Some

DYNAMIC ECONOMIC MODELS OF THE FIRM

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Elon V. Ekman

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DYNAMIC ECONOMIC MODELS OF THE FIRM

A Microeconomic Analysis with Emphasis on Firms that Maximize Other Goals than Profit Alone

EFI THE ECONOMIC RESEARCH INSTITUTE
AT THE STOCKHOLM SCHOOL OF ECONOMICS

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PREFACE

This report will shortly be submitted as a doctor's thesis at the Stockholm School of Economics. The research has been carried out at the Economic Research Institute at the Stockholm School of Economics, but the author has been entirely free to conduct his research in his own way as an expression of his own ideas.

The Institute is grateful for the financial support which has made this research possible.

Stockholm, November 1978

THE ECONOMIC RESEARCH INSTITUTE
AT THE STOCKHOLM SCHOOL OF ECONOMICS

Karl-Erik Wärneryd
Director of the Institute

Bertil Näslund
Program Director
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FOREWORD

The most important theme of this microeconomic study is the dynamic optimization analysis of firms with other objectives than profit alone, for instance the managerial firm and the labor-managed firm. Dynamic processes such as financing, capital investment, marketing investments and labor planning are studied in optimal control theory models.

The work has been mainly inspired by Professor Bertil Näslund, who has provided the frame of reference and who has spent considerable time discussing the subject matter. By far, most of my gratitude goes to him for his stimulating support, knowledgeable advice and great interest in the work.

Also, I should especially like to thank Professor Paulsson Frenckner of the University of Stockholm for spending considerable time and effort especially in the initial phases of this project. Without his keen interest and support, this study would never have been done.

Docent Ingolf Ståhl has led a number of seminars where most parts of the study have been discussed. He deserves many thanks for this and for his interesting and knowledgeable comments. Also, I should especially like to acknowledge with gratitude the many stimulating comments received from all colleagues at the Institute during these seminars.

I further want to sincerely thank Professor Karl-Göran Mäler for reading and commenting on the draft of this book and for his kind interest and skilled advice during the later stage of this project.

I have had the advantage of discussing part of this project with Professor A. Bensoussan and other members of the faculty of the European Institute for Advanced Studies in Management (EIASM) in Brussels, Belgium. I should like to express my sincere thanks for this. Professors H. Albach, Germany and T.H. Naylor, USA and Dr. J. Ylä-Liedenpohja, Finland have kindly commented on parts of the study.

Special thanks also go to Peter Johansson at the Royal Institute of Technology in Stockholm for generous and valuable aid with computer runs for the numerical examples. Also I mention with gratitude Claes Trygger, at the same school, for stimulating mathematical discussions.

I am very grateful to Mrs. Birgitta Mossberg for typing the final version of this report and to Mrs. Barbro Orrung for skillful and efficient administration of the production of this book. I also thank Mrs. Nancy Adler very much for checking and improving on the English language although the opportunity was not given for her to check the final modifications of the draft.

Perhaps it is appropriate to mention here that of the two graduates of the Stockholm School of Economics with the author's full name, father and son, the present author is not the graduate from 1922.

Finally, I would like to express my deepest gratitude to my wife, Sophie, who has not only whole-heartedly supported the work but also typed and retyped all initial drafts. I would also like to include our children, the small dynamic models Louise and Philip, in my gratitude. To all three I dedicate this book.

Stockholm, October 1978

Elon V. Ekman

CONTENTS

1 BACKGROUND, SCOPE AND AIM

1.1 THEORIES OF THE FIRM IN GENERAL

1.1.1 Many Theories of the Firm

There are many theories and models of the firm, developed within various disciplines for various purposes and with various characteristics. A number of surveys and classifications of these theories and models have appeared, especially during the last decade. Some of these are Boulding & Spivey (1960, Chapters 1 and 6), McGuire (1964), Cohen & Cyert (1965), Machlup (1967), Naylor & Vernon (1969, Chapters 1 and 5), Horowitz (1970, Chapters 1, 10-12, 15), Näslund & Wadell (1971), Archibald (1971), Cyert & Hedrick (1972), Hawkins (1973) and Crew (1975).

No attempt will be made here to provide yet another survey or yet another review of various definitions and concepts of the firm. Rather the intention is to describe briefly the frame of reference and concept of the firm used here and relate them to the different disciplines within which the behavior of the firm is studied. In doing this we shall refer to some of the authors mentioned above.

1.1.2 Holistic and External Behavior

McGuire (1964, p. 18) distinguishes between holistic and behavioral concepts of the firm. Naylor & Vernon (1969) mention:

> "Holistic models view the firm as a 'unified acting entity or organism' (McGuire, 1964, p. 18) in which input and output decisions are made simultaneously in the light of some given objective of the firm as a whole and in the light of given product-demand information, production-technology information and factor-supply information. Behavioural models conceive of the firm as the 'confluence of several streams of interrelated behaviour' (McGuire, 1964, p. 18)."
>
> (Naylor & Vernon, 1969, p. 109)

We shall limit ourselves mainly to a holistic view of the firm rather than a behavioral one. The holistic description of the firm quoted above, may also serve as a broad definition of the firm for our present purpose.

Thus we shall concentrate here on the firm as a whole instead of on its respective parts and functions, and also we shall concentrate on the external behavior of the firm rather than the internal behavior as studied for example in orgainization theory and in the behavioral theory of the firm of Cyert & March (1963). We shall discuss other aspects of organization theory later in this chapter.

1.1.3 Positive Economic Theories of the Firm

Theories of the firm in the above sense were studied at an early stage within the discipline of economics.

> "... economists use their theory of the firm as a means to predict resource allocation among competing uses. The theory of the firm, in this interpretation, forms a part of this larger schema but must not be isolated or studied apart from its larger setting." [1)]
>
> (McGuire, 1964, p. 9)

Our frame of reference and concept of the firm is similar to the one of the economist's in the above sense. In other words we take an external view of the firm, seeing

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1) McGuire (1964, p. 9) in referring to the thesis advanced in Krupp, S., Pattern in Organization Analysis: A Critical Examination. Philadelphia, Pa, 1961, Chapter 1.

it as one of the building blocks to help us understand the functioning of the economy as a whole, rather than to help us understand the exact functioning of the firm per se and for its own purpose. This is a usual position taken by economists although there is growing uneasiness about it.[1)]

We shall thus concentrate on the pure economic theory of the firm in the above sense, as opposed to other theories of the firm.

However, we do not want to consider the firm as a pure theoretical construct [2)] as is the case in the classical theory of the perfectly competitive economy. Rather we prefer to modify some of the assumptions in the traditional economic theory of the firm, and thus to arrive at other and possibly better predictions - and perhaps explanations - of external firm behavior in certain instances.[3)] But we want to do this, as stated above, mainly from the point of view of economics and economic policy rather than from the point of view of the exact functioning of the firm per se. In a sense we shall be studying a representative firm, to use the terminology of Marshall (1890).

It follows that we shall concentrate mainly on a non-normative or positive approach rather than on a normative one.[4)] However, in spite of the limiting assumptions of our models, it will be seen that they are not perhaps completely void of normative content, especially as regards general ideas on dynamic optimization. Also they may be used as building blocks for other more complete and detailed models of the firm which might have a normative content.

The criticism generally directed at the traditional economic theory of the firm is equally valid, where applicable, to the models suggested here. But so is the defense

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1) Cyert & Hedrick (1972, pp. 408-409).
2) Machlup (1967, p. 9).
3) Machlup (1967, pp. 9-10).
4) McGuire (1964, pp. 11-12) and Friedman (1953, Part I).

of it. We shall not go into this debate here, as it is well described in the references mentioned in Section 1.1.1.

1.1.4 Extensions of the Economic Theory of the Firm

One of many recent verbal descriptions of the traditional economic model of the firm, gives some of the assumptions as follows:

> "1) The universe is static. Tomorrow will be like today. Time is not considered except as its passage is necessary to allow shifts to occur from one equilibrium to another. 2) The firm seeks maximum short-run profits. 3) The firm is merely a computing point without financial needs. 4) The quantity produced equals the quantity sold. There is no inventory;...7) Nonprice variables either do not affect market demand or do not change;...10) Whatever the demand and supply functions may be, they are known by the entrepreneur for a range of input or output quantities;..."
>
> (Oxenfeldt & Tennant, 1963, pp. 196-197)

The same authors mention that,

> "...more adequate models might take into account some of the important matters assumed away...The pressure of financial needs...might be included. More attention could be given to the nonprice aspects of demand...Other aspects of business decisions besides the adjustment of price-output levels might be included...The dynamic problems of adaptation through time...should be incorporated into the model if they can be."
>
> (Oxenfeldt & Tennant, 1963, p. 200)

A great deal of work has been done along these lines both before the above was written in 1963 and, more particularly, since.

It is our aim to study some extensions of the traditional economic theory of the firm as described above.

Three important areas of development in the economic theory of the firm over the last two decades have been the dynamics of the firm, the objectives of the firm, and uncertainty.[1] Typically the second of these mainstreams was still not receiving much emphasis at the time of the above

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1) We use uncertainty interchangeably as a term for risk and/or uncertainty.

quotation from 1963. We shall discuss these respective areas in the following three sections 1.2-1.4.

Combinations of the first two of these areas of development are illustrated in Figure 1:1 as regards deterministic economic models of the Firm.

	Static (and Steady-State) Models	Dynamic Models
Profit-Maximizing Objective	(1) Traditional Economic Theory of the Firm	(2) Dynamic Investment, Finance and Marketing (and Other) Models of the Firm (Chapter 5)
Objectives Implying Maximization of Other Variables Than Profit Alone	(3) Managerial and Labor Managed (and Other) Theories of the Firm	(4) Dynamic Managerial and Labor-Managed (and Other) Models of the Firm (Chapters 2-4)

Figure 1:1. Some Areas of Development in the Deterministic Economic Theory of the Firm.

Whereas the literature in areas (1) and (3) in Figure 1:1 is relatively well developed, the literature in area (2) is now in development. However, the literature in area (4) is relatively insignificant, which is surprising, because firms with objectives of maximizing other variables than solely profit would appear to have at least as great a need to invest and grow dynamically as profit-maximizing firms. However, an explanation of this situation is probably that the employed dynamic optimization methods are relatively new and therefore will be used in area (4) as soon as area (2) in Figure 1:1 has been sufficiently studied. As will be discussed later on in this chapter, the present study mainly belongs to area (4) in Figure 1:1, which, as suggested, is a relatively new and explorative area of research.

For the sake of completeness we shall mention here the seminal work of Coase (1937) on the nature of the firm. Coase (1937) is concerned with why a coordinating device such as the firm is needed in the economy in addition to the price mechanism. His answer is that the firm exists because it can organize services internally at a lower cost than if each service is purchased in the market. Coase' (1937) concept of the firm is related to the theoretical construct of pure neoclassical economics in that it regards the firm as a device in the price mechanism. However, the pure neoclassical firm does not have to correspond to actual firms, whereas Coase' (1937) firm has the advantage of relating to actual firms as well as being a device in the price mechanism. Based on the Coase (1937)-related concept that the firm incurs costs when internally reorganizing production resources for growth, Söderström (1977) has developed a dynamic model of the pure neo-classical firm. This leads us now to discuss the economic dynamics of the firm.

1.2 DYNAMIC ECONOMIC THEORIES AND METHODS OF THE FIRM

In this study we usually define a system as *dynamical* if its behavior over time is determined by functional equations in which variables at different points of time are involved in an essential way.[1)] Where no misunderstanding is possible steady-state growth models and kinematic models are also included in the term dynamic models in order to distinguish them from purely static models.

1.2.1 Macro- and Microeconomic Growth

Over the last two decades interest in macroeconomic growth problems has grown considerably, as discussed by e.g. Burmeister & Dobell (1970) and Sen (1970).

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1) Cf. Frisch (1936), Samuelson (1947, pp. 311-317) and Gandolfo (1971, pp. 1-3).

This tendency to study macroeconomic growth problems naturally also encouraged economists to study microeconomic growth problems of the firm at about the same time as in the work of Penrose (1959), Baumol (1962), Marris (1964), O.E. Williamson (1964) and J.H. Williamson (1966). Recent surveys with new models have been presented by Marris & Wood (1971), Eriksson (1975) and Herendeen (1975).

An important reason for studying growth behavior at the level of the firm is that growth in society is determined to a significant extent by firms.[1)]

Growth models of the firm were also studied at an early stage in the financial literature, for instance by Williams (1938), Durand (1957), Miller & Modigliani (1961), Gordon (1962) and others.

All these studies of the firm are devoted mainly to an analysis of constant rates of growth of the firm. However, the assumption of balanced steady-state growth, i.e. growth of all firm variables at the same constant rate, is normally a very strong assumption, especially in the initial phases of firm growth. Also the problem is to find the optimal initial size from which to start growing at a constant rate.[2)]

This dissatisfaction with the steady-state growth theory of the firm led a number of authors to study more dynamic theories of the firm. A contributing factor was the new development of dynamic optimization methods, such as dynamic programming by Bellman (1957) and the Maximum Principle by Pontryagin et al. (1962).

This trend towards more dynamic economic models of the firm was also influenced by a similar and earlier development in macroeconomics.[3)]

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1) Näslund & Sellstedt (1977, p. 219).
2) Solow (1971) and Bensoussan, Hurst & Näslund (1974, pp. 75-76).
3) An example in macroeconomics as of 1967 is the collection of essays by Shell (1967). A more recent example, in the macroeconomic field of environmental economics, is Mäler (1974, Chapter 3).

Fully dynamic economic optimization models of the firm have been developed mainly during the last decade. No comprehensive survey is yet available but the works of Phelps (1970), Hughes (1970), Jacquemin & Thisse (1972), Bensoussan, Hurst & Näslund (1974), H.T. Söderström (1974) and Brechling (1975) each review some of the work done.

As mentioned above, the study of dynamic economic optimization models of the firm has been closely linked with the development of dynamic optimization methods. We shall now discuss some of these methods.

1.2.2 Two Dynamic Optimization Methods

Many studies of fully dynamic economic optimization models of the firm have utilized Pontryagin's Maximum Principle [1)] rather than Dynamic Programming[2)]. One reason is that the former provides a logical dynamic counterpart to traditional static optimal differential calculus. Another reason is that Dynamic Programming, although not unsuited to analytical economics, is especially useful in numerical analysis on account of its recursive equations. A third reason, related to the first, is that models of economic theory have traditionally been formulated mainly in continuous terms and dynamic programming, in its best-known form, is a discrete time technique.

These newer dynamic optimization techniques are actually developments of the classical calculus of variations originated by mathematicians such as Euler in the eighteenth and nineteenth centuries. However, they add the important element of being able to handle inequality constraints. Also the Maximum Principle implies a new way of formulating dynamic optimization problems, using a technique for the transition equations that is similar to the method

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1) Pontryagin et al., (1962).
2) Bellman (1957).

of Lagrangian multipliers in static optimization problems with constraints. The adjoint variables - belonging to the transition equations - of the Maximum Principle are of special interest in economics because they represent the value - at optimum - of imputing a unit of the variable describing the state of the system studied. This makes the Maximum Principle especially interesting to dynamic economics in the same way that shadow values and the dual problem formulation of mathematical programming make the mathematical programming technique of special interest to many-variable constrained static economics.

1.2.3 Optimal Control Theory

It is often convenient to formulate a dynamic problem as a control problem. Two types of variables may be distinguished, state variables and control variables. State variables describe the state of the system at each point in time, while control variables are used to guide the state of the system towards the desired goal. If this goal is the optimization of some objective function, we may speak of optimal control theory. The fully dynamic optimization models in the recent macro- and microeconomic research mentioned above, are often formulated in terms of optimal control theory.

1.2.4 Mathematical Programming

Another way of analyzing dynamic optimization behavior makes use of mathematical programming.

Pioneering work in mathematical programming in the area of the financial behavior of the firm has been done independently by Weingartner (1963) in the USA and by Albach (1962) in Germany. Later Jääskeläinen (1966) and others have elaborated such models, while Näslund (1966) and others have introduced risk into them.

A typical mathematical programming formulation of the optimal financial behavior of the firm is

$$\text{Max } U(d_o, d_1, d_2, \ldots\ldots d_T, \sum_{i=1}^{n} w_i\, x_i)$$

subject to

$$\sum_{i=1}^{n} a_{it}\, x_i + d_t = b_t \qquad (t = 0,1,\ldots\ldots,T)$$

$$0 \leqslant x_i \leqslant c_i \qquad (i = 1,\ldots\ldots,n)$$

$$d_t \geqslant 0 \qquad (t = 0,1,\ldots\ldots,T)$$

where dividends at time t are d_t and U is a utility function. a_{it} is the net outlay caused by investment project i at time t. Net receipts caused by a project imply negative a_{it}. Financing projects are thus also incorporated into the model. b_t represents the available cash at time t from projects other than those studied. x_i is the amount of project i chosen, and is the decision variable. w_i is the terminal value at time T of project i and is exogenously given. There are n possible projects and T time periods. This model is discussed further in Hax (1972, Chapter 3.2).

Thus, the problem is to maximize the utility of intertemporal dividends and project end values subject to liquidity constraints in each time period and to the non-negativity of the quantity of each project and of the dividend for each time period. Also, the magnitude of each project i is limited by c_i, an exogenous constant.

The liquidity constraint implies that for each time period, the money available for dividends d_t is equal to the available cash b_t plus the net proceeds of all the investment and financing projects

$$\sum_{i=1}^{n} (-a_{it})x_i\,.$$

By assuming, for example, that $a_{jo} = a_{j1} = a_{j2} = 0$, it is possible to include a project j that does not start until the third period. Thus the model is quite versatile.

Dynamic models within the framework of optimal control theory will be formulated at a later stage in this paper, but it can be mentioned already here that the mathematical programming approach offers relatively equal possibilities and that in many respects, such as shadow values and dual formulations, it is similar to the optimal control theory approach. For example, optimal control theory models may be transformed into mathematical programming form, for solution by computer calculations. The reasons for choosing optimal control theory rather than mathematical programming for the dynamic analyses presented in this study are similar to those mentioned in Section 1.2.2 above when discussing the dynamic programming approach.

1.2.5 Stochastic Models of Size Distributions and Growth of Firms

Gibrat (1931) formulated a "law of proportionate effect", arguing that the proportionate change in the size of the firm is independent of its absolute size. Such a law of growth generates a lognormal size-distribution of firms, a distribution which resembles distributions of firms observed in practice.

Hart (1962) tends toward the view that there is a large stochastic component in the forces determining the growth of firms, which surpasses the influence from economic forces and makes it difficult to adopt a deterministic explanation of firm growth.

On the other hand, Eatwell (1971) notes,

> "Even if Gibrat's Law, or some variation thereof, had been found to be appropriate for the description of size distributions, the real-world causality underlying the generation of these distributions (as distinct from the mathematical causality) would not be known. Stochastic models are useful for studying the concentration process and as a 'check' on other models, but they do not answer questions about the internal dynamics of corporate growth."
>
> (Eatwell, 1971, p. 418)

The dynamic models we analyze in this work subscribe more to Eatwell's view than Hart's view, although both seem clearly relevant to the study of the behavior of firms.

A recent discussion which includes stochastic process models of firm distributions and other size distributions is to be found in Näslund (1977), which surveys the field and discusses new approaches.

1.2.6 Organizational Growth Theories, Biological Analogies and Systems Theory

Although we are focusing mainly on economic dynamic theories of the firm, mention must also be made of various related dynamic frames of reference, such as organizational growth theory as discussed for example in Filley & House (1969, Chapter 18), biological analogies as discussed for instance in Penrose (1952) and Haire (1959), and general systems and complex adaptive systems theory as surveyed for example in Näslund & Wadell (1971, Chapters 9 and 10).

1.2.7 Simulation, Industrial Dynamics and Strategic Planning

Somewhat related to the satisficing school of thought on the firm - Section 1.3.3 below - are the dynamic simulation models of the firm, the Industrial Dynamics approach introduced by Forrester (1961), and the framework of strategic planning.

In dynamic simulation models of the firm, assumptions are made about parameters and independent variables, such as sales over time for example. Using postulated relations, the dependent variables - profit, for example - are calculated. By repeating these calculations many times on a computer with many different values for the parameters and independent variables, it is possible to get some idea by trial and error about, for example, various ways of increasing profit.

Simulation models of this kind are often quite detailed and rather complex, which has the advantage that it is also possible to make them more realistic. On the other hand, the large number of specific variables and relations often makes it difficult to draw general conclusions and to generate general hypotheses about firm behavior. In the case of the dynamic optimization model, once an optimum has been found, it is reasonably often possible to perform a sensitivity analysis (comparative dynamics) of the optimum. However, if an optimum is found in a simulation model by a trial and error process, then a small parameter shift would call for a new trial and error process. Thus a sensitivity analysis is more normally and readily performed in an otpimization model.

A further account of these issues of dynamic simulation models in general is given by Naylor (1971) and Schrieber (1970) among others, and in the field of financial planning by Gershefski (1968) and others. Other interesting examples of firm growth simulation models are Albach (1967) and Burrill & Quinto (1972). Simulation models are also used for evaluating the effect of various budget alternatives for a firm, as discussed by Gavatin (1975) for example.

A special type of dynamic simulation model appears in the field of Industrial Dynamics introduced by Forrester (1961). The firm's activities are represented by stocks and flows of for instance orders, goods, personnel and money.

By inserting delivery times for goods, credit time for customers etc it is possible to introduce non-trivial time lags and to simulate, for example, the resulting cash flow and profit for the firm given a certain flow of incoming orders to the firm. The effect over time of a sudden large incoming order can also be analyzed. The Industrial Dynamics approach is influenced by traditional non-optimal control and servo-mechanism theory. Its merits and drawbacks are similar to those discussed above with respect to general simulation models.

Somewhat related to the area of simulation and Industrial Dynamics is the framework of practical long-term strategic business planning. This area is fairly familiar to practical businessmen and we shall not go into detail here but rather refer to two analytical studies on the subject, one "early" and one recent, Ansoff (1965) and Eliasson (1976).

1.3 THE OBJECTIVES OF THE FIRM IN ECONOMIC THEORIES OF THE FIRM

Some well-known objectives of the firm will now be discussed - Sections 1.3.1 to 1.3.4 - after which some other related frameworks will be reviewed. These latter frameworks are mentioned in order to set our own frame of reference in perspective.

1.3.1 The Profit Objective

In economics, the firm has traditionally been regarded as being run by an entrepreneur who is both owner and manager and who maximizes profits. In markets with perfect competition this leads to a convenient general equilibrium analysis.[1)] This further strengthened the use of the profit objective in economics.

1.3.2 The Managerial Objective

The entrepreneurial concept was questioned first by Berle and Means (1932) and later in particular by for instance Frenckner (1953), Baumol (1959), Marris (1964) and O.E. Williamson (1964). It was now being argued instead - in particular by the first and latter three authors - that in large firms with thousands of variously small and uninterested shareholders, control of the firm lay with management rather than with the owners.

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1) Walras (1874) and later developments by Arrow & Debreu (1954) and many others.

The managers were assumed to be more interested in sales or growth than in profit. This appeared to be a rational assumption, looking at it from the managers' point of view. They were often seeking security, status, prestige and power, and it was assumed that these motives were more closely correlated to size than to profits. Also there were indications that managers' salaries were more often tied to sales than to profits, cf. for instance Roberts (1959) and McGuire, Chiu & Elbing (1962).[1)] Thus Baumol (1959) assumed that firms maximize sales, subject to a minimum profit constraint. Marris (1964) assumed that firms maximize a utility function of growth rate and of a valuation ratio, which expresses the relation between share value and the book value of net assets. O.E. Williamson (1964) assumed that firms maximize a utility function of profit and discretionary expenditures on staff and managerial emoluments.

Galbraith (1967) has argued that the best way to ensure the autonomy of the technostructure is to maintain a high rate of sales growth. The technostructure is Galbraith's term for the highly skilled technical and managerial personnel of the firm. In order to ensure survival, Galbraith argues for an acceptable level of dividends - and retained earnings - which sets a limit on the rate of sales growth.

For a further discussion of these and other managerial theories of the firm, reference is made to recent reviews by Wildsmith (1973) and Herendeen (1975).

1.3.3 The Satsficing and Behavioral Objective

Another line of development as to the objective of the firm has questioned whether firms do in fact try to maximize

1) Contrary evidence, viz. that executive compensation is more dependent on profit or equity share value than on sales, has been brought forward by e.g. Lewellen & Huntsman (1970) and Lewellen (1971).

anything at all, or whether they try to *satisfy* a number of aims. This has been discussed by Simon (1959) and Cyert & March (1963). The behavioral theory of the firm (Cyert & March, 1963) has provided a fairly adequate description of real decision making processes in some specific firms. But, even after 15 years, it has still not made any great impact on economics. Perhaps the main reason for this is the difficulty of generalizing beyond the specific firms studied. Another reason, although an irrational one, may be that the satisficing assumption is less tractable to mathematical analyses than the optimizing assumption. However, economic analyses that allow for non-optimizing behavior are becoming increasingly common. An example is Näslund & Sellstedt (1977), which will be discussed later in this chapter.

1.3.4 The Labor-Influenced Objective

Another recent trend, that goes beyond the managerial and behavioral modifications of the traditional economy theory of the firm, explicitly introduces labor objectives. As late as Machlup (1967), there was little or no mention of any concern about labor influence on the firm. It was perhaps felt that this area lay outside the scope of traditional market economics. Presently, however, there is a trend towards worker influence and worker management in a number of market economies. The best-known example is Yugoslavia, which has made an attempt to realize a labor-managed market economy. A number of Western industrialized countries are also heading in a similar direction.

In economics the extreme form of labor influence, viz. complete management by labor[1], has been studied by Ward (1958), Domar (1966) and Vanek (1970) for example. These and also other authors assume that the objective of the

1) We shall use labor also as a term for all employees since not only blue-collar employees but also white-collar employees participate in the trend towards greater employee influence.

pure labor-managed firm is maximization of labor income per unit of labor. In spite of criticism of this objective, for example by Horvat (1975) and Jan Vanek (1972, Chapter 6), many, especially Western, authors have adopted it in their economic analyses. When referring to a pure labor-managed firm in this study we mean the Ward-Domar-Vanek type of firm.

Recently Meade (1972) and Cars (1975) among others have studied models of the firm, with objectives lying somewhere between those of the profit-maximizing firm and the pure labor-managed firm.

We shall choose to study the pure labor-managed firm. This is partly because, on account of its extreme form of labor influence, it allows us to capture many of the tendencies of the other less extreme types of labor-influenced firms. Another reason is that the pure labor-managed firm occupies a central place in the economic literature. A third reason is that it is reasonably tractable to economic and mathematical analysis.

One difference, among others, between the pure labor-managed firm and the profit-maximizing firm is the following: in the profit-maximizing firm the capital-owners receive the residual revenue after each production factor has received its remuneration; in the labor-managed firm the opposite is true, i.e. labor receives the residual revenue after the other production factors, including capital, have received their predetermined remuneration.

Labor-managed firms have also been variously designated illyrian firms [1], co-operative [2] and state co-operative firms [3] in the literature.

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1) Ward (1958).
2) Meade (1972).
3) Cars (1975).

1.3.5 The Attenuation of Property Rights. Social Theories as a Background

The tendency, just discussed, to study the dissipation of the power of the owner of the firm and the consequent increase in power of other groups in the firm such as managers and labor, has encouraged an interest in developing a general theory of property rights and in using broader social theories as a starting point for economic analysis. Pejovich (1971) provides an example of the former, and Näslund & Sellstedt (1977) of the latter.

When the owner to some extent loses control over what he owns, we may say that his property rights have become attenuated, as Pejovich (1971), Alchian (1965) and others have discussed in more detail.

In order to analyze this process in the area of labor influence more comprehensively and on a macroeconomic scale, Näslund & Sellstedt (1977) extend the base of the analysis to various social theories and underlying paradigms.

> "One paradigm is based on the concept of class and is derived from Marx. A second paradigm is based on the concept of organization and has its roots in Max Weber and others. A third paradigm is based on the concept of the individual, representing an alternative stemming from social psychology. Whereas power is a central issue in the first two paradigms, the third paradigm has very little to say about power. What has just been said about paradigms is important. If the third paradigm is taken as a starting point, labor influence issues are by definition more or less excluded. Taking the paradigm based on organization as a starting point, ownership issues, for example, become less important. The problem is, however, that the choice of paradigm can very rarely be made by reference to facts or reality. Different paradigms interpret facts in different ways."

(Näslund & Sellstedt, 1977, pp. 15-16; translated here)

In the first paradigm the main decision making unit is the collective of individuals - cf. Marx - whereas in the second paradigm it is the organization or, possibly, so-called élites - cf. Max Weber - and in the third the individual.[1)]

There is an important difference between theories of class (the first paradigm) and theories of the élite (the second paradigm). In the former there is antagonism between classes, and one question to be considered is: how can the antagonism be abolished and the classless society obtained? In theories of the élite, on the other hand, the antagonism or competition between élites is seen as a guarantee of democracy.[2)]

An interesting example of the way different paradigms imply different interpretations of a given situation is provided by the analysis of American society to be found in Baran & Sweezy (1966) and Galbraith (1967). The former may be said to be based on a class paradigm and the latter on an organization paradigm. Galbraith (1967), in stressing that the power over large firms is being substantially transferred from the owners to the so-called technostructure, i.e. the specialists and management, implicitly assumes that a class-theoretical starting point is not applicable to the developed capitalistic societies of today. A possible conclusion in such a context is that if labor wants more influence over the firms, ownership is not the main thing to strive for.

This point of view has been criticized from a class-theoretical perspective, becuase it fails to distinguish between the sources of power - which in class theory terms means ownership - and the way power is actually expressed in firms. According to Marx' theory, which is based on the class paradigm and on the collective rather than the individual, it is the ownership of the means of production that has to be altered, if labor influence is to be changed in any significant way.[3)]

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1) Näslund & Sellstedt (1977, pp. 60-61).
2) Näslund & Sellstedt (1977, p. 63).
3) Näslund & Sellstedt (1977, pp. 60-62).

Much of the neo-classical economic theory is based on the paradigm of the individual. Consumers are assumed to maximize utility. Firms are identified in terms of individuals - the owners or, in more modern versions, the managers. In the economic theory of the pure labor-managed economy, the firm is identified in terms of the employees. The goal of the firm in these individual-oriented theories is identical with the goal of the individuals concerned, be they owners, managers or labor.

It is within this third paradigm, based on the individual, that we shall study the behavior of firms. In a sense it could be said that, as regards the formulation of our models, we are limiting ourselves, in this present study, to marginalist models in the neo-classical tradition, although we study goals other than the traditional neo-classical profit-maximizing goal.

1.3.6 Post-Keynesian Economic Theory

In order to analyze the effect on the economy of increased labor influence and labor ownership, Näslund & Sellstedt (1977, Chapters 7-9) have turned to post-Keynesian economic theory, which is based on both the class and the organization paradigms. The main reason for this is that income distribution and power are both essential issues in such a context.[1)]

On the micro level, traditional neo-classical economic theory and the modifications introduced in this present study are both more or less based on a concept of the individual that is compatible with the first stage in the evolution of orgainization theory, namely scientific management as developed by Taylor and others[2)]. The post-Keynesian micro theory used by Näslund & Sellstedt (1977, Chapters 7-9) makes less specific assumptions about man. This means that

1) Näslund & Sellstedt (1977, p. 141).
2) Näslund & Sellstedt (1977, p. 117).

many of the more recent organization theories based on human factors, human relations and technological and organizational factors can be embraced by the post-Keynesian micro theory. The effect is, however, that on certain microtheoretical questions less specific statements may be made.[1)]

The post-Keynesian framework used by Näslund & Sellstedt (1977, Chapter 7) at the level of the firm has some special characteristic elements:

First, there is the above-mentioned organization-theory basis which allows for the limited rationality of the decision maker and the fact that the firm is subject to uncertainty. Also this framework is designed so that future changes in the theory of organizational development, which are a possible effect of greater labor influence, can be taken into account. Thus the theory emphasizes the dynamic aspects of organizational behavior. The firm is not regarded as a holistic concept and two levels of the firm are distinguished: a higher managerial level where investment decisions are made, and a lower level where price and production decisions are made.

Second, as regards the marketing side, it is assumed that firms set their price as a mark-up on prime costs or direct manufacturing costs (cost-plus). This is considered to be more consistent with the real behavior of firms than the traditional marginalist behavior of letting marginal revenue be equal to marginal cost. It is assumed that an increase in demand, which especially in the case of consumer goods may be generated by advertising, involves a change in output rather than in price. It is further assumed that imperfect competition prevails in the product market, and that competition between firms arises in decisions on capital investment. And it is assumed that in the long run the profit rate, or rate of return on capital, tends to be equal for different firms.

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1) Näslund & Sellstedt (1977, p. 140).

Third, price and investment decisions are linked, in that a higher mark-up means greater internally generated funds and this in turn favours the financing of investments.

Fourth, in the short-run, marginal costs are assumed to be more or less constant until full capacity is reached, after which they increase rapidly.

Fifth, wages are not assumed necessarily to reflect the marginal productivity of labor. Rather, it is assumed that money wages are determined by negotiations between the parties on the labor market.

Sixth, the financing of investments is assumed to be highly dependent on the generation of internal funds.

An interesting implication of this post-Keynesian type of model of the firm is that, because of the firm's relatively extensive control over its own decisions about such things as prices and investment, the instruments of monetary and fiscal economic policy are probably less effective than in traditional marginalist economic theory.

The aim of this relatively general survey of a post-Keynesian type of theory of the firm is to illustrate an economic theory that provides an alternative to the marginalist neo-classically oriented theory used in this present study. A feature that is interesting in the present context is the emphasis on investment and the growth of the firm.

1.3.7 Organization Theory

We can now look briefly at one particular modern organization theory and its relation to our frame of reference, namely the approach of Simon (1957), March & Simon (1958) and others. In this approach the firm is seen as a coalition of individuals or groups who make contributions to the firm and receive corresponding inducements from it. Each stakeholder group in the firm tries through negotiations to get its "share of the cake", and the relations between the participants are regarded as symmetrical with management as an arbitrator. Among the stakeholder groups in the firm are the owners, managers, labor, consumers, suppliers and society as a whole. It is assumed that the decision makers behave with limited rationality.

If we compare this with the modern marginalist economic theory of the firm as used in this present study, we see two essential differences.

First, in our economic models the owner or manager or employee is a perfectly rational decision maker or "economic man", as opposed to the limited rational decision maker or "administrative man" of organization theory.

Second, in the modern marginalist economic theory of the firm either the owner or the manager or the employee occupies an active central place, while other participants such as customers, suppliers and society appear as passive constraints to which the main decision maker - owner, manager or labor - adjusts in order to achieve optimal behavior. In the organization theory of March & Simon (1958), however, the relation between the stakeholders in the firm is more symmetrical and interest is directed more towards survival than towards optimization.

The participant theory of the firm of March & Simon (1958) and others, in which management is seen as an arbitrator between the diverging interests of the other stakeholders, has been criticized by the adherents of the class paradigm. They normally do not accept the participant theory as a valid description of reality, claiming that the managers and capitalist owners have far more power and control over the firm than that of simple arbitrators between the conflicting interests of participant groups. Cf. for instance Näslund & Sellstedt (1977, p. 130).

1.4 UNCERTAINTY

A third important area of development in the economic theory of the firm, alongside those discussed in the last two sections, is uncertainty.[1)]

The consideration of uncertainty influences practically all aspects of the economic theory of the firm, including the objectives of the firm.

Especially when analyzing dynamic economic problems, it is essential to take the uncertainty of the future into consideration. However, dynamic stochastic optimization models involve quite advanced mathematics. This is one reason why we prefer to study deterministic dynamic models and rather include uncertainty only in certain instances in deterministic forms. Another reason is that the present study is explorative, and the inclusion of uncertainty constitutes a possible extension of the analyses.

The following, to mention only a few, are some recent surveys of related areas where uncertainty is discussed and studied: in the traditional theory of the firm, Horowitz (1970); in the finance and growth of the firm and in the study of capital asset markets, Lintner (1971), Fama & Miller (1972) and Ekern (1973); in stochastic economics and stochastic optimal control, Tintner & Sengupta (1972) and Bensoussan, Hurst & Näslund (1974); and as regards the firm in intertemporal general equilibrium analysis, Svensson (1976).

We now proceed to review the literature on the growth and dynamics of firms that maximize other objectives than profit alone. First, the managerial firm will be discussed in Section 1.5 and then the labor-managed firm will be discussed in Section 1.6.

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1) We use uncertainty interchangeably as a term for risk and/or uncertainty.

1.5 DYNAMIC OBJECTIVES AND MODELS OF A MANAGERIAL FIRM

1.5.1 General Dynamic Models

As was mentioned briefly in Section 1.3.2, different authors have proposed different dynamic objective functions for the managerial firm.

In the constant growth rate case, Baumol (1962) mentions maximization of the growth rate of sales.

Marris (1964) proposes maximization of a utility function of growth and a valuation ratio, representing market share value in relation to the book value of net assets.

J.H. Williamson (1966) analyzes the maximization of growth and maximization of discounted sales.

Solow (1971) analyzes a steady-state growth model of the firm, which comprises an analysis of the growth objectives of Marris (1964) and J.H. Williamson (1966).

Herendeen (1975, Chapter 7) suggests that the firm maximizes growth, subject to a valuation constraint.

To summarize: many managerial steady-state growth models of the firm assume that the firm maximizes growth or discounted sales. Often a constraint that requires a minimum value of the firm is assumed, in order to guard the firm against take-over.

As yet there has been only rather limited study of a fully dynamic optimization theory of the managerial firm.

Leland (1972) has extended Baumol's static model (1959) to provide a fully dynamic version as regards capital and labor, and to a lesser extent as regards debt.

Bensoussan, Hurst & Näslund (1974, pp. 94-95) analyze a model similar to Leland's (1972).

Both models assume that the firm maximizes discounted sales, subject to a minimum profit or dividend constraint. An unusual assumption which they both make is that perfect competition prevails in the product market, which is normally regarded as a situation in which managerial firms are least likely to exist. Lesourne (1977) has recently discussed the subject of managers' behavior and perfect competition.

1.5.2 Financial Dynamic Models

Often an important factor in the growth of the firm is financing and the interaction between capital investment and financing.

Many of the steady-state growth models mentioned above discuss and study the problem of financing firm growth. Another recent and lucid economic model of the firm, where the financial behavior of the firm lies at the heart, is Wood (1975), which also extends into macroeconomic theory. Important elements, apart from finance, in Wood's (1975) theory of the firm are growth, price mark-up and the distribution of income (on the macro level). In these respects his work shows similarities with post-Keynesian economic theory (cf. Section 1.3.6). However, whereas Wood's (1975) theory revolves around the relationship between profits and the availability of finance, this issue is tangential to neo- or post-Keynesian theory.[1)]

It lies in the nature of the financing - and investment - process of the firm that it is fully dynamic and thus is best analyzed in a fully dynamic framework. Therefore recently a number of fully dynamic financial optimal control models of the profit-maximizing firm have appeared in the literature. Some of these are Hochman et al. (1973), Lesourne (1973), Bensoussan, Hurst & Näslund (1974, Chapter 4), Ylä-Liedenpohja (1976) and Ludwig (1976). The last two references include surveys and discussions of work done until recently in this area.

However, there has been little study of any fully dynamic financial model of a managerial firm in the economic

1) Wood (1975, p. 14).

literature. This is notable because capital investment and financial considerations are probably particularly important in attaining a goal such as size (and/or growth), which is characteristic of the managerial firm. Practically the only example is the brief but compact model in Leland (1972, pp. 383-384). Thus it seems interesting to develop such a model, which is the purpose of Chapter 2.

1.6 DYNAMIC MODELS OF A LABOR-MANAGED FIRM

1.6.1 General Dynamic Models

There has not yet been many studies regarding a dynamic optimization theory of the labor-managed firm.

Vanek (1970, Section 8.4 and 14.6-14.7) has briefly discussed investment decisions of the labor-managed firm but does not study any dynamic optimization models of the firm.

In the constant growth rate case, Atkinson (1973) used a model-building approach similar to Solow (1971). He discussed the steady-state growth behavior of the labor-managed firm and compared this with the growth of a corresponding profit-maximizing firm. He also examined growth in managerial versions of both types of firms, as well as self-financing in a labor-managed firm.

Steinherr & Peer (1975) briefly commented on Atkinson (1973), and received a brief reply from Atkinson (1975).

Eriksson (1975, pp. 147-150), using a steady-state growth model similar to Atkinson's (1973), analyzed briefly but compactly the financial behavior of the labor-managed firm.

In the fully dynamic case, Furubotn (1971 and 1976) described a model of investment behavior in which the objective is to maximize a multiperiod utility index of labor income and the firm's working climate, subject to restrictions. This is done in a quasi-concave programming framework. Stephen & Smith (1975) commented on and extended Furubotn (1971).

Litt, Steinherr & Thisse (1975) studied internally financed investment decisions of a labor-managed firm in an optimal control theory framework. They also made comparisons in this respect with profit-maximizing firms. However, general results were difficult to obtain in this initial explorative analysis.

Apart from these studies, little else has been presented in connection with a dynamic optimization theory of the labor-managed firm.

