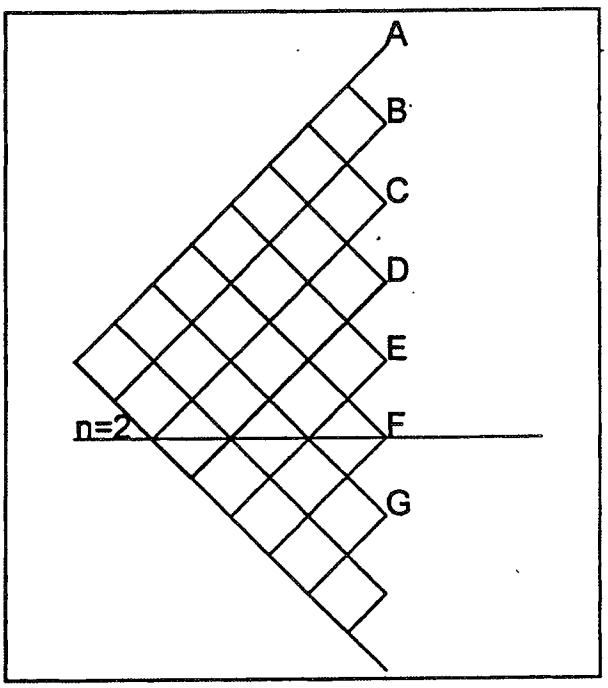

Figure 11

The binomial coefficients for the three highest nodes, A,B and C do not have to be corrected, since the paths to these can never pass through the barrier, n=2. The probability of reaching these nodes are represented by the first term in the summation of [2], that is:

$$\text{Prob}(A)+\text{Prob}(B)+\text{Prob}(C) = p^8q^0\binom{8}{0}+p^7q^1\binom{8}{1}+p^6q^2\binom{8}{2} = \sum_{k=0}^{2} p^{8-k}q^k\binom{8}{k} \quad (15)$$

Node D in the figure has one path that is "forbidden". This path is represented by the bold line and is "forbidden" since it passes through the barrier. Hence, this path must not be included in the probability of reaching node D. Therefore we subtract this path in the formula. For node E we have 8 "forbidden" paths. For node F we have 28 "forbidden" paths. Node G, and the nodes below G, are not possible to reach at all, since they are located below the barrier. So, the probability of reaching nodes D, E and F is given by:

$$\text{Prob}(D)+\text{Prob}(E)+\text{Prob}(F) = p^5q^3\left[\binom{8}{3}-\binom{8}{3-1-2}\right]+p^4q^4\left[\binom{8}{4}-\binom{8}{4-1-2}\right]+$$

$$+\ p^3q^5\left[\binom{8}{5}-\binom{8}{5-1-2}\right] = \sum_{k=2+1}^{\left\lfloor\frac{8+2}{2}\right\rfloor} p^{8-k}q^k\left[\binom{8}{k}-\binom{8}{k-1-2}\right] \quad (16)$$

This is the second term of [2] for t=8 and n=2.

We now conclude by giving the expression for the probability of not having passed through the lower barrier n=2 up to time t=8. This is the sum of expression [15] and [16], that is:

$$\sum_{k=0}^{2} p^{8-k}q^k\binom{8}{k} + \sum_{k=2+1}^{\left\lfloor\frac{8+2}{2}\right\rfloor} p^{8-k}q^k\left[\binom{8}{k}-\binom{8}{k-1-2}\right] \quad (17)$$

This is expression [2] for t=8 and n=2.

A1

In this section we show the continuous correspondence of q' and p'. Since p' = 1-q' it is enough to investigate q'. We know from [8] that q' = q(d/R) = (d/R) (R-u)/(d-u). When we go to the limit, i.e. from [8] to [9], the first and the second moments of the stock price must be well defined.[24] Therefore we make the following definitions:

Definition 6: Let $h \equiv T/t$ represent the elapsed time between successive trading events. T is a fixed lenght of calendar time, expressed on an annual basis and t is the number of trading events during T.

Definition 7: Let $u-1 \equiv \exp(\sigma\sqrt{h})-1$ and $d-1 \equiv \exp(-\sigma\sqrt{h})-1$ represent changes in the asset price if there is an upward and downward movement respectively, over the time-interval h.

Definition 8: Let $R^h \equiv (1+rh)$ be the return on the riskless asset over the time-period h. r is interest rate expressed on an annual basis. The total return over the time-interval [0,T] is therefore $R^{ht}=R^T$. When the number of trading events t goes to infinity, that is $t \to \infty$, then the total return $R^T \to e^{rT}$. Hence, $T\ln R = rT$.

Substituting definitions 6-8 into the expression for q' gives:

$$q' = \frac{d}{R^h}\frac{(R^h - u)}{(d-u)} = \frac{e^{(h\ln R-\sigma\sqrt{h})} - 1}{e^{(h\ln R-\sigma\sqrt{h})} - e^{(h\ln R+\sigma\sqrt{h})}} =$$

$$= \frac{h\ln R - \sigma\sqrt{h} + \frac{\sigma^2}{2}h}{-2\sigma\sqrt{h}} = \frac{1}{2} - \frac{1}{2}\left[\frac{\ln R + \frac{\sigma^2}{2}}{\sigma}\right]\sqrt{h} \qquad (18)$$

The second equality in [18] is given after a Taylor expansion up to terms of second order and we have omitted h-terms of higher order than one.

Analogous we find that:

[24] This means that neither expected value nor variance is allowed to go to zero or infinity when $T \to \infty$.

$$q = \frac{1}{2} - \frac{1}{2}\left[\frac{\ln R - \frac{\sigma^2}{2}}{\sigma}\right]\sqrt{h} \tag{19}$$

The expected asset price, $E[X_t]=E[X_0u^{t-k}d^k]$, after t trading events can be written in terms of expected growth as:

$$E\left[\ln\frac{X_t}{X_0}\right] = \left[E[k]\ln\frac{d}{u} + t\ln u\right] = \left[q\ln\frac{d}{u} + \ln u\right]t \equiv \mu t \tag{20}$$

where we have used that $E[k]=tq$ and $E[\bullet]$ is the expectation operator.

If we substitute the expressions for u and d from definition 7 and q from expression [19] into [20]. We see that:

$$\mu = \left(r - \frac{1}{2}\sigma^2\right)h \tag{21}$$

When the number of trading events goes to infinity, i.e. $t \to \infty$, we can express the expected growth in X, for a given T, as:

$$E\left[\ln\frac{X_T}{X_0}\right] = \left(r - \frac{1}{2}\sigma^2\right)ht = \left(r - \frac{1}{2}\sigma^2\right)T \tag{22}$$

The major difference between [20] and [22] is as follows: In [20] the time interval, T and the number of trading events, t are identical. In [22], we have the continuous correspondence of [20] with an infinite number of trading events.

Taking the expected value with q' instead of q we have that:

$$E'\left[\ln\frac{X_t}{X_0}\right] = \left[E'[k]\ln\frac{d}{u} + t\ln u\right] = \left[q'\ln\frac{d}{u} + \ln u\right]t \equiv \mu' t \tag{23}$$

If we substitute the expressions for u and d from definition 7 and q' from expression [18] into [23], we have that:

$$\mu' = \left(r + \frac{1}{2}\sigma^2\right)h \tag{24}$$

According to the same pattern the variance of the growth is:

$$\begin{aligned} \mathrm{Var}\left[\ln\frac{X_t}{X_0}\right] &= q(1-q)\left[\ln\frac{d}{u}\right]^2 t = \\ &= \left(\frac{1}{4} - \frac{1}{4}\frac{\left(r - \frac{1}{2}\sigma^2\right)^2}{\sigma^2}h\right)4\sigma^2 ht = \sigma^2 T - \left(r - \frac{1}{2}\sigma^2\right)^2 \frac{T^2}{t} \end{aligned} \tag{25}$$

When the number of trading events goes to infinity, i.e. $t \to \infty$, the last term of [25] disappears and the variance of the growth is:

$$\mathrm{Var}\left[\ln\frac{X_T}{X_0}\right] = \sigma^2 T \tag{26}$$

The corresponding variance under q' is:

$$\begin{aligned} \mathrm{Var}'\left[\ln\frac{X_t}{X_0}\right] &= q'(1-q')\left[\ln\frac{d}{u}\right]^2 t = \\ &= \left(\frac{1}{4} - \frac{1}{4}\frac{\left(r + \frac{1}{2}\sigma^2\right)^2}{\sigma^2}h\right)4\sigma^2 ht = \sigma^2 T - \left(r + \frac{1}{2}\sigma^2\right)^2 \frac{T^2}{t} \end{aligned} \tag{27}$$

We will use the expressions for the expected value and the variance below, when we take the limit of the binomial distribution.

A2

In order to go from [8] to [9] we must derive the limit expressions of $\Phi[a,t;q']$, $\Phi[a-n-1,t;q']$, $\Phi[a,t;q]$ and $\Phi[a-n-1,t;q]$ respectively.

We know from definition 4 that $\Phi[a,t;q']$=prob[$k \leq a$]. In order to standardize this expression we subtract the expected value and divide by the standard deviation. Then we have that:

$$\text{prob}\left[\frac{k - tq'}{(tq'(1-q')^{\frac{1}{2}}} \leq \frac{\dfrac{\ln\frac{X_0}{E} + t\ln u}{\ln\frac{u}{d}} - \varepsilon - tq'}{(tq'(1-q'))^{\frac{1}{2}}}\right] \tag{28}$$

Multiply the numerator and the denominator of the right hand side by ln(u/d) and the numerator and the denominator of the left hand side by ln(d/u) and use the expressions for $\mu't$ and $\sigma't$:

$$\text{prob}\left[\frac{\ln\frac{X_t}{X_0} - \mu't}{\sigma'\sqrt{t}} \leq \frac{\ln\frac{X_0}{E} + \mu't - \varepsilon\ln\frac{u}{d}}{\sigma'\sqrt{t}}\right] \tag{29}$$

In [28], ε is a real valued scalar between zero and one. The purpose of this factor is to make **a** integer valued. When $t \to \infty$, $\ln(d/u) \to 0$, $\mu't \to (r + 0.5\sigma^2)T$ and $\sigma'\sqrt{t} \to \sigma\sqrt{T}$. Therefore we can write [29] as:

$$\text{prob}\left[\frac{\ln\frac{X_T}{X_0} - \mu' T}{\sigma'\sqrt{T}} \leq \frac{\ln\frac{X_0}{E} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right] \equiv$$

$$\equiv N[Z] = N\left[\frac{\ln\frac{X_0}{E} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right] \tag{30}$$

Thus $\Phi[a;t,q']$ converges to $N[Z]$ in [31] as $t \to \infty$.

For $\Phi[a\text{-}n\text{-}1;t,q']$ we proceed in the same way and get:

$$\text{prob}\left[\frac{\ln\frac{X_T}{X_0} - \mu' T}{\sigma'\sqrt{T}} \leq \frac{\ln\frac{X_0}{E} + \left(r + \frac{\sigma^2}{2}\right)T + 2\ln\frac{y}{X_0}}{\sigma\sqrt{T}}\right] \equiv$$

$$\equiv N[Z] = N\left[\frac{\ln\frac{y^2}{X_0 E} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right] \tag{31}$$

Thus $\Phi[a\text{-}n\text{-}1;t,q']$ converges to $N[Z]$ in [31] as $t \to \infty$.

The limit expressions for $\Phi[a;t,q]$ and $\Phi[a\text{-}n\text{-}1;t,q]$ are analogous.

A3

In this section we derive the continuous correspondence of $(q'/p')^{n+1}$ and $(q/p)^{n+1}$ in [8], which appear in the continuous option formula [9]

Proposition: The factors $(q'/p')^{n+1}$ and $(q/p)^{n+1}$ in equation [8] have the limits $(X_0/y)^{-2r/\sigma^2-1}$ and $(X_0/y)^{-2r/\sigma^2+1}$ respectively when $t \to \infty$.

Proof:

We divide this proof into two parts, in **I** and **II**, proving the limits of $(q'/p')^{n+1}$ and $(q/p)^{n+1}$ respectively.

I

Use definition 3 of **n** and definition 6 of **d** and **u** and the expressions for q' and p'=1-q' respectively, given by [18]. Then we have:

$$\left(\frac{q'}{p'}\right)^{n+1} = \left[\frac{1 - \left(\dfrac{\ln R + \dfrac{\sigma^2}{2}}{\sigma}\right)\sqrt{h}}{1 + \left(\dfrac{\ln R + \dfrac{\sigma^2}{2}}{\sigma}\right)\sqrt{h}}\right]^{\frac{\ln\frac{X_0}{y}}{\sigma\sqrt{h}}+1} \tag{32}$$

Let $\gamma \equiv \ln(X_0/y)$ and $\xi \equiv \gamma/\sigma\sqrt{h}$ and substitute these into the expression [32]:

$$\left(\frac{q'}{p'}\right)^{n+1} = \left[\frac{1 - \left(\frac{\ln R + \frac{\sigma^2}{2}}{\sigma}\right)\frac{\gamma}{\sigma\xi}}{1 + \left(\frac{\ln R + \frac{\sigma^2}{2}}{\sigma}\right)\frac{\gamma}{\sigma\xi}}\right] \frac{\left[1 - \left(\frac{\ln R + \frac{\sigma^2}{2}}{\sigma}\right)\frac{\gamma}{\sigma\xi}\right]^{\xi}}{\left[1 + \left(\frac{\ln R + \frac{\sigma^2}{2}}{\sigma}\right)\frac{\gamma}{\sigma\xi}\right]^{\xi}} \tag{33}$$

If we let $t \to \infty$ then $h \to 0$, which implies that $\xi \to \infty$.

Use the following limit formula for the exponential constant e = 2.718..:

$$\lim \xi \uparrow \infty \left(1 + \frac{\alpha}{\xi}\right)^{\xi} = e^{\alpha} \tag{34}$$

Expression [33] is of the same form as [34]. Therefore we can write [33] as:

$$1 \cdot e^{-2\left(\gamma \frac{\ln R}{\sigma^2} + \frac{\gamma}{2}\right)} = e^{\gamma\left(-\frac{2r}{\sigma^2} - 1\right)} = \left[\frac{X_0}{y}\right]^{-\frac{2r}{\sigma^2} - 1} \tag{35}$$

This is the first part of the second term in formula [9].

II

Now we proceed with expression $(q/p)^{n+1}$. With same substitutions as above we have:

$$\left(\frac{q}{p}\right)^{n+1} = \left[\frac{1 - \left(\frac{\ln R - \frac{\sigma^2}{2}}{\sigma}\right)\frac{\gamma}{\sigma\xi}}{1 + \left(\frac{\ln R - \frac{\sigma^2}{2}}{\sigma}\right)\frac{\gamma}{\sigma\xi}}\right] \frac{\left[1 - \left(\frac{\ln R - \frac{\sigma^2}{2}}{\sigma}\right)\frac{\gamma}{\sigma\xi}\right]^{\xi}}{\left[1 + \left(\frac{\ln R - \frac{\sigma^2}{2}}{\sigma}\right)\frac{\gamma}{\sigma\xi}\right]^{\xi}} \tag{36}$$

Let $t \rightarrow \infty$ and we have:

$$1 \cdot e^{-2\left(\gamma\frac{\ln R}{\sigma^2} - \frac{\gamma}{2}\right)} = e^{\gamma\left(-\frac{2r}{\sigma^2}+1\right)} = \left[\frac{X_0}{y}\right]^{-\frac{2r}{\sigma^2}+1} \tag{37}$$

This completes the proof.

A4

This appendix shows the probability expressions for upper and lower barriers.

Since the discounted risk neutral value of the rebate, z-E, is the discounted value weighted by the risk neutral probability of reaching the upper or lower absorbing barrier at a certain point of time, given that it has never been crossed before, we have to use the first passage time probability distribution. The aim of this appendix is to derive this distribution for a Brownian motion from a binomial process.

We take the continuous time version of formula [3], that is, the probability that the process is alive at a certain time t, as our starting point. If we calculate this probability one instant later, t + dt, it will be clear from the continuous version of [3] that this probability is smaller. The decrease in the probability over the interval (t,t+dt) is the probability that the barrier has been hit during that period of time.

For the case of an upper barrier:

Let z be an upper barrier and suppose that a stochastic process is given by:

$X_t \quad = \quad X_0 u^k d^{t-k}$, where $X_0 < z$

Then the conditional probability of not passing through z up to time t is:

$$R_t(X,z) = \left[\Phi\left(\frac{t+n}{2}, t;p\right) - \left(\frac{p}{q}\right)^{n+1} \Phi\left(\frac{t-n-2}{2}, t;p\right)\right] \tag{38}$$

where n is analogous to definition 3 but with an upper barrier, z.

