Reappraisal of Market Efficiency Tests Arising from Nonlinear Dependence, Fractals, and Dynamical Systems Theory

Gun-Ho Cha

AKADEMISK AVHANDLING

som för avläggande av ekonomie doktorsexamen
vid Handelshögskolan i Stockholm
framlägges till offentlig granskning
torsdagen den 27 maj 1993
kl. 10.15 i sal Torsten
å högskolan, Sveavägen 65

STOCKHOLM 1993

Reappraisal of Market Efficiency Tests Arising from Nonlinear Dependence, Fractals, and Dynamical Systems Theory

// EFI, EKONOMISKA FORSKNINGSINSTITUTET
Handelshögskolan i Stockholm

Adress: Sveavägen 65, Box 6501, 113 83 Stockholm. Tel. 08-736 90 00

---

* grundades år 1929
* är en vetenskaplig forskningsinstitution vid Handelshögskolan i Stockholm
* bedriver teoretisk och empirisk forskning inom företagsekonomins och samhällsekonomins olika områden och ger i samband därmed avancerad utbildning
* söker välja forskningsobjekt efter forskningsområdenas behov av teoretisk och praktisk vidareutveckling, projektens metodologiska intresse samt problemställningarnas generalitet
* offentliggör alla forskningsresultat; detta sker i institutets skriftserier och seminarier, genom tidskriftsartiklar samt via undervisning vid Handelshögskolan och annorstädes samt ofta också i artikelform
* består av ett 100-tal forskare, främst ekonomer men också psykologer, sociologer och statistiker
* har organiserat sin forskningsverksamhet i följande sektioner och fristående program

A	Företagslednings- och arbetslivsfrågor
B	Redovisning och finansiering
C	Kostnadsintäktsanalys
CFR	Centrum för riskforskning
CHE	Centrum för hälsoekonomi
D	Distributionsekonomi, strukturekonomi och marknadspolitik
ES	Ekonomisk statistik
F	Förvaltningsekonomi
FDR	Fonden för handels- och distributionsforskning
FI	Finansiell ekonomi
I	Information Management
IEG	Internationell ekonomi och geografi
P	Ekonomisk psykologi
S	Samhällsekonomi
PMO	Forskningsprogrammet Människa och organisation
PSC	Forskningsprogrammet Systemanalys och planering

Ytterligare information om EFI:s pågående forskning och publicerade forskningsrapporter återfinns i institutets projektkatalog, som kan rekvieras direkt från EFI.

Reappraisal of Market Efficiency Tests Arising from Nonlinear Dependence, Fractals, and Dynamical Systems Theory

Gun-Ho Cha

STOCKHOLM SCHOOL OF ECONOMICS
THE ECONOMIC RESEARCH INSTITUTE

A Dissertation for the
Doctor's Degree in Philosophy
Stockholm School of Economics 1993

© EFI and the author
ISBN 91-7258-365-7

JEL Classification: C12; E44; F14.

Keywords:
Market efficiency;
Nonlinear dependence;
U-statistics;
Chaos;
Fractals;
Long term memory.

Distributed by:
EFI, Stockholm School of Economics
Box 6501, S-113 83 Stockholm, Sweden
Graphic Systems AB, Stockholm 1993

ACKNOWLEDGEMENT

My sincere thanks go to Professor Bertil Näslund, Chairman of my degree committee, who offered invaluable advice and support in conducting this research and my academic career at the Stockholm School of Economics (SSE).

Equally acknowledged are the invaluable help and comments from Dr. Ragnar Lindgren and the introduction to the stock market efficiency tests given by Professor Peter Jennergren. Special thanks to the Statistics Department of the SSE for timely comments.

I also wish to thank some professors outside the committee who have helped me to clarify many new concepts. Among those, Professor W. Brock (University of Wisconsin) and Dr. Mizrach (Federal Reserve Bank of New York) provided me with their invaluable papers. Dr. Baek (Iowa State University) allowed me to take part in his program.

Any errors found in this research, however, are solely mine.

Finally, my greatest debt is to my family, Ok-Jong, Yu-Hae, and You-Jin, for their support and understanding, which made it possible for me to do this research.

This research was partially supported by Non Directed Research Fund, Korea Research Foundation (1991) and Bankforskningsinstitutet, Sweden (1992).

February 1993

TABLE OF CONTENTS

LIST OF TABLES

Table

Appendix Table

LIST OF FIGURES

Figure

Appendix Figure

INTRODUCTION

Problem Statement

The efficient market hypothesis [EMH] has long been perceived as the cornerstone of modern finance theory. However, the efficient market hypothesis has also recently been dismissed as "the most remarkable error in the history of economic theory" (Wall Street Journal, Oct. 23, 1987) in response to the combined phenomenon of the October 1987 Crash and a proliferation of empirical anomalies reported in financial economics and statistics journals. Whatever strong reservations we may hold in the efficient market hypothesis debate, the hypothesis has still been central to fundamental policies of investment practice and corporate financial management. Can we really believe that financial markets are efficient?

In fact, while a voluminous literature has developed supporting the hypothesis, much research has also been conducted on the various violations of the traditional financial models and assumptions as well as on the tests against the EMH in the financial literature from the 1960s. A considerable amount of research has also focused on the alternative approaches or models which cope with these violations.

It is generally believed that stock index returns do not hold a Gaussian distribution. They are dependent and nonlinear in the second moment. Consequently, most financial models and tests, assuming independent and identical distributions for price changes and the linear structure between them, might display misleading results. In particular, the concept of nonlinear dependence structure in the second moment became more important in testing the EMH in the late 1980s. Recent developments follow two research tracks. One research track focuses on testing the random walk hypothesis under nonlinear structure.

The other research track focuses on verifying the nonlinear structure between returns. In both tracks, the research relies on detecting the evidence against the EMH.

With regard to nonlinear dependence between returns, some more dynamic models have appeared. Taylor (1986) proposed that the product process model would give more accurate results under the nonlinear structure. Bollerslev (1986) has suggested an extended GARCH model in which the disturbance term follows a conditional t-distribution with time varying variance. More recent studies tend to detail nonlinear dependence in the variance. Some researchers apply the R/S (Rescaled range) model, Bispectrum model, and Fractals to detect long term memory and patterns, which provide evidence against the EMH (Lo (1988); Hinich and Patterson (1992); Li (1991); Peter (1992)). Moreover, some researchers make use of a new class of statistical tests and estimates, arising out of the nonlinear dynamical systems theory to test for the EMH and to describe the nonlinear structure between returns (Brock (1991); Hsieh (1991); Scheinkman and LeBaron (1989); Willey (1992)).

As far as stock markets are concerned, these new concepts have mainly been applied to thick markets: the USA, the UK, and Japan. Most of the findings are evidence against the EMH, that is, there exist persistent patterns; there are significant autocorrelations; there might exist long memory; there exists low dimension-structure. Our major contribution in this study is to apply such new concepts (models) to two relatively small stock markets: the Swedish market and the Korean market, both of which are well known by the author. From the studies, we intend to find answers to the following questions:

1. Are there any differences between the previous results analyzed under the linear and independent generating process and our findings?

2. Are there any differences between the results for some thick markets and our findings for the sample markets?

3. Can we really believe that the sample markets are efficient under the nonlinear and dependent generating process? If we can not, "why"?

4. If the sample markets are not efficient, can volatility be predicted?

5. What are the suggestions for future study in this area?

Procedures

There are many alternative approaches (models) for the EMH under the dependent and nonlinear generating process, which cope with the various violations of traditional financial models and assumptions as well as tests. Generally, they can be classified into 4 categories. No one can say which is the best. Hence we apply one approach from each category.

The first category focuses on applying familiar economic models and conventional statistical tools based on a normal (Gaussian) distribution in a time domain methodology. Our study follows Taylor's model, the product process model, in particular. We will still use the conventional statistical tools but introduce a new alternative hypothesis against the EMH, "Excessive Response Hypothesis."

The second category focuses on frequency analyses. We apply the power law function to the sample markets, since this is very useful in detecting whether long term memory and patterns exist or not. The spectral density gives an estimate of the mean square fluctuations at frequency f and, consequently, of the variations over a time scale of order $1/f$. We will use a logarithmic slope of the spectral density in detecting what kind of noises exist in the sample processes.

The third category focuses on detecting long term memory and patterns with a technique, "R/S Analysis", which comes from Fractals. There are two approaches to the selection of lags and starting points: F Hurst and G Hurst.

Since the bias and variability of estimates of F Hurst is less than G Hurst, our study will make use of F Hurst. Moreover, we will also apply the modified R/S, developed for the application to financial time series.

The last category focuses on determining whether the generating process is stochastic or deterministic and on testing the EMH, with a new concept, "correlation integral", which is a key concept of the dynamical systems theory. The deterministic chaos hypothesis will be tested to see whether the process generating returns is really stochastic. Moreover, we will decide the correlation dimensions for the sample sequences. This test also validates the adequacy of the stochastic approach to describe the sample data if rejected.

The data set consists of 2534 daily AFGX series for the Swedish stock market and 3374 daily KCSPI series for the Korean stock market.

Organization of Study

This study is organized as follows:

Chapter 1 reviews the challenges to stock market efficiency. It focuses on the modern evidence of inefficiencies: excessive volatility, stationarity issues, speculative bubbles, fads, mean reversion, GARCH model, bispectrum model, the product process model, and mixed diffusion jump process. The nonlinear dynamic models which can capture the nonlinear dependence that is sometimes not uncovered in the above models, are then discussed.

Chapter 2 criticizes traditional financial models and assumptions and furthermore reviews various alternative approaches. To date, numerous models have been considered. Even though past research is not decisive, some findings are noteworthy. First, the variance of price change is not constant but is changing over time. Second, a very recent alternative model with a deterministic system can lead to better forecasting and control of state variables of the economy and identify the nonlinear structure between returns. Third, the long term memory concept is also essential for theoretical and practical reasons in daily price changes. These are discussed in this chapter. Moreover, we suggest that it would be better if the tests for the EMH should be done under a new model which considers the time domain concept, the frequency domain concept, the long term memory concept, and the deterministic chaos concept.

Chapter 3 discusses the sample data. Details of each data set and summary statistics are provided. In discussion of the characteristics and statistics, evidence of non-normality (in particular, dependence, and nonlinear structure between returns) is clearly demonstrated. This implies that conventional statistical tests based on the independently and identically distributed (i.i.d.) normality or linear structure between returns may not work well empirically.

Chapter 4 presents the estimated results under the product process model for the sample markets, and compares them with the previous results analyzed on the i.i.d. normality and linear process as well as with the results for some thick markets. As test statistics, we take the first autocorrelation statistic, the Box-Pierce statistic, and price trend statistic. In particular, the excessive response statistic, an extension of the price trend model, is also introduced and tested. Since all the tests are based on the estimated autocorrelation coefficients, this chapter is concerned with a time domain methodology of using the autocorrelation function.

Chapter 5 is concerned with a frequency domain method. In contrast to the general spectral analysis relevant for economic studies, the focus will be on the low power spectrum (law), which implies that the spectrum peaks at lower frequencies are due to the large contributions from the slow varying, non periodic components. It will be demonstrated that the power law function works well empirically for the sample series. Furthermore, this study verifies that although $1/f$ noise is common in nature, the time series of price changes are much closer to $1/f^2$-noise with self similar patterns.

Chapter 6 investigates the distinct but non periodic cyclical patterns that typify plots of economic aggregations over time. In the finance literature, detecting the presence of long term dependence in stock prices is often viewed as evidence against the market efficiency. The test statistics, called Rescaled range (R/S) analysis, a method favored and used extensively by Mandelbrot, will be investigated. Furthermore, Lo's modified R/S statistic under the null of strong mixing is also applied to the sample series in this chapter. Here, it will be demonstrated that the sample series have persistent patterns, but not for long lags.

Chapter 7 deals with the market inefficiency arising from nonlinear dynamical systems theory. First, the recent developments and issues concerning this area will be reviewed. Furthermore, the recent developments will be divided into the 3 issues: (1) the import-ance of low power problems of standard (conventional) test; (2) identification of the source of model misspecification; and (3) detection and description of nonlinear structure. Second, a relatively new concept, the correlation integral, will be described. In particular, some of its properties will be presented: U-statistics and the connection to dimension measures. Moreover, the BDS test for detecting nonlinear dependence will be introduced. Finally, this model will be applied to the sample series. It will be shown that the sample markets have low dimensions. It will also be established that the sample series is not chaotic, but stochastic.

Chapter 1

Challenges to Stock Market Efficiency: A Modern Overview

The efficient market hypothesis [EMH] has been one of the most intensely researched topics in the investments field in the past few decades. In fact, much of the theoretical basis for current monetary and financial theory rests on the economic efficiency of financial markets. Considerable effort has been expended to test the EMH, usually in the form of the random walk model for stock prices. The fall of the 1990 finally saw the efficient market school's ultimate hour of triumph. Three celebrated figures in the field, Harry Markowitz, William Sharpe and Merton Miller, received the Nobel Prize in economics.

Most of the early research was concerned with detecting the efficiencies or inefficiencies under the random walk hypothesis by autocorrelation tests, runs tests, and filtering tests. In general, the inefficiencies detected are relatively small. Recently, however, historical price series have been used to severely question the EMH. Much research has been concerned with the apparent detection of anomalies, such as size or earning anomalies or seasonal effects. Another field covered in this topic is the existence of excess volatility, and rational bubbles in stock prices. In both of these areas, the evidence suggests the existence of some inefficiencies. Moreover, more recently, there has been an explosion of research activity to detect inefficiencies in the general area which we will call "nonlinear science." This perspective includes chaos theory, a part of statistical mechanics which deals with interacting particle systems, self-organized criticality, spin glass models and stochastic approximation theory as well as typical nonlinear time series models.

This chapter addresses the question of the efficiency of the stock market - do stock prices correctly reflect available information about future fundamentals, such as dividends and interest rates? It provides an overview of the efficient markets hypothesis and alternatives to the efficient markets view, concentrating not only on recent research but also on traditional approaches. In section 1.1, the basic concept for the efficient stock market paradigm is described. Section 1.2 reviews the efficient market hypothesis in relation to stock market anomalies. In the following section, we briefly present modern evidence of inefficiencies. In particular, we focus on excessive volatility, stationarity issues, speculative bubbles, fads, mean reversion, the GARCH model, the bispectrum model, the product process model and the mixed diffusion jump process model. In section 1.4, the chaotic models of nonlinear systems are discussed. These models capture the nonlinear dependence between returns which is sometimes not uncovered in the above models. Recent results on detecting inefficiencies under the dynamical systems models are also reported. Finally, we conclude with a brief summary and some comments in section 1.5.

1.1 The Efficient Stock Market Paradigm

Practitioners are interested in the stock market paradigm, since it is their bread and butter. Academic economists are interested for a very different reason: for them, the stock market provides an excellent laboratory for the evaluation of microeconomic theory. Common stocks are highly standardized products traded in an active auction market with very easy exit and entry of both producers (firms issuing equity) and consumers (investors purchasing shares); as a result, the prices of common stocks should conform to the implications of the theory of competitive markets.

The notion of market efficiency paradigm has been defined in many ways. Fama (1965; 1991) presents a thorough discussion of both theoretical issues and modern empirical tests of this proposition. In the development below, it is assumed that the required expected rate of return on the security is equal to a constant, r, which is known with certainty. In fact, as has frequently been observed, standard tests of market efficiency are really joint tests of efficiency and a model specifying expected returns. So, the assumption made here that the ex ante return is known and constant makes it possible to focus only on the test of market efficiency.[1)]

Also assume that the security in question yields a sequence of cash flows, D_t. These may be thought of as dividends if the security is a stock, or coupons if the security is a bond. If the security has a finite maturity, T, then D_T may be taken to represent its liquidation value, and all subsequent values of D_t may be taken to equal zero. One statement of the hypothesis of market efficiency holds that:

$$P_t = E_t[P_t^*] = E\left[\sum_{s=t}^{\infty} \frac{D_s}{(1+r)^{s-t}} \mid \Omega_t\right] \tag{1.1}$$

where $E_t(\bullet \mid \Omega_t)$ denotes the mathematical conditional expectations operator; Ω_t represents the set of information available to market participants at time t; r denotes the opportunity cost of capital. In fact, this is not the form in which the hypothesis is usually tested. Equation (1.1) is mathematically equivalent to the statement that, for all t:

$$P_t = E\left[\frac{P_{t+1}}{1+r}\right] + E\,[D_t] \tag{1.2}$$

or the equivalent statement that

1) Since we assume that the model generating expected returns is known with certainty, it will overestimate the power of available statistical tests. Recent theoretical work suggests that the particular model of ex-ante returns considered here cannot be derived rigorously. What is crucial is that the discussion is carried on assuming full knowledge of the model characterizing ex-ante returns.

$$E[R_t] = E\left[\frac{P_{t+1}}{P_t} - 1 + \frac{(1+r)D_t}{P_t}\right] = r \tag{1.3}$$

where R_t denotes the real return on stocks at time t. Note that once a transversality condition is imposed on the difference equation (1.3), it implies equation (1.1). Furthermore, let $\delta=(1+r)^{-1}$. Then, on rearranging (1.1), we get

$$P_t = \sum_{k=0}^{\infty} \delta^{k+1} E_t[D_{t+k}] \tag{1.4}$$

Equation (1.3) also implies that:

$$R_t = r + e_t \tag{1.5}$$

where e_t is serially uncorrelated and orthogonal to any element of Ω_t. Normally, the market efficiency is tested by adding regressors drawn from Ω_t to (1.5) and testing the hypothesis that their coefficients are equal to zero, or by testing the hypothesis that e_t follows a white noise process. The former represent tests of "semistrong" efficiency while the latter are tests of "weak " efficiency.

In short, the EMH paradigm provides two definitions:

The EMH, Definition 1: Prices Are Optimal Forecasts.

The EMH, Definition 2: Risk-Adjusted Returns Are Equalized.

Of course, there are some caveats in the EMH paradigm. Both forms of the EMH rest on a strong assumption: the market equilibrium of asset prices is independent of the distribution across investors of the two basic raw materials of investment: information and wealth. All those things that make different investors evaluate assets differently are treated as of negligible importance. Among these "irrelevant" factors are differences in probability assessments, differences in transaction costs, and differences in tax rate paid by investors. If these factors can be ignored, the prices or returns on assets will be determined solely by the fundamentals. But if they are important, prices can deviate, perhaps persistently, from fundamental values.

1.2 Stock Market Anomalies and the EMH

A voluminous literature has developed to support this hypothesis. But skepticism about the efficiency of the stock market has also been bolstered by several recent challenges to the empirical validity of the efficient market paradigm. These challenges are of three kinds. First, some researchers report instances in which stock returns do not behave according to the predictions of the EMH since investors can use available information to earn "excess" profits. Second, other researchers argue that stock price changes are by far too great to be compatible with the efficient markets model. In the USA, it has even been suggested that if "excess volatility" characterizes stock prices, the Federal Reserve should reduce volatility through open market operations in the stock market.[2] Third, some complexity scientists argue that stock price movements follow a deterministic mechanism, and furthermore, they show the low dimension structure underlying the stock price changes.

There are also numerous studies showing that stock prices are random walks in the sense that stock prices provide no useful information in predicting future stock prices. However, these studies do not represent the preponderance of the evidence for two reasons. First, gross inefficiencies can coexist with random walks in stock prices, as in the case of rational bubbles. Second, and more important, by the 1980s a vast literature on stock market anomalies had developed. These anomalies, defined as departures from efficient markets that allow economic agents to enjoy unusually high(risk-adjusted) returns, appeared to lead to rejection of the EMH. This section reviews some of the major anomalies in stock price determination.

Some important anomalies which are among the list of causes that would appear against the market efficiency are as follows:

---

2) See Fisher and Merton (1984).

(1) *The Small Firm Effect:* The common stocks of small capitalization companies have, on average, exhibited unusually high rates of returns throughout most of this century. According to the EMH, the small firm effect should be due solely to higher beta coefficients for small stocks; in other words, the higher rate of return is solely due to higher risks. The evidence suggests, however, that the higher return on small-cap stocks cannot be explained by higher risk.

(2) *Weekend and January Effects:* Many researchers have reported that average stock returns are lower on Mondays and higher on Fridays than other days of the week. This difference is an anomaly, since the efficient market paradigm cannot account for this systematic effect. The EMH would predict, if anything, that returns should be higher on Mondays since Monday's return is for three days rather than for one. A plausible explanation of the weekend effect is that firms and governments release good news during market trading, when it is readily absorbed, and store up bad news for after the close on Friday, when investors cannot react until the Monday opening. In recent years, the January effect has received considerable attention; the rate of return on common stocks appears to be unusually high during January. The primary explanation is the existence of tax-loss selling at year end: investors sell their losing stocks before year end in order to obtain the tax savings from deducting those losses from capital gains realized during the year. However, this explanation is not consistent with the EMH, according to which investors with no capital gains taxes, such as pension funds, should identify any tendency toward abnormally low prices in December and should become buyers of stocks oversold in late December. This means that tax-loss selling should affect the ownership of shares but not their price. Keim (1983) has shown that the January effect appears to be due largely to price behavior in the first five trading days of January; it is really an Early-January Effect. Also, Reinganum (1983) has found that the January effect and the small firm effect are mixed: the

January effect appears to exist primarily for small firms and, in fact, much of the small-firm effect occurs in January.

(3) ***The Value Line Enigma:*** The Value Line Investment Survey, the largest advisory firm in the United States, produces reports on 1700 publicly traded firms. Semistrong form efficiency predicts that investors should not benefit from the Value Line recommendations. Several studies have documented, however, that investors following the Value Line recommendations would have earned abnormally high returns. Moreover, it is noteworthy that this advantage existed even when adjustments were made for both transaction costs and risk (beta) (see Copeland and Mayers (1982); Holloway (1981); Stickel (1985)).

(4) ***Money Supply:*** One test of whether money supply growth can be used to predict stock returns is to estimate the relationship between stock returns and past money growth rates.If the stock market is semistrong form efficient, there should be no statistically significant association between stock returns and past money supply movements. Some researchers have reported that there is a relationship between money growth and stock returns (see Sprinkel (1964); Homa and Jaffee (1971)). However, Rozeff (1984), and Davidson and Froyen (1982) reach the opposite conclusion.

So far, we have briefly reviewed some anomalies, defined as departures from the EMH. The difficulty is that even though there has appeared so much strong empirical evidence against the efficient market model by finding some anomalies, there are still several reports which support the EMH by finding no anomalies. For example, some evidence that the small firm effect has disappeared and that the daily patterns of stock returns have grown weak has been reported in recent years.[3)]

1.3 Modern Evidence of Inefficiency

The previous section reported the results of "traditional" approaches to assessing the EMH: examination of specific examples of departures from the EMH, called anomalies. We also indicated that no distinct conclusion could be drawn regarding the anomalies. So, during the 1980s several "modern" approaches have been developed. These are discussed in this section.

1.3.1 Excess Volatility of Stock Prices

Under the null hypothesis of the EMH, the market price must vary by no more than the fundamental price. Any "excess" volatility is, therefore, a symptom of market inefficiency. Leroy and Porter (1981), and Shiller (1981) have asserted that the observed amount of stock price volatility is too great to be consistent with the EMH.

The logic of the excess volatility argument is based on a property of statistical theory: the optimal forecast of a random variable should, on average, vary by no more than the amount of variation in the random variable being forecasted. Thus, if the market price is an optimal forecast of the fundamental value - as the EMH implies - it should vary less than the fundamental value.

A formal statement of the excess volatility argument is that the relationship between the fundamental price under the actual state (s) and the optimal forecast of the fundamental price is

---

3) For the VWT series, re-estimating the model over five-year subperiods indicates that the weekend effect disappeared after 1975. For the EWT series, the effect remains but becomes less significant. Note that VWT is the rate of return on a portfolio of all stocks on the New York Stock Exchange and American Stock Exchange in which the return on each stock is value weighted by the size of the company, and EWT is the rate of return on the same portfolio but with each stock equally weighted.

$$P_s^* = E(P^* \mid \Omega_t) + e_s \tag{1.6}$$

where e_s is a random variable that measures the deviation between the fundamental value for the state which actually occurs, P^*_s, and the optimal forecast of the fundamental value, $E(P^* \mid \Omega_t)$. If the forecast is optimal, these deviations must be random and uncorrelated with the forecast itself. Now, the EMH implies that $P=E(P^*_s \mid \Omega_t)$, which leads to

$$P_s^* = P + e_s \qquad (1.6)'$$

In other words, the correct price conditional on knowing the true state is equal to the market price plus a random term, denoted by e_s, which measures the surprise resulting when the true state is known. This random term must be uncorrelated with P, because P is the optimal forecast and, therefore, already reflects any systematic information.