1.6.2 Advertising [1] Dynamic Models

A main focus of interest in the Western literature on the economic theory of the labor-managed firm has been the comparison of the labor-managed firm with the profit-maximizing firm. One point that has been discussed is whether market expenditures such as advertising[1] are smaller in the labor-managed case than in the profit-maximizing case. If this is so, it could be argued that in this respect the labor-managed situation is socially more desirable.[2]

This has been studied in a static framework by Vanek (1970), Steinherr (1975)[3] and Ireland & Law (1977)[3], and has been commented on by Meade (1972, p. 413, fn. 2).

Advertising has an inherent growth-generating property and furthermore may have a carry-over effect on future sales.[4] The advertising process of the firm is thus actu-

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1) What we term advertising may be considered as a larger and more important class of expenditures including sales promotion, product improvement, product quality or, in general, most firm internal expenditures that shift the demand curve of the firm. Cf. Jacquemin (1972).

2) Cf. Vanek (1970, p. 123).

3) The articles by Steinherr (1975) and Ireland & Law (1977) came to our knowledge after our own analysis had been carried out. However, they do not proceed to a dynamic analysis as we do.

4) Vidale & Wolfe (1957), Telser (1962) and others provide empirical support for a carry-over effect of advertising on future sales.

ally dynamic and may therefore in such cases best be analyzed in a dynamic framework. Therefore recently a number of fully dynamic optimal advertising models of the profit-maximizing firm have appeared in the literature. Most of these, apart from the seminal article by Nerlove & Arrow (1962), date from the 1970's and have been surveyed in Sethi (1973), for instance Gould (1970), Schmalensee (1972) and Jacquemin (1972 and 1973).

However, no fully dynamic advertising model of a labor-managed firm has been presented in the economic literature, even though this may perhaps provide interesting results on for instance the issue of comparative advertising behavior mentioned above. Thus it seems interesting to develop such a model, which is the purpose of Chapter 3.

1.6.3 Dynamic Models of a Price Change

Another much discussed issue is the non-normal reaction of a pure labor-managed firm to a price change.[1)] This is actually a fully dynamic optimization problem, as the reaction is often said to be different in the short run and the long run.[1)] However a non-verbal dynamic model of this adjustment process has not been reported in the economic literature. Thus it seems interesting to develop such a model, which is the purpose of Chapter 4.

1.7 THE DISCOUNT RATE IN DYNAMIC MODELS

In dynamic, as compared to static economic models, a need arises to evaluate entities at various point in time. This is often accomplished by means of a weighting process and often by an exponential discounting process. The discount rate in the latter case is generally assumed to be an exogenous constant. In Chapter 5 we shall discuss some modifications of this assumption within the general framework of

1) Cf. Vanek (1970) and others.

models using an exponential discounting factor, and we shall analyze the effects of such modifications.

1.8 THE AIM OF THE STUDY

The background and scope of this study has been discussed above in this chapter and leads to the following aim of the work.

The general aim of this study is to develop dynamic economic models of firms maximizing other goals than profit alone - and also to an extent profit-maximizing firms - in a marginalist-type framework, in order to gain some insight into the dynamic economic behavior of such firms and how this behavior differs from that of a profit-maximizing firm.

Profit-maximizing in a dynamic sense is interpreted as the maximization of the discounted present value of a firm. The term profit-maximizing firm will be used in this sense in the following even when the context is one of dynamics.[1)] The term non-profit maximizing will be used in the sense of maximization of an objective that is not profit alone.

In Chapter 2 the dynamic optimal financial behavior of a managerial firm will be studied and in Chapter 3 the dynamic optimal advertising behavior of a pure labor-managed firm will be studied. The object of study in Chapter 4 is the dynamic optimal employment adjustment behavior of a competitive pure labor-managed firm to a change in output price.

The aim of Chapter 5 is to develop dynamic optimal economic models of profit-maximizing firms in a marginalist-type framework in which the discount rate is not constant but is either exogenously or endogenously variable. The purpose is to gain some insight into the effect of such non-constant discount rates compared to constant discount rates in certain cases.

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1) Cf. for instance Williamson, J.H. (1966)

Finally, in Chapter 6, the study is summarized and the main conclusions and possible extensions are discussed.

1.9 SUMMARY

In this chapter different types of theories of the firm have been discussed and the scope of this study was narrowed down to the discipline of economics and especially the economic theory of the firm. Three important areas of development in the economic theory of the firm were mentioned: the dynamics of the firm, the objectives of the firm and uncertainty. These areas were discussed and the scope of the inquiry further narrowed down to the first two areas, namely the dynamics and objectives of the firm. Finally the literature and issues in these areas were discussed and the scope was further narrowed down to the subjects that will be studied in Chapters 2-5.

2 A DYNAMIC FINANCIAL MODEL OF A MANAGERIAL FIRM

2.1 INTRODUCTION

The aim of this chapter is to develop a dynamic financial model of a managerial-type firm in a marginalist-type framework in order to gain some insight into the dynamic financial behavior of such a firm and how this behavior differs from that of a profit-maximizing firm.

In Section 2.1 a background and review of the literature is given as an introduction to Section 2.2 which presents the main model of the chapter. In Section 2.3 the optimal dynamic behavior of the firm is deduced and Section 2.4 provides a comparative dynamics analysis of the optimal solution. The main part of the chapter is summarized in Section 2.5, and Sections 2.6 and 2.7, which are appendices, provide a further analysis of a special case of the main model.

2.1.1 Finance in the Economic Theory of the Firm

There is a growing tendency to include financial considerations in the economic theory of the firm. One rather early example of this is Carlson (1939, Chapter IV). On a static model level, Vickers (1968) has been among the proponents of this view. Vickers (1968) points out that financial limitations will constrain both the size and the rate of growth of firms. He also mentions that the financial requirements of the factors of production will influence the firm's optimum factor mix.

Which are the most important limits to firm size and growth has been a subject much debated, and candidates have been organizational, managerial and financial factors as well as factors related to production and market. Penrose (1959), for example, speaks for managerial time and abilities as an essential limiting factor. Here, however, we follow such authors as Scitovsky (1951) and Vickers (1968) in studying the financial limits to size and growth. A recent survey of the limits to the size and growth of firms has been provided by Herendeen (1975, Chapter 6).

As regards steady-state firm growth models with financial content, we have already mentioned some of the more important (in Sections 1.2.1 and 1.5 in Chapter 1). Another lucid model and survey, from the field of corporate finance, is provided by Lerner & Carleton (1966). These authors include a macro-economic outlook (in Chapter 11), and on the question of the goal of the firm they limit themselves to share value maximization.

There is a very interesting and promising line of research in the financial behavior of the firm represented, among others, by Donaldson (1969) in the USA and Gandemo (1976) in Sweden. On the basis of empirical studies and induction, these authors claim that the main financial concern of the firm is to adapt its long-range financial planning to a rapidly changing environment and that the question of an optimal debt leverage, for example, does not occupy a central place in this planning.

This may be relevant to a study of specific management problems in the firm. However, in studying the firm from the point of view of predicting its behavior in society, we cannot conclude that studies of dynamic optimal behavior of the type presented in the present work and elsewhere are without value. This is a similar argument as that put forward in defense of the traditional economic theory of the firm.

2.1.2 Financial Models of the Firm of Optimal Control Type

Non-steady-state financial growth models of the firm of optimum control type have been developed fairly recently (cf. Section 1.5.2 in Chapter 1). The aim of these models is to study the financing and investment process of the firm without assuming *a priori* that the growth process is a steady-state-process. The common dynamic problem in these models is to determine, over a period of time, the capital of the firm and how it should be financed.

The firm is typically assumed to maximize the present value of dividends or utility of dividends, subject to a minimum constraint on dividends and subject to some type of limit on debt or debt cost. Some studies include the possibility of raising external equity, but most assume that retained earnings and debt are the only financial sources.

Without claiming to provide a complete list, we should like to mention, in addition to the references given above (in Chapter 1, Section 1.5.2), some other interesting studies in this area, namely: Magill (1970), Davis (1970), Davis & Elzinga (1971), Krouse (1972, 1973), Inselbag (1973), Bensoussan (1973), Krouse & Lee (1973), Senchak (1975), Elton et al. (1975), Lev & Pekelman (1975) and Näslund (1975).

Wong (1975) is a particularly interesting paper, in which managerial objectives such as growth and sales are shown to satisfy the characteristics of a profit-maximizing firm during its convergence path to a stationary state. Wong (1975) assumes internal financing only.

2.1.3 The Managerial Firm

The above-mentioned optimal control theoretic approaches to a dynamic theory of the firm assume that the firm maximizes its share value, expressed as the present value of future dividends. However, this dynamic objective, which is a counterpart to the classical static profit-maximizing objective, has been open to debate especially in the case of manager controlled and manager-owner controlled firms. A managerial theory of the firm has emerged, as we saw in Chapter 1 (Sections 1.3.2 and 1.5.1).

There have been a number of empirical studies lately, trying to determine whether manager-oriented firms behave differently from owner-oriented firms and, if so, how. Several authors have empirically confirmed a difference and to some extent in a way that agrees with the predictions of the managerial theories of the firm. Examples are: Monsen, Chiu & Cooley (1968), Radice (1971), Palmer (1972, 1973), McEachern (1975, 1976) and Round (1976). But other authors have found the opposite. Among these are Kamerschen (1968), Larner (1970), Hindley (1970), Elliot (1972), Sorensen (1974) and Kania & McKean (1976). Nevertheless Sorensen (1974) admits that his result does not mean that all firms are profit-maximizers. He also notes, along with others, that if the modern corporation has abandoned profit maximization as its primary goal, the reason for this may lie in the sheer size of the firm or the possession of market power rather than in a change in corporate control (Sorensen, 1974, p. 148). In the present study we shall refer to a firm with an element of sales-maximizing in its goal as a managerial-type firm. In this we follow the tradition and the argument of Baumol (1959) and others (cf. our Chapter 1, Sections 1.3.2 and 1.5.1).

Yet another author, Cooper (1977), questions the criteria that are used empirically to distinguish between manager-controlled and owner-controlled firms. If they are not valid, then of course the results of the empirical studies will to some degree also be incorrect. This in turn is questioned by Lawriwsky & Round (1977).

Thus, together with Näslund (1977, p. 77) and others, we may say that it is still an open question whether managerial firms behave differently from those that are owner-controlled (see e.g. Nichols, 1969, Chapter IX and Baran & Sweezy, 1966).

As this is an area which is currently the subject of scientific inquiry and in particular of empirical study, we shall only refer here, for a summary and review of the question, to some of the most recent, and mainly empirical, accounts mentioned above: McEachern (1975, 1976), Kania & McKean (1976), Round (1976), Cooper (1977) and Lawriwsky & Round (1977).

2.1.4 The Goal of a Managerial Firm

Granted that we cannot exclude the existence of managerially controlled firms, and that we cannot completely dismiss the notion that they perform differently from owner-controlled firms, the question remains of the possible goals of managerial firms. This brings us back to the issues underlying the emergence of various managerial economic theories of the firm at the end of the 1950's and, particularly, during the 1960's. We have discussed this earlier (in Chapter 1, Sections 1.3.2 and 1.5.1) and we can now add some supplementary views.

Alchian (1965, pp. 30-31) notes that Baumol's (1959) model of sales maximization subject to a minimum profit constraint, runs up against the objection that it implies that the firm will not make *any* sacrifice in sales, as long as the minimum profit constraint is met with, no matter how great the increment in profit may be. Alchian (1965, p. 37) favors instead a utility maximizing framework similar to O.E. Williamson's (1964), where the incentive to increase sales is treated not as a single criterion for maximization, but as a means for the manager to increase his earnings, which together with the firm's profit constitute two parts of the manager's utility function.

Peston (1959) preceded both O.E. Williamson (1964) and Alchian (1965) in examining a more general framework of the utility maximizing type, in order to put Baumol's (1959) sales maximization hypothesis in perspective. In Peston's (1959) framework, the firm's static objective is given by the following preference function

$$U(\Pi, R) \tag{2-1}$$

where

Π = short-run profit

R = gross revenue or sales

U = utility function.

Ames (1965) used a similar framework in which the firm maximizes

$$U(\Pi, q) \tag{2-2}$$

where q = output quantity.

Ames (1965) assumes a linear utility function,

$$U = (1-\alpha)\cdot\Pi + \alpha\cdot q \tag{2-3}$$

where α = constant, and $0 \leqslant \alpha \leqslant 1$.

Hawkins (1973, p. 68) mentions that "revenue maximization is certainly one of the most important - some would argue *the* most important - of the alternatives to profit maximization".

Smyth, Boyes & Peseau (1975, p. 79) conclude that the firm has a utility function that includes both sales and profits.

Up to now, in this section, we have mainly been discussing static goals of the managerial firm. We can now look at their dynamic counterparts. J.H. Williamson (1966) mentions,

> It seems quite plausible to suppose that managements which derive utility from the size of the undertaking they conrol will similarly discount future sales. After all, most managers must anticipate retirement or coronary thrombosis in the less than infinitely far distant future, so that it is reasonable to suppose that they will prefer an increase in sales in the present to an equal increase in the future. We therefore assume that management applies a discount rate to future sales, ...

Discounted or long-run sales R_1 over the time interval from time 0 to terminal time T may be expressed as

$$R_1 = \int_0^T e^{-rt} \cdot R(t)\ dt \qquad (2\text{-}4)$$

where r = discount rate and R(t) = gross revenue or sales at time t.

An interesting result in J.H. Williamson (1966) is that Baumol's (1959) static sales-maximization model can be derived from long-run sales maximization:

$$\text{Max } R_1 \qquad (2\text{-}5)$$

A natural extension of Peston's (1959) static objective of the firm (2-1) into a dynamic one is

$$\text{Max } U(M, R_1) \qquad (2\text{-}6)$$

where M = equity share value and R_1 = discounted sales, cf. (2-4).

The objective (2-6) implies that the firm derives utility from equity share value and long-run sales, and that there is a trade-off between the two. Equity share value is one of the most commonly used dynamic extensions of the static criterion of short-run profit. Other feasible dynamic goals of a managerial firm such as Leland's (1972) and Bensoussan, Hurst & Näslund's (1974, pp. 94-95) have been discussed earlier (in Chapter 1, Section 1.5.1 and partly Section 1.3.2).

2.2 A DYNAMIC FINANCIAL MODEL OF A MANAGERIAL FIRM

2.2.1 Specification of Objective and Assumptions

From our discussion in the previous section it appears that a reasonable dynamic objective for a less extreme form of the managerial firm than Leland's (1972) is

$$\text{Max } U(M, R_1) \qquad (2\text{-}6)$$

where

M = equity share value

R_1 = discounted sales

U = utility function.

Following many authors, we can express the firm's share value M as the present value of future dividends and terminal book value of equity, thus:

$$M = \int_0^T e^{-i_1 t} d(t)dt + e^{-i_1 T} \cdot y(T) \tag{2-7}$$

where

$d(t)$ = dividends

i_1 = discount rate for dividends

T = time period considered

$y(t)$ = equity, at book value

t = time.

Further, we may express discounted sales R_1 as

$$R_1 = \int_0^T e^{-i_2 t} R(t)\, dt \tag{2-8}$$

where R = current sales or gross revenue and i_2= discount rate for sales.

With the formulation (2-6) above for the objective of the firm, it is conceivable that the utility function U may be maximized over a time period T, although dividends d may be largely negative at some point in time. It is therefore reasonable, following many authors, to introduce a minimum constraint on dividends at each point in time. We shall return to this below.

The firm has two important financial decisions to make: (i) how large should investments and dividends be, and (ii) how should they be financed to achieve the firm's objective.

If the firm decides on large dividends in the short run then there will be less cash left for investment in the firm and long run profits will be small. In this situation the firm can of course borrow, but as leverage increases the interest rate will rise and interest costs will become prohibitive. Thus optimum investment and debt values may perhaps be found.

In order to pursue this question, we need to specify our model and assumptions further.

We can make the following assumptions:

A 1 Investments and dividends are financed either by retained earnings or by borrowing. Share capital is kept constant.

A 2 Operation profit before depreciation, interest and tax is concave in capital.

A 3 Dividends are constrained, so that they are at least as large as a certain percentage of the after-tax operating profit.

A 4 Operating profit is proportional to sales revenue.

A 5 The average interest rate on debt is a non-decreasing function of the firm's financial leverage, i.e. the debt's proportion of total capital.

A 6 Net working capital is zero.

A 7 The utility function in the objective (above) is linear.

A 8 The discount rates for dividends and sales are assumed to be identical.

A 9 The discount rate is assumed to be exogenously given and, for instance, independent of the corporate income tax rate and the general interest rate.

A 10 The firm has initially a given equity.

A 11 The firm does not experience technological development (in the operating profit function).

In the dynamic economic models of the firm of optimal control type mentioned in Section 2.1.2 above, similar assumptions to A 1 - A 11 are made.

Assumption A 1 is not especially restrictive because financing by new share issues is normally rather insignificant (cf. for instance Wood, 1975, p. 53).

Assumption A 2 represents a limit to size and growth of the firm because each additional unit of capital implies a lower marginal return on capital.

Assumption A 3 is commented on below in Section 2.2.5.

Assumption A 4 is special for our model because we study a managerial firm and require an expression for sales in the model. A 4 together with A 2 put certain restrictions on the underlying production function.

Assumption A 5 represents a limit to size and growth of the firm (as A 2 does) because as leverage increases, the interest cost will gradually become prohibitive.

Normally we assume that the interest rate tends to infinity as debt leverage tends to unity. Assumption A 5 has been used in a static context by for instance Jensen & Johansson (1969, Chapter 10), who obtain similar results in a static model as we do in the dynamic model in Appendix 2.1. Ludwig (1976, pp. 5-6) surveys eight restrictions on debt capital (limited debt, limited new debt, increasing interest rates etc) that have appeared in the optimal control literature of which assumption A 5 is one.

Assumption A 6 is a common assumption in economic theory and it will be adopted here too. If the firm is considered to be mainly a production unit operating in perfect certainty in the input and output market, A 6 is a reasonable assumption. Also A 6 may be used as a first approximation of a more realistic situation. The model in this chapter is, in essential aspects and cases, also valid for the case when net working capital is proportional to fixed capital.

Assumptions A 7 and A 8 are first approximations and may in a later stage be modified in order to obtain more general results.

Assumption A 9 is also a first approximation. Actually the discount rate may well be a function of for instance leverage, tax rates and the general interest rate level. This implies that the results obtained in this chapter are of a partial nature and may be subject to modification for more general assumptions. In Chapter 5 we shall analyze the effect of modifying the assumption of a given constant discount rate (A 9) in some specific cases.

Assumption A 10 implies a given state of the firm at the beginning of the time period. This assumption is commonly used in the literature (Section 2.1.2).

Assumption A 11 is also commonly used in the literature (Section 2.1.2). However, especially in dynamic models, technological development is of great interest but the dynamic models will then be of a non-autonomous nature and considerably more difficult for analysis.

Assumption A 7 means that

$$U = (1-\beta)\cdot M + \beta R_1 \tag{2-9}$$

where β = weighting factor, $0 \leqslant \beta < 1$.

The managerial coefficient β is assumed to be small in relation to unity in order to preserve profit as the main element in the objective of the managerial-type firm and also to avoid mathematical intricacies in the limit cases of β approaching unity.

The objective function of our model is thus

$$\text{Max } U = \int_0^T e^{-it} \left[(1-\beta)\cdot d(t) + \beta\cdot R(t)\right] dt + e^{-iT} (1-\beta)y(T) \tag{2-9a}$$

where according to assumption A 8,

$$i_1 = i_2 = i \tag{2-9b}$$

and where i is the common discount rate.

2.2.2 Equity Growth and Dividends

Given our assumption A 1 of no new share issues, the growth in equity as an accounting identity is equal to net profit less dividends, thus

$$\dot{y} = (1-\tau)\left[f(K) - i'\cdot w - \alpha K\right] - d \tag{2-10}$$

where

y = equity, at book value

τ = income tax rate, $0 \leqslant \tau < 1$

$f(K)$ = operating profit, before depreciation, interest and taxes

K = capital = debt + equity

i' = interest rate on debt

w = debt

α = rate of depreciation of capital, $0 \leq \alpha < 1$

d = dividends.

The dot over a variable, e.g. $\dot{y}$, symbolizes time derivation, i.e. $\dot{y} = \frac{dy}{dt}$. Operating profit f in (2-10) is a concave function of capital K according to assumption A 2:

$$f'(K) > 0 \qquad (2\text{-}10a)$$

$$f''(K) \leq 0 \qquad (2\text{-}10b)$$

We use capital K and debt w as variables in a financial theory of the firm, instead of output price and quantity as in traditional price theory.

Net profit in (2-10) is equal to operating profit f(K) minus interest cost on debt i'w, minus depreciation of capital αK, minus income tax $\tau \cdot (f(K) - i'w - \alpha K)$.

In order to simplify expression (2-10) we may define a new variable v thus:

$$v = \dot{y} + (1-\tau)(i'w + \alpha K) \qquad (2\text{-}11)$$

The economic meaning of the new variable v is retained earnings plus after-tax interest cost and after-tax depreciation. v is thus partly an investment variable. In the no-tax case, v is internally financed investment and interest, cf. (2-11).

If the expression for v in (2-11) is inserted into (2-10), we have for dividends the reduced expression

$$d = (1-\tau)f(K) - v \qquad (2\text{-}12)$$

2.2.3 State Equation

The interest rate i' may be expressed according to assumption A 5 as

$$i' = i'(m) \qquad \text{with } \frac{di'}{dm} > 0 \text{ and } \frac{d^2(i'm)}{dm^2} > 0 \qquad (2\text{-}13)$$

where m = debt leverage, or debt/capital.

The firm has initially a given equity according to assumption A 10,

$$y(0) = y_o \qquad (2\text{-}14)$$

where y_o = initial equity, a given constant.

We may now study the behavior of the firm in terms of the investment variable v and the leverage variable m and see how the equity of the firm y will develop in response to changes in them. To use the terms of control theory, the investment variable v and the leverage variable m are control variables and equity y is a state variable.

We may transform all previous expressions into the variables equity y, "investment" v and leverage m, by expressing capital K and debt w in equity y and leverage m.

By definition capital is equal to equity plus debt, and debt is equal to leverage times capital, i.e.

$$K = y + w \tag{2-15}$$

$$w = mK \tag{2-16}$$

We thus have

$$K = \frac{y}{1-m} \tag{2-17}$$

$$w = \frac{m}{1-m} \cdot y \tag{2-18}$$

By inserting expressions (2-17) and (2-18) for capital K and debt w into (2-11), we obtain a state equation for equity y expressed in the control and state variables

$$\dot{y} = v - (1-\tau) \cdot h(m) \cdot y \tag{2-19}$$

where h(m) is defined as

$$h(m) = \frac{\alpha + i'(m) \cdot m}{1-m} \tag{2-20}$$

In other words, h(m) is depreciation and interest cost per equity dollar.

2.2.4 Sales Revenue and the Objective Function

Sales revenue is assumed to be proportional to operating profit, according to assumption A 4, thus:

$$R(t) = \gamma \cdot f(K) \tag{2-21}$$

where γ = constant, $\gamma \geq 1$.

By inserting dividends (2-12) and sales revenue (2-21) into the objective function (2-9a), we obtain

$$U(m,v,y) = \int_{0}^{T} e^{-it} \left[\delta \cdot f(K) - \eta \cdot v\right] dt + e^{-iT} \eta \cdot y(T) \qquad (2\text{-}22)$$

where, by definition

$$K = \frac{y}{1-m} \qquad (2\text{-}17)=(2\text{-}23)$$

and where δ and η are defined such that

$$\delta = (1-\beta)(1-\tau) + \beta\gamma \qquad (2\text{-}24)$$

$$\eta = 1-\beta \qquad (2\text{-}25)$$

2.2.5 Constraints on Leverage and Dividends

We shall assume that the firm requires some equity, i.e.

$$m < 1 \qquad (2\text{-}26)$$

and that total debt is positive or zero, but not negative, i.e.

$$m \geqslant 0 \qquad (2\text{-}27)$$

A firm with a negative debt could be interpreted as an all-equity firm with not only productive assets represented by capital K, but also purely financial assets (-w) generating a rate of return of i'. This possibility is treated in a similar context by Hochman, E., Hochman, O. & Razin, A. (1973).

The dividend constraint in assumption A 3 mentioned above means that

$$d \geqslant (1-b) \cdot (1-\tau) f(K) \qquad (2\text{-}28)$$

where b = retention factor, $0 < b \leqslant 1$.

This is a dividend constraint based on gross profit rather than on net profit, which in some form is assumed by other authors (Section 2.1.2). Which type of dividend assumption is most valid is an empirical question. However, a dividend constraint based on gross profit is possible, considering that firms may continue paying dividends at least in the short run, even though they have a net loss. From (2-12) and (2-28), the dividend constraint is

$$b(1-\tau) f(K) - v \geqslant 0 \qquad (2\text{-}29)$$

In the tax case this dividend constraint (2-29) is relatively more manageable than the constraints used in the financial control-theory models mentioned earlier in this chapter, which is another reason for choosing it here.

2.2.6 Summary of the Model

We can thus state our model as

$$\max_{m,v} U(m,v,y) \qquad \text{(objective function)} \qquad (2\text{-}30)$$

subject to

$$\dot{y} = v - (1-\tau)\cdot h(m)\cdot y \qquad \text{(state equation)} \qquad (2\text{-}19)=(2\text{-}31)$$

$$b(1-\tau)f(\tfrac{y}{1-m}) - v \geqslant 0 \qquad \text{(dividend constraint)} \qquad (2\text{-}29)=(2\text{-}32)$$

$$m \geqslant 0 \qquad \text{(leverage constraint)} \qquad (2\text{-}27)=(2\text{-}33)$$

$$y(0) = y_o \qquad \text{(initial equity)} \qquad (2\text{-}14)=(2\text{-}34)$$

where $U(m,v,y)$ is specified in (2-22) and (2-23). Leverage m and the internal investment and interest cost variable v are control variables, while equity y is state variable.

2.3 DYNAMIC OPTIMUM

2.3.1 Necessary Conditions for an Optimum

The dynamic financial model of the firm (2-30)-(2-34) is well suited for analysis by Pontryagin's Maximum Principle. Cf. Pontryagin et al. (1962).

The dynamic Lagrangian L is

$$L\cdot e^{it} = \delta\cdot f(\tfrac{y}{1-m}) - \eta v + \lambda\left[v - (1-\tau)h(m)y\right] + \mu_1\left[b(1-\tau)f(\tfrac{y}{1-m}) - v\right] + \mu_2 m \qquad (2\text{-}35)$$

where $\lambda(t)$ is an adjoint variable representing the instant value of an imputed equity dollar and $\mu_1(t)$ and $\mu_2(t)$ are dynamic Lagrangian multipliers.

The necessary conditions for an optimum are that

$$L_v = 0 \tag{2-36}$$

$$L_m = 0 \tag{2-37}$$

$$L_y = -\frac{d}{dt}(\lambda e^{-it}) \tag{2-38}$$

$$L_\lambda = \dot{y} \tag{2-39}$$

$$\mu_1\left[b(1-\tau)f(\tfrac{y}{1-m}) - v\right] = 0 \tag{2-40}$$

$$\mu_2 \cdot m = 0 \tag{2-41}$$

$$\lambda(T) = \eta \tag{2-42}$$

$$y(0) = y_o \tag{2-43}$$

$$\mu_1 \geqslant 0 \tag{2-44}$$

$$\mu_2 \geqslant 0 \tag{2-45}$$

We thus have a system of six equations (2-36)-(2-41), in six variables v, m, y, λ, μ_1 and μ_2. If a solution exists, we have a sufficient number of equations to solve the problem.

For the two first-order differential equations (2-38) and (2-39), the initial and terminal values (2-42) and (2-43) specify a two-point boundary value problem.

The dynamic Lagrangian is linear in the control variable v and, since v is not bounded by fixed limits, we set the coefficient for v in L equal to zero as in (2-36). Also, we constrain the state variable y to be continuous, which rules out infinite v values, from (2-31).

2.3.2 Different Policies

From (2-40) and (2-41) we get four possible policies for the firm.

Policy A: $\mu_1 > 0$, $\mu_2 = 0$

Minimum dividends. Debt financing.

Policy B: $\mu_1 > 0$, $m = 0$

Minimum dividends. All equity firm.

Policy C: $\mu_1 = 0$, $\mu_2 = 0$

Free dividends. Debt financing.

Policy D: $\mu_1 = 0$, $m = 0$

Free dividends. All-equity firm.

We shall discuss these different policies later.

2.3.3 Analysis of Necessary Conditions

We shall now analyze the necessary conditions (2-36)-(2-39), which in turn are

$$-\eta + \lambda - \mu_1 = 0 \qquad (2\text{-}46)$$

$$\delta f'\cdot K_m - \lambda(1-\tau)\, h'\cdot y + \mu_1 b(1-\tau)f'\cdot K_m + \mu_2 = 0 \qquad (2\text{-}47)$$

$$\delta f'\cdot K_y - \lambda(1-\tau)\, h + \mu_1 b(1-\tau)f'\cdot K_y = i\lambda - \dot{\lambda} \qquad (2\text{-}48)$$

$$v - (1-\tau)h(m)\cdot y = \dot{y} \qquad (2\text{-}49)$$

or respectively

$$L_1\ (\lambda,\mu_1) = 0 \qquad (2\text{-}50)$$

$$L_2\ (m,y,\lambda,\mu_1,\mu_2) = 0 \qquad (2\text{-}51)$$

$$L_3\ (m,y,\lambda,\dot{\lambda},\mu_1) = 0 \qquad (2\text{-}52)$$

$$L_4\ (m,y,\dot{y},\dot{v}) = 0 \qquad (2\text{-}53)$$

where L_1, L_2, L_3 and L_4 are functions, and the alphabetical subscripts represent partial differentiation with respect to the subscripted variable.

By eliminating μ_1 and y from (2-50)-(2-52), we obtain

$$\dot{\lambda} = \left[i - (1-\tau)(i'm)_m\right]\cdot\lambda + \mu_2\,\frac{1-m}{y} \qquad (2\text{-}54)$$

By eliminating μ_1 from (2-50) and (2-51) we obtain

$$\lambda = \lambda(m,y,\mu_2) \qquad (2\text{-}55)$$

Now we may analyze the various policies more closely.

2.3.4 Policy A. Minimum Dividends. Debt Financing

The condition for this policy $\mu_1 > 0$ implies from (2-40) that

$$v = b(1-\tau)f\left(\frac{y}{1-m}\right)$$

Thus, from (2-49),

$$\dot{y} = (1-\tau)\left[bf\left(\frac{y}{1-m}\right) - h(m)\cdot y\right] \tag{2-56}$$

With $\mu_2 = 0$ for this policy, and using (2-54), we get

$$\dot{\lambda} = \left[i - (1-\tau)(i'm)_m\right]\cdot\lambda \tag{2-57}$$

2.3.4.1 Economic Interpretation of the Adjoint Variable

From (2-57) the adjoint variable λ is an exponential function of time with a non-linear exponent. Also the terminal value of λ is η from (2-42). We thus have

$$\lambda(t) > 0 \tag{2-58}$$

which means that the instant value of an imputed dollar of equity is positive.

However, (2-44) and (2-46) imply further for an optimum, that

$$\lambda(t) \geqslant \eta \tag{2-59}$$

In other words in the profit maximizing case of $\beta = 0$ and, thus, $\eta = 1$, the instant value of an imputed dollar of equity is at least one dollar, which seems reasonable.

In the managerial case $\beta > 0$, the instant value of an imputed dollar of equity need only be larger than something less than unity (η) from (2-59), because other dollars, for example of debt, are also of value to the managerial firm by increasing the size component of the firm's objective function.

The condition (2-59) must be checked for a specific optimum policy A.

The necessary condition (2-57) implies

$$(1-\tau)(i'm)_m = -\frac{\dot{\lambda}_1}{\lambda_1} \tag{2-59a}$$

where

$$\lambda_1 = \lambda\cdot e^{-it} \tag{2-59b}$$

and λ_1 is the value at time $t = 0$ of an additional dollar of equity imputed at time t.

In other words, (2-59a) implies that at optimum the marginal after-tax weighted cost of debt capital with respect to leverage is equal to the relative depreciation of the value of an imputed dollar of equity.

2.3.4.2 Derivation of Necessary Condition

From (2-55), we have

$$\lambda = \lambda(m,y) \tag{2-60}$$

We may eliminate λ by differentiating (2-60) with respect to time and inserting this into (2-57) and using (2-56). We thus obtain

$$\dot{m} = \dot{m}(m,y) \tag{2-61}$$

or

$$\dot{m} = \frac{\left[(1-m)^2h'-bf'\right]\cdot\left[(1-\tau)(i'm)_m - i\right]-(1-\tau)(bf-hy)\,\frac{(-f'')}{f'}\cdot(1-m)h'}{(1-m)(i'm)_{mm} + y\,\frac{(-f'')}{f'}\,h'} \tag{2-62}$$

Thus $\dot{y}$ and $\dot{m}$ are, from (2-56) and (2-62) respectively, expressed only in m and y, so that we may draw a phase diagram in the m-y plane.

It is interesting to note that for the special case treated in Appendix 2.1, i.e. for $f(K) = \mu K$, $f'(K) = \mu$, $f''(K) = 0$ and $b = 1$, (2-56) reduces to (2-107) and (2-62) reduces to (2-114). The second terms in the numerator and denominator respectively of (2-62) are non-vanishing in the general case, on account of the concavity of the operating profit function, i.e. because $f''(K) \neq 0$. Thus the optimum with increasing leverage $\dot{m} > 0$ in the special case in Appendix 2.1 (2-130), may very well be reversed to a decreasing leverage $\dot{m} < 0$ in the more general case treated here. Also intuitively, we may expect that a linear operating profit f(K) may place a premium on leverage growth, whereas a concave operating profit f(K) will probably discourage a growing leverage because of diminishing returns for operating profit f(K).

2.3.4.3 Phase Diagram for Policy A

By analyzing the first-order differential equations for $\dot{y}$ and $\dot{m}$, (2-56) and (2-62), we obtain the phase diagram in the m-y plane in Figure 2:1. By linearly expanding (2-56) and (2-62) around the stationary point (m_A^+, y_A^+) where $\dot{y} = 0$ and $\dot{m} = 0$, we find that the eigen-values of the system are real and of opposite signs, so that (m_A^+, y_A^+) is a saddle point. We thus have the trajectory movement around (m_A^+, y_A^+) as depicted in Figure 2:1[1]. The thick curve, ending at (m_C^+, y_C^+), in the left part of Figure 2:1 may be neglected for the moment.

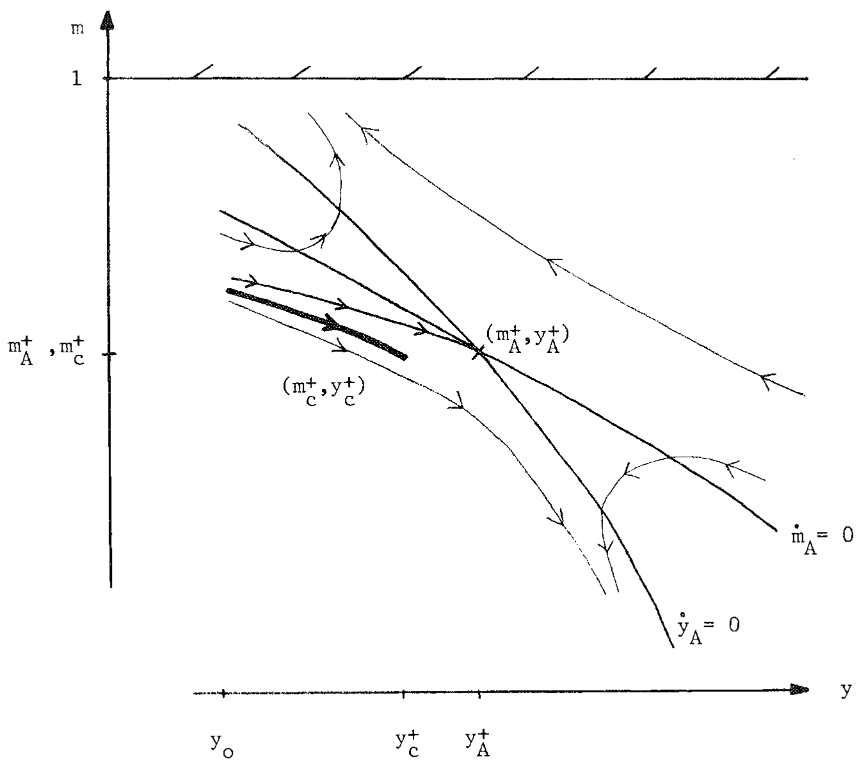

Figure 2:1. Phase Diagram for m and y. m = Debt Leverage. y = Equity. (m_A^+, y_A^+) = Intersection of the Curves $\dot{m}_A = 0$ and $\dot{y}_A = 0$. (m_C^+, y_C^+) = Stationary Equilibrium (Policy C). y_o = Initial Equity.

1) Cf. Appendix 2.3.

2.3.5 Policy B. Minimum Dividends. All-Equity Firm

Given the conditions for this policy $\mu_1 > 0$ and

$$m = 0 \tag{2-63}$$

and using (2-40), we get

$$v = b(1-\tau)f(y) \tag{2-64}$$

and thus from (2-49) and (2-63), we get

$$\dot{y} = (1-\tau)\left[bf(y) - \alpha y\right] \tag{2-65}$$

From (2-65), we have

$$y = y(t) \tag{2-66}$$

and it is thus, in principle, possible explicitly to solve this policy.

2.3.6 Policy C. Free Dividends. Debt Financing

Given the conditions for this policy $\mu_1 = 0$ and $\mu_2 = 0$, and using (2-46) and (2-54), we get

$$\lambda = \eta \tag{2-67}$$

$$i = (1-\tau)(i'm)_m \tag{2-68}$$

Thus, from (2-68), we find that

$$m = m_c^+ = \text{constant} \tag{2-69}$$

From (2-67), (2-69) and either of (2-47) or (2-48) we find that

$$y = y_c^+ = \text{constant} \tag{2-70}$$

and, thus, from (2-69), (2-70) and (2-49) that

$$v = v_c^+ = \text{constant} \tag{2-71}$$

We thus have a stationary equilibrium for the firm, which according to (2-67) coincides with the terminal condition for the adjoint variable λ (2-42).

We must check that the constraints (2-32) and (2-33) are satisfied, and we start with (2-33). We set

$$i'(0) = i'_o = \text{constant} \tag{2-72}$$

and, following assumption A 5, have

$$i'_m \equiv \frac{di'}{dm} > 0 \tag{2-73}$$

where i'_o = interest rate for the all-equity case.

Thus for the constraint (2-33) to be satisfied we find from (2-68) that

$$i \geqslant (1-\tau)i_o' \tag{2-74}$$

i.e. the all-equity after-tax interest rate must be lower than the discount rate. This is rather intuitive, because debt is unlikely to occur in a stationary equilibrium when the all-equity after-tax interest rate is higher than the discount rate.

The constraint (2-32) is fulfilled for

$$b(1-\tau)f(K_c^+) - v_c^+ \geqslant 0 \tag{2-75}$$

From (2-49) and (2-70) we have

$$v_c^+ = (1-\tau)\cdot h(m_c^+)\cdot y_c^+ \tag{2-76}$$

which, inserted into (2-75), gives us

$$(1-\tau)\left[bf(K_c^+) - h(m_c^+)y_c^+\right] \geqslant 0$$

or

$$b \geqslant \frac{h(m_c^+)y_c^+}{f(K_c^+)} \equiv b_c \tag{2-77}$$

where the constant b_c is defined in (2-77). From (2-77) we see that the retention factor b must be sufficiently large for an interior solution such as Policy C to be feasible.

From the definitions of (m_A^+, y_A^+) in (2-62) and (2-56) and of (m_c^+, y_c^+) in (2-68) and the constraint (2-77), we may deduce

$$m_A^+ = m_c^+ \tag{2-78}$$

$$y_A^+ \geqslant y_c^+ \tag{2-79}$$

Thus we have deduced the position of the stationary equilibrium (m_c^+, y_c^+) in the m-y phase diagram in Figure 2:1.

2.3.7 Policy D. Free Dividends. All-Equity Firm

Given the conditions for this policy $\mu_1 = 0$ and

$$m = 0 \tag{2-80}$$

and using (2-46), we obtain

$$\lambda = \eta \tag{2-81}$$

and also using (2-48), we obtain

$$\frac{\delta}{\eta} f'(y) = i + \alpha(1-\tau) \tag{2-82}$$

From (2-82) we find that

$$y = y_D^+ = \text{constant} \tag{2-83}$$

and thus from (2-49) that

$$v_1 = v_D^+ = \text{constant} \tag{2-84}$$

We thus have a stationary equilibrium for the firm, which according to (2-81) coincides with the terminal condition for the adjoint variable λ (2-42).

However, we have to check that the necessary condition $\mu_2 \geqslant 0$ (2-45) is satisfied. (2-45), (2-47), (2-48), (2-80) and (2-81) imply

$$i \leqslant (1-\tau)i'_o \tag{2-86}$$

i.e. the all-equity after-tax interest rate must be higher than the discount rate. This is rather intuitive, because otherwise a debt situation is likely to occur in a stationary equilibrium.