Now, let the process go to the limit in accordance with A1-A3. When $t \to \infty$ then formula [38] has the following limit:

$$R_T(X,z) = N\left(\frac{\ln\frac{z}{X_0} - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \left(\frac{z}{X_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(\frac{\ln\frac{X_0}{z} - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \tag{39}$$

This is the probability of not having hit the upper barrier z on the interval [0,T]. Therefore, the complementary event of having hit z on that interval is:

$$G_T(X,z) \equiv 1 - R_T(X,z) = N\left(\frac{\ln\frac{X_0}{z} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + \left(\frac{z}{X_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(\frac{\ln\frac{X_0}{z} - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \tag{40}$$

In order to calculate the probability of hitting the upper barrier z at an arbitrary point of time provided that it has not been reached before, we differentiate $G_T(X,z)$ with respectto the time variable T. This gives us:

$$\frac{d[G_T(X,z)]}{dT} = \ln \frac{z}{X_0}\left[\sigma^2 T^3 2\pi\right]^{-\frac{1}{2}} \exp - \frac{\left[\ln \frac{X_0}{z} + \left(r - \frac{1}{2}\sigma^2\right)T\right]^2}{2\sigma^2 T} \qquad (41)$$

A lower barrier:

Now, let y be a lower barrier and let the stochastic process be:

$X_t = X_0 u^{t-k} d^k$, where $X_0 > y$

As before the conditional probability of not passing through the lower barrier y is given by:

$$R_t(X,y) = \left[\Phi\left(\frac{t+n}{2}, t;q\right) - \left(\frac{q}{p}\right)^{n+1} \Phi\left(\frac{t-n-2}{2}, t;q\right)\right] \qquad (42)$$

where n is given by definition 3.

If we proceed as before and take the limit of [42] we have:

$$R_T(X,y) = N\left(\frac{\ln \frac{X_0}{y} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \left(\frac{y}{X_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(\frac{\ln \frac{y}{X_0} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \qquad (43)$$

Carrying out the same procedure as in the upper barrier case we have

$$G_T(X,y) \equiv 1 - R_T(X,y) =$$

$$= N\left(\frac{\ln \frac{y2}{X_0} - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + \left(\frac{y}{X_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(\frac{\ln \frac{y}{X_0} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \qquad (44)$$

The corresponding time derivative of $G_T(X,y)$is:

$$\frac{d[G_T(X,y)]}{dT} = \ln\frac{X_0}{y}\left[\sigma^2T^3 2\pi\right]^{-\frac{1}{2}} \exp - \frac{\left[\ln\frac{y}{X_0} - \left(r - \frac{1}{2}\sigma^2\right)T\right]^2}{2\sigma^2 T} \tag{45}$$

A5

In this appendix we show the derivation of C_1, given by [11], that is, the value of the rebate, z-E, which is paid out if and only if the ***upper*** barrier z is hit. In continuous-time this probability is given by $G_T(X,z)$, which is the risk-neutral probability measure, shown in A4 above. Then, dG/ds is the probability of passing z at time s on condition on that this is the first passage time. Thus we multiply z-E by this probability for all intervals, ds, and discount it by the corresponding time-factor exp(-rs). Then we sum these factors over the total interval [0,T] and arrive at:

$$C_1 = (z-E)\int_0^T e^{-rs}\, G'(s|\cdot)ds \tag{46}$$

Substitute G', which is given by [41], into the integral and we get:

$$C_1 = (z-E)\int_0^T e^{-rs}\ln\frac{z}{X_0}\left[\sigma^2 s^3 2\pi\right]^{-\frac{1}{2}} \exp\frac{-\left[\ln\frac{X_0}{z} + \left(r - \frac{1}{2}\sigma^2\right)s\right]^2}{2\sigma^2 s}\, ds =$$

$$= (z-E)\left(\frac{X_0}{z}\right)\int_0^T \ln\frac{z}{X_0}\left[\sigma^2 s^3 2\pi\right]^{-\frac{1}{2}} \exp\frac{-\left[\ln\frac{X_0}{z} + \left(r + \frac{1}{2}\sigma^2\right)s\right]^2}{2\sigma^2 s}\, ds = \tag{47}$$

$$= (z-E)\left[\left(\frac{X_0}{z}\right)N\left(\frac{\ln\frac{X_0}{z} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + \left(\frac{z}{X_0}\right)^{2\frac{r}{\sigma^2}} N\left(\frac{\ln\frac{X_0}{z} - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right]$$

This is expression [11].

A6

In this appendix we derive the value of C_2, given by formula [12]. In order to value the terminal pay-off, we use the risk neutral measure Q, denoted E^Q in the first row in formula [48] below, and thereby discount the future payments by the risk-free rate of interest. Under this martingale measure Q, all discounted price-processes in the economy are martingales.

$$C_2 = e^{-rT} E^Q \left[\max\left[X_T - E, 0\right] \mid X_s < z, \forall s < T\right] = e^{-rT} \int_{\ln\frac{E}{X_0}}^{\ln\frac{z}{X_0}} f_T(X,z)(X_T - E)dx \quad (48)$$

where

$$f_T(X,z) = \frac{dR_T[X,M]}{dX} = \frac{1}{X\sigma\sqrt{T}} e^{\frac{m\ln X}{\sigma^2} - \frac{m^2 T}{2\sigma^2}} \left[\varphi\left(\frac{\ln X}{\sigma\sqrt{T}}\right) - \varphi\left(\frac{2\ln\frac{Z}{X_0} - \ln X}{\sigma\sqrt{T}}\right)\right] \quad (49)$$

$$m \equiv r - \frac{1}{2}\sigma^2$$

and φ is the standard normal density function.

If we substitute [49] into [48] and carry out the integration, the desired result [12] is obtained.

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Paper 5: Valuation of Risky Debt in the Presence of Jumps, Safety Barriers and Collaterals

1. Introduction

Since Black and Scholes (1973) and Merton (1974) formalized the relationship between valuation of options and discount debt with default risk, there have been numerous contributions on the pricing of more complex corporate securities.[1] The main principle of this valuation is to view equity, in the presence of a zero-coupon risky debt, as a European call option written on the firm value, with an exercise price equal to the nominal value of debt. A preference-free valuation of the debt may then be given by the balance sheet relation between equity, debt and assets.[2] The present paper investigates different aspects of the pricing of risky debt in the presence of jumps, safety barriers and collaterals.

The first purpose of the paper is to analyze a debt contract in a jump-diffusion model. In this sense we apply a model similar to Merton (1976) on valuation problems discussed in Merton (1974). This specification captures extraordinary changes in a borrower's asset portfolio.

The second purpose of the paper is to value a risky debt contract when a safety covenant is included. Safety covenants, of different kinds, are common features of debt contracts aiming to limit the risk of a creditor. These give the creditor the right to nullify the contract, when certain conditions are fulfilled. The condition of interest for the present analysis is when the borrower's wealth or asset process decreases to a prespecified lower level, called a safety barrier. Black and Cox (1976) is an early studie of the effects of

[1] Comprehensive presentations of the different directions in the contingent claims analysis literature of corporate securities valuation are given in Smith (1980), Merton and Mason (1985) and Ingersoll (1987).

[2] A preference-free valuation is obtained if the market is complete.

safety covenants on the valuation of corporate securities. In, for example, Smith (1980), it is argued that such observed provisions in loan contracts, are efficient control mechanisms which address the borrower-lender conflict rather than being evidence of disequilibrium in the lending market, as argued in the credit rationing literature. In the present paper the impact on the debt value from changes in the barrier level is investigated in more detail. In order to see the impact from a barrier, we compare the results with Merton (1974).[3]

The third purpose is to allow for collaterals or guarantors in a debt contract. Such arrangements will limit the risk taken by the creditor and are common features on the commercial loan market. For example, a bank will often demand some kind of collateral or co-signer for a personal loan to be granted. Previous studies in this area are Smith (1980) and Johnson and Stulz (1987). Johnson and Stulz analyze how options subject to default risk and debt guarantees should be valued. However, they do not end up with explicit pricing formulas. The present paper applies the same approach on a debt contract and gives an explicit pricing formula.

The fourth purpose of the paper is to allow for the inclusion of *both* collaterals and safety covenants in a debt contract. This contract specification is, by nature, quite complex and can be regarded as an extension of the studies mentioned above. The analysis can be used to compute relative values of collaterals and safety covenants, which is important information for the contractual parts when designing the debt contract.[4]

The paper is organized as follows:
Section 2 is devoted to a brief presentation of the principle of valuing risky debt with contingent claims analysis. In Section 3, we value a risky debt when the underlying

[3] Another early study in this area is Mason and Bhattacharya (1981). They allowed for discontinuous sample paths and were in this respect extending Black and Cox (1976). Longstaff and Schwartz (1993), Nielsen et al.(1993) and Kim et al.(1993) are recent studies, incorporating both default and interest-rate risk.

[4] There is a vast literature on the design of financial securities in general, and corporate liabilities in particular focusing on the consequences of agency relationships. Jensen and Meckling (1976) is an important early contribution. One interesting attempt to integrate this framework with the methods of risky debt valuation à la Merton (1974) can be found in Anderson and Sundaresan (1992). Since our paper deals with a complete market and perfect information economy, we do not consider this strand of research.

asset follows a jump-diffusion process. Section 4 contains an analysis of debt valuation in the presence of safety barriers. In Section 5 the valuation is carried out for a situation when the borrower has collaterals or when a third part guarantor is involved. An extension of these sections is given in Section 6 by allowing for both stochastic collaterals *and* safety covenants in the debt contract. Concluding remarks are given in Section 7.

2. A Short Presentation of the Contingent Claims Approach to Corporate Securities Valuation

In this section we present the principle of valuation of risky debt with contingent claims analysis.[5]

Consider a firm, financed by equity and a single issue of zero-coupon debt with a promised final payment of E dollars. At the maturity date of the debt, T, the equity holders will pay off the debt holders if they can, that is if the value of the firm, V_T, is greater than or equal to E. If not, the ownership of the firm passes to the debt holders. Thus, the pay-off structure of the equity holders, f(T,v), and the debt holders, D(T,v), at time T is:

$$f(T,v) = \max[v - E,0]^{6}, \text{ for all } v.$$

and

$$D(T,v) = \min[v,E] = v - \max[v - E,0] = v - f(T,v), \text{ for all } v.$$

Hence, according to this pay-off structure, which is known as the balance sheet relation, it follows that:

$$v = f(T,v) + D(T,v), \text{ for all } v.$$

If the debt is risk-free, its value at time t<T is:

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[5] The section is also motivated by the definition of the notation used in later sections.

[6] We know from Black and Scholes (1973) and Merton (1974) that the equity can be valued as a European call option on the firm with the promised payment of the debt as the strike price.

$$D(t,v) = \exp[-r(T-t)]E$$

where r is the risk-free rate of interest.
That is, the promised payment at time T, E, is discounted by the risk-free rate.

However, since the debt is risky, in terms of default, we must discount E by a factor larger than r. We denote this factor R. Therefore, the value of the debt at time t<T is:

$$D(t,v) = \exp[-R(T-t)]E$$

Thus, R, which is the yield-to-maturity, can be written as:

$$R = -[1/(T-t)]\log[D(t,v)/E]$$

We know from the balance sheet relation that D(t,v) can be written as: D(t,v) = v - f(t,v). Therefore, since V_t is definitely known at time t, it is enough to calculate the the value of the equity f(t,v) as a call otpion to get the value of the debt, D(t,v), at time t.

Thus, the value of the debt in terms of yield-to-maturity is:

$$R = -[1/(T-t)]\log[v/E - f(t,v)/E]$$

This yield will primarily depend on:

(i) the risk-free rate of interest, r
(ii) the stochastics of the firm value, V_t
(iii) the time-to-maturity, T-t
(iv) the promised payment, E
(v) the various restrictions contained in the debt contract, like safety covenants, collaterals and seniorities.

We conclude this section by refering to a discussion in Merton (1974) of how valid R is as a measure of risk. Merton uses the standard deviation of the rate of the return on the bond as the measure of risk. Thus, in Merton's sense, if a bond is riskier, that is, has a higher standard deviation than another bond, it must have a larger risk premium, R-r. Therefore, we require the risk premium to move in same direction as the standard

deviation, in response to changes in the underlying variables. However, Merton concludes that R-r is a valid measure of risk premium only for bonds with same maturities. For bonds with different maturities it is not clear if a higher standard deviation implies a higher risk premium.[7] In the sections below we will use the measure R-r in order to evaluate the pricing formulas.

3. Valuing Risky Debt in a Jump-Diffusion Model

In this section we value a risky debt when the underlying asset process, V_t, follows a jump-diffusion process. This is motivated by empirical studies. For example, Ball and Tourous (1983) find significant evidence of jumps in the stock returns on New York Stock Exchange in terms of "fatter tails". Further, the mixture of jumps and diffusion makes it possible to create return distributions which capture these features.

Merton (1976), Jones (1984) and Naik (1990) allow for jump-diffusion processes. However, these studies do not consider a complete market. Therefore, it is a priori not possible to construct a portfolio strategy which replicates a claim contingent on the underlying asset. In order to get a valuation, which all investors will agree up on, independently of their preferences towards risk, additional assumptions have to be made. Merton uses an equilibrium framework and assumes the jump risk to be perfectly diversifiable. This facilitates a creation of a hedge porfolio, which replicates the contingent claim "on average". The deviation of the portfolio value from the claim is not priced in equilibrium. Jones uses options with different strike prices to achieve a hedge portfolio. However, he *exogenously* specifies the price processes of these options, which may be strong assumption. Naik uses equilibrium arguments to fix option prices and then computes the hedge portfolio containing these options.

Another method of achieving a preference-free valuation is simply to complete the market by assuming the existence of as many securities as there are risk sources.[8] Thus, if there are a finite number of jump sizes, n, we must have n securities taking care of

[7] See Merton (1974) for an explanation of this.

[8] We are aware that this may be a naive approach.

these in order to create a perfect hedge portfolio.[9] For simplicity, we assume only one jump size and are thereby in need of one additional security, which is represented by security G_t below.

The purpose of the model is to derive the value of a risky debt, whose value is contingent on the underlying asset process, V_t, which follows a jump-diffusion process. From the discussion in Section 2, we know that at time T, the expiration date, the creditor will receive:

$\min[v,E] = v - \max[v - E,0]$, for all v.

The usual assumptions in this type of analysis are as follows:

Assumption 1: Capital markets are perfect in the sense that there are no transactions costs or taxes and that trading takes place continuously. All agents have free access to all available information.

Assumption 2: The agents have homogeneous beliefs about the stochastic behavior of the value of the borrower's assets. This excludes the possibility that the borrower alters his behaviour once the contract is signed.

Assumption 3: The dynamic behavior of the value of the assets is independent of the probability of default.

Assumption 4: There are no costs, for example, bankruptcy costs, associated with default.

Assumption 5: Short sales of all assets, with the full use of proceeds, are unrestricted.

Assumption 6: Borrowing and lending, at the same rate of interest, are unrestricted.

Suppose that the process V_t, representing the value of the borrower's assets, has the following dynamics:

[9] Given that each of these n securities does not take care of more than one jump size.

$$dV = \alpha_V V dt + \sigma_V V dW + k_V V dN \tag{1}$$

where α_V = the instantaneous expected return per unit of time on the asset between the jumps, assumed to be constant.

σ_V = the instantaneous standard deviation of the return of the asset, assumed to be constant.

k_V = the proportional decrease in V in case of a jump, with $-1<k_V<0$.

W = a standard Brownian motion.

N = a Poisson process with intensity parameter λ.

The assets of the borrower is thereby represented by one security. Note that we could have represented it by an asset *portfolio* as well, which would not have changed the results.

The Poisson process jumps upwards and since k_V is negative, V_t jumps downwards with a constant proportional size of k_V. The restriction, $-1<k_V<0$, guarantees non-negative values of V_t. In Merton (1976) the jump sizes are assumed to be lognormal distributed. Therefore, in his framework, V_t can take both upward and downward jumps. We assume that only downward jumps can take place. This reflects extraordinary losses in the borrower's wealth due to macro events. Thus, this assumption captures only the lower "fat tail" of the return distribution according to the discussion in the introduction of this section.

There is also a riskless asset, B_t, with the following dynamics:

$$dB = rBdt \tag{2}$$

where $r>0$ is the risk-free rate of interest, assumed to be constant.[10]

Suppose that this economy also consists of a third security, G_t, with the following

[10] This assumption of constant interest rates is of course difficult to justify since we know that interest rates in reality are stochastic. However, since our objective is to value risky debt in the presence of jumps, safety barriers and collaterals the relaxation of this assumption would make our models very complex with loss of intuition as a result. Examples of studies incorporating stochastic interest rates in risky debt valuation problems are Nielsen et al. (1993), Kim et al. (1993) and Longstaff and Schwartz (1993).

dynamics:

$$dG = \alpha_G G dt + k_G G dN \tag{3}$$

where α_G = the instantaneous expected return between the jumps, assumed to be constant.

k_G = the proportional decrease in G_t in case of a jump,with $-1<k_G<0$, assumed to be constant.

N = a Poisson process with intensity parameter λ.

The G_t-process jumps *exactly* at the same instant as the V_t-process. The existence of such a security in reality may be disputable, but we can think of it as a security reflecting some kind of macro risk.[11]

Now, let $f(t,v) \in C^{1,2}$ be the price, at time t, of a contingent claim, which gives its holder the right to buy asset V_t for E dollars at time T. The value of the contingent claim at time T is therefore $f(T,v) = \max[v-E,0]$, for all v.