Equation (1.6)' implies that the variance of the fundamental price is equal to the variance of the market price plus the variance of the surprise. Turning this around produces the following relationship:

$$var(P_t) = var(P_t^*) - var(e_t) \tag{1.7}$$

Since variances must be non-negative, the variance of the market price must be no greater than the variance of the fundamental value, if the EMH is valid:

$$var(P_t) \leq var(P_t^*) \tag{1.8}$$

or

$$var(P_t) / var(P_t^*) \leq 1 \qquad (1.8)'$$

The variance inequality or variance ratio, has become an extremely popular way of testing the EMH model of stock prices.

The conclusion that excess volatility exists has been criticized for a number of reasons, each of which can be seen as a criticism of the test. Marsh and Merton

(1986) have disputed one of the assumptions underlying Shiller's test - that dividends are a stationary time series - and have shown that if the process by which dividends are set is non-stationary, the EMH test is reversed: under the EMH, market prices should be more volatile than fundamental values. Kleidon (1986) has also criticized the excess volatility test on statistical grounds, arguing that the Shiller test is an asymptotic test, assuming a very large sample of observations over time, and that the data available are necessary finite, hence small-sample biases can weaken the test. In addition, the power of the test against reasonable alternative hypothesis is quite low, meaning that the test is not likely to reject the EMH when it should be rejected.

Whatever the validity of the excess volatility tests is, they do provide an additional reason - other than observed anomalies - to doubt the validity of the EMH, and they have had a significant effect on the state of academic thinking about market efficiency.

1.3.2 Stationarity Issues

This sub-section concerns the stationarity assumption invoked by Shiller. Both Kleidon (1986) and Marsh and Merton (1986) argue that the results noted above may simply be a reflection of the assumption, implicitly made in Shiller's 1981 paper, that the price and dividend series do not contain unit roots. However, when allowance is made for stochastic non-stationarity, by using arithmetic or logarithmic differences of D_t and P_t, the inequality (1.8) still tends to be violated, although by a lesser magnitude (Kleidon (1986) and Shiller (1983)). Perhaps one of the neatest, and currently most popular, ways in which the non-stationarity issue has been handled is that proposed by Campbell and Shiller (1987). Their insight is to note that if both D_t and P_t series contain unit roots, then (1.4) implies that D_t and P_t must be cointegrated (Engle and Granger (1987)) with a cointegration parameter $\gamma=\delta(1-\delta)^{-1}$; that is, the variable $P_t-\gamma D_t$ is stationary. The intuition for this result is that if both D_t and P_t contain unit arithmetic roots, subtracting some multiple of D_t from P_t removes the linear trend in P_t and gives a stationary random variable.

For example, on subtracting γD_t from both sides of (1.4), and on defining $S_t=P_t-\gamma D_t$ as the spread, we obtain

$$S_t = E_t[S_t^*] \tag{1.9}$$

where

$$S_t^* = \gamma \sum_{k=1}^{\infty} \delta^k \Delta D_{t+k} \tag{1.10}$$

In words, (1.9) simply states that S_t is an optimal forecast of a geometric weighted sum of future values of ΔD_t, conditional on the agent's full information set. Campbell and Shiller (1987) propose testing this version of the model in a number of ways. First, if D_t and P_t are cointegrated it is well known that S_t and ΔD_t (or ΔP_t) will have a vector autoregressive (VAR) representation.[4] Standard restriction tests may be performed on this system in order to gauge the validity of the hypothesis captured by (1.4) or, equivalently, (1.9) and (1.10). Furthermore, the VAR representation allows one to test, in a straightforward fashion, the idea that S_t should Granger-cause ΔD_t (i.e., if S_t is an optimal forecast of future values of ΔD_t it should have explanatory power, over and above the past history of ΔD_t (see Campbell and Shiller (1987)). A second advantage of the VAR framework is that it can be used to generate measures of the model's economic importance (in addition to its statistical significance as described by the restrictions tests). Such measures can be implemented by constructing the theoretical spread S'_t, which is defined as the unrestricted VAR forecast of the present value of future changes in dividends ((1.9) and (1.10)), and comparing it graphically with the actual spread; if the model is valid the two series should be closely related. Additionally, the variance ratio $var(S_t)/var(S'_t)$ may be computed and should, as before, equal unity if the model is valid; if the ratio is

4) See Engle and Granger (1987).

larger than one, then the spread is too volatile relative to information about future dividends.

Campbell and Shiller (1987) implement the VAR model and find that although the spread Granger-causes the change of dividend, the model is rejected at standard significance levels. Additionally, the computed variance ratios for the spreads are dramatically different from one and the theoretical spread does not track the actual spread in a satisfactory way. Hence even accounting for non-stationarity in this rather sophisticated way does not appear to save the simple efficient markets model.

1.3.3 Speculative Bubbles

The notion of a "bubble" is a familiar one: a bubble reflects a difference between the fundamental value of an asset and its market price. Here we discuss speculative bubbles and attempt to assess whether they do indeed save the EMH model.

The bubble concept has been powerful because of the notion of self-fulfillment: bubbles are self-fulfilling departures of prices from fundamental values which continue until, for some reason, the conditions of self-fulfillment disappear. What do we mean by self-fulfilling bubbles? We return, for simplicity, to the constant returns version of the model.

It is well known from the rational expectations literature that, in the absence of an appropriate transversality condition, equation (1.2) has multiple solutions each of which may be written in the form

$$P_t = E_t[P_t^*] + b_t \tag{1.11}$$

where b_t is the rational bubble term. It is straightforward to demonstrate that if $E_t[b_{t+1}]=\delta^{-1}b_t$, then (1.11) is a solution to (1.2). If b_t is a constant equal to b, then (1.11) says that, even if dividends are constant, the stock price will grow at the rate $b\delta^{-1}$. This is a pure capital gain unrelated to fundamentals and there are an infinite number of such paths, one for each value of b. If everyone believes that share prices will rise at some common rate, unrelated to fundamentals, the price will go up by that amount; prices can rise through speculation and, moreover, those expectations will be fulfilled.

The deterministic bubble b is perhaps the simplest example, but casual empiricism would suggest that it is not very realistic. The South Sea bubble and the Tulipmania bubble (see Garber (1989)) are both examples of asset price movements which are not consistent with a deterministic process; furthermore, those bubbles eventually burst. A more attractive family of bubbles has been suggested by Blanchard (1979). For example, a collapsing bubble may be modelled from the following structure:

$$b_{t+1} = \begin{cases} b_t(\delta\pi)^{-1} & \text{with probability } \pi \\ 0 & \text{with probability } 1-\pi. \end{cases}$$

This structure satisfies $E_t[b_{t+1}]=b_t$ and therefore provides a solution to (1.2). The bubble has a probability of collapse at any point in time, and once it collapses, it is over since all future b_t's are zero. However, with probability π it does not collapse and continues growing.

The existence of bubbles can explain the empirical finding that variance ratios are violated. Since there is no a priori reason to exclude the possibility that P_t and b_t are positively correlated, inequalities like (1.8) cannot be derived from (1.11); evidence of excess volatility is "prima facie" evidence for the presence of speculative bubbles. However, a number of theoretical issues and also empirical evidence lead us to seriously doubt that it is bubbles which explain the rejection of the EMH model. From a theoretical perspective, the existence of bubbles is somewhat questionable. First, it is theoretically impossible to have negative bubbles (i.e., $b_t<0$). This is because such a bubble would imply a stock price below its fundamental value, a capital gain when the bubble bursts and therefore the requirement of a potential capital loss as the bubble moves downwards. However, since stock prices cannot be negative, there must be some (low) price which precludes further capital loss and which therefore must also be inconsistent with a bubble; by backward deduction, any higher price must also be inconsistent (since it inevitably leads to the low price) and therefore there cannot be a negative bubble. Positive bubbles, which are the kind usually discussed in the literature, also have a rather thin theoretical basis. Tirole (1982) has demonstrated, in an infinite horizon model, that positive bubbles cannot arise. The idea is that an agent who sells a stock at a price higher than its fundamental value and leaves the market passes on a stock with a negative present value; this is clearly not an attractive investment for a rational maximizing agent. In finite horizon, overlapping generations, perfect foresight models, positive bubbles may exist in steady state equilibrium as long as the rate of growth in the economy is greater than the return on the stock (see, for example, Tirole (1986)). However, West (1988) indicates that this relationship does not find support in the US data.

Furthermore, in testing for bubbles a financial econometrician is immediately faced with the problem of distinguishing them from other phenomena. For example, Hamilton (1986) gives some illustrations of how an econometrician might incorrectly conclude that there exists a speculative bubble when there is an expected regime change of which he is unaware (i.e., the so-called peso problem first noted by Krasker (1980)).

How should one go about testing the role of rational bubbles? This question is difficult to answer, since a rational bubble will not affect the sequence of prices until it breaks. The analysis of such low probability events is called the "peso problem": market prices will not reflect the effects of very low probability events even if they should have dramatic effects when they appear. Hence, it would be impossible to uncover a rational bubble as long as it exists. However, the disappearance of a bubble, such as a major decline in stock prices, can be examined to determine whether it was preceded by a speculative bubble in price.

1.3.4 Fads

We have reviewed that rational speculative bubbles do not seem to provide a viable way of explaining excess volatility findings. Thus, some attention has focused on the idea that uninformed, naive or noise traders may be at least partially to blame for this empirical finding. Anecdotal evidence on noise trading is widespread; the most famous discussion is probably Keynes's (1936) analogy of the stock market with a beauty contest. More recently, DeBondt, Werner and Thaler (1985) and Shiller (1981), among others, have argued that noise trading may be at the root of excess asset price volatility.

A simple basic equation for a fads model would be

$$P_t = E_t[P_t^*] + f_t \tag{1.12}$$

where f_t denotes a fads term. This will, in general, lead to a violation of the variance inequality (1.8) in the same way as the presence of bubbles. Superficially, this is indeed very similar to the case of rational bubbles, as discussed above. The important difference, however, is that the fads term is assumed to be mean-reverting: the price will have a tendency to return to fundamentals. For example, we might have

$$f_t = \kappa f_{t-1} + e_t$$

where $|\kappa| < 1$ and e_t is stationary white noise. Another, more sophisticated example would be (West (1988))

$$f_{t+1} = \begin{cases} (\kappa/\pi) f_t + e_{t+1} & \text{with probability } \pi \\ e_{t+1} & \text{with probability } (1-\pi) \end{cases}$$

where $|\kappa| < 1$ and e_{t+1} is stationary white noise and, in particular, $E_t[e_{t+1}]=0$. Thus $E_t[f_{t+1}]=\kappa f_t$, and the deviation from fundamentals is stationary or mean-reverting. Such a model would seem to capture behavior which, although mean-reverting, is similar in some ways to that which would be generated by speculative bubbles.

One way of reconciling the concept of fads with market rationality is to argue that, although a large proportion of trading is done by naive speculators or so-called noise traders, a significant proportion is in fact undertaken by informed speculators. Informed speculators, in turn, do drive the market price to a risk adjusted equilibrium, but the extra risk which is generated by the presence of noise traders is not captured by standard models.[5)]

Apart from anecdotal evidence on the presence of fads, formal evidence at the level of individual equity prices is provided by DeBondt and Thaler (1985,1987), Lehmann (1987), and Camerer (1987). These papers demonstrate that abnormally high returns can be generated by decreasing holdings of stocks which have recently performed well and by buying stocks which have recently performed poorly - thus providing evidence of mean reversion in prices.

At the aggregate market level, evidence of mean-reverting behavior in stock prices is provided by, inter alia, Lo and McKinley (1987), Fama and French (1988), Poterba and Summers (1988), and Campbell and Shiller (1988). In particular, these studies demonstrate that the fads term is predictable, using variables such as lagged dividend-price ratio or earnings.

However, as noted by West (1988), evidence of mean reversion and predictability in stock prices is at best suggestive of the presence of fads; it does not prove that they are present. Indeed, the most compelling argument presently available in support of the fads hypothesis is that they are the major viable alternative explanation of excess volatility findings, given that neither traditional models nor rational bubbles appear to be satisfactory in this respect. Further research on fads should thus concentrate on more direct, testable implications of fads models.

---

5) See e.g. De Long et al. (1987).

1.3.5 Mean Reversion in Stock Returns

As mentioned in (1.3.4), mean reversion can be thought of as another anomaly. The phenomenon of mean reversion is the tendency for stocks that have enjoyed high (low) returns to exhibit lower (higher) returns in the future; that is, returns appear to regress toward the mean. Its explanations center on fads, noise traders, rational speculative bubbles, and time-varying expected returns.

To test mean reversion in stock returns, sample variance ratio methodology is introduced. A close relative of variance ratios is the regression coefficient calculated by Fama and French (1988; 1989) and Fama (1990). They regressed the cumulate return from period t to period t+k on the return from t-k to t. Fama and French have found evidence of mean reversion for holding periods longer than 18 months.

While we have some skepticism about the Fama-French tests, (the results appear to be due primarily to the inclusion of the 1930s in the sample period), the phenomenon of mean reversion has been supported by other tests. Poterba and Summers (1988) have found that the variances of holding period returns do not increase in proportion to the length of the holding period, an indication of mean reversion. Their logic is simple. Suppose that the one-period rate of return on stocks is approximated by the change in the logarithm of the price. Suppose further that - as many financial studies assume - the change in the logarithm can be represented by a constant plus a random error term, so $\log P_{t+1} - \log P_t = \mu + e_{t+1}$. Then the average return over N periods is approximately $\log P_{t+N} - \log P_t = N\mu + (e_{t+1} + e_{t+2} + \ldots + e_{t+N})$. If the EMH is correct, the e's are identically and independently distributed. Denoting the variance of e as σ^2, the variance of the N period return is $\mathrm{var}(\log P_{t+N} - \log P_t) = N\sigma^2$: the variance of returns is proportional to the period over which the returns are experienced. If, as Summers and Poterba conclude, the variance increases less than in proportion to the period, the return on stocks is mean-reverting.

However, there are also questions on the findings of mean reversion. Kim, Nelson and Startz (1991) show that evidence of mean reversion is overstated due to the assumption of normally distributed returns. Richardson (1989) shows that by ignoring the interdependence among measures of mean reversion at different return horizons, interpretations that focus on individual horizon statistics can be erroneous. Richardson and Stock (1989) note the inadequacy of the usual approximating asymptotic distributions in multi-year return tests and develop a better asymptotic distribution for finite samples, which yields less evidence against the random walk. More recently, McQueen (1992) suggests that observed mean reversion in prior tests, based on OLS generated first-order autocorrelations and asymptotically consistent standard errors, is overstated due to: a) the inefficiency of OLS estimates that give undue weight to the Depression/World War II period; b) the reliance on asymptotically consistent standard errors with small sample sizes resulting from the use of long-horizon returns; and c) improper focus on the most negative results after testing over multiple horizons.

1.3.6 The Distribution Model of Changes

Another line of research on the EMH is to measure the correct distributions for price movements. It is believed that knowing the correct distributions for price movements is very important in testing the random walk hypothesis and in structuring the efficient market model. It is also believed that the distributions are not normal but leptokurtic, and in addition, that evidence of nonlinear dependence (serial dependence in the second or higher moments) exists for stocks.

If the distribution is not normal but leptokurtic, statistical tests, including the efficient market hypotheses tests, based on i.i.d. normality assumptions, are likely to give misleading results by underestimating sample variances and standard errors. It was not until recently that applied researchers in financial econometrics started explicitly modeling the variation in second - or higher - order moments and leptokurtosis. Here we review some of the most prominent tools: the mixed diffusion jump process of Merton (1976), and Akgiray and Booth (1986); the ARCH process of Engle (1982) and its various extensions; the product process of Taylor (1986); and the bispectrum model of Hinich (1992).

1.3.6.1 The Mixed Diffusion Jump Process

The efficient market hypothesis suggests that adjustments to new information are instantaneous (Fama (1965)). Thus, an asset price may change by a large amount in a very short time period when important news become public. Yet, most financial models employ continuous stochastic processes in which the probability of such discrete jumps is zero. If discontinuous jumps exist in a return generating process, it is not surprising that a normal or continuous diffusion process gives a poor fit. To allow both continuous and discontinuous changes in prices, mixed processes have been suggested. A mixed process can be modelled by combining the continuous Brownian motion with the Poisson jump process (Merton (1976) and Akgiray (1986)). The reason for choosing the Brownian and Poisson processes is simply because both are the limiting cases to which their continuous and discontinuous families converge.

Let q(t) denote the asset price at time t. Then in stochastic differential equation form, the mixed diffusion-jump process can be expressed as:

$$dq(t)/q(t) = \alpha dt + \sigma dB(t) + J(t)dP(t) \qquad (1.13)$$

where B(t) is the geometric Brownian motion with instantaneous expectation α, and instantaneous variance σ^2 per unit time. P(t) is the Poisson counting process with intensity parameter $\lambda > 0$, which indicates the number of jumps occurring during the time period. The jump amplitude, J(t), is a normal random variable with mean μ and variance σ_j^2, measuring the size of the Poisson jump. The amplitude represents the logarithm of one plus the percentage change in the price at time t. The jump process is assumed to be identical and independent of the Brownian motion. Oldfield et al. (1977) have found the solution to the differential equation (1.13) using Ito's lemma. Dividing the solution by the previous period price, q(t-1), and taking the natural logarithms, (ln), of the division yields the one-period return as:

$$ln[q(t)/q(t-1)] = [\alpha - (\sigma^2/2)] + \sigma B(t) + \sum_{n=1}^{P(t)} lnJ(t)_n \quad (1.14)$$

where $J(t)_n$ is the size of the n^{th} jump. In this form, the return is composed of three parts. The first two parts are due to the continuous diffusion process, the last to the cumulative size of the discontinuous jump process. If there is no jump, the return is normally distributed with mean $(\alpha-\sigma^2/2)$ and variance σ^2. Therefore, a normal distribution is a special case of this mixed process. The probability distribution function of the return, x(t)=ln[q(t)/q(t-1)] can be obtained by mixing distribution functions of the Brownian motion and the jump process. By differentiating the probability function term by term the probability density function is derived, which is the same as the distribution function except the normal density function replaces the normal distribution function. The density function for the mixed process is

$$g(x|\theta) = \sum_{n=0}^{\infty} [(e^{-\lambda}/n!)\lambda^n] \bullet N\left[x|\alpha - \sigma^2/2) + n\mu, \sigma^2 + n\sigma_j^2\right] \quad (1.15)$$

where θ is the parameter vector composed of diffusion mean and variance, jump intensity, and jump mean and variance. N(•) is the normal density function. All moments exist for this density function, and this function is skewed if μ is not zero and the direction of skewness is the same as the sign of μ. Also, the distribution is leptokurtic if λ is greater than zero. If the random variable, x is standardized by setting $\alpha=\mu=0$ and $\sigma^2+\lambda\sigma^2_j=1$, then the distribution is more peaked and fat-tailed than a standard normal distribution. It approaches, however, the normal distribution as the jump intensity, λ, or the jump amplitude, J(t), decreases, that is, with less frequent or less volatile jumps.

Since the diffusion jump process captures skewness as well as leptokurtosis, and stock prices have been found to be indeed skewed (see Singleton and Wingerder (1986)), some financial econometricians use this process in testing the EMH.

Akgiray and Booth (1986) have found that the addition of the Poisson jump process to the Brownian motion process increases the stock price model's descriptive power significantly. However, since the constant variance with respect to time assumed in the diffusion-jump process is inconsistent with the empirical finding of nonlinear dependence, the conditional heteroskedastic models or the product process models are used more often in testing the EMH and in constructing the index returns process.

1.3.6.2 (G)ARCH Model

If we can find more complicated dynamic structures for the time-varying conditional second-order moments in stock returns, we can also propose relevant economic variables driving stock volatilities. The ARCH model can be viewed as a reduced form of such dynamic structures. Thus, finding the widespread existence of ARCH effects shows that there is substantial evidence that excess returns on the stock market are predictable on the basis of information available at the moment the investment is undertaken. As Fama and French (1988b) stress, this could be an indication either of market inefficiency or of risk averse behaviour.

Engle (1982) has developed a parsimonious but effective time series model which can explain the observed leptokurtosis. The autoregressive conditional heteroskedastic (ARCH) model is designed to allow the conditional variance (conditioned on past information) to change over time leaving the unconditional variance constant. The process is autoregressive because the conditional variance is specified as a linear function of past realizations of the squared past disturbance terms. Thus, large past realizations of disturbance are modelled to increase the current variability. In other words, this process models the fact that large disturbances occur together, so that a large disturbance today increases the chance of a large disturbance tomorrow.

The generalized ARCH(GARCH) process was developed by Bollerslev (1986) to give a more flexible lag structure in the conditional variance equation. The GARCH process allows current and lagged conditional variances as well as past realizations of the disturbance term to affect the mean return process.

The GARCH(p,q) process is modelled as follows. Let y_t be a series of sample observations, x_t a vector of dependent variables and β a vector of unknown parameters. Suppose a simple linear data generating model is:

$$y_t = x_t\beta + \epsilon_t \qquad (1.16)$$

The random shock ϵ_t is normally distributed and conditioned on past information with zero mean, and follows the GARCH(p,q) process with the conditional variance, h^2_t:

$$\epsilon_t \mid \phi_{t-1} \sim N(0,h_t^2) = (2\pi h_t^2)^{1/2} \exp(-\epsilon_t^2/2h_t^2), \text{ and} \qquad (1.17)$$

$$E(\epsilon_t^2 \mid \phi_{t-1}) = h^2_{t|t-1} = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon^2_{t-i} + \sum_{j=1}^{p} \beta_j h^2_{t-j|t-j-1}, \qquad (1.18)$$

$$\alpha_0 > 0,\ \alpha_i \geq 0 \text{ and } \beta_j \geq 0.$$
$$i = 1,\ldots\ldots, q,$$
$$j = 1,\ldots\ldots, p.$$

E(•) is the expectation operator, and ϕ_{t-1} is the set of all information available at time t-1. If p=0, the process reduces to the ARCH process. If p=q=0, the conditional variance is the same as the unconditional variance, that is, there is no heteroskedasticity, and the disturbance is a Gaussian white noise.

As with the ARCH process, Bollerslev (1986) has showed that the GARCH process generates data with fatter tails than the normal distribution. In addition, the GARCH process implies the nonlinear dependence in the second moment. Therefore, the GARCH process is expected to fit the sample distributions well, since it not only models changing variances but also captures the leptokurtosis.

ARCH effects have generally been found to be highly significant in stock markets. For example, highly significant test statistics for ARCH have been reported for individual stock returns by Engle and Mustafa (1992), and for index returns by Akgiray (1989). In addition, most empirical implementations of GARCH(p,q) models adopt low orders for the lag lengths p and q, though there are some exceptions (see Attanasio (1991)) to this low-order rule in the ARCH specification.[6] It is very interesting and important for further research to note that such small numbers of parameters seem sufficient to model the variance dynamics over very long sample periods, which implies that the generating mechanism of stock returns can be a kind of chaotic dynamics of nonlinear systems.

1.3.6.3 Product Process

The product process is another model characterizing daily stock returns to be of low autocorrelation, to be of approximately symmetric distribution having long tails and high kurtosis for long series, and to be nonlinearly dependent.

Differently from the ARCH, this process is expressed in the general form:

$$X_t = \mu + V_t U_t$$

or (1.19)

$$X_t - \mu = V_t U_t$$

with $\{U_t\}$ a standardized process, so $E[U_t]=0$, and $var[U_t]=1$ for all t, and $\{V_t\}$ a process of positive random variables, usually having $var[X_t|V_t]=V^2_t$; also $E[X_t]=\mu$ for all t. Normally we call this model to be a product process, since $\{X_t-\mu\}$ is the product of stochastically independent processes $\{U_t\}$ and $\{V_t\}$.