2.3.8 The Optimal Path in the $i \geqslant (1-\tau)i'_o$ Case

We shall study the $y_o \leqslant y_C^+$ and $i \geqslant (1-\tau)i'_o$ case. The other cases can be studied in the same way.

The case $y_o \leqslant y_C^+$ means that we study an expansion path of equity of the firm rather than a contraction path. The case $i \geqslant (1-\tau)i'_o$ means that the discount rate is larger than the after-tax all-equity interest rate. This appears to be a plausible case since $(1-\tau)i'_o$ is the *minimum* after-tax interest rate for debt and it seems natural that the owners and managers of the firm employ a higher discount rate than this minimum.

In the $i \geqslant (1-\tau)i'_o$ case we know from the above analysis of the different policies that Policy C satisfies all the necessary conditions while Policy D does not.

Also Policy C, which is the point (m_C^+, y_C^+) in the m-y phase diagram in Figure 2:1, satisfies the transversality conditon (5-42). We would therefore expect either of the Policies A and B to end at the point (m_C^+, y_C^+). However, Policy B is a no-debt policy $m = 0$ and its y-m trajectory is therefore confined to the y-axis in Figure 2:1. Since our problem-formulation implies continuous m and y, Policy B is ruled out in this case. Thus *the optimal path is Policy A followed by Policy C in the* $i \geqslant (1-\tau)i'_0$ *case*. A requirement for this is that the time period T is sufficiently long, so that the firm will have sufficient time to expand to the point (m_C^+, y_C^+).

Of all possible trajectories in the y-m phase diagram in Figure 2:1, the optimal one is thus the most heavily marked one ending at point (m_C^+, y_C^+). We can see from Figure 2:1 that leverage decreases over time

$$\dot{m} \leqslant 0 \qquad (2\text{-}87)$$

as opposed to the opposite result obtained in the special case in Appendix 2.1, cf. (2-130).

From (2-57), (2-58), (2-68), (2-69), (2-73) and (2-87) we have

$$\dot{\lambda} < 0 \qquad (2\text{-}88)$$

which together with (2-42) means that

$$\lambda(t) \geqslant \eta \qquad (2\text{-}89)$$

and thus that the necessary condition (2-59) for Policy A is fulfilled in the optimal enchainment of policies A + C proposed here.

The optimal behavior of the firm is thus initially to choose a high leverage in order to exploit the high return on capital soonest possible, and then gradually to consolidate the firm as it grows.

We also see from Figure 2:1, as expected, that equity increases over time, i.e.

$$\dot{y} \geqslant 0 \qquad (2\text{-}90)$$

and from (2-23), (2-56) and (2-62) we can deduce that

$$\dot{K} \geqslant 0 \qquad (2\text{-}91)$$

The size of the firm thus grows in spite of decreasing leverage.

These results were obtained from the *necessary* conditions (2-36)-(2-45). From the economic interpretation of the model it appears likely that these conditions are also *sufficient* for a maximum, despite the non-convexity of the dividend constraint (2-32). We shall not here go further towards proving that the necessary conditions obtained are also sufficient, as this would be mathematically rather tedious.

It is interesting to mention here that suggestive and heuristic results of the type just mentioned regarding sufficiency - rather than results deduced by a strict mathematical proof - are rather common in the economic optimal control literature according to Sydsaeter (1978). We do not propose this as a defense of our procedure but only note that our study is similar to other published studies in this respect.

It should be mentioned here, in this respect, that the linear expansion around the stationary point in Section 2.3.4.3 is valid only in a small neighborhood around this point. The slopes of the curve $\dot{m}_A = 0$ and to an extent the curve $\dot{y}_A = 0$ in the phase diagram Figure 2:1 are subject to a similar limitation. However the results may nevertheless be valid also in an extended neighborhood around the stationary point (cf. for instance Magill, 1970, pp. 39-40).

2.4 COMPARATIVE STATICS AND COMPARATIVE DYNAMICS IN THE $i \geqslant (1-\tau)i'_o$ CASE

We can now look at some comparative static and comparative dynamic effects in our model in the $i \geqslant (1-\tau)i'_o$ case.

In this section we shall study the effects of increasing the managerial coefficient β in the objective function, while in the special case in Appendices 2.1 and 2.2 we shall examine the effect of increasing the general interest rate level i'_o and the corporate income tax rate τ.

The comparative dynamics analysis of an increase in the managerial coefficient β is of special interest here because it constitutes a comparative study of a managerial-type firm compared to a profit-maximizing firm (in the case of $\beta = 0$).

From the conditions determining the points (m_A^+, y_A^+) and (m_c^+, y_c^+) mentioned above, we can deduce

$$\frac{dm_A^+}{d\beta} = 0 \tag{2-92}$$

$$\frac{dy_A^+}{d\beta} = 0 \tag{2-93}$$

$$\frac{dK_A^+}{d\beta} = 0 \tag{2-94}$$

$$\frac{dd_A^+}{d\beta} = 0 \tag{2-95}$$

and

$$\frac{dm_c^+}{d\beta} = 0 \tag{2-96}$$

$$\frac{dy_c^+}{d\beta} > 0 \tag{2-97}$$

$$\frac{dK_c^+}{d\beta} > 0 \tag{2-98}$$

$$\frac{dd_c^+}{d\beta} = \begin{cases} \geqslant 0 \text{ for } \beta \leqslant \beta_o \\ \leqslant 0 \text{ for } \beta \geqslant \beta_o \end{cases} \tag{2-99}$$

where β_o is a coefficient dependent on i, i', α, τ and γ.

The reason why the managerial coefficient β does not affect the stationary equilibrium leverage m_c^+ (2-96), is that the objective function (2-9a) is linear in dividends and sales. The *cost* side of the objective function, which determines the stationary equilibrium leverage, cf. (2-68), is unaffected by a linear addition via β of *sales* to the objective function. We may thus expect adjustments to the y-m phase diagram as in Figure 2:2.

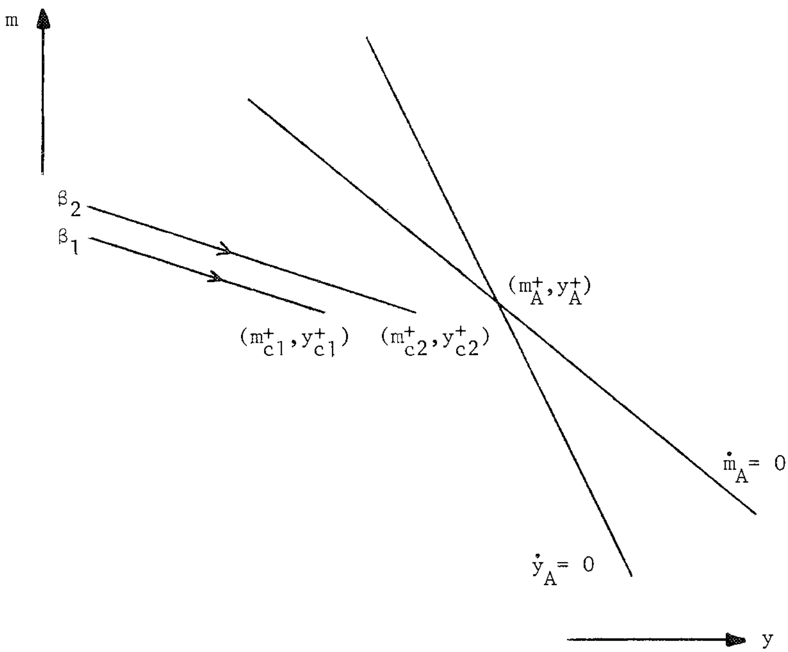

Figure 2:2. Comparative Dynamics Effect of a Shift in the Managerial Coefficient β from β_1 to β_2, where $\beta_2 > \beta_1$. m = Debt Leverage. y = Equity. (m^+_A, y^+_A) = Intersection of the Curves $\dot{m}_A = 0$ and $\dot{y}_A = 0$.
(m^+_{c1}, y^+_{c1}) and (m^+_{c2}, y^+_{c2}) are the Stationary Equilibriums (Policy C) for β_1 and β_2 respectively.

It is interesting to note that optimal final capital increases as the firm takes a step towards "managerialness" (2-98). This result is expected.

By studying the mathematical expressions for the slopes of the β_1 and β_2 paths in Figure 2:2, as in Treadway (1969), we can deduce that β_1 and β_2 trajectories may have a common tangency point. We cannot thus be quite sure that the relation of the β_1 and β_2 trajectories is as in Figure 2:2. But we would expect heuristically from (2-97) and this discussion that

$$\frac{dm^o}{d\beta} > 0 \qquad (2\text{-}100)$$

where the superscript o denotes the instantaneous equilibrium of the firm[1]. If (2-100) is the case, which should be checked more closely, we obtain the important result that the comparative dynamics effect of an increased managerial influence on the firm (2-100) is different from the comparative statics effect (2-96). As firms are perhaps more often than not out of stationary equilibrium, we would therefore expect comparative-dynamics results such as (2-100) to be just as important empirically as comparative-static results such as (2-96).

The comparative dynamics result (2-100) can be checked by using the methods described in Oniki (1973). However, we shall not pursue this matter here, as the mathematics becomes rather tedious.

2.5 SUMMARY AND CONCLUSIONS

The aim of this chapter has been to develop a dynamic financial model of a managerial-type firm in a marginalist-type framework, in order to gain some insight into the dynamic financial behavior of such a firm and into the way its behavior differs in this respect from that of a profit-maximizing firm.

The purpose is also to gain some insight into the adjustment of the firm to a final state rather than into the continuous growth of the firm at a steady rate. In Appendix 2.1 the aim is to study a more conventional continuous growth case but of the non-steady-state growth type.

A feature that distinguishes the model in this chapter from most other dynamic financial models of the firm is that it analyzes a managerial type of firm. The goal of the managerial-type firm studied here is a linear function of equity share value and long-term sales. The long-term sales component in the objective function is assumed relatively small.

1) Cf. Appendix 2.3.

As we are mainly interested in directions of change and relative comparisons, this latter assumption is not all too restrictive. Also the model differs from other similar models in that it includes the corporate income tax and - in Appendices 2.1 and 2.2 - a study of the comparative dynamics of a change in the tax rate (as well as in the general interest rate). Appendix 2.2 is interesting because it provides a comparative dynamics analysis for the *whole* time period according to a general method by Oniki (1973).

The factors that limit the growth of the firm in the model studied are an increasing interest cost and diminishing operating returns to capital. Other models[1), instead, assume for instance an upper limit on debt instead of an increasing interest cost.

The model is studied in an optimal control theoretic framework and utilizes the Pontryagin Maximum Principle. The Lagrangian-type dynamic multiplier for the dynamic equity transition equation is discussed from an economic point of view. The multiplier is given an economic interpretation as the imputed value at optimum of an equity dollar. As in mathematical programming - and its corresponding shadow values - this economic interpretation is an interesting reason for using the method of optimal control theory and the Pontryagin Maximum Principle in economics.

The model studied predicts that the relation between the discount rate for dividends - and, in our case, also sales - and the after-tax all-equity interest rate, is of importance for type of phases or policies of development of the firm. This result is supported by results of other authors[1)]. Such a result is also expected as the model deals with optimal financing, and the relation between marginal interest rates is one rather natural determinant of financial policies in a marginalist-type model of the firm (Carlson,

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1) Bensoussan, Hurst & Näslund (1974, Chapter 4) as an example.

1939, pp. 124-125, and others). Should government fiscal or monetary policy measures change the relation between the interest rates mentioned, the model predicts that the firm may switch its whole policy to a completely different path of development. A dynamic model of the type studied here is thus of interest in analyzing shifts in the structural dynamics of the firm due to economic policy measures [1)]. This is perhaps one of the most interesting reasons for studying economic models of the firm in an optimal control theoretic framework.

In the case of a small enough initial equity and in the case of the discount rate being greater than the after-tax all-equity interest rate, which would appear to be a plausible case, the managerial-marginalist-type firm of the studied model exhibits growth in both total capital and equity. The model studied also implies that the firm initially chooses a high leverage in order to exploit the high return on capital soonest possible, and then immediately proceeds to gradually decrease leverage and consolidate the firm as it grows. This result is supported by the work of Jacquemin (1972, p. 132) and Jacquemin & Thisse (1972, pp. 67 and 83). They state that in dynamic economic models of this type, the optimal behavior of the firm is to use its control policy most heavily in the initial periods and continually decrease the effort as the final state is approached.

The decreasing leverage implies that the binding minimum dividends, being dependent on capital - in the studied model - grow more slowly than equity. Thus the model normally implies a dividend lag behind earnings, which is equivalent to a fairly rapid internal increase in equity and most likely a fall in leverage. This phenomenon of decreasing leverage due to a dividend lag behind earnings, can be observed,

1) Bensoussan, Hurst & Näslund (1974, Chapter 4) as an example.

according to Miller (1977, p. 264), during a period of expansion in the economy. The dynamic model studied here, and in particular the dividend constraint, thus furnishes a possible explanation for this.

As mentioned above the optimum dynamic policy implies an increasing equity. Equity increases, and earnings are retained, as long as the imputed relative value of a dollar of equity ($-\dot{\lambda}_1/\lambda_1$) is greater than the loss in shareholder utility of retaining the marginal dollar (i)[1]. Gradually the imputed relative value of a dollar of equity will decline and no longer be greater than the loss in shareholder utility of retaining the marginal dollar. At such a point in time, as expected, a *stationary* equilibrium is reached - which we have called Policy C[2].

In order to compare the managerial-type firm, with a profit-maximizing firm, a comparative-statics and a comparative-dynamics analysis of shifts in the managerial coefficient (in the utility function) has been performed. It was found that an increase in the "managerialness" of the firm most likely would call for an increase in leverage during the adjustment of the firm towards its final state (comparative dynamics), whereas the final state itself is unaffected (comparative statics), given the assumption of a linear objective function. This comparative dynamics result is supported by economic reasoning because increased "managerialness" places a premium on sales and thereby leverage rather than on profit and interest costs.

It is especially interesting to note that the comparative-dynamics effect of an increase in managerial influence on the firm is probably different from the comparative-statics effect. As firms are perhaps more often than not *out* of stationary equilibrium, one may therefore expect comparative-dynamics results deduced from dynamic economic models of the type studied here, to be just as important as a basis

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1) This follows from (2-59a), (2-68) and (2-87).
2) Cf. (2-59a) and (2-68).

for empirical studies, as comparative-statics results deduced from either static models or steady-state growth models.

The dynamic financial model in this chapter may be used for other comparative dynamics predictions concerning e.g. tax shifts and general interest rate shifts. Such predictions may also be deduced for other variables than leverage. In the general model these predictions become, to an extent, indeterminate and also mathematically cumbersome. However, in the special case in Appendices 2.1 and 2.2 further specific predictions along these lines are deduced.

APPENDIX 2.1

2.6 A DYNAMIC FINANCIAL MODEL OF A MANAGERIAL FIRM. SPECIAL CASE

We shall now study a special case of the model in this chapter which can be solved analytically and therefore provides more specific results.

Let us modify assumptions A2 and A3 as follows:

A2'. Operating profit before depreciation, interest and tax is linear in capital.

A3'. Dividends are constrained to be non-negative.

In other words, from A2', the operating profit can be written:

$$f(K) = \mu K \tag{2-101}$$

where μ is a positive constant. This is an assumption used fairly often in financial and growth theories of the firm, for instance in Gordon (1962) and Lesourne (1973).

From A3', the dividend constraint is $d(t) \geq 0$, which is equivalent to setting

$$b = 1 \tag{2-102}$$

in our general model.

We shall study the same case as in the general model $y_o \leq y_c^+$ and $i \geq (1-\tau)i_o'$, which implied an optimal behavior of Policy A followed by Policy C.

The assumption A2' implies that the dynamic Lagrangian (2-35) is linear in the state variable y and that Policy C is indeterminate in equity y and the investment variable v. We shall therefore study the other case Policy A, the minimum dividends policy, closer in this Appendix.

2.6.1 Optimal Leverage, Equity and Equity Growth

2.6.1.1 Discussion

The static return on equity before tax r_e is equal to operating profit less depreciation and interest, in relation to equity. Thus

$$r_e = \frac{\mu-\alpha-i'm}{1-m} \tag{2-103}$$

It follows from (2-103) that

$$(1-m)\,\frac{dr_e}{dm} = r_e - \frac{d(i'm)}{dm} \tag{2-104}$$

In our special case, (2-60) becomes

$$[1-m]\left[r_e - \frac{d(i'm)}{dm}\right] = -\frac{\mu\ \beta\ \gamma}{\lambda\cdot(1-\tau)} \tag{2-105}$$

which is independent of equity y.

(2-58), (2-104) and (2-105) imply

$$\frac{dr_e}{dm} \leqslant 0\ . \tag{2-106}$$

From (2-20), (2-56) and (2-103) we find that

$$\dot{y} = (1-\tau)\cdot r_e y \tag{2-107}$$

that is

$$y(t) = y_o\cdot e^{(1-\tau)\cdot\int_o^t r_e(m(u))du} \tag{2-108}$$

i.e. the growth rate of equity is the after-tax return on equity, as expected when dividends are zero.

For the non-managerial case $\beta = 0$, and following (2-105), leverage m is a constant, which we shall denote $\hat{m}$.

For the managerial case $\beta > 0$, (2-106) gives us

$$m_{opt}(t) > \hat{m} \tag{2-109}$$

and (2-106), (2-108) and (2-109) give us

$$y_{opt}(t) < \hat{y}(t) \tag{2-110}$$

where $\hat{y}(t)$ is the optimum in the profit-maximizing steady-state growth case $\beta = 0$ and m_{opt} and y_{opt} denote the optimum in the managerial case $\beta > 0$.

Following (2-107), the growth rate of equity g can be written

$$g = \frac{\dot{y}}{y} = r_e(1-\tau) \tag{2-111}$$

For the non-managerial case $\beta = 0$, (2-104) and (2-105) give us

$$\frac{dr_e}{dm} = 0 \ . \tag{2-112}$$

As $r_e(m)$ is concave over a relevant interval around $m = \hat{m}$, the rate of return on equity r_e has a maximum at $m = \hat{m}$. The concavity of $r_e(m)$ over a relevant interval around $m = \hat{m}$, can be derived from (2-103) using (2-104) for simplification and making the plausible interest rate assumption

$$\frac{d^2(i'm)}{dm^2} > 0 \ . \tag{2-112a}$$

Thus, as the optimum m_{opt} in the managerial case $\beta > 0$ differs from the leverage $\hat{m}$ that maximizes the rate of return on equity r_e, we have from (2-106) and (2-109)

$$r_e(m_{opt}) < r_e(\hat{m}) \tag{2-112b}$$

and thus from (2-111)

$$g_{\beta>0} < g_{\beta=0} \tag{2-113}$$

i.e. the equity growth rate in the managerial case is lower than in the non-managerial case. This is expected because the no-dividend case studied here implies that the equity growth rate is equal to the after-tax rate of return on equity (2-111) which intuitively should have its maximum in the profit-maximizing case. Equivalently we may expect the managerial-type firm to sacrifice some return on equity in order to obtain greater size, i.e. equity *plus debt*.

2.6.1.2 Result

The main result of this section is that leverage is greater (2-109), equity smaller (2-110) and the equity growth rate lower (2-113) along the optimal time path in the managerial firm case $\beta > 0$ compared with the profit-maximizing firm case $\beta = 0$. The first two results are reasonably intuitive in the corresponding static case - which is not reported here - and are apparently also valid in the present dynamic growth case.

2.6.2 Leverage. Analytical Solution

We shall now attempt to find an analytical solution to our special case. The general expression (2-62) reduces to

$$\dot{m}(t) = \left[\frac{d(i'm)}{dm} - r_e(m)\right]\cdot\left[(1-\tau)\frac{d(i'm)}{dm} - i\right]\Big/\frac{d^2(i'm)}{dm^2} \tag{2-114}$$

Set the inverse of the right-hand side of (2-114) equal to $\phi(m(t))$. Thus

$$\phi(m(t)) = \frac{\dfrac{d^2(i'm)}{dm^2}}{\left[\dfrac{d(i'm)}{dm} - r_e(m)\right]\cdot\left[(1-\tau)\dfrac{d(i'm)}{dm} - i\right]} \tag{2-115}$$

(2-114) and (2-115) give us

$$\dot{m} = 1/\phi(m) \tag{2-116}$$

Thus

$$\int_t^T dt = \int_m^{m_T} \phi(u)du \tag{2-117}$$

where $m = m(t)$ and $m_T = m(T)$.

From (2-117) we get

$$t = T - \int_m^{m_T} \phi(u)du \tag{2-118}$$

Define $\Phi(u)$ from

$$\Phi'(u) = \phi(u) \tag{2-119}$$

Then (2-118) becomes

$$t = T - \Phi(m_T) + \Phi(m) \tag{2-120}$$

which is the explicit solution for the control variable m, although stated in inverse form, i.e. time as a function of m.

$m_T = m(T)$ is a constant which can be calculated from (2-105) and (2-42).

Assuming $i'(m)$ is a general rational function, i.e. a quotient of two polynomials, $\phi(m)$ in (2-115) will then also be a general rational function and by splitting $\phi(m)$ in partial fractions, $\Phi(m)$ may be explicitly calculated in terms of general rational functions, logarithm functions and arctg functions. This case covers a reasonably large class of feasible interest rate/leverage relationships. The optimal control variable m(t) can be analyzed a step further in such cases.

Assuming

$$\lim i'(m) = \infty \qquad \text{only if } m \to 1 \tag{2-121}$$

then

$$\lim_{m \neq 1} \phi(m) = \infty \qquad \text{iff}$$

$$\frac{d(i'm)}{dm} - r_e(m) = 0 \text{ and/or } (1-\tau)\frac{d(i'm)}{dm} - i = 0 \tag{2-122}$$

In other words the denominators in the partial fractions of $\phi(m)$ are readily found from the optimal condition in the $\beta = 0$ case and the minimum condition for the tax-adjusted cost of capital.

We shall now analyze the sign of $\dot{m}$ using (2-114). From (2-58) we have

$$\lambda(t) > 0 \tag{2-123}$$

Thus from (2-105)

$$\frac{d(i'm)}{dm} - r_e > 0 \tag{2-124}$$

In other words, from (2-104) and (2-124), the marginal rate of return on equity with respect to leverage is negative.

A plausible assumption as to the interest rate function is

$$\frac{di'}{dm} > 0 \tag{2-125}$$

$$\frac{d^2 i'}{dm^2} \geqslant 0 \tag{2-126}$$

which implies

$$\frac{d^2(i'm)}{dm^2} > 0 \qquad \text{(2-112a)=(2-127)}$$

which is equivalent to (2-112a).

Assuming the marginal tax-adjusted cost of capital is positive at optimum, we have

$$(1-\tau)\,\frac{d(i'm)}{dm} - i > 0 \tag{2-128}$$

This implies from (2-42) and (2-57) that

$$\lambda \geqslant \eta \tag{2-129}$$

which fulfills the necessary condition (2-59) and thus justifies (2-128).

In other words, from (2-114), (2-124), (2-127) and (2-128), we find that

$$\dot{m}(t) > 0 \tag{2-130}$$

i.e. there is increasing leverage.

In the profit maximizing case $\beta = 0$, (2-128) becomes

$$(1-\tau)\left[\frac{d(i'm)}{dm}\right]_{m=\hat{m}} > i \tag{2-131}$$

Following (2-105) for $\beta = 0$, (2-131) may be expressed as

$$(1-\tau)\cdot r_e(\hat{m}) > i \tag{2-132}$$

In other words, in the profit-maximizing case $\beta = 0$, a positive net economic return (2-132) implies that the minimum dividends policy (Policy A) studied here, is optimal as expected.

In the case of the managerial-type firm, the relation

$$(1-\tau)r_e(m_{opt}) > i \tag{2-133}$$

implies that (2-128) and thus the necessary condition (2-129) is satisfied, following (2-124).

In other words, a positive net economic return (2-133) implies that the minimum dividends policy (Policy A) studied here is optimal also in the case of a managerial-type firm. Actually, as long as the *maximum* net economic return is positive (2-132), the necessary condition (2-129) is satisfied in the case of a managerial-type firm. This follows from (2-131) being equivalent to (2-132) and from (2-109) and (2-112a).

Our finding as regards increasing leverage over time

(2-130) for a firm with $\beta > 0$, offers another explanation for this leverage phenomenon which is commonly observed in, for instance Sweden. Cf. for example Bertmar & Molin (1977). The explanation in our case is that the firm subscribes not only to a profit goal but also to a size goal. We do not suggest that this is a main reason, but it is interesting that our model, with all its simplified assumptions, comes to this conclusion.

A prerequisite for this result is that the discount rate i is not too high, in which case leverage is already big to start with, after which it decreases. This follows from (2-114), (2-124) and (2-127).

2.6.3 Accelerating Leverage

The sign of the change in increase in leverage $\ddot{m}$ may also be investigated. Differentiation of (2-114) with respect to time gives us

$$\ddot{m}/(1-\tau)\dot{m} = \left[\frac{d(i'm)}{dm} - r_e\right] + \left[\frac{d(i'm)}{dm} - \frac{i}{1-\tau}\right]\cdot\left[1 - \frac{dr_e}{dm}\Big/\frac{d^2(i'm)}{dm^2}\right] - \dot{m}\,\frac{\dfrac{d^3(i'm)}{dm^3}}{\dfrac{d^2(i'm)}{dm^2}}\cdot\frac{1}{1-\tau} \tag{2-134}$$

The first two terms are positive from (2-106), (2-124), (2-127) and (2-128), whereas following (2-127) and (2-130) and assuming

$$\frac{d^3(i'm)}{dm^3} > 0, \tag{2-135}$$

the third term is negative. However for small β-values the third term is in many cases of an order of magnitude smaller than the second term, because of the occurrence of two relatively small factors in the numerator of $\dot{m}$ in (2-114). Thus, in many cases, for small β

$$\ddot{m}(t) > 0 \tag{2-136}$$

and leverage m thus accelerates.

2.6.4 Optimal Equity. Analytical Solution

From (2-108) we find that optimal equity y can be written

$$y(t) = y_o \cdot e^{(1-\tau)\int_o^t r_e(m(u))du} \qquad (2\text{-}108)=(2\text{-}137)$$

In order to calculate the integral in (2-137), we observe from (2-116) that

$$dt = \phi(m)dm \qquad (2\text{-}138)$$

and thus, with a new integration variable ξ,

$$\int_t^T r_e(m(u))du = \int_m^{m_T} r_e(\xi)\phi(\xi)d\xi \qquad (2\text{-}139)$$

where $m_T = m(T)$ is a constant which can be calculated from (2-42) and (2-105).

In the special case where $i'(m)$ is a general rational function, $r_e(\xi)$ and $\phi(\xi)$ are also general rational functions and the integral in (2-139) may be calculated analytically as discussed in the optimal m case above.

2.6.5 Optimal Capital

We can now study the development of the firm's capital, which may also be conceived of as a measure of the size of the firm. From (2-23) we may deduce by time differentiation that

$$\frac{\dot{K}}{K} = \frac{\dot{y}}{y} + \frac{\dot{m}}{1-m} \qquad (2\text{-}140)$$

In the case studied above, from (2-107) with $r_e > 0$ and from (2-130), we obtain

$$\dot{K} > 0 \qquad (2\text{-}141)$$

In other words it is optimal for the firm to accumulate capital. From (2-109), (2-113), (2-130) and (2-140) it is not immediately evident whether the growth rate of capital is larger in the case of the managerial firm than in the case of the profit-maximizing firm.

However, we may take this analysis one step further, by observing from (2-103), (2-107), (2-114) and (2-140) that the growth rate of capital $\dot{K}/K$ is a function of leverage m, which we denote here G(m). Thus, from (2-140) we find that

$$G(m) = \frac{\dot{K}}{K} = (1-\tau)\cdot r_e(m) + \frac{L_1(m)}{1-m} \tag{2-142}$$

where we have defined

$$\dot{m} = L_1(m) \tag{2-143}$$

The functional form of $L_1(m)$ is given in (2-114).

As we know from (2-109), the optimum leverage in the case of the managerial firm is larger than the optimum leverage in the case of the profit-maximizing firm. Thus we may study the sign of $\frac{dG}{dm}$ at $m = \hat{m}$, and from that draw conclusions about the relation between the capital growth rates in the two types of firms. By differentiation of (2-142), we get

$$G'(m) = (1-\tau)\,\frac{dr_e}{dm} + \frac{L_1'(m)}{1-m} + \frac{L_1(m)}{(1-m)^2} \tag{2-144}$$

and thus

$$G'(\hat{m}) = \frac{L_1'(\hat{m})}{1-\hat{m}} \tag{2-145}$$

because

$$\left(\frac{dr_e}{dm}\right)_{m=\hat{m}} = 0 \tag{2-146}$$

from (2-112), and because

$$L_1(\hat{m}) = 0 \tag{2-147}$$

from (2-105), (2-114) and (2-143), for $\beta = 0$.

From (2-114) we may, after some calculation and using (2-146) and (2-147), determine $L'_1(\hat{m})$, so that from (2-145) we get

$$G'(\hat{m}) = \frac{1}{1-\hat{m}}\cdot\left[(1-\tau)\left(\frac{d(i'm)}{dm}\right)_{m=\hat{m}} - i\right] \tag{2-148}$$

From (2-131) and from (2-148), we obtain

$$G'(\hat{m}) > 0 \tag{2-149}$$

In other words for m_{opt} close to $\hat{m}$ and, thus, for small β, we may conclude that

$$G_{\beta>0} > G_{\beta=0} \tag{2-150}$$

i.e. the capital growth rate for the managerial firm is larger than the capital growth rate for the profit-maximizing firm.

It is interesting to note from (2-148) that this conclusion rests on the condition that the optimum (=maximum) growth rate of equity in the profit-maximizing case is larger than the equity discount rate, cf. (2-111), (2-131) and (2-132). In the infinite horizon case this would cause problems of convergence, but not in our case where a finite time horizon is used in the model.

From (2-23), (2-34) and (2-109) we may conclude that

$$K(0)_{\beta>0} > K(0)_{\beta=0} \tag{2-151}$$

in other words the initial capital of the firm in our special model is larger for the managerial firm than for the profit-maximizing firm. We shall investigate these matters more closely in a further specification of our special model in Section 2.6.8 of this Appendix.

2.6.6 Comparative Dynamics at End of Time Period

2.6.6.1 Tax Change

We now want to determine how a tax change will affect the leverage and equity growth rate of the firm at time T.

We thus differentiate (2-105) totally with respect to the tax rate τ for t=T.

$$\frac{dm(T)}{d\tau} = \frac{\mu\ \beta\ \gamma}{\eta(1-\tau)^2(1-m)\ \dfrac{d^2(i'm)}{dm^2}} \tag{2-152}$$

For

$$\frac{d^2(i'm)}{dm^2} > 0$$

or (112a), and for $\beta > 0$, we get from (2-152)

$$\frac{dm(T)}{d\tau} > 0 \qquad (2\text{-}153)$$

From (2-152) we find that if the tax rate on profit increases, the profit-maximizing firm ($\beta = 0$) will not increase its leverage. This is because our profit-maximizing firm maximizes the return on equity after tax with respect to leverage, which as expected gives the same result irrespective of changes in the multiplicative tax rate.

In the managerial case the negative effect of a tax increase on after-tax profits can be partly offset by the increase in sales resulting from an increase in leverage. Cf. (2-153).

The optimum equity growth rate g is obtained from (2-111):

$$g = \frac{\dot{y}}{y} = r_e(1-\tau) \qquad (2\text{-}111)=(2\text{-}154)$$

For sufficiently small β we may expect from (2-112b) and (2-132) that

$$(1-\tau)\cdot r_e(m_{opt}) > i \qquad (2\text{-}133)=(2\text{-}154a)$$

and thus especially

$$r_e(m_{opt}) > 0 \qquad (2\text{-}154b)$$

which is equivalent to a positive equity growth rate according to (2-154).

Following (2-154) we get

$$\frac{dg}{d\tau} = -\, r_e + (1-\tau)\,\frac{dr_e}{dm}\cdot\frac{dm}{d\tau} \qquad (2\text{-}155)$$

and for $t = T$, (2-106), (2-153) and (2-154b) give us

$$\frac{dg(T)}{d\tau} < 0 \qquad (2\text{-}156)$$

In other words for $t = T$, the optimum growth rate of equity decreases as the tax rate increases, for both the profit-maximizing and the managerial firm. This is what would be expected.

Also from (2-106), (2-154) and (2-155) we obtain

$$\left|\frac{dg(T)}{d\tau}\right|_{\beta>0} = r_e + (1-\tau)\cdot\left|\frac{dr_e}{dm}\right|\cdot\frac{dm(T)}{d\tau} \qquad (2\text{-}157)$$

and

$$\left|\frac{dg(T)}{d\tau}\right|_{\beta=0} = r_e \tag{2-158}$$

which implies, following (2-153), that

$$\left|\frac{dg(T)}{d\tau}\right|_{\beta>0} > \left|\frac{dg(T)}{d\tau}\right|_{\beta=0} \tag{2-159}$$

In other words for $t = T$, the managerial firm is more affected by a profits-tax change than the profit-maximizing firm, with regard to a change in equity growth rate.

This is because, in the profit-maximizing case, the equity growth rate, which is the return on equity after tax $r_e(1-\tau)$, is directly decreased by the tax change, whereas in the managerial case, for $t = T$, the return on equity *before* tax (r_e) also decreases because of the effect of an increasing leverage (2-106) and (2-153).

2.6.6.2 Interest Rate Change

We also want to determine how a change in the external interest rate will affect the financing and net worth or equity of the firm at time T.

As regards the interest rate i'

$$i' = i'(i_o, m) \tag{2-160}$$

where

$$\frac{\partial i'}{\partial i_o} > 0 \tag{2-160a}$$

and where i_o is a coefficient in the interest rate function which is influenced by the general interest rate level of the economy.

By differentiating (2-105) totally with respect to the external interest rate level i_o for $t = T$, we get

$$\frac{dm(T)}{di_o} = - \frac{\dfrac{\partial i'}{\partial i_o} + m(1-m)\,\dfrac{\partial^2 i'}{\partial i_o \partial m}}{(1-m)\,\dfrac{\partial^2 (i'm)}{\partial m^2}} \tag{2-161}$$

For

$$\frac{\partial^2(i'm)}{\partial m^2} > 0$$

or (112a), and given a positive numerator in (2-161), which appears to be a plausible case,

$$\frac{dm(T)}{di_o} < 0 \qquad (2\text{-}162)$$

In other words for t = T, both our profit-maximizing and managerial firms will decrease their leverages in response to an external interest rate increase. This is to be expected.

From (2-154) we find that

$$\frac{1}{1-\tau}\,\frac{dg}{di_o} = \frac{\partial r_e}{\partial i_o} + \frac{\partial r_e}{\partial m}\,\frac{dm}{di_o} \qquad (2\text{-}163)$$

The first term in (2-163) is negative from (2-103) and (2-160a) and the second term is positive for t = T from (2-106) and (2-162).

For the profit-maximizing firm,(2-112) and (2-163) give us

$$\left.\frac{dg(T)}{di_o}\right|_{\beta=0} < 0 \qquad (2\text{-}164)$$

In other words the equity growth rate of the profit-maximizing firm at terminal time is reduced for an external interest rate increase. This is intuitive, since in this special case the equity growth rate is equal to the return on equity (2-154) and the direct effect of an increase in the interest rate on the return on equity (r_e) is of course negative (2-103).

In the managerial case, however, the interest rate increase also has an indirect positive effect on the equity growth rate (g) because of the lower leverage for t = T. Cf. (2-162) and (2-163). Whether this effect implies less response on the part of the managerial firm compared with the profit-maximizing firm to an external interest rate increase at the horizon date, depends on the specific interest rate function $i'(m)$. However for a small managerial coefficient β we may expect, from (2-104) and (2-105), the second term in (2-163) to be small, so that the equity growth rate is reduced for an increase in the external interest rate also in the managerial case.

2.6.6.3 Results and Discussion

We have studied the effect of economic policy measures, such as changes in tax rate (τ) and interest rate (i_o), on the financing and net worth or equity of the firm at the horizon date. We can now summarize the results.

Responses at the Horizon Date to Economic Policy Measures. The Profit-Maximizing Firm Model:

	Parameter	
Variable	τ	i_o
m	0	-
g	-	-

Responses at the Horizon Date to Economic Policy Measures. The Managerial Firm Model:

	Parameter	
Variable	τ	i_o
m	+	-
g	-	-(+)

The managerial firm model provides an alternative prediction of firm behaviour in the respect that an increase in the profits tax rate causes an increase in leverage to counteract the tax loss. This is contrary to the result obtained in the profit-maximizing firm model.

This is somewhat similar to the difference in the price-output decision in the static case between a profit-constrained sales maximizing firm and a profit-maximizing firm, when the tax rate is changed (Baumol, 1959).

The above results are valid at the horizon date. However, they are most likely also valid immediately before the horizon date, due to continuity. Furthermore, for the profit-maximizing firm they are valid at all points in time because the optimum profit-maximizing firm is in a situation of steady-state growth. Because of their plausibility, they may also be valid for other points in time in the case of the managerial firm.

In order to test this last proposition, an analysis influenced by Oniki (1973) is undertaken in Appendix 2.2. The conclusion of Appendix 2.2 is that the results in the two tables in this section for the horizon date also hold for the whole time period, at least for small values of the managerial coefficient β.

The comparative dynamics analysis in Appendix 2.2 is interesting also because this type of analysis (Oniki (1973)) is only little reported yet in the economics literature and because it apparently in special cases may provide useful results.

2.6.7 A Further Specification of the Special Model

In order to further analyze the special model studied in this appendix, we shall specify the interest rate function $i'(m)$ and derive an analytical solution.

Let us assume an interest rate i' of the form (2-160) and with

$$\lim_{m \to 1} i'(m) = \infty \tag{2-165}$$

and

$$i'(0) = i_o \tag{2-166}$$

where i_o is a constant.

The simplest i'(m) of this type is

$$i'(m) = i_o + b_1 \cdot \frac{m}{1-m} \tag{2-167}$$

where b_1 is a constant, governing how the interest rate increases with leverage.

The interest rate function (2-167) satisfies (2-125)-(2-127) and (2-135). Inserting (2-167) into (2-115) and calculating the partial fractions and integrating, we obtain the optimum (2-120) explicitly

$$t = T + \frac{2b_1}{B_o(A_o+2b_1)} \{C_1 \cdot \ln \frac{m-\nu_1}{m_T-\nu_1} + C_2 \cdot \ln \frac{m-\nu_2}{m_T-\nu_2} + $$

$$+ C_3 \cdot \ln \frac{m-\nu_3}{m_T-\nu_3} \} \tag{2-168}$$

where

$$A_o = \mu-\alpha-i_o \tag{2-169}$$

$$B_o = i_o - b_1 - \frac{i}{1-\tau} \tag{2-170}$$

$$C_i = \frac{1-\nu_i}{(\nu_{i+2}-\nu_i)\ (\nu_{i+1}-\nu_i)} \quad i = 1,2,3; \nu_4=\nu_1; \nu_5=\nu_2 \tag{2-171}$$

$$\nu_{1,2} = 1 \pm \sqrt{\frac{b_1}{-B_o}} \quad \text{(The m-values that give extrema of the cost of capital)} \tag{2-172}$$

$$\nu_3 = \frac{A_o}{A_o + 2b_1} \quad \text{(Optimum m for the } \beta=0 \text{ case)} \tag{2-173}$$

$$m_T = m(T) = \frac{A_o + C_o}{A_o + C_o + 2b_1} \tag{2-174}$$

$$C_o = \frac{\beta\ \gamma\ u}{(1-\beta)(1-\tau)} \tag{2-175}$$

$$\beta > 0 \tag{2-176}$$

Optimal equity y is calculated as mentioned earlier in Section 2.6.4 and results after integration in a long expression which will not be reproduced here.

Instead the result of these calculations in a numerical example will be discussed.

2.6.8 Numerical Example

Assume the following constants:

$\mu = 30\%$	$b_1 = 2\%$	$y_o = 1$
$\alpha = 13\%$	$i = 6\%$	
$\gamma = 3$	$T = 10$	

and the following cases, for tax rates and general interest rate levels:

I	$\tau = 40\%$	and	$i_o = 8\%$
II	$\tau = 50\%$	and	$i_o = 8\%$
III	$\tau = 60\%$	and	$i_o = 8\%$
IV	$\tau = 50\%$	and	$i_o = 7{,}5\%$
V	$\tau = 50\%$	and	$i_o = 8{,}5\%$

By calculating optimal behavior in these cases for $\beta = 0{,}01$ and $\beta = 0$ respectively, the effect of economic policy levels on the long-run behavior of managerial-type firms and profit-maximizing firms may be compared.

The optimum leverage and after-tax rate of return on equity for the profit-maximizing firm $\beta=0$ for alternative II become $\hat{m} = 0{,}69$ and $(1-\tau)\cdot r_e(\hat{m}) = 13{,}6\%$ respectively.

The interest cost of delaying the imputation of an extra equity dollar by one time period is $|\dot{\lambda}/\lambda - i|$ which, for $\beta = 0$ and following (2-57), is equal to $(1-\tau)\left[(i'm)_m\right]_{m=\hat{m}} = 13{,}6\%$. As economic reasoning leads us to expect, this is equal to the after-tax rate of return on equity for the profit-maximizing firm above.