If we make a Taylor expansion of $f(t,V_t)$ and apply Ito's lemma, we end up with the following dynamics:

$$df = \alpha_f f dt + \sigma_f f dW + k_f f dN \tag{4}$$

where

$$\alpha_f = \frac{\frac{1}{2}\sigma_v^2 V^2 f_{vv} + \alpha_v V f_v + f_t}{f}$$

$$\sigma_f = \frac{\sigma_v V f_v}{f} \tag{5}$$

$$k_f = \frac{f(t, V + k_v V) - f(t,V))}{f}$$

[11] *If* the jumps are a reflection of information, which is firm specific, as in Merton (1976), then the jump component of the asset's return will represent non-systematic risk. In that case, we would not need security G_t to conduct a risk neutral valuation, since the risk is diversifiable and will not be priced in equilibrium.

In order to achieve a preference-free valuation of the contingent claim, we construct a replicating hedge portfolio, P, containing the assets V_t, G_t, and B_t. Since this portfolio will have the same dynamics as the contingent claim, it follows that dP-df = 0 in order to avoid arbitrage. The result of this arbitrage argument, which is the value of the contingent claim at time t, f(t,v), is:[12]

$$f(t,v) = \sum_{N=0}^{\infty} \frac{\left(\left(\frac{r-\alpha_G}{k_G}\right)\tau\right)^N}{N!} e^{-\left(\frac{r-\alpha_G}{k_G}\right)\tau} \left[v(1+k_V)^N e^{-\left(\frac{r-\alpha_G}{k_G}\right)k_v\tau} \Phi(d_N) - e^{-r\tau} E\Phi(d_N - \sigma_v\sqrt{\tau}) \right] \tag{6}$$

where

$$d_N = \frac{\ln\frac{v}{E} + N\ln(1+k_v) + \left(r - \left(\frac{r-\alpha_G}{k_G}\right)k_v + \frac{1}{2}\sigma_v^2\right)\tau}{\sigma_v\sqrt{\tau}} \tag{7}$$

and

τ = T-t, time-to-maturity

$\Phi(\bullet)$ = the standardized normal distribution

Note that [6] is independent of the security G_t, which is only used in the hedge portfolio. However, the effects from the existence of security G_t appears in the pricing formula through the Poisson intensity $(r-\alpha_G)/k_G$.[13]

One interesting feature of the model is that when the jump component, k_V, is changed we do not have to alter the intensity parameter in order to maintain the no-arbitrage condition. This will not be the case, as we shall see in Section 5, when we introduce a stochastic collateral, which follows a jump process. In such a case the intensity will decrease as the jump component increases and therefore the effect on the debt value will be smaller than in the present model.

[12] The derivation of this expression by construction of a hedge portfolio is given in appendix A1. Appendix A2 illustrates the corresponding derivation by the use of the martingale representation technique, formalized by Harrison and Kreps (1979).

[13] This intensity is derived in appendix A1.

Since our aim is to value the risky debt, we apply the analysis in Section 2, where we concluded that the value of a risky debt at time t<T could be expressed as:

$D(t,v) = v - f(t,v)$,

where f(t,v) is given by [6].

Figures 1-3 below show the effects on the debt value, D(t,v) from changes in the underlying asset v, the jump size k_V and time-to-maturity t respectively. All figures are plotted for $r = 0.1$, $\sigma = 0.2$, $E = 1$.

Figure 1 is plotted for different initial values of the underlying process, v and $\tau = 1$, $k_V = -0.1$ and a Poisson intensity $\delta \equiv (r-\alpha_G)/k_G = 2$.

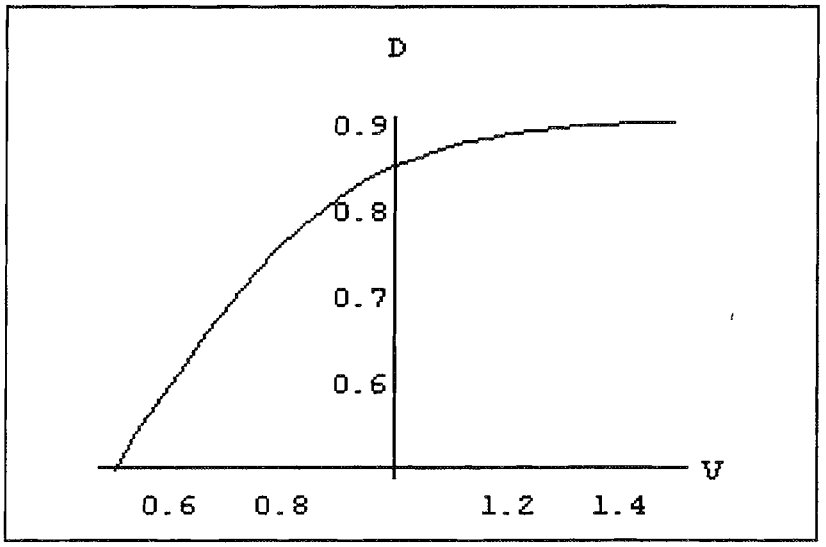

Figure 1

We can see from Figure 1 that, as expected, when v goes to infinity the bond value, D(t,v), goes to the value of a risk-free bond.

It is common to express bonds in terms of yields rather than prices. Therefore we use the results from Section 2 and compute the risk premia.

Thus, we have R(T) = [-1/(T-t)]log[(v - f(t,v))/E], where R(T) is the yield-to-maturity of a promised payment of E dollars at time T, *given* that the payment can take place, that is, $V_T \geq E$.

In Figure 2 below, the risk premium, R(T)-r, is plotted for a jump-size range of $k_V \in [-0.3,0]$ and initial asset values of $v \in [0.9,1.2]$.

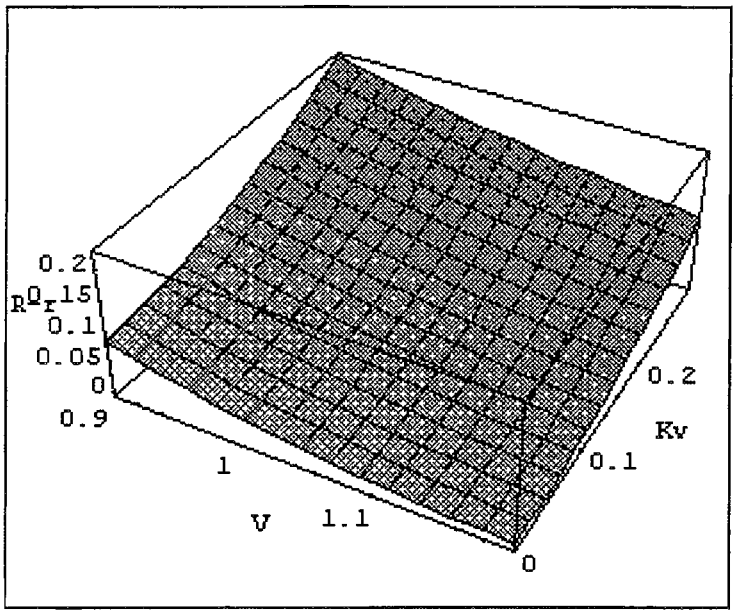

Figure 2

The risk premia for $k_V = 0$ correspond to the no-jump case, which is derived in for example Merton (1974). This is shown as the front contour in Figure 2. The risk premia vary between 0.5% and 18.8%.

In Figure 3 below, which is plotted for v = 1.2, k_V = -0.1 and δ = 1, the risk premium,R(T)-r, is plotted for different time-to-maturities. The upper curve represents this relation for a risky bond with jumps and the lower curve for a bond without jumps like in Merton (1974). Thus, the difference between the upper and the lower curve represents the value of the jumps. We see that the existence of jumps significantly increases the risk premium. This is discussed further in connection to Figure 4 below.

The intuition of the shape of these graphs, which is valid for asset values strictly greater than the promised payment of E=1, is as follows:

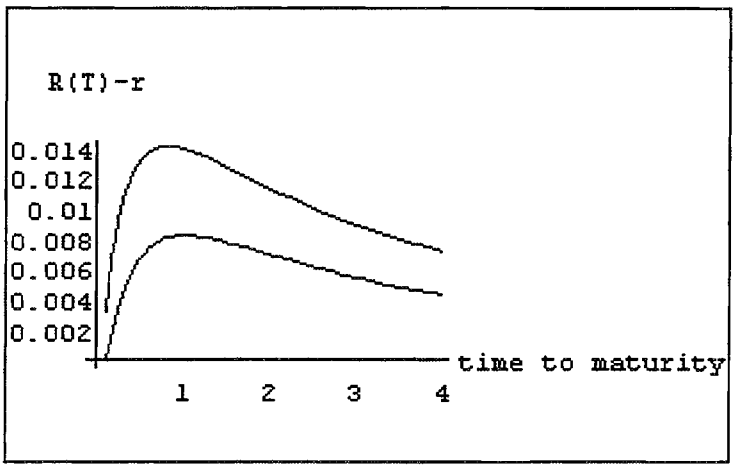

Figure 3

The lower curve:

This curve is valid for the no-jump case, similar to Merton (1974). The reason for the curve to start at a zero risk premium when the time-to-maturity is zero is that the promised payment of E = 1 will take place with probability one, since v = 1.2. For short maturities an additional increase of time-to-maturity will increase the probability that V_T will end up below the promised payment of E dollars, that is, the borrower defaults on the loan. This will increase the risk premium and explaines the upward sloping segment. After a certain point of time, approximately at t = 1 in the figure, when the time-to-maturity becomes larger this probability declines. This is because infinity is the upward potential for the asset process but the lognormal property restricts downward potential. This explains the downward sloping segment. The location of the peak has mainly to do with the loaction of v compared to the promised payment of E. For v<< E, the curve is downward sloping for all time to maturities. For v>> E, the peak exists and is located more to the right the larger v is.

The upper curve:

This is the jump case. The explaination of the shape of the curve is similar to the lower curve. The reason for this curve to be located strictly above the lower curve is the impact from jumps.

The difference between the two curves, explained by the impact from jumps, is plotted in Figure 4 below. The upper curve is plotted for $\sigma_V = 0.2$ and the lower curve for $\sigma_V = 0.1$. The rest of the parameter values are the same as before.

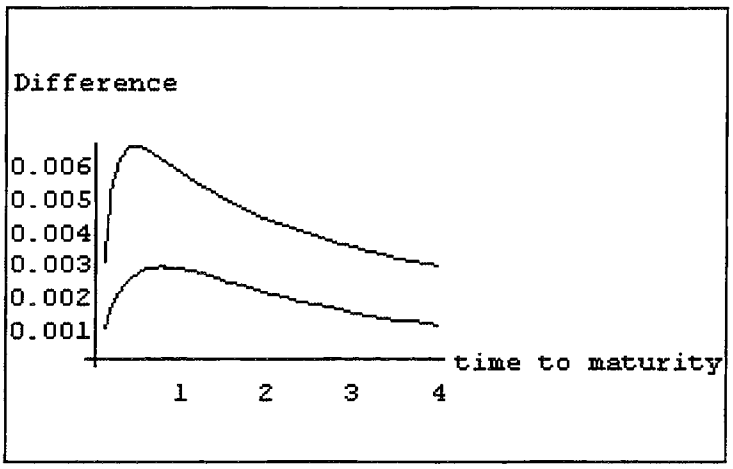

Figure 4

We observe that the difference in risk premia, between the jump case and the no-jump case, increases in the standard deviation of the underlying asset. However, we observe from Figure 5 below, that this is true only for certain intervals of standard deviations.

The following figure is plotted for v = 1.2, r = 0.1, E = 1, δ = 1, t = 1 and k_V = -0.1 respectively.

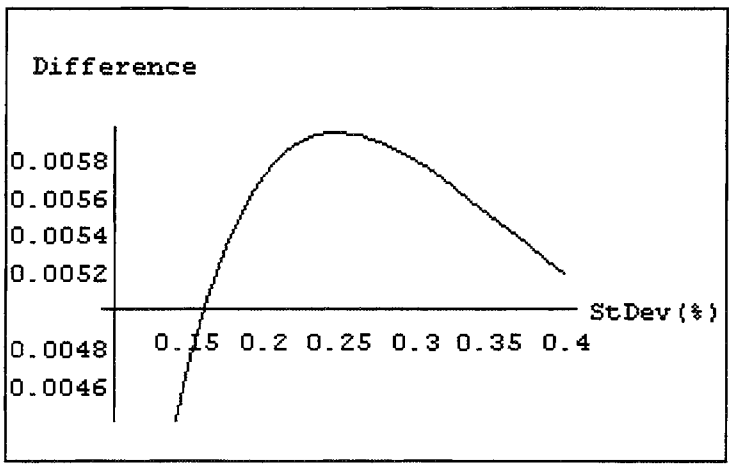

Figure 5

The shape of Figure 5 indicates that the difference between the jump case and the no-jump case, that is, the Merton (1974) model, increases in standard deviation up to approximately 0.25 and then decreases. The reason for this comes from the impact from the standard deviation on the jump components in formulas [6] and [7].

In this section we have investigated the impact from jumps in the underlying asset value on the risk premium, R(T)-r. We have also seen, by comparing with results in Merton (1974), that this impact is significant. In the next section we will study the impact on the risk premium from the presence of a safety barrier.

4. Loans with a Safety Barrier

In the introduction we briefly mentioned that exogenously given safety covenants could be included in a debt contract in order to limit the risk of the contract. This contract provision gives the creditor the right to act in a certain way, prior to maturity, if the wealth of the borrower hits a lower prespecified barrier. This safety barrier could be thought of as a default boundary. In this model we assume that the lender receives the borrower's remaining wealth when this event occurs.[14] The valuation problem is no longer path-independent since there is a positive probability of a premature exercise of the contract. Therefore, the valuation of such debt contract can be carried out by the use of a down-and-out call option.

In this section we will analyze the impact of such a lower horizontal barrier, M, in a debt contract.[15]

Let the borrower's asset process be represented by the following dynamics:

$$dV = \alpha V dt + \sigma V dW \tag{8}$$

[14] Thus, we assume no bankruptcy costs and disregard bargaining processes among corporate claimants during the bankruptcy.

[15] Thus, we assume a constant safety barrier. Nielsen et al.(1993) use a stochastic default boundary based on the firm's inability to raise new funds for its due payments.

where α = the instantaneous expected return per unit of time on the asset, assumed to be constant.

σ = the instantaneous standard deviation of the return of the asset, assumed to be constant.

Assume that the borrower's initial asset, v, is financed by equity and a single issue of zero coupon debt with a safety provision of the above-mentioned type. Then we know that the borrower's equity can be valued as a down-and-out call option. The value of this option is:[16]

$$f^D(t,v) = v\left[\Phi\left(\frac{\ln\frac{v}{E}+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) - \left(\frac{v}{M}\right)^{-\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\ln\frac{M^2}{vE}+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)\right] - \tag{9}$$

$$- Ee^{-r\tau}\left[\Phi\left(\frac{\ln\frac{v}{E}+\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) - \left(\frac{v}{M}\right)^{-\frac{2r}{\sigma^2}+1}\Phi\left(\frac{\ln\frac{M^2}{vE}+\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)\right] -$$

Following the analysis of Section 3, the value, at time t, of the debt contract is:[17]

$$D(t,v) = v - f^D(t,v)$$

The value of the debt for different v-values is plotted in the figures below. The parameter values are $r = 0.1$, $\sigma = 0.2$, $\tau = 1$ and $E = 1$.

The following Figure 6 is plotted for a barrier M=0.75.

---

16 For a derivation see Dahlfors and Jansson (1994a). Notice that this derivation makes use of definition 2-5 in Section 6 in the present paper.

17 Notice that we have derived this debt value in a slightly different way in Section 6. Formula [23] in that section is the complete expression.

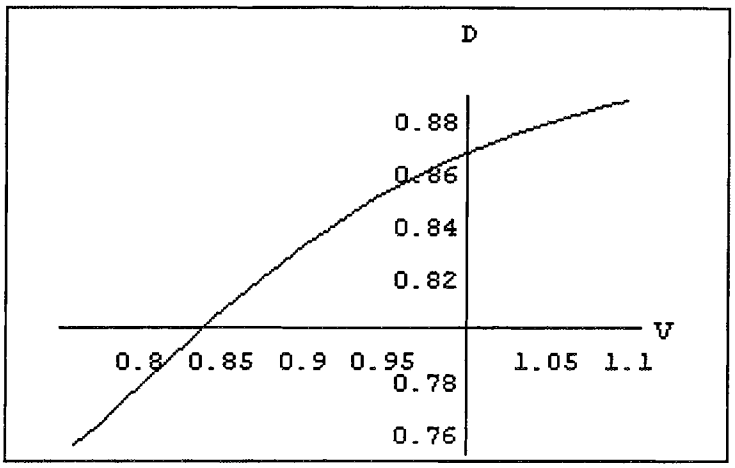

Figure 6

The relationship between v and D in Figure 6 is what we could expect, namely that the value of the debt is increasing in v. However, as we will see in the next figure, this pattern is far from unambigous, but will depend on the barrier's location in relation to the discounted, by the risk-free rate, nominal value of the debt.