6) Typically GARCH(1,1), GARCH(1,2) or GARCH(2,1) models are adopted.

This model suggests that an approximation to the standardized return $\{U_t=(X_t-\mu)/V_t\}$ is to be used for the test of the EMH, instead of $\{X_t\}$ Taylor (1986) has shown that the random walk hypothesis under this process is rejected in the USA stock returns. Takeaki Kariya (1989) has also tested the random walk hypothesis for the individual Tokyo stock returns under this process, and has found that the EMH is usually rejected for most stock returns.

1.3.6.4 Bispectrum Model

It has recently become more common to use the estimated bispectrum to formally test the null hypothesis that the generating mechanism of a stock returns time series is linear. The original form of this process is due to Subba Rao and Gabr (1980), which has been improved by Hinich and Patterson (1992). In fact, many tests on the EMH assume that the generating mechanism of a stock returns series is linear. So, if we find that this assumption is not true, the test results based on this assumption might be erroneous.

Let $\{X(t)\}$ denote a third-order stationary time series,[7] where t is an integer. To simplify exposition, let $E[x(t)]=0$. The third order cumulant function is defined to be

$$c_{xxx}(m,n) = E[x(t+n)x(t+m)x(t)] \tag{1.20}$$

The bispectrum at frequency pair (f_1,f_2) is the double Fourier transform of $c_{xxx}(m,n)$:

7) The time series $\{x(t)\}$ which satisfies the following three conditions is said to be third-order stationary:

(i) $E(x(t)) = \mu$, independent t,

(ii) $Cov(x(t), x(t+m))=R(m)$, a function of m only,

(iii) $Cum(x(t), x(t+m), x(t+n))$

$= E[(x(t)-\mu)\ (x(t+m)-\mu)\ (x(t+n)-\mu)]$

$= C(m,n)$ is a function of m and n only.

$$B_x(f_1,f_2) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{xxx}(m,n)\, exp\left[-i2\pi\,(f_1 m + f_2 n)\right] \quad (1.21)$$

It is a spatially periodic complex function whose principal domain is the triangular set $\Omega=\{0<f_1<½, f_2<f_1, 2f_1+f_2<1\}$. A rigorous treatment of the bispectrum is given by Brillinger and Rosenblatt (1967).

Suppose that $\{x(t)\}$ is linear, that is, it can be expressed as

$$x(t) = \sum_{n=0}^{\infty} a(n)\,u(t-n) \quad (1.22)$$

where $\{u(t)\}$ is purely random (i.e. stationary and serially independent) and the weights $\{a(n)\}$ are fixed. Assuming that $\Sigma^{\infty}{}_{n=0}|a(n)|$ is finite, the bispectrum of $\{x(t)\}$ is

$$B_x(f_1,f_2) = \mu_3 A(f_1) A(f_2) A^*(f_1+f_2) \quad (1.23)$$

where

$$\begin{aligned} \mu_3 &= E[u^3(t)], \\ A(f) &= \sum_{n=0}^{\infty} a(n)\exp(-i2\pi f n) \end{aligned} \quad (1.24)$$

and $A^*(f)$ is its complex conjugate. Since the spectrum of $\{x(t)\}$is

$$S_x(f) = \sigma_u^2\,|A(f)|^2 \quad (1.25)$$

it follows from (1.23) that

$$\Psi^2(f_1,f_2) \equiv \frac{|B_x(f_1,f_2)|^2}{S_x(f_1)\,S_x(f_2)\,S_x(f_1+f_2)} = \frac{\mu_3^2}{\sigma_u^6} \quad (1.26)$$

for all f_1 and f_2 in Ω.

The left hand side of (1.26) defines the squared skewness function of $\{x(t)\}$, $\Psi(f_1,f_2)$. Furthermore, the squared skewness function is a constant if $\{x(t)\}$ is linear. Hence, the $\Psi(f_1,f_2)$ is also a constant, if $\{x(t)\}$ is linear. This property is the basis for the Hinich linearity test. Hinich and Patterson (1992) have rejected the martingale difference hypothesis, which could be evidence that daily returns from the CRSP (Center for Research in Security Prices) at the University of Chicago are not generated by the linear model proposed in economic theory. Moreover, they claim that the results on the random walk hypothesis based on the traditional linear process between returns might be erroneous.

1.4 Chaotic (Nonlinear) Dynamics Model

It is generally believed that no strong statistical evidence confirming the efficient market hypothesis or random walk hypothesis has been offered, and also that since the variance of stock returns is not constant over time, stock returns deviate from the random walk model. In fact, such a finding has already been pointed out by Mandelbrot (1963). He noted that although stock returns appeared ncorrelated, large changes tended to be followed by large changes, and small changes tended to be followed by small changes. This has led to the development of the conditional variance models such as the GARCH and the product process model. These models mainly attempt to capture the changing variance in a time series of stock returns.

Before these models, most investigations focused on the linear predictability of stock return changes. Under the linear paradigm, we can find that, at best, the data suggest that stock return changes are uncorrelated. This is not sufficient to prove statistical independence, in view of the non-normality of $\{X_t\}$. It is possible for stock returns changes to be linearly uncorrelated and nonlinearly dependent. Theoretically, there is no reason to believe that economic systems should be intrinsically linear. So, the models based on the nonlinearity between returns are of considerable use in detecting inefficiencies or in testing the random walk hypothesis.

More recently, some financial econometricians have seemed skeptical of the use of autocorrelations. They discovered from dynamical systems theory that the autocorrelation functions of many economic time series decay rather slowly to zero; i.e. the spectrum has a peak at the origin and displays a downward slope as frequency increases. That is to say that there is a lot of spectral power at the lower frequencies and returns volatility series appear to have autocorrelation functions which die off roughly with power law tails, in contrast to the exponential die-off which is typical in ergodic stationary processes. So, they claim that traditional tests focused on finding linear dependence by using autocorrelation coefficients might not be effective in detecting nonlinear dependence. Furthermore, in a certain process, the process appears to be a pure random process to a researcher calculating its autocorrelation, even though the process is generated by a deterministic equation (see Sakai and Tokumaru (1980)).

Now a new method is developing , which does not depend on independence, normality, or finite variance, but includes fractals and nonlinear dynamics whose characteristics appear to conform more closely to observed behavior. The method makes use of the idea of the "correlation integral", used in the natural sciences by Grassberger and Procaccia (1983). Given a time series $\{X_t: t=1,2,.....,T\}$ of D-dimensional vectors, the correlation integral is defined as

$$C(e) = \lim_{T \to \infty} \frac{2}{T(T-1)} \sum_{i<j} I_e(x_i, x_j),$$

where $I_e(x,y)$ is an indicator function that equals one if $\|x-y\| < e$, and zero otherwise, where $\| \bullet \|$ is the sup-norm.[8)]

Physicists use this concept to distinguish between chaotic deterministic systems and stochastic systems, despite the lack of a proper statistical theory. However, Brock, Dechert, and Scheinkman (1987) propose to test the null hypothesis that the data are independently and identically distributed, using a procedure with the

---

8) The correlation integral C(e) measures the fraction of the pairs of points $\{X_t\}$ that are within a distance of e from each other.

correlation integral, instead of trying to distinguish a chaotic system from a stochastic system, referred to as the BDS test.

Scheinkman and LeBaron (1989) discovered the presence of nonlinear dependence in weekly returns by using the BDS test. They used the data of continuously compounded weekly returns from CRSP at the University of Chicago. More recently Willey (1992) presented the results of the BDS test for nonlinear dependence in the daily prices of the S&P 500. He also found the existence of an underlying nonlinear relationship. However, he has concluded the series is stochastic, not chaotic.

1.5 Summary of Chapter

This chapter assesses the current state of tests of the efficient market hypothesis, which was the conventional wisdom among academic economists in the 1970s and most of the 1980s. We have mainly reviewed the empirical evidence. We have also noticed that some overwhelming cases against the efficient market hypothesis have been offered.

First, we have reviewed the existence in the form of a number of well established anomalies - the small firm effect, the January effect, the weekend effect, the Value Line enigma, and money supply. In the earlier stages these anomalies were more profound. However, some EMH-supporters have suggested that many of these anomalies exist only among small firm stocks, not among large firm stocks. Nonetheless, analysis of the S&P 500, which is dominated by large firms, also shows important anomalies such as a weekend effect. More recently, some researches (for example, Pearce (1987)) have shown that the evidence of anomalies against the efficient market model is not sufficient to reject the efficient market model.

Second, we have reviewed the excess volatility, stationary issues, speculative bubbles, fads, mean reversion, and some distributions for stock price changes.

Generally, these concepts are evaluated to be useful in providing an additional reason - other than observed anomalies - to doubt the validity of the EMH, while some studies of these concepts have been found to be flawed. However, some criticisms of this evidence have also been offered. For example, some researchers have reported that when transaction costs are taken into account, many of the apparent deviations from the predictions of the efficient market model are too small to allow investors to earn excess returns.

Third, we have reviewed some distributions of price changes, which can capture the variation in second or higher order moments and leptokurtosis: the mixed diffusion jump model, the ARCH model and its various extensions, the product process model, and the bispectrum model. It is found that these distributions are highly significant in stock markets and very effective in testing the EMH.

Finally, this chapter has reviewed a new method which is not based on independence, normality and autocorrelation coefficients: the BDS test. Since standard statistical tests based on autocorrelation functions and their Fourier transforms may fail to detect hidden order, it is expected that the BDS test is very effective in testing the EMH and in uncovering an underlying nonlinear process of stock index returns.

Our fundamental conclusion is that the efficient market hypothesis is having a near-death experience and is very likely to succumb unless new technology, as yet not known, can revive it. Furthermore, we believe that nonlinear dependence models should widely be employed in testing the efficient market hypothesis.

Chapter 2

Alternative Approaches for Violations of Assumptions on the Efficient Market Models

Many empirical investigations on the efficient market hypothesis in the financial literature are concerned with the statistical process that governs security returns and volatilities. So, identifying the process that generates asset returns is essential for designing valid financial models and exact statistical tests. Linear models based on the independently and identically distributed normality (hereafter i.i.d. normality) assumption have preferably been used to describe stock returns and to detect volatilities (Fama (1965) and Granger and Morgenstern (1970)). In the earlier days, it was in general believed that the i.i.d. normality fitted well and that the small magnitude of serial correlations could not be used to obtain significant excess returns. Hence, it ruled out any profitable trading rules based on forecasts of future prices from past price movements, and the efficient market models were regarded as an adequate description of asset returns.

However, in chapter 1, we have observed various challenges to the market efficiency paradigm and we have also noticed that even though linear models have preferably been employed in the literature, recently several authors[1] have begun to study nonlinear models to describe the dynamic generating process between returns and to detect the evidence against the market efficiency, which might not be captured under the linear generating process. Now, some findings rejecting the i.i.d. normality assumption have been offered. Two findings, in

1) Some examples are Granger and Anderson (1978), Engle (1982), Taylor (1986), Bollerslev (1987), French, Schwert, and Stambaugh (1987), Scheinkman and LeBaron (1989), Tong (1990), Hsieh (1991), Bera and Higgins (1991), and Hinich and Patterson (1992).

particular, are fatal to the equilibrium models such as the capital asset pricing model, Black and Scholes' option pricing model, and Fama's efficient market models. The first is that the distribution of speculative price change has more observations around the mean and in the extreme tails. The second is that successive price changes are not independent. These cause the statistical tests which are based on the i.i.d. normality assumption as well as the equilibrium models which use the variance as a measure of risk to be biased.

This chapter aims at criticizing some assumptions on the efficient market models and furthermore reviewing various alternative approaches for the violations of the assumptions. In section 2.1, some assumptions on the efficient market models are discussed. In section 2.2, it is presented what kind of concepts should be considered for a new process of price changes. Section 2.3 details various alternative approaches for a new return generating process. The last section summarizes this chapter.

2.1 Assumptions on the Efficient Market Models

The efficient market concept is the basis for most of the traditional financial equilibrium models such as the capital asset pricing model, Black and Scholes' option pricing model, and portfolio models of asset allocation, used to explain the behavior of speculative prices. Fama (1965) has defined an efficient market as one where prices fully reflect available information. Jensen (1978) has given a broader definition that a market is efficient with respect to information if no one can make an economic profit by trading on the basis of past information. Empirical tests of the efficient market hypothesis have usually tested the random walk model following Fama's more restrictive definition and usually employed linear models mainly because

(1) they provide good first-order approximations to many processes;

(2) statistical theory is well developed for linear Gaussian models;

(3) computational tools for linear models, such as model specification, estimation, and diagnostics are readily available;

(4) they are satisfied with the property of independence, i.e., the total response to an action is equal to the sum of the results of the values of the separate factors;

(5) they are satisfied with the property of proportionality, i.e., the response of the action of each separate factor is proportional to its value.

The random walk model implies that knowledge of the past behavior of price changes cannot increase the expected return today. This model is based on two assumptions about asset price behavior:

(1) successive price changes are independent;

(2) the price changes are drawn from the same probability distribution (Fama (1970)).

More rigorously, prices follow a random walk if successive price changes are strictly white noise, so that the sequence of the past returns is of no consequence in assessing the distribution of future returns. A stochastic process is called to be stationary if both the mean and the autocovariance are not dependent on time, and in particular, the stationary process is white noise if successive observations are uncorrelated. Zero autocorrelation, however, is not sufficient to show independence. Therefore, for the process to be strictly white noise or purely random, any successive observations should be statistically independent, that is, no serial correlation even in higher moments. Furthermore, if a data series is strictly white noise, so are its absolute and squared values. Independence is the key assumption for the random walk model. Fama's definition of efficiency, however, only suggests that prices follow a more general martingale process. The martingale process only assumes no autocorrelation. It does not require independence. Thus, it is believed that past tests of the random walk model are not appropriate tests of market efficiency.

A Gaussian white noise process is always linear (a linear function of other strict white noise process) and purely random. This is an important reason why most of the efficient market models prefer the normal distribution to any other alternatives. In addition, a normal distribution has convenient properties such as

(1) symmetry about the mean;

(2) having finite second and higher moments;

(3) a limiting distribution of most general distributions, so that the classical central limit theorem holds;

(4) more importantly, describing the distribution completely only with the mean and variance.

Consequently, the efficient market models assume independent and identical normal distribution for price changes.

The distributions for daily price changes are usually assumed to be i.i.d. normal, but as discussed in chapter 1, evidence against this assumption is voluminous. The most serious violations of the assumption are the presence of leptokurtosis and lack of independence. It is believed that the leptokurtosis causes dispersion under the i.i.d. normality to be underestimated,[2)] and also causes standard hypothesis tests using t- and F-distributions to be biased. Thus, previous tests for

---

2) Fama (1965) clearly illustrated that:
"....., if the population of price changes is strictly normal, on the average for any given stock we would expect an observation greater than 4 standard deviations from the mean about once every fifty years. In fact observations this extreme are observed about four times in every five year-period. Similarly, under the Gaussian hypothesis for any given stock an observation more than five standard deviations from the mean should be observed about once every 7,000 years. In fact such observations seem to occur about once every three to four years."

the capital asset pricing models and efficient market hypothesis under the i.i.d. normality assumption can give misleading results. The lack of independence is also a serious violation of the i.i.d. normality assumption. Dependence in the price changes increases the forecasting ability of alert investors and immediately invalidates the efficient market models.

2.2 *Violations of Assumptions on the Efficient Market Models*

As shown in 2.1, the efficient markets models assume the i.i.d. normality for price changes. However, some factors violating the i.i.d. normality assumption have appeared, e.g.

2.2.1 *Autocorrelation*

The existence of autocorrelation in asset prices has been extensively studied, since it is typically used to test the efficient market hypothesis. This dependence implies that asset prices behave with some patterns so that alert speculators are able to make an abnormal profit by trading the patterns if no transaction costs occur.

Fama (1970) and Akgiray (1989) have found some evidence of linear dependence in daily stock price returns. Friedman and Vandersteel (1982) have found similar results for foreign exchange rates. Trends have also been found in some futures prices (Rocca (1969); Cargill and Rausser (1975)). The presence of serial correlation in daily asset price changes can be explained by market disequilibrium (Lukac et al. (1988)). This theory is based on the notion that prices do not quickly adjust to information shock, and, thus, markets are in short-run disequilibrium. Summers (1986) has argued that there is no ground to assume either that irrational traders will be eliminated by market forces, or that they will be unable to move market prices. Taylor (1985) has pointed out that irrational traders, or rational traders unable to interpret all the information quickly, may cause trends. Even if the linear dependence is not large enough to reject the weak form of the market efficiency hypothesis after considering transaction costs (Lukac et at. 1988), modelling possible autocorrelation is necessary to determine the correct probability distribution and to get efficient estimators.

Let Y_t be a series of sample observations, X_t a vector of independent variables and β a vector of unknown parameters. Suppose a simple linear data generating model is $Y_t=X_t\beta+\epsilon_t$. The conventional assumption on the estimation of the regression model in this equation is that the disturbance ϵ_t is i.i.d. normal with zero mean and constant variance. When this assumption is violated, the least squares estimator of β loses efficiency. Moreover, if the violation is serious, alternative methods of estimation which give more satisfactory results should be developed. We will develop a new process which copes with the time-varying variance problem under which the ordinary least squared estimators lose all their efficiency properties. The autocorrelated errors cause the same problems. It is really important to consider it in order to obtain precise test results on the efficient market hypothesis. For these reasons, we must allow for this possibility in the estimation model and in the test methodology.

Furthermore, more recently, some dynamical systems scientists claim that the autocorrelation function is even useless in the nonlinear structure. They suggest to replace it with a new concept, the correlation integral.[3)] For example, let's consider the trajectories generated by

$$X_{t+1} = \frac{1 - |2X_t - 1|}{1 - |2a - 1|} \quad 0 \leq X_t \leq 1 \quad \textit{and} \quad 0 < a < 1 ,$$

as shown by Sakai and Tokumaru (1980). They yield the same autocorrelation function as the AR(1) process $X_{t+1}=(2a-1)X_t+\epsilon_t$, where ϵ_t is a white noise error term. Hence if a is equal to ½, $\{X_t\}$ generated by the above equation appears to be a pure random process to a researcher calculating its autocorrelation function, even though it must be a deterministic process.

3) The concept of correlation integral is detailed in chapter 7.

2.2.2 *Day-of-the-Week Effects*

If information reaches markets evenly for all days of the week, the distribution of asset price changes should be identical for all days. French (1980) has hypothesized that if returns are generated continuously in calendar time, the returns for Monday will be three times as large as those for other days of the week. If returns are generated in trading time, returns are, on the average, the same for all days of the week. He has found, however, that the average return of the S&P composite on Monday is significantly negative while the averages for the other four days are positive. The so-called 'day-of-the-week' effect was first reported by Cross (1973). He has found that the S&P 500 index composite performed better on Friday than on Monday. Gibbons and Hess (1981) have confirmed the day-of-the-week effect for the Treasury bill index. This anomaly has also been found in the futures (Chiang and Tapley (1983)), spot commodity (Chang and Kim (1988)) and foreign exchange markets (McFarland et al. (1982)), which generally confirm that the return for Monday is negative while those for other days of the week are not. Of course, there are also some studies that do not support the existence of anomalies. (See Pearce (1987).)

While the existence of the day-of-the-week or weekend effect prevails in both financial and commodity markets, the causes are not clearly explained. French (1980) has proposed that the information released over the weekend tends to be unfavorable. Keim and Stambaugh (1984) hypothesized that Friday closing price quotes were systematically biased. Unfortunately, no hypothesis is supported by empirical data. Explaining the cause of this anomaly is not the purpose of this section. However, the effects should be modelled by a time-varying variance model since returns for all days of the week are not drawn from an identical distribution. Furthermore, those distributions are changing over time. For example, Smirlock and Starks (1986) ascertained that intra-day patterns related to the day-of-the-week effect were changing over time for the Dow-Jones Industrial Average index. Most studies investigating the day-of-the-week effect tried to capture the heteroskedasticity over trading days (French (1980); Gibbons and Hess (1981); Chiang and Tapley (1983); Flannery and Protopapadakis (1988); Chang and Kim (1988)). They, however, have failed to consider the fact the distribution changes over time. In other words, to correct the different variances for all days of the week, they standardized the dummy variable models by the estimated standard deviations for each day of the week. But heteroskedasticity also exists within the same day of the week. For example, the distribution for Monday returns may not be constant but changes over time. It is needed to develop a new process which captures this violation.

2.2.3 Seasonality

In a competitive economy, price is determined where supply is equal to demand. If a commodity is annually harvested but the demand is about the same over the year, price volatility at harvest time can be different from the remaining period. This systematic volatility pattern is called 'seasonality'. Anderson (1985) has asserted that such volatility is relatively high during periods when significant amounts of supply or demand uncertainty are resolved. Most agricultural products, by their biological nature, show seasonal variations which result in price instability from month to month. Seasonality is especially apparent in agricultural futures and spot commodity markets. Anderson (1985) has examined the volatility of daily price changes for nine futures commodities over 1966 to 1980. He has confirmed that seasonality emerged as an important determinant of volatility of futures prices over time. Kenyon et al. (1987) have also found that grain futures price volatility is affected by the season of the year. Brorsen and Irwin (1987) have found similar evidence of seasonality in futures markets.

Seasonal variation in production is not exclusively confined to agricultural commodities. Chang and Pinegar (1988) found that monthly growth rates in industrial production showed significant systematic patterns. This seasonality in industrial production may affect price volatility in stock markets and foreign exchange markets. So far, there have been few attempts to model seasonality to explain volatility in stock prices and foreign exchange rate changes. However, voluminous research examined the seasonality in the means of stock prices. Rozeff and Kinney (1976) have found that stock returns are higher in January and July than in other months of the year. Tinic and West (1984) have shown that January is the only month to show a positive risk premium.

A new process should also be able to capture the seasonality as closely as possible. Past studies have employed dummy variables to model seasonal variations. The dummy variable model, however, catches only discrete variations. Moreover, discrete jumps can distort the data when re-scaling for diagnostic tests.

2.2.4 Maturity Effect

It is believed that futures price volatility is not constant with respect to time to maturity. The hypothesis is that futures price changes per unit of time are more volatile as delivery date approaches, assuming that the spot price follows a stationary first order autoregressive process and the futures price is the unbiased predictor of the cash price. An argument for this maturity effect is that as time to maturity decreases, more information is released progressively, which results in more active trading. Another possible explanation is that, because the maturity date is not adjusted to individual investor's needs or strategies, futures investment is riskier near maturity. Even though the maturity effect hypothesis is based on some strong assumptions, Anderson (1985) has found that the inverse relation between volatility and time to maturity is a general property in futures markets. Milonas (1986) confirmed that maturity effect adjusted to the contract month effect, using broad data sets which included financial instruments and metals. So (1987), however, found no significant relationship in currency futures.
The contract month effect tells that, since futures contracts mature in different calendar months which might have different price volatility, the maturity effect can be different even if the time to maturity is the same. Milonas (1986) suggested normalizing the variance to make the futures price volatility dependent on only time to maturity. However, a new process can preferably adjust to the contract month effect without a normalization procedure.

2.2.5 *Standard Deviation in Mean*

As the degree of uncertainty in returns varies over time, the compensation (risk premium) required by a risk averse trader for holding that asset must also vary. This time dependent relation can not be captured by the efficient market models based on the i.i.d. normality assumption. French et al. (1987) have found that the expected market risk premium is positively related to the predictable volatility of stock returns.

A new process should be able to cope with the time-varying relation between the risk and returns. Modelling standard deviation in the mean equation has two implications. First, the day-of-the-week, seasonality and maturity effects, if any, directly influence the mean generating process. Second, this confirms the random beta model (Fabozzi and Francis (1978); Simonds et al. (1986)) if the estimated coefficient of the standard deviation is significant.

Dusak (1973) found that the systematic risk for wheat, corn and soybean futures contracts over the period 1952 to 1967 was not statistically significant. So (1987) confirmed Dusak's finding for the 1953-1976 period, but found that the systematic risk was nonstationary.