We also see that in this numerical example $i > (1-\tau)i_o$, because $i=6\%$ and $(1-\tau)i_o=4\%$ in alternative II above.

From the above calculations and (2-109) and (2-125) it is evident that (2-128) is valid, which verifies (2-130), i.e. we have an increasing leverage $\dot{m} > 0$ in this numerical example in the managerial firm case.

Also, the above calculations regarding marginal interest rates verify (2-88), i.e. we have a decreasing imputed instant value for an extra equity dollar $\dot{\lambda} < 0$ for both the managerial-type firm and the profit-maximizing firm.

The growth of the firm, given various tax rates and interest rates, can be seen in Table 2:1.

The calculations in Table 2:1 show that the growth rate for the managerial-type firm is larger than for the profit-maximizing firm, as we should expect from (2-150).

The results in Table 2:1 also show that new higher tax-rate levels and interest-rate levels have a greater absolute effect on the growth rate of the managerial-type firm than on the growth rate of the profit-maximizing firm. In both cases the growth rate decreases.

However, the relative effect of such policy measures on the growth rate of the firm is greater for the profit-maximizing firm[1]. This may be because growth in the profit-maximizing case is a result of profit-maximizing and is therefore more sensitive to cost increases from interest rates and taxes. On the other hand, growth in the managerial case is a result also of non-profit maximizing behavior, and it is therefore relatively less sensitive to cost components such as interest rates and taxes.

Numerical computer calculations based on the analytical solutions (2-168)-(2-176) have been made for this special case for other optimal stock variables such as equity and debt, as well as for optimal flow variables such as investments and their financing by external or internal means, and also for return on equity. These calculations provide the same information for each of these variables as Table 2:1 provides for capital.

The analytical results obtained in this appendix are confirmed by the numerical example just studied. For example the acceleration of leverage (2-136) is confirmed. Also the analytically deduced absolute effects of shifts in the corporate income tax and general interest rate are confirmed by the numerical example at all points in time.

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1) The difference in the numerical example is moderate, cf Table 2:1. This may depend on the small value of the managerial coefficient $\beta = 0{,}01$ used in the example.

Table 2:1. Firm growth given various tax rates and general interest rates and for different types of firm. K_o = Initial Capital. K_T = Terminal Capital. i_o = General Interest Rate. τ = Corporate Income Tax Rate. β = Managerial Coefficient.

A. Firm Growth $\frac{K_T - K_o}{K_o}$ for $i_o = 8\%$ and various τ.

	$\beta = 0$		$\beta = 0{,}01$	Relative Decline in Growth Rate $\beta = 0$		Relative Decline in Growth Rate $\beta = 0{,}01$
$\tau = 0{,}4$	409,11%		445,78%			
$\tau = 0{,}5$	288,16%		316,06%			
Diff. $\tau = 0{,}4 \rightarrow 0{,}5$	120,95%	<	129,72%	29,56%	>	28,43%
$\tau = 0{,}6$	195,94%		215,82%			
Diff $\tau = 0{,}5 \rightarrow 0{,}6$	92,22%	<	100,24%	32,00%	>	31,72%

B. Firm Growth $\frac{K_T - K_o}{K_o}$ for $\tau = 50\%$ and various i_o.

	$\beta = 0$		$\beta = 0{,}01$	Relative Decline in Growth Rate $\beta = 0$		Relative Decline in Growth Rate $\beta = 0{,}01$
$i_o = 7{,}5\%$	311,26%		341,08%			
$i_o = 8{,}0\%$	288,16%		316,06%			
Diff. $i_o = 7{,}5\% \rightarrow 8\%$	23,10%	<	25,02%	7,42%	>	7,34%
$i_o = 8{,}5\%$	267,50%		293,67%			
Diff. $i_o = 8\% \rightarrow 8.5\%$	20,66%	<	22,39%	7,17%	>	7,08%

2.6.9 Summary and Conclusions of Appendix 2.1

The aim of Appendix 2.1 has been to analyze a special case of the dynamic financial model of a managerial-type firm in the main part of this chapter in order to gain further insight into the dynamic financial behavior of such a firm and how this behavior differs from that of a profit-maximizing firm.

The special assumptions are that operating profit is a certain fixed percentage of total capital and that dividends are only restricted to be non-negative. Also, as in the general case, the discount rate is assumed greater than the after-tax all-equity interest rate, the managerial coefficient β is assumed to be small and the limit to growth is assumed to be the rising interest cost as leverage increases. Furthermore it is assumed that the interest cost in relation to capital, increases with leverage at an increasing rate (2-112a). Appendix 2.1 is limited in scope to the analysis of the case of minimum dividends and non-zero debt (Policy A).

The necessary condition for Policy A to be optimal is satisfied if the maximum after-tax rate of return on equity is greater than the discount rate (2-132). This is a rather plausible result, as such a case puts a premium on capital accumulation - and thus minimum dividends - by the generation of positive net economic profits.

The necessary condition for an optimum also implies that the managerial-type firm grows continuously in the studied time period. Whereas the profit-maximizing firm exhibits steady-state-growth, the managerial-type firm grows at a non-steady-state rate.

As is expected the model predicts that the managerial-type firm grows faster and is more heavily levered with debt than the profit-maximizing firm.

The model also predicts an increasing leverage over time for a managerial-type firm compared to a constant leverage over time for a profit-maximizing firm. This offers another explanation for the phenomenon of increasing leverage found in e.g. Sweden. Cf. for instance Bertmar & Molin (1977). The explanation offered by the model is that the firm not only subscribes to a profit goal but also to a size goal. It is not proposed here that this is a main reason, but it is interesting that the model, with all its simplified assumptions, comes to this conclusion.

The model also predicts that at the horizon date the managerial-type firm increases debt leverage in response to an increase in the corporate income tax rate whereas the profit-maximizing firm does not. This result is somewhat similar to the difference in the price-output decision in the static case between a Baumolian sales-maximizing firm and a profit-maximizing firm when the corporate income tax rate is changed (Baumol, 1959).

It should be stressed here - with taxes as an example - that the model of the firm in this chapter is partial. For instance, the model does not take into consideration the possibility that the discount rate may be affected by a tax shift. Neither does the model take explicit consideration to personal income taxes and capital gains taxes. These considerations may be built into the model at a later stage.

As expected, the model predicts - at the horizon date - a lower debt leverage in response to an increase in the general interest rate level for both a managerial-type firm and a profit-maximizing firm.

The just mentioned predictions of the effects of changes in fiscal and monetary policies may well, due to their plausibility, be valid at other points in time than at the horizon date. In order to test this, an analysis according to Oniki (1973) is undertaken in Appendix 2.2. The conclusion of Appendix 2.2 is that the comparative dynamics results for the horizon date also hold for the whole time period for small values of the managerial coefficient β.

In order to further analyze the special model studied in this Appendix 2.1, the interest rate function $i'(m)$ has been specified here and a numerical example has been studied using computer calculations based on a derived analytical solution. The numerical example confirms the analytical results in Appendix 2.1 and also provides comparative dynamics predictions for capital. In the numerical example, the growth of the firm is reduced in response to a tax rate increase or an interest rate increase both in the case of a managerial-type firm and a profit-maximizing firm. However, the relative effect on the growth of the firm is relatively smaller in the managerial firm, in the numerical example[1), probably because the managerial firm, compared with the profit-maximizing firm, is less sensitive to cost increases from interest rates and taxes. This in turn would depend on the non-profit component of the managerial firm's objective function.

1) The difference in the numerical example is moderate, cf Table 2:1. This may depend on the small value of the managerial coefficient $\beta = 0{,}01$ used in the example.

APPENDIX 2.2

2.7 ECONOMIC POLICY AND THE GROWTH AND FINANCING OF A MANAGERIAL FIRM. SPECIAL CASE.

In this Appendix 2.2 we shall study the effect of a profits tax rate change and interest rate change on the entire optimal time path of the firm in the special case treated in Appendix 2.1. Previously these tax and interest rate shifts have been analyzed at the horizon date in Appendix 2.1, Section 2.6.6. The aim here is to analyze if the results at the horizon date are valid also for the whole time period.

The method of Oniki (1973) appears useful for this analysis and will be used here. One reason is that even though an analytical solution is derived in Appendix 2.1, Section 2.6.2, the comparative dynamics result is not readily found from it. Another, and more important, reason for the choice of the method of Oniki (1973) is that it is rather general and does not presuppose an analytical solution. An explicit analytical solution is rather uncommon in optimal control theory.

2.7.1 Optimum Necessary Conditions

We shall summarize the necessary conditions for a maximum in the special case treated in Appendix 2.1.

$$\dot{\lambda} = -\lambda\left[(1-\tau)\frac{d(i'm)}{dm}-i\right] \tag{2-177}$$

$$\dot{y} = r_e(m)\cdot(1-\tau)y \tag{2-178}$$

$$m = m(\lambda,\tau) \tag{2-179}$$

$$y(0) = y_o \tag{2-180}$$

$$\lambda(T) = \eta \tag{2-181}$$

where

y = equity; the state variable

m = leverage; the decision or control variable. Leverage is debt in proportion to debt and equity

i' = interest rate on debt
i = discount rate
r_e = return on equity; cf. (2-103)
λ = shadow value for an extra dollar of equity
τ = profits tax rate
T = terminal date
y_o = initial equity
η = weighting factor of share value in the objective function.

The expression (2-179) is a short-hand notation for the optimum condition for leverage (2-105).

The expression (2-178) or (2-107) states that at optimum, equity grows at a rate equal to the return on equity after tax. This is self-evident when no dividends are paid out, which is the case studied here (Policy A in Section 2.3.4).

The expression (2-177) is equivalent to (2-57) and an economic interpretation is given in Section 2.3.4.1.

The initial condition on equity is (2-180) and the terminal condition for the shadow value is (2-181) from (2-42).

We thus have a two-point boundary value problem with three equations (2-177)-(2-179) and the three variables leverage m, equity y and the shadow value λ. Formally a solution to our problem may thus be found from (2-177)-(2-181).

Corporate income tax and profits tax will be used interchangably as denominations.

2.7.2 Comparative Dynamics of a Corporate Income Tax Rate Change

Our problem is one of comparative dynamics or sensitivity analysis of a dynamic solution with respect to a marginal profits tax rate change. We must thus find the derivatives

$$\frac{dy^x}{d\tau} = ? \qquad \frac{d\lambda^x}{d\tau} = ? \qquad \frac{dm^x}{d\tau} = ?$$

where we use the asterisk to denote the optimal solution of the two-point boundary value problem (2-177)-(2-181), when necessary. A first step is to find the effects of a profits tax rate change on the initial and end values.

By differentiation of (2-180) and (2-181) we have immediately

$$\frac{dy^x(0)}{d\tau} = 0 \qquad (2\text{-}182)$$

$$\frac{d\lambda^x(T)}{d\tau} = 0 \qquad (2\text{-}183)$$

As we consider the variables a function of the two variables time t and profits tax rate τ in this comparative dynamics analysis and as we assume the variables continuously differentiable, we have in general:

$$\left.\frac{dy(t)}{d\tau}\right|_{t=t'} = \frac{dy(t')}{d\tau} \qquad (2\text{-}184)$$

$$\left.\frac{d\lambda(t)}{d\tau}\right|_{t=t'} = \frac{d\lambda(t')}{d\tau} \qquad (2\text{-}185)$$

In other words, to find the marginal effect at time t' of a marginal change in the profits tax rate, we can set t=t' *before* we differentiate. Thus from (2-182)-(2-185),

$$\left.\frac{dy^x(t)}{d\tau}\right|_{t=0} = 0 \qquad (2\text{-}186)$$

$$\left.\frac{d\lambda^x(t)}{d\tau}\right|_{t=T} = 0 \qquad (2\text{-}187)$$

The critical value to determine now is,

$$\left.\frac{d\lambda^x(t)}{d\tau}\right|_{t=0} = ? \qquad (2\text{-}188)$$

i.e. the initial value of the shadow value or rather its derivative with respect to the profits tax τ. Once this is found, the total time development of $\frac{d\lambda}{d\tau}$, $\frac{dy}{d\tau}$ and $\frac{dm}{d\tau}$ may be found from (2-177)-(2-179) or rather the derivatives of them, which after some calculus from (2-177) and (2-178) respectively are

$$\frac{d\dot{\lambda}}{d\tau} = - \left[(1-\tau)r_e - i_1\right]\cdot\frac{d\lambda}{d\tau} + r_e\lambda \qquad (2\text{-}189)$$

$$\frac{d\dot{y}}{d\tau} = (1-\tau)y(1-m)\,\frac{\left[\frac{dr_e}{dm}\right]^2}{\lambda\cdot\frac{d^2(i'm)}{dm^2}}\cdot\frac{d\lambda}{d\tau} + r_e(1-\tau)\cdot\frac{dy}{d\tau} - r_e y -$$

$$- y(1-m)\,\frac{\left[\frac{dr_e}{dm}\right]^2}{\frac{d^2(i'm)}{dm^2}} \qquad (2\text{-}190)$$

In order to see the signs of the coefficients better, we may write (2-189) and (2-190) for short

$$\frac{d}{dt}\left[\frac{d\lambda}{d\tau}\right] = \left[-\right]\frac{d\lambda}{d\tau} + \left[+\right] \qquad (2\text{-}191)$$

$$\frac{d}{dt}\left[\frac{dy}{d\tau}\right] = \left[+\right]\frac{d\lambda}{d\tau} + \left[+\right]\frac{dy}{d\tau} + \left[-\right] \qquad (2\text{-}192)$$

for the assumed case of $\frac{d^2(i'm)}{dm^2} > 0$, i.e. (2-112a), and for the economic assumption $(1-\tau)r_e - i > 0$ which appears plausible from (2-132) and for small β. Cf. (2-133)=(2-154a).

Also in the left-hand members we have used,

$$\frac{d}{dt}\left[\frac{d\lambda}{d\tau}\right] = \frac{d}{d\tau}\left[\frac{d\lambda}{dt}\right] \qquad (2\text{-}193)$$

$$\frac{d}{dt}\left[\frac{dy}{d\tau}\right] = \frac{d}{d\tau}\left[\frac{dy}{dt}\right] \qquad (2\text{-}194)$$

which follow from similar reasons as (2-184)-(2-185).

Returning to the problem (2-188), we have

$$\left.\frac{d\lambda^x(t)}{d\tau}\right|_{t=0} = -\frac{d\lambda(T)}{d\tau}\Big/\frac{d\lambda(T)}{d\lambda(0)} \qquad (2\text{-}195)$$

according to Oniki (1973, p. 279, formula (62 a)).

In general, the method of solution in this Appendix 2.2 is directly influenced by Oniki (1973).

In (2-195) and the following, where necessary, the variables *without* an asterisk represent a solution to

(2-177)-(2-179) based on given *initial* values y_o and $\lambda(0)$ respectively.

We shall denote this initial value for the shadow value λ^o which is considered a constant in the analysis:

$$\lambda(0) = \lambda^o \tag{2-195a}$$

In order to find the denominator of the right-hand side of (2-195), we differentiate (2-177)-(2-178) with respect to the initial value $\lambda(0)$:

$$\frac{d\dot{\lambda}}{d\lambda(0)} = -\left[(1-\tau)r_e - i\right] \cdot \frac{d\lambda}{d\lambda(0)} \tag{2-196}$$

$$\frac{d\dot{y}}{d\lambda(0)} = (1-\tau)y(1-m)\frac{\left[\frac{dr_e}{dm}\right]^2}{\lambda \cdot \frac{d^2(i'm)}{dm^2}} \cdot \frac{d\lambda}{d\lambda(0)} + r_e(1-\tau)\frac{dy}{d\lambda(0)} \tag{2-197}$$

As with (2-191) and (2-192), we may write (2-196) and (2-197) respectively

$$\frac{d}{dt}\left[\frac{d\lambda}{d\lambda(0)}\right] = \left[-\right]\frac{d\lambda}{d\lambda(0)} \tag{2-198}$$

$$\frac{d}{dt}\left[\frac{dy}{d\lambda(0)}\right] = \left[+\right]\frac{d\lambda}{d\lambda(0)} + \left[+\right]\frac{dy}{d\lambda(0)} \tag{2-199}$$

The initial values are from (2-180) and (2-195a) respectively

$$\left.\frac{d\lambda(t)}{d\lambda(0)}\right|_{t=0} = \frac{d\lambda(0)}{d\lambda(0)} = 1 \tag{2-200}$$

$$\left.\frac{dy(t)}{d\lambda(0)}\right|_{t=0} = \frac{dy(0)}{d\lambda(0)} = \frac{dy_o}{d\lambda(0)} = 0 \tag{2-201}$$

We may thus draw the following phase diagram:

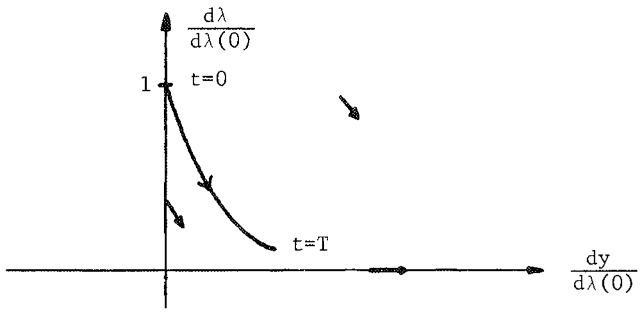

Figure 2:3. Phase Diagram for $\frac{d\lambda}{d\lambda(0)}$ and $\frac{dy}{d\lambda(0)}$. λ = Instant Value of an Imputed Dollar of Equity (Adjoint Variable). $\lambda(0)$ = The Value of the Adjoint Variable $\lambda(t)$ at Initial Time t=0. y = Equity.

The small arrows indicate the direction of the curve and are deduced from the signs of the time derivatives in the left-hand members of (2-198) and (2-199). For example in the first quadrant and on the ordinate, (2-198) and (2-199) imply

$$\frac{d}{dt}\left[\frac{d\lambda}{d\lambda(0)}\right] < 0 \qquad \text{and} \qquad \frac{d}{dt}\left[\frac{dy}{d\lambda(0)}\right] > 0 .$$

This means that $\frac{d\lambda}{d\lambda(0)}$ decreases and $\frac{dy}{d\lambda(0)}$ increases over time, as indicated by the small arrows in Figure 2:3.

The full-drawn curve in Figure 2:3 is based on the initial values (2-200)-(2-201) and the small direction arrows just mentioned. The curve is a rough approximation of the optimal time path. As the end point of the optimum time path lies in the first quadrant in Figure 2:3, we have

$$\frac{d\lambda(T)}{d\lambda(0)} > 0 \tag{2-202}$$

In order to find the numerator in (2-195) we shall now construct a phase diagram similar to Figure 2:3 but for the equations (2-191) and (2-192).

The initial values are from (2-180) and (2-195a) respectively

$$\left.\frac{d\lambda(t)}{d\tau}\right|_{t=0} = \frac{d\lambda^{o}}{d\tau} = 0 \tag{2-203}$$

$$\left.\frac{dy(t)}{d\tau}\right|_{t=0} = \frac{dy_o}{d\tau} = 0 \qquad (2\text{-}204)$$

We thus have the following phase diagram:

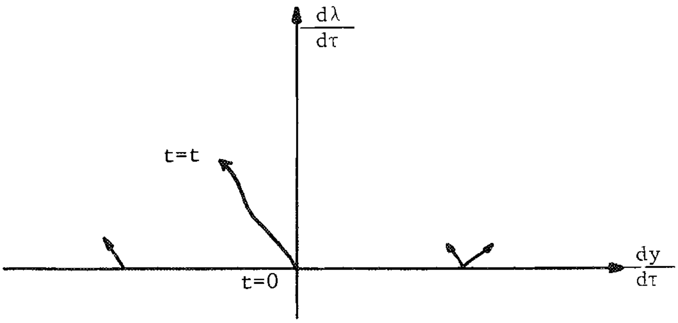

Figure 2:4. Phase Diagram for $\frac{d\lambda}{d\tau}$ and $\frac{dy}{d\tau}$. λ = Instant Value of an Imputed Dollar of Equity (Adjoint Variable). y = Equity. τ = Corporate Income Tax Rate.

The small direction arrows have an analogous interpretation to those in Figure 2:3 and are based on (2-191)-(2-192).

We see from Figure 2:4 that the curve will start in the second quadrant and never pass into the third and fourth quadrants. Thus

$$\frac{d\lambda(T)}{d\tau} > 0 \qquad (2\text{-}205)$$

From (2-195), (2-202) and (2-205), we get

$$\left.\frac{d\lambda^x(t)}{d\tau}\right|_{t=0} < 0 \qquad (2\text{-}206)$$

We are now in a position to draw the following phase diagram, using (2-186), (2-187) and (2-206):

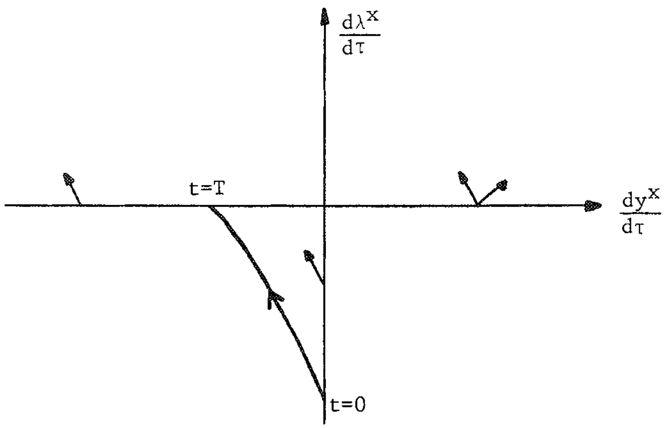

Figure 2:5. Phase Diagram for $\frac{d\lambda^x}{d\tau}$ and $\frac{dy^x}{d\tau}$. λ^x = Optimal Instant Value of an Imputed Dollar of Equity (Adjoint Variable). y^x = Optimal Equity. τ = Corporate Income Tax Rate.

The small direction arrows are obtained from (2-191) and (2-192), which also are valid for the variables with asterisk.

We see from Figure 2:5, that

$$\frac{d\lambda^x}{d\tau} \leqslant 0 \qquad \text{for all } t \tag{2-207}$$

$$\frac{dy^x}{d\tau} \leqslant 0 \qquad \text{for all } t \tag{2-208}$$

From (2-179), we get

$$\frac{dm^x}{d\tau} = \frac{\partial m^x}{\partial \lambda^x}\frac{d\lambda^x}{d\tau} + \frac{\partial m^x}{\partial \tau} \tag{2-209}$$

From (2-105), by inversely differentiating, we have

$$\frac{\partial m^x}{\partial \lambda^x} < 0 \tag{2-210}$$

$$\frac{\partial m^x}{\partial \tau} > 0 \tag{2-211}$$

Thus (2-207), (2-209)-(2-211) imply

$$\frac{dm^x}{d\tau} > 0 \qquad \text{for all } t \tag{2-212}$$

From (2-106), we have

$$\frac{dr_e(m^x)}{dm^x} \leqslant 0 \quad \text{for all } t \tag{2-213}$$

Thus (2-212) and (2-213) inserted into (2-155) imply

$$\frac{dg^x}{d\tau} < 0 \qquad \text{for all } t \tag{2-214}$$

We have thus in (2-212) and (2-214) obtained the main result of this section, that the corresponding results obtained for the *terminal time* in Appendix 2.1, Section 2.6.6 also are valid at *all points in time for small* β. This result was not unexpected considering the strong assumptions of the model but also not completely self-evident.

The economic interpretation of the obtained results are the same as already discussed in the section mentioned. We should add here that not only the optimum growth of equity (2-214) but also the optimum equity itself (2-208) decreases if the profits tax rate is raised.

The comparative dynamics approach of Oniki (1973) to optimal dynamic economics, appears to be an interesting parallel to the traditional comparative statics approach of optimal static economics.

2.7.3 Comparative Dynamics of an Interest Rate Change

In the previous section we studied the comparative dynamics effect of a profits tax change on the financial model of Appendix 2.1. Here we shall study in a similar way the effect of an interest rate change on the same model of the firm.

The optimal conditions (2-177)-(2-181) are the same, except that we now write the optimal leverage (2-179) as

$$m = m(\lambda, i_o) \tag{2-215}$$

Also we recall from (2-160) that the interest rate is,

$$i' = i'(i_o, m) \tag{2-216}$$

where i_o is a coefficient in the interest rate function which depends directly on the general interest rate level of the economy.

The problem is to find the derivatives of equity, shadow value and leverage respectively with respect to the rate of interest for all dates t:

$$\frac{dy^x}{di_o} = ? \qquad \frac{d\lambda^x}{di_o} = ? \qquad \frac{dm^x}{di_o} = ?$$

This means that we seek the effect of a change from one constant interest rate level for the time period to another marginally increased constant interest rate level for the time period. This marginal shift in the constant interest rate level is valid from the initial time. The analysis is not fully dynamic, because the optimum *transient* response to a jump in the interest rate level at t=0 is not studied.

A first step is to find the effects of an interest rate change on the initial and end values.

By differentiation of (2-180) and (2-181), we have immediately from (2-184) and (2-185)

$$\left.\frac{dy^x(t)}{di_o}\right|_{t=0} = \frac{dy^x(0)}{di_o} = 0 \tag{2-217}$$

$$\left.\frac{d\lambda^x(t)}{di_o}\right|_{t=T} = \frac{d\lambda^x(T)}{di_o} = 0 \tag{2-218}$$

In analogy to (2-195) we have a third boundary value:

$$\left.\frac{d\lambda^x(t)}{di_o}\right|_{t=0} = -\frac{d\lambda(T)}{di_o}\bigg/\frac{d\lambda(T)}{d\lambda(0)} \tag{2-219}$$

From (2-202) we have

$$\frac{d\lambda(T)}{d\lambda(0)} > 0 \tag{2-220}$$

Studying comparative dynamics for i_o instead of τ does not alter this result.

To find the numerator of (2-219) we derivate (2-177) and (2-178) with respect to i_o and obtain after some calculus respectively

$$\frac{d\dot{\lambda}}{di_o} = -\left[(1-\tau)r_e - i\right]\frac{d\lambda}{di_o} + \lambda(1-\tau)\frac{m}{1-m}\;\frac{\partial i'}{\partial i_o} \tag{2-221}$$

$$\frac{d\dot{y}}{di_o} = (1-\tau)y(1-m)\,\frac{\left[\frac{\partial r_e}{\partial m}\right]^2}{\lambda\cdot\frac{\partial^2(i'm)}{\partial m^2}}\cdot\frac{d\lambda}{di_o} + r_e(1-\tau)\cdot\frac{dy}{di_o} - (1-\tau)yA \tag{2-222}$$

where

$$A = \frac{m}{1-m}\cdot\frac{\partial i'}{\partial i_o} - \frac{\partial r_e}{\partial m}\;\frac{\partial m}{\partial i_o} \tag{2-223}$$

As in (2-160a), economic reasoning implies

$$\frac{\partial i'}{\partial i_o} > 0 \tag{2-224}$$

Also, at optimum (2-106) implies

$$\frac{\partial r_e}{\partial m} < 0\,. \tag{2-225}$$

After some calculus we obtain from (2-105)

$$\frac{\partial m}{\partial i_o} = -\frac{\frac{\partial i'}{\partial i_o} + m(1-m)\,\frac{\partial^2 i'}{\partial i_o\,\partial m}}{(1-m)\,\frac{\partial^2(i'm)}{\partial m^2}} \tag{2-226}$$

For (2-112a), i.e.

$$\frac{\partial^2(i'm)}{\partial m^2} > 0$$

and a positive numerator in (2-226), which appears plausible considering (2-224), we get from (2-226)

$$\frac{\partial m}{\partial i_o} < 0 \tag{2-227}$$

We notice the equivalence between (2-161) and (2-226) which is natural because in (2-161) we have also kept λ constant.

For the profit-maximizing firm (2-112), (2-223) and (2-224) imply

$$A_{\beta=0} > 0 \tag{2-228}$$

Continuity implies that A > 0 also for small β:

$$A_{small\beta} > 0 \tag{2-229}$$

However for large β, A may be negative.

In order to see the signs of the coefficients in (2-221) and (2-222) better, we may now write them for short

$$\frac{d}{dt}\left(\frac{d\lambda}{di_o}\right) = \left[-\right]\frac{d\lambda}{di_o} + \left[+\right] \tag{2-230}$$

$$\frac{d}{dt}\left(\frac{dy}{di_o}\right) = \left[+\right]\frac{d\lambda}{di_o} + \left[+\right]\frac{dy}{di_o} + \left[\mp\right] \tag{2-231}$$

where we have assumed $(1-\tau)r_e - i > 0$, which appears plausible from (2-132) and for small β. Cf. (2-133)=(2-154a).

The top sign in (2-231) represents small β and the bottom sign represents large β.

The initial values are from (2-180), (2-184), (2-185) and (2-195a):

$$\left.\frac{d\lambda}{di_o}\right|_{t=0} = \frac{d\lambda(0)}{di_o} = \frac{d\lambda^o}{di_o} = 0 \tag{2-232}$$

$$\left.\frac{dy}{di_o}\right|_{t=0} = \frac{dy(0)}{di_o} = \frac{dy_o}{di_o} = 0 \tag{2-233}$$

It is now possible to construct the following phase diagram from (2-230)-(2-233):

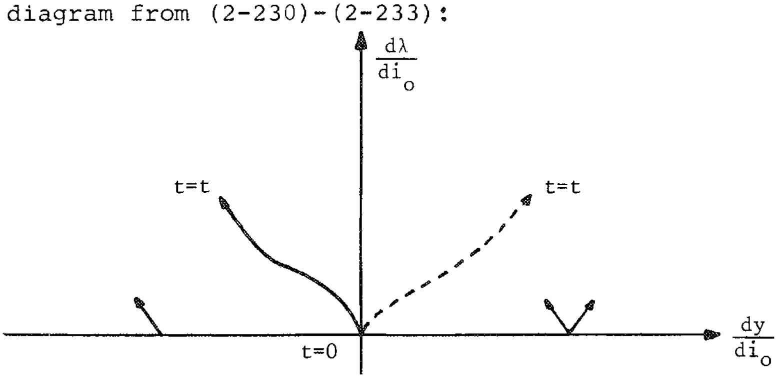

Figure 2:6. Phase Diagram for $\frac{d\lambda}{di_o}$ and $\frac{dy}{di_o}$. λ = Instant Value of an Imputed Dollar of Equity (Adjoint Variable). y = Equity. i_o = General Interest Rate Level.

The full-drawn curve in Figure 2:6 represents the case of small β and the dashed curve represents the case of large β.

We see from Figure 2:6 that the curves will start in the first or second quadrant and never pass into the third and fourth quadrants.

Thus

$$\frac{d\lambda(T)}{di_o} > 0 \tag{2-234}$$

From (2-219), (2-220) and (2-234), we get

$$\left.\frac{d\lambda^x(t)}{di_o}\right|_{t=0} < 0 \tag{2-235}$$

We are now in a position to draw the following phase diagram, using (2-217), (2-218) and (2-235):

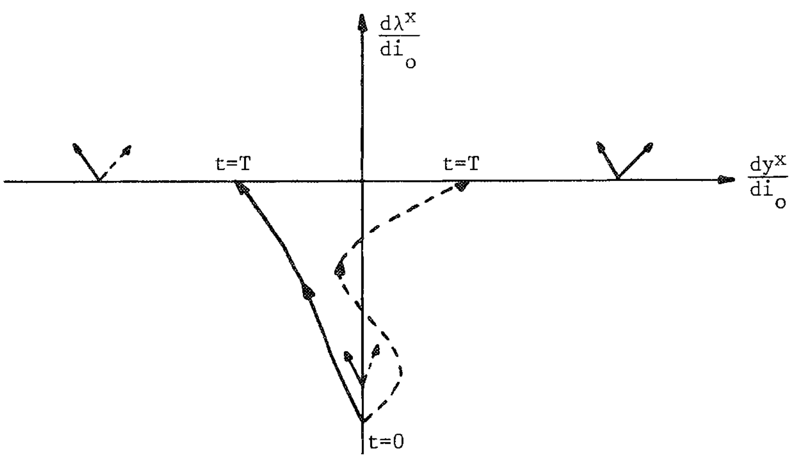

Figure 2:7. Phase Diagram for $\frac{d\lambda^x}{di_o}$ and $\frac{dy^x}{di_o}$. λ^x = Optimal Instant Value of an Imputed Dollar of Equity (Adjoint Variable). y^x = Optimal Equity. i_o= General Interest Rate Level.

The small direction arrows in Figure 2:7 are obtained from (2-230) and (2-231), which also are valid for the variables with asterisk. The full-drawn small direction *arrows* are valid for small β and for large β *also* the dashed ones are possible.

The full-drawn *curve* in Figure 2:7 represents the case of small β and the dashed curve is an example of how an optimal time path perhaps may look for large β. In any event, for large β we can not, in a general analysis like this, state for sure that the optimal time path will confine itself to the third quadrant in Figure 2:7.

However, we see from Figure 2:7 that

$$\frac{d\lambda^x}{di_o} \leqslant 0 \qquad \text{for all } t \tag{2-236}$$

and

$$\frac{dy^x}{di_o} = \begin{cases} \leqslant 0 \text{ for all } t\text{, for small } \beta; & (2\text{-}237) \\ \lesseqgtr 0 \text{ for large } \beta; & (2\text{-}238) \end{cases}$$

From (2-215), we get

$$\frac{dm^x}{di_o} = \frac{\partial m^x}{\partial \lambda^x}\,\frac{d\lambda^x}{di_o} + \frac{\partial m^x}{\partial i_o} \tag{2-239}$$

Also from (2-105) and (2-213), we find that

$$\frac{\partial m^x}{\partial \lambda^x} = 1/\frac{\partial \lambda^x}{\partial m^x} = (1-m^x)\,\frac{\dfrac{\partial r_e(m^x)}{\partial m^x}}{\lambda^x \cdot \dfrac{\partial^2(i'm^x)}{\partial m^{x2}}} < 0 \tag{2-240}$$

Thus (2-218), (2-227), (2-236) and (2-240) imply,

$$\frac{dm^x}{di_o} = \begin{cases} < 0 \text{ for } t=T \text{ and its vicinity}; & (2\text{-}241) \\ < 0 \text{ for all } t\text{, for small } \beta; & (2\text{-}242) \\ \lesseqgtr 0 \text{ for large } \beta; & (2\text{-}243) \end{cases}$$

In a general analysis like this, we can not for sure state that $\frac{dm^x}{di_o}$ is negative for large β.

From (2-103), we get

$$\frac{\partial r_e(m^x)}{\partial i_o} = -\frac{m^x}{1-m^x}\,\frac{\partial i'}{\partial i_o} < 0 \tag{2-244}$$

Thus from (2-163), (2-213), (2-240)-(2-242) and (2-244),

$$\frac{dg^x}{di_o} = \begin{cases} < 0 \text{ for all } t, \text{ for small } \beta; & (2\text{-}245) \\ \lesseqgtr 0 \text{ for large } \beta; & (2\text{-}246) \end{cases}$$

We have thus in (2-242) and (2-245) obtained the main result of this section that the corresponding results obtained for the *terminal time* in Appendix 2.1, Section 2.6.6.2 also are valid at *all points in time for small* β.

Small β represents a firm which basically is a profit maximizer but also has a small element of sales maximization in its objective. As mentioned earlier in this chapter, such a firm is a type of managerial firm.

The obtained result was not unexpected considering the strong assumptions of the model but also not completely self-evident. The economic interpretation of the obtained results is the same as already discussed in Appendix 2.1, Section 2.6.6. We should add here that

$$\frac{dg^x}{di_o} < 0$$

is valid for the managerial-type firm according to (2-245) and not only for the profit-maximizing firm.

We should also like to add here that not only the optimum growth of equity of the managerial-type firm (2-245), but also the optimum equity itself (2-237) decreases if the general interest rate level is raised. As the initial equity is given and as (2-245) is valid for all points in time, (2-237) follows intuitively from (2-245).

2.7.4 Summary of Appendix 2.2

The aim of Appendix 2.2 has been to analyze the effect of shifts in fiscal and monetary policies on the optimum growth of equity and optimum financing of a managerial firm of the type studied in Appendix 2.1.

The comparative dynamics method of Oniki (1973) was employed and proved useful.

The results obtained for the horizon date in Appendix 2.1, Section 2.6.6 were confirmed at all points in time for small values of the managerial coefficient β.

The comparative-dynamics analysis in this Appendix 2.2 is interesting also because this type of analysis (Oniki, 1973) is only little reported yet in the economic literature and because it apparently may provide useful results.

APPENDIX 2.3

2.8 DYNAMIC ANALYSIS OF THE MINIMUM DIVIDENDS POLICY A

In this appendix we shall provide the deductions underlying the conclusions in Section 2.3.4.3 regarding the phase diagram of Policy A in Figure 2:1. Also, we shall analyze the comparative dynamics of a shift in the managerial coefficient β, as discussed in Section 2.4.

The methods used correspond to those used by Treadway (1969) and Gould (1970).

2.8.1 The (m,y) Phase Diagram of Policy A

The system of motion in Policy A is described by the equations (2-56) and (2-62):

$$\dot{m} = M(m,y) \tag{2-247}$$

$$\dot{y} = Y(m,y) \tag{2-248}$$

where M(m,y) and Y(m,y) are defined as the right-hand sides of (2-62) and (2-56) respectively. We shall not use the index A everytime to denote that Policy A is studied.

The slopes of the loci $\dot{m} = 0$ and $\dot{y} = 0$ at the stationary point (m_A^+, y_A^+) in the (m,y) phase diagram Figure 2:1 are respectively

$$\left.\frac{dm}{dy}\right|_{\dot{m}=0} = - \frac{M_y(m_A^+, y_A^+)}{M_m(m_A^+, y_A^+)} < 0 \tag{2-249}$$

and

$$\left.\frac{dm}{dy}\right|_{\dot{y}=0} = - \frac{Y_y(m_A^+, y_A^+)}{Y_m(m_A^+, y_A^+)} < 0 \tag{2-250}$$

which confirm the signs of the slopes at (m_A^+, y_A^+) in Figure 2:1. This follows from

$$M_m(m_A^+, y_A^+) > 0 \tag{2-251}$$

$$M_y(m_A^+, y_A^+) > 0 \tag{2-252}$$

$$Y_m(m_A^+, y_A^+) < 0 \tag{2-253}$$

$$Y_y(m_A^+, y_A^+) < 0 \tag{2-254}$$

which have been calculated straight-forward from the definitions of M(m,y) and Y(m,y), using afterwards the relations

$$M(m_A^+, y_A^+) = 0 \tag{2-255}$$

$$Y(m_A^+, y_A^+) = 0 \tag{2-256}$$

$$(1-m_A^+)\cdot f\left(\frac{y_A^+}{1-m_A^+}\right) > y_A^+\cdot f'\left(\frac{y_A^+}{1-m_A^+}\right) \tag{2-257}$$

where expression (2-257) is the condition for concavity of $f(\cdot)$.

From (2-251)-(2-254) we find that $\dot{m} > 0$ to the right of the locus $\dot{m} = 0$ in Figure 2:1 and $\dot{y} < 0$ to the right of the locus $\dot{y} = 0$ in Figure 2:1. By assuming a point on one of the loci $\dot{m} = 0$ or $\dot{y} = 0$ in Figure 2:1, we obtain the relation between the slopes of the two loci and thus the depicted movement in the phase diagram in Figure 2:1. In order to ascertain this motion we shall study the nature of the stationary point (m_A^+, y_A^+) in the phase diagram Figure 2:1. This will be done by linearly expanding the expressions (2-247) and (2-248) for $\dot{m}$ and $\dot{y}$ respectively, around the stationary point (m_A^+, y_A^+).

The linear part of the Taylor series expansion about the point (m_A^+, y_A^+) is

$$\begin{bmatrix} \dot{m} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} M_m & M_y \\ Y_m & Y_y \end{bmatrix} \begin{bmatrix} m-m_A^+ \\ y-y_A^+ \end{bmatrix} \tag{2-258}$$

where M_m, M_y, Y_m and Y_y are evaluated at the critical point (m_A^+, y_A^+).

If the main matrix in (2-258) is denoted a, the eigenvalues ν are obtained from the determinant equation

$$|a - \nu I| = 0 \tag{2-259}$$

where I is the unity matrix.

The eigen-values are

$$\nu_{1,2} = \frac{M_m + Y_y}{2}\left\{1 \pm \sqrt{1+4\,\frac{M_y Y_m - M_m Y_y}{(M_m + Y_y)^2}}\right\} \tag{2-260}$$

where $M_y Y_m - M_m Y_y > 0$ at the critical point (m_A^+, y_A^+).

Thus the eigen-values are real and of opposite signs, so that (m_A^+, y_A^+) is a saddle point. The trajectory movement around (m_A^+, y_A^+) in Figure 2:1 is thus confirmed. However, this result is only valid in a neighborhood of (m_A^+, y_A^+) such that linear expansion is reasonably accurate.

From the saddle-point property follows that among the paths to the left of the locus $\dot{m} = 0$ in Figure 2:1 there exists only one path approaching the critical point (m_A^+, y_A^+). This one is illustrated in Figure 2:1.