This figure is plotted for M=0.95

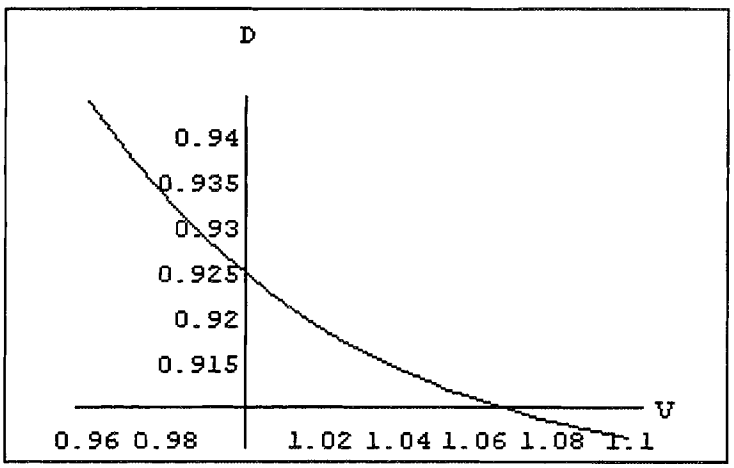

Figure 7

The following figure is plotted for $M=Ee^{-0.1}$

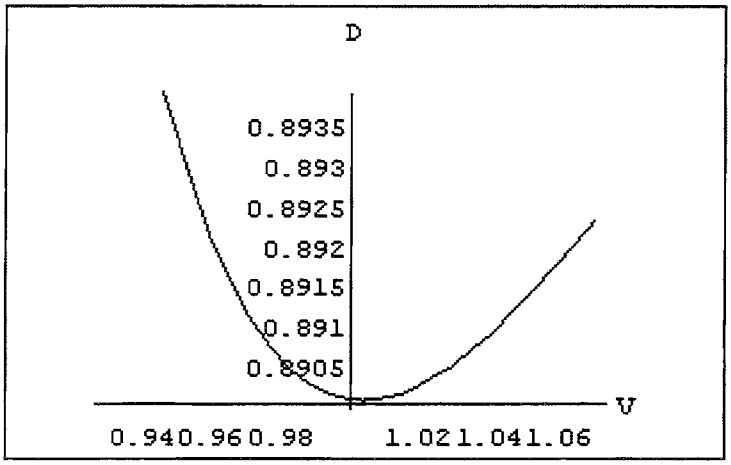

Figure 8

From Figure 7 we can observe two interesting features of the debt value. First, the value of the risky debt exceeds the value of a risk-free debt, Ee^{-rT}, which in this case is 0.9048. Second, the debt value is actually decreasing in v. This second feature may seem puzzling with respect to what we know from Merton (1974), where the value of a debt contract is always increasing in v.

The first feature is possible in contracts including barriers located in the interval $M \in (Ee^{-rT}, E]$. The reason for this is that in this barrier interval, the lender receives an amount of M at the hitting time t, which, for $t \in [0, (\log M + rT)/r]$, exceeds the nominal value of a risk-free debt, that is, $Ee^{-r(T-t)}$.

The second feature is explained as follows: We can decompose the debt value into two parts:[18]

(i) a value of early exercise if the barrier is hit.
(ii) a value of terminal payment if the barrier is never hit.

For relatively high barriers, in relation to the promised payment, E, the first part dominates the second and for low barriers the second part dominates the first.[19] The

[18] This decomposition is analyzed in detail in Dahlfors and Jansson (1994a).

[19] More formally, the sign of the derivative of debt value with respect to V depends on whether the derivative of the down-and-out option, see expression [9], with respect to V is greater or less than 1. Recall that $D_t = V_t + f_t^D$.

early exercise value is decreasing in v and the terminal payment is increasing in v. Thus, we would expect the debt value to decrease in v for sufficiently high values of M. These effects explain the shapes of Figures 6,7 and 8 above. In Figure 6 the second part dominates the first. In Figure 7 the first part dominates the second. In Figure 8 the first part dominates the second for initial asset values, v, between M and approximately one, whereas the second part dominates for v>1. Of course, for all barriers $M \in [0,E]$, the value of the debt equals the risk-free value, when $v \to \infty$. To conclude this discussion, we argue that it is important take these effects into consideration when valuing debt contracts with barrier features.[20] We will come back to the importance of this in Section 6.

In order to see the relationship between v, M and D simultaneously, we present the following three-dimensional figure. This figure, which can be viewed as a summary of the discussion above, is plotted for barriers in the interval $M \in [0.75,0.90]$ and initial asset values of $v \in (0.90,1.1]$. The risk premia, R(T)-r, are shown on the vertical axis. The parameter values are $r = 0.1$, $\sigma = 0.2$, $\tau = 1$ and $E = 1$.

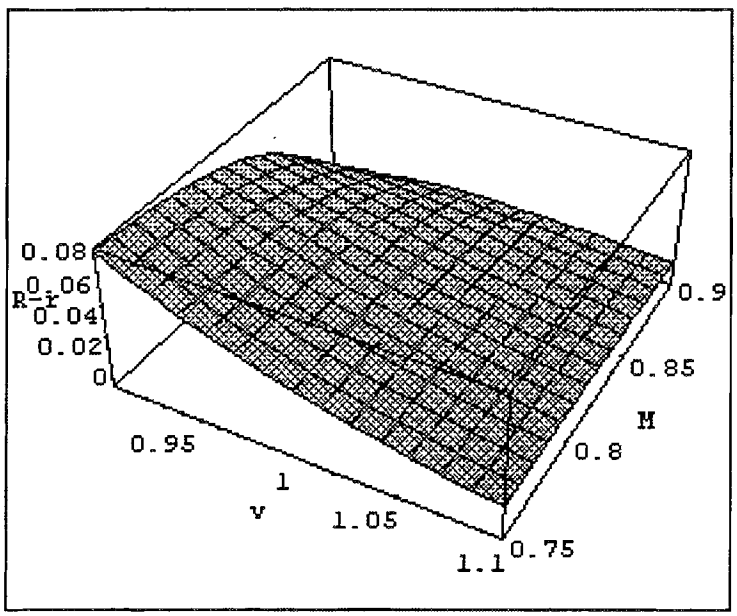

Figure 9

[20] These effects from the barrier levels have, to our knowledge, not been discussed in the previous literature.

In Figure 10 below we have plotted the difference in risk premia between a debt contract without a barrier, like in Merton (1974), and debt contract including a safety barrier. The figure is plotted for $v = 1$, $E = 1$, $r = 0.1$, $\tau = 1$ and $\sigma = 0.2$.

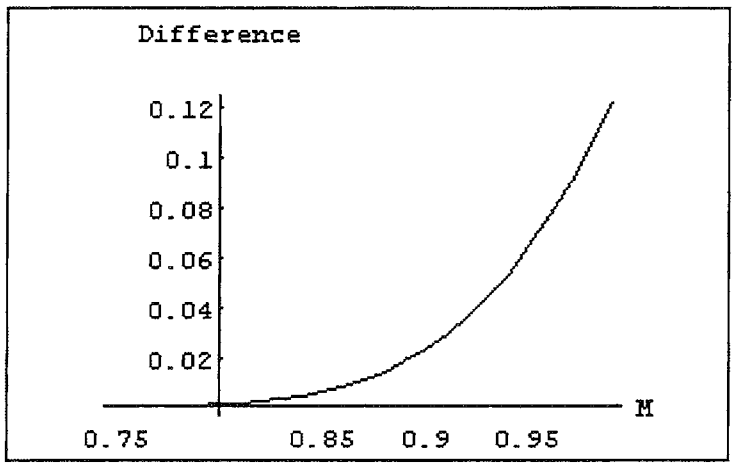

Figure 10

The graph in the figure can be viewed as the value of the barrier, since the debt contract in Merton (1974) does not include any barrier. The difference in risk-premium between the two contracts increases in the barrier level. The explanation of this follows from the decomposition result discussed above.

5. Loans with Stochastic Collaterals

5.1 The Concepts of Collaterals and Guarantees

When a collateral is included in a debt contract the default risk of the loan is reduced. The obvious reason for this is that the debt contract is backed by an additional security. Furthermore, a collateral is often indisposable for the borrower during the life of the debt contract.[21] This could legitimate a distinction between the assets used as collaterals and the rest of the borrowers wealth.[22] A collateral could also be used as an indicator in a safety covenant contract. For example, if a firm borrows for an investment in real

[21] For example, if real estate is used as collateral, the borrower may not be free to sell or let it unless permission is given by the lender.

[22] This separation is represented by the G_t and V_t processes.

estate, this may serve as the collateral for the loan. In this case a safety covenant could be a lower barrier at which the loan should be renegotiated if hit by the value process of the real estate.

Thus, collaterals *reduce* but do not *eliminate* the default risk of the debt contract. But what if the debt contract is combined with a guarantee? Suppose a third part guarantor signs a legal commitment to assume the responsibility of the debt.[23] Then the value of the debt contract, D_t, at time $t<T$, would be:[24]

$$D_t = \exp[-r(T-t)]E - p_t$$

where E = the promised payment at the maturity date T.
p_t = the value of the guarantee, expressed as a put option with exercise price equal to E.

But again, this valuation requires that the writer of the put option, that is, the guarantor, will fulfil his obligations with probability one. In principle, this can never be the case.[25] Instead, it is more reasonable to consider guarantees issued by banks, firms and persons as non-credible. These guarantees must be valued as so-called vulnerable put options, analyzed in Johnson and Stulz (1987).

Now, consider a risky debt, which is contingent on the borrower's asset process, V_t and the guarantor's asset process, G_t. At the maturity date, T, the value of the risky debt is $D(T,v,g) = \mathrm{Min}[E,v+g]$, for all v and g.

---

[23] It is a common feature of the commercial loan market that an agent in need of financing borrows reputational capital from a third part guarantor when issuing debt in the market.

[24] This can be verified by the put-call parity for options, which is $f - p = V - \exp[-r(T-t)]E$, i.e. the pay-off of a portfolio of a long position in a European call and a short position in a European put, both with exercise price E, is replicated by a long position in the underlying asset less the proceeds from a loan of $\exp[-r(T-t)]E$. Further, since $V = f + D$, we can write $f - p = f + D - \exp[-r(T-t)]E$ or $D = \exp[-r(T-t)]E - p$. The similarities between a guarantee and a put option was first analyzed in Merton (1977).

[25] Exceptions are governmental guarantees, for example deposit insurance and guarantees of loans made to private corporations as parts of public policy programs. Merton (1977) and Jones and Mason (1980) study the pricing of fully credible financial guarantees using contingent claims analysis.

We end this section by concluding that the semantic difference between collaterals and guarantors is that a collateral is "material" and cannot therefore be said to default in relation to the promised payment. The guarantor, on the other hand, who is a legal or physical person, can be said to *default* on his commitment. However, the valuation will be invariant if the collateral is used or if the assets of the guarantor are used as a back up for the borrower. This is of course valid only if their properties, in all senses, can be regarded as equal. Therefore, in this paper we use these concepts interchangeably.

5.2 The Model

From the previous sections we have seen that a bond holder who is promised a future payment of E dollars, values this bond to D_t at time t. At the maturity date, T, the value is:

$$D(T,v) = \text{Min}[E,v] = v - \max[v-E,0].$$

His compensation for taking the risk of holding this bond is R, which is strictly greater than the risk-free rate, r.

Now suppose that the contract is backed by a *non-credible* guarantee or combined with a collateral, whose price is G_t. This means that the value of the bond at time T is:

$$D(T,v,g) = \text{Min}[E,v+g] = v + g - \max[v+g-E,0].$$

Let the processes V_t and G_t have the dynamics of [10] and [11]:

$$\begin{aligned} dV &= \alpha_v V dt + \sigma_v V dW \\ V_0 &= v \end{aligned} \tag{10}$$

$$\begin{aligned} dG &= \alpha_G G dt + k_G G dN \\ G_0 &= g \end{aligned} \tag{11}$$

where $-1 < k_G < 0$
See Section 3 for notation.

The solution to [11] is:

$$G_T = g(1+k_G)^N e^{\alpha_G T} \tag{12}$$

Definition 1: Let N^c be the maximum number of downward jumps in G_t such that $G_T > E$.

A necessary condition for N^c to take positive values is g>E.
The definition of N^c combined with [12] implies:

$$N^c = \text{integer of} \left[\frac{\ln\frac{E}{g} - \alpha_G T}{\ln(1+k_G)} \right] \tag{13}$$

Hence for $N \leq N^c$, $D_T = E$ at time T, that is, $D_0 = Ee^{-rT}$. Note that this case implies that the debt contract will be fulfilled irrespectively of the value of V_T. On the other hand, when $N > N^c$, the collateral has decreased so much that V_T affects the value of the loan. Therefore, the value at time T is the weighted probabilities of the two cases.

Thus, we can divide the value of the debt in two parts. Its value at the maturity date, T, is:

$$\begin{aligned} D(T,v,g) &= \text{prob}\left[N \leq N^c\right] E + \left(1 - \text{prob}\left[N \leq N^c\right]\right) \min\left[E, v+g\right] = \\ &= \text{prob}\left[N \leq N^c\right] E + \left(1 - \text{prob}\left[N \leq N^c\right]\right)\left(v+g-\max[v+g-E,0]\right) \end{aligned} \tag{14}$$

Following the same procedure as in Section 3, we end up with a price of the debt, at time t=0, according to:[26]

[26] The derivation is given in appendix A3.

$$D(0,v,g) = \sum_{N=0}^{N^c} Po(\cdot)e^{-rT} E + \sum_{N=N^c+1}^{\infty} Po(\cdot) \Big[v + g(1+k_G)^N e^{(\alpha_G - r)T} -$$

$$- \Big[v\Phi(d_N) - e^{-rT}\big(E - g(1+k_G)^N e^{\alpha_G T}\big) \Phi\big(d_N - \sigma_v\sqrt{T}\big)\Big]\Big] \qquad (15)$$

where

$$d_N = \frac{\ln v - \ln\big(E - g(1+k_G)^N e^{\alpha_G T}\big) + \left(r + \frac{1}{2}\sigma_v^2\right)T}{\sigma_v\sqrt{T}} \qquad (16)$$

and

$\Phi(\bullet)$ = the standardized cumulative normal distribution

$Po(\bullet)$ = the Poisson distribution under the probability measure Q with intensity $T(r-\alpha_G)/k_G$.

In order to express the value of the debt in terms of yield to maturity, we divide [15] by E and obtain $D/E = e^{-R(T)T}$, where R(T) is the yield to maturity. Hence, $R(T)=(-1/T)\ln[D/E]$.

Figure 11 below shows the risk premium, R(T)-r, for $g \in [0,0.4]$ and $v \in [0.5,0.7]$, where r $r = 0.1$, $\sigma = 0.2$, $k_G = -0.2$, $\alpha_G = 0.15$, $T = 1$ and $E = 1$.

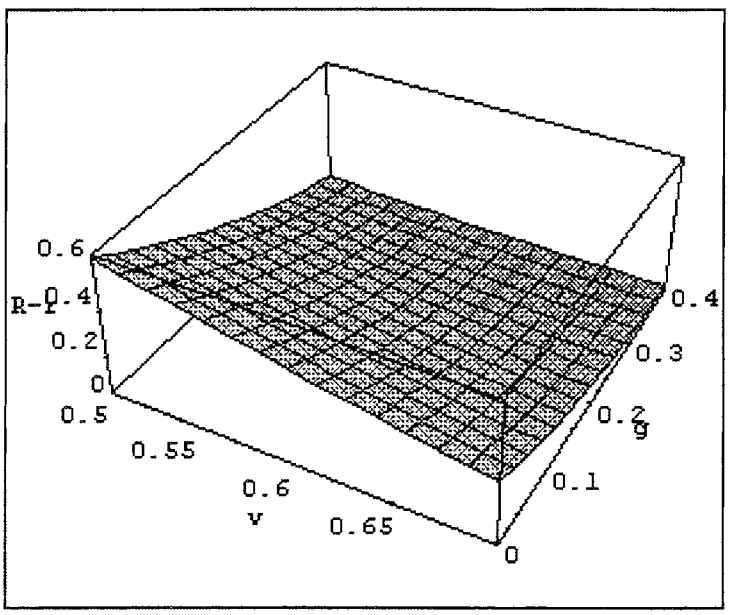

Figure 11

One question is, for a given initial value, for example, v+g=1, what combination of v and g gives the highest debt value? The answer will depend on the stochastic features of V_t and G_t. To solve this problem one selects positive weights of the two assets, which summarize to one and maximize the debt value with respect to the weights. We wish, however, to merely point out this fact and will not discuss it any further in this paper.

6. Loans with a Stochastic Collateral and a Safety Barrier

In this section we value a risky debt in presence of *both* a safety barrier, which was analyzed in Section 4, and a stochastic collateral, which was analyzed in Section 5.

The contract specifies that if the wealth process of the borrower, V_t, defined by [8], hits a lower safety barrier, denoted M, at time τ, the creditor will receive:

$$D(\tau,M,g) = \min[M+g, Ee^{-r(T-\tau)}], \text{ for all } g.$$

where the process of either a guarantor or a certain asset, used by the borrower as a collateral, for example real estate, is defined by [9].