2.2.6 *Deterministic Chaos*

There is increasing interest in the proposition that economic fluctuations are generated within deterministic economic systems, rather than by stochastic shocks (Stutzer (1980); Day (1982); Benhabib and Day (1982); Brock (1986)). A predominant assumption about the structure of the economy has been that the real characteristics of the economy such as taste, endowments, productivities and technologies are not constant, but erratic over time. Such exogenous disturbances have been believed to cause the business cycle and market fluctuations. However, there is no prior reason to rely solely on the assumption that real characteristics are stochastic, to explain economic fluctuations. Indeed, many economists tried to identify the internal mechanisms that could explain the observed variations in price movements (Hayek (1933); Schumpeter (1939)).

In contrast to earlier macroeconomic and financial economic studies, an explicit model from deterministic dynamical systems theory has been considered. Such a nonlinear dynamic model is deterministic with respect to initial conditions, but errors in estimating parameters and initial conditions, no matter how tiny, will exponentially accumulate into forecasting errors (Day (1982); Brock (1986)). This makes the process look random. Deterministic processes that look stochastic are referred to as 'deterministic chaos'. The deterministic process is distinguished from the stochastic process neither by naked eyes, nor by statistical tests such as the spectral test or autocovariance test (Brock and Dechert (1988)). Hence, the previous results on the efficient market hypothesis which were derived under the stochastic process might be erroneous.

The hypothesis that time series data might be generated by a nonlinear deterministic chaotic mechanism is plausible, especially for macroeconomics and finance. Deterministic chaos implies better forecasts and controls of financial time series data and return generating processes in speculative asset markets. Näslund (1988) has shown that it is theoretically possible for the adjustment process in markets to have a chaotic pattern. Some empirical papers have also appeared. Brock (1986) and Brock and Sayers (1988) conducted tests for the null hypothesis of a deterministic chaotic explanation on the U.S. macroeconomic data. Similarly, Frank and Stengos (1988) tested the hypothesis on Canadian macroeconomic data. Neither found evidence of deterministic chaos. However, Frank and Stengos (1989) and Barnett and Chen (1987) found more positive results for gold and silver rates of returns and the U.S. monetary aggregates, respectively. Scheinkman and LeBaron (1989) also found some nonlinear dependence in the U.S. stock markets using the residuals from ARCH models. These findings have significant implications on financial models that are based on the assumption of stochastic processes. The concept of deterministic chaos surely violates the underlying assumptions of the efficient markets models. It is desirable that a new process should be able to cope with such a violation.

2.3 *Alternative Approaches for Violations of Assumptions*

In section 2.2, we detailed the various violations of assumptions on the efficient market models. In fact, much research has also been done for the alternative approaches which cope with these violations. In a practical sense, considerable research has tried mainly either to remove the observed leptokurtosis through a transformation of the data, or to model the exact sample distribution. The first alternative which was proposed by Mandelbrot (1963) and Fama (1965) to explain the observed leptokurtosis is that the distributions follow a symmetric stable Paretian law. A stable Paretian distribution is, by definition, invariant under addition.[4)] That is, sums of independent stable variables will also be stable with the same form as the individual variable. This type of distribution has infinite second and higher moments, and can model the leptokurtosis. Some studies, however, have shown evidence against this hypothesis through the tests either based on the log-likelihood ratio (Blattberg and Gonedes (1974); Tucker and Pond (1988)) or on the stability-under-addition property of stable distribution. Intuitively, the infinite variance characterized by this law seems to exaggerate the sample distribution. Officer (1972) has suggested that an analytic distribution function for which the second moment is finite may be a more appropriate model than a stable law. Perry (1983) has rejected the hypothesis that security return distributions have infinite variances. Nonetheless, more recently, this distribution is evaluated to be appropriate to stock price changes with the development of fractals theory, since this distribution can explain a power-type long tail in a frequency domain (Montroll and Schlesinger (1984)). In this study we also confirm that this distribution can explain more dynamics for stock price changes.

4) See Appendix C for a formal discussion of stable distribution.

As an alternative to the stable Paretian distribution, we can consider the student t-distribution. This distribution allows the variance to be stochastic, and has fatter tails which is consistent with the observed leptokurtosis. Praetz (1972), and Blattberg and Gonedes (1974) have suggested the evidence in favor of the student t-distribution over the stable distribution for stock price changes.

Another hypothesis which is consistent with the results of the stability-under-addition tests is that each sample is drawn from two or more different normal distributions with different means and variances, which lead to different skewnesses and different kurtoses. Such a mixture of normals avoids the infinite variance of the stable Paretian distribution. Kon (1984) has found that a mixture of normal distributions is more descriptive of stock market returns than the student-t distribution. Also, Akgiray and Booth (1988) and Tucker and Pond (1988) showed similar evidence for exchange rates changes. They, however, suggested a different type of distribution, a so-called mixed diffusion-jump process. They found that the mixture of a Brownian motion and an independent and homogeneous Poisson process could explain exchange rate movements better than the stable distribution, student t-distribution and mixture of normal distributions. Therefore, the mixed diffusion-jump process appeared to be the best among distributions that assumed independence.

These mixed distributions can partly explain leptokurtosis, but since they assume that successive observations are independent, the mixed distributions are inconsistent with the empirical work that has found the linear dependence between returns or the nonlinear dependence between returns. Perry (1983) has found that while the variance of stock returns is finite, it changes over time in a complex fashion. Friedman and Vandersteel (1982) examined three hypotheses in foreign exchange rate movements:

(1) i.i.d. stable Paretian;

(2) i.i.d., but mixture of two normal;

(3) normal with time-varying parameters.

They found evidence in favor of the third hypothesis for the movements. That is, both the trend and volatility of exchange rate movements are affected by changing economic and institutional factors over time, which can lead to serial dependence in means and variances.

If the observed leptokurtosis can be best explained by a normal distribution with variance changing over time, theoretical models based on normality still hold if investors' horizons are short enough to avoid the variance change. Furthermore, a correction for heteroskedasticity may result in a normal distribution and permit use of familiar economic models and statistical tools. These kinds of approaches have appeared mainly to the statistical community. They focus on the nonlinear dependence in the second moment. Taylor (1986) has proposed that the rescaled data (original data divided by its forecast standard deviation; see Chapter 4) can give more accurate results. McCulloch (1985) tried to remove heteroskedasticity in interest rates by using an ACH (adaptive conditional heteroskedastic) model. Bollerslev (1987) has suggested an extended GARCH (generalized autoregressive conditional heteroskedastic) model in which the disturbance term follows a conditional t-distribution with the variance changing over time. He has found that this model is more likely to describe foreign exchange rates and stock price index movements than the GARCH process with conditional normal errors. Hsieh (1989) modelled an ARCH process considering the day-of-the-week effect in both mean and variance of foreign exchange rate movements. The ARCH model removed the observed leptokurtosis to a large extent. Akgiray (1989) found the GARCH process could explain the observed leptokurtosis for some stock indices satisfactorily. While a GARCH or similar time varying models look promising, past research on these models has used limited data sets, has not considered all factors which can permit variance to change, and,in general, has not tested the adequacy of the models rigorously.

More recent studies tend to detail the nonlinear dependence in the variance. One branch aims to develop test statistics for detecting long term memory. Generally, they have utilized the R/S (Rescaled range) statistic and the Hurst coefficient (H) which are popular in hydrology, and have suggested that long term memory processes such as fractionally differenced random process better explain the real financial price changes. Greene and Fielitz (1977) have concluded that a large number of series with H estimates far enough from H=0.05 suggest that long term dependence in stock returns is a non-trivial problem.[5] When applied to daily and intraday commodity futures price data and foreign exchange rates data, Booth, Kaen and Koveos (1982), and Helms, Kaen and Rosenman (1984) have shown the similar conclusion, i.e., long term dependence exists significantly. On the contrary, recent study by Lo (1988) has found no evidence of long term dependence when R/S statistic is applied to weekly, monthly, and annual stock returns indices over several different time periods although few exceptions are found from daily returns. He has used a modified rescaled analysis which considers the short term autocorrelations' weights. Another branch focuses on the analyses by using the frequency-domain methods. The bispectrum analysis has been preferred by Hinich and Patterson (1989; 1992). They have presented the results of an investigation into the linear and nonlinear behavior of the continuous-time stock return generating process. Their findings are that linearity is rejected more often for daily returns. Furthermore, they strongly suggest that nonlinear dynamics (deterministic or stochastic) should be an important feature of any empirically plausible macroeconomic and financial model. The frequency-domain methods have also been preferred in detecting patterns by some fractals scientists. They have focused on the power law function. Li (1991) has performed the power spectral analysis on the daily closing value of the Dow Jones Industrial average and has found that the aggregate stock price follows a fractional Brownian process, which implies that the aggregate stock price movements have fractal patterns.

---

5) H=0.5 indicates the pure random walk process.

All the alternative approaches discussed above, in common, aim at modelling stochastic distributions. This would seem a natural approach to be taken by most of the financial econometricians who are used to dealing with stochastic time series processes, but recently a literature has begun to develop the model which solves the question of whether such series might be generated by nonlinear deterministic laws of motion. This has been prompted by findings, in the natural sciences, of completely deterministic processes that generate behavior which looks random under standard statistical tests: the processes that are termed 'deterministic chaos'. Brock (1986) suggests that asset price returns may not follow a stochastic process. Rather, it might be generated by deterministic chaos in which the forecasting error grows exponentially so that the process looks stochastic. If this is the case, we should test the market efficiency hypothesis, after distinguishing stochastic generating process from deterministic generating process for price changes. Then, all that has to be done is to determine the structure of the dynamic deterministic system. Brock, Dechert and Scheinkman (1987) have developed a statistic, so called 'the BDS-statistic', by which we can determine whether the generating process is stochastic or chaotic. Hsieh and LeBaron (1988), and Ramsey and Yuan (1989) have expanded the finite distribution of the BDS statistic. Frank and Stengos (1989) found evidence of nonlinear structure for gold and silver markets. Scheinkman and LeBaron (1989) also found some support for the stock market following a chaotic dynamics of nonlinear systems.

2.4 Summary of Chapter

Identifying the process that generates asset returns is essential for exact statistical tests as well as for designing valid financial models. We discussed the issue in this chapter.

We detailed the various violations of assumptions on the efficient markets models: significant autocorrelations; day-of-the-week-effect; seasonality; maturity effect; standard deviation in mean; and deterministic chaos. Furthermore, we have reviewed the various alternative approaches which can capture factors violating the i.i.d. normality assumption.

Even though there have been offered some alternative approaches for the efficient market tests, they can be classified into the following branches:

(1) The approach branch, which still focuses on permitting familiar economic models and conventional statistical tools based on a normal distribution in a time domain methodology. Basically they use autocorrelation function.

(2) The approach branch, which focuses on using a frequency domain methodology based on the power law function and the bispectrum. Basically they use the Fourier transform of autocorrelation function.

(3) The approach branch, which focuses on detecting long term memory and patterns. Basically they use a new portable concept, the Hurst coefficient.

(4) The approach branch, which focuses on determining whether the generating process is stochastic or chaotic. Basically they use a new concept, the correlation integral.

So, it would be desirable that the tests on the market efficiency hypothesis should be carried out by a new approach which considers time-domain concept, frequency-domain concept, long-term memory and patterns concept, and deterministic chaos concept.

Chapter 3

Sample Data

This chapter generally describes the sample data employed in the present study and discusses its characteristics and descriptive statistics. We have selected the indices of two thin stock markets representing two geographical areas: Europe and the Far East. To be more specific, we have chosen the Swedish stock market and the Korean stock market, since the author is familiar with both markets. The daily market indices are collected. In section 3.1, daily returns and the random walk hypothesis under nonlinear process are defined. Section 3.2 details the descriptions of sample data and provides some features. Section 3.3 reviews the dependence and the nonlinear structure between returns in the sample markets to check whether the conventional statistical tests are applicable directly to the sample markets.

3.1 The Definitions

3.1.1 The Definition of Daily Returns

In statistical terms, there are several ways to define the random walk hypothesis. In every case, the best forecast of tomorrow's price requires today's price but not previous prices. In fact, direct statistical analysis of financial prices is difficult, since consecutive prices are highly correlated and the variance of prices generally increases over time. Consequently it is more convenient to analyze change in prices. Let P_t be the price on trading day t and let dt be the dividend (if any) during day t. Then stock index returns are calculated as the difference in natural logarithm of the index value for two consecutive days:

$$X_t = \ln(P_t) - \ln(P_{t-1})$$

or (3.1)

$$X_t = \ln(P_t + dt) - \ln(P_{t-1}).$$

Fama (1965) provides the following reasons to use this construction for the return. First, the change in log price is the continuously compounded yield from holding the asset for that day. Second, log price change is independent of the price level. Third, for changes within 15% in the absolute term, log price changes are approximately the same as the percentage price change. In this study the return is defined as the log price change, which follows nearly all past research in this area.

3.1.2 The Random Walk Hypothesis under Nonlinear Process

The random walk hypothesis is a statement that price changes in some way are random and that prices wander in an entirely unpredictable way. Accordingly, when prices follow a random walk, the only relevant information in the series of present and past prices is the most recent price. Market participants have already made perfect use of the information on past prices. Following Fama (1970), academicians call this market efficient since the prices fully reflect available information.[1] Thus the market efficiency requires the random walk hypothesis.

Under the random walk hypothesis prices wander ('walk') in an entirely unpredictable way. Consequently, forecasts based on today's price cannot be improved by also using the information in previous prices. Alternatively, the mean square error cannot be reduced by using previous prices.

---

1) In this paper we will only consider the information available in the present price and past prices. So, to be more precise, these statements reflect the so-called weak form of the hypothesis.

There have been offered four definitions for the random walk hypothesis. Generally, if the return process of (3.1) meets to one of the following conditions, the process is said to be a random walk:

(1) $\{X_t\}$ should be independent and identical Gaussian (IID). (Bachelier (1900));

(2) $\{X_t\}$ is independent and identical, but doesn't require necessarily a Gaussian distribution. (Fama (1965));

(3) $\{X_t\}$ is independent. (Granger and Morgenstern (1970));

(4) $\{X_t\}$ is uncorrelated. (Granger and Morgenstern (1970))[2)]

Under the condition of (1) or (2) we assume the constant variance between returns. However, under the condition (3) or (4), we assume the time varying variance between returns. Furthermore, it is important to appreciate that even though $\{X_t\}$ is white noise, X_i and X_j are not necessarily independent for $i \neq j$. Zero correlation is sufficient, for practical purpose under non linear structure, to ensure that out-of-date prices are irrelevant when forecasting. So, we follow the condition (4), i.e., our random walk hypothesis is defined by the zero correlations between the price changes for any pair of different days.

Definition

The process $\{\ln P_t\}$ (or simply $\{P_t\}$) is said to be a random walk if $\{X_t\}$ in (3.1) is an uncorrelated process, i.e.;

$$\text{Covariance}\ (X_t, X_s) = 0 \quad \text{for } t \neq s$$

2) By definitions we can induce the following relationship among them:
$(i) \subset (ii) \subset (iii) \subset (iv)$.

3.2 *The Data*

3.2.1 *The Data*

For this study, we have selected the indices of two thin stock markets representing two geographical areas: Europe and the Far East: i.e., the Swedish stock market and the Korean stock market. Specifically we choose the AFGX (Affärsvärldens General Index) for the Swedish stock market and the KCSPI (Korean Composite Stock Price Index) for the Korean market.

The initial data set consists of 2534 daily AFGX series $\{P_t\}$ for the period of January 2, 1980 to March 6, 1990 and 3374 (including Saturdays' tradings) daily KCSPI series $\{P_t\}$ on the aggregate market value covering January 4, 1980 to July 16, 1991. From these initial data sets we construct two daily log returns series $\{X_t\}$ as in (3.1), where P_t denotes the t-th stock index. Regretfully, we can not update the AFGX data from 1990. The sample period includes the period of the crash in October, 1987. There is no reason to exclude these outliers since they reflect the nature of the market.

Data has been obtained from the Stockholm School of Economics and the Korean Stock Exchange respectively. These are the standard sources used in past studies. Note that we use the X_t excluding dividends in these series. Since dividends are typically small, both index returns, measuring only capital gains, should not be erroneous. The untransformed stock index, the log transformed stock index and the index returns for the sample series are plotted in Figure 3.1, Figure 3.2 and Figure 3.3 respectively. The Figure 3.1 and the Figure 3.2 clearly show that the stock indices for the sample markets are non-stationary.

3.2.2 *Statistical Analysis*

This sub-section contains a general analysis of the time-series properties of the index returns in the sample. To extract some distribution features of the process $\{X_t\}$ of the daily index returns in (3.1), we observe their average value (X_a), standard deviation (s.d.), skewness (b) and kurtosis (k), which are listed in Table 3.1. These statistics show some parameters for the returns generating process.

Table 3.1: Summary Statistics of Log Price Changes ($ln(P_t / P_{t-1})$)

Statistics	*Sweden*	*Korea*
Sample Size (n)	*2534*	*3374*
Average ($10^4 X_a$) *($\mu=0$)*	*+9.6939* *(4.41)*	*+5.5059* *(2.87)*
S.D.(10^3s.d.)	*+10.5876*	*+11.1349*
Skewness (b) *(standardized b)*	*-0.8963* *(-18.42)*	*+0.07931* *(1.88)*
Kurtosis (k) *(standardized k)*	*+13.124* *(104.03)*	*+7.25* *(50.39)*

(note) . The values were obtained from the statistical package "Statgraphic."

. The standardized values are given in parenthesis for the null hypothesis of mean=0, skewness=0, and kurtosis=3.

. The statistics are calculated by the following formulas:

$X_a = (1/n)\Sigma X_i$,

$s.d. = [(1/n\text{-}1)\Sigma(X_i\text{-}X_a)^2]^{½}$,

$b = [n\Sigma(X_i\text{-}X_a)^3]/[(n\text{-}1)(n\text{-}2)(s.d.)^3]$,

$k = [n(n+1)\Sigma(X_i\text{-}X_a)^4]/[(n\text{-}1)(n\text{-}2)(n\text{-}3)(s.d.)^4]$.

The results seem to confirm the well known fact that daily stock index returns are not normally distributed, but are skewed and peaked, whatever country is concerned. Both distributions exhibit high level of kurtosis, meaning that they are more peaked than a normal distribution. However for the skewness they exhibit different figures. The distribution for the AFGX returns has highly negative skewness, indicating that it is not symmetric, while the distribution for the

KCSPI returns has slightly positive skewness.[3] Some sample statistics from international stock indices are summarized in Table 3.2 in order to make an international comparison. The frequency distributions are also plotted in Figure 3.4, compared with the frequency from a normal distribution. The distributions seem to be fairly symmetric but is clearly peaked.

Table 3.2: Summary Statistics of Log Price Changes - international

Statistics	*USA*	*UK*	*Japan*	*Italy*	*France*
Sample Size(n)	2803	2803	2803	2803	2803
Average(10^4X_a) (μ=0)	+3.72 (1.81)	+5.11* (2.99)	+4.14* (2.27)	+6.82* (2.52)	+4.70* (2.39)
S.D.(10^3s.d.)	10.84	9.04	9.64	14.29	10.36
Skewness(b)	-3.52*	-1.69*	-1.70*	-0.99*	-1.51*
Kurtosis(k)	+75.0*	+19.7*	+36.7*	+11.7*	+16.9*

(note) . *The asterisked values are significant at 5 % level.*

. *The sample period is from 1/1/1980 to 30/9/1990.*

. *Used indices are value weighted indices for the U.S(Standard & Poor's Composite), the UK (FT All-shares), France (CAC General), Japan (Nikkei Dow Jones) and Italy (Milan Banca).*

. *The source is Corhay and Rad (1991).*

From these tables, the following observations can be made:

(1) As expected, the average X_a values are small enough, compared to standard deviation (s.d.). They can be relatively negligible.

(2) Somewhat surprisingly, the standard deviation is not much larger than those on the international markets. As an index on a thin market, none of the samples exhibit higher volatility than indices on large and liquid markets such as the USA, the UK and Japan.

---

3) We cannot conclude from these statements that the Korean stock market is much stabler than the Swedish stock market, since the values are sensitive to the sample period.

(3) Under the null hypothesis that $\{X_t\}$ is IID, the t-values of the meanare significant at $\alpha=0.01$ for the sample markets. So, it can be inferred that the average value might not be zero.[4)]

(4) From Figure 3.4, the sample distributions seem to be fairly symmetric. However skewness statistics in Table 3.1 assess the symmetry of distributions more precisely. The estimate of b for the Swedish stock market shows negative skewness, like those for other international markets, while the estimate of skewness for the Korean stock market exhibits slightly positive skewness. Nonetheless, the skewness is not extreme in any way for both markets. From Table 3.2 we can infer that the returns on the sample markets are less skewned than most of the international stock returns. When we formulate hypothesis H_0: $\{X_t\}$ is of IID normal skewness, H_0 is rejected for the Swedish market, but not for the Korean market. However, these tests for zero skewness using the standardized b are not formally valid, since our sample series exhibit serial correlation as discussed in the next sub-section, even though we have tested the hypothesis with the modification that the variance is inflated due to the serial correlation.[5)]

(5) From Figure 3.4, it is obvious that the sample distributions are more peaked than a normal distribution. We can also clearly verify from Table 3.1 that the sample markets represent far greater estimates of the kurtosis (k), considering that normal distributions have kurtosis equal to 3. It is very clear that the returns generating processes for the sample markets are not even approximately Gaussian. When we formulate hypothesis H_0: $\{X_t)\}$ is of IID normal kurtosis, H_0 is rejected for the sample markets. We have tested the hypothesis with the modification that the variance is inflated due to the serial correlation.[6)]

---

4) Since the t-values of the average could be biased, due to positive heteroskedasticity, we cannot claim formally that the average value is not zero.

5) The standard error of an estimate b is $\sqrt{(6/n)}$ for Gaussian white noise. However, when serial correlation is present, the variance is inflated to $\text{var}(b)=(6/n)\Sigma\rho(\tau)^3$ (see Harvey (1989)).

6) If $\{X_t\}$ is IID normal and n is large, then asymptotically $K \rightarrow N(0, 24/n)$. However, when the serial correlation is present, the variance is inflated to $(24/n)\Sigma\rho(\tau)^4$ (see Harvey (1989)).