2.8.2 Comparative Dynamics of a Shift in the Managerial Coefficient

Figure 2:2 indicates that a shift in the managerial coefficient $\beta(\beta_2 > \beta_1)$ will increase leverage along the optimal adjustment path of the firm. However the two curves β_1, β_2 can change position if they intersect at some point, i.e. if

$$\left.\frac{dm^o}{dy}\right|_{\beta_2} \gtrless \left.\frac{dm^o}{dy}\right|_{\beta_1} \tag{2-261}$$

where the superscript o denotes the instantaneous (non-stationary) equilibrium of the firm.

We may also write (2-261) as

$$\frac{\partial}{\partial \beta}\left[\frac{dm^o}{dy}\right]_P \gtrless 0 \tag{2-262}$$

at the point of intersection P.

As we have

$$\frac{dm^o}{dy} = \frac{\dot{m}}{\dot{y}} \tag{2-263}$$

$\frac{dm^o}{dy}$ is independent of the managerial coefficient β, following (2-56) and (2-62), and thus from (2-262), we get

$$\frac{\partial}{\partial\beta}\left[\frac{dm^o}{dy}\right] = 0 \qquad (2\text{-}264)$$

Thus the curves most likely do not intersect and therefore (2-100) probably is valid, i.e.

$$\frac{dm^o}{d\beta} > 0 \qquad (2\text{-}100)=(2\text{-}265)$$

Q.E.D.

3 A DYNAMIC ADVERTISING MODEL OF A LABOR-MANAGED FIRM

The economics of the labor-managed firm has been the subject of increasing attention. This was discussed in Chapter 1, Sections 1.3.4 and 1.6, to which reference is made.

In particular the optimal price-output decision and the related input decision of the pure labor-managed firm have been analyzed in the literature and compared with the corresponding optimal behavior of the profit-maximizing firm (e.g. Ward, 1958 and Domar, 1966).

The object of this chapter is however to analyze the dynamic advertising behavior of the labor-managed firm as discussed in Chapter 1, Section 1.6.2.

More specifically the aim of this chapter is to develop a dynamic advertising model of a labor-managed firm in a marginalist-type framework, in order to gain some insight into the dynamic advertising behavior of such a firm and how its behavior in this respect differs from that of a profit-maximizing firm.

First a general analysis of the advertising behavior of a labor-managed firm will be studied in Section 3.1. In order to obtain more specific results a special case is studied in Section 3.2. Finally Section 3.3 contains a summary.

3.1 GENERAL ANALYSIS

First the static case will be discussed as a background.

3.1.1 Static Case

The profit-maximizing firm (PM) maximizes the economic profit of the owners after labor has received a fixed wage rate. The labor-managed firm (LM), on the other hand, maximizes the return accruing to labor after the capital owners have received a fixed income rate.

Thus the objective in the PM-case can be written

$$\text{Max } \{pq - r \cdot K - w_o \cdot L - a\} \tag{3-1}$$

and the objective in the LM-case

$$\text{Max } \{\frac{pq - rK - a}{L}\} \tag{3-2}$$

where

p = price of output

q = output quantity

K = capital input

r = user cost of capital

L = labor input

w_o = average wage rate

a = advertising expenditure.

Following Vanek (1970, p. 1), the term "labor" is used here to include everyone working in the economic unit studied.

In order to avoid double calculation the objective function may be summarized as

$$\Pi = \frac{p \cdot q - r \cdot K - c_o \cdot w_o L - a}{c_1 \cdot L + c_2} \tag{3-3}$$

where the objective function Π is defined from (3-3) and

$$c_o = c_2 = 1, \ c_1 = 0 \text{ for the PM-case} \tag{3-4}$$

and

$$c_o = c_2 = 0, \ c_1 = 1 \text{ for the LM-case} \tag{3-5}$$

In other words, for both types of firm, the problem is

$$\max_{p,a,L,q} \Pi(p,q,a,L) \tag{3-6}$$

subject to a production function of capital and labor and subject to a demand function of price and advertising:

$$q = f(K,L) \quad \text{(production function)} \tag{3-7}$$

$$q = g(p,a) \quad \text{(demand function)} \tag{3-8}$$

Capital K is regarded as an exogenously given constant. Denoting the Lagrange coefficients for (3-7) and (3-8), λ_1 and λ_2 respectively, the problem is to maximize the Lagrangian M:

$$M = \Pi + \lambda_1 \cdot [q-f(K,L)] + \lambda_2 \cdot [q-g(p,a)] \tag{3-9}$$

Differentiating M partially with respect to p, a, L and q respectively gives us the following necessary conditions for a maximum:

$$q/(c_1 L + c_2) - \lambda_2 g_p = 0 \tag{3-10}$$

$$- 1/(c_1 L + c_2) - \lambda_2 g_a = 0 \tag{3-11}$$

$$- (c_o \cdot w_o + \Pi \cdot c_1)/ (c_1 L + c_2) - \lambda_1 f_L = 0 \tag{3-12}$$

$$p/(c_1 L + c_2) + \lambda_1 + \lambda_2 = 0 \tag{3-13}$$

where alphabetical indices represent partial differentiation.

Division of (3-11) through (3-10) and using (3-8) gives us

$$g + g_p/g_a = 0 \tag{3-14}$$

The optimal condition (3-14) may be expressed in terms of elasticities, defined as:

$$\eta =- pg_p/g \tag{3-15}$$

$$\Theta = ag_a/g \tag{3-16}$$

where η is the price elasticity of demand, and Θ is the advertising elasticity of demand.

(3-14), (3-15) and (3-16) give us

$$\frac{a}{pg} = \frac{\Theta}{\eta} \tag{3-17}$$

which represents a statement of the Dorfman-Steiner theorem (Dorfman & Steiner, 1954).

(3-17) states that the ratio of advertising to sales is equal to the quotient of the advertising elasticity and the price elasticity.

It is interesting to note that the Dorfman-Steiner theorem is unaltered in the case of a pure labor-managed firm compared with the case of a profit-maximizing firm. This follows from (3-17), as (3-17) is independent of the constants c_o, c_1 and c_2. This similarity stems from the general aim in both types of firms for the efficient use of non-labor resources (Vanek, 1970).

Although the Dorfman-Steiner theorem is applicable to both types of firm, the optimal advertising/sales ratio will most likely differ because the quotient of the elasticities Θ and η will normally differ at the respective optimums of the two types of firm.

However, for constant price and advertising elasticities η and Θ, the relative advertising behavior - advertising/sales - is the same for both types of firm, from (3-17). Thus this is an important case, in which the social favorableness of the labor-managed firm claimed by Vanek (1970, pp. 120-123) does not necessarily hold. Meade (1970, p. 413, fn. 2) and Ireland & Law (1977) have also questioned Vanek's (1970) position in this respect.

The respective optimums may be calculated by solving the system of equations (3-7)-(3-8) and (3-10)-(3-13).

Insert (3-10) and (3-12) into (3-13),

$$c_o w_o + \Pi\, c_1 = (p + g/g_p)\cdot f_L \qquad (3\text{-}18)$$

which represents the regular marginal value product condition that an input factor, in this case labor, is employed up to the point where its cost rate is equal to its marginal value product.

(3-7) and (3-8) give us

$$f(K,L) = g(p,a) \qquad (3\text{-}19)$$

Thus we have three equations (3-14), (3-18) and (3-19) and three unknown variables p, a and L. First solve L from (3-19) and insert L into (3-18), after which only two equations (3-14) and (3-18) and the two unknown variables p and a remain. In general a solution may therefore be found.

A requirement for a maximum is also that second-order sufficiency conditions are satisfied. This is analyzed further by Ireland & Law (1977).

3.1.2 Dynamic Case

It is now interesting to ask whether a dynamic counterpart to the Dorfman-Steiner theorem can be found for the labor-managed firm.

3.1.2.1 Dynamic Advertising Model

To try to answer this question, the same formulations as in the static case above may be used. However, we must also take account of the carry-over effect of advertising,

$$\dot{A} = a - bA \tag{3-20}$$

where

A = accumulated goodwill of past advertising [1)]

b = the decay rate of A if no new advertising (a) is undertaken.

The objective in this case can be written

$$\underset{p,a,L,q}{\text{Max}} \int_0^\infty e^{-\rho t}\, \Pi \, dt \tag{3-21}$$

subject to (3-3), (3-7), (3-8) and (3-20). ρ is the discount rate of the labor collective and capital K is assumed constant.

The assumption that capital K is constant is questionable in a dynamic analysis over a long time period. However, the main aim is to analyze advertising *ceteris paribus* and to provide a comparative advertising study of the labor-

1) Schmalensee (1972, Chapter 2) has pointed out the difficulty of observing advertising goodwill A, and therefore has proposed other types of dynamic advertising models.

managed and profit-maximizing firm, rather than to incorporate "best" possible assumptions. Also, the analysis here is explorative and may be extended to account for other assumptions at a later stage.

The dynamic objective (3-21) is a somewhat arbitrary, yet plausible, objective of a pure labor-managed firm. The choice of it is not intended to give us the "best" objective. It is rather a first exploratory step. The dynamic objective (3-21) is supported by the studies of Atkinson (1973) and Litt, Steinherr & Thisse (1975), who use objectives similar in form. In the Appendix of this chapter we have mentioned some alternative dynamic objectives of a pure labor-managed firm.

Following Pontryagin's Maximum Principle (Pontryagin et al., 1962) the Hamiltonian H is

$$H \cdot e^{\rho t} = \Pi(p,a,L,q) + \lambda_1\left[q-f(K,L)\right] + \lambda_2\left[q-g(p,a,A)\right] + \\ + \lambda_3(a-bA) \tag{3-22}$$

where demand in the dynamic case is also dependent on the accumulated advertising goodwill A, and λ_3 is the adjoint variable.

p, a, L and q are control variables and A is a state variable. The necessary conditions for a maximum are, in correspondance to (3-10)-(3-13),

$$q/(c_1L + c_2) - \lambda_2 g_p = 0 \tag{3-23}$$

$$- 1/(c_1L + c_2) - \lambda_2 g_a + \lambda_3 = 0 \tag{3-24}$$

$$- (c_o w_o + \Pi c_1)/(c_1L + c_2) - \lambda_1 f_L = 0 \tag{3-25}$$

$$p/(c_1L + c_2) + \lambda_1 + \lambda_2 = 0 \tag{3-26}$$

$$- \lambda_2 g_A - \lambda_3 b = \rho\lambda_3 - \dot{\lambda}_3 \tag{3-27}$$

where expression (3-27) is new in comparison with the static case and represents the necessary condition for the adjoint variable λ_3. λ_3 has the interesting property of representing the imputed value of an extra unit of advertising goodwill A.

The transversality conditions are

$$\lim_{t\to\infty} e^{-\rho t}\cdot\lambda_3(t) \geqslant 0 \tag{3-27a}$$

$$\lim_{t\to\infty} e^{-\rho t}\cdot\lambda_3(t)\cdot A(t) = 0 \tag{3-27b}$$

and the initial condition is assumed to be

$$A(0) = A_o \tag{3-27c}$$

where A_o is a positive constant.

We shall now study the case of a profit-maximizing firm and thereafter study the case of a labor-managed firm.

3.1.2.2 Profit-Maximizing Firm

In the case of a profit-maximizing firm $c_o = c_2 = 1$, $c_1 = 0$ and (3-23) and (3-24) give us

$$\frac{a}{pq} = \frac{\theta}{\eta(1-\lambda_3)} \tag{3-28}$$

By inserting (3-23) into (3-27), we obtain

$$\frac{A}{pq} = \frac{\varepsilon}{\eta\left[\lambda_3(\rho+b)-\dot{\lambda}_3\right]} \tag{3-29}$$

where $\varepsilon = Ag_A/g$ is the elasticity of demand with respect to advertising goodwill.

(3-28) is a dynamic counterpart to the Dorfman-Steiner theorem (Dorfman & Steiner, 1954) and has been developed by Jacquemin (1973). $\theta/(1-\lambda_3)$ in (3-28) may be conceived of as the long-term advertising elasticity of demand, which for the normal case of $0 < \lambda_3 < 1$ is greater than the short-term elasticity θ, as is to be expected.

(3-29) is an extension of the Nerlove-Arrow theorem (Nerlove & Arrow, 1962), which treats the special case of $\lambda_3 = 1$. This generalization of the Nerlove-Arrow theorem for the profit-maximizing firm (3-29) has been developed by Jacquemin (1973). (3-29) states that the ratio of advertising goodwill to sales is equal to the quotient of the elasticities of advertising goodwill and price. However, the elasticity of advertising goodwill is adjusted by the factor $1/[\lambda_3(\rho+b)-\dot{\lambda}_3]$, representing the effect of a non-unit shadow value for advertising goodwill.

3.1.2.3 Labor-Managed Firm

In the case of a labor-managed firm $c_o = c_2 = 0$, $c_1 = 1$ and (3-23) and (3-24) give us

$$\frac{a}{pq} = \frac{\Theta}{\eta(1-\lambda_3 L)} \tag{3-30}$$

By inserting (3-23) into (3-27), we obtain

$$\frac{A}{pq} = \frac{\varepsilon}{\eta L[\lambda_3(\rho+b)-\dot{\lambda}_3]} \tag{3-31}$$

In other words, in contrast to the static case, the Dorfman-Steiner theorem (3-28) is modified in the dynamic case for a labor-managed firm (3-30).

The extended Nerlove-Arrow theorem (3-29) is also modified for a labor-managed firm (3-31).

The modified theorems (3-30) and (3-31) are not so unexpected because $\lambda_3 L$ in the case of the labor-managed firm corresponds formally to λ_3 in the case of the profit-maximizing firm, due to the L in the denominator of the objective function in the labor-managed case.

As in the static case, the formal similarity between (3-28) and (3-30) and between (3-29) and (3-31) does not mean that the advertising/sales ratio and advertising goodwill/sales ratio, respectively, are equal for the two types of firm, since optimal elasticities, labor force and the adjoint variable may take on different values for the two types of firm.

As in the static case, the optimal solution may be calculated from the necessary conditions and constraints above. However, they now involve differential equations over time instead of variables constant in time. This may complicate an analytical solution considerably.

3.1.2.4 Steady-State Solution

After a certain passage of time, a steady-state solution will probably be obtained in the optimum. This steady state solution could be expected to have properties similar to the static case.

To deduce the steady-state solution we let $\dot{A} = 0$ and $\dot{\lambda} = 0$. The necessary conditions (3-23)-(3-27) and transversality conditions (3-27a)-(3-27b) are satisfied in this case. By eliminating the adjoint variable from (3-28)-(3-29) and (3-30)-(3-31) respectively, the same expression is obtained in the PM-case as in the LM-case, namely

$$\frac{a}{pq} = \frac{1}{\eta} \{\theta + \frac{\varepsilon}{1+\rho/b}\} \qquad (3\text{-}32)$$

where the expression in parentheses is the "enlarged" long-term elasticity of advertising.

As expected, the steady-state solution (3-32) - due to the similarity with the static case - is the same, formally, for the profit-maximizing and the labor-managed firm. This formal similarity most likely stems from the general aim in both types of firm to use non-labor resources efficiently.

Although (3-32) is the same in form for both the PM-firm and LM-firm the actual value of the advertising/sales ratio is different because the optimum elasticity values η, θ and ε are different in the two cases. However, for a demand function with constant elasticities with respect to price, advertising and goodwill, (3-32) implies the same steady-state advertising/sales ratio for the labor-managed firm as for the profit-maximizing firm.

(3-32) implies the interesting result that even though the firm is in a static situation, the regular static Dorfman-Steiner theorem is not valid. This is due to the assumed carry-over effect of advertising and the dynamic model formulation. For $\varepsilon = 0$, which means that demand is unaffected by previous advertising, (3-32) is reduced to the regular static Dorfman-Steiner theorem (3-17) as expected.

The steady-state advertising/sales ratio in (3-32) is positively influenced by advertising elasticity Θ, goodwill elasticity ε and goodwill decay coefficient b and negatively influenced by price elasticity η and discount rate ρ. This is as we may expect.

3.1.2.5 Summary and Comments

A static and a dynamic microeconomic model of the optimal advertising behavior of a labor-managed firm have been discussed in Section 3.1 of this chapter. The optimal advertising behavior of a labor-managed firm was also compared with that of a profit-maximizing firm.

An interesting conclusion was that the static Dorfman-Steiner theorem is valid also for the labor-managed firm. It came to our knowledge after the contents of this chapter was written that especially Ireland & Law (1977) and also Steinherr (1975) had performed a similar static analysis. However, it seems that neither author has proceeded to a dynamic analysis as in this chapter.

The dynamic Dorfman-Steiner and extended Nerlove-Arrow theorems are valid for the labor-managed firm provided that $\lambda_3 L$ for constant L is substituted for λ_3. λ_3 is the dynamic shadow value for advertising goodwill and L is labor quantity. However, this similarity in mathematical form does not imply similarity in magnitude. Rather, in the general case, because of different optimum elasticities, goodwill shadow value and employment, we can expect different optimal dynamic advertising behavior for the two types of firm.

As expected, the steady-state advertising/sales ratio - due to the similarity with the static case - is the same, formally, for the profit-maximizing and the labor-managed firm. Furthermore for a demand function with constant elasticity with respect to price, advertising and goodwill, the steady-state advertising/sales ratio is the same for the labor-managed firm as for the profit-maximizing firm. The steady-state advertising/sales ratio is greater than the static one, for constant elasticities of demand, because of the carry-over effect of advertising.

The results are limited by the assumptions of the analysis, such as regarding the objective function of the labor-managed firm (cf. Section 3.1.2.1 and the Appendix). The assumption that the labor-managed firm and the profit-maximizing firm have the same discount rate may be questioned (Atkinson, 1973 and 1975, Steinherr & Peer, 1975). For instance if, as is often argued, the rate of time preference is higher for labor than for capital owners then future income streams may be more heavily disounted in a labor-managed firm (Litt, Steinherr & Thisse, 1975, p. 5).

Some of the results are dependent on the questionable assumption that both firms have the same level of capital input and are subject to the same production function. Differences in the economic environment of the two types of firm might further complicate the picture.

The assumptions of this study are used mainly for *ceteris paribus* reasons and because we are interested in a comparative analysis of two types of firm, rather than in the "best" possible assumptions for one type of firm. Another reason for using simplified assumptions is that the analysis is of an explorative nature. As a next step, the assumptions used can be modified and the results compared to those found here. The analysis in this study may be considered a first approximation to such extended studies.

3.2 SPECIAL ANALYSIS

From the general results deduced in Section 3.1 it is difficult to get a direct idea of the difference in optimal dynamic advertising behavior between a labor-managed firm and a profit-maximizing firm. We shall therefore assume specific demand and production functions in order to gain some further insight into the matter. However, we shall first make some small modifications in the dynamic model used above.

First, since capital is assumed to be fixed, it will be appropriate to assume a finite instead of an infinite time horizon. Second, the structure and analysis of the model can be simplified, without loss of generality, by assuming advertising and labor rather than advertising and price to be independent variables. Thus, in our modified model, advertising expenses allow the firm to charge a higher price, while output level is determined independently by optimal labor input.

3.2.1 Demand Function

In order to keep the analysis to manageable proportions, we assume a simple price-elastic revenue function that is combined additively and multiplicatively separable, and which can be written

$$R = pq = \beta_1 q^{\alpha_1} A^{\alpha_2} + \beta_2 a^{\alpha_3} \tag{3-33}$$

where

R	= revenue
p	= price of output
q	= output quantity
a	= current advertising expenditure, $a > 0$
A	= accumulated advertising goodwill
$\alpha_1, \alpha_2, \alpha_3$	= positive coefficients less than unity
β_1, β_2	= positive coefficients.

This is equivalent to a demand function of the conventional price-elastic decreasing hyperbolic type:

$$p = \beta_1 q^{\alpha_1 - 1} A^{\alpha_2} + \beta_2 a^{\alpha_3}/q \tag{3-34}$$

Changes in advertising a and advertising stock A are equivalent to a shift in the demand curve.

3.2.2 Static Case

We shall first study the static case and assume a revenue function such as in (3-33). However, we shall let A be equal to an exogenously given constant A_s.

Following (3-33) the profit function (3-1) is then

$$\Pi_{PM} = \beta_1 q^{\alpha_1} A_s^{\alpha_2} + \beta_2 a^{\alpha_3} - rK_o - a - w_o L \tag{3-36}$$

where

w_o = wage per labor quantity

$q = q(L)$ = output quantity, $q(0) = 0$

r = user cost of capital

K_o = capital

It is assumed that output increases with labor but at a decreasing rate:

$$q_L > 0 \text{ and } q_{LL} < 0 \tag{3-36a}$$

Capital is assumed to be an exogenously given constant K_o.

Setting

$$\frac{\partial \Pi_{PM}}{\partial a} = 0$$

the optimum advertising outlay is

$$a_{PM} = (\beta_2\alpha_3)^{\frac{1}{1-\alpha_3}} \tag{3-37}$$

Due to diminishing marginal revenue products of labor and advertising, and the separability of the profit function (3-36), sufficiency conditions for a maximum are satisfied. We can also see this by partially differentiating (3-36) twice with respect to a and L:

$$\frac{\partial^2 \Pi_{PM}}{\partial a^2} = -\beta_2\alpha_3(1-\alpha_3)\cdot a^{\alpha_3-2} < 0 \tag{3-37a}$$

$$\frac{\partial^2 \Pi_{PM}}{\partial L^2} = -\beta_1\alpha_1(1-\alpha_1)q^{\alpha_1-2} q_L^2 A_s^{\alpha_2} + \beta_1\alpha_1 q^{\alpha_1-1} q_{LL} A_s^{\alpha_2} < 0 \tag{3-37b}$$

$$\frac{\partial^2 \Pi_{PM}}{\partial L \partial a} = 0 \tag{3-37c}$$

In the case of the labor-managed firm, the objective function (3-2) can be written

$$\Pi_{LM} = \frac{\beta_1 q^{\alpha_1} A_s^{\alpha_2} + \beta_2 a^{\alpha_3} - rK_o - a}{L} \tag{3-38}$$

and, because of the separability of the revenue function (3-33), the same partial optimum as in the profit-maximizing case is obtained

$$a_{LM} = (\beta_2 \alpha_3)^{\frac{1}{1-\alpha_3}} \tag{3-39}$$

To check sufficiency conditions, we shall analyze the optimum labor quantity.

By equating the marginal revenue product of labor with the average labor income, or using normal first-order conditions, we can find the optimum labor quantity. From (3-38), we get

$$\frac{\partial \Pi_{LM}}{\partial L} = \frac{1}{L} (\beta_1 \alpha_1 A_s^{\alpha_2} q^{\alpha_1 - 1} q_L - \Pi_{LM}) \tag{3-40}$$

Thus, by letting this equal zero, we obtain the following conditon for the optimum labor quantity:

$$\beta_1 \alpha_1 A_s^{\alpha_2} q^{\alpha_1 - 1} q_L = \Pi_{LM} \tag{3-41}$$

By inserting the advertising optimum (3-39), which is independent of L, into (3-38) we obtain from (3-40) at optimum

$$\frac{\partial^2 \Pi_{LM}}{\partial L^2} = \frac{1}{L} \Big[-\beta_1 \alpha_1 (1-\alpha_1) A_s^{\alpha_2} q^{\alpha_1 - 2} q_L^2 +$$

$$+ \beta_1 \alpha_1 A_s^{\alpha_2} q^{\alpha_1 - 1} q_{LL} - 2 \frac{\partial \Pi_{LM}}{\partial L} \Big] < 0 \tag{3-42}$$

which ensures that the sufficiency condition for a local maximum is satisfied.

However, a corner optimum at L = 0 is in principle possible if the function

$$c(a) = a - \beta_2 a^{\alpha_3} + r\,K_o \tag{3-43}$$

is negative in which case Π_{LM} tends to infinity.

c(a) has a minimum at

$$a_{LM} = (\beta_2 \alpha_3)^{\frac{1}{1-\alpha_3}}$$

and

$$c(a_{LM}) = r\,K_o - (\beta_2 \cdot a_{LM}^{\alpha_3} - a_{LM}).$$

In order to have a specific well-behaved case, we shall assume that

$$r\,K_o > \beta_2\, a_{LM}^{\alpha_3} - a_{LM} \tag{3-44}$$

for which the local maximum derived above is also a global maximum. Assumption (3-44) implies that, at optimum, the capital rent is greater than a function of the optimal advertising outlay. By moving the term a_{LM} to the left-hand side of (3-44), we see that (3-44) is equivalent to assuming that the current advertising term in the revenue function (3-33) is less than capital rent and advertising expenditure at optimum.

In this special case, the advertising optimum for the labor-managed firm is the same as for the profit-maximizing firm according to (3-37) and (3-39). Normally, however, the labor-managed firm optimally advertises less in the short run than the profit-maximizing firm (e.g. Vanek, 1970, pp. 120-123).

However, it is also interesting to analyze the relative advertising behavior, i.e. advertising in relation to output and revenue (cf. Meade, 1972, p. 413, fn. 2, Steinherr, 1975 and Ireland & Law, 1977).

From (3-36) and (3-38) we have

$$\frac{\Pi_{PM}}{L} = \Pi_{LM} - w_o \qquad (3\text{-}45)$$

At the optimum for Π_{PM}, a small decrease in employment L causes the left-hand side of (3-45) to increase, because Π_{PM} changes very little in the neighborhood of its optimum. As w_o is a constant, (3-45) implies that Π_{LM} increases and thus in the case of unique maximums that

$$L_{LM}^{opt} < L_{PM}^{opt} \qquad (3\text{-}46)$$

Thus the optimum labor-managed monopolistic firm is smaller, in the short-run, than the optimum profit-maximizing monopolistic firm. However, this is only valid when $\Pi_{PM} > 0$ at the PM-optimum. In the reverse case, the opposite of (3-46) is valid, as can be seen from (3-45).

From (3-33), (3-36a), (3-37), (3-39) and (3-46) we find that

$$\left[\frac{a}{q}\right]_{LM}^{opt} > \left[\frac{a}{q}\right]_{PM}^{opt} \qquad (3\text{-}47)$$

$$\left[\frac{a}{pq}\right]_{LM}^{opt} > \left[\frac{a}{pq}\right]_{PM}^{opt} \qquad (3\text{-}48)$$

This means that the monopolistic labor-managed firm advertises more heavily in relation to output and revenue than the monopolistic profit-maximizing firm (Ireland & Law, 1977, p. 233). However, this was deduced for our special case where the optimum advertising levels for the two types of firms are equal, from (3-37) and (3-39). This in turn is because the revenue function (3-33) is additively separable in current advertising expenditure a.

In the more general case

$$a_{LM}^{opt} < a_{PM}^{opt} \qquad (3\text{-}49)$$

as is discussed in e.g. Vanek (1970, pp. 120-123).

As regards the relative advertising behavior in the more general case, (3-46) suggests that it is an open question which type of firm advertises most heavily. This depends in each case on the specific demand function, as Ireland & Law (1977) have elaborated on. Steinherr (1975) has shown that for many normal demand functions the opposite of (3-47) and (3-48) is valid.

Let us now turn to the dynamic case.

3.2.3 Dynamic Case

Before we go into the dynamics of the *special* case, we shall state the *general* model of Section 3.1 in a form corresponding to the framework of this Section 3.2.

3.2.3.1 Dynamic Advertising Model

The dynamic advertising problem we shall study is

$$\max_{a,L} \int_{o}^{T} e^{-\rho t}\, \Pi \, dt \tag{3-50}$$

subject to

$$\dot{A} = a - bA \tag{3-51}$$

$$A(0) = A_o$$

where

ρ = discount rate of the labor collective

b = depreciation of advertising goodwill

A_o = positive constant

and where the goal function Π is different in the profit-maximizing Π_{PM} and labor-managed firm Π_{LM} case.

We shall, as before, derive the necessary conditions for an optimum by using Pontryagin's Maximum Principle.

The Hamiltonian H for our problem (3-50) and (3-51) can be written

$$He^{\rho t} = \Pi + \lambda(a - bA) \tag{3-52}$$

where current advertising a and labor L are control variables, advertising goodwill A is state variable and λ is the adjoint variable.

By letting $H_a = 0$ and $H_L = 0$ we obtain necessary conditions for the control variables:

$$\Pi_a = -\lambda^{1)} \tag{3-53}$$

$$\Pi_L = 0 \tag{3-54}$$

Also by letting

$$H_A = -\frac{d}{dt}(\lambda e^{-\rho t}) \tag{3-55}$$

we obtain the necessary condition for the state variable

$$-\dot{\lambda} = \Pi_A - (\rho+b)\lambda \tag{3-56}$$

To interprete (3-56), we note that the shadow value λ embodies at time t the future earnings power of an additional unit of advertising goodwill at time t. This earnings power decreases, according to (3-56), at the rate that marginal income minus interest and depreciation are generated. This corresponds to the optimum relation for an investment in capital goods.

Optimal a, L, A and λ time functions may be formally derived from the four independent equations (3-51), (3-53), (3-54) and (3-56). However, it is difficult in practice to derive an analytical solution even with simple demand and production functions in Π. We can nevertheless analyze the solution a little further by assuming a revenue function of the type we have in (3-33).

3.2.3.2 Profit-Maximizing Firm

For the profit-maximizing firm our special case implies that

$$\Pi_{PM} = \beta_1 q^{\alpha_1} A^{\alpha_2} + \beta_2 a^{\alpha_3} - a - w_o L \tag{3-57}$$

---

1) This result (3-53) is interesting because it states that, for $\lambda > 0$, the marginal revenue of advertising is *less* than marginal cost. Thus, the *dynamic* effect of advertising offers an alternative hypothesis to the sales maximization theory of Baumol (1959). Cf. Jacquemin (1973, pp. 202-203).

Unlike the static case (3-36), we have now omitted the constant capital rent rK_o for the sake of simplicity, because it does not affect the optimum state and control.

From (3-54) follows that a necessary condition for a maximum is that

$$\frac{\partial \Pi_{PM}}{\partial L} = \beta_1 \alpha_1 q^{\alpha_1 - 1} q_L A^{\alpha_2} - w_o = 0 \tag{3-58}$$

As the first term in (3-58) is decreasing in L, the partial sufficiency condition for a maximum in L is satisfied. (3-58) gives L as an implicit function of A. By formally inserting this into (3-57) we get Π_{PM} as a function of the advertising variables a and A.

3.2.3.3 Labor-Managed Firm

For the labor-managed firm our special case implies, in analogy to (3-38), that

$$\Pi_{LM} = \frac{\beta_1 q^{\alpha_1} A^{\alpha_2} + \beta_2 a^{\alpha_3} - r K_o - a}{L} \tag{3-59}$$

where capital K_o is an exogenously given constant.

From (3-54) follows that a necessary condition for a maximum is that

$$\frac{\partial \Pi_{LM}}{\partial L} = \frac{1}{L} (\beta_1 \alpha_1 q^{\alpha_1 - 1} q_L A^{\alpha_2} - \Pi_{LM}) = 0 \tag{3-60}$$

The partial sufficiency condition for labor is satisfied which is confirmed by the similarity of (3-60) and (3-40), and the result of (3-42).

(3-60) gives L as an implicit function of a and A. By formally inserting this into (3-59) we get Π_{LM} as a function of the advertising variables a and A.

3.2.3.4 Advertising at the Horizon Date

From (3-53), (3-57) and (3-59) we obtain the necessary conditions for current advertising in the PM-case and LM-case respectively, i.e.

$$\lambda_{PM} = 1 - \beta_2\alpha_3 \cdot a^{\alpha_3 - 1} \tag{3-61}$$

$$\lambda_{LM} = \frac{1}{L}\,(1 - \beta_2\alpha_3 \cdot a^{\alpha_3 - 1}) \tag{3-62}$$

The transversality condition for our problem is that

$$\lambda_{PM}\,(T) = 0 \tag{3-63}$$

$$\lambda_{LM}\,(T) = 0 \tag{3-64}$$

This implies, following (3-61)-(3-64), that

$$a_{PM}\,(T) = (\beta_2\alpha_3)^{\frac{1}{1-\alpha_3}} \tag{3-65}$$

$$a_{LM}\,(T) = (\beta_2\alpha_3)^{\frac{1}{1-\alpha_3}} \tag{3-66}$$

which, as expected, coincide with the static solutions (3-37) and (3-39).

We can also calculate analytically the stationary part of the trajectories by letting $\dot{A} = 0$ in (3-51) and $\dot{\lambda} = 0$ in (3-56). However, instead we prefer to further specify our model in order to obtain some more detailed results on the complete optimal time-trajectories.

3.2.3.5 Production Function

Although there are many well-known production functions in the literature, such as Cobb-Douglas, CES (Constant Elasticity of Substitution) and others[1)], we shall choose the following one:

$$q = f_1\,(K) \cdot \frac{L}{1+L} \tag{3-67}$$

$$q_L > 0,\ q_{LL} < 0 \tag{3-68}$$

This type of microeconomic production function implies that, for a given capital input K_o, production is technologically limited to $f_1\,(K_o)$, or q_m as we shall denote it, no matter how much labor is employed. Other production func-

1) Cf. for instance Naylor & Vernon (1969, p. 81).

tions theoretically allow unlimited output even in the short run, i.e. for constant capital, (3-67) does not. Technological development is assumed away in (3-67).

The reason for choosing the relatively simple production function (3-67) is that it is specific and keeps the analysis to relatively manageable proportions without severely limiting the generality of the results. Also, originally the idea was to assume a production function limited from above in order to eliminate possible piecewise infinite optimal solutions. As a next step, other production functions may be used, but as our main interest is in a comparative analysis of the profit-maximizing and labor-managed firm, the comparative results may not differ all too much because of the choice of production function.

3.2.3.6 Optimum Conditions

We shall now proceed to specify the optimum labor conditions and the income functions, first for the labor-managed firm and then for the profit-maximizing firm.

By inserting the production function (3-67) into (3-60), we obtain for the labor-managed firm the necessary labor condition

$$c(a) \cdot A^{-\alpha_2} = \beta_1 q_m^{\alpha_1} \frac{L^{\alpha_1}(1-\alpha_1+L)}{(1+L)^{\alpha_1+1}} \tag{3-69}$$

c(a) is defined in (3-43).

Since the right-hand side of (3-69) is an increasing function of L, and since (3-69) represents

$$\frac{\partial \Pi_{LM}}{\partial L}$$

we get

$$\frac{\partial^2 \Pi_{LM}}{\partial L^2} < 0$$

and thus the partial sufficiency condition for labor is satisfied.

Using (3-69), the goal function (3-59) of the labor-managed firm can be simplified thus

$$\Pi_{LM} = \alpha_1 \cdot \frac{c(a)}{L(1-\alpha_1 + L)} \tag{3-70}$$

(3-69) gives the necessary condition for a partial local optimum in employment for the labor-managed firm, and gives labor as an implicit function of the advertising variables. This implicit function can, as mentioned above, be used in principle to express the goal function Π_{LM} as a function of the advertising variables a and A only.

Let us also specify more closely the optimum conditions for the *profit-maximizing* firm.

After inserting the production function (3-67) into the optimum condition (3-58), we obtain

$$\beta_1 \alpha_1 q_m^{\alpha_1} A^{\alpha_2} = w_o L^{1-\alpha_1} \cdot (1+L)^{\alpha_1+1} \tag{3-71}$$

We can also simplify the profit function (3-57) somewhat by inserting (3-71) into (3-57), such that

$$\Pi_{PM} = \frac{w_o}{\alpha_1} (1-\alpha_1+L)L + \beta_2 a^{\alpha_3} - a \tag{3-72}$$

The expressions (3-69)-(3-72) are of interest for calculations later on, but first we shall briefly discuss some sufficiency conditions.

We shall not deduce comprehensive sufficiency conditions here, but only mention three plausible aspects. First, in order to have a well-behaved limited interior optimum we may expect, from (3-59), that $\alpha_1 + \alpha_2 < 1$. Second, we may assume, for the same reason, the exponent of advertising goodwill α_2 to be small. Third, in order to avoid a corner optimum at L = 0 for the labor-managed firm, we assume (3-44) to be valid.

In order to analyze our problem further and come to detailed results regarding the optimal time-trajectories, we shall now study a specific example.

3.2.3.7 Specific Example

We shall specify some parameter values and proceed to a numerical solution.

Numerical Solution

Algorithms based on the idea of differential dynamic programming [1)] seem appropriate to our case with, basically, one state variable and one control variable.

Computer tests were run with the following parameters for the demand function:

$\alpha_1 = \frac{1}{2}$ $\beta_1 = 1$

$\alpha_2 = \frac{1}{8}$ $\beta_2 = 1$

$\alpha_3 = \frac{1}{2}$

and with the following other parameters:

$T = 60$ $\rho = 0{,}1$

$b = 0{,}2$ $A_o = 0{,}75$

$q_m = 1$

For the profit-maximizing firm we further assume $w_o = 1$ and for the labor-managed firm that $rK_o = 0{,}45$, which satisfies (3-44).

Using (3-50), (3-51), (3-69) and (3-70) we obtain the optimum trajectories shown in Figures 3:1 - 3:4 for the labor-managed firm.

For the profit-maximizing firm, the computer tests were run with the expression (3-50), (3-51), (3-71) and (3-72), which resulted in the trajectories to be found in Figures 3:1 - 3:4.

Labor was found to be $L_{LM} = 0{,}134$ and $L_{PM} = 0{,}173$ in the steady-state phase, as expected from (3-46), and thus, following (3-67), output quantity is well below its assumed

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1) Cf. Jacobson & Mayne (1970) and Johnsson & Nordlund (1974).

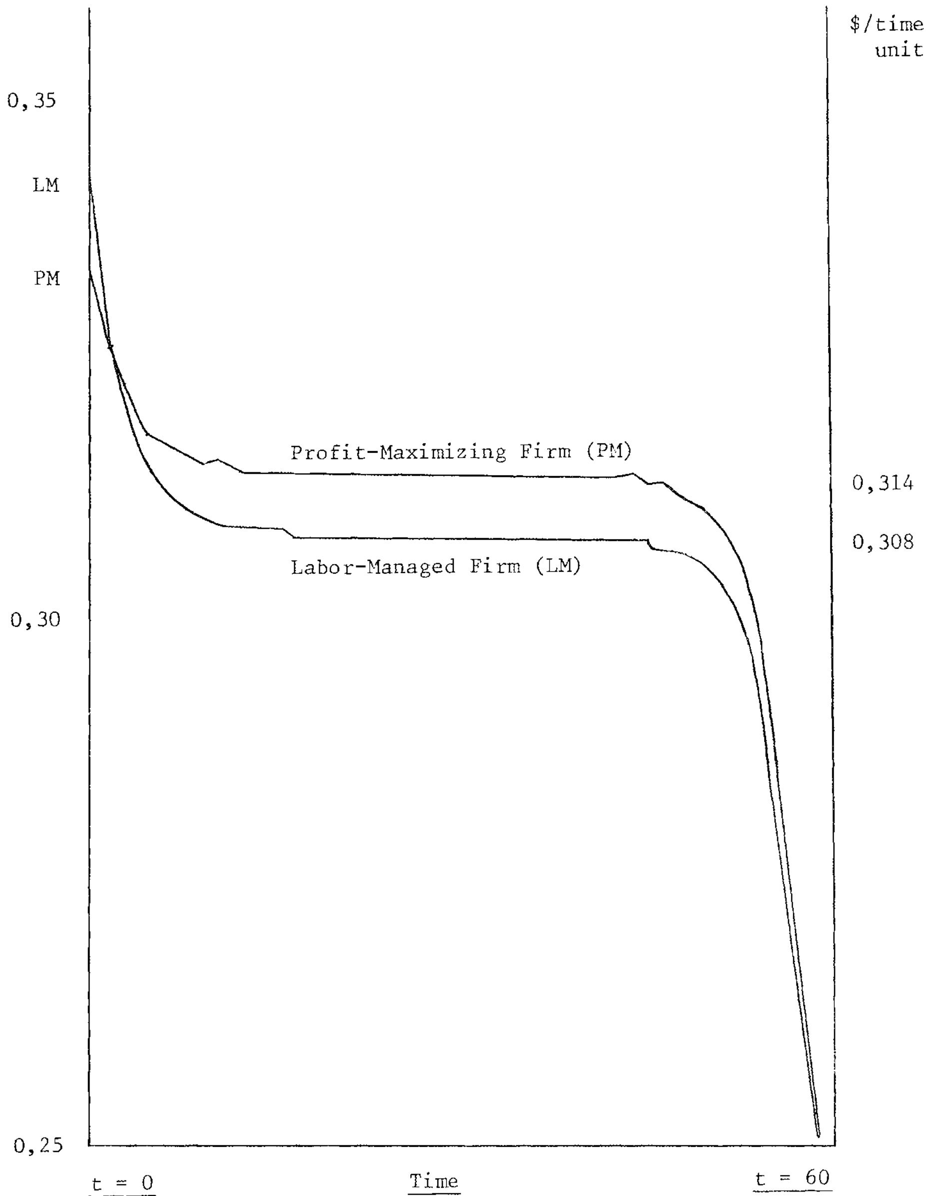

Figure 3:1. Optimal Advertising Costs a. Computer-Calculated Curves. $r\ K_o = 0{,}45$ and $w_o = 1$.