If V_t never hits M prior to T, the pay-off structure of the creditor at the maturity date, T, is:

$$D(T,v,g) = \min[v+g,E] = v+g - \max[v+g-E, 0].$$ [27]

The derivation of the value of this contract is done in appendix A4.

Definition 2: Let $\tau \equiv \inf\{t \in [0,T] : V_\tau = M\}, V_t > M, \forall_t : 0 \leq t \leq \tau$. That is, τ is a stopping time.

The boundary conditions of the debt contract are:

[27] Note that we could alter the definitions of the processes by denoting V_t as the collateral process. This setup could indicate a situation where the creditor has the right to nullify the contract when the value of the collateral, for example, real estate, decreases to a certain level.

(i) $D(\tau,M,g) = \min[M+g, Ee^{-r(T-\tau)}]$, for $\{V_\tau = M \text{ and } \forall s: V_s > M, s < \tau\}$ and all g.

(ii) $D(T,v,g) = \min[v+g, E]$, for $\{V_t > M, \forall t: 0 \le t \le T\}$

We will now present some probability functions, which are crucial when analyzing barrier contracts and will thereby be used in the derivation of the debt value:[28]

Definition 3: Let $F_T(M,M) \equiv \text{Prob}[\inf V_t > M, \forall t; 0 \le t \le T]$ be the one-sided first passage time probability distribution for V_t, that is:

$$F_T[M,M] = \Phi\left(\frac{\ln\frac{v}{M} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \left(\frac{v}{M}\right)^{-\frac{2r}{\sigma^2}+1} \Phi\left(\frac{\ln\frac{M}{v} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \quad (17)$$

where Φ is the standard normal distribution function.

This is the probability that V_t is greater than the safety covenant M, at the maturity date T, of the loan *and* that V_t has always been above M for all $t<T$.

Definition 4: Let $H_T(M,M) \equiv 1 - F_T(M,M)$ and let $f_t[M,M]$ be the density of $H_T(M,M)$ given by:

$$f_T[M,M] = \frac{dH_t[M,M]}{dt} = \ln\frac{v}{M}\left[\sigma^2 t^3 2\pi\right]^{-\frac{1}{2}} \exp - \left(\frac{\left[\ln\frac{M}{v} - \left(r - \frac{1}{2}\sigma^2\right)t\right]^2}{2\sigma^2 t}\right) \quad (18)$$

This is the first passage time density function, that is, the probability that V_t hits the lower barrier M for the first time at time t.

Definition 5: Let $f_T(V,M)$ be the density function of V_T given that infinum of $V_t > M, \forall t; 0 \le t \le T$. $\varphi(z)$ is the standard normal density function, that is:

[28] A heuristic derivation of these definitions are given in Dahlfors and Jansson (1994a). For a more rigorous derivation, see Karlin and Taylor (1975).

$$f_T[V,M] = \frac{dF_t[V,M]}{dV} = \frac{1}{V\sigma\sqrt{T}} e^{\frac{m\ln V}{\sigma^2} - \frac{m^2 T}{2\sigma^2}} \left[\varphi\left(\frac{\ln V}{\sigma\sqrt{T}}\right) - \varphi\left(\frac{2\ln\frac{M}{v} - \ln V}{\sigma\sqrt{T}}\right)\right] \quad (19)$$

where $m = r - 0.5\sigma^2$

This is the density function of V_T at time T, given that V_t has always been above M for all $t<T$.[29]

Definition 6: Let $N^M(t)$ be the maximum number of jumps in G_t such that $G_t \geq Ee^{-r(T-t)} - M$, for $0 \leq t \leq T$, that is:

$$N^M(t) = \text{integer of} \left[\frac{\ln\left(\frac{Ee^{-r(T-t)} - M}{g}\right) - \alpha_G t}{\ln(1+k_G)} \right] \quad (20)$$

where $g > Ee^{-rT} - M$.

Definition 6 implies that if the number of downward jumps in G_t is less than N^M, the debt holder always receives $Ee^{-r(T-t)}$ dollar, which is the risk-free discounted value of the debt.

If we use these definitions, the value of the debt contract at time t=0, derived according to appendix A4, is:

$$D(0,v,g) = \int_0^T f_t[M,M]e^{-rt}\left[\sum_{N=0}^{N^M(t)} Po(\cdot)Ee^{-r(T-t)} + \sum_{N=N^M(t)+1}^{\infty} Po(\cdot)\left[M + g(1+k_G)^{N(t)}e^{\alpha_G t}\right]\right]dt +$$

$$+ e^{-rT}\left[F_T[M,M]\sum_{N=0}^{N^M(T)} Po(\cdot)E + \sum_{N=N^M(T)+1}^{\infty} Po(\cdot)\left(F_T[M,M]G_T + \right.\right. \quad (21)$$

$$\left.\left. + \int_{\ln\frac{M}{v}}^{\infty} f_T[V_T,M]V_T dV - \int_{\ln\frac{M}{v}}^{\infty} f_T[V_T,M]\max\left[V_T + G_T - E, 0\right]dV\right)\right]$$

where Po( ) = the Poisson frequency function with intensity $\lambda(1+g)$, which is defined by [31]

[29] Recall that V_t is lognormal distributed.

in appendix A2.

The first part of [21] gives the risk-neutral discounted expected value of the debt given that the contract is nullified, that is, V_t hits the lower barrier M. The second part gives the risk neutral discounted expected value of the debt given that the barrier has never been hit. Solving [21] yields the following valuation formula of a debt contract at time t = 0:

$$D(0,v,g) = \int_0^T f_t[M,M]e^{-rt}\left[\sum_{N=0}^{N^M(t)} Po(\cdot)Ee^{-r(T-t)} + \sum_{N=N^M(t)+1}^{\infty} Po(\cdot)\left[M + g(1+k_G)^{N(t)}e^{\alpha_G t}\right]\right]dt +$$

$$+ e^{-rT}F_T\,[M,M]\sum_{N=0}^{N^M(T)} Po(\cdot)E + \sum_{N=N^M(T)+1}^{\infty} Po(\cdot)\left[F_T[M,M]g(1+k_G)^{N(T)}e^{(\alpha_G - r)T} + \right.$$

$$+ \left[v\left(\Phi\left(\frac{\ln\frac{v}{M} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \left(\frac{v}{M}\right)^{-\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\ln\frac{M}{v} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right)\right.$$

$$- \left(v\left[\Phi\left(\frac{\ln\frac{v}{L_N} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \left(\frac{v}{M}\right)^{-\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\ln\frac{M^2}{vL_N} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right] - \right.$$

$$\left.\left.\left. - L_N e^{-rT}\left[\Phi\left(\frac{\ln\frac{v}{L_N} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \left(\frac{v}{M}\right)^{-\frac{2r}{\sigma^2}+1}\Phi\left(\frac{\ln\frac{M^2}{vL_N} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right]\right)\right]\right] \tag{22}$$

Where $L_N = E - G_T$.

The first row of [22] gives the discounted expected value of the debt, *given* that the debt contract is exercised prior to maturity. The probability of this is given by definition 4. The first term in the second row is the value from the probability of receiving E at time T, independent of V_T, on condition that M has not been hit by V_t up to time T. The second term of row two and the third row are the expected discounted value of $G_t + V_t$, when $G_T < E$. The last two rows correspond to a down-and-out call option with exercise price equal to L and a lower barrier at level M, for different number of jumps in G_t. This comes from the max-expression in [21].

Since the first row in this expression involves some complexities, it may be convenient to get a better understanding of the formula by a simplification. Therefore, we let $g \rightarrow 0$, that is, the initial collateral value goes to zero value. Then the boundary conditions will be:

(i) $D(\tau,M) = \min[M, Ee^{-r(T-\tau)}]$

and

(ii) $D(T,v) = \min[v,E]$

Dependent on where the barrier is located in relation to $Ee^{-r(T-\tau)}$, we will have three different cases, which we present below.

Case 1

If the barrier M is chosen to be $M \leq Ee^{-rT}$, then (i) will be $D(\tau,M)=M$, that is, the payment at the hitting time is constant and formula [22] converges to:

$$
\begin{aligned}
D(0,v) = {} & v\left(\Phi\left(\frac{\ln\frac{M}{v}-\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)+\left(\frac{v}{M}\right)^{-\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\ln\frac{M}{v}+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right)+ \\
& +\left[v\left(\Phi\left(\frac{\ln\frac{v}{M}+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)-\left(\frac{v}{M}\right)^{-\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\ln\frac{M}{v}+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right)-\right. \\
& -\left(v\left[\Phi\left(\frac{\ln\frac{v}{E}+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)-\left(\frac{v}{M}\right)^{-\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\ln\frac{M^2}{vE}+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right]-\right. \\
& \left.\left.-Ee^{-rT}\left[\Phi\left(\frac{\ln\frac{v}{E}+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)-\left(\frac{v}{M}\right)^{-\frac{2r}{\sigma^2}+1}\Phi\left(\frac{\ln\frac{M^2}{vE}+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right]\right)\right]
\end{aligned}
\tag{23}
$$

This formula is identical to the debt value expression in Section 4, where no collateral was involved. The first row now corresponds to the value from early exercise, that is, when V_t hits M. The remaining rows are identical to the three last rows in [22]. The reason for presenting [23] is that the debt value is expressed in a different way compared to Section 4, where the debt value was expressed as the firm value minus a down-and-out call option.

Case 2

If the barrier M is chosen to be M=E, then (i) will be $D(\tau,M)=Ee^{-r(T-\tau)}$, that is, the payment at the hitting time is time-dependent. In this case formula [22] will not converge to expression [23]. Instead, the down-and-out part has to be calculated as a down-and-out option with a rebate given by $R(\tau)=M-Ee^{-r(T-\tau)}$. The reason for this is that the value of the borrower's assets must equal the value of debt plus equity.

Case 3

If the barrier M is chosen to in the interval $M \in [Ee^{-rT}, E]$ then (i) will be $D(\tau,M)=Ee^{-r(T-\tau)}$ for $\tau \in [0,t^*]$, where $t^*=(\log(M/E) + rT)/r$ and $D(\tau,M)=M$ for $\tau \in [t^*,T]$. This debt value can be calculated by value the equity as a down-and-out call option which pays a time-dependent rabate up to time t* and thereafter zero.

One motivation for presenting these three cases is to address the significance of the location of the barrier. This was also discussed in Section 4.

7. Concluding Remarks

In this paper we have analyzed aspects of risky debt in terms of contingent claims analysis. From this starting point we have first shown the impact on debt value from the existence of jumps in the underlying asset process. After that we have derived the preference-free valuation of a risky debt and the risk premium for different kinds of debt contracts. The debt contract specification involves both safety barriers and collaterals. These objects are not only tools for the creditor to limit his risk; The borrower also benefits from the inclusion of these devices by their important feature of resolving agency conflicts. The major contributions of this paper are the derivation of the impact on debt value from the existence of jumps and the inclusion of contractual provisions like collaterals and safety barriers. One example of an observation we have made is that the debt value is actually decreasing in underlying asset value for a certain range of safety barriers levels close to the exercise price.

One issue, not discussed in the paper, is the meaning of the existence of the risk-free bond together with the assumption that short selling of this is unrestricted. Why should a firm finance itself through risky bonds if it has access to a market where it can always borrow at the risk-free rate?

Another question, not discussed is how valid the contingent claims approach is to value an additional issue of debt. We know from the formulas in the paper that one input that determines the value of the debt is the initial asset value. But a new loan will affect the asset value, via the balance sheet relationship, and thereby the value of the loan for a given promised payment. Therefore, the asset value and the value of the debt will determine each other.

Appendix

A1

The traditional approach:

This appendix derives the explicit hedge portfolio in order to make a preference-free valuation of the debt according to price formula [6].

The economy is governed by two sources of uncertainty, a Brownian motion and a Poisson process. If the capital market consists of only one risky asset, given by equation [1], and one risk-free asset, given by equation [2], this economy does not exhibit completeness, since the event space is not spanned by the number of assets. The introduction of another risky asset, G_t, given by equation [3], enables us to complete the market.

Let P_t be the value of a self financing portfolio with a relative weight of u^V in asset V_t, u^G in asset G_t and $u^B = 1-u^V-u^G$ in asset B_t. Thus, $u^V = h^V V/P$ and $u^G = h^G G/P$, where h^V= the number of shares invested in asset V_t and h^G= the number of shares invested in asset G_t. This portfolio has the following dynamics:

$$\frac{dP}{P} = \left[r + (\alpha_v - r)\, u^v + (\alpha_G - r) u^G\right] dt + \sigma_v u^v dW + \left[k_v u^v + k_G u^G\right] dN \qquad (24)$$

where we have substituted $u^B = 1-u^V-u^G$.

If this portfolio mimics the contingent claim, f(t,V), the following must be true:

$$P_t = f(t, V_t) \qquad (25)$$

Recall that the dynamics of the contigent claim, f(t,V) is given by [4]:

$$\frac{df}{f} = \alpha_f df + \sigma_f dW + k_f dN \qquad (26)$$

where α_f, σ_f och k_f are given by [5].

If the portfolio is to mimic the contigent claim, its stochastic terms must be identical. Therefore the portfolio weights, u^V and u^G, must be chosen according to:

$$u^v = \frac{Vf_v}{f}$$

$$u^G = \frac{f(t,V+k_vV)-f(t,V)-k_vVf_v}{k_Gf} \tag{27}$$

Substitute these weights into the portfolio dynamics, [24]:

$$\frac{dP}{P} = \left[r+(\alpha_v-r)\left(\frac{Vf_v}{f}\right) + (\alpha_G-r)\left(\frac{f(t,V+k_vV)-f(t,V)-k_vVf_v}{k_Gf}\right)\right]dt +$$

$$+\sigma_v\left(\frac{Vf_v}{f}\right)dW + \left[k_v\left(\frac{Vf_v}{f}\right) + k_G\left(\frac{f(t,V+k_vV)-f(t,V)-k_vVf_v}{k_Gf}\right)\right]dN \tag{28}$$

Comparing [28] and [26] we see that the stochastic parts are the same. Now, subtract [28] from [25], use expression [25], that is, P=f and multiply by f:

$$0 = \frac{1}{2}\sigma_v^2V^2f_{vv} + \left[r-\left(\frac{r-\alpha_G}{k_G}\right)k_v\right]Vf_v + f_t - rf + \left(\frac{r-\alpha_G}{k_G}\right)\left[f(t,V+k_vV)-f(t,V)\right] \tag{29}$$

The boundary condition is f(T,v) = max[v-E,0], for all v.

The solution is given by [6].

A2

The martingale approach:

This appendix derives the preference-free valuation of [6] using the martingale approach. It is closely connected to appendix A1.

Consider the probability space $(\Omega, \Im, P, \{\Im_t\})$ and the price processes given by [1] - [3]. $\{\Im_t\}$ is the filtration which has been generated by W_t and N_t, where W_t is a standard Wiener process and N_t is a Poisson process with intensity λ, that is,
$\Im_t = \sigma\{W_s, N_s; 0 \leq s \leq t\}$.

Define the discounted price system $Z^B = B/B = 1$, $Z^V = V/B$, $Z^G = G/B$
The differentials, under probability measure P, are given by:

$$dZ^B = 0$$

$$dZ^V = (\alpha_V - r)Z^V dt + \sigma_V Z^V dW + k_V Z^V dN$$

$$dZ^G = (\alpha_G - r)Z^G dt + k_G Z^G dN$$

Assume that there is an absolute continuous measure Q such that $L_T = dQ/dP$ and $E_P[L_T] = 1$, where L_T is the Radon-Nikodym derivative. Then Girsanov's theorem states that under probability measure Q:

(i) $dW = hdt + dW^Q$, where W^Q is a Q-Wiener process

(ii) $dN = \lambda(1+g)dt + dM^Q$, where M^Q is a Q-martingale

Substitute these into the discounted price processes:

$$dZ^V = (\alpha_V - r + \sigma_V h + \lambda(1+g)k_V)Z^V dt + \sigma_V Z^V dW^Q + k_V Z^V dM^Q$$

$$dZ^G = (\alpha_G - r + \lambda(1+g)k_G)Z^G dt + k_G Z^G dM^Q$$

The price system is arbitrage free if and only if the expectation of these increments under the measure Q is zero. Since both dW^Q and dM^Q are martingales under Q we only

have to choose h and g in order to eliminate the drift term. Therefore we must solve the following equation system, which gives a unique pair of h and g:

$$\begin{bmatrix} \alpha_v & \lambda k_v \\ 0 & \lambda k_G \end{bmatrix} \begin{bmatrix} h \\ (1+g) \end{bmatrix} = \begin{bmatrix} r-\alpha_v \\ r-\alpha_G \end{bmatrix} \rightarrow \tag{30}$$

$$\rightarrow \begin{bmatrix} h \\ (1+g) \end{bmatrix} = \begin{bmatrix} \dfrac{(r-\alpha_v)k_G-(r-\alpha_G)k_v}{\sigma_v k_G} \\ \dfrac{(r-\alpha_G)}{\lambda k_G} \end{bmatrix}$$

Since $\lambda(1+g)$ is the intensity of N under Q, we have a requirement on the second row of [30] to be non-negative, that is:

$$\lambda(1+g) = \frac{(r-\alpha_G)}{k_G} > 0 \tag{31}$$

Since $k_G < 0$, we must have that $\alpha_G > r$.