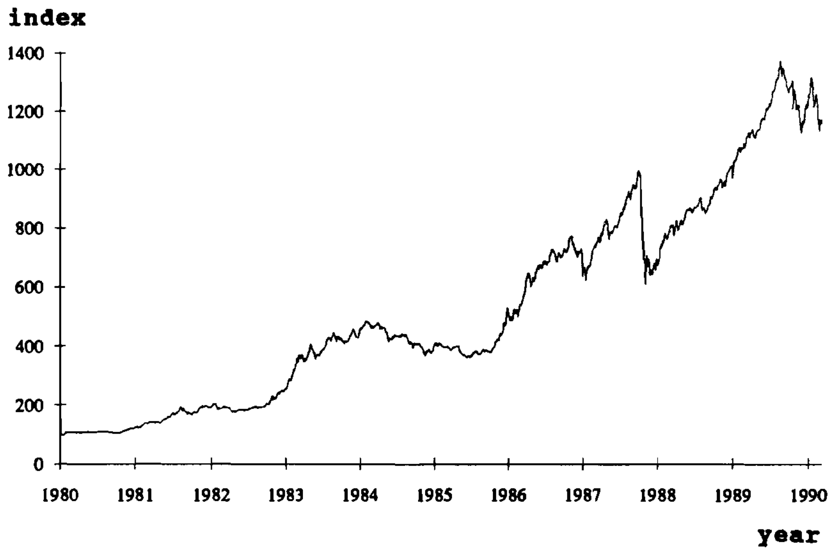


Figure 3.1(a): The Swedish Untransformed AFGX (Daily)
(1980.1.2--1990.3.6)

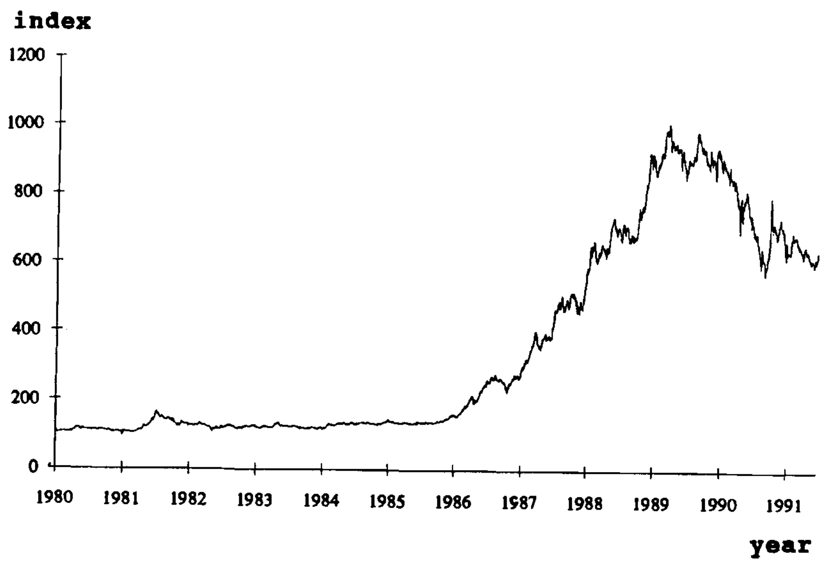


Figure 3.1(b): The Korean Untransformed KCSPI (Daily)
(1980.1.4--1991.7.16)

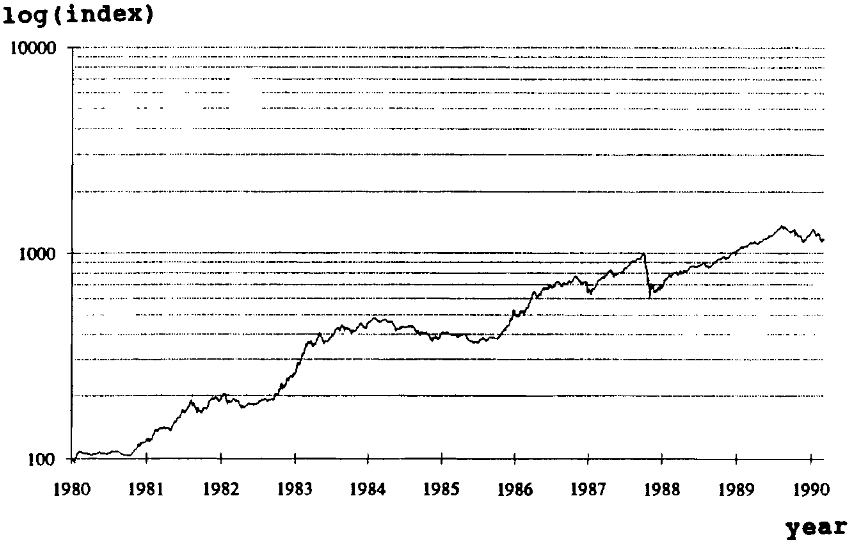


Figure 3.2(a): Log-Transformed AFGX

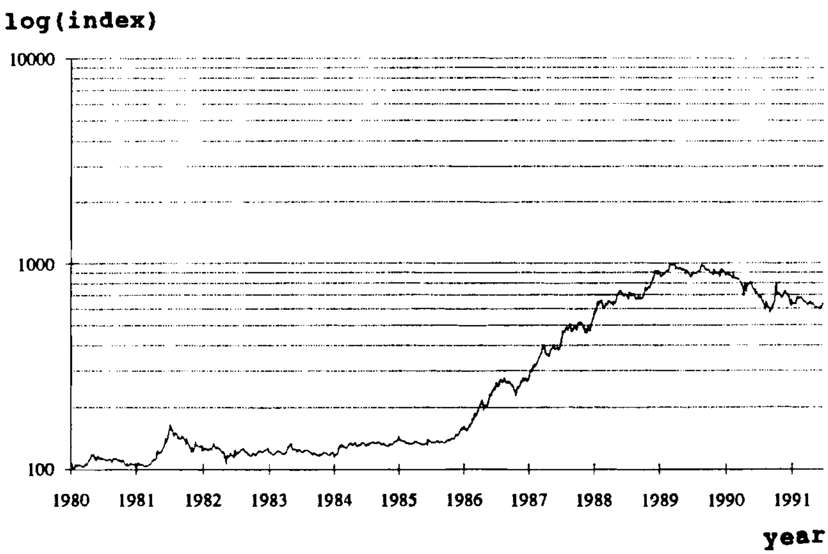


Figure 3.2(b): Log-Transformed KCSPI

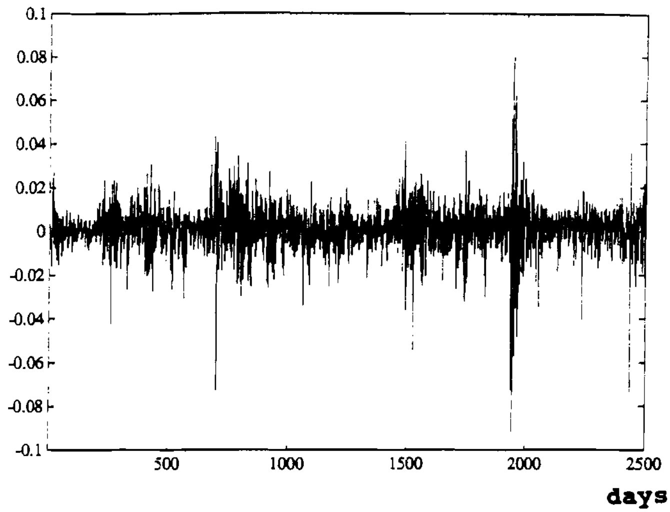


Figure 3.3(a): The AFGX Daily Returns ($\log P_t - \log P_{t-1}$)

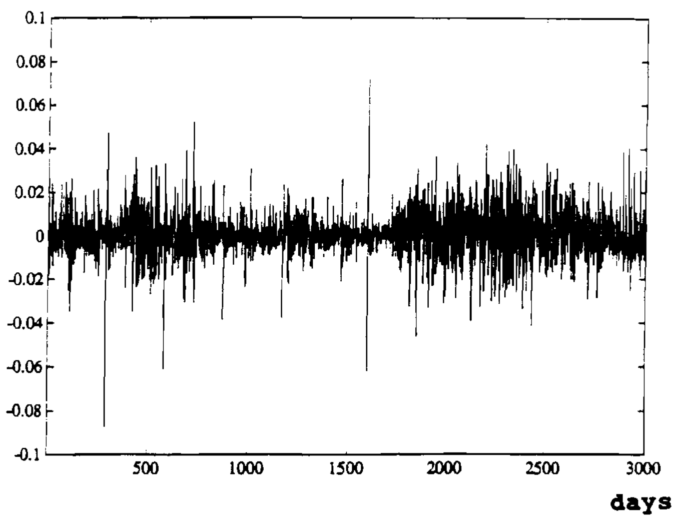


Figure 3.3(b): The KCSPI Daily Returns ($\log P_t - \log P_{t-1}$)

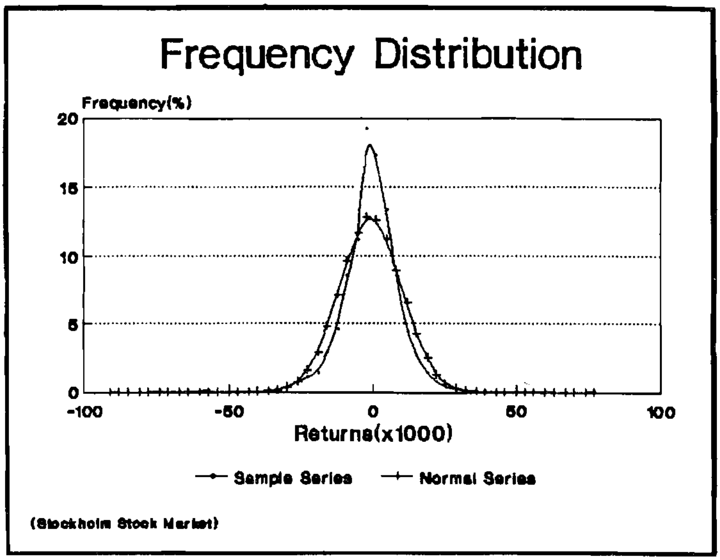


Figure 3.4(a): Empirical Distribution of the AFGX Returns Compared to the Normal Distribution

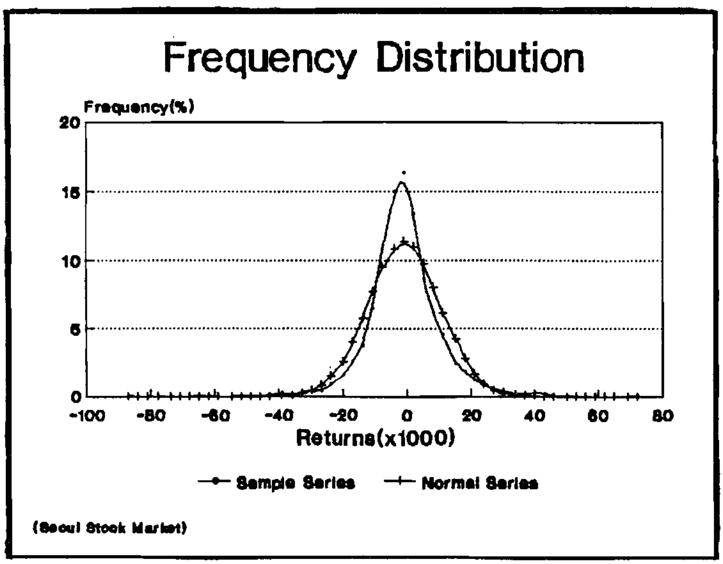


Figure 3.4(b): Empirical Distribution of the KCSPI Returns Compared to the Normal Distribution

Even though we have tested the departures from normality only by moment tests, we can claim that the returns for both the AFGX and the KCSPI are not normally distributed.[7] However, the sample markets do not seem to deviate more from the normality than other international stock markets. Also, by our findings we might consider that even if the average is significant for the hypothesis of $\mu=0$, the variation of μ_t will be small, compared to the variation of the standard deviations. This implies that the variation of risk premiums in a risk premium model, or in an efficient market model, is much smaller than that of the unexplained part or error term. The hypothesis that the distributions of the sample markets are normally distributed could not be supported by the descriptive statistics.

3.3 *The Process $\{X_t\}$ by Autocorrelation*

3.3.1 *Dependence between Returns*

In this section, we will verify that the sample markets' data are dependent between returns. We will use the asymptotic distribution of sample autocorrelation coefficients. The correlation between returns separated by a time-lag of τ days can be estimated from n observations $\{X_1, X_2, ...,X_n\}$ by the sample autocorrelation coefficient defined by

$$R = R_\tau (X) = \sum_{t=1}^{n-\tau} (X_t - E[X]) (X_{t+\tau} - E[X]) \,/\, \sum_{t=1}^{n} (X_t - E[X])^2, \tau \geq 1 \qquad (3.2)$$

Let $r_{\tau,x}$ be the realized value of a random variable $R_\tau(X)$. The estimates r_τ are often used to test hypotheses about the theoretical autocorrelations ρ_τ. We need to know the distribution of the variable R_τ to perform the tests.

7) There are a plethora of tests for judging departures from normality; moment tests, Chi-square test, the Kolmogorov-Smirnov test, probability plots, combined moment tests, ECDF(empirical cumulative distribution function)tests, Shapiro-Wilk test, D'Agostino tests, and so on. However there is not one test that is optimal for all possible deviations from normality (see Landry and Lepage (1992)).

Detailed results about the distribution of the R_τ are available for large samples from the linear process (Anderson and Walker (1964)). However, comparable results are not known for the nonlinear process. They provide a very general theorem about the multivariate distribution of $(R_1, R_2, \ldots.. R_k)$ for the linear process. To discover the dependence of process $\{X_t\}$, we use their findings.

Consider a process $\{X_t\}$ defined by

$$X_t - \mu = \sum_{j=0}^{\infty} b_j \varepsilon_{t-j}$$

having innovations ϵ_t independently and identically distributed with finite variance and also $\Sigma|b_j|$ and $\Sigma j b_j^2$ finite (summing over $j \geq 0$). Then the asymptotic distribution, as $n \to \infty$, of

$$\sqrt{n}\,(R_1 - \rho_1, R_2 - \rho_2, \ldots\ldots, R_k - \rho_k)$$

is multivariate normal with all mean zero and covariance matrix W_k determined by the complete sequence ρ_τ, $\tau > 0$. All useful linear processes with finite variance satisfy the conditions of this theorem. In particular, for independently and identically distributed X_t (strict white noise), possessing finite variance, W_k is simply the k x k identity matrix. Therefore, for large n,

$$R_\tau \sim N(o, 1/n) \text{ approximately,}$$

and furthermore R_i and R_j are approximately independent for all $i \neq j$, where $N(\mu, \sigma^2)$ indicates the normal distribution of which mean and variance are μ and σ^2, respectively.[8] This is the key point of their findings.

Here we will make use of their findings with the procedure of transforming returns and then finding autocorrelations.[9] Now let $R_\tau(|X|)$ and $R_\tau(X^2)$ be the τ-lag

---

8) Note that the above mentioned standard result is generally false for a nonlinear and uncorrelated process, even if it is white noise with finite variance. Such processes can have $var(R_\tau)$ far greater than 1/n. Thus if sample autocorrelations are significantly different from zero, then we cannot correctly reject the hypothesis that the X_t are uncorrelated. All we could say is that the X_t are not independently and identically distributed.

9) This idea was suggested by Granger and Anderson (1978).

coefficients of observed absolute returns $|X_t|$ and squared returns X^2, respectively. If $\{X_t\}$ is white noise, then, both $\{|X_t|\}$ and $\{X_t^2\}$ are also white noise. Moreover, for $n \to \infty$,

$$R_\tau(X) \to N(0, 1/n),$$
$$R_\tau(|X|) \to N(0, 1/n),$$
$$R_\tau(X^2) \to N(0, 1/n),$$

should be satisfied. Accordingly, the sample autocorrelation coefficients of $R_\tau(|X|)$ and $R_\tau(X^2)$ as well as $R_\tau(X)$ should be within an interval of $[-1.96/\sqrt{n}, +1.96/\sqrt{n}]$ in case of 5% significance level, provided $\{X_t\}$ is white noise.

Autocorrelation coefficients $R_\tau(X)$, $R_\tau(|X|)$ and $R_\tau(X^2)$ have been calculated for all lags τ between 1 and 30 trading days inclusive (see Appendix A). To summarize the signs and magnitudes of the correlations each is assigned to one of six classes.[10] These are

(1) $R < -0.1$
(2) $-0.1 \leq R < -0.05$
(3) $-0.05 \leq R < 0$
(4) $0 \leq R < 0.05$
(5) $0.05 \leq R < 0.1$
(6) $0.1 \leq R$.

Table 3.3(a), Table 3.3(b) and Table 3.3(c) give, by country, the coefficient at a lag of one day and the number of the 30 coefficients in each class. (For a comparison with thick markets, two thick markets frequencies appear below.)

From the tables, the following can be observed:

---

10) This classification follows Taylor's (1986) research.

Table 3.3(a):Autocorrelations $R_T(X)$ for Returns - frequencies by class

country	$R_1(X)$	class (1)	class (2)	class (3)	class (4)	class (5)	class (6)
Sweden	0.169 (0.02)	0 (0%)	0 (0%)	13 (43%)	13 (43%)	3 (10%)	1 (4%)
Rep.Korea	0.135 (0.02)	0 (0%)	0 (0%)	14 (46%)	15 (50%)	0 (0%)	1 (4%)
USA(15 stocks)	0.086 (NA)	(0%)	(2%)	(52%)	(42%)	(3%)	(1%)
Japan	0.033 (NA)	(0%)	(13%)	(37%)	(40%)	(7%)	(3%)

Table 3.3(b): Autocorrelations $R_T(|X|)$ for Absolute Returns - frequencies by class

country	$R_1(\|X\|)$	class (1)	class (2)	class (3)	class (4)	class (5)	class (6)
Sweden	0.338 (0.02)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	3 (10%)	27 (90%)
Rep.Korea	0.287 (0.02)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	14 (46%)	16 (54%)
USA(15 stocks)	0.165 (NA)	(0%)	(0%)	(2%)	(24%)	(49%)	(25%)
Japan	0.386 (NA)	(0%)	(0%)	(0%)	(13%)	(23%)	(64%)

Table 3.3(c): Autocorrelations $R_T(X^2)$ for Squared Returns - frequencies by class

country	*$R_T(X^2)$*	*class (1)*	*class (2)*	*class (3)*	*class (4)*	*class (5)*	*class (6)*
Sweden	*0.453 (0.02)*	*0 (0%)*	*0 (0%)*	*0 (0%)*	*6 (20%)*	*6 (20%)*	*18 (60%)*
Rep.Korea	*0.30 (0.02)*	*0 (0%)*	*0 (0%)*	*1 (4%)*	*18 (60%)*	*4 (13%)*	*7 (23%)*
USA(15 stocks)	*0.149 (NA)*	*(0%)*	*(0%)*	*(6%)*	*(37%)*	*(37%)*	*(20%)*
Japan	*0.294 (NA)*	*(0%)*	*(0%)*	*(13%)*	*(67%)*	*(17%)*	*(3%)*

(Note) . *The data of the USA and Japan are obtained from S. Taylor 1986) and T. Kariya (1989).*
. *Standard errors are given in parenthesis of the 2nd column.*
. *NA denotes the "Not Available."*

(1) All the first-lag coefficients are positive for both stock returns. Moreover, they are significant under the null hypothesis that $\{X_t\}$ is IID. This indicates the dependence between at least the observed returns at consecutive days.

(2) In an international comparison, the first order autocorrelation estimates for the AFGX returns as well as for the KCSPI returns are not small.

(3) $R_T(X)$'s in Appendix A are generally small and most of them are in the 95% confidence interval $(-1.96/\sqrt{n}, 1.96/\sqrt{n})$. To the contrary, for the $R_T(|X|)$ and the $R_T(X^2)$, most of them are outside the given interval.[11)]

11) The intervals are [-0.0389, 0.0389] for the AFGX returns and [-0.0337, 0.0337] for the KCSPI returns.

(4) For both index returns, the autocorrelations of absolute returns, $R_\tau(|X|)$'s, and those of squared returns, $R_\tau(X^2)$'s, are great and significant for first several τ's, which implies that the sample processes are not IID.

(5) Coefficients $R_\tau(|X|)$ and $R_\tau(X^2)$ for lags τ up to 30 days are almost always positive and larger than $R_\tau(X)$, which can be seen in Appendix A. In particular, Table 3.3(a) tells us that less than 5% of autocorrelation estimates for the index returns of the Korean stock market and less than 15% for the Swedish stock returns are outside the range -0.05 to +0.05, for $\tau=1,2,\ldots,30$. By contrast, the proportions of the coefficients which are greater than +0.05 for $|X_t|$ and for X^2 are high; i.e., 100% for $|X_t|$ and 37% for X^2 on the Korean market and 100% for $|X_t|$ and 80% for X^2 on the Swedish market. This means that the distribution for the next absolute (or squared) returns can depend on several previous absolute (or squared) returns,first noticed by Fama (1965).[12)]

The facts (4) and (5) deny that $\{X_t\}$ is IID. Hence we can consider that $\{X_t\}$ is not independent, $\{X_t\}$ is not identically distributed, or $\{X_t\}$ is neither independent nor identically distributed. However, even if $\{X_t\}$ is not identically distributed, Anderson and Walker's theorem will hold approximately via Lindberg-Feller Central Limit Theorem so long as X_t's are independent. Furthermore $R_\tau(|X|)$ is significant for long lags. Consequently, it is natural that we should conclude that the returns processes for the AFGX as well as for the KCSPI are dependent.

3.3.2 Nonlinear Structure between Returns

Many hitherto standard methods of financial research appear unreliable once we appreciate that long series of returns are not generated by either a strict white noise or a linear process. Now we will verify whether the AFGX returns and the KCSPI

---

12) Fama (1965) observed that large absolute returns are more likely than small absolute returns to be followed by an absolute return.

returns follow a linear process or a nonlinear process. Recall that a process $\{X_t\}$ is defined to be linear if there exist an IID process $\{\epsilon_t\}$ with mean zero and finite variance, a series of means $\{\mu_t\}$ and a series of constants $\{b_j\}$ with $\Sigma b_j^2 < \infty$ such that

$$X_t - \mu_t = \sum_{i=0}^{\infty} b_i \varepsilon_{t-i}, \tag{3.3}$$

In financial research, it is extremely important to distinguish strong white noise from white noise, and to distinguish a linear stationary process from a nonlinear stationary process. Nonetheless, we often neglect it in empirical research.

Assume that X_t has finite kurtosis (k), $k=E(X_t-\mu)^4/[E(X_t-\mu)^2]^2$. Also assume that $E[X_t^4]$ is finite. Then without further loss of generality, we can assume $b_0=1$ and $E[\epsilon_t^2]=1$. Let $\lambda=E[\epsilon_t^4]$ and define

$$S_t = (X_t - \mu)^2 = \sum_i b_i^2 \varepsilon_{t-i}^2 + 2 \sum\sum_{i<j} b_i b_j \varepsilon_{t-i} \varepsilon_{t-j},$$

and

$$S_{t+\tau} = \sum_k b_k^2 \varepsilon_{t+\tau-k}^2 + 2 \sum\sum_{k<m} b_k b_m \varepsilon_{t+\tau-k} \varepsilon_{t+\tau-m}.$$

Then $E[S_t] = \Sigma b_i^2$. To find the autocorrelation of $\{S_t\}$ we need $E[S_t S_{t+\tau}]$. It can be shown, by straightforward algebra,that for all $\tau \geq 0$,[13]

$$Cov(S_t, S_{t+\tau}) = E[S_t S_{t+\tau}] - E[S_t]^2$$

$$= (\lambda - 3) \sum_{i=0}^{\infty} b_i^2 b_{i+\tau}^2 + 2\left(\sum_{i=0}^{\infty} b_i b_{i+\tau}\right)^2.$$

Consequently, the autocorrelations of $\{S_t\}$ are given by

$$\rho_{\tau,s} = \frac{(\lambda - 3)\ \Sigma b_i^2 b_{i+\tau}^2 + 2\ (\Sigma b_i b_{i+\tau})^2}{(\lambda - 3)\ \Sigma b_i^4 + 2\ (\Sigma b_i^2)^2}$$

Furthermore, Taylor (1986) has shown that if $\{X_t\}$ has autocorrelations $\rho_{\tau,x}$, $\rho_{\tau,s}$ is given by weighted averages for certain constants α_τ determined by the b_i;

13) See Taylor (1986), p. 60.

$$\rho_{\tau,s} = \frac{(k-3)}{k-1}\alpha_\tau + \frac{2}{k-1}\rho^2_{\tau,x} \approx \frac{(k-3)}{k-1}\alpha_\tau + \frac{2}{k-1}R^2_\tau(X) \tag{3.4}$$

where α_T are autocorrelations of the process $\{X^*_t\}$, defined by

$$X^*_t = \Sigma\, b_i^2\, \epsilon_{t-i} \tag{3.5}$$

Therefore, when $\{X_t\}$ is Gaussian, k=3 and $\rho_{T,s} = \rho^2_{T,x}$. Let θ be the proportional reduction in mean square error obtained by optimal forecasts, so

$$\theta = \{\, var(X_t) - var(\epsilon_t)\,\} \;/\; var(X_t) \tag{3.6}$$

Then the constants α_T are constrained by

$$\alpha_\tau \geq 0 \;(all\ \tau \succ 0),\ and\ \Sigma\, \alpha_\tau \leq \theta \,/\, (1-\theta)^2 \tag{3.7}$$ [14]

Consequently, when the kurtosis of X_t is greater than 3,

$$0 \leq \sum_{\tau=1}^{k} R_\tau(S) \leq MAX\left[\sum_{\tau=1}^{k} R_\tau^2(X), \theta/(1-\theta)^2\right] \tag{3.8}$$

To determine the size of θ, consider the MSE (mean square errors) of an observed forecast $\hat{X}_t$, which is defined by

$$MSE(\hat{X}_t) = E[X_t - \hat{X}_t] \tag{3.9}$$

Also let X^b_t be the best predictor, which is defined by

$$X^b_t = \mu_t + \sum_{j=1}^{\infty} b_j \epsilon_{t-j} \tag{3.10}$$

Then, they provide

$$var\ (X_t) = MSE\ (\mu_t) \tag{3.11}$$

14) This is proved in Taylor (1986), pp 60-62.

$$var\ (\epsilon_t) = MSE\ (X_t^b) \tag{3.12}$$

and

$$\theta = 1 - \left[MSE(X_t^b) / MSE(\mu_t) \right] \tag{3.13}$$

Now we can interpret that the value of θ measures the efficiency of simple predictor μ_t for X_t relative to the best predictor $X^b{}_t$. Of course, $0 \leq \theta \leq 1$. It is generally believed that the security can be an object of investment and it will fluctuate rather information- efficiently. Hence, the MSE of the simple predictor μ_t and that of the best predictor $X^b{}_t$ will not differ greatly. It is very difficult to determine the optimal size of θ. All the investors believe that their predictors will be the best. Now, for returns we may safely assume that the relative efficiency is at best 0.2:

$$0 \leq \theta \leq 0.2, \qquad (3.13)^{15)}$$

which implies that we assume that no one has been able to find a forecast anywhere near 20% more accurate than the random walk forecast.