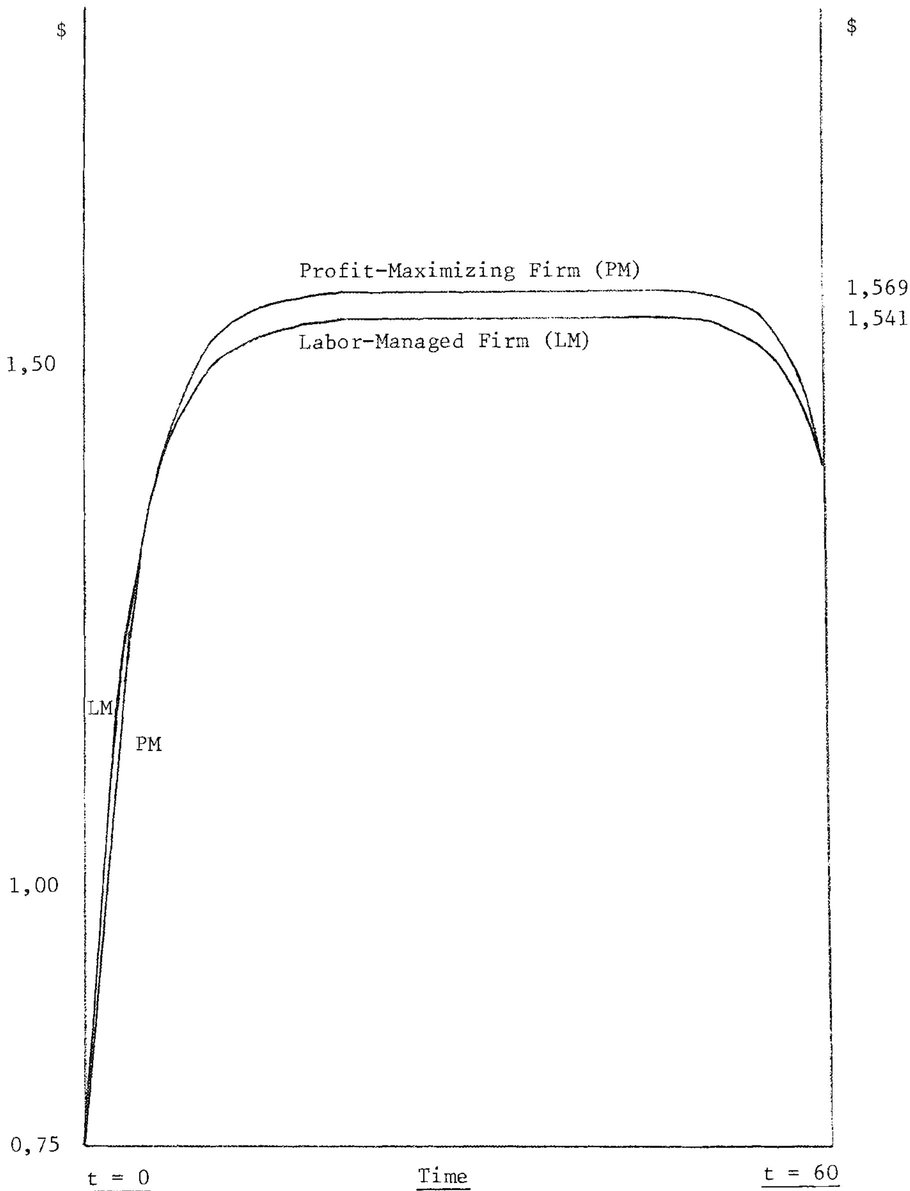

Figure 3:2. Optimal Advertising Goodwill A. Computer-Calculated Curves. r K_o = 0,45 and w_o = 1.

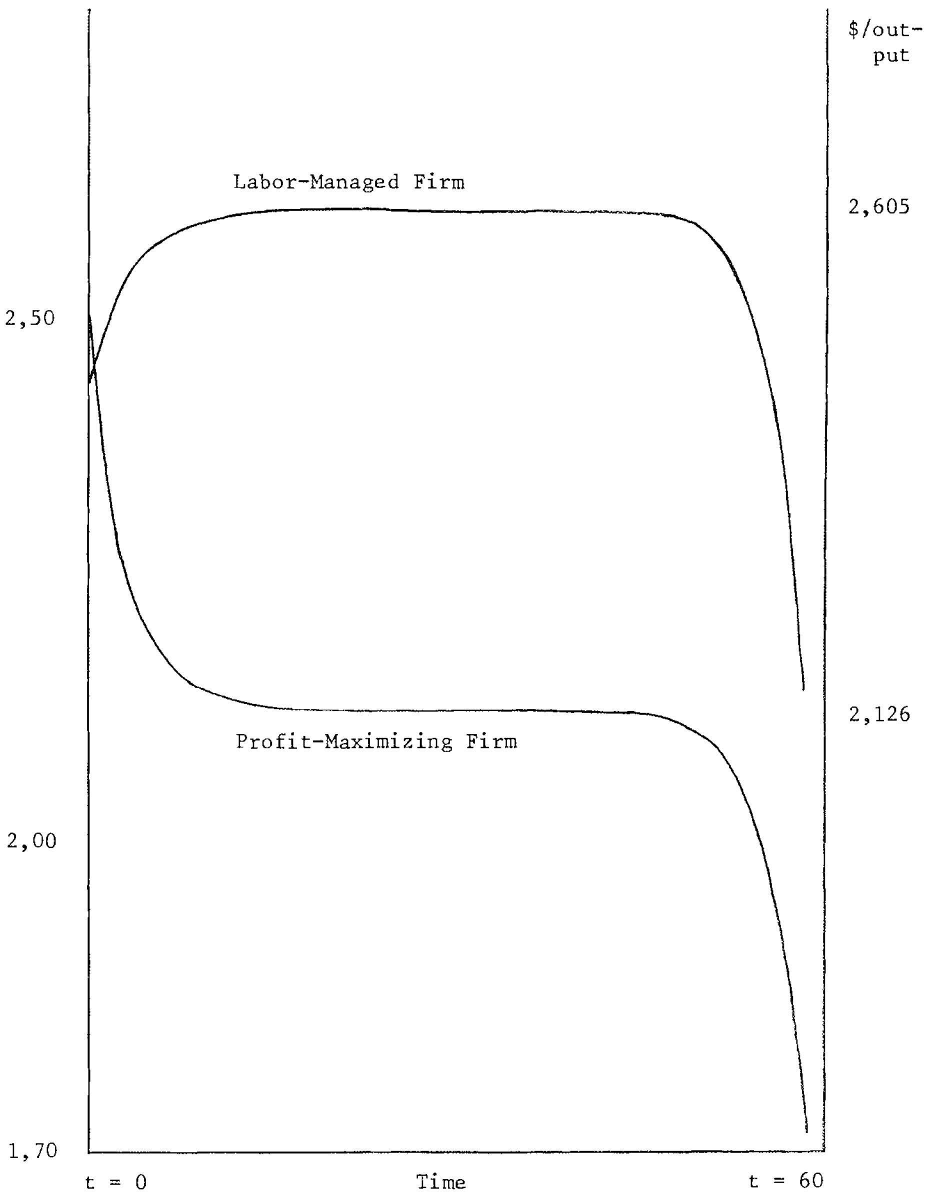

Figure 3:3. Optimal Advertising/Output Ratio $\frac{a}{q}$. Computer-Calculated Curves. r K_o = 0,45 and w_o = 1.

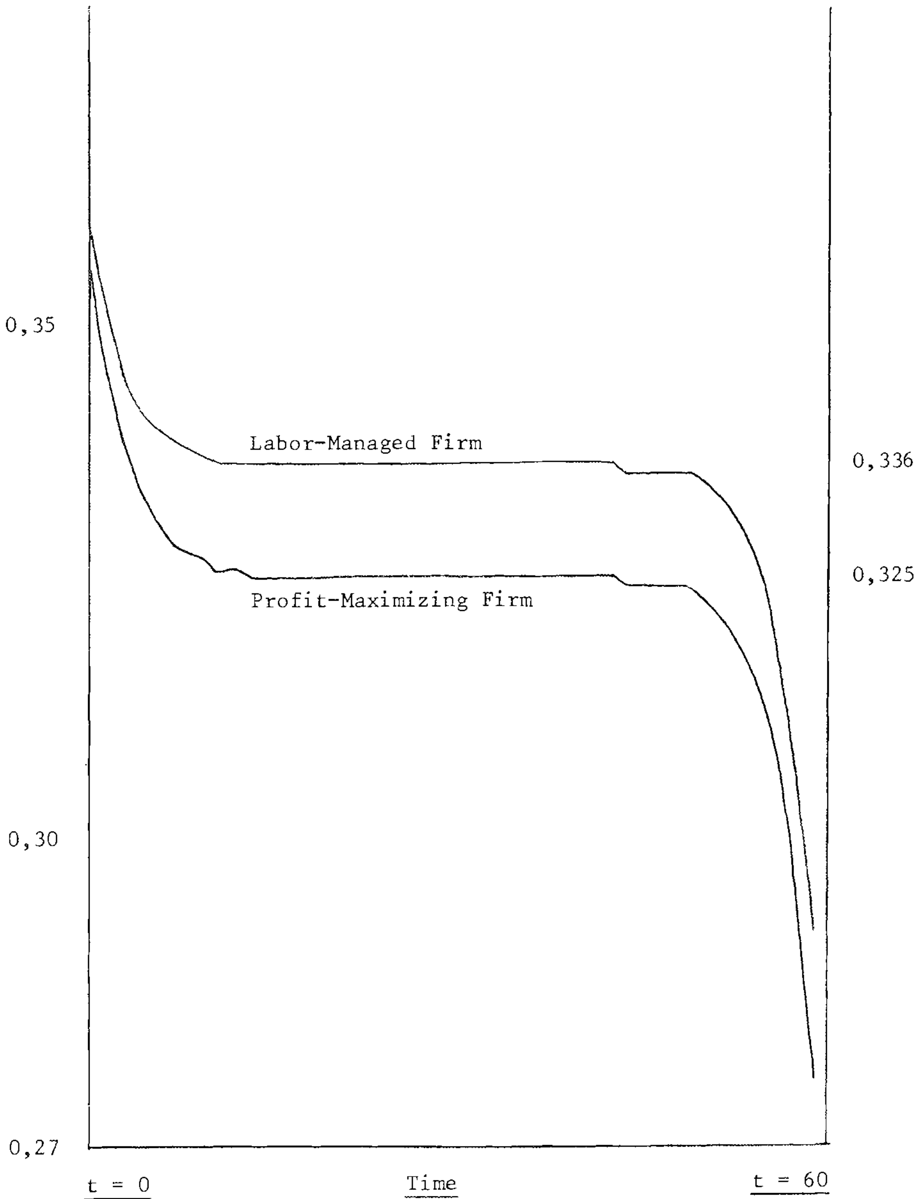

Figure 3:4. Optimal Advertising/Sales Ratio $\frac{a}{pq}$. Computer-Calculated Curves. r K_o= 0,45 and w_o= 1.

limit. Except for initially, the relation (3-46) was found valid.

The analytical results obtained above in Section 3.2 are confirmed by the numerical example.

Discussion of Results

From Figures 3:1 - 3:4 we can see that the results from the static case hold in the dynamic case at the horizon date in our example. However, an interesting new insight is obtained from the dynamic example. It exemplifies a case where, in spite of the larger absolute advertising activities of the profit-maximizing firm, the relative current advertising of the profit-maximizing firm is smaller than for the labor-managed firm. Thus, as indicated in Section 3.1.1, the social advantage claimed in this respect by Vanek (1970, pp. 120-123) and debated by others, is perhaps an open question that requires further research.

A rather long time period, for the assumed constant capital, was chosen in the numerical case, to be sure of obtaining a stationary equilibrium. The main results would perhaps not be very different if a shorter time period were chosen.

As a next interesting step, the computer tests may be run with other parameter values in order to obtain comparative-dynamics predictions.

We can see in Figure 3:1 that it is optimal for the labor-managed firm to advertise much initially and then to gradually decrease advertising until a steady-state level is reached. This is similar to the result obtained by Jacquemin (1972, p. 132) and Jacquemin & Thisse (1972, pp. 67 and 83) in the case of a profit-maximizing firm. We have actually extended their results to the case when labor is an extra control variable, in this numerical example. We can also see in Figures 3:1 - 3:4 that the model implies a turnpikeproperty (Shell, 1967 and Mäler, 1974, Chapter 3) for the advertising variables. Although we have not done a formal analysis, this discussion suggests that the labor-managed firm has a phase diagram, in the a,A-plane, that is similar to the one envisaged by Jacquemin (1972 and 1973) and Jacquemin & Thisse (1972), and which we shall analyze further in Chapter 5, Sections 5.4 and 5.5.

3.3 SUMMARY AND CONCLUSIONS

The aim of this chapter has been to develop a dynamic advertising model of a labor-managed firm in a marginalist-type framework, in order to gain some insight into the dynamic advertising behavior of such a firm and how its behavior in this respect differs from that of a profit-maximizing firm.

A dynamic advertising model is constructed and a deduction of the dynamic counterpart to the Dorfman-Steiner theorem and to the extended Nerlove-Arrow theorem is provided for the labor-managed firm. This, it seems, has not previously appeared in the literature. The dynamic Dorfman-Steiner theorem implies, for constant price and advertising elasticities, that the dynamic labor-managed firm has a greater advertising/sales ratio than the static labor-managed firm. The dynamic firm advertises more than the static one in order to build up advertising goodwill for the future. This behavior is common to both the labor-managed firm and profit-maximizing firm and illustrates their common general interest to use non-labor resources efficiently.

Put in another way, an important result of the dynamic model is that the marginal revenue of advertising is *less* than marginal cost. Thus, the *dynamic* effect of advertising offers an alternative hypothesis to the sales maximization theory of Baumol (1959). This has been pointed out by Jacquemin (1973). In fact this is a special case of the more general hypothesis that marginal revenue is less than marginal cost when long-run effects are taken into account (Jacquemin & Thisse, 1972). This general result offers an alternative, within the area of profit-maximization, to the managerial theories of the firm. This general result also illustrates the hypothesis that the firm will sacrifice some current profit in order to obtain larger profits in the future.

Another conclusion is that the dynamic Dorfman-Steiner theorem and extended Nerlove-Arrow theorem are valid also for the labor-managed firm under the qualification that λL for constant L is substituted for λ. λ is the dynamic shadow value for advertising goodwill and L is labor quantity. This could be expected, because the maximization of total labor income - which is equal to average labor income multiplied by labor - is equivalent to the maximization of profit, for constant labor. However, this similarity in mathematical form does not imply similarity in magnitude. Rather, due to different optimum elasticities, goodwill shadow value and employment, different optimal dynamic advertising behavior of the labor-managed firm and profit-maximizing firm is expected.

In a steady-state situation for the firm, the model indicates that the value of the advertising/sales ratio normally is different for the labor-managed firm and the profit-maximizing firm. However, for a demand function with constant elasticities with respect to price, advertising and goodwill, the model studied predicts that, in the steady state, the advertising/sales ratio for the labor-managed firm is the same as for the profit-maximizing firm. The steady-state situation is most likely an optimum state in the studied model after a certain passage of time.

In order to obtain detailed results regarding the optimum time-trajectories, assumptions have been made about the structure of the demand function and production function, and about the numerical values of the parameters.

An interesting new insight has been obtained from the dynamic example. It represents a case where, in spite of greater absolute advertising activities of the profit-maximizing firm, the relative current advertising of the profit-maximizing firm is smaller than for the labor-managed firm. Thus the social advantage of the labor-managed firm that is claimed in this respect by some authors, and debated by others, perhaps remains an open question requiring further research.

Finally, it should be stressed that the above results rest on a number of limiting assumptions, for instance as regards structure of model employed, goodwill transition equation, objective function, discount rate, demand and production function, social-economic environment and the assumption of constant capital input. Also, the models are partial and do not explicitly take into consideration competition and reactions from other firms in the product market. However, the analysis in this chapter nevertheless provides some interesting insight and may provide a basis for modification and further inquiries.

APPENDIX 3.1

3.4 GOAL FORMULATION OF THE LABOR-MANAGED FIRM

There are many possible dynamic goal formulations for our advertising model of the labor-managed firm with finite time horizon. For example,

$$\max_{a,L} \int_o^T e^{-\rho t}\, \Pi_{LM}\, dt \tag{3-73}$$

$$\max_{a,L} \int_o^T e^{-\rho t}\, \Pi_{LM}\, dt \qquad \text{subject to } A(T) \geq A_c \tag{3-74}$$

$$\max_{a,L} \left\{\int_o^T e^{-\rho t}\, \Pi_{LM}\, dt + A(T)\cdot e^{-\rho T}\right\} \tag{3-75}$$

$$\max_{a,L} \left\{\int_o^T e^{-\rho t}\, \Pi_{LM}\, dt + \frac{A(T)}{L(T)}\cdot e^{-\rho T}\right\} \tag{3-76}$$

where A_c= constant, and where for instance $A_c \geq A_o$(=initial advertising goodwill).

The last of these goal formulations (3-76) also suggests that the relevant state variable may be advertising goodwill per unit of labor A/L rather than advertising goodwill A.

In Section 3.2 we have chosen the first of these goal formulations (3-73), at this explorative stage, in order to keep the analysis as simple as possible. Another reason is that our main aim here has been to compare advertising behavior of the two types of firm, rather than to analyze the absolute best - and in some sense most realistic - model of the labor-managed firm.

The goal formulation (3-75) implies the same necessary conditions as the goal formulation (3-73) except that

$$\lambda_{PM}(T) = \lambda_{LM}(T) = 1 \tag{3-77}$$

This means, following (3-61) in Section 3.2, that

$$\lim_{t\to T} a_{PM} = +\infty \tag{3-78}$$

and the problem of an infinite control thus arises. The goal formulation (3-76) generally calls for an even more complex analysis than the other goal formulations do.

4 THE DYNAMIC ADJUSTMENT OF A COMPETITIVE LABOR-MANAGED FIRM TO A PRICE CHANGE

4.1 INTRODUCTION

A peculiarity of the competitive labor-managed firm in its pure static Illyrian form is its low short-run elasticity of supply in comparison with the competitive profit-maximizing firm. In the pure static case of the one-product firm in which labor is the only variable factor, the short-run supply curve of a competitive labor-managed firm even bends backwards, as many authors from Ward (1958) onwards have shown.

However, in the longer term when capital is also variable, the long-run tendency towards capital expansion and corresponding labor expansion may eventually counteract the short-run contraction in labor that is caused by a price increase. In the present chapter this process of adjustment to a price increase will be discussed first in static terms in Section 4.2 and then in dynamic terms in Section 4.3.

More specifically the aim of this chapter is to develop a marginalist-type model of the dynamic adjustment of a competitive labor-managed firm to a change in output price. The purpose is to gain some insight into the dynamic employment behavior of such a firm and how its behavior in this respect differs from that of a profit-maximizing firm.

4.2 STATIC CASE

The goal of a competitive pure labor-managed firm (Ward, 1958, Vanek, 1970) is to maximize income per unit of labor, i.e.

$$\Pi^{LM} = \frac{pq-rK}{L} \tag{4-1}$$

where

Π^{LM} = income per unit of labor

p = price of output (or value added per unit of output)

q = output quantity

r = user cost of capital

K = capital input

L = labor input

Following Vanek (1970, p. 1) the term "labor" is used here to include everyone working in the economic unit studied.

Let us first analyze the short-run case and thereafter the long-run case.

4.2.1 Short Run

In the short run, i.e. with fixed capital, labor is expanded until the marginal value product of labor is no longer greater than the average labor income, i.e.

$$pq_L = \Pi^{LM} \tag{4-2}$$

where alphabetical subscripts denote partial differentiation.

Relation (4-2) is represented graphically in Figure 4:1 for two price levels p_1 and p_2 where $p_2 > p_1$.

For a given L (and K), it is quite clear from (4-1) and (4-2) that a price increase from p_1 to p_2 will shift pq_L less than Π^{LM} and thus the intersection L^+ of the pq_L-curve and Π^{LM}-curve in Figure 4:1 will move to the left, i.e. optimum employment will decrease, $L_2^+ < L_1^+$. As output in the short run is directly determined by employment in the model, the supply curve will thus surprisingly have negative price elasticity (Ward, 1958, Vanek, 1970).

Assuming diminishing returns to labor in the production function, i.e. $q_{LL} < 0$, the second-order conditions for a maximum are satisfied because the pq_L-curve in Figure 4:1 then cuts the Π^{LM}-curve from above, and the Π^{LM}-curve will have the general increasing-decreasing shape shown in Figure 4:1.

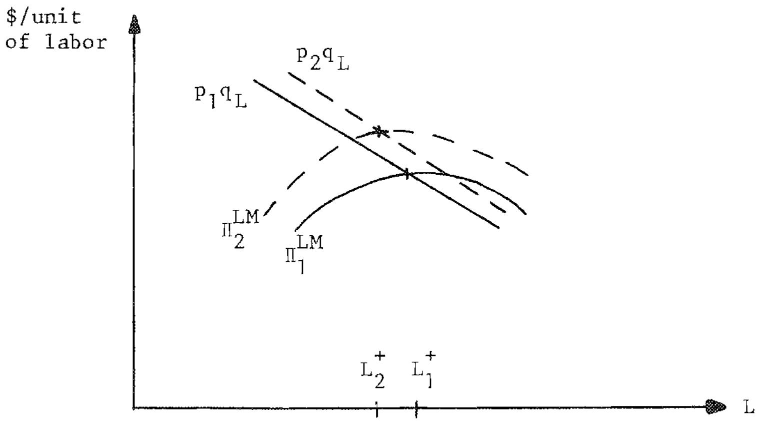

Figure 4:1. Labor-Managed Firm: The Effect of a Price Change on Short-Run Optimum Employment (Labor).
p = output price ($p_2 > p_1$)
q = output quantity
L = labor
L_1^+, L_2^+ = optimum labor for price p_1, p_2 respectively
Π^{LM} = income per unit of labor.

4.2.2 Long Run

In the long run, capital is no longer fixed but capital is employed up to the point where the marginal value product of capital is equal to the user cost of capital, i.e.

$$pq_K = r \tag{4-3}$$

where the production function depends on capital and labor, i.e.

$$q = q(K,L) \tag{4-4}$$

The problem of maximizing average labor income in (4-1) in the long run is thus a problem of maximizing a function of two variables K and L.

The two necessary conditons (4-2) and (4-3) may be written,

$$F(K,L) \equiv pq_L - \Pi^{LM} = 0 \tag{4-6}$$

$$f(K,L) \equiv pq_K - r = 0 \tag{4-7}$$

where the functions F(K,L) and f(K,L) are defined by (4-6) and (4-7).

The long-run optimum is obtained by finding the intersection of the two curves (4-6) and (4-7) in the K,L-plane. The slopes of these curves are found by differentiating (4-6) and (4-7), i.e.

$$\left[\frac{dL}{dK}\right]_{f=0} = \frac{-q_{KK}}{q_{KL}} \tag{4-8}$$

$$\left[\frac{dL}{dK}\right]_{F=0} = \frac{q_{KL}}{-q_{LL}} + \frac{\Pi^{LM}_{K}}{pq_{LL}} \tag{4-9}$$

For $q_{KL} > 0$, $q_{KK} < 0$ and $q_{LL} < 0$, which is a reasonable case, the f-curve is positively sloped according to (4-8). The first term in (4-9) is then also positive and, as the second term is zero at optimum, the F-curve is also expected to be positively sloped, at least in a neighborhood around the optimum, due to continuity. Thus, schematically, we can draw the F-curve and the f-curve in a neighborhood around the optimum as in Figure 4:2.

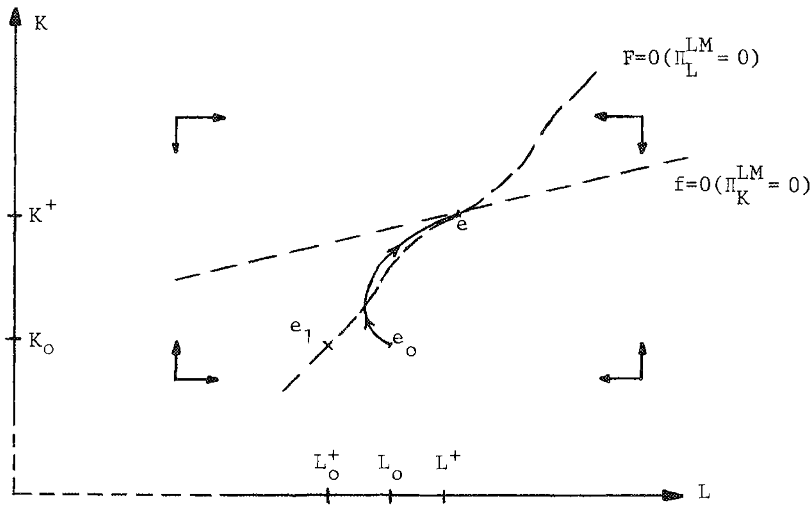

Figure 4:2. Labor-Managed Firm: Neighborhood of Long-Run Capital and Labor Optimum.
K = capital input
L = labor input
e = long-run optimum (K^+,L^+)
e_o = initial state (K_o,L_o)
e_1 = short-run labor optimum of initial state (K_o,L_o^+)
The double arrows represent the sign of Π_L^{LM} and Π_K^{LM} in the respective quadrants.

The intersection of the two curves in Figure 4:2 at e is the long-run optimum of the labor-managed firm which we shall denote (K^+,L^+).

As

$$\Pi_L^{LM} = \frac{1}{L}\cdot F(K,L) \tag{4-10}$$

$$\Pi_K^{LM} = \frac{1}{L}\cdot f(K,L) \tag{4-11}$$

second-order conditions imply that $\Pi_L^{LM} > 0$ to the left of the F-curve in Figure 4:2, indicating a tendency there towards an increase in L, and vice versa to the right of the F-curve in Figure 4:2. In the same way, $\Pi_K^{LM} > 0$ below the f-curve, implying a tendency there towards an increase in K, and vice versa above the f-curve in Figure 4:2. These adjustment tendencies are indicated in Figure 4:2 by the double arrows in the four quadrants defined by $F = 0$ and $f = 0$. The general configuration of the two curves in relation to each other in Figure 4:2 is a necessary one for a maximum.

4.2.3 Adjustment Process

We are now in a position to study the long-run effect of a price change on a competitive pure labor-managed firm.

Assume that the short-run optimum for constant capital K_o and output price p_o is e_o in Figure 4:2. The corresponding F-curve and f-curve have not been drawn in Figure 4:2.

A price increase from p_o to, let us say, p is assumed to shift the F-curve and f-curve to those shown in Figure 4:2. The new short-run equilibrium of the labor-managed firm is thus e_1, implying a decrease in employment, as discussed in Section 4.2.1.

However, in the long run when capital is variable, we can expect capital to increase up to its long-run equilibrium level e, so that the first adjustment will be in a direction north-west from e_o. This is also consistent with the double arrows in the quadrant for e_o. However, if the firm could invest sufficiently much and increase labor sufficiently fast without incurring too high adjustment costs, the adjustment process could actually move north-east from e_o directly towards the long-run equilibrium e, ignoring the intermediate short-run tendencies. The magnitude of the adjustment costs for capital and labor and the shape of the three-dimensional production "mountain" over the K,L-plane, are main determinants of the path which the firm will take from e_o to e.

We are thus led to the study of the adjustment process as a dynamic process.

4.3 DYNAMIC CASE

Following from the above, the analysis may be furthered by making assumptions regarding the dynamic adjustment process.

4.3.1 Assumptions and Model

A common type of adjustment process in economics is

$$\dot{K} = I - \delta_1 K \tag{4-12}$$

$$\dot{L} = l - \delta_2 L \tag{4-13}$$

where

K = capital input

I = investment quantity

L = labor input

l = net employed labor input per time period

δ_1 = depreciation of capital

δ_2 = quit rate of labor

A dot over a variable denotes time differentiation.

The former adjustment process (4-12) is well known from capital and growth theory and the latter (4-13) is rather well-known from dynamic labor economics as in for example Salop (1973).

However, as was mentioned above, there are limits to the speed with which the adjustment may take place without causing significant costs for the firm. One way of taking this into account is to set upper and lower limits to investment I and new labor hires l. Another way is to let I and l vary freely, but to introduce an extra adjustment cost term $C(I,l)$ into the objective function, income per unit of labor, i.e.

$$\Pi^{LM} = \frac{pq - rK - C(I,l)}{L} \tag{4-14}$$

where $C(I,l)$ rises, possibly at an increasing rate, with increasing absolute values of I and l. A recent review of this second method as applied to a profit-maximizing firm is to be found in Söderström (1974, Chapters IV and V).

The above two methods can also be combined.

As a first exploratory step we can choose the first method and set upper and lower bounds on investment I and new labor hires l, i.e.

$$I_{inf} \leqslant I \leqslant I_{sup} \tag{4-15}$$

$$l_{inf} \leqslant l \leqslant l_{sup} \tag{4-16}$$

where I_{inf} and I_{sup} are constants representing the lower (infimum) and upper (supremum) bound of investment and l_{inf} and l_{sup} are constants representing the lower (infimum) and upper (supremum) bound of new labor hires.

Let us assume also that the dynamic objective of the pure labor-managed firm is to maximize the present value of income per unit of labor using the discount rate of the labor collective. This is a somewhat arbitrary, yet plausible, dynamic objective of a pure labor-managed firm. The choice of it is not intended to give us the "best" objective. It is rather a first exploratory step. Our dynamic objective is supported by the studies of Atkinson (1973) and Litt, Steinherr & Thisse (1975), who use similar objectives. Some alternative dynamic objectives of a pure labor-managed firm are mentioned in Chapter 3, Section 3.4 of this study.

Our problem is thus

$$\underset{I,l}{\text{Max}} \int_{o}^{T} e^{-\rho t} \Pi^{LM} \, dt \tag{4-17}$$

subject to (4-12)-(4-16) with $C(I,l) \equiv 0$ and the following initial and terminal conditions,

$$K(0) = K_o = \text{constant} \tag{4-18}$$

$$L(0) = L_o = \text{constant} \tag{4-19}$$

$$K(T) \geqslant K_o \tag{4-20}$$

$$L(T) \geqslant L_o \tag{4-21}$$

The end point of the time period is T and the discount rate is ρ.

The initial conditions (4-18) and (4-19) are represented by the point e_o in Figure 4:2 and we assume $K_o < K^+$ and $L_o < L^+$.

The terminal conditions (4-20) and (4-21) state that capital and labor should not be lower at the end of the time interval than at the beginning. This is an arbitrary assumption but, on the other hand, as we shall see, it has little significance in the case we are studying, i.e. the optimal adjustment process from e_o to e in Figure 4:2.

Our problem (4-12)-(4-21) is a problem of optimal control with investment I and new labor hires l as bounded control variables, and capital K and labor L as free state variables, apart from the initial and terminal conditions (4-18)-(4-21).

Assuming concavity in $\Pi^{LM}(K,L)$ in a neighborhood around its maximum, including the initial point e_o in Figure 4:2, the sufficiency conditions for an optimum (maximum) adjustment path are satisfied.[1)] We are thus sure that if we find a solution to the necessary conditions of an optimum adjustment path, this solution will be a maximum.

The next step is thus to obtain the necessary conditions for an optimum. Problem (4-12)-(4-21) lends itself to analysis by the Maximum Principle as developed by Pontryagin et al. (1962).

The Hamiltonian H can be written

$$H \cdot e^{\rho t} = \Pi^{LM}(K,L) + \lambda_1(I-\delta_1 K) + \lambda_2(1-\delta_2 L) \qquad (4\text{-}22)$$

where λ_1 and λ_2 are adjoint variables.

The Maximum Principle states that

$$\max_{I,1} H \qquad (4\text{-}23)$$

are necessary conditions for a maximum.

1) Kamien & Schwartz (1971) and Seierstad & Sydsaeter (1977).

As H is linear in the controls I and l, we have a bang-bang solution. The controls that maximize H depend on the signs of λ_1 and λ_2. We have three cases for I, viz.

Path 1: $\lambda_1 > 0 \implies I = I_{sup}$ (4-24)

Path 2: $\lambda_1 < 0 \implies I = I_{inf}$ (4-25)

Path 3: $\lambda_1 = 0 \implies I =$ determined below (4-26)

and three cases for l, viz.

Path A: $\lambda_2 > 0 \implies l = l_{sup}$ (4-27)

Path B: $\lambda_2 < 0 \implies l = l_{inf}$ (4-28)

Path C: $\lambda_2 = 0 \implies l =$ determined below (4-29)

Path 1 represents, from (4-12), an increasing concave exponential capital growth curve for

$$I_{sup} > \delta_1 K^+ \quad (4\text{-}30)$$

where (K^+, L^+) represents the maximum of $\Pi^{LM}(K,L)$ i.e. point point e in Figure 4:2.

Path 2 represents, from (4-12), a decreasing convex capital contraction curve for

$$I_{inf} < \delta_1 K_o \quad (4\text{-}31)$$

Path A represents, from (4-13), an increasing concave exponential labor growth curve for

$$l_{sup} > \delta_2 L^+ \quad (4\text{-}32)$$

Path B represents, from (4-13), a decreasing convex exponential labor contraction curve for

$$l_{inf} < \delta_2 L_o^+ \quad (4\text{-}33)$$

where (K_o, L_o^+) represents the short-run maximum of $\Pi^{LM}(K,L)$ for the initial capital K_o, i.e. point e_1 in Figure 4:2. We shall assume (4-30)-(4-33) to be valid, since this does not essentially limit our analysis.

The Maximum Principle also implies that the state variable conditions

$$H_K = -\frac{d}{dt}(\lambda_1 e^{-\rho t}) \quad (4\text{-}34)$$

$$H_L = -\frac{d}{dt}(\lambda_2 e^{-\rho t}) \tag{4-35}$$

and the transversality conditions

$$\lambda_1(T) = 0 \tag{4-36}$$

$$\lambda_2(T) = 0 \tag{4-37}$$

are necessary conditions for an optimum, although in the case of the transversality conditions this is only so when there is inequality in (4-20)-(4-21). This is most likely in our case, but should be checked afterwards.

For Path 3, (4-34) reduces to

$$H_K = \Pi_K^{LM} = \frac{1}{L} \cdot f(K,L) = 0 \tag{4-38}$$

i.e. the f-curve in Figure 4:2.

For Path C, (4-35) reduces to

$$H_L = \Pi_L^{LM} = \frac{1}{L} \cdot F(K,L) = 0 \tag{4-39}$$

i.e. the F-curve in Figure 4:2.

We have now analyzed the various time paths and need to link them together, i.e. to synthesize them, so that the objective function is maximized.

4.3.2 Synthesis of Time Paths

It appears likely that the labor-managed firm will end up at the maximum point e in Figure 4:2, provided - as we shall assume - that T is large enough. Also, it is likely that the sooner it reaches e the better, but not at the expense of too low an average for labor incomes Π^{LM} during the period of adjustment.

We shall not embark here on a rigorous analysis of the synthesized time paths that are optimal under various assumptions regarding adjustment rates l_{inf}, l_{sup}, I_{inf}, I_{sup} and production function $q = q(K,L)$. Rather we shall only indicate some plausible possibilities. From the configuration of Figure 4:2 it seems plausible that capital should be increased as fast as possible up to the long-run equilibrium level K^+. In other words the optimum capital adjustment would be as shown in Figure 4:3.

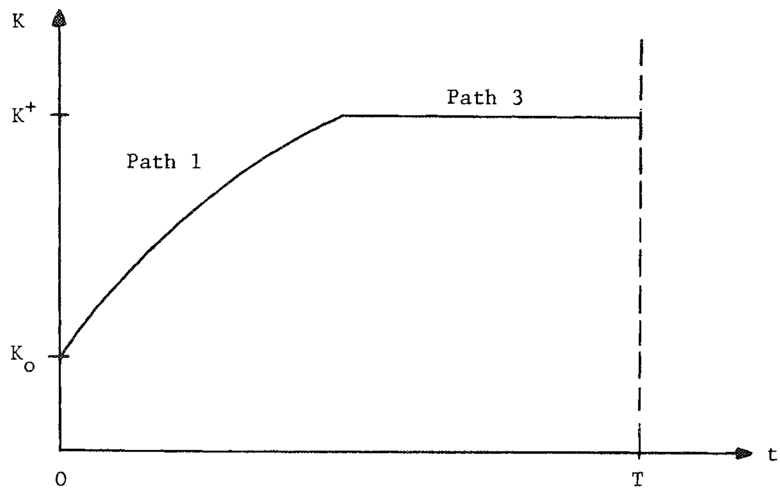

Figure 4:3. Labor-Managed Firm: The Adjustment of Capital K to its Final State, in the Case of an Increase in Price.
K^+= steady-state capital
K_o= initial capital
T = terminal time.

This is in agreement with the capital adjustment behavior of the competitive profit-maximizing firm, as is to to be expected as regards the non-labor inputs of the labor-managed firm.[1)]

Should maximum initial growth of capital $I_{sup} - \delta_1 K_o$ be small in relation to maximum initial contraction of labor $\delta_2 L_o - l_{inf}$, and should initial marginal labor income of capital $\Pi_K^{LM}(K_o, L_o)$ be small in relation to initial marginal labor income of decreasing labor $|\Pi_L^{LM}(K_o, L_o)|$, then we may expect $\dot{L}(0) < 0$ and, thus, an optimum adjustment process for labor as illustrated in Figure 4:4 and as in Figure 4:2.

1) Bergström (1973, p. 109).

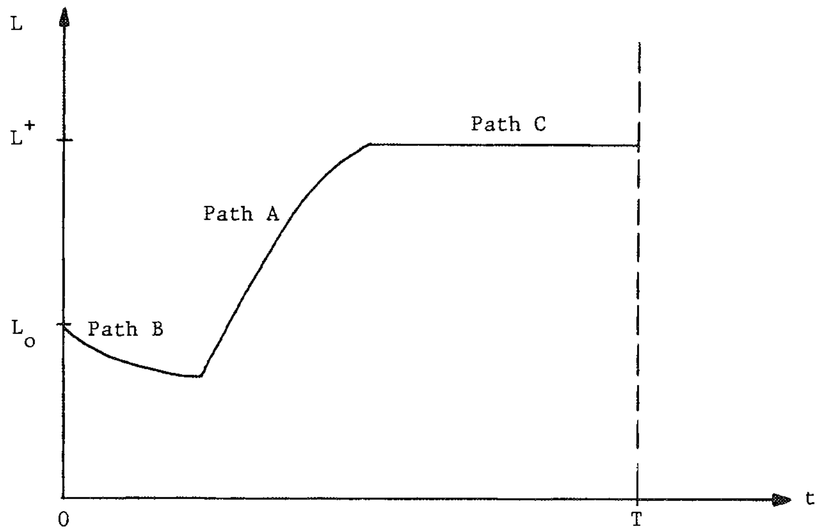

Figure 4:4. Labor-Managed Firm: The Adjustment of Labor L to its Final State, in the Case of an Increase in Price.
L^{+}= steady-state labor
L_o= initial labor
T = terminal time.

We shall discuss the just mentioned statement leading to $\dot{L}(0) < 0$ in more detail here.

If maximum investment I_{sup} is only little larger than initial depreciation $\delta_1 K_o$, then initial net capital increase $\dot{K}(0)$ will be small for Path 1. This follows from the equation of motion (4-12).

Also, if initial departure of labor $\delta_2 L_o$ is large in relation to minimum new labor employment l_{inf}, then initial net labor decrease $\dot{L}(0)$ will be large for Path B. This follows from the equation of motion (4-13).

We may illustrate this in another way. The direction of the trajectory at the initial point e_o in the phase diagram Figure 4:2, is determined by

$$\left[\frac{dK}{dL}\right]_{t=0} = \frac{\dot{K}(0)}{\dot{L}(0)} \qquad (4\text{-}40)$$

The less negative the slope

$$\left[\frac{dK}{dL}\right]_{t=0}$$

the more horizontal the trajectory at e_o in Figure 4:2. A large negative $\dot{L}(0)$ and a small $\dot{K}(0)$ will achieve this, from (4-40), and this will strengthen the proposed optimality of Path B at t = 0 and thus $\dot{L}(0) < 0$.

The adjustment process in Figure 4:4 indicates the short-run tendency for a pure labor-managed firm to decrease employment immediately after a price increase (Path B). However, after a while the long-run expansion tendency takes over and employment increases again (Path A) up to its long-run equilibrium level (Path C).

This behavior is distinct from that of the competitive profit-maximizing firm, which will increase employment in the short-run in response to a price increase and will follow a monotonously increasing path towards long-run equilibrium.

4.3.3 Comments

The long-run process of adjustment to a price increase in the competitive labor-managed firm has been qualitatively described by Bergström (1973, pp. 130-131). His description corresponds to the heuristically proposed optimum adjustment process illustrated in Figures 4:3 and 4:4.

Figure 4:4, referring to the labor-managed firm, suggests that, in the short run, labor adjustment is rather rigid in response to an increase in industry demand and price during a recession. Also, greater short-run stability in employment when industry demand and price decrease in a depression may be expected compared to a profit-maximizing firm (Bergström, 1973, p. 130).