The dynamics of V_t and G_t under the measure Q is:

$$dV = \left(r - k_v\left(\frac{r-\alpha_G}{k_G}\right)\right)Vdt + \sigma_v VdW^Q + k_v VdN^Q \tag{32}$$

$$dG = \alpha_G Gdt + k_G GdN^Q \tag{33}$$

Hence, under the measure Q, both V_t and G_t have the local drift of r, since the Poisson process has the intensity given by [31]. Therefore, the discounted price processes are all martingales and the price system is free from arbitrage. Furthermore, since the Q measure is unique, the price of the contingent claim f(t,v) is unique as well. Its value is

given by the expected discounted value of the pay-off structure at time T, where the expectation is taken under probability measure Q and the discounting factor is the risk-free rate. Thus, the value of the contingent claim at time t is:

$$f(t,v) = e^{-r(T-t)}E^{Q}\left[\max\left[v-E,0\right]\right] \tag{34}$$

where the expected value is taken under the measure Q. This means that the dynamics of V_t is given by [32]. The solution to [34] is given by formula [6].

A3

This appendix is connected to Section 5, where the value of a loan contract, in presence of a stochastic collateral, is derived.

The price processes in this valuation problem is given by [10] and [11]. Proceed in the same way as in appendix A2. The unique pair of h and g which martingalise the discounted price processes, Z^V and Z^G, is given by:

$$\begin{bmatrix} h \\ (1+g) \end{bmatrix} = \begin{bmatrix} \dfrac{r-\alpha_v}{\sigma_v} \\ \dfrac{r-\alpha_G}{\lambda k_G} \end{bmatrix} \tag{35}$$

The dynamics of V_t and G_t under measure Q is:

$$dV = rVdt + \sigma_v VdW^Q \tag{36}$$

$$dG = \alpha_G Gdt + k_G GdN^Q \tag{37}$$

According to the same arguments as in appendix A2, the value of the debt D(t,v,g) at time t is:

$$D(t,v,g) = e^{-r(T-t)}E^{Q}\left[\min\left[E,v+g\right]\right]^{*} \tag{38}$$

where the "star" implies a decomposition of the expectation expression with respect to whether N is greater or less than N^C. This decomposition is explained by [14]. The solution of [38] is given by the pricing formula [15].

A4

This appendix deals with the derivation of [21] and [22], which involves a lower absorbing barrier, M. The main difference from appendix A3 is that when the process V_t hits this barrier, an additional boundary condition is activated. This is given by:

$D_\tau = \min[V_\tau + G_\tau, Ee^{-r(T-\tau)}]$, for $\{V_t = M \text{ and } \forall_S; Vs > M, s < \tau\}$.

In order to achieve a preference-free valuation we follow the technique in appendix A3. The value of the debt at time t is then :

$$D(t,v,g) = e^{-r(T-t)}E^Q\left[\min\left[E, v+g\right]\right] \tag{39}$$

When solving [39] we use the probability functions defined in definitions 3-5. These are needed to take the path dependence of V_t into consideration, that is, the barrier constraint, M. We also use the decomposition of the pay-off according to whether N is greater or less than $N^M(t)$. That decomposition is explained by definition 6. The solution is then given by [22].

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Paper 6: Portfolio Selection and the Pricing of Personal Loan Contracts

1. Introduction

The contingent claims literature that follows Black and Scholes (1973) has mainly taken the underlying asset dynamics for given. This may be an appropriate approach when valuing stock options. However, there are many types of contracts where this can be considered as too simplifying an assumption. For example, when valuing a personal loan contract it is obvious that the borrower's consumption-investment behaviour affects the process, on which the contract is contingent. Moreover, the existence of penalty clauses or reputational costs in case of default will probably imply less risky actions and a more cautious consumption rule by the borrower. The main purpose of this paper is to incorporate an individual's portfolio selection behaviour into the valuation of a personal loan contract. In this sense, our study is an attempt to synthesize Merton (1969) and Merton (1974).

Merton (1969) was first to provide a model for an individual's intertemporal consumption and portfolio selection in continuous time. It was shown that, for special choices of utility functions, closed form solutions for optimal consumption and portfolio selection could be obtained. In Merton (1974) the isomorphic price relation between levered equity and a call option was recognized. This made it possible to attain a preference-free valuation of risky corporate debt.[1] Besides strategic interactions between borrower and lender, the assumption of exogenous firm value dynamics in Merton's model may be resonable.[2] However, as mentioned above, if the purpose is to value a personal loan contract, as in the case of Smith (1980), the underlying wealth

[1] The term "risky" refers to the probability of default, i.e. that the promised payment in the debt contract may not take place, and not on random changes in the term structure in general.

[2] Examples of other studies concerning pricing of corporate debt, that take the underlying process for given are Black and Cox (1976), Mason and Bhattacharya (1981), Longstaff and Schwartz (1993) and Nielsen et al. (1993).

process should not be assumed to be exogenous. We believe that it would be more realistic if it were determined by a dynamic optimization behaviour, as in Merton (1969). Therefore, we apply the Merton (1969) model and use the outcome to determine the optimal wealth dynamics of the borrower. This wealth process will then be used to determine the value of a personal loan contract along the lines of Merton (1974). This is also the major contribution of the paper.

We are aware that the underlying process in our model is in fact exogenous at the settlement of the contract, but its dynamics are consistent with a utility maximizing agent. We must also state that we do not attempt to determine the *amount* borrowed by the individual, which is also assumed to be exogenous. Further, we do not deal with endogenously determined contract terms, moral hazard behaviour or possible renegotiations of the contract.[3] However, the model captures the borrower's consumption and portfolio selection response on exogenously given contract provisions in a perfect capital market framework. Even if our model, in some sense, is a blend of two seminal studies we have not found any previous attempts to combine these two frameworks.

The rest of the paper is organized as follows. In Section 2 we model the optimal consumption and investment rules of the borrower and incorporate his preferences for repaying the loan. The design of the boundary condition captures the features of contractual penalty clauses and/or reputation costs in case of default on the loan. The portfolio rules determine the optimal dynamics of the borrower's asset process, which is used as the underlying asset in the debt contract. The valuation of this debt contract is undertaken in Section 3. In Section 4 we discuss the impact from parameters reflecting the type of borrower, with respect to credit risk, on the value the debt. Concluding remarks are given in Section 5.

[3] Note that there is a large and fast growing literature on the design of corporate debt contracts in an *imperfect* information framework. Jensen and Meckling (1976) is an early and important contribution within this literature. Anderson and Sundaresan (1992) and Gorton and Kahn (1994) are examples of more recent studies in this area.

2. Optimal Consumption and Portfolio Rules[4]

Consider a utility maximizing agent with an initial asset value of P_0, financed by equity and a zero coupon debt with a promised payment of E dollars at time T. For this agent, we derive optimal consumption-investment rules. This specification implies that the amount borrowed by the agent will not explicitly be determined by the model. However, as will be seen, it is an attempt to derive optimal conditions for an agent, who is constrained from borrowing at the risk-free rate of interest. If the agent only receives utility from consumption, optimality conditions usually force final assets to zero at time T. Hence, the promised payment at time T will not take place. However, we can rule out this trivial outcome by introducing some kind of bequest function. This function, formulated in definition 3 below, captures the borrower's utility from having assets left at the maturity of the loan. We believe that the introduction of this is realistic since people seem to be concerned about repayment of loans, for different reasons. The existence of such a function will give rise to a more cautious consumption rule, thereby decreasing the default risk. Moreover, this function, as will become clear in Section 3, will play an important role in determining the value of a risky loan.

The economy is assumed to contain one risky asset, with the price X and one risk-free asset, with the price B. If the borrowing agent does not exhibit a risk-less behaviour, he is restrained from borrowing at the risk-free rate.[5] The agent receives income only from capital gains on assets.[6] It is further assumed that trading takes place continuously and frictionlessly on a perfect market in the usual sense.

The dynamics of the risky asset, are given by:

$$dX = \alpha X dt + \sigma X dZ \tag{1}$$

where α = the instantaneous expected return of X.
σ = the instantaneous standard deviation of the rate of return of X.
Z = a standard Brownian motion.

[4] This section is essentially along the lines of Merton (1969).

[5] For reasonable parameter values it will be shown that the agent's optimal behaviour must be risky, which restrains him from risk-free borrowing.

[6] For simplicity, we do not consider stochastic wage incomes.

The dynamics of the risk-free asset, are given by:

$$dB = rBdt \tag{2}$$

where r>0 is the risk-free rate, assumed to be constant.

The Budget Equation

We will now present the budget equation and refer to Merton (1971) for the derivation. The agent is assumed to allocate his wealth according to the following definition:

Definition 1: Let P be the value of a self financing portfolio consisting of the risky asset and the risk-free asset, with a fraction ω of the portfolio value invested in the risky asset and $1-\omega$ invested in the risk-free asset. From this portfolio the agent consumes at the rate C per unit time.

The dynamics of this portfolio, which is the agent's budget equation, is:

$$dP = \left[\left((\alpha - r)\omega + r\right)P - C\right] dt + \omega\sigma P dZ \tag{3}$$

The drift-part of the portfolio consists of two components:

(i) A positive impact from the returns of the investment in the risky asset and the risk-free asset respectively. Since it is assumed below that $\omega \in [0,1]$ and $\alpha > r$, this term is never less than the risk-free rate of return, r.

(ii) A negative impact, due to withdrawals from the portfolio in terms of consumption.

The diffusion-part is dependent on how much is invested in the risky asset.

Furthermore, since the portfolio is exposed to risk, it is reasonable to assume that the individual is restrained from borrowing at the risk-free rate, that is, $(1-\omega)$ must not be negative. Thus, the fraction invested in the risky asset must be $\omega \in [0,1]$.

In the optimization program below, the optimal ω and C will be determined and thereby the optimal portfolio dynamics.

The Utility Specification

The agent is assumed to receive utility from consumption according to the following definition:

Definition 2: Let the utility function be of a form, which yields isoelastic marginal utility, defined by $U(C) \equiv C^b/b$,[7] where $b<1$ and $b \neq 0$. This implies that 1-b is the Pratt (1964) measure of relative risk aversion.

This utility function is time-additive and state-independent and is strictly concave in consumption, C. Furthermore, it implies a constant relative risk aversion, and therefore a decreasing absolute risk aversion.

The motivation for this special choice of utility function is that explicit solutions can be obtained. See Merton (1969) and Merton (1971) for a discussion concerning the choice of utility functions.

The Bequest Function

Since we want the agent to have incentives to pay back the loan at the end of the planning horizon, we must introduce some kind of bequest function. This is given by the following definition:

Definition 3: At the end of the planning period, T, the agent is assumed to receive utility according to a bequest function given by the following expression:

$$\Psi(T,P_T) = \gamma \frac{e^{-\beta T}}{b} \left[\frac{P_T}{E}\right]^b \tag{4}$$

The parameter $\gamma \geq 0$ should be regarded as the agent's willingness to repay the promised payment of E dollars and β is a subjective discounting factor.

[7] This is a special case of the more general class of utility functions called the HARA-family.

Although the chosen bequest function [4] has the desired feature of being increasing and concave in P_T, the marginal utility for $P_T>E$ may be too high, since it is the willingness to repay the loan that is the objective. It would perhaps be more appropriate if the utility increased fast in P_T for $0<P_T\leq E$, but were non-increasing for $P_T>E$. That is, we would like the function to be more concave in P_T or represented by some kind of jump-function. However, even if such a specification were preferable, it would induce a more complex solution.

Therefore, the bequest function must be interpreted in the following sense: It gives the agent both utility from repaying the loan *but also* utility from real bequest motives, which in many cases may be realistic.

We will now discuss the meaning of the γ parameter. This parameter should be interpreted as how the borrower perceives defaulting on a loan. Thus, a high value on γ could correspond to a situation where the cost of default on the loan is considered to be substantial by the borrower for different reasons. Therefore, γ is a function of the borrower's nature and his response to several factors as, for example, the existing institutional or sociological environment and, of course, the contractual terms. These factors are assumed to be constant during the time-horizon. Hence, γ is also a constant and not a strategic parameter used by the borrower to get better contract terms. Hence, moral-hazard behaviour is ruled out.

In practice, it may be difficult to observe γ, but the lender could use instrumental variables to reveal its true nature. Examples of instrumental variables are: sex, age, health records, history of debt service, employment status et cetera. The degree to which the borrower suffers from legal penalty and reputational costs due to default on the loan will also have impact on γ.

In reality, commercial lenders, as banks, actually estimate the "type" of the borrower, which in our vocabulary corresponds to γ. Finally, we want to point out that the estimation problems of the borrower's type is not unique to the present paper. These problems are also present in many studies within the agency literature of banking such as, for example, Gorton and Kahn (1994).

The Optimization Program

We will now turn to the optimization problem where optimal consumption-portfolio rules are derived. These will be used to determine the agent's optimal portfolio dynamics.

The objective for the agent is to maximize his discounted expected utility from consumption and bequest by choosing C and ω according to:

$$J(t,P_0) \equiv \max_{c,\omega} E^*\left[\int_0^T e^{-\beta t}\frac{C^b}{b}\,dt + \gamma\frac{e^{-\beta T}}{b}\left[\frac{P_T}{E}\right]^b\right]$$

$$\text{s.t.} \quad dP = \left[\left((\alpha - r)\omega + r\right)P - C\right]dt + \omega\sigma P dZ \tag{5}$$

$$J(T,P_T) = \gamma\frac{e^{-\beta T}}{b}\left[\frac{P_T}{E}\right]^b$$

where E* is the expectation operator.

This program can be transformed to the basic equation for stochastic optimal control. This equation, given by expression [6] below, which is often refered to as the Hamilton-Jacobi-Bellman equation, gives necessary conditions for optimal control.[8]

$$-J_t = \max_{c,\omega}\left[e^{-\beta t}\frac{C^b}{b} + J_P\left[(\alpha - r)\omega P + rP - C\right] + \frac{1}{2}\omega^2\alpha^2P^2J_{PP}\right]$$

$$J(T,P_T) = \gamma\frac{e^{-\beta T}}{b}\left[\frac{P_T}{E}\right]^b \tag{6}$$

As mentioned above the agent is not allowed to borrow at the risk-free rate. Therefore, the optimal fraction invested in the risky asset must satisfy $\omega^* \in [0,1]$. This could be handled by the usual Kuhn-Tucker methods with resulting inequalities. However, we

[8] See e.g. Øksendal (1992) for sufficient conditions.

will use parameter values satisfying that restriction.[9] Basically, we assume the agent to be sufficiently risk averse to satisfy $\omega^* \in [0,1]$.

First order conditions of the right-hand side of [6] give optimal consumption-portfolio rules according to:

$$C^* = \left[e^{\beta t} J_P\right]^{\frac{1}{b-1}}$$
$$\omega^* = \frac{(r-\alpha)J_P}{\sigma^2 P J_{PP}} \tag{7}$$

Substitution of these back in [6] yields:

$$-J_t = [J_P]^{\frac{b}{b-1}} \left(\frac{1-b}{b}\right) e^{\frac{-\beta t}{b-1}} + rPJ_P - \frac{(r-\alpha)^2 [J_P]^2}{2\sigma^2 J_{PP}}$$
$$J(T,P_T) = \gamma \frac{e^{-\beta T}}{b} \left[\frac{P_T}{E}\right]^b \tag{8}$$

The solution to [8] is:[10]

$$J(t,P) = \frac{A(t)}{b} e^{-\beta t} P^b \tag{9}$$

where

$$A(t) = \left[\left(\gamma^{\frac{1}{1-b}} E^{-\frac{b}{1-b}} - \frac{1-b}{\mu}\right) e^{\frac{\mu}{1-b}(t-T)} + \frac{1-b}{\mu}\right]^{1-b} \tag{10}$$

and

---

[9] In appendix A3 we derive the optimal consumption-investment rules when the no-borrowing restriction *is* binding.

[10] The derivation is given in detail in appendix A1.

$$\mu \equiv \beta - rb - \frac{(r-\alpha^2)b}{2\sigma^2(1-b)} \tag{11}$$

Taking the derivatives of [9] and substitute them into [7] gives the explicit consumption-portfolio rules according to:

$$C(t)^* = \frac{P(t)}{\left[\left(\gamma^{\frac{1}{1-b}} E^{-\frac{b}{1-b}} - \frac{1-b}{\mu}\right) e^{\frac{\mu}{1-b}(t-T)} + \frac{1-b}{\mu}\right]}$$

$$\omega^* = \frac{(\alpha - r)}{\sigma^2(1-b)} \tag{12}$$

Hence, from [10] and [12], we see that $A(t)^{-1/1-b}$ is the marginal propensity to consume from the portfolio at each instant of time. The optimal consumption rule is a time-dependent fraction of the portfolio value and the investment rule is a constant, independent of the portfolio value. Since the agent is restrained from borrowing at the risk-free rate, we must have $\omega^* \leq 1$ as the optimal proportion of the portfolio invested in the risky asset. This ensures the fraction of the portfolio, invested in the risk-free asset to be $(1-\omega^*) \geq 0$. Note that this in fact implies a restriction on the degree of risk aversion, b, of the borrower.[11]

If we let the planning horizon be infinite, that is, $T \to \infty$, we observe that the limit of $A(t)^{-1/1-b}$ is $\mu/(1-b)$. That is, in this case the marginal propensity to consume is constant.[12]

[11] For parameter values used in Section 3 below, i.e. $\alpha=0.15$, $\sigma=0.2$ and $r=0.1$, the restriction on b becomes $b \leq -0.25$. As an example, for a relative risk aversion of 2, i.e. $b=-1$, the fraction invested in the risky asset will be 0.625 and accordingly the investment in the risk-free asset is 0.375.