By applying (3.13),

$$\theta/(1-\theta)^2 \leq 0.312.$$

On the other hand, in the Appendix A, we can observe that all the estimates of $R_T(X)$ are smaller than 0.17 for the Swedish market and 0.14 for the Korean market. Hence, we may assume that

$|R_T(X)| < 0.17$ for the Swedish market

and

$|R_T(X)| < 0.14$ for the Korean market.

Then, by applying (3.8),

15) Taylor (1986) assumed the $\theta \leq 0.05$, while Kariya (1989) assumed that $\theta \leq 0.2$, as we assume in this study.

$$\sum^{30}_{\tau=1} R_\tau(S) < \fallingdotseq 0.867 \text{ for the Swedish market}$$

and (3.14)

$$\sum^{30}_{\tau=1} R_\tau(S) < \fallingdotseq 0.588 \text{ for the Korean market.}$$[16)]

For stock index daily returns, since average(X_a) is small enough, compared to standard deviation (s.d.) , we may regard $S_t \rightarrow (X_t - X_a)^2 \rightarrow X_t^2$ when the variation of S_t is considered. Hence $\Sigma R_\tau(S)$ can be estimated by $\Sigma R_\tau(X^2)$. We can easily verify that both markets go far over the upper bound in (3.14).[17)] Therefore a linear process cannot generate the observed returns. Any reasonable model for the sample stock markets must cope with nonlinear structure.

3.4 Summary of Chapter

The returns are constructed as the first difference of natural logarithms of daily closing prices for the Swedish stock market and for the Korean stock market. It has been observed that the sample returns exhibit high kurtosis. Furthermore, it has also been observed that both returns have the tendency of dependence between returns as well as of nonlinear generating structure between returns by analyzing the auotcorrelations of index returns, absolute returns and squared returns. These results imply that it is necessary to extend the analysis further and that a new analysis should be based on the nonlinear generating structure and dependence between returns. Some results derived from the linear generating process and independence between returns might be erroneous.

---

16) These values are arbitrary numbers. For example, for the Swedish market the value is calculated by $(0.17)^2 \times 30$.

17) The calculated values of $\Sigma R_\tau(X^2)$ are 4.588 for the AFGX returns and 1.843 for the KCSPI returns.

Chapter 4

Market Efficiency under Product Process

Many hitherto standard methods of financial research appear unreliable once we appreciate that long series of index returns are not generated by either a strict white noise or a linear process. Hence, tests for zero autocorrelations between returns should realize that the process will be strict white noise even if it is uncorrelated.

In the previous chapter, we have characterized the Swedish stock index returns and the Korean stock index returns to be lowly autocorrelated, to be distributed with long tails and high kurtosis, and to be generated nonlinearly between returns. This calls the previous results for the sample series into question, which were derived from the assumptions of linear generating process and independence between returns.

The product process model is one which is consistent with these features of the variations of returns, i.e.,

(1) leptokurtotic distribution;

(2) dependent process;

(3) and nonlinear generating process.

Section 4.1 describes the product process model for returns having either non-stationary variance or conditional variance dependent on past observations and additional variables. The model copes with nonlinear process and high kurtosis for returns as well as positive autocorrelations between squared returns. In section 4.2, the concept of the standardized returns (rescaled returns) is introduced to obtain a series possessing a reasonably homogeneous variance. Random walk hypothesis [RWH] should be tested with this series. In the next section, the random walk hypothesis against some alternatives is tested under the product process model for the AFGX rescaled returns as well as for the KCSPI rescaled returns. The findings are also compared with the previous results analyzed under the linear and independent process. In the last section, this chapter is summarized.

4.1 Product Process Model

4.1.1 Product Process Model

Generally, the index return x_t can be calculated at the end of the day t. This number can always be interpreted as being the realization of a random variable x_t. Before day t begins, the index return x_t cannot be known and so a probability distribution for x_t can be used to describe the possible outcomes x_t. Then by the end of the day, various participants will collectively have helped to determine a particular outcome x_t, depending mainly on new information and interpretations thereof. Similarly, we can suppose that by the end of day t the market has determined a conditional standard deviation v_t so that, for fixed v_t, x_t is an observation from a distribution having variance v_t^2. New information about the goods traded or other sources of economic and political information could partially determine v_t, as also could the changing preferences of investors for different goods. During day t-1, it is reasonable for us to assume v_t is not known exactly although good forecasts may be available. Consequently, v_t can be viewed as the realized value of a random variable v_t.

The product process model is one of the models consistent with the preceding discussion of a conditional standard deviation.[1)] The model can be written in

1) Before Taylor (1986), several researchers dealt with this idea: Granger and Morgenstern (1970), Praetz (1972), Clark (1973), Epps and Epps (1976), Ali and Giaccotto (1982), Engle (1982), Tauchen and Pitts (1983), etc.

the general form:

$$x_t = \mu + v_t u_t$$

or (4.1)

$$x_t - \mu = v_t u_t \ .$$

with $\{u_t\}$ a standardized process, so $E[u_t]=0$ and $var[u_t]=1$ for all t, and $\{v_t\}$ a process of positive random variables, usually having $var(x_t|v_t)=v_t^2$; also $E[x_t]=\mu$ for all t. Many financial researchers have assumed the$\{u_t\}$ is satisfied with normal distribution, and furthermore they suppose process $\{v_t\}$ and process $\{u_t\}$ to be stochastically independent.[2)] Under the assumptions the conditional distribution of $(x_t-\mu)$ given to v_t is expressed as

$$(x_t - \mu) \ | \ v_t \sim N(0, v_t^2) \tag{4.2}$$

where v_t^2 is the conditional variance of $\{x_t\}$.

In 4.1, the mean of $\{x_t\}$ is assumed to be constant, i.e., $E[x_t]=\mu$; however, as observed in chapter 3, t-values against $\mu_t=0$ are significant. Nevertheless, since we have also observed in chapter 3 that the standard deviation of $\{x_t\}$ is greater than the mean for both indices returns, the effects of μ_t's variation can be disregarded.

Under the product process model, it is easy to obtain the covariance between x_t and x_s:

$$cov(x_t, x_s) = E[u_t u_s] \ E[v_t v_s] \tag{4.3}$$

Hence, the statement that the x_t are uncorrelated for $\tau > 0$ is equivalent to the statement that u_t are uncorrelated for $\tau > 0$. The random walk hypothesis of stock process $\{x_t\}$, therefore, can be represented as

---

2) Engle (1982) has shown that stochastic independence is sufficient but not necessary to ensure $var(x_t|v_t)=v_t^2$.

$$H_0 : correl(u_t, u_s) = 0 \quad (t \neq s) \tag{4.4}$$

by the process of $\{u_t\}$.

In this chapter, we make an estimate of the process $\{u_t\}$ which is needed to test the hypothesis H_0 in (4.4), and furthermore review some characteristics of the process $\{u_t\}$. Taylor (1986) has introduced the easiest way to explain how market forces can make v_t a function of events occurring on day t by expressing x_t-μ as the sum of intra-day price movements:

$$x_t - \mu = \sum_{i=1}^{W_t} \omega_{it} \tag{4.5}$$

where, every ω_{it}, and W_t are random variables and ω_{it} represents the i-th variation factor on returns occurring during the day on day t. Suppose that for a fixed W_t, $\{\omega_{it}$: i=1,2,.....,$W_t\}$ are independently and identically distributed as $N(0,\sigma_\omega^2)$. Then, given an observed number W_t of intra-day price movements, the conditional variance is simply

$$v_t^2 = \sigma_\omega^2 W_t \tag{4.6}$$

Returns will display nonlinear behavior if the W_t are autocorrelated.

Tauchen and Pitts (1983) have described an economic model for the reactions of individual traders to separate items of information. They assume $\mu=0$ and identify W_t with the number of relevant information items during day t. They also assume $\{W_t\}$ is strict white noise. This must, however, be inappropriate, since then $\{x_t\}$ would also be strict white noise and we know this is not so.

Generally, we can identify v_t with a level of market activity covering: the amount and importance of new information, trading volume, the number of active traders, interest in the market relative to others and perhaps also seasonal factors. Many of these activity measures can only be assessed at the end of the day's business. Thus, v_t is realized at the end of day t, as x_t is. We cannot observe v_t at the end of day (t-1).

Since we suppose that prices do not cause the conditional standard deviations, it is reasonable to assume $\{v_t\}$ and $\{u_t\}$ are stochastically independent. We can easily prove that this is true for model (4.5) with $\omega_{it} \sim N(0, \sigma_\omega^2)$. Let $v_t = \sigma_\omega \sqrt{W_t}$ and by applying (4.1) and (4.5)

$$u_t = \frac{x_t - \mu}{v_t} = \frac{1}{\sigma_\omega \sqrt{W_t}} \sum_{i=1}^{W_t} \omega_{it}. \tag{4.7}$$

Given any realization v_t, W_t is v_t^2/σ_ω^2, so $\Sigma\omega_{it} \sim N(0,v_t^2)$ and $u_t|v_t \sim N(0,1)$. As the conditional distribution of u_t is the same for all v_t. So, $u_t \sim N(0,1)$ and v_t and u_t are independent. Likewise, the vectors $u' = (u_1,u_2,\ldots\ldots,u_n)$ and $v' = (v_1,v_2,\ldots\ldots,v_n)$ are independent for all n, establishing stochastic independence, providing all sets of variables ω_{it} are independent of v; i.e., individual intra-day price changes are stochastically independent of how many price changes there are. Accordingly, we can treat (4.5) as one of the models which are satisfied by (4.1).

4.1.2 Applicability of the Product Process Model

We first discuss whether the model (4.7) can be applicable to the AFGX returns and (or) to the KCSPI returns, by confirming the result $K_x > K_u$ in (4.7), where K_x and K_u denote the kurtosis of the x_t and the u_t, respectively.

The process $\{v_t\}$ could be stationary or non-stationary. If we suppose both $\{v_t\}$ and $\{u_t\}$ to be stationary, and suppose that both v_t and u_t have finite fourth moments, results for stationary $\{v_t\}$ and $\{u_t\}$ can be obtained quickly by applying their assumed stochastic independence. As $E[u_t]=0$ and $E[u_t^2]=1$, so $K_u=E[u_t^4]$. Consequently for normal u_t, $K_u=3$ and $\delta=\sqrt{(2/\pi)} \simeq 0.798$, where δ denotes the mean absolute deviation of the u_t, i.e., $\delta=E[|u|]$. Furthermore, by assumption of stochastic independence

$$E[(x_t-\mu)^2] = E[v_t^2 u_t^2] = E[v_t^2]E[u_t^2] = E[v_t^2] \qquad (4.8)$$

Thus, the kurtosis of the X_t is given by

$$K_x = \frac{E[(x_t-\mu)^4]}{E[(x_t-\mu)^2]^2} = \frac{E[v_t^4 u_t^4]}{E[v_t^2]^2} = \frac{K_u E[v_t^4]}{E[v_t^2]^2} \qquad (4.9)$$

Since $E[v_t^4] > E[v_t^2]^2$ for all positive variables v_t, having positive variance, $K_x > K_u=3$, is satisfied.[3] In chapter 3, it was verified that the sample series has greater kurtosis than 3. The model (4.7), therefore, can be applicable to both the AFGX returns and the KCSPI returns.

Next, we review the applicability of the product process model form the aspect of the following observed in chapter 3:

(1) Sample autocorrelations of $\{x_t\}$ are generally small.

(2) Sample autocorrelations of the absolute returns process $\{|x_t|\}$ are positive and they are greater than the auotcorrelations of $\{x_t\}$.

(3) Sample autocorelations of the squared process $\{x_t^2\}$ are also greater than the autocorrelations of $\{x_t\}$.

Let $\rho_{T,u}$, $\rho_{T,v}$, and $\rho_{T,s}$ denote the autocorrelations of the u_t, v_t, and $S_t=(x_t-\mu)^2$, respectively. We assume $\{u_t\}$ to be white noise. So, $\{x_t\}$ is also white noise. However, the adjusted squares S_t can be autocorrelated;

$$E[S_t] = E[v_t^2], \qquad (4.10)$$

and

3) The result $K_x > K_u$ confirms that mixing distributions having different variances increase the kurtosis. (See Taylor (1986), p. 67.)

$$E[S_t S_{t+\tau}] = E[v_t^2 v_{t+\tau}^2] E[u_t^2 u_{t+\tau}^2] = E[v_t^2 v_{t+\tau}^2], \quad (for\ \tau > 0) \tag{4.11}$$

Thus,

$$\begin{aligned} cov(S_t, S_{t+\tau}) &= E[S_t S_{t+\tau}] - E[S_t]^2 \\ &= E[v_t^2 v_{t+\tau}^2] - E[v_t^2]^2 = cov(v_t^2, v_{t+\tau}^2) \end{aligned} \tag{4.12}$$

Consequently, the autocorrelations $\rho_{\tau,s}$ of $\{S_t\}$ are related to the autocorrelations $\rho_{\tau,v}{}^2$ of $\{v_t^2\}$ by the linear relationship:

$$\begin{aligned} \rho_{\tau,s} &= \frac{cov(S_t, S_{t+\tau})}{var(S_t)} = \frac{cov(v_t^2, v_{t+\tau})}{var(S_t)} \\ &= \frac{\rho_{\tau,v}^2 var(v_t^2)}{var(S_t)} = \rho_{\tau,v}^2 \left[var(v_t^2) / var(S_t)\right] \end{aligned} \tag{4.13}$$

We observed that most of the sample autocorrelations $R_\tau(x^2)$ and $R_\tau(s)$ are positive among observed squares x_t^2 and adjusted squares $s_t=(x_t-x_a)^2$ for the sample series (see Appendix A). The sample series, therefore, can be consistent with equation (4.13) if the process of conditional variances $\{v_t^2\}$ is positively autocorrelated.

Moreover, let $e_r = E[v_t^r]$. Then $var(v_t^2) = e_4-e_2^2$ and var $(S_t) = E[S_t^2]-E[S_t]^2 = K_u(e_4-e_2^2)$. Now, (4.13) can be changed into for any distribution of the v_t:

$$\rho_{\tau,s} = \rho_{\tau,v}^2 \left[(e_4 - e_2^2) \ / \ (K_u(e_4 - e_2^2))\right]$$

or (4.14)

$$\frac{\rho_{\tau,s}}{\rho_{\tau,v}^2} = \frac{(e_4 - e_2^2)}{K_u(e_4 - e_2^2)} = \frac{1}{K_u}$$

The absolute returns $A_t = |x_t-\mu|$ have mean $E[A_t] = E[v_t|u_t] = \delta E[v_t]$. Recall that δ denotes the mean absolute deviation of the u_t, i.e., $\delta = E[|u_t|]$. Also, $E[A_t^2] = E[v_t^2]$ and so it is easy to get $var(A_t)$. For $\tau > 0$,

$$E[A_t A_{t+\tau}] = E[v_t v_{t+\tau}] \ E[|u_t u_{t+\tau}|] = \delta^2 E[v_t v_{t+\tau}], \tag{4.15}$$

and

$$cov(A_t, A_{t+\tau}) = E[A_t A_{t+\tau}] - E[A_t^2] = \delta^2 \ cov(v_t, v_{t+\tau}) \tag{4.16}$$

Consequently the A_t have autocorrelations

$$\rho_{\tau,A} = \frac{cov(A_t, A_{t+\tau})}{var(A_t)} = \frac{\delta^2 \ cov(v_t, v_{t+\tau})}{var(A_t)}$$
$$= \frac{\delta^2 \cdot \rho_{\tau,v} var(v_t)}{var(A_t)} = \delta^2 \ \rho_{\tau,v}[var(v_t) \ / \ var(A_t)] \tag{4.17}$$

In terms of $e_r = E[v_t^r]$, $var(v_t) = e_2 - e_1^2$, and $var(A_t) = e_2 - \delta^2 e_1^2$. Since $\delta = 0.798$, $var(v_t) \leq var(A_t)$, giving

$$0 \leq \rho_{\tau,A} \ / \ \rho_{\tau,v} \leq \delta^2. \tag{4.18}$$

From (4.14) and (4.18), we can infer that the statement that observed $s_t = (x_t - x_a)^2$ and $a_t = |x_t - x_a|$ have positive autocorrelations is equivalent to the statement that v_t and v^2_t have positive autocorrelations; i.e.,

$$\rho_{\tau,s} > 0 \Leftrightarrow \rho_{\tau,v2} > 0$$

and (4.19)

$$\rho_{\tau,A} > 0 \Leftrightarrow \rho_{\tau,v} > 0.$$

The positive autocorrelations of v_t and v^2_t are verified in Appendix A.

4.2 *Rescaled Returns*

4.2.1 *Selection of the Process $\{v_t\}$*

The high variance of conventional autocorrelation coefficients are almost certainly caused by the non-constant conditional variances of the returns. To be accurate in financial research we need a series possessing a reasonably homogeneous variance. Ideally, this would be the series $\{u_t\}$ in (4.20):

$$u_t = (x_t - \mu) / v_t \qquad (4.20)$$

Since u_t is the realization of exact or approximated strict white noise $\{u_t\}$, coefficients calculated from the $\{u_t\}$ would have orthodox variances, i.e., $var(R_{\tau,u}) \simeq 1/n$. Furthermore, we have stated in the previous section that the random walk hypothesis of (H_0: $correl(x_t,x_s) = 0$ $(t \neq s)$), is equivalent to the hypothesis of (H_0: $correl(u_t,u_s) = 0$ $(t \neq s)$) under the product process. The difficulty is that realized conditional standard deviations v_t are not observable so u_t cannot be observable either.

An approximation to the standardized return u_t can be given by substituting estimates for μ and $\hat{v}_t$. Using x_a for μ and forecast $\hat{v}_t$ made at time (t-1) for v_t gives the rescaled return defined as

$$y_t = (x_t - x_a) / \hat{v}_t \simeq (x_t - \mu) / \hat{v}_t \qquad (4.21)$$

So, here we discuss how to obtain a good forecast of $\hat{v}_t$.

The conditional standard deviation v_t, in fact, is not directly related to past returns x_{t-j} $(j>0)$; it rather indicates the level of market activities on day t; and it is realized on the end of the day t as x_t is. Hence, v_t's optimal estimate is a forecast at time (t-1), when both v_{t-1} and x_{t-1} are realized together. At time (t-1) neither x_t nor v_t are realized. However, if we assume that $\{v_t\}$ follows a specified process, an optimal estimate $\hat{v}_t$ at time (t-1) can be regarded as a realized value at time t. Here we follow such a logic.

To get an optimal forecast, $\hat{v}_t$, we need specify the process $\{v_t\}$. We use as a criterion the following sampled mean square errors-denoted MSE and defined

$$MSE(\hat{v}_t) = \sum_{t=n_1}^{n} (v_t - \hat{v}_t)^2 / (n - n_1) \qquad (4.22)$$

where $\hat{v}_t$ is a forecast of v_t at time (t-1), and n_1 is the number of data used for obtaining the initial value. It is natural that (4.22) is an estimate of the population

$$MSE\ (\hat{v}_t) = E[v_t - \hat{v}_t]^2 \tag{4.23}$$

Since the process $\{v_t\}$ is usually not observed directly under the assumptions:

$$|x_t - \mu| = |u_t| v_t;$$

$$u_t \sim N(0,1);$$

and

$$\delta = E[|u_t|] = 0.798,$$

we regard the equation (4.24) as an estimate of v_t

$$v_t^* = a_t / \delta \tag{4.24}$$

where $a_t = |x_t - x_a|$.

Similarly, the forecast $\hat{v}_t$ is defined by

$$\hat{v}_t = \hat{a}_t / \delta \tag{4.25}$$

where $\hat{a}_t$ is a forecast of a_t.[4] Hence, to find the best forecast of v_t, it suffices to obtain the optimal forecast $\hat{a}_t$ and then define $\hat{v}_{t+1} = \hat{a}_{t+1}/\delta$. To the followings, we assume $a_t = \delta v_t$.

It is generally noted that if the $\{v_t\}$ is a lognormal AR(1) process, the autocorrelations of $\{A_t\}$ can be modelled, exactly or approximately, by $\rho_{\tau,A} \simeq D(\beta)\psi^{\tau}$ with $0<D<1$, $0<\psi<1$, as $\beta \to 0$. These are the same autocorrelations as those of the ARMA(1,1) process:[5]

4) Taylor (1986) has shown that an optimal forecast minimizing MSE satisfies the linear relationship $\hat{A}_t = \delta\hat{V}_t$.

5) Taylor (1986), p.102.

$$\acute{A}_t - \mu_A - \psi(\acute{A}_{t-1} - \mu_A) = \epsilon_t - \theta\epsilon_{t-1} \; (1 > \psi > 0)$$

with $\{\epsilon_t\}$ uncorrelated, providing θ is chosen to give

$$D = (1-\Psi\theta)(\Psi-\theta)/\{(1-2\Psi\theta+\theta^2)\}.$$

The optimal forecast $\hat{A}_{t+1}$ from among the class of linear unbiased forecasts

$$\mu_A + \sum_{i=0}^{\infty} m_i (A_{t-i} - \mu_A) \tag{4.26}$$

is generally determined by the autocorrelations $\rho_{\tau,A}$. Thus $\{A_t\}$ and $\{\acute{A}_t\}$ have the same best sequence m_i. Furthermore it is believed that it follows that the optimal linear forecast of A_{t+1} is

$$\hat{A}_{t+1} = \mu_A + (\psi-\theta)(A_t-\mu_A) + \theta(\hat{A}_t - \mu_A) \qquad (4.27)^{6)}$$

To produce a forecast $\hat{a}_{t+1}$ it is necessary to replace μ_A, ψ and θ by estimates. At time $t > n_1$, μ_A has been estimated by $\hat{\mu}_{A,t+1}$ while ψ and θ have been estimated from observations 1 to n_1. A forecast $\hat{a}_{t+1}$ derived from (4.27) depends on the latest observation a_t, the previous forecast $\hat{a}_t$ made at time t-1, and the estimates, thus:

$$\hat{a}_{t+1}^{(1)} = (\hat{\psi} - \hat{\theta})\, a_t + \hat{\theta}\,\hat{a}_t + (1 - \hat{\psi})\,\hat{\mu}_{A,t+1}$$

or

$$\hat{a}_t^{(1)} = (\hat{\psi} - \hat{\theta})\, a_{t-1} + \hat{\theta}\,\hat{a}_{t-1} + (1 - \hat{\psi})\,\hat{\mu}_{A,t} \tag{4.28}$$

where $\hat{\psi}$ and $\hat{\theta}$ are estimates of parameters in the ARMA(1,1) and $\hat{\mu}_{A,t}$ indicates

---

6) The optimal linear forecast of x_{t+1} for the process $\{x_t\}$ following the ARMA(1,1) is
$\mu + (a+b)\Sigma(-b)^{i-1}(x_{t-i+1} - \mu)$
where a is a parameter for AR(1) and b is a parameter for MA(1).

$$\hat{\mu}_{A,t} = \sum_{s=1}^{t-1} a_s / (t-1) \tag{4.29}$$

as an estimate of μ_A at time (t-1).