However, as mentioned before, the adjustment process depends on the allowed adjustment rates for I and l, and on the slopes of the production function. Thus other optimum adjustment processes for labor, such as Path A + Path C with switching point at L^+ or with switching point below L^+, are also conceivable. Also, the optimum capital adjustment may be Path 1 + Path 3 with switching point below K^+. The relation in time between the switching points of the optimum L(t)-curve and the optimum K(t)-curve is also an open question. A full analysis of the optimum synthesis of the time paths may be done by inserting the different alternatives in the objective function (4-17) and then maximizing with respect to the switching points, and choosing the alternative that gives the highest value to the objective function[1].

Litt, Steinherr & Thisse (1975) have studied the investment decisions of a labor-managed firm in an optimal control theoretic framework similar to ours. However, they assume instantaneous adjustment of labor rather than continuous adjustment as in our case (4-13) and (4-16). They also assume that investment is financed out of retained earnings rather than externally as we do. In spite of these differences, it is interesting to note that Litt, Steinherr & Thisse (1975), in one of their cases, come to the conclusion that employment decreases until it reaches an assumed minimum limit, which is somewhat similar to our Path B.

Ward (1958), Vanek (1970) and others have shown that the reaction of a pure *static* labor-managed firm to a change in output price may be modified when including many variable factors in the model instead of only one (labor) as in our case. We may expect similar modifications in the *dynamic* case studied here but we shall not elaborate on this.

4.4 SUMMARY AND CONCLUSIONS

The aim of this chapter has been to develop a marginalist-type model of the dynamic adjustment of a competitive labor-managed firm to a change in output price. The purpose

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1) Alternatively we may analyze the shadow values λ_1 and λ_2 in (4-34)-(4-37) starting backwards from t=T, using the continuity of λ_1 and λ_2 and then finding the condition for $\lambda_2(0) < 0$ (Path B).

has been to gain some insight into the dynamic employment behavior of such a firm and how its behavior in this respect differs from that of a profit-maximizing firm.

A peculiarity of the competitive labor-managed firm in its pure static form is its low short-run elasticity of supply, as compared with the competitive profit-maximizing firm. In the pure static case of a one-product firm with labor as the only variable factor, the short-run supply curve of a competitive labor-managed firm will even bend backwards as many authors have shown. However, in the longer term when capital is also variable, the long-run tendency towards capital expansion and a corresponding expansion in labor might counteract the short-run contraction in labor that is caused by an increase in price. The short-run and long-run adjustment behavior of the labor-managed firm to a change in price have been linked together in this study in an optimal control-theory framework. The problem does not seem to have been analyzed in this way before.

In the one-product, one non-capital factor case, a possible prediction of the model in this chapter is that an increase in price implies a decrease in employment at first in the labor-managed firm, to be followed by an increase in employment up to the long-run equilibrium level. In other words, the labor-managed firm would follow different structural paths at different times in its adjustment process. The profit-maximizing firm of neo-classical economics would however normally react to a price increase by continuously expanding employment both in the short-run and long-run.

From the model studied it seems plausible that the adjustment process depends on the allowed adjustment rates for investment and employment and on the marginal labor income of capital and labor. Although a strict proof is not given for the result in the previous paragraph it seems safe to assume, from the discussion in this chapter that there exists a set of parameter values for which the stated result is valid. However, similarly, there also probably exists a set of parameter values for which the result in the previous paragraph is not valid. The main object of this chapter has been to form a framework for analysis of the problem rather than to obtain all-conclusive results.

Further insight into the dynamic adjustment of a competitive labor-managed firm to a change in output price may be gained from the proposed model by specifying the parameters or relations between the parameters and further analyzing the synthesis of the various paths.

5 A DYNAMIC DISCOUNT RATE OF THE FIRM

In dynamic as compared to static economic models it is of interest to evaluate entities at various points in time. This is often accomplished by a weighting process and often by an exponential discounting process. The discount rate in the latter case is generally assumed to be an exogenous constant. In this chapter we shall discuss some modifications of this assumption within the general framework of models using an exponential discounting factor, and we shall analyze the effects of such modifications.

More specifically the aim of this chapter is to develop dynamic economic models of profit-maximizing firms in a marginalist-type framework, in which the discount rate is not constant but is either exogenously or endogenously variable. The purpose is to gain some insight into the effects of such non-constant discount rates compared with constant discount rates in specific cases.

In Section 5.1 we study the effect of an exogenously variable discount rate in certain instances, while in Section 5.2 we analyze the effect of an endogenously variable cost of capital in a Fisher-Hirshleifer framework (Hirshleifer, 1970).

In Section 5.3 a dynamic counterpart of the Fisher-Hirshleifer framework of Section 5.2 is formulated and in Section 5.4 we study the effect of an endogenously variable discount rate on the dynamic semi-investment behavior of the firm. The results found in Section 5.4 will then be applied in Section 5.5 to some well-known advertising theorems. Section 5.6 contains a summary and conclusions.

5.1 AN EXOGENOUSLY VARIABLE DISCOUNT RATE OF THE FIRM

We shall in this section assume that the firm exists in a perfect capital market, but that the market interest rate varies over time $r = r(t)$.

Our aim is to inquire into the effects of such an assumption, as compared with the assumption of a time-constant market interest rate.

5.1.1 Specific Model

In order to obtain concrete results, we shall study a specific model of the firm

$$M = \int_0^T d(t) \cdot e^{-\int_0^t r(u)du} dt + y(T) \cdot e^{-\int_0^T r(u)du} - I \qquad (5\text{-}1)$$

where

M = net share value, i.e. share value of the firm over and above the net book value of the firm

d(t) = dividends

y(T) = terminal value of the firm

I = net book value of the firm (a positive constant)

T = terminal time

r(u) = market interest rate at time u.

We shall assume the functions d(t) and y(T) to be exogenously given, e.g. given by some other optimization process (since they normally interact with one another).

We shall study a linearly increasing and cyclically fluctuating interest rate respectively:

$$r(t) = r_o + r_y \cdot t \qquad (5\text{-}2)$$

$$r(t) = r_m + r_a \cdot \sin w(t-t_o) \qquad (5\text{-}3)$$

where r_o, r_y, r_m, r_a, w, t_o are given positive constants.

We may insert the respective interest rates (5-2) and (5-3) in our model (5-1) and perform the time integrations and obtain the share value M. However, to do this we require an explicit expression for dividends d(t). We assume dividends d(t) to be exponentially decreasing, as

$$d(t) = B_o \cdot e^{-bt} - B_1 \tag{5-4}$$

where $B_1 << B_o$ and B_o, B_1 and b are positive constants.

We also specify the terminal value of the firm y(T) to be exponentially decreasing, i.e.

$$y(T) = S_o \cdot e^{-sT} \tag{5-5}$$

where S_o and s are positive constants.

The time integration in the linear interest case (5-2) is carried out in Appendix 5.1 for $B_1 = 0$ and results in expression (5-61) in that appendix.

In the cyclical interest rate case (5-3) the result after time integration is that in the first approximization

$$M = C(B_o,b) - C(B_1,0) + S_o\, e^{-h(T)} - I \tag{5-6}$$

where

$$h(T) = -(s+r_m)T + \frac{r_a}{w}\left[\cos w(T-t_o) - \cos wt_o\right]$$

and $C(B_o,b)$ and $C(B_1,0)$ are functions of T and derived in Appendix 5.2 in (5-68).

As we have derived explicit solutions for the net share value of the firm M, we may now numerically calculate the effect of a time-variable interest rate r(t) in the two cases (5-2) and (5-3), as compared with the constant interest rate case.

Alternatively we may maximize the net share value of the firm M with respect to terminal time T and thereafter calculate the net share value $M(T_{max})$ for the optimal time period T_{max}, and compare the variable and constant interest rate cases. We shall choose this latter alternative, which is somewhat more complex than the first alternative.

In order to obtain concrete results let us assume specific numerical values, viz. in the linear interest case (5-2),

$$d(t) = 15 \cdot e^{-0,13t} \tag{5-7}$$

$$y(T) = 50 \cdot e^{-0,1T} \tag{5-8}$$

$$r(t) = 0,1 + r_y t \tag{5-9}$$

$$I = 50 \tag{5-10}$$

i.e. B_o=15, B_1=0, b=0,13, S_o=50, s=0,1 and r_o=0,1.

In the cyclical interest case (5-3) let us assume that

$$d(t) = 15 \cdot e^{-0,13t} - 1,5 \tag{5-11}$$

$$y(T) = 50 \cdot e^{-0,1T} \tag{5-12}$$

$$r(t) = 0,1 + r_a \sin \frac{2\Pi}{10} (t-t_o) \tag{5-13}$$

$$I = 50 \tag{5-14}$$

i.e. B_o=15, B_1=1,5, b=0,13, S_o=50, s=0,1, r_m=0,1 and $w=\frac{2\Pi}{10}$.

In the case of maximizing net share value M with respect to the time horizon T, we obtain the results in the linear interest rate case in Figure 5:1 and in the cyclical interest rate case in Figure 5:2. These results were calculated by Ekman & Wejke (1964), who interpreted the model (5-1) as a machine investment problem. The model in this section may be interpreted as a model of a firm whose main activity concerns an investment project that is limited in time. A possible example is a firm whose object is to construct a building or similar.

5.1.2 Discussion

We can see from Figure 5:1 that for positive slopes r_y of the linear interest rate, the optimal net share value of the firm decreases as expected and that the decrease compared to the constant interest rate case r_y= 0 is not insignificant.

In the cyclical interest rate case in Figure 5:2, with a ten year interest rate cycle starting with an interest rate increase (cf. (5-13) for t_o= 0), the optimal net share value

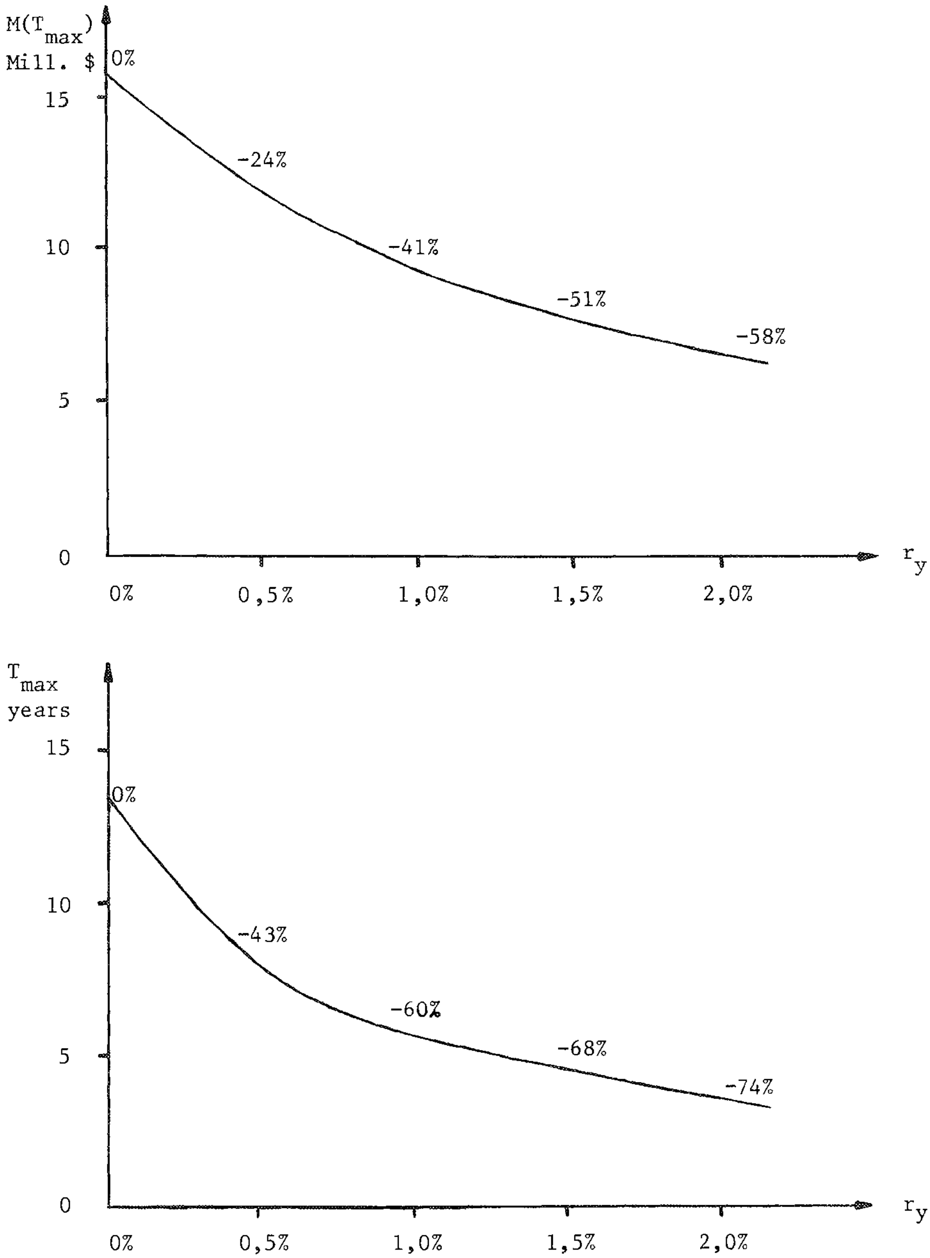

Figure 5:1. Optimum Net Share Value $M(T_{max})$ and Optimum Terminal Time T_{max} as a Function of the Gradient r_y of a Linearly Variable Discount Rate in a Numerical Example of a Firm. $r_o = 10\%$.

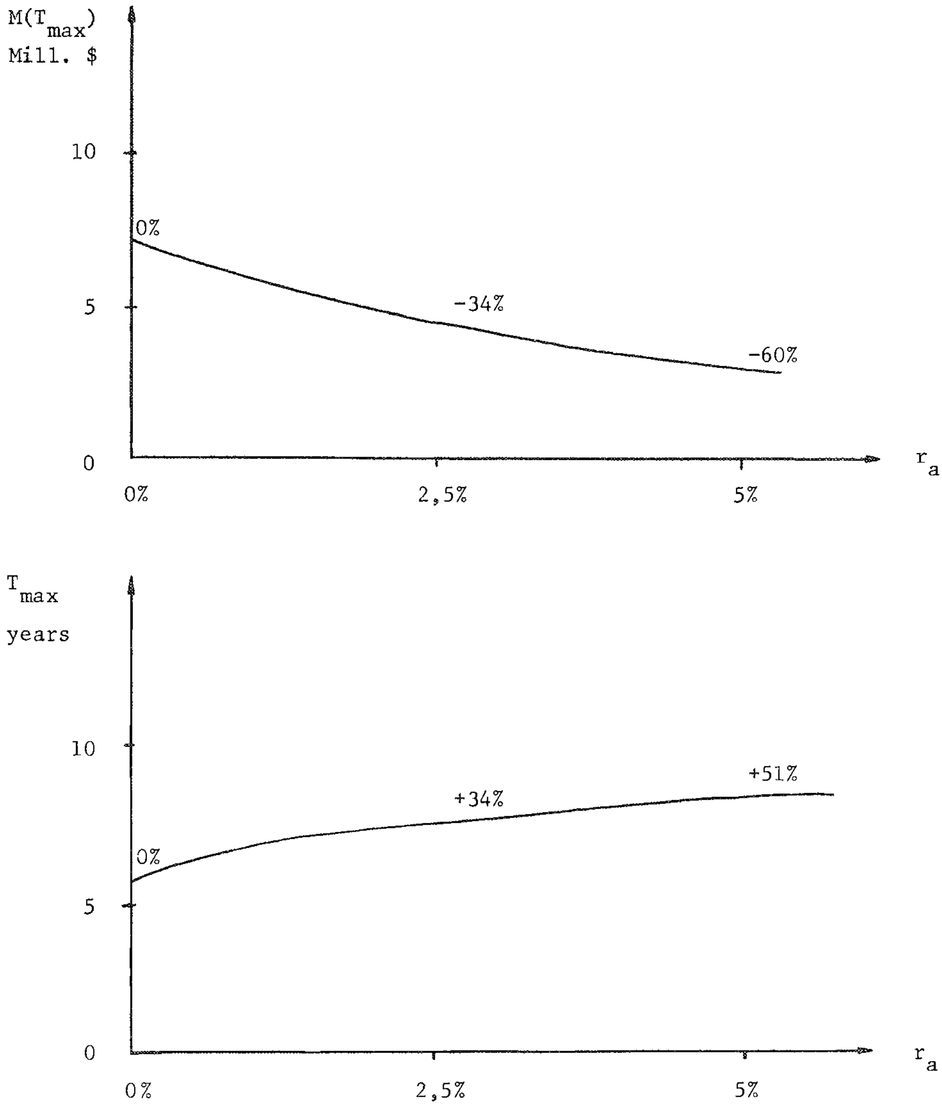

Figure 5:2. Optimum Net Share Value $M(T_{max})$ and Optimum Terminal Time T_{max} as a function of the Amplitude r_a of a Cyclical Discount rate in a Numerical Example of a Firm. $r_m = 10\%$, $t_o = 0$. $w = \frac{2\Pi}{10}$ (i.e. 10 year cycle).

of the firm decreases as expected. We also see from Figure 5:2 that the decrease is not insignificant.

The linearly increasing interest rate case is interesting, because it illustrates Gordon's (1963) concept of discounting more heavily in the future to take into consideration increasing uncertainty.

The analysis of a cyclical interest rate is also interesting because it illustrates the effect of business cycles on the value of the firm. By varying the frequency of the cycles w and the starting point of the cycles t_o, we may derive how the value of the firm changes under different assumptions regarding the business cycle. There is some discussion of this in Ekman & Wejke (1964, pp. 5:14-5:18).

However, in interpreting the economic results of this section we should remember that the analysis is partial and that other variables in our model apart from the interest rate may be affected by the business cycle and other environmental effects.

In spite of the special nature of the case studied in this section, the general conclusions above may well hold for more general models and numerical examples because the model is only used as a basis for comparison and because the numerical example is a non-extreme case.

We shall not elaborate on the effects of an exogenously variable discount rate here but rather in the following sections also study an endogenously variable discount rate.

5.2 THE FISHER-HIRSHLEIFER PROBLEM

The firm's investment and finance problem may be characterized by the standard Fisher-Hirshleifer production-exchange problem (Hirshleifer, 1970) as illustrated in Figure 5:3. P represents the production opportunity curve and M the market exchange line. c_o is consumption in period zero and c is the discrete perpetuity consumption level from period one to infinity (Hirshleifer, 1970). c_o and c

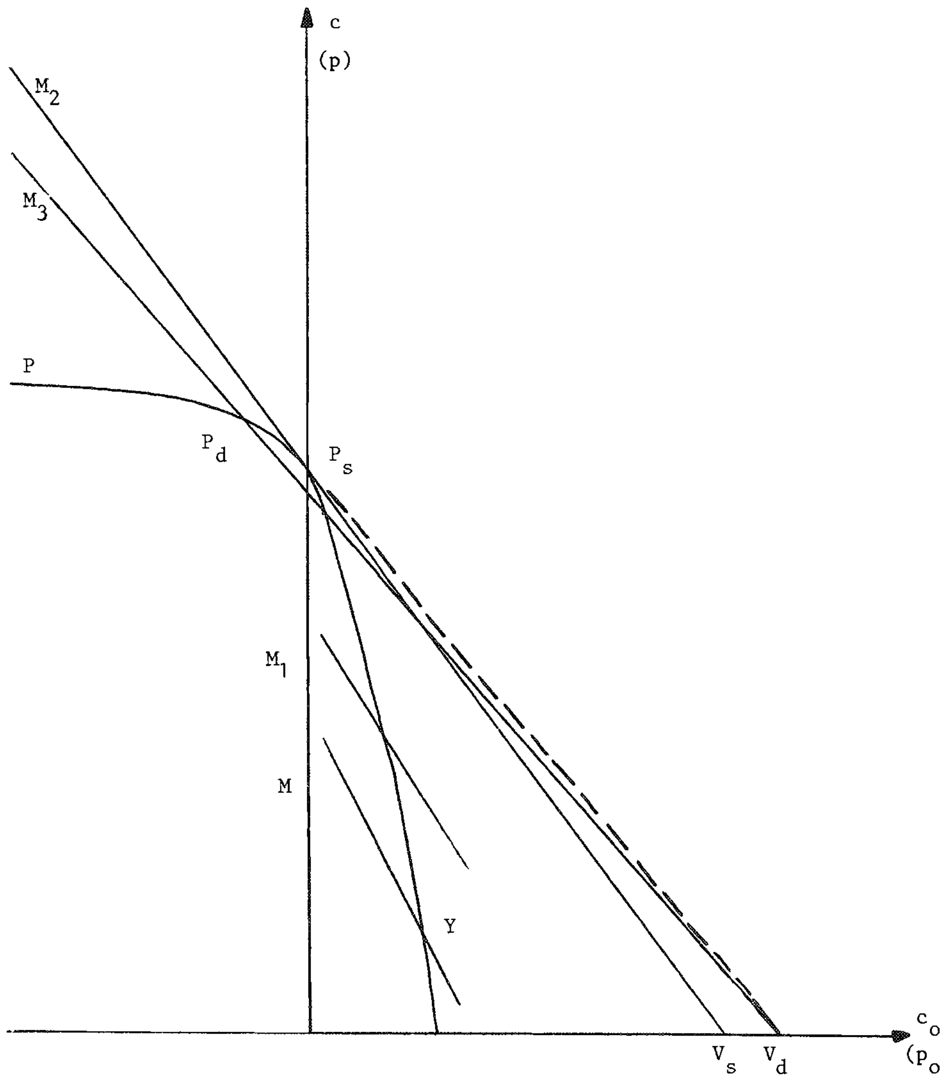

Figure 5:3. Productive and Consumptive Opportunities. P is production opportunity curve. M, M_1, M_2 and M_3 are market exchange lines. Y is endowment. c_o,c denote present and future consumption. p_o,p denote present and future production. P_s is the point where the market exchange line is tangential to the production opportunity curve. V_s is the corresponding productive present value. P_d is production point with maximum productive present value V_d. The dashed curve between P_s and V_d, represents the envelope of the dominating market exchange lines in the first quadrant.

may be interpreted as payments from the firm to the capital holders.

Given an endowment Y in Figure 5:3, the problem is to find the point which the firm should choose on its production opportunity curve, i.e. how much of current production, which is denoted p_o, should be sacrificed for future production, which is denoted p.

Let us limit ourselves to a firm which has a large number of profitable productive investment projects in relation to its endowment in period zero. More specifically let us assume that all the firm's endowment in period zero can be invested in future production at a rate of return higher than the firm's cost of capital. If we assume that the last dollar of endowment in period zero can be invested in production at exactly the firm's cost of capital, the c-axis will pass through point P_s where the market line M_2 is tangent to the production opportunity frontier in Figure 5:3. The following analysis is also valid for all positions of the c-axis to the right of the point P_s. This is because the market exchange line, rather than the production opportunity curve, will form the efficient frontier of the total opportunity set in this case.

Let us assume now that instead of being in a perfect capital market, the firm exists in an imperfect capital market in the sense that a larger firm enjoys a lower cost of capital than a smaller firm. Although this assumption is plausible, there are grounds for assuming the opposite also. However, the main object of this chapter is not to make the best or most realistic assumptions but rather to make such plausible assumptions that enable us to do explorative analyses of variable discount rates or costs of capital. Let us also assume that the size and the productive capital of a firm are correlated. We shall further assume that the firm can obtain external funds at the same cost of capital, regardless of the quantity of funds supplied. This may be accomplished, for instance, by keeping the debt/equity ratio of the supplied funds constant and equal to the debt/equity ratio of the endowment.

The previous paragraph implies that the market line is linear but that its slope decreases for greater p-values on the production opportunity frontier in Figure 5:3.

This is indicated by the line M_1 in Figure 5:3 as compared to line M. The maximum productive present value V_d is obtained at P_d (with the market line M_3), whereas the maximum productive present value in an assumed constant discount rate case V_s is obtained at P_s (with the market line M_2), which unlike P_d is a tangency point. This requires second-order conditions to be satisfied.

From this discussion follows that point V_d is the largest attainable c_o-value for the total opportunity set, and point P_s is the largest attainable c-value in Figure 5:3, if we limit ourselves to the first quadrant in Figure 5:3.

For intermediary c_o,c-values the total opportunity frontier will be represented intuitively by the envelope of the market lines in the first quadrant of Figure 5:3, as the productive point moves from P_s to P_d. The dashed curve from P_s to V_d represents this envelope curve. We can see from Figure 5:3 that the opportunity frontier set is no longer convex, but concave.

The consumptive optimum is obtained at that point on the dashed envelope curve P_sV_d, where the highest indifference curve is reached. The indifference curves have not been drawn in Figure 5:3, but are assumed convex, as usual, and confined to the first quadrant. We make the further assumption that the indifference curves are more convex at points of tangency with the opportunity frontier, than the opportunity frontier curve itself. This seems to be a possible assumption, given reasonable limits to the change in the cost of capital due to changes in the firm's capital. Given this assumption, the consumptive optimum will be reached at the point of tangency between the dashed envelope curve and the undrawn indifference curves. The productive optimum is the top intersection between the market line that is tangential to the envelope curve at the consumptive optimum on the one hand, and the production opportunity curve on the other hand.

The indifference curves may also be such that no tangency with the opportunity frontier exists, and corner points V_d or P_s are optimal. The former is the case, for instance, for a linear indifference curve with a rate of time preference greater than that represented by the slope of a straight line through P_s and V_d in Figure 5:3 and vice versa for the latter. V_d is also optimal for more general indifference curves which at the intersection with the c_o-axis, have a rate of time preference greater than that represented by the slope of a straight line through P_s and V_d in Figure 5:3. Should the indifference curves intersect the c-axis, and should they have a rate of time preference there that is lower than that of a straight line through P_s and V_d, then the corner point P_s is the optimal solution.

It should be mentioned here that Figure 5:3 is only illustrative and does not necessarily reflect correct magnitudes or proportions. Figure 5:3 is only used in order to clarify certain aspects in principle.

Having briefly discussed the corner solutions above we shall now analyze the optimal interior solution mathematically.

Our problem is to

Maximize $U = v(c_o) + v(c)/\eta$ (5-15)

subject to

$p_o+f^{-1}(p) = Y_o$	(the production opportunity curve)	(5-16)
$c-p = -r(K)(c_o-p_o)$	(the market exchange line)	(5-17)
$p = f(K)$	(the production or operating profit function)	(5-18)

where we have assumed an additively separable utility function with

$v(c)$ = one-period utility function, $v'(c) > 0$, $v''(c) < 0$

η = discount rate for future utility (a constant).

We have denoted present and future production p_o and p, capital K, the cost of capital $r(K)$ where $r'(K) < 0$ in our assumed case, present and future consumption c_o and c, operating profit $f(K)$ and endowment Y_o (a constant) which is assumed to represent a point on the c_o-axis in Figure 5:3, in this analysis. Depreciation is assumed zero.

We want to maximize U in (5-15) with respect to c_o and c, subject to the constraints (5-16)-(5-18), which can be reduced to a single constraint in c_o, c and p. This means that the optimum consumptive point (c_o,c) automatically determines the optimum productive point (p_o,p).

We thus maximize the Lagrangian L:

$$L = U(c_o,c) + \lambda_1\left[Y_o - p_o - f^{-1}(p)\right] + \lambda_2\left[c-p+r(K)(c_o-p_o)\right] + \\ + \lambda_3\left[f(K)-p\right]$$

The necessary conditons are

$L_{c_o} = 0,\ L_c = 0,\ L_{p_o} = 0,\ L_p = 0,\ L_K = 0$:

$$v_{c_o} + \lambda_2 r = 0 \tag{5-19}$$

$$v_c/\eta + \lambda_2 = 0 \tag{5-20}$$

$$-\lambda_1 - \lambda_2 r = 0 \tag{5-21}$$

$$-\lambda_1 f_p^{-1} - \lambda_2 - \lambda_3 = 0 \tag{5-22}$$

$$\lambda_2 r_K(c_o-p_o) + \lambda_3 f_K = 0 \tag{5-23}$$

The consumptive optimum is obtained from (5-19) and (5-20)

$$r = \frac{v_{c_o}}{v_c}\eta \tag{5-24}$$

which implies tangency between the market exchange line and the indifference curve.

The productive optimum is obtained from (5-21)-(5-23)

$$p_K = r + r_K \frac{p-c}{r} \tag{5-25}$$

which depends on the consumptive optimum c and, consequently, on the indifference curves.

For $c = p$, (5-25) implies $p_K = r$ which corresponds to a productive optimum at point P_s in Figure 5:3.

For $c = 0$, (5-25) implies $p_K = r + r_K \frac{p}{r}$ which corresponds to a productive optimum at point P_d in Figure 5:3. This case corresponds to the maximization of the productive present value.

Values of c intermediate to these two extreme cases correspond to c-values on the dashed envelope curve $P_s V_d$ in Figure 5:3. The corresponding productive optimums are on the arc from P_s to P_d on the production opportunity curve.

The main result from (5-25) is that the introduction of a decreasing endogenous cost of capital implies a further expansion of the firm's productive capital, in comparison with the constant cost of capital case, because $p_K < r$ in (5-25) for $r_K < 0$ (and $p > c$). This is because if we start from a comparable optimum in the constant cost of capital case, point P_s in Figure 5:3, we find that the positive effect of a lower cost of capital on the total volume of supplied capital, when expanding productive capital, is greater than the negative effect of the fact that the rate of return on the marginal productive investment is lower than the cost of capital. This net positive effect disappears, however, as productive capital is expanded more and an optimum is achieved. This requires second-order conditions to be satisfied, which however shall not be elaborated on here.

The main conclusion - that the optimum is greater in the dynamic cost of capital case than in the constant cost of capital case - is valid for a certain class of indifference curves, viz. all those for which an interior consumptive optimum along P_sV_d (the dashed envelope curve) in Figure 5:3 is obtained. This also requires second-order conditions to be satisfied, in particular as regards the indifference curve (5-15). We shall however not elaborate on this here but rather state a dynamic generalization of the analysis, in the next section.

5.3 THE DYNAMIC INVESTMENT AND FINANCE BEHAVIOR OF THE FIRM

A formulation of the dynamic investment and finance behavior of the firm in the general case comparable to the two-period analysis in the previous section, is

$$\max_{I,M} \; U = \int_0^\infty v(c)e^{-\eta t}\,dt \tag{5-26}$$

where

$$c = f(K) - I - N \tag{5-27}$$

and subject to

$$\dot{K} = I - bK \tag{5-28}$$

$$\dot{B} = r(K)B - N \tag{5-29}$$

where

$B \geqslant 0$, $c \geqslant 0$, $K \geqslant 0$

B = new capital funds (accumulated)

N = cash flow from the firm to the holders of new capital

f(K) = cash flow or operating profit

K = total capital

I = investment

r(K) = cost of capital, dependent on the size of total capital

b = depreciation rate

c = consumption or withdrawals for the holders of initial capital funds.

The expression (5-26) corresponds to (5-15) and to the indifference curves in the two-period case. The accounting identity (5-27) and the time-development expressions of total capital and new capital funds (5-28) and (5-29) respectively, together correspond to (5-16)-(5-18) and to the total opportunity set (not the envelope curve) in the two-period case. We have added depreciation in (5-28).

The problem is to maximize utility U in (5-26) with respect to I and N as control variables and K and B as state variables. It is interesting to observe that the model of (5-26)-(5-29) has similarities in form to the model of Hochman, E., Hochman, O. & Razin, A. (1973).

5.4 THE EFFECT OF AN ENDOGENOUS DISCOUNT RATE ON THE DYNAMIC SEMI-INVESTMENT BEHAVIOR OF THE FIRM

By semi-investments we mean such expenditures, normally reported in the accounts as costs, that have a current demand-increasing or cost-decreasing effect on the firm and which also have a positive carry-over effect on profits in subsequent years. Examples are marketing expenditures such as advertising and sales promotion, and expenditures on research and development, industrial engineering, machine maintenance and personnel development. They are thus of considerable importance to the firm.

Referring to the Fisher-Hirshleifer case in Section 5.2 and Figure 5:3, let us assume that the indifference curves (not shown in Figure 5:3) are such that the corner point V_d in the total opportunity set is optimal. This implies present value maximization with r(K) as discount rate.

Following the assumption in the first paragraph above, let us add a positive effect of semi-investment I on current operating profit, such that f = f(K,I).

A dynamic model similar to the one in Section 5.3 may be constructed. However, as mentioned above, the productive present value is maximized rather than the consumptive utility:

$$\max_{I(t)} V = \int_0^\infty p(K,I)\cdot e^{-R}\,dt \tag{5-30}$$

subject to

$$\dot{K} = I - b\,K; \qquad K(0) = K_o \text{ (a constant)} \tag{5-31}$$

$$\dot{R} = r; \qquad R(0) = 0 \tag{5-32}$$

where

V = present value of cash flow

p = cash flow (or production in Fisher-Hirshleifer terms, cf. Section 5.2)

I(t) = semi-investment

K(t) = (semi-)capital

b = depreciation rate

r = discount rate

and where R is a new variable defined by (5-32) and K and R are state variables and I is a control variable and

$p = p(K,I) = f(K,I) - I$

$r = r(K)$

and $K = K(t)$, $I = I(t)$ and $R = R(t)$ and the initial capital is K_o.

Following Uzawa (1968) in a similar context, we change from normal time t to "psychological" time R in order to eliminate one of the constraints.

According to (5-32)

$$dt = dR/r \tag{5-33}$$

and thus our problem reduces to

$$\underset{I(R)}{\text{Max}}\ V = \int_o^\infty \frac{p}{r}\, e^{-R}\, dr \tag{5-34}$$

subject to

$$\frac{dK}{dR} = (I - bK)\,/r; \qquad K(0) = K_o \tag{5-35}$$

where "psychological" time R is now the time argument of the variables involved, instead of normal time t.

Let us follow Pontryagin et al. (1962) and formulate the Hamiltonian H such that

$$H = \frac{p}{r}\, e^{-R} + \lambda e^{-R}(I-bK)\,/r \tag{5-36}$$

where $\lambda(R)e^{-R}$ is the adjoint variable and represents a shadow value, viz. the present imputed value of a unit of capital at "time" R.

Necessary conditions for an optimum are $H_I = 0$ and $H_K = -\frac{d}{dR}(\lambda e^{-R})$ where subscripts in general refer to partial differentiation with respect to the subscript.

This leads to

$$p_I + \lambda = 0 \tag{5-37}$$

$$-r\lambda_R = p_K - \lambda(r+b) - \frac{r_K}{r}\left[p + \lambda(I-bK)\right] \tag{5-38}$$

We make the plausible assumptions $c_I \gtrless 0$ (first increasing and then decreasing), $c_{II} < 0$ and $c_K \geqslant 0$, $c_{KK} \leqslant 0$.

Assuming a positive shadow price, i.e. $\lambda > 0$, (5-37) implies that, along the optimum path, the semi-investments or costs I should be expanded until the marginal revenue is lower than marginal cost. This apparent contradiction to the traditional theory of the firm is due to the positive carry-over effect of the semi-investments on future profits. This conclusion is similar to the results of Jacquemin (1972, pp. 130-131) and Jacquemin & Thisse (1972, pp. 66-67 and 83).[1)]

To interpret (5-38), we note that the shadow value λ embodies at "time" R the future earnings power of an extra unit of capital at "time" R. This earnings power decreases, according to (5-38), at the rate that marginal operating profit minus interest and depreciation plus the positive effect of a decreased discount rate ($r_K < 0$), are generated. The last term in (5-38) is the correction term due to the dynamic discount rate and is equal to $-r_K \cdot He^R$. It represents the marginal "interest" gain at "time" R, due to a marginal decrease in the discount rate caused by a marginal increase in capital.

If $\max_I H$ is concave in K for given λ and R, then a policy satisfying (5-35), (5-37) and (5-38) and the transversality conditions

$$\lim_{R \to \infty} e^{-R}\lambda(R) \geqslant 0; \quad \lim_{R \to \infty} e^{-R}\lambda(R)K(R) = 0$$

1) As expected, this general result is unaffected by the assumption of an endogenously variable discount rate.

is optimal, if such an optimum exists. Cf. Arrow & Kurz (1970, p. 49).

After partial differentiation twice of (5-36) with respect to K, it can easily be seen that a convex discount rate function, i.e. $r_{KK} > 0$ will increase the concavity of H for $H > 0$.

To find solutions to the optimization problem we eliminate λ and λ_R in (5-37) and (5-38) and obtain together with (5-35) the following. We have assumed $f(K,I)$ to be additively separable in K and I.

$$rI_R = \frac{p_K}{p_{II}} + \frac{p_I}{p_{II}}(r+b) - \frac{r_K}{p_{II}r}\left[p-p_I(I-bK)\right] \tag{5-39}$$

$$rK_R = I - bK \tag{5-40}$$

This is a system of two first order differential equations in the two unknown functions $I(R)$ and $K(R)$ with the initial condition $K(0) = K_o$.

In general no analytical solution can be found to this problem and a numerical analysis is required.

However, we can study the steady state $I_R = 0$ and $K_R = 0$ which implies that

$$p_K + p_I(r+b) - \frac{r_K}{r}p = 0 \tag{5-41}$$

$$I = bK \tag{5-42}$$

and which satisfies the necessary, sufficient and transversality conditions. We note the similarity between (5-25) and (5-41) for $b = 0$, $p_I = -1$ and $c = 0$.

In the constant discount rate case this reduces to

$$p_K + p_I(r_m+b) = 0 \tag{5-43}$$

$$I = bK \tag{5-44}$$

where r_m is a constant average discount rate $r(K_d) < r_m < r(K_o)$ and K_d is the steady state optimum of the dynamic discount rate case and $r_K < 0$ and $K_o < K_d$.

We should like to find the relation between the steady state optimum in the constant and dynamic discount rate case respectively.

Inserting (5-44) in (5-43) and denoting the steady state optimum in the constant discount rate case K_c, we obtain

$$p_K(K_c) + p_I(bK_c)(r_m+b) = 0 \tag{5-45}$$

In general let us call the function in the left hand side of (5-45) g(K). That is

$$g(K) = p_K(K) + p_I(bK)(r_m+b) \tag{5-46}$$

$$g'(K) = p_{KK}(K) + bp_{II}(bK)(r_m+b) < 0 \tag{5-47}$$

From (5-41) for $r_K < 0$, $p > 0$

$$p_K(K_d) + p_I(bK_d)(r(K_d) +b) < 0 \tag{5-48}$$

For $\lambda > 0$ we have $p_I < 0$ in the vicinity of an optimum according to (5-37) and since $r(K_d) < r_m$, (5-46) and (5-48) imply that

$$g(K_d) < 0 \tag{5-49}$$

From (5-45) and (5-46)

$$g(K_c) = 0 \tag{5-50}$$

Thus (5-47), (5-49) and (5-50) imply that

$$K_d > K_c \tag{5-51}$$

and from (5-42) and (5-44)

$$I_d > I_c \tag{5-52}$$

where the indices for I have the same meaning as for K.

Thus the steady state optimum for $K_o < K_c$, $K_o < K_d$ and $r_K < 0$ is larger as regards both capital and semi-investment in the dynamic discount rate case than in the constant discount rate case. This result is quite intuitive and stems from the fact that when a single discount rate is decided upon, it will normally constitute some sort of average over the economic life of the problem. If the firm's capital is inadequate to start with, then as an appropriate capital level is reached the discount rate will decrease below the constant "average" rate. A lower instantaneous discount rate is generally favorable to investments and to capital formation, which helps explain the result we have obtained.

At this point we may ask whether any general statement can be made about how the optimum "time" paths of K(R) and I(R) respectively are related in the dynamic and the constant discount rate cases. The equations (5-39) and (5-40) provide the answer in each specific situation, but general statements are difficult to make. However, it is probable that, due to continuity, K(R) and I(R) are larger in the dynamic discount rate case than in the constant discount rate case for some "time" interval preceding the steady state. This "time" interval may not be small, and it can be computed from (5-39) and (5-40) for each specific situation. This can also be seen from the phase diagram in Figure 5:4, which illustrates the case when $K_o < K_d$ and $r_K < 0$. We can see that for a given initial capital K_o, there is only one stable trajectory $I_{od}I_d$ that describes the optimum path. The proof of this will not be elaborated on here. The small arrows in Figure 5:4 indicate the sign of K_R and I_R respectively, and we note that $I_R < 0$ for the optimum path $I_{od}I_d$, which means that optimum semi-investment I is greatest at the start and is reduced as the steady state I_d is approached. The line of reasoning and results to be drawn from the phase diagram for the dynamic discount rate case are thus similar to those in the constant discount rate case discussed by Jacquemin (1972) and Jacquemin & Thisse (1972). The constant discount rate case is illustrated in Figure 5:5, where the level of the curve $I_R = 0$ in the dynamic discount rate case (dashed line) shows higher K(R) and I(R) values in the "time" interval preceeding the steady state, in the way we have already discussed.

Figures 5:4 and 5:5 are only illustrative and do not necessarily reflect correct magnitudes or proportions. They are only used in order to clarify certain principle aspects.