[12] To get a better feel of what is going on, we can observe that the analgous non-stochastic problem, i.e. the classical Ramsey-problem, with the discounting factor β set equal to the risk-free rate r, implies that $\mu/(1-b)=r$. This implies that the agent consumes the return on the capital at each instant, i.e. the portfolio value remains constant. For $\beta > r$, the agent will consume more than the return and therefore the portfolio value will deteriorate at each instant.

The Optimal Portfolio Dynamics

Since we have derived the optimal consumption-portfolio rules, given by [12] above, we can substitute these into the portfolio dynamics, given by [3], and end up with a fully specified portfolio dynamics under optimality, according to:

$$dP = \Omega(t)Pdt + \Gamma PdZ \tag{13}$$

where

$$\Omega(t) = \frac{(\alpha - r)^2}{\sigma^2(1-b)} + r - A(t)^{-\frac{1}{1-b}}$$

$$\Gamma = \frac{(\alpha - r)}{\sigma(1-b)} \tag{14}$$

$\Omega(t)$ is a time-dependent drift term. The first two terms in $\Omega(t)$ is the weighted average of the expected return on the risky and risk-free asset respectively. The third term, which is time-dependent, is the withdrawals from the portfolio in terms of consumption.

Γ is the standard deviation of the portfolio. If the standard deviation of the risky asset s increases, we observe that Γ decreases monotonically. At first sight this may seem puzzling. However, we see from [12] that the reason for this comes from a lower investment in the risky asset as σ increases. We can also observe that when $b \rightarrow 1$, then $\Gamma \rightarrow \infty$, that is, the standard deviation of the portfolio becomes infinite when the relative risk aversion goes to zero. Further, when $b \rightarrow -\infty$, then $\Gamma \rightarrow 0$, that is, the portfolio becomes risk-free. This is because the agent is so risk averse that it is optimal for him to invest nothing in the risky asset.

We can also provide critical values of b, which relate the standard deviation of the portfolio Γ to the standard deviation of the risky asset σ. If $b>(r-\alpha+\sigma^2)/\sigma^2$, then $\Gamma > \sigma$. This also implies that the optimal fraction invested in the risky asset will be greater than one and will therefore induce net borrowing. For $b=(r-\alpha+\sigma^2)/\sigma^2$, then $\Gamma > \sigma$ and if $b<(r-\alpha+\sigma^2)/\sigma^2$, then $\Gamma < \sigma$. These two last critical values of b implies that the agent will not find it optimal to borrow at the risk-free rate, even if he had the opportunity to do so. This is the assumption made in the present paper.

Further, the optimal portfolio has a log-normal distribution, which will be a useful property in the subsequent section, where we value a personal loan contract contingent on this portfolio dynamics.

3. Valuation of a Personal Loan Contract

In this section we value a personal loan contract, which is contingent on the borrower's portfolio dynamics and time-to-maturity. The analysis takes as a starting point the Merton (1974) model, where it was found that option valuation methods could be used to arrive at a preference-free valuation of corporate liabilities. It basically relies on the isomorphic price relation between levered equity of a firm and a call option. Smith (1980) extends this analysis to the valuation of personal loans. He argues that the value of the loan should be determined by the stochastic features of the borrower's assets. The dynamics of these assets are exogenously specified. However, since these assets are controlled by the borrower, who probably maximizes his utility, such a specification may be too simplifying. It would be preferable if it had taken account of the agent's optimal consumption-portfolio decision. These observations have motivated us to value a personal loan contingent on a portfolio process, which is *consistent* with a utility maximizing agent's behaviour.

Hence, instead of taking the portfolio process as given, we use the dynamics from the outcome of the optimization program in Section 2. That is, the optimal portfolio dynamics, given by expression [13], are used as the dynamics of the underlying asset in the personal loan contract. Before continuing with the analysis we make the following assumptions:

Assumption 1: Capital markets are perfect in the sense that there are no transaction costs or taxes and that trading takes place continuously. All agents have free access to all available information.

Assumption 2: The agents have homogeneous beliefs about the stochastic behaviour of the value of the borrower's assets.

Assumption 3: The dynamics of the value of the borrower's assets are independent of the probability of default.

Assumption 4: There are no costs, for example, liquidation costs associated with default.

Assumption 5: The agent is restrained from borrowing at the risk-free rate.

Assumption 6: Time-to-maturity of the loan contract is equal to the time-horizon in the optimization problem in Section 2.

Assumption 7: The borrower's initial wealth, which consist of own capital and borrowed capital, is exogenously given. Hence, the valuation that is conducted below, merely decomposes the initial wealth into two parts: one part determining the value of the borrower's own capital, and one part determining the value of the loan.

Assumption 8: The personal loan contract, considered below, is a pure zero-coupon debt with a promised payment of E dollars at the maturity date T. Thus, no debt service will take place during the life of the loan contract.

Let F(t,P) be the value of a personal loan contract contingent on the borrower's portfolio P, whose dynamics are given by [13], and time t. Taylor expansion of F(t,P) and the use of Itô's lemma give the dynamics of the loan contract:

$$dF = \left[F_t + \Omega(t)PF_P + \frac{1}{2}\Gamma^2P^2F_{PP}\right]dt + \Gamma PF_P dZ \tag{15}$$

At the maturity date T, the value of the loan contract F_T, is:

$F_T = \text{Min}[P_T,E] = P_T\text{-Max}[P_T\text{-}E,0]$

In order to achieve a preference-free valuation, we construct a self financing portfolio, which replicates the dynamics of the loan contract.[13]

The value of the debt at time t=0 is then:

[13] This is done in appendix A2.

$$F = P_0 e^{-A'(T)} N\left(\frac{\ln\left(\frac{E}{P_0}\right) - \left(r + \frac{\Gamma^2}{2}\right)T + A'(T)}{\Gamma\sqrt{T}}\right) + Ee^{-rT} N\left(\frac{\ln\left(\frac{P_0}{E}\right) + \left(r - \frac{\Gamma^2}{2}\right)T - A'(T)}{\Gamma\sqrt{T}}\right) \quad (16)$$

Where $A'(T)$ is the accumulated propensity to consume, defined by expression [29] in appendix A2.

If $A'(T)$ is set equal to a constant s, which reflects debt service, then the value of the loan is identical to the corresponding value in Smith (1980).

An interesting feature, that may contradict the feeling, is that when the standard deviation in the risky asset σ, increases, the value of the debt also increases. This follows from inspection of [14]. The intuition is that a risk averse individual chooses to hold less risky assets in his portfolio when σ increases, ceteris paribus. This in turn decreases the portfolio standard deviation and thereby increases the debt value.

Because it is common in discussions of debt pricing to use yields rather than prices, we rewrite [16] as:

$$R(T) = \left(-\frac{1}{T}\right) \ln\left(\frac{F}{E}\right) \quad (17)$$

Where $R(T)$ is the yield-to-maturity of the debt contract.

Figure 1 below illustrates how the yield-to-maturity $R(T)$, depends on the repayment parameter γ (Gamma) and the underlying value P_0. The figure is plotted for $\gamma \in [0.1, 2]$ and $P_0 \in [0.9, 2]$. The other parameter values are $r=0.1$, $\beta=0.15$, $\sigma=0.2$, $\alpha=0.15$, $b=-1$, $E=1$ and $T=1$.

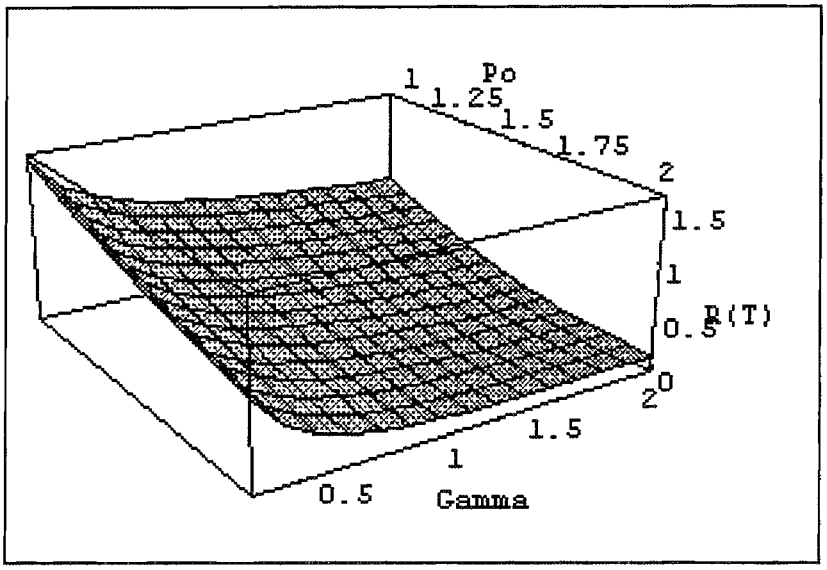

Figure 1

We can observe that the yield-to-maturity, R(T), varies between just above the risk-free rate of 10 percent up to 150 percent. The yield-to-maturity R(T), responds more to changes in P_0 for low values of γ (Gamma) than for high values. We also observe that R(T) is decreasing in γ. The explanation is that when γ increases, the incentives for paying back the loan increases, and therefore the required rate of return on the loan decreases. For larger values of γ the response of R(T) to changes in P_0 converges to the Merton (1974)-framework.[14] For a further discussion of this issue, see Section 4.

In Table 1 below we give some numerical values of the risk premium R(T)-r. The values are given for different g and P_0/E, where the other parameter values are the same as in Figure 1.

[14] The reason for this is that when $\gamma \to \infty$ the optimal consumption $C^* \to 0$. So, if the standard deviation of the portfolio in our model is set equal to the standard deviation of the firm value in Merton (1974) the derivative of R(T) with respect to the underlying process will be the same.

Table 1

The Risk Premium, R(T)-r, as a function of γ and P_0/E

b=-1	γ=1	γ=2	γ=5
P_0/E=2.0	0.02477	0.00180	0.00003
P_0/E=1.8	0.07326	0.01120	0.00042
P_0/E=1.6	0.16402	0.04783	0.00466
P_0/E=1.4	0.29129	0.13755	0.03150
P_0/E=1.2	0.44500	0.28145	0.12268

The values in the table support the discussion of Figure 1. As will become clear in Section 4, the rather high risk premia are due to withdrawals from the portfolio in terms of consumption and not because of a specially high portfolio risk. In fact, for the given parameter values, the standard deviation of the portfolio Γ is equal to 0.125, whereas the standard deviation of the risky asset σ is 0.2.

In the next section we will see how the value of the loan depends on the relative risk aversion of the individual (1-b) and his incentives for paying back the loan γ.

4. A Discussion of the Model Parameters

In this section we examine how the value of the loan contract is affected by changes in some central parameters, namely the incentives of the borrower to repay the loan (γ) and the relative risk aversion of the borrower (1-b). Thus, in the figures below we have plotted the relationship between F and γ (gamma) and F and b respectively. For all figures we have $E = 1$, $r = 0.1$, $\beta = 0.15$, $\alpha = 0.15$, $\sigma = 0.2$ and $P_0 = 1.2$.

The graph in Figure 2 below is valid for b=-1, which implies a relative risk aversion of (1-b)=2.

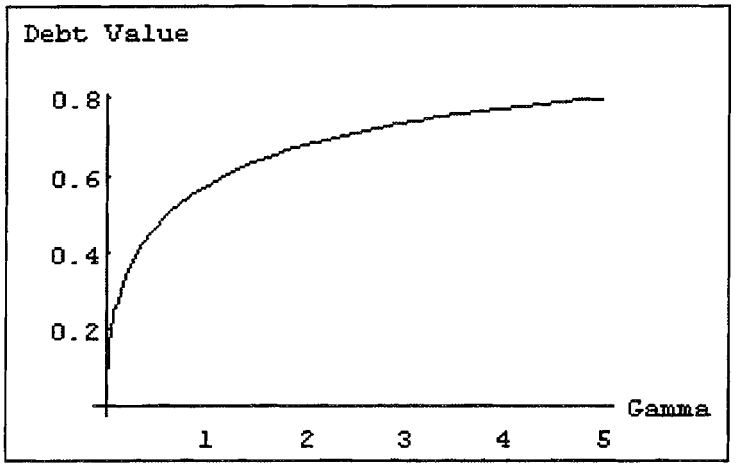

Figure 2

As discussed in Section 3, the debt value is increasing in γ and is, for very large γ, converging to the value of a risky debt according to the Merton (1974)-framework, where the underlying process is taken as given.[15] This comes from the fact that the only impact on F through γ is via the consumption variable A'(T), plotted in Figure 3 below, which goes to zero for $\gamma \to \infty$. This follows from expression [29] in appendix. We can also observe that when $\gamma \to 0$, that is, when the borrower's utility from having assets left at T goes to zero, the debt value too goes to zero, which is also clear from expression [29].

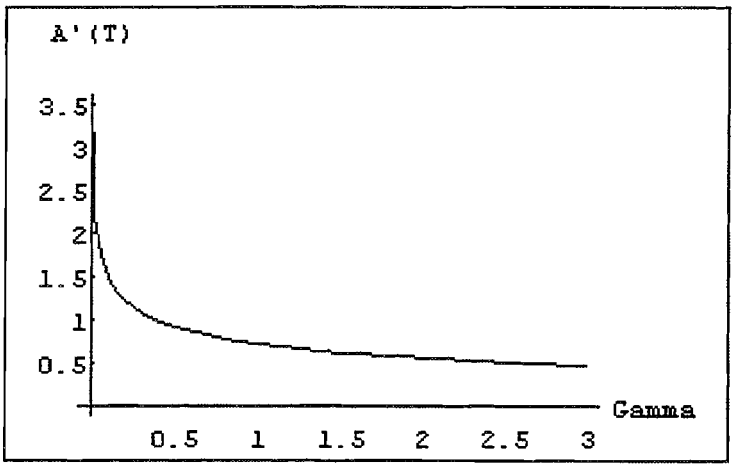

Figure 3

[15] This comparison with Merton (1974) is valid for a portfolio standard deviation, Γ, equal to the standard deviation of the underlying asset, σ.

Thus, when $\gamma \to 0$ the limit of A'(T) is infinity. This comes from the fact that the marginal rate of consumption $[C^*(t)/P(t)] \to \infty$ as $t \to T$, since zero utility is associated with positive wealth for $t>T$. That is, the optimal solution drives $P(t) \to 0$ as $t \to T$, in the absence of bequest motives. Therefore, since A'(T) is the integral of the marginal rates, it also becomes infinite. When $\gamma \to \infty$ we have that $A'(T) \to 0$ because utility from "bequest" dominates utility from consumption.

Figure 4 below illustrates the effect on the debt value from changes in the relative risk aversion of the borrower, 1-b, for $b \in [-6,-0.25]$.[16] The upper curve is drawn for g = 2 and the lower curve is drawn for $\gamma = 0.5$.

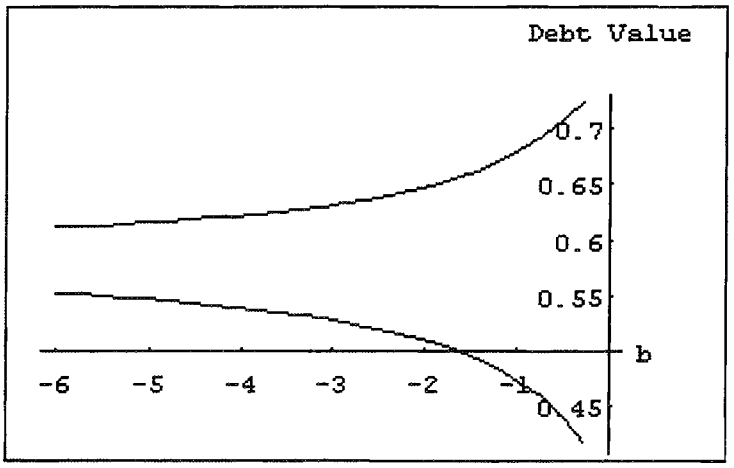

Figure 4

The explanation of the shapes of the curves is as follows: When $b \to -0.25$, F increases for $\gamma>1$ and decreases for $\gamma \leq 1$. This comes from the following two effects on the debt value from a change in b:

(i) the consumption variable A'(T), is increasing in b for $\gamma \leq 1$ and decreasing in b for $\gamma>1$.

(ii) the portfolio standard deviation, Γ, is increasing in b.