The ARMA(1,1) forecast described so far assumes a stationary process for returns. This causes the assumed stationary mean μ_A to appear in (4.27). Indeed, (4.27) can be generalized to show that the best $\hat{a}_{t+n}$, as $n \rightarrow \infty$, is μ_A. However, it is also perfectly reasonable to question the assumption of stationarity, especially for long series.

Estimates of the future mean $E[A_{t+1}]$ calculated from a_s, $s \leq t$, will then be unreliable. A simple and practical way to avoid using μ_A is to set $\psi = 1$ in (4.27), giving

$$\hat{A}_{t+1} = \sum_{i=0}^{\infty} (1-\theta)\theta^i A_{t-i} = (1-\theta)A_t + \theta \hat{A}_t \tag{4.30}$$

This defines an EWMA (Exponentially-Weighted Moving Average). From (4.30), actual forecasts can be calculated using

$$\hat{a}_{t+1}^{(2)} = (1-\hat{\theta})a_t + \hat{\theta}\hat{a}_t$$

or

$$\hat{a}_t^{(2)} = (1-\hat{\theta})a_{t-1} + \hat{\theta}\hat{a}_{t-1} \tag{4.31}$$

Taylor (1986) has compared various forecasts and has concluded that (4.28) is best for a stationary model for returns and (4.31) for a non-stationary model for returns. Furthermore, he has shown that the difference between them is usually very small because the forecasts obtained are similar when ψ is close to 1 for a stationary model. Hence, he recommended (4.31) because of its simplicity in calculation.[7]

7) Taylor (1986) compared various forecasts by RMSE (relative mean square errors).

Following Taylor's suggestion, we will calculate an actual forecast of a_t, using (4.31). In particular by letting $\gamma=1-\theta$, from (4.25)[8)]

$$\hat{v}_t = (1-\gamma)\hat{v}_{t-1} + \gamma\,|x_{t-1}-x_a|\,/\,\delta \quad (t \geq 21)$$

$$v_{20} = \sum_{i=1}^{20} |x_t - x_s|\,/\,20\,\delta \tag{4.32}$$

For the AFGX returns and for the KCSPI returns, we estimate the smoothing parameter by the Brown exponentially smoothing model and MSE as a criterion. The results are in Table 4.1.

Table 4.1: Best Smoothing Constant (γ)

series	γ
AFGX Returns	0.11
KCSPI Returns	0.20

After computing the autocorrelations $R_\tau(\hat{v})$ of $\hat{v}_t$, we have observed that $nR_\tau(\hat{v})/(n-\tau)$ is well fitted by $\rho_{\tau,\hat{v}} = D\psi^\tau$ with $D>0$ and $\psi>0$ ($\tau=1,2,\ldots,30$), which is an ARMA(1,1) type autocorrelation. Furthermore, the estimate of ψ is close to 1 (over 0.95) for the sample series. Hence, the process $\{v_t\}$ will be approximated by an ARMA(1,1) model with ψ close to 1.[9)]

4.2.2 Asymptotic Limit

In a product process model, the rescaled return $u_t=(x_t-\mu)/v_t$ is a stationary Gaussian Process, which implies that the process $\{u_t\}$ can be expressed as

---

8) γ, called a smoothing parameter, indicates how much the information on the deviation of returns at time (t-1) is reflected at time t.

9) $\rho_{\tau,\hat{v}} = (0.9249)(0.9654)^\tau$ -----(for Sweden),

$\rho_{\tau,\hat{v}} = (0.7913)(0.9576)^\tau$ -----(for Rep. Korea).

$$u_t = \sum_{j=0}^{\infty} b_j \varepsilon_{t-j} \qquad (b_0 = 1) \tag{4.33}$$

$$where,\ \{u_t\} \sim N(0, \sigma^2),\ \sigma^2 = \left(\sum_{j=0}^{\infty} b_j^2\right)^{-1}.$$

Hence if the process $\{u_t\}$ is directly observed , the sample autocorrelations of the $\{u_t\}$, $R_{\tau,u}$, can be used in testing various hypotheses by the following theorem:

Theorem 10)

Assume that process $\{w_t\}$ is a linear process, defined by

$$w_t = \sum_{j=0}^{\infty} b_j \varepsilon_{t-j}$$

having innovations ϵ_t independently and identically distributed with finite variance and also $\Sigma|b_j|$ and $\Sigma j b_j^2$ finite (summing over $j \geq 0$). Then the asymptotic distribution, as $n \to \infty$, of

$$\sqrt{n}\,(R_{1,w}-\rho_1, \ldots\ldots, R_{k,w}-\rho_k) \rightarrow N(0, \Omega_k)$$

where ρ_τ are population autocorrelations for w_t and $R_{\tau,w}$ are sample autocorrelations for w_t, is multivariate normal with all means zero and covariance matrix $\Omega_k = (\omega_{\alpha\beta})$ determined by the complete sequence ρ_τ, $\tau > 0$. In particular, for independently and identically distributed w_t(strict white noise), possessing finite variance, Ω_k is simply the k x k identity matrix. Therefore, for large n,

$$\sqrt{n}\, R_{\tau,w} \sim N(0,1), \quad \text{approximately} \tag{4.34}$$

and $R_{i,w}$ and $R_{j,w}$ are approximately independent for all $i \neq j$. Moreover, the asymptotic covariance is given to

10) The proof is given by Anderson (1971) and Anderson and Walker (1964).

$$\omega_{\alpha\beta} = \lambda_{\alpha-\beta} + \lambda_{\alpha+\beta} + 2(\lambda_0\rho_\alpha\rho_\beta - \lambda_\alpha\rho_\beta - \lambda_\beta\rho_\alpha) \tag{4.35}$$

$$with \quad \lambda_i = \sum_{j=-\infty}^{\infty} \rho_j \rho_{i+j} .$$

The more important thing is that this theorem is applicable only to a linear process and not to a nonlinear returns process. We will see the instability of the variance of $\sqrt{n}R_{\tau,x}$ for the raw returns $\{x_t\}$ and verify that the sample autocorrelations for the rescaled returns $\{u_t\}$, $R_{\tau,u}$ follow this theorem. In fact, the $\{u_t\}$ is not observable, so, we make use of the sample rescaled returns y_t, defined by

$$y_t = (x_t - x_a) \, / \, \hat{v}_t . \tag{4.36}$$

Suppose now that $a_{\tau,n}(x)$ denotes the estimate of $var(R_\tau(x))$ based on n observations, given by equation (4.37)

$$var\left(R_{\tau,n}(x)\right) = a_\tau(x) = \sum_{t=1}^{n-\tau} (x_t - x_a)^2 (x_{t+\tau} - x_a)^2 \, / \left[\sum_{t=1}^{n} (x_t - x_a)^2 \right]^2 \tag{4.37}$$

As the expected value of every x_t is assumed to be zero, the variance of (4.37) can be shown to be

$$a_\tau(x) = \sum_{t=1}^{n-\tau} x_t^2 x_{t+\tau}^2 \, / \left(\sum_{i=1}^{n} x_t^2 \right)^2 \tag{4.37'}$$

Moreover, let $b_{\tau,n}(x) = n(a_{\tau,n}(x))$. Then,

$$b_{\tau,n} = \left(\frac{1}{n} \sum_{t=1}^{n-\tau} x_t^2 x_{t+\tau}^2 \right) / \left(\frac{1}{n} \sum_{t=1}^{n} x_t^2 \right)^2$$

and therefore, by the law of large numbers,

$$b_{\tau,n}(x) \rightarrow \beta_\tau(x) = E[x_t^2 x_{t+\tau}^2] \, / \, E[x_t^2]^2 \tag{4.38}$$

as $n \rightarrow \infty$. A more convenient formula for $\beta_\tau(x)$ can be found by applying the result that

$$cov(x_t^2, x_{t+i}^2) = E[x_t^2 x_{t+i}^2] - E[x_t^2]^2$$

firstly with $i=\tau$ and secondly with $i=0$. This gives us the result that β_τ depends on the kurtosis of the x_t, denoted K_x, and the autocorrelations of x^2_t [11)]

$$b_{\tau,n}(x) = n\, a_{\tau,n}(x) \rightarrow \beta_\tau(x) = 1 + (K_x - 1)\,\rho_\tau(x_t^2) \quad (4.39)$$

Clearly $\beta_\tau(x)=1$ for any strict white noise process, even if it is not symmetric, since $\rho_\tau(x^2_t)=0$. In other words, if the $\{x_t\}$ were i.i.d. normal, $b_{\tau,n}(x)$ would scatter around 1. Hence, if the $\{y_t\}$ of (4.36) is a good estimate of $\{u_t\}$, it is at least necessary that the estimates $b_{\tau,n}(y)$ should scatter around 1. We can observe these results for the AFGX returns series and for the KCSPI returns series in Figure 4.1.

Consequently, it will be at least approximately valid to regard the $\{y_t\}$ as an estimate of $\{u_t\}$ and to assume that $R_\tau(y)$ is asymptotically normal N(0,1/n). That is, the processes $\{y_t\}$ for both the AFGX returns and the KCSPI returns do not contradict with the usual convergence theorem and is valid for any test of the random walk hypothesis.

In Table 4.2, $R_\tau(y)$ for the samples are shown.[12)]

---

11) $\beta(x) = E[x^2_t x^2_{t+\tau}] \,/\, E[x^2_t]^2$.

since $E[x^2_t x^2_{x+\tau}] = cov(x^2_t, x^2_{t+\tau}) + E[x^2_t]$,

$$\beta(x) = \frac{cov(x_t^2, x_{x+\tau}^2) + E[x_t^2]}{E[x_t^2]^2} = \frac{cov(x_t^2, x_{t+\tau}^2)}{var[x_t^2]}\,\frac{E[x_t^4] - E[x_t^2]^2}{E[x_t^2]^2} + 1$$

$$= \rho_{\tau,x2}(E[x_t^4]/E[x_t^2]^2 - 1] + 1 = \rho_{\tau,x2}(K_x - 1) + 1 .$$

12) Details are given in Appendix A.

Table 4.2: Autocorrelations $R_\tau(y)$ for Rescaled Returns - frequencies by class

country	$R_1(y)$	class (1)	class (2)	class (3)	class (4)	class (5)	class (6)
Sweden	0.209 (0.019)	0 (0%)	0 (0%)	9 (30%)	20 (66%)	0 (0%)	1 (4%)
Rep. Korea	0.107 (0.017)	0 (0%)	0 (0%)	12 (40%)	17 (56%)	0 (0%)	1 (4%)

(Note) . The six classes are the same as in Table 3.3.
. Standard errors are given in parenthesis of the 2nd column.

4.3 Testing the Random Walk Hypothesis

4.3.1 Null Hypothesis and Test Methodology

Several definitions of the random walk hypothesis have been offered, as noted in chapter 3. It is generally believed that the hypothesis that index returns have independent and identical distributions has not been proved. Changes in either variance or conditional variance can suffice to explain the rejection of the i.i.d. normality hypothesis.

Under the product process model a more general null hypothesis can be defined by firstly replacing identical distributions by identical means and secondly replacing independent distributions by uncorrelated distributions, giving

$$H_0 : E[y_t]=E[y_{t+\tau}] \text{ and } \mathrm{cov}(y_t,y_{t+\tau})=0$$

or (4.40)

$$H_0 : \rho_\tau = 0,$$

for all t and $\tau > 0$.

Figure 4.1(a): Estimates of the Variances of Autocorrelation Coefficients.

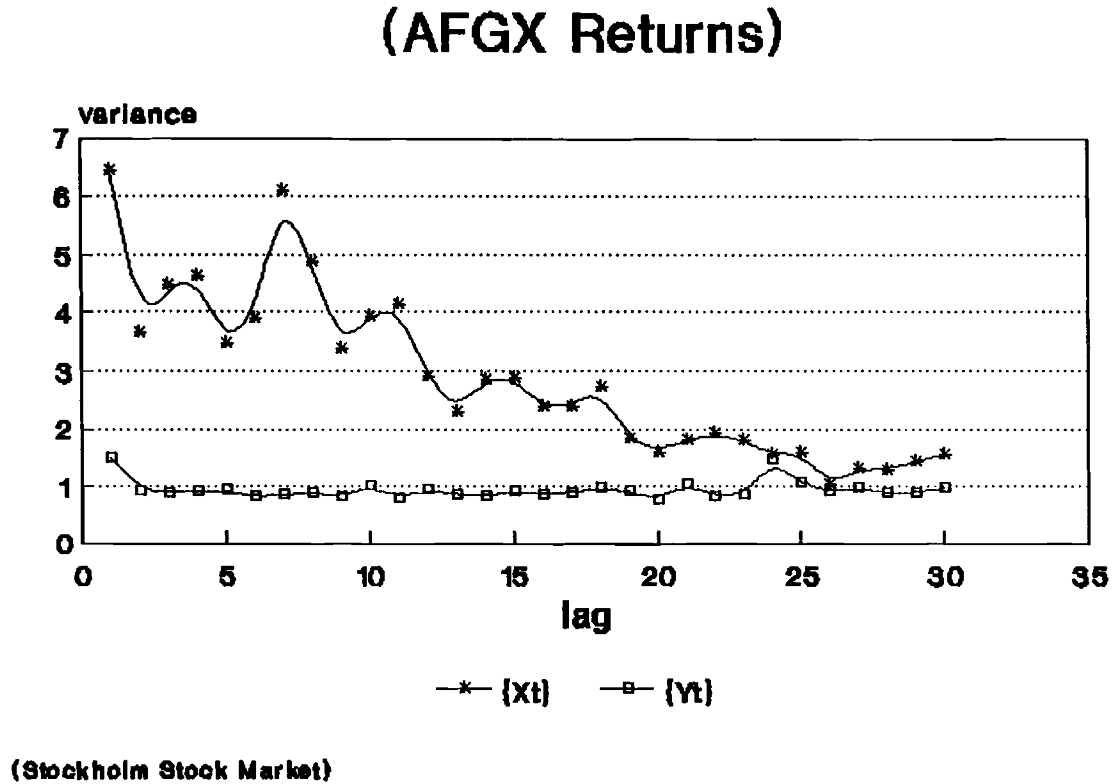
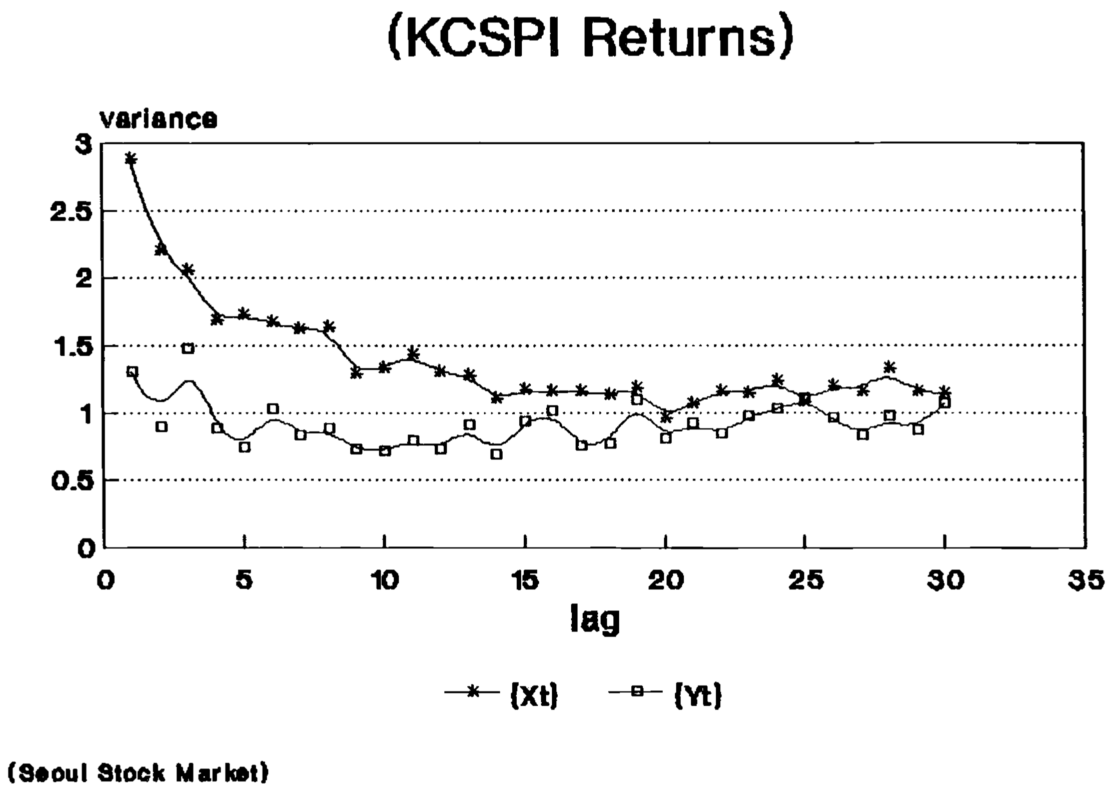


Figure 4.1(b): Estimates of the Variances of Autocorrelation Coefficients.



This is our definition of the random walk hypothesis. Note that H_0 does not require the process $\{x_t\}$ to be stationary.

Rejection of the random walk hypothesis is not sufficient to refute the efficient market hypothesis. Trading costs can prevent the exploitation of statistical dependence and then the random walk hypothesis is false but the efficient market hypothesis is not. It is easier to test for randomness than for efficiency so it is best to test the random walk hypothesis first.

Correct distribution for the sample autocorrelations should be established to derive reliable tests of the random walk hypothesis, since the random walk hypothesis H_0 against an alternative hypothesis is tested on the basis of the sample autocorrelation coefficients. Otherwise, the significance level will be erroneous. In particular, the significance level is underestimated when the autocorrelation variance is greater than the values given by large sample theory for a strict white noise process. However, since rescaled returns are used here, it can be avoided.

Choice of significance level is always arbitrary to some degree. A 5 per cent level is used throughout this chapter.

Information about the distribution of the sample autocorrelation at lag τ, R_τ, for true and false null hypotheses is needed to obtain powerful random walk tests. We have already reviewed the asymptotic results, as $n \to \infty$, for strict white noise in the previous section:

$$\text{as } n \to \infty \quad , \sqrt{n}\, R_\tau(Y) \sim N(0,1).$$

4.3.2 A Selection of Test Statistics

Several random walk test statistics are available so it is tempting to perform several tests. As a matter of fact, there is no consensus about an appropriate set of test statistics in financial literature. We will take, among them, the first autocorrelation test, the Box-Pierce test, price-trend autocorrelation test, and excessive response test.

(1) The First Autocorrelation Test: $\sqrt{n}R_1(Y)$

This is a popular and simple test, rejecting H_0 at the 5 per cent significance level if $\sqrt{n}|R_1(Y)| > 1.96$. This test is based on the theorem that as $n \to \infty$, $\sqrt{n}R_1(X) \to N(0,1)$ approximately, when H_0 is true. An alternative hypothesis of this test is,

$$H_1 = \rho_1 \neq 0. \tag{4.41}$$

(2) Box-Pierce Test: Q_k

The Box-Pierce test is used to test some coefficients of $R_\tau(y)$. Under this test, the null hypothesis is

$$H_0 = \rho_1 = \ldots\ldots = \rho_k = 0$$

and an alternative hypothesis is

$$H_1 = \rho_i \neq 0 \quad (for\ any\ i,\ \ 1 \leq i \leq k). \tag{4.42}$$

A natural way to combine k coefficients into a single test statistic is given by

$$Q_k = n \sum_{\tau=1}^{k} R_\tau^2(y) \tag{4.43}$$

This statistic is based on the fact that when H_0 is true,

$$n(R_1^2(y) + \ldots\ldots + R_k^2(y)) \to \chi_k^2, \text{ approximately,}$$

under the assumption of independence of $R_i(y)$ and $R_j(y)$, $i \neq j$. The null hypothesis is rejected for sufficiently high values of Q_k. Results will be given for two tests, using k=10, and k=30. The respective tail areas in which H_0 is rejected are $Q_{10} > 18.31$ and $Q_{30} > 43.77$.

(3) Price-Trend Test: T^* and T^{**}

A price trend is essentially a general movement of prices in a fixed direction, up or down. However, this was not used so often in the random walks tests before Taylor(1982). In the price trend hypothesis, trends would imply that prices do not adjust fully and instantaneously when new information becomes available. Instead, some new information would have to be incorporated slowly into prices. Taylor (1986) has stated that trends will occur if information is used imperfectly, for example, if enough people are irrational or rational but unable to interpret all the information quickly and correctly.[13)]

When a particular item of information is interpreted slowly, such a slow interpretation of the information item will cause several returns to be partially determined by the same information. The fundamental trend idea is that when a particular information item causes several returns, these returns are all influenced in the same way, either towards a positive conditional mean or towards a negative conditional mean. Thus, trends will cause positive autocorrelations. Furthermore, since it is believed that the impact of that current information which is not fully reflected in the current price, upon future returns, should diminish as time goes on, the autocorrelations should decrease as the lag increases.

Taylor (1986) defined the price-trend hypothesis by

$$H_0 : \rho_\tau = 0 \ (\tau > 0)$$

vs.

$$H_1 : \rho_\tau = A\phi^\tau \ (A,\phi,\tau > 0) \tag{4.44}$$

13) Stevenson and Bear (1970) and Leuthold (1972) offered some evidence for trends. However, Praetz (1976) criticized the method.

There are two parameters in H_1. Parameter A measures the proportion of information not reflected by prices within one day. Parameter ϕ measures the speed at which imperfectly reflected information is incorporated into prices. As $A \rightarrow 0$, or $\phi \rightarrow 0$, information is reflected perfectly.

Powerful tests of the random walk hypothesis H_0 against the price-trend hypothesis H_1 can be constructed, using the theoretical distribution of sample autocorrelations. Taylor (1982) has derived that when H_0 is true and standard asymptotic results are assumed,

$$T_{k,\phi} = \sum_{\tau=1}^{k} \phi^{\tau} R_{\tau}(y)$$

has mean zero and variance $\Sigma\phi^{2\tau}/n$. Consequently as $n \rightarrow \infty$, $T \sim N(0, \Sigma\phi^{2\tau}/n)$.

Furthermore, he has suggested the test statistic

$$\begin{aligned} T^{*} &= T_{30,0.92} \,/\, \sum_{\tau=1}^{30} (0.92^{2\tau}/n)^{1/2} \\ &= 0.4247\sqrt{n} \sum_{\tau=1}^{30} (0.92)^{\tau} R_{\tau}(y) \end{aligned} \tag{4.45}$$

is an observation from N(0,1) if H_0 is true. We accept H_0 if $T^* < 1.65$ and accept H_1 if $T^* > 1.65$ at the 5 per cent significance level.

Sometimes revised test statistic T^{**} would be used because of data errors. Price series often contain errors. Naturally, we can correct data errors by checking large returns against another source. However, small errors may remain. The primary consequence of errors is to decrease $R_1(y)$, since an error in the price P_t causes errors in the two returns x_t and x_{t-1} one being positive and the other negative.

A model for price error is

$$log(P_t) = \log(P_t^*) + \delta_t$$

with P_t, P_t^* and δ_t generating the analyzed price, the true price, and an error respectively with a very high chance that $\delta_t=0$. Assuming $\{\delta_t\}$ is white noise, stochastically independent of true returns $\{x_t^*\}$, with $\Gamma = var(\delta_t)/var(x_t^*)$, it can be shown that the theoretical autocorrelations ρ_T of the analyzed returns are related to those of the true returns ρ_T^* by

$$\begin{aligned} \rho_1 &= (\rho_1^* - \Gamma) / (1 + 2\Gamma) \\ \textit{and} \quad \rho_\tau &= \rho_\tau^* / (1 + 2\Gamma), \; \textit{if } \tau \geq 2. \end{aligned} \tag{4.46}$$

In fact, it is very difficult to estimate Γ. But it is clear that the major impact of errors occurs at the first lag.

Taylor (1982) has shown that T^* should be replaced by

$$T^{**} = 0.4649\sqrt{n} \sum_{\tau=2}^{30} (0.92)^\tau R_\tau(Y) \tag{4.47}$$

when errors are suspected. As with T^*, the asymptotic distribution of T^{**} is N(0,1) when the random walk hypothesis is true.