Although the analysis in this section assumes $r'(K) < 0$, i.e. a decreasing discount rate, the analysis may well be performed with the alternative assumption

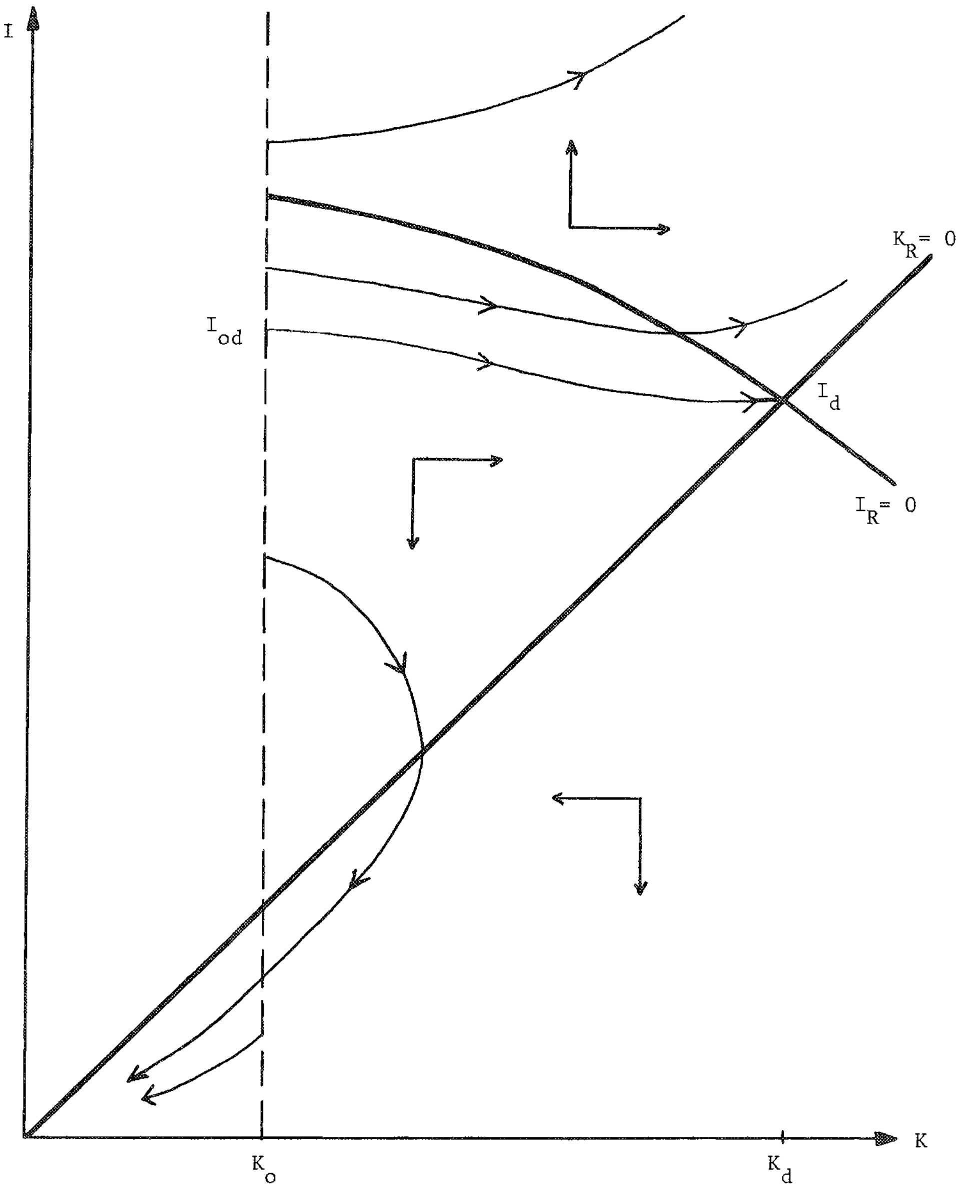

Figure 5:4. Phase Diagram for Semi-Investment I and (Semi-)Capital K in the *Dynamic* Discount Rate Case.
I_{od}, K_o represent initial values. I_d, K_d represent the steady-state solution. $I_R = 0$ and $K_R = 0$ represent the curves for stationary semi-investment and (semi-)capital respectively. The straight arrows represent the directions of change in the phase diagram.

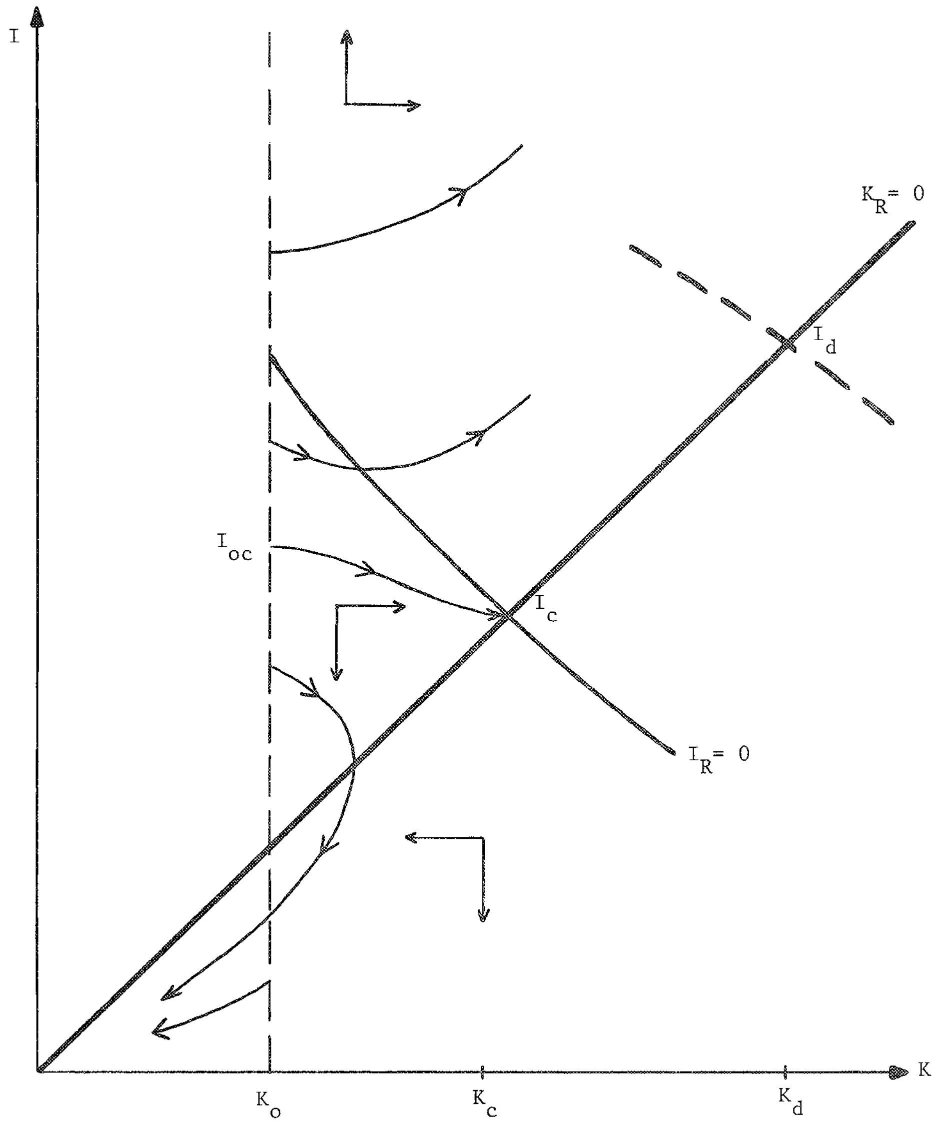

Figure 5:5. Phase Diagram for Semi-Investment I and (Semi-)Capital K in the *Constant* Discount Rate Case.
I_{oc}, K_o represent initial values. I_c, K_c represent the steady-state solution. $I_R = 0$ and $K_R = 0$ represent the curves for stationary semi-investment and (semi-)capital respectively. I_d, K_d represent the steady-state solution in the dynamic discount rate case. The straight arrows represent the directions of change in the phase diagram.

$r'(K) > 0$ also. However the main interest in this chapter is on an analysis of a variable discount rate rather than on the absolute best assumption of this variability. This viewpoint is also taken generally as regards the chosen models and assumptions in this chapter. It lies in the nature of the subject studied that the analysis in this chapter is explorative.

5.5 THE DORFMAN-STEINER [1] AND NERLOVE-ARROW [2] THEOREMS

One application of the model presented in Section 5.4 is to the problem of the carry-over effect of advertising. Cf. Chapter 3. The operating profit may be specified as

$$f(K,I) = hq - C(q) \tag{5-53}$$

where

h = output price

q = output quantity

C(q) = costs of the firm apart from advertising expenditure.

Also

$$q = q(h,I,K) \tag{5-54}$$

where

$q_h < 0$, $q_I > 0$, $q_K > 0$, $q_{II} < 0$ and $q_{KK} < 0$

I = current advertising expenditure

K = stock of advertising goodwill of the firm, summarizing the effect of past advertising expenditures.

By formulating the Hamiltonian H as in Section 5.4 and regarding price and advertising expenditure as control variables and advertising goodwill as state variable, a new necessary condition $\frac{\partial H}{\partial h} = 0$, i.e. $\frac{\partial p}{\partial h} = 0$ is added.

$$hq_h + q - C_q q_h = 0 \tag{5-55}$$

(5-37) gives us

$$(h - C_q)\, q_I - 1 + \lambda = 0 \tag{5-56}$$

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1) Dorfman & Steiner (1954).
2) Nerlove & Arrow (1962).

From (5-38) we get

$$- r\lambda_R = (h - C_q)\, q_K - \lambda(r+b) - r_K He^R \tag{5-57}$$

where the subscripts refer to partial differentiation.

By inserting $h - C_q$ from (5-55) in (5-56) and introducing the price elasticity of demand $\eta = - q_h \cdot \frac{h}{q}$ and the advertising elasticity of demand $\theta = q_I \cdot \frac{I}{q}$, we obtain

$$\frac{I}{hq} = \frac{\theta}{\eta(1-\lambda)} \tag{5-58}$$

along the optimum path.

(5-58) is a dynamic counterpart to the Dorfman-Steiner theorem [1] and has been developed by Jacquemin (1973) in the constant discount rate case. $\frac{\theta}{1-\lambda}$ may be considered as the long-term advertising elasticity of demand, which for the normal case of $0 < \lambda < 1$ is greater than the short-run elasticity θ, as is to be expected.

We have shown here that the Dorfman-Steiner theorem in the dynamic discount rate case has the same form according to (5-58) as in the constant discount rate case. However, the ratio of advertising to sales will normally be different because of a different λ. By inserting $h - C_q$ from (5-55) in (5-57) and introducing the elasticity of demand with respect to the level of advertising goodwill $\varepsilon = q_K \frac{K}{q}$, we obtain

$$\frac{K}{hq} = \frac{\varepsilon}{\eta\{\lambda(r+b) - r\lambda_R + r_K He^R\}} \tag{5-59}$$

(5-59) is a generalization of the Nerlove-Arrow theorem [2] and has been developed by Jacquemin (1973) in the constant discount rate case.

We have shown here that the generalized Nerlove-Arrow theorem in the dynamic discount rate case includes in the denominator for $r_K < 0$ an extra negative term $r_K He^R$ representing the "interest" gain due to a marginal unit of good-

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1) Dorfman & Steiner (1954).
2) Nerlove & Arrow (1962).

will. The ratio of advertising goodwill to sales will thus normally be different in the dynamic discount rate case as compared with the constant discount rate case. However due to different values of λ in the two cases it is an open question how the advertising goodwill ratio will change.

5.6 SUMMARY AND CONCLUSIONS

The aim of this chapter has been to develop dynamic economic models of profit-maximizing firms in a marginalist-type framework, in which the discount rate is not constant but is either exogenously or endogenously variable. The purpose has been to gain some insight into the effects of such non-constant discount rates compared to constant discount rates in certain cases.

First the effect of an exogenously variable discount rate in a specific share-value model of the firm was studied, after which the effect of an endogenously variable cost of capital in a Fisher-Hirshleifer framework was analyzed. A dynamic counterpart of the Fisher-Hirshleifer analysis was stated and after that the effect of an endogenously variable discount rate on the dynamic semi-investment behavior of the firm was studied. Finally these latter results were applied to some well-known advertising theorems.

In the case of an exogenously variable discount rate, one interesting conclusion is that a cyclical discount rate, compared with its constant average, altered the present value in certain instances to a not insignificant extent.

The linearly increasing interest rate case is interesting because it illustrates Gordon's (1963) concept of discounting more heavily in the future to take into consideration increasing uncertainty.

In the case of an endogenously variable discount rate, it was shown how some dynamic semi-investment and advertising models of the firm are modified.

First the effect of a cost of capital that decreases with firm size was analyzed in a two-period Fisher-Hirshleifer framework. As expected, the firm was found to expand its activities more than in the constant cost of capital case in a number of instances.

The same result was also obtained as regards the dynamic semi-investment behavior of the firm in the steady state, as well as along the optimal trajectory near the steady state (due to continuity).

The general result, in the type of dynamic models studied in this chapter, that marginal revenue is *less* than marginal cost in the dynamic case, is unaffected by the assumption of an endogenously variable discount rate, as expected. The same is valid as regards the general dynamic hypothesis that it is optimal for the firm to use its policy most heavily in the initial periods and thereafter gradually decrease the effort.

Also, modified dynamic Dorfman-Steiner and generalized Nerlove-Arrow theorems were deduced. An interesting result is that the dynamic Dorfman-Steiner theorem is invariant in form when an endogenously variable discount rate is introduced. The reason is that the Dorfman-Steiner theorem mainly expresses instantaneous price and advertising and therefore is not explicitly dependent on the discount rate. The generalized Nerlove-Arrow theorem is modified with a new term in the denominator when an endogenously variable discount rate is introduced. This term represents the "interest" effect of increasing advertising goodwill one unit, and is a natural extension. As regards both of these dynamic advertising theorems, the direction of the change is not self-evident from the general expressions deduced.

Of special interest in the case of an endogenously variable discount rate is the change-over in the solution from regular time to "psychological" time (Uzawa, 1968).

The analysis in this chapter appears to confirm that, as soon as the assumption of a stationary perfect capital market as regards the discounting process is dropped, the analysis becomes relatively complicated both conceptually and analytically. It follows from the nature of the subject studied that the analysis performed is explorative. However, the cases studied in this chapter provide interesting insight and may also provide building blocks for further inquiries. Such inquiries may be especially timely, since fluctuating discount rates and interest rates are increasingly common in many economies today.

APPENDIX 5.1

5.7 NET SHARE VALUE FOR A LINEAR VARIABLE DISCOUNT RATE

Let us insert the linear interest rate (5-2), dividends (5-4), assuming $B_1 = 0$, and the terminal value (5-5) into our model (5-1). First we derive the discount factor

$$e^{-\int_o^t r(u)du} = e^{-\int_o^t (r_o+r_y u)du} = e^{-r_o \cdot t - r_y t^2/2}$$

Our model (5-1) can now be written

$$M = \int_o^T B_o\, e^{-bt-r_o t-r_y t^2/2}\, dt + S_o\, e^{-sT-r_o T-r_y T^2/2} - I \qquad (5\text{-}60)$$

We denote the first term in (5-60) $B_o D$ and thus obtain

$$D = e^{\frac{(r_o+b)^2}{2r_y}} \cdot \int_o^T e^{-\frac{r_y}{2}\left(t+\frac{r_o+b}{r_y}\right)^2} dt$$

which, after the substitution

$$u = \sqrt{r_y}\left(t+\frac{r_o+b}{r_y}\right)$$

can be written

$$D\, e^{-\frac{(r_o+b)^2}{2r_y}} = \frac{1}{\sqrt{r_y}} \cdot \int_{\frac{r_o+b}{\sqrt{r_y}}}^{\sqrt{r_y}\left(T+\frac{r_o+b}{r_y}\right)} e^{-u^2/2}\, du =$$

$$= \sqrt{2\Pi/r_y} \cdot \left[\, G\left(\sqrt{r_y}\left(T+\frac{r_o+b}{r_y}\right)\right) - G\left(\frac{r_o+b}{\sqrt{r_y}}\right) \right]$$

where by definition

$$G(x) = \frac{1}{\sqrt{2\Pi}} \cdot \int_{-\infty}^{x} e^{-u^2/2}\, du$$ [1]

(5-60) can now be written

$$M = \sqrt{2\Pi/r_y} \cdot B_o \cdot e^{\frac{(r_o+b)^2}{2r_y}} \cdot \left[G\left(\sqrt{r_y}\left(T+ \frac{r_o+b}{r_y}\right)\right) - G\left(\frac{r_o+b}{\sqrt{r_y}}\right) \right] +$$

$$+ S_o\, e^{-sT-r_oT-r_yT^2/2} - I \qquad (5\text{-}61)$$

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[1] G(x) is thus the normal distribution function which can be found in tabulated form.

APPENDIX 5.2

5.8 NET SHARE VALUE FOR A CYCLICAL DISCOUNT RATE

Let us insert the cyclical interest rate (5-3), dividends (5-4) and the terminal value (5-5) into our model (5-1). First we derive the discount factor

$$e^{-\int_o^t r(u)du} = e^{-\int_o^t \left[r_m + r_a \cdot \sin w(u-t_o)\right] du} =$$

$$= e^{-r_m t + \frac{r_a}{w} \cos w(t-t_o) - \frac{r_a}{w} \cos wt_o} \tag{5-62}$$

The dividend term in our model (5-1) then becomes

$$\int_o^T B(t) e^{-\int_o^t r(u)du} dt = B_o e^{-\frac{r_a}{w} \cos wt_o} \int_o^T e^{-(r_m+b)t + \frac{r_a}{w} \cos w(t-t_o)} dt -$$

$$- B_1 e^{-\frac{r_a}{w} \cos wt_o} \int_o^T e^{-r_m t + \frac{r_a}{w} \cos w(t-t_o)} dt =$$

$$= C(B_o,b) - C(B_1,0) \tag{5-63}$$

where

$$C(B_o,b) = B_o e^{-\frac{r_a}{w} \cos wt_o} \int_o^T e^{-(r_m+b)t + \frac{r_a}{w} \cos w(t-t_o)} dt$$

and

$$C(B_1,0) = B_1 e^{-\frac{r_a}{w} \cos wt_o} \int_o^T e^{-r_m t + \frac{r_a}{w} \cos w(t-t_o)} dt$$

Let us introduce F as

$$F = \frac{C(B_o,b)}{B_o} e^{\frac{r_a}{w} \cos wt_o} = \int_o^T e^{-(r_m+b)t} e^{\frac{r_a}{w} \cos w(t-t_o)} dt \tag{5-64}$$

In order to solve this last integral F, we expand the second exponential factor in the integrand in series [1] and obtain, using the first two terms:

$$F = \int_o^T e^{-(r_m+b)t}\, dt + \int_o^T e^{-(r_m+b)t} \cdot \frac{r_a}{w} \cos w(t-t_o)\, dt \qquad (5\text{-}65)$$

We define $q = r_m + b$ and denote the first term in (5-65) F_1 and the second term F_2. F_1 can be readily integrated as

$$F_1 = \int_o^T e^{-qt}\, dt = \frac{1-e^{-qT}}{q} \qquad (5\text{-}66)$$

In order to integrate F_2 we use complex numbers

$\cos w(t-t_o) = \mathrm{Re}\; e^{j\, w(t-t_o)}$

where j = imaginary unit.

$$F_2 \cdot \frac{w}{r_a} = \int_o^T e^{-qt} \cos w(t-t_o)\, dt = \mathrm{Re} \int_o^T e^{-qt+jw(t-t_o)}\, dt =$$

$$= \mathrm{Re}\, \frac{1}{jw-q} \Big[e^{-qt+jw(t-t_o)} \Big]_o^T =$$

$$= -\, \mathrm{Re}\, \frac{jw+q}{w^2+q^2} \Big[e^{-qt} \left[\cos w(t-t_o) + j \sin w(t-t_o)\right] \Big]_o^T =$$

$$= -\frac{1}{w^2+q^2} \Big[e^{-qt} \left[q \cos w(t-t_o) - w \cdot \sin w(t-t_o)\right] \Big]_o^T =$$

$$= \frac{1}{w^2+q^2} \{ w\left[e^{-qT} \sin w(T-t_o) + \sin wt_o\right] -$$

$$- q\left[e^{-qT} \cos w(T-t_o) - \cos wt_o\right] \} \qquad (5\text{-}67)$$

As $F = F_1 + F_2$, we get from (5-64)

$$C(B_o,b) = B_o\, e^{-\frac{r_a}{w} \cos wt_o} \cdot (F_1+F_2) \qquad (5\text{-}68)$$

---

1) $e^z = 1 + z + z^2/2 + \ldots..$

where F_1 and F_2 are stated explicitly in (5-66) and (5-67) respectively. We shall not here write the complete expression. Our problem is thus basically solved.

We obtain (5-6) by inserting (5-62) and (5-63) in our model (5-1). The dominating term in C is F_1, which is basically the expression for the constant discount rate case, whereas F_2 is a correction term which disappears in the constant discount rate case $r_a = 0$.

The error we commit in (5-65) by including only two terms in the series expansion, is of the same order of magnitude as the first term omitted, i.e.

$$\frac{B_o}{2} e^{-\frac{r_a}{w} \cos wt_o} \cdot \left[\frac{r_a}{w}\right]^2 \int_o^T e^{-qt} \cos^2 w(t-t_o)\, dt \leqslant$$

$$\leqslant \frac{B_o}{2} e^{-\frac{r_a}{w} \cos wt_o} \left[\frac{r_a}{w}\right]^2 \left[1-e^{-(r_m+b)T}\right] \frac{1}{r_m+b}$$

This latter expression follows from $\cos^2 w(t-t_o) < 1$ and (2-66).

In normal cases the error in the net share value M is of the order of magnitude 2-4 %.

6 SUMMARY AND CONCLUSIONS

This microeconomic study of the firm addresses dynamic processes such as financing, capital investment, marketing investments and labor planning. It lies in the nature of these activities that they are fully dynamic and therefore best analyzed in a fully dynamic frame of reference. A comprehensive dynamic optimal treatment has only been feasible the last decade or so with the development of new mathematical techniques, such as optimal control theory.

The study also addresses the current issue of the dissipation of the power of the owners of the firm and the consequent increase in power of other groups in the firm such as managers and labor.

The main objective of the study has been to develop some microeconomic models of the dynamic economic behavior of firms that maximize other goals than profit alone. The managerial and labor-managed firms have been studied in this respect in an optimal control theory framework, and the results have been compared with those of the profit-maximizing firm.

Also the effect of non-constant discount rates on some dynamic optimal economic models of the firm, has been studied.

An important reason for analyzing growth and dynamic economic behavior at the level of the firm is that growth in society is determined to a significant extent by firms.

More specifically, the aim of this study has been to develop dynamic economic models of firms maximizing other goals than profit alone - and also to an extent profit-maximizing firms - in a marginalist-type framework, in order to gain some insight into the dynamic economic behavior of such firms and how this behavior differs from that of a profit-maximizing firm.

A possible use of such insight is to provide elements of firm behavior that perhaps may be used in a broader macroeconomic or general equilibrium framework. In general, hypotheses of firm behavior may be generated. New insight into the dynamic optimal economic behavior of the firm may, apart from this, be of interest in that it, hopefully, adds to the general stock of knowledge on optimal firm behavior as such.

The background, scope and aim of the work was discussed in Chapter 1. In Chapter 2 the dynamic optimal financial behavior of a managerial firm was studied and in Chapter 3 the dynamic optimal advertising behavior of a pure labor-managed firm was studied. The object of study in Chapter 4 was the dynamic optimal employment adjustment behavior of a competitive pure labor-managed firm to a change in output price.

The aim of Chapter 5 was to develop dynamic optimal economic models of profit-maximizing firms in a marginalist-type framework in which the discount rate is not constant but is either exogenously or endogenously variable. The purpose was to gain some insight into the effect of such non-constant discount rates compared with constant discount rates in certain cases.

In Chapter 1 different types of theories of the firm have been discussed and the scope of the study was narrowed down to the discipline of economics and especially the economic theory of the firm. Three important areas of development in the economic theory of the firm were mentioned: the dynamics of the firm, the objectives of the firm and uncertainty. These areas were discussed and the scope of the inquiry further narrowed down to the first two areas, namely the dynamics and objectives of the firm. Finally the literature and issues in these areas were discussed and the scope was further narrowed down to the subjects that have been studied in chapters 2-5.

The aim of Chapter 2 was to develop a dynamic optimal financial model of a managerial-type firm in a marginalist-type framework, in order to gain some insight into the dynamic optimal financial behavior of such a firm and into the way its behavior differs in this respect from that of a profit-maximizing firm.

The purpose was also to gain some insight into the adjustment of the firm to a final state rather than into the continuous growth of the firm at a steady rate. In Appendix 2.1 the aim was to study a more conventional continuous growth case but of the non-steady-state growth type.

The firm has two important financial decisions to make: (i) how large should investments and dividends be, and (ii) how should they be financed to achieve the firm's objective. It lies in the nature of this financing - and investment-process of the firm that it is fully dynamic and thus is best analyzed in a fully dynamic framework. Therefore recently a number of fully dynamic financial optimal control models of the firm have appeared in the literature.

A feature that distinguishes the model in Chapter 2 from most other dynamic financial models of the firm is that it analyzes a managerial type of firm. There has been little study of any fully dynamic financial model of the managerial firm in the economic literature. This is notable because capital investment and financial considerations are probably particularly important in attaining a goal such as size (and /or growth), which is characteristic of the managerial firm. The goal of the managerial-type firm studied here is a linear function of equity share value and long-term sales. The long-term sales component in the objective function is assumed relatively small. As we are mainly interested in directions of change and relative comparisons, this latter assumption is not all too restrictive. Also the model differs from other similar models in that it includes the corporate income tax and - in Appendices 2.1 and 2.2 - a study of the comparative dynamics of a change in the tax rate (as well as in the general interest rate). Appendix 2.2 is interesting because it provides a comparative dynamics analysis for the *whole* time period according to a general method.

The factors that limit the growth of the firm in the model studied are an increasing interest cost and diminishing operating returns to capital. Other models, instead, assume for instance an upper limit on debt instead of an increasing interest cost.

The time preference is introduced in two forms (a) via the discount rate and (b) via a minimum constraint on dividends. This appears plausible since firms probably have some "subsistence" contribution which must be paid to the owners and above that level they presumably have more freedom.

The model is studied in an optimal control theoretic framework and utilizes the Pontryagin Maximum Principle. The Lagrangian-type dynamic multiplier for the dynamic equity transition equation is discussed from an economic point of view. The multiplier is given an economic interpretation as the imputed value at optimum of an equity dollar. As in mathematical programming - and its corresponding shadow values - this economic interpretation is an interesting reason for using the method of optimal control theory and the Pontryagin Maximum Principle in economics.

The model studied predicts that the relation between the discount rate for dividends - and, in our case, also sales - and the after-tax all-equity interest rate, is of importance for type of phases or policies of development of the firm. This result is supported by results of other authors. Such a result is also expected as the model deals with optimal financing, and the relation between marginal interest rates is one rather natural determinant of financial policies in a marginalist-type model of the firm. Should government fiscal or monetary policy measures change the relation between the interest rates mentioned, the model predicts that the firm may switch its whole policy to a completely different path of development. A dynamic model of the type studied here is thus of interest in analyzing shifts in the structural dynamics of the firm due to economic policy measures. This is perhaps one of the most interesting reasons for studying economic models of the firm in an optimal control theoretic framework.

In the case of a small enough initial equity and in the case of the discount rate exceeding the after-tax all-equity interest rate, which would appear to be a plausible case, the managerial-marginalist-type firm of the studied model exhibits growth in both total capital and equity. The model studied also implies that the firm initially chooses a high leverage in order to exploit the high return on capital soonest possible, and then immediately proceeds to gradually decrease leverage and consolidate the firm as it grows. This result is supported by the work of other authors. They state that in dynamic economic models of this type, the optimal behavior of the firm is to use its control policy most heavily in the initial periods and continually decrease the effort as the final state is approached.

The decreasing leverage implies that the binding minimum dividends, being dependent on capital - in the studied model - grow more slowly than equity. Thus the model normally implies a dividend lag behind earnings, which is equivalent to a fairly rapid internal increase in equity and most likely a fall in leverage. This phenomenon of decreasing leverage due to a dividend lag behind earnings, can be observed, according to some authors, during a period of expansion in the economy. The dynamic model studied here, and in particular the dividend constraint, thus furnishes a possible explanation for this.

As mentioned above the optimum dynamic policy implies an increasing equity. Equity increases, and earnings are retained, as long as the imputed relative value of a dollar of equity is greater than the loss in shareholder utility of retaining the marginal dollar. Gradually the imputed relative value of a dollar of equity will decline and no longer be greater than the loss in shareholder utility of retaining the marginal dollar. At such a point in time, as expected, a *stationary* equilibrium is reached.

The optimal debt policy is such that the imputed relative value of a dollar of equity is equal to the marginal after-tax weighted cost of debt capital with respect to leverage. The former relative value decreases and in the steady-state, which corresponds to the static case, the normal static marginal condition is valid, viz. that the marginal weighted cost of equity capital is equal to the marginal after-tax weighted cost of debt capital with respect to leverage.

In order to compare the managerial-type firm, with a profit-maximizing firm, a comparative-statics and a comparative-dynamics analysis of shifts in the managerial coefficient (in the utility function) was performed. It was found that an increase in the "managerialness" of the firm most likely would call for an increase in leverage during the adjustment of the firm towards its final state (comparative dynamics), whereas the final state itself is unaffected (comparative statics), given the assumption of a linear objective function. This comparative dynamics result is supported by economic reasoning because increased "managerialness" places a premium on sales and thereby leverage rather than on profit and interest costs.

It is especially interesting to note that the comparative-dynamics effect of an increase in managerial influence on the firm is probably different from the comparative-statics effect . As firms are perhaps more often than not *out* of stationary equilibrium, one may therefore expect comparative-dynamics results deduced from dynamic economic models of the type studied here, to be just as important as a basis for empirical studies, as comparative-statics results deduced from either static models or steady-state growth models.

The dynamic financial model in Chapter 2 may be used for other comparative dynamics predictions concerning e.g. tax shifts and general interest rate shifts. Such predictions may also be deduced for other variables than leverage. In the general model these predictions became, to an extent, indeterminate and also mathematically cumbersome. However, in the special case in Appendices 2.1 and 2.2 further specific predictions along these lines were deduced. Reference is made to the Summary in Section 2.6.9.

The aim of Chapter 3 was to develop a dynamic optimal advertising model of a labor-managed firm in a marginalist-type framework, in order to gain some insight into the dynamic optimal advertising behavior of such a firm and how its behavior in this respect differs from that of a profit-maximizing firm.

A main focus of interest in the Western literature on the economic theory of the labor-managed firm has been the comparison of the labor-managed firm with the profit-maximizing firm. One point that has been discussed is whether market expenditures such as advertising are smaller in the labor-managed case than in the profit-maximizing case. If this is so, it could be argued that in this respect the labor-managed situation is socially more desirable.

Advertising has an inherent growth-generating property and furthermore may have a carry-over effect on future sales. The advertising process of the firm is thus actually dynamic and may therefore in such cases best be analyzed in a dynamic framework. Therefore recently a number of fully dynamic optimal advertising models of the profit-maximizing firm have appeared in the literature.

However, no fully dynamic advertising model of a labor-managed firm has been presented in the economic literature, even though this may perhaps provide interesting results on for instance the issue of comparative advertising behavior mentioned above.

A dynamic advertising model was therefore constructed and a deduction of the dynamic counterpart to the Dorfman-Steiner theorem and to the extended Nerlove-Arrow theorem was provided for the labor-managed firm. This, it seems, has not previously appeared in the literature. The dynamic Dorfman-Steiner theorem implies, for constant price and advertising elasticities, that the dynamic labor-managed firm has a greater advertising/sales ratio than the static labor-managed firm. The dynamic firm advertises more than the static one in order to build up advertising goodwill for the future. This behavior is common to both the labor-managed firm and profit-maximizing firm and illustrates their common general interest to use non-labor resources efficiently.

Put in another way, an important result of the dynamic model is that the marginal revenue of advertising is *less* than marginal cost. Thus, the *dynamic* effect of advertising offers an alternative hypothesis to the sales maximization theory of Baumol. This has been pointed out by other authors. In fact this is a special case of the more general hypothesis that marginal revenue is less than marginal cost when long-run effects are taken into acount. This general result offers an alternative, within the area of profit-maximization, to the managerial theories of the firm. This general result also illustrates the hypothesis that the firm will sacrifice some current profit in order to obtain larger profits in the future.

Another conclusion was that the dynamic Dorfman-Steiner theorem and extended Nerlove-Arrow theorem are valid also for the labor-managed firm under the qualification that λL for constant L is substituted for λ. λ is the dynamic shadow value for advertising goodwill and L is labor quantity. This could be expected, because the maximization of total labor income - which is equal to average labor income multiplied by labor - is equivalent to the maximization of profit, for constant labor. However, this similarity in mathematical form does not imply similarity in magnitude. Rather, due to different optimum elasticities, goodwill shadow value and employment, different optimal dynamic advertising behavior of the labor-managed firm and profit-maximizing firm is expected.

In a steady-state situation for the firm, the model indicates that the value of the advertising/sales ratio normally is different for the labor-managed firm and the profit-maximizing firm. However, for a demand function with constant elasticities with respect to price, advertising and goodwill, the model studied predicts that, in the steady state, the advertising/sales ratio for the labor-managed firm is the same as for the profit-maximizing firm. The steady-state situation is most likely an optimum state in the studied model after a certain passage of time.

In order to obtain detailed results regarding the optimum time-trajectories, assumptions were made about the structure of the demand function and production function, and about the numerical values of the parameters.

An interesting new insight was obtained from the dynamic example. It represents a case where, in spite of greater absolute advertising activities of the profit-maximizing firm, the relative current advertising of the profit-maximizing firm is smaller than for the labor-managed firm. Thus the social advantage of the labor-managed firm that is claimed in this respect by some authors, and debated by others, perhaps remains an open question requiring further research.

Finally, it should be stressed that the above results of Chapter 3 rest on a number of limiting assumptions, for instance as regards structure of model employed, goodwill transition equaition, objective function, discount rate, demand and production function, social-economic environment and the assumption of constant capital input. Also, the models are partial and do not explicitly take into consideration competition and reactions from other firms in the product market. However, the analysis in Chapter 3 nevertheless provides some interesting insight and may provide a basis for modification and further inquiries.

The aim of Chapter 4 was to develop a marginalist-type model of the dynamic optimal adjustment of a competitive labor-managed firm to a change in output price. The purpose was to gain some insight into the dynamic optimal employment behavior of such a firm and how its behavior in this respect differs from that of a profit-maximizing firm.

A peculiarity of the competitive labor-managed firm in its pure static form is its low short-run elasticity of supply, as compared with the competitive profit-maximizing firm. In the pure static case of a one-product firm with labor as the only variable factor, the short-run supply curve of a competitive labor-managed firm will even bend backwards as many authors have shown. However, in the longer term when capital is also variable, the long-run tendency towards capital expansion and a corresponding expansion in labor might counteract the short-run contraction in labor that is caused by an increase in price. The short-run and long-run adjustment behavior of the labor-managed firm to a change in price have been linked together in this study in an optimal control-theory framework. The problem does not seem to have been analyzed in this way before.

In the one-product, one non-capital factor case, a possible prediction of the model in Chapter 4 is that an increase in price implies a decrease in employment at first in the labor-managed firm, to be followed by an increase in employment up to the long-run equilibrium level. In other words, the labor-managed firm would follow different structural paths at different times in its adjustment process. The profit-maximizing firm of neo-classical economics would however normally react to a price increase by continuously expanding employment both in the short-run and long-run.

From the model studied it seems plausible that the adjustment process depends on the allowed adjustment rates for investment and employment and on the marginal labor income of capital and labor. Although a strict proof is not given for the result in the previous paragraph it seems safe to assume, from the discussion in this chapter that there exists a set of parameter values for which the stated result is valid. However, similarly, there also probably exists a set of parameter values for which the result in the previous paragraph is not valid. The main object of this chapter has been to form a framework for analysis of the problem rather than to obtain all-conclusive results.

Further insight into the dynamic adjustment of a competitive labor-managed firm to a change in output price may be gained from the proposed model by specifying the parameters or relations between the parameters and further analyzing the synthesis of the various paths.

The aim of Chapter 5 was to develop dynamic economic models of profit-maximizing firms in a marginalist-type framework, in which the discount rate is not constant but is either exogenously or endogenously variable. The purpose was to gain some insight into the effects of such non-constant discount rates compared to constant discount rates in certain cases.

First the effect of an exogenously variable discount rate in a specific share-value model of the firm was studied, after which the effect of an endogenously variable cost of capital in a Fisher-Hirshleifer framework was analyzed. A dynamic counterpart of the Fisher-Hirshleifer analysis was stated and after that the effect of an endogenously variable discount rate on the dynamic semi-investment behavior of the firm was studied. Finally these latter results were applied to some well-known advertising theorems.

In the case of an exogenously variable discount rate, one interesting conclusion was that a cyclical discount rate, compared with its constant average, altered the present value in certain instances to a not insignificant extent.

The linearly increasing interest rate case is interesting because it illustrates the concept of discounting more heavily in the future to take into consideration increasing uncertainty.

In the case of an endogenously variable discount rate, it was shown how some dynamic semi-investment and advertising models of the firm are modified.

First the effect of a cost of capital that decreases with firm size was analyzed in a two-period Fisher-Hirshleifer framework. As expected, the firm was found to expand its activities more than in the constant cost of capital case in a number of instances.

The same result was also obtained as regards the dynamic semi-investment behavior of the firm in the steady state, as well as along the optimal trajectory near the steady state (due to continuity).

The general result, in the type of dynamic models studied in Chapter 5, that marginal revenue is *less* than marginal cost in the dynamic case, is unaffected by the assumption of an endogenously variable discount rate, as expected. The same is valid as regards the general dynamic hypothesis that it is optimal for the firm to use its policy most heavily in the initial periods and thereafter gradually decrease the effort.

Also, modified dynamic Dorfman-Steiner and generalized Nerlove-Arrow theorems were deduced. An interesting result is that the dynamic Dorfman-Steiner theorem is invariant in form when an endogenously variable discount rate is introduced. The reason is that the Dorfman-Steiner theorem mainly expresses instantaneous price and advertising and therefore is not explicitly dependent on the discount rate. The generalized Nerlove-Arrow theorem, on the other hand, is modified with a new term in the denominator when an endogenously variable discount rate is introduced. This term represents the "interest" effect of increasing advertising goodwill one unit, and is a natural extension. As regards both of these dynamic advertising theorems, the direction of the change is not self-evident from the general expressions deduced.

Of special interest in the case of an endogenously variable discount rate is the change-over in the solution from regular time to "psychological" time.

The analysis in Chapter 5 appears to confirm that, as soon as the assumption of a stationary perfect capital market as regards the discounting process is dropped, the analysis becomes relatively complicated both conceptually and analytically. It follows from the nature of the subject studied that the analysis performed is explorative. However, the cases studied in Chapter 5 provide interesting insight and may also provide building blocks for further inquiries. Such inquiries may be especially timely, since fluctuating discount rates and interest rates are increasingly common in many economies today.

Following this summary of the various chapters, a few concluding comments would be appropriate.

Perhaps the most important theme of this study is the *dynamic* optimization analysis of firms with *other objectives* than profit alone.

Due to the nature of the subject, and to the relative novelty of the area studied and the methods used, the whole study may be considered as an exploratory investigation. Many of the assumptions made could be substituted for other and more realistic assumptions. However, the choice of assumptions was often motivated on grounds of simplicity in order to have a manageable analysis. Although many assumptions may be considered unrealistic, perhaps they can be regarded as a first approximation which may be modified in a further analysis. Some of the assumptions are: the firm is studied in complete isolation from the rest of the economy, uncertainty does not enter explicitly into the analysis, inflation is absent, the parameters are constant in time and no technological development occurs in the firm.

Also, it should be mentioned that the main result of the study is perhaps not the specific conclusions about optimal dynamic firm behavior in the cases studied, but rather the optimal dynamic analysis of firms with other objectives than solely profit, as such. In other words the analysis to achieve the results is perhaps equally important as the results themselves.

Several areas of inquiry could or should be added. An important one concerns the further testing of the models and results against existing empirical evidence. However, as mentioned above, the models include various limiting assumptions that often preclude the possibility of testing the models and results against existing empirical evidence.

Various hypotheses posed in the study, for example that relative advertising expenditure in the labor-managed firm is greater in comparison with the profit-maximizing firm, may of course be subjected to testing in new empirical studies designed for the purpose. Other hypotheses, such as the significance of a fluctuating discount rate on the equity share value of the firm, may be more difficult to test empirically because of the relative theoretical abstraction inherent in the discount rate; also, even if that could be overcome, there would still be the difficulty of finding an appropriate fluctuating time series for the discount rate.

Apart from adding more realistic assumptions and/or more empirical testing, the study could also be extended by developing the theoretical analysis and results both in economic and mathematical terms. Especially, the economic interpretation of the analysis may be extended and compared to other theoretical economic studies. More work may be done in clarifying how the obtained results depend on the various assumptions and which alternative assumptions may be used. Also, further sensitivity analyses (comparative dynamics) and numerical calculations could be undertaken. This may in turn facilitate empirical testing.

There are therefore many areas calling for further investigation, and many questions left open, which in a sense means that the study has been worthwhile.

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