The consumption variable A'(T), drawn in Figure 5 below for $\gamma=0.5$ (the upper curve) and $\gamma=2$ (the lower curve), which represents the accumulated consumption rates, is

[16] The reason for b to be bounded above by -0.25 is to ensure that the optimal weight invested in the risky asset is less than one, as discussed in Section 2.

decreasing in b for $\gamma>1$ and gives a positive effect on the debt value.[17] This effect dominates the negative effect from the portfolio standard deviation Γ, given by [14], which is increasing in b and therefore has a negative impact on the debt value. Hence, for $\gamma>1$, the debt value is increasing in b. This explains the upper curve in Figure 4.

For $\gamma\leq1$, the consumption variable A'(T) increases in b and therefore induces a negative impact on the debt value. Γ, as mentioned before, always increases in b and induces a negative impact on the debt value. Since these two effects work at the same direction, the debt value is decreasing in b. This explains the lower curve in Figure 4.

Finally, as $b\rightarrow -\infty$ the standard deviation of the portfolio $\Gamma\rightarrow0$, since the optimal fraction of the portfolio invested in the risky asset, $\omega^*\rightarrow 0$. Further, for the given parameter values, A'(T) and the debt value converges to 0.71867 and $P_0e^{-A'(T)}= 0.58469$ respectively. This is clear from the inspection of the formulas [29] and [16]. Thus, no matter how risk averse the borrower is, the debt will never have a risk-free value since the consumption is converging towards a positive fraction and therefore deteriorates the portfolio value.

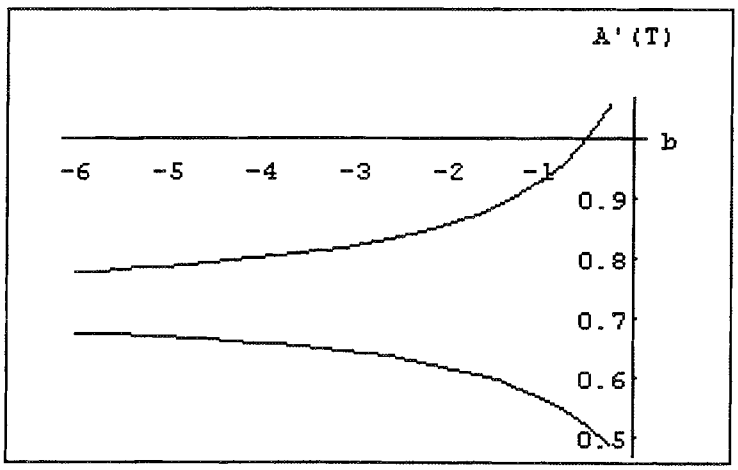

Figure 5

To summarize the effects of γ and b on the debt value F(t,P), we present the following three-dimensional figure. The plot range is $\gamma\in[0.01, 2]$ and $b\in[-3, -0.25]$.

[17] This follows from inspection of [29], which explains the accumulated consumption behaviour.

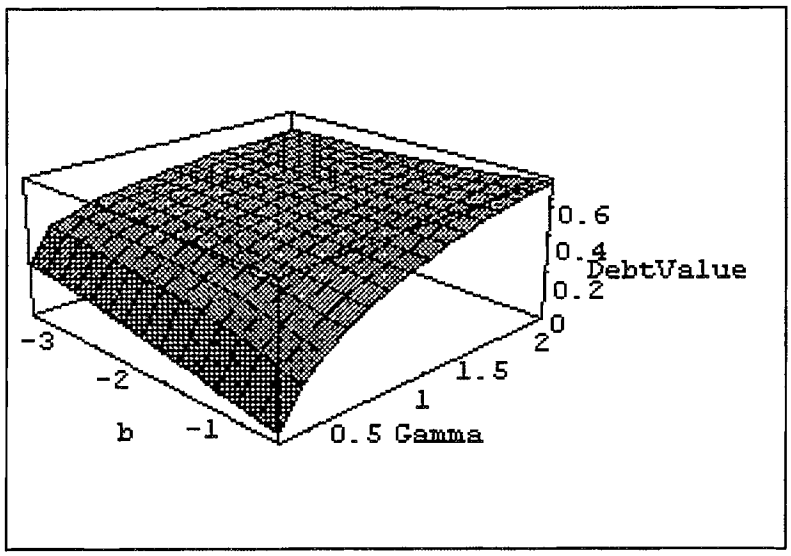

Figure 6

If it is possible to estimate the parameters γ and b, then the relationship shown in the figure can be used by, for example, a bank in the position of pricing a loan to a certain person. It is especially important to know when estimation errors of the parameters can lead to a serious mispricing. For example, if the true parameter values of the borrower are $\gamma>1.5$ and $b<-2$, small estimation errors will not cause any grave pricing errors. On the other hand for $\gamma<1.5$ and $b>-2$ we can observe from the figure that even small estimation errors can be serious.

5. Concluding Remarks

In this paper we have focused on the importance of considering a borrower's consumption-portfolio selection when valuing a personal loan contract. The optimal wealth dynamics, derived from an optimization program, is used as the underlying state variable when pricing the loan contract with contingent claims analysis. Thereby we have been able to study the impact on the debt value from the borrower's type, which is characterized by the pair of relative risk aversion and utility from repaying the loan. This is the major contribution of the paper. The advantage of the paper, over the previous contingent claims valuation studies of personal loan contracts, is that the dynamics of the wealth process are *consistent* with utility maximization.

Appendix

A1

This appendix provides proof that [9] is a solution to [8].

To solve [8], try the following solution:

$$J(t,P) = \frac{A(t)}{b} e^{-\beta t} P^b \tag{18}$$

Take the relevant partial derivatives of this solution and substitute them into [8]. A necessary condition that [18] in fact is a solution is then that A(t) must satisfy the following ordinary differential equation:

$$\frac{dA}{dt} = A\left[\beta - rb - \frac{(r-\alpha)^2 b}{2\sigma^2(1-b)}\right] - (1-b)A^{\frac{b}{b-1}} \tag{19}$$

$$A(T) = \gamma E^{-b}$$

The solution of [19], A(t), fully specifies [18]. In order to find this, try the following solution:

$$A(t) = \left[Be^{\frac{\mu}{1-b}t} + C\right]^{1-b} \tag{20}$$

where B and C are integration constants to be determined and μ is defined by:

$$\mu \equiv \beta - rb - \frac{(r-\alpha)^2 b}{2\sigma^2(1-b)} \tag{21}$$

Now, substitute [20] and the derivative of [20] with respect to t into [19]. Then we get the following equation from which the integration constants can be determined:

$$B\mu e^{\frac{\mu t}{1-b}}\left(Be^{\frac{\mu t}{1-b}}+C\right)^{-b} = \mu\left(Be^{\frac{\mu t}{1-b}}+C\right)^{1-b} - (1-b)\left(Be^{\frac{\mu t}{1-b}}+C\right)^{-b} \qquad (22)$$

This equation, together with the boundary condition in [19], determines B and C according to:

$$B = e^{-\frac{\mu}{1-b}T}\left(\gamma^{\frac{1}{1-b}}E^{-\frac{b}{1-b}} - \frac{1-b}{\mu}\right) \qquad (23)$$

$$C = \frac{1-b}{\mu}$$

Substitution of these constants into [20] gives the solution of A(t) and therefore also the solution of J(t,P).

A2

In this appendix we construct a self-financing replicating portfolio in order to derive a preference-free valuation of the contingent claim, F(t,P).

Let V(t) be the value of a self-financing portfolio with a relative weight of u^0 invested in the risk-free asset and relative weight of u^1 invested in the individual's wealth. Thus, $u^0=h^0B/V$ and $u^1=h^1P/V$, where h^0= the number of shares invested in the risk-free asset and h^1= the number of shares invested in the individual's wealth. Furthermore, the dynamics of B and P are given by [2] and [13] respectively. The dynamics of the portfolio, V, are then:

$$dV = V\left[r + u^1\left(\Omega(t) - r + A(t)^{-\frac{1}{1-b}}\right)\right]dt + u^1V\Gamma dz \qquad (24)$$

The dynamics of the contingent claim F(t,P), are given by [15].

In order to replicate the contingent claim we chose:

$$u^1 = \frac{PF_P}{V} \tag{25}$$

Substitute this weight into [24] and conclude that if V(t) is replicating F(t,P), then dF-dV =0, that is, the following differential equation must hold in order to avoid arbitrage:

$$\frac{1}{2}\Gamma^2 P^2 F_{PP} + \left(r - A(t)^{\frac{1}{1-b}}\right)PF_P + F_t - rF = 0 \tag{26}$$

$$F(T,P) = \mathrm{Min}[P,E]$$

Solving [26] by the method of Feynman-Kac gives the following price of the debt at time t:

$$F(t,P) = e^{-r(T-t)}E^Q \mathrm{Min}[P_T,E] = e^{-r(T-t)}E^Q\left[P_T - \mathrm{Max}[P_T - E, 0\ ]\right] \tag{27}$$

where E^Q is the expectation operator under the risk-neutral probability measure Q. The solution of P_T under probability measure Q is:

$$P_T = P_t \exp\left[\left(r - \frac{1}{2}\Gamma^2\right)(T-t) - \int_t^T A(s)^{-\frac{1}{1-b}} ds + \Gamma\left(Z(T) - Z(t)\right)\right] \tag{28}$$

where

$$\int_t^T A(s)^{-\frac{1}{1-b}} ds = \int_t^T \left[\left(\gamma^{\frac{1}{1-b}} E^{-\frac{b}{1-b}} - \frac{1-b}{\mu}\right) e^{\frac{\mu}{1-b}(s-T)} + \frac{1-b}{\mu}\right]^{-1} ds =$$

$$= \left[\frac{T\mu}{1-b} - \ln\left(\gamma^{\frac{1}{1-b}} E^{-\frac{b}{1-b}}\right)\right] - \tag{29}$$

$$-\left[\frac{t\mu}{1-b} - \ln\left(\left(\gamma^{\frac{1}{1-b}} E^{-\frac{b}{1-b}} - \frac{1-b}{\mu}\right) e^{\frac{\mu}{1-b}(t-T)} + \frac{1-b}{\mu}\right)\right] \equiv A'(T-t)$$

Substitution of [29] into [28] and [28] into [27], and then calculating the expected value, gives the value of the debt at t=0 as:

$$F = P_0 e^{-A'(T)} N\left(\frac{\ln\left(\frac{E}{P_0}\right) - \left(r + \frac{\Gamma^2}{2}\right)T + A'(T)}{\Gamma\sqrt{T}}\right) + Ee^{-rT} N\left(\frac{\ln\left(\frac{P_0}{E}\right) + \left(r - \frac{\Gamma^2}{2}\right)T - A'(T)}{\Gamma\sqrt{T}}\right) \tag{30}$$

This is expression [16].

A3

In Section 2 we solved the consumption-investment rules as if the agent was not restrained from borrowing at the risk-free rate. However, in the explicit solution we assumed the parameter values to satisfy $\omega^* \leq 1$, that is, the no-borrowing restriction was not violated. In this appendix we derive the consumption-portfolio rules when the no-borrowing restriction <u>is</u> binding, that is, when the optimal investment in the risky asset is $\omega^*=1$.

The problem can be handled by the Kuhn-Tucker methods with resulting inequalities. When the no-borrowing is not binding the consumption-investment rules are identical to the ones derived in Section 2. When the restriction is binding, the optimal consumption-investment rule will be as follows:

$$C^*(t) = \frac{P(t)}{\left[\left(\gamma^{\frac{1}{1-b}} E^{-\frac{b}{1-b}} - \frac{1-b}{\rho}\right) e^{\frac{\rho}{1-b}(t-T)} + \frac{1-b}{\rho}\right]} \tag{31}$$

$$\omega^* = 1$$

where

$$\rho \equiv \beta - \alpha b - \frac{1}{2}\sigma^2 b(1-b) \tag{32}$$

For the parameter values r=0.1, β=0.15, α=0.15 and σ=0.2 the ρ-parameter is identical

to the μ-parameter, given in expression [11] in Section 2, for b=-0.25. This is because b=-0.25 is the critical value at which the optimal solution of the unconstrained maximization problem is identical to its constrained counterpart.
The dynamics of the optimal portfolio will be:

$$dP = (\alpha P - C^*(t))dt + \sigma P dZ \tag{33}$$

If we conduct the same calculation as in appendix A2 above, we end up with a value of the loan contract according to:

$$F = P_0 e^{-A'(T)} N\left(\frac{\ln\left(\frac{E}{P_0}\right) - \left(r + \frac{\sigma^2}{2}\right)T + A'(T)}{\Gamma\sqrt{T}}\right) + Ee^{-rT} N\left(\frac{\ln\left(\frac{P_0}{E}\right) + \left(r - \frac{\sigma^2}{2}\right)T - A'(T)}{\Gamma\sqrt{T}}\right) \tag{34}$$

where A'(T) is identical to expression [29] except that μ is replaced by ρ, defined by expression [32].

For $\sigma = \Gamma$ and $\rho = \mu$ the value of this loan coincides with the value of the loan derived in Section 2, given by expression [16].

In Table 2 below we give some numerical values of the risk premium, R(T)-r. The parameter values are $r = 0.1$, $\beta = 0.15$, $\alpha = 0.15$, $\sigma = 0.2$ and $b = 0.5$.

Table 2

The Risk Premium, R(T)-r, as a function of γ and P_0/E

b=0.5	γ=1	γ=2	γ=5
P_0/E=2.0	0.05488	0.00021	5.83×10^{-6}
P_0/E=1.8	0.10532	0.00105	0.00005
P_0/E=1.6	0.18568	0.00473	0.00033
P_0/E=1.4	0.30008	0.01875	0.00221
P_0/E=1.2	0.44796	0.06258	0.01266

In Table 3 below risk premia are given for a less risk averse individual, with b = 0.9.

Table 3

The Risk Premium, R(T)-r, as a function of γ and P_0/E

b=0.9	γ=1	γ=2	γ=5
P_0/E=2.0	0.05408	2.46×10^{-6}	2.40×10^{-6}
P_0/E=1.8	0.10410	2.09×10^{-5}	2.05×10^{-5}
P_0/E=1.6	0.18404	0.00017	0.00017
P_0/E=1.4	0.29813	0.00128	0.00127
P_0/E=1.2	0.44588	0.00833	0.00823

In Table 3 the risk premia are significantly lower than in Table 2. This may be surprising, since in Table 3 the agent is much more tolerant to risk (b=0.9) than in Table 2 (b=0.5). However, since he is restrained from borrowing at the risk-free rate, he cannot exploit this risk tolerance, and his weight in the risky asset is equal to unity in both cases, which also implies that the standard deviation of the optimal portfolio is identical in both cases, namely $\Gamma = \sigma = 0.2$. Therefore, the only impact of different risk aversion on the risk premium is through the consumption behaviour, which is reflected in A'(T). For $\gamma>1$, the risk premium will decrease in b due to this effect. For $\gamma\leq1$, the risk premium will increase in b. These effects can be seen in the following two figures:

The following figure is plotted for γ=2 and P_0=1.2.

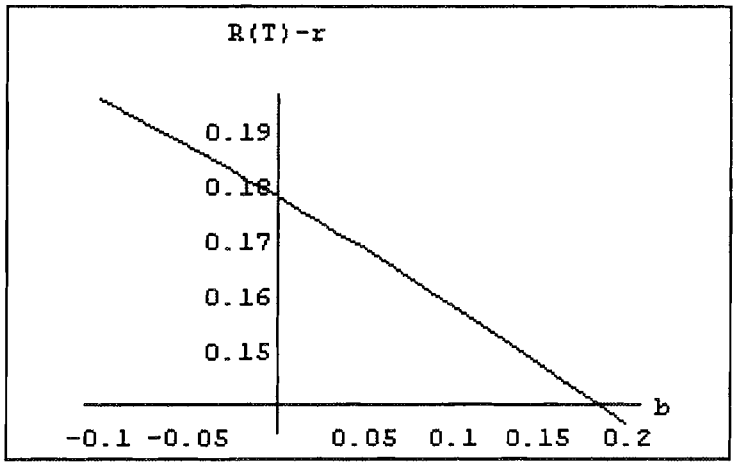

Figure 7

The following figure is plotted for $\gamma = 0.9$ and $P_0 = 1.2$.

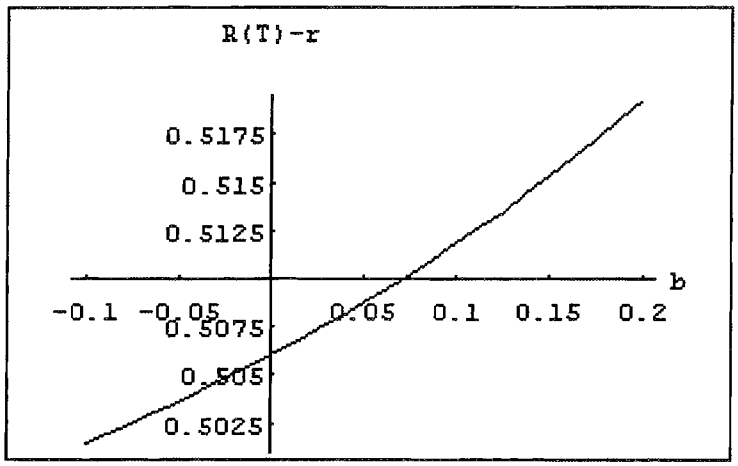

Figure 8

We can conclude that γ has a significant impact on the risk premium.

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