(4) Excessive Response Test: E^* and E^{**}

As a powerful test of the random walk hypothesis, a special hypothesis - excessive response hypothesis can be constructed. Suppose that the information that a firm has developed a new product arrives on the stock market and that the information has sufficient value to upgrade the firm's stock price to Z^t. Under the excessive response hypothesis, the impact of new information makes the price move to more than (less than) Z^t. But this also becomes new information. Hence, the next time the price moves a little down (up). After a few days' fluctuations, the price converges towards Z^t.

This hypothesis can be defined by

$$H_0 : \rho_T = 0$$

vs. (4.48)

$$H_1 : \rho_T = \pm A\phi^T.$$

The test statistic for this hypothesis is given to

$$E^* = 0.4274n \sum_{\tau=1}^{30} (0.92)^{2\tau} R_\tau(y)^2 \tag{4.49}$$

and if $E^* < 4.12$, H_0 is accepted.

To be similar to T^{**} in (4.47), we can also get a revised test statistic E^{**}:

$$E^{**} = 0.4649n \sum_{\tau=2}^{30} (0.92)^{2\tau} R_\tau(y)^2 \tag{4.50}$$

and if $E^{**} < 3.79$, H_0 is accepted.[14]

4.3.3 Numerical Results

In Table 4.3, the values of the test statistics for the AFGX returns and for the KCSPI returns are listed. Our observations for each test statistic are as follows.

(1) The first autocorrelation test shows for both samples that the RWH is rejected. Furthermore, the first autocorrelation coefficient is positive.

(2) The Box-Pierce test statistic Q_{10} as well as Q_{30} rejects H_0 for both markets. Comparing the first lag autocorrelation coefficient with those of other lags, we can guess that the latter are negligible. So, we might infer that Box-Pierce test results are dependent on the first autocorrelation test.[15]

---

14) The critical region for excessive response test is derived in Appendix B.

15) For more accurate conclusions, it is needed to calculate test statistics against each individual stock and to investigate how many stocks are rejected in Box-Pierce test.

(3) The price trend test statistic T^* rejects the RWH strongly for the samples.

(4) The adjusted trend test statistic T^{**} also rejects the RWH for both markets. But not strongly, compared to T^*. If there are no data errors, this result means that the trend alternative H_1 might hold largely due to the largeness of the first autocorrelation coefficient, not due to its own patterns. This observations is strengthened by (1).

(5) The excessive responsive test E^* strongly rejects the RWH for the samples. However the adjusted excessive response test statistic E^{**} does not reject the RWH. We can judge that the first autocorrelation coefficient influences the tests strongly, which implies that the excessive response hypothesis is not effective, neither on the Swedish stock market nor on the Korean stock market.

(6) The least squares estimates of ϕ's based on $\{y_t\}$ are also listed in Table 4.3. We can verify that the estimates of ϕ's for the samples are nearly consistent with the value $\phi=0.92$ adopted for the construction of the approximation.

4.3.4 Comparisons

With our findings, in this sub-section, we compare not only the results reported for some thick markets but also the previous results analyzed under the linear generating process for the sample markets. As a relatively young model, the product process model has not been applied to many international stock markets. So far, the results for only three thick stock markets have been reported. They are as follows:

Taylor (1986) has studied daily prices for 15 US stocks, which are the first fifteen stocks in the Dow Jones Industrial Average, covering the period from

Table 4.3: Values of the Random Walk Test Statistics

test statistics	*AFGX returns*	*KCSPI returns*
$R(1)$	10.49*	6.23*
Q_{10}	128.14*	48.62*
Q_{30}	140.24*	70.79*
T^*	7.16*	4.15*
T^{**}	3.30*	1.85*
E^*	43.07*	16.34*
E^{**}	3.48	2.56
ϕ	0.94	0.96

(note) Asterisks indicate significance at the 5 percent level.

January, 1966 to December, 1976. Taylor (1986) has also studied the Financial Times 30-share index series, which is the geometric average of the prices of thirty leading UK shares between 1975 and 1982, for the UK market. More recently, Kariya (1989) has applied the product process model to the Tokyo stock market. He has studied daily Nikkei Dow Jones Index series, covering the period from January, 1983 to December, 1987. Table 4.4 shows their results.

When comparing our findings with the values of the test statistics for the thick markets, given to Table 4.4, we can observe:

(1) The first autocorrelation test shows the RWH is rejected for the sample markets as well as for the thick markets. Furthermore, all markets have positive first lag autocorrelations. Note that the positiveness of the 1st lag autocorrelation should not be overestimated. The average of individual stock

Table 4.4 Values of the Random Walk Test Statistics (the USA, the UK and Japan)

series (market)	$R_{1,Y}$	R_1 1.96	Q_{10} 18.3	Q_{30} 43.7	T^* 1.65	T^{**} 1.65	E^* 4.1	E^{**} 3.7
(USA) Allied	0.07	3.77	23.5	54.2	0.73	-0.82	NK	NK
Alco	0.18	9.77	102.9	124.4	2.87	-1.06	NK	NK
Am.Can	0.13	6.94	58.3	85.4	1.84	-0.97	NK	NK
ATT	0.09	5.11	40.8	64.5	0.19	-1.98	NK	NK
Am.Bra.	0.09	4.94	44.6	63.2	-0.74	-2.92	NK	NK
Anacon.	0.04	2.43	14.8	23.1	-0.31	-1.37	NK	NK
Bethle.	0.10	5.54	45.6	61.4	-0.26	-2.65	NK	NK
Chrysl.	0.04	2.37	26.78	51.1	-0.03	-1.05	NK	NK
Dupont	0.13	6.82	52.0	66.1	3.22	0.58	NK	NK
Kodark	0.05	2.84	19.9	37.3	-0.42	-1.68	NK	NK
G.Elec.	0.10	5.30	38.9	59.6	1.24	-0.92	NK	NK
G.Food	0.11	5.92	46.1	67.3	0.79	-1.67	NK	NK
G.Motor	0.05	2.86	25.8	51.0	-0.23	-1.47	NK	NK
G.Tele.	0.06	3.40	20.0	40.3	-0.28	-1.76	NK	NK
Harves.	0.07	4.15	24.6	42.1	0.89	-0.81	NK	NK
(UK) FT30	0.05	2.13	20.3	39.3	1.41	0.62	NK	NK
(Jap.) Nikkei	0.18	5.65	36.3	50.4	1.88	-0.38	12.5	1.0

(note) . NK denotes the "Not Known:"
. Critical value at the 5 % significance level for each test statistic is given in the first row.

analysis can sometimes show the inverse result on the positiveness of the first lag autocorrelation.

(2) For the Box-Pierce test statistic, Q_{10}, all the markets reject the null hypothesis. Moreover, even for the Q_{30}, all the markets except the UK reject the null hypothesis. Comparing the first lag autocorrelation with those of long lags and the values of Q_{10} with those of Q_{30}, we can induce that the Box-Pierce test results are largely dependent on the relative size of the 1st lag autocorrelation for all the markets.

(3) While the USA and the UK do not show the price trend alternative, the Tokyo market follows the price trend alternative. However, considering that the test statistic T^{**} for the Tokyo market accepts the RWH, we can infer that the price trend alternative might hold, largely due to the largeness of the first lag autocorrelation, not due to the market's patterns. So, we cannot say that the Tokyo stock market follows the price trend model in a really meaningful sense. Even though the sample markets also reject the RWH against the price trend alternative by the T^{*} as well as T^{**}, the values of the T^{**} are relatively small, compared to the T^{*}. Hence, the sample markets do not follow the price trend model in a really meaningful sense.

(4) For the excessive response hypothesis, we can make a comparison only with the Tokyo market. The sample markets show the similarity to the Tokyo stock market: The Swedish stock market and the Korean stock market as well as the Tokyo stock market accept the excessive response model by the E^{*}. However, they reject the model on the E^{**}. We can infer that the acceptance of the excessive response hypothesis is largely due to the size of the first lag autocorrelation, not due to the market's patterns. This implies that three stock markets really do not agree with the excessive response model.

Generally, there seems to be no difference between the sample markets and the thick stock markets. This implies that the first autocorrelation test seems to influence on the other test statistics largely for all the stock markets. Hence, under the time domain methodologies using the autocorrelation concept, we might be unable to detect long term memory and patterns, because of the relative greatness of the first lag autocorrelation.

Next, we compare our findings with the previous results for the sample markets, analyzed under the linear and independent generating process between returns. Table 4.5 shows the previous tests of the random walk hypothesis for the sample markets.

From the Table 4.5, we can characterize the sample markets as follows:

(1) significant first lag autocorrelation,

(2) significant dependence between returns,

(3) deviations from normality,

(4) rejection the RWH over daily intervals,

(5) positive first lag autocorrelation.

Comparing our findings with the above, over daily intervals, it can be verified that there is no difference between the results obtained under the linear and independent generating process between returns and those under the nonlinear and dependent generating process between returns. In fact, one of the advantages from the nonlinear and dependent generating models between returns is that long term memory and patterns can be detected by the models. Nonetheless, we cannot find them neither for the sample markets nor for some thick markets under the product process. Naturally, there can be no long term memory and patterns on the stock markets. However, we can also question that the time domain methodologies are not effective in detecting long term memory and patterns. This causes many financial econometricians to research further under the frequency domain methodologies.

Table 4.5: Previous Tests of the RWH for the Swedish Stock Market and the Korean Stock Market - Daily

country (author) (year)	period (year)	tests	conclusion
Sweden (Jennergren) (1975)	67-71	. Runs . Serialcorrelation of lags from 1 to 10 . Filter rule	. reject the RWH over daily intervals . maybe inefficient . deviation from normality
Sweden (Claesson) (1987)	78-84	. Runs . Filter rule . Autocorrelation test	. positive autocorrelation . dependence . relatively efficient
Korea (Lee) (1989)	83-87	. Runs . Autocorrelation test(lag= 1,...,10)	. dependence . positive auto-correlation

4.4 Summary of Chapter

The product process model, as one of the models which cope with a conditional standard deviation, was discussed in this chapter. After reviewing the applicability of this model for the AFGX returns and the KCSPI returns, we have found that raw index returns are not appropriate in testing the random walk hypothesis. Instead, we proposed to use the rescaled returns and discussed the procedure of estimation for them.

The null hypothesis of i.i.d. normality was tested by some powerful tests: the first autocorrelation test; the Box-Pierce test; price-trend test; and excessive response test. Generally, the sample markets rejected the random walk hypothesis. In particular, we have found that the effect of the first autocorrelation is strong for both markets. We have compared our findings with the results for some thick stock markets as well as with the previous results for the sample markets, analyzed under the linear and independent generating process. No difference between them has been found. All the stock markets have the significant first lag autocorrelation. Furthermore, they are positive. Long term memory and patterns cannot be found, since all the stock markets do not follow the price trend model or the excessive response model in a really meaningful sense. So, it might be questionable whether long memory and patterns can be detected under the time domain methodologies using the autocorrelation concept, because of the relative greatness of the first lag autocorrelation. Further research is still required. These will be discussed in the following chapters.

Chapter 5

Power Spectra in Stock Index Returns

In the previous chapters, the importance of noise was reviewed and the AFGX returns and the KCSPI returns were tested statistically in a time domain. We have also concluded that daily returns for the sample stock markets do not follow random walks under the nonlinear generating process. However, even though the sample stock markets showed significant dependence between returns, we could not uncover whether long term memory and patterns existed in them. Significance of only the first lag autocorrelation was identified. Hence, this chapter aims at diagnosing whether it is possible to uncover patterns for the sample markets under the frequency domain methodologies. Among many frequency domain methodologies, we specially focus on the "power-law function", since this is very useful in explaining the grounds for the fractal structure of patterns in the price changes.

Noise can be also classified in a frequency domain according to its power spectrum — a decomposition of the time series into components with different frequencies by their contribution to power or variance. In fact, the power spectrum has been one of the best known and most frequently applied statistical measures to characterize complex time series and it has been obtained for numerous physiological variables.

Typically the power spectrum at a given frequency is proportional to the sequence of the coefficient of the sine wave of that frequency and has one or more peaks corresponding to the main frequencies present in the signal. In addition to these main peaks,other frequencies may be present but at a lower amplitude.[1] Sometimes, due to the large contributions from the slow-varying, nonperiodic components in the time series, the spectrum peaks at lower frequencies. The increase of the power spectrum at lower frequencies can be modelled as a power-law function, $1/f^{\alpha}$, where f is the frequency.[2] As mentioned above, our concerns are concentrated to this power law function.

Furthermore, concerning the power law function's application to the stock market, there is one hot issue: Does the stock market follow the f^{-1} or f^{-2}? Unlike white noise and $1/f^2$ noise, which can be easily generated by a sequence of uncorrelated random variables and a sequence with uncorrelated random increment (random walk or Brownian motion), the theory of $1/f$ noise, which is believed to lead to "self organized criticality" introduced by Per Back at Brookhaven National Laboratory, is wide open. Although the origin of $1/f$-noise remains a mystery after more than 60 years of investigation, it represents the most common type of noise found in nature, and it has replaced most of the noises which have been treated as a Brownian noise or white noise. Some scientists also introduce stock exchange price as an example of $1/f$-noise (Bak,et.al. (1988)). However, although $1/f$ noise is very common in nature, there is little empirical literature on whether stock market prices follow $1/f$ noise. More recently, Li (1991) performed the power spectral analysis on the daily closing value of the Dow Jones Industrial average. However, Li's results show that this time series is much closer to $1/f^2$ noise, even after the trend is removed.

In fact, it is very important to distinguish $1/f$-noise from $1/f^2$-noise for stock price movements, since it gives us the information on long range correlation in time. So, it is natural that some financial econometricans should take interest in the fractal randomness in time for stock price movements.

---

1) Granger and Hatanaka (1964) have already described spectral theory relevant for economic studies and Praetz (1979) highlighted practical problems encountered.

2) For example with $\alpha=2$, the time series is called $1/f^2$ noise, and with $\alpha=1$, the series is called $1/f$ noise or flicker noise.

The existence of structures in very large scales results in long range correlations. These long range correlations can be detected by examining the two point correlation function, that is, by seeing whether it decays slower than an exponential function, or whether it reaches the zero value at a very large distance. In particular, if the two point correlation function decays as a power law, we have a "scaling phenomenon". The power spectrum P(f), which is the Fourier transformation of the correlation function, will also be a power law function. If the two point correlation function decays even slower than a power law, such as the case of logarithmic function, the spectrum is then exactly inversely proportional to the frequency, i.e., P(f) $\sim 1/f$, which represents the most common type of noise found in nature.

This chapter is organized as follows. Section 5.1 presents some properties of the power law function. In particular, noises are classified according to the power exponent and power exponent $\alpha=2$ is established for random walks. In section 5.2, power law function on the speculative market is reviewed. Section 5.3 details the results for the sample data, and section 5.4 summarizes the findings.

5.1 Some Properties of the Power Law Function

5.1.1 Classification of Noises

Assuming that stock price movements follow the power law function, it can be verified that changes in time have many of the same similarities at different scales. Among the many domains where self-similar power laws flourish, statistics ranks very high. Especially, the power spectra (squared magnitude of the Fourier transform) of statistical time series, often known as noises, seem addicted to simple, homogeneous power laws in the form $f^{-\alpha}$ as functions of frequency. Thus, noises can be classified according to α values.

Prominent among noises is white noise, with a spectral exponent $\alpha=0$. So, the power spectrum of white noise is independent of frequency. In a white noise process, every value of the process is completely independent of its past. But white noise, that is, a noise with a constant or flat power spectrum, is a convenient fiction - a little white lie. Just like white light (hence the name white noise), the spectrum of white noise is flat only over some finite frequency range. Nevertheless, white spectra provide a supremely practical paradigm, modelling untold processes across a wide spectrum of disciplines. The increments of Brownian motion and numerous other innovation processes, the learned name for a succession of surprises belong to this class (see Figure 5.1(a)).

If we integrate a white noise over time, we get a "brown" noise, such as the projection of a Brownian motion onto one spatial dimension. Brown noise has a power spectrum that is proportional to f^{-2} over an extended frequency range. In contrast to a white noise process, in "brown noise" only the increments are independent of the past, giving rise to a rather boring tune (see Figure 5.1(c)).

Even though white noise and brown noise are more popular in the sciences, they are far from exhausting the spectral possibilities: between white and brown there is pink noise with an f^{-1} spectrum. Many natural scientists have discovered that the exponent found in most phenomena of sciences is near the middle of this range, giving rise to the hyperbolic power law f^{-1}.

Such time functions are called pink noise or flicker noise, since they are intermediate between brown(ian) (f^{-2}) and white (f^{-0}) (see Figure 5.1(b)).

Beyond brown, black noise lurks, with a power spectra proportional to $f^{-\alpha}$ with $\alpha > 2$. Figure 5.1(d) shows a waveform of black noise with $\alpha = 3$. This Black noise phenomena govern natural and unnatural catastrophes like floods and droughts in the natural science and bear markets in finance. Because of their black spectra, such disasters often come in clusters.

So far we have classified noises into 4 categories: white noise, brown noise, pink noise and black noise. All of these phenomena share an important trait: their power spectra are homogeneous power functions of the form $f^{-\alpha}$ over some respectable range of frequencies, with exponent α running the gamut from 0 to

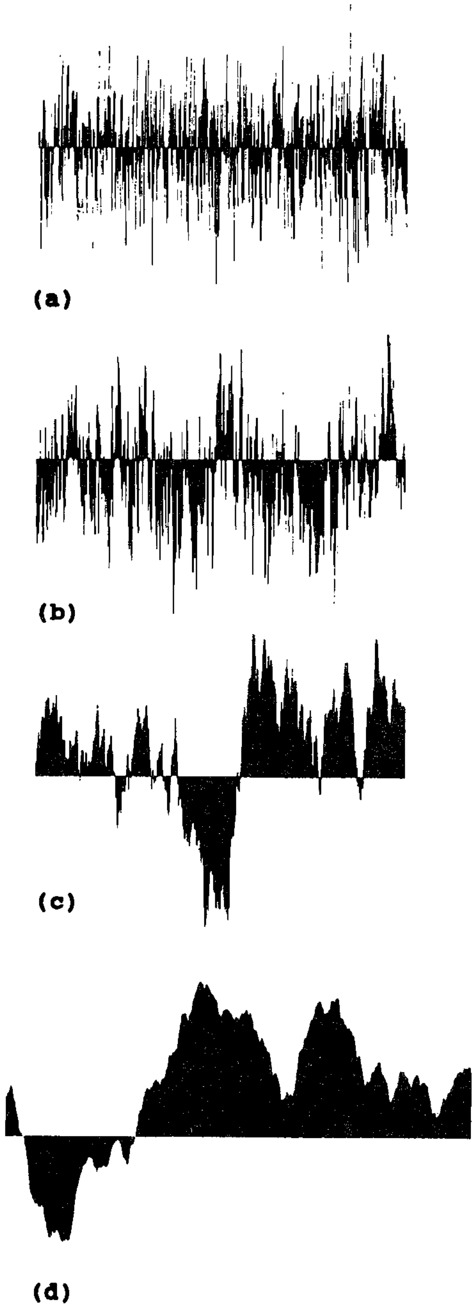


Figure 5.1: Sample of (a) white nosie with f^0 power spectrum; (b) "pink" noise with $1/f$ power spectrum; (c) "brown" noise with $1/f^2$ power spectrum; and (d) "black" noise with $\alpha=3$.

4. However, it is now believed the more important noises in financial economics are pink noise and brown noise.

Theoretically, the power spectrum of many noises in natural science is proportional to f^0. However, they have been found by experiment to be incorrect for small values of f; a $1/f$ noise is observed at small f. So, pink noise is now being regarded as a more important concept under the frequency domain. Some social scientists are also doubtful of some phenomena known to be random walk processes.

Pink noise has equal power in any constant intervals on a logarithmic frequency scale. So, this power spectrum indicates that noise has a strong correlation with time. Intuitively, this may be hard to accept, but it has been confirmed that the form of the spectrum does not change regardless of how long the observation period might be.

It is perhaps more interesting that if stock price movements follow the power law function $1/f^{\alpha}$, there might exist patterns, long correlations in time, and the long term period depends on the value of the coefficient α. Generally, Brownian motion has much shorter correlated patterns. Intuitively, it is very difficult to catch the meaning. However, it is possible to use fractals in finding the meaning. For example, Figure 5.2 shows samples of "music" generated from the three characteristic types of "noise" shown in Figure 5.1. It can be easily verified that Brownian motion is too correlated compared to other noises.

5.1.2 α Coefficient for Random Walks

There is a long standing literature on using the spectral distribution to test various hypothesis, for example, Granger and Hatanaka (1964), Anderson (1971), Priestiley (1981) and Harvey (1981). The power spectrum $P(f)$ of a stationary stochastic process can be obtained for a finite number of data by the following discrete Fourier transformation:

$$P(f) = N|A(f)|^2,$$

$$with \quad A(f) = (1/N) \sum_{j=1}^{N} x_j \, exp(i2\pi fj/N) \tag{5.1}$$

where f is the frequency.

Thus, the power spectrum may be viewed as a decomposition of the variance of the process in terms of frequency.

In section 5.1.1, noises were classified according to the value of a spectral exponent α. Since many financial price series have been considered to follow random walks, here we derive the fact the random walk time series have $1/f^2$ power spectra; i.e., $\alpha=2$.

The power spectrum P(f) of a continuous time series x(t) is defined as

$$P(f) = |A(f)|^2 = \left[\int_{-\infty}^{\infty} x(t) \, exp(2\pi ift) \, dt \right]^2 \tag{5.2}$$

If the time series is not stationary, the power spectra is not well defined. Since the random walk series are nonstationary, we assume that the function is defined only for $-\infty < t_1 < t < t_2 < \infty$, but zero when $t < t_1$ or $t > t_2$.

A random walk series is defined as the series whose derivative $\epsilon(t) = dx(t)/dt$ is uncorrelated (white noise). The Fourier transform of x(t) is given to

$$\begin{aligned} A(f) &= \int_{t1}^{t2} x(t) \, e^{i2\pi ft} dt \\ &= \frac{1}{i2\pi f} \int_{t1}^{t2} x(t) \, d(e^{i2\pi ft}) \\ &= \frac{1}{i2\pi f} \left[x(t) \, e^{i2\pi ft} \Big|_{t1}^{t2} - \int_{t1}^{t2} e^{i2\pi ft} \varepsilon(t) \, dt \right] \\ &= \frac{1}{i2\pi f} \left[x(t_2) \, e^{i2\pi ft_2} - x(t_1) \, e^{i2\pi ft_1} - \int_{t1}^{t2} e^{i2\pi ft} \varepsilon(t) \, dt \right] \end{aligned} \tag{5.3}$$

In particular, the last term is a constant since $\epsilon(t)$ is white noise, which implies that its Fourier transformation is a constant. The first two terms are highly oscillatory functions of f, if $|t_2|$ and $|t_1|$ are large. It leads to

$$A(f) = \frac{1}{i2\pi}\left[x(t_2)e^{i2\pi f t_2} - x(t_1)e^{i2\pi f t_1} - C\right]\frac{1}{f}.$$

Let

$$G(f) = (1/i2\pi)\left[x(t_2)e^{i2\pi f t_2} - x(t_1)e^{i2\pi f t_1} - C\right] \qquad (5.4)$$

where C is a constant. This item contains only the fast variations of Fourier components. Then,

$$P(f) = |A(f)|^2 \sim |G(f)|^2/f^2. \qquad (5.5)$$

We can easily verify that the power spectrum of a random walk series has a form of $1/f^2$.

5.2 *Power Laws on the Speculative Markets*

The prediction of the future constitutes a central function of economics, and numerous methods have been developed for this purpose; one of them is the chart method. The so-called chartists plot the past suitably and proclaim that they can predict the future from the geometry of the resultant chart. Bachelier (1964) has counterclaimed that charting is useless because successive price changes are statistically independent. He has asserted that as the first approximation, any competitive price follows a one-dimensional Brownian motion, B(t), defined as

$$B(t) = Z(t+s) - Z(t) \qquad (5.6)$$

based on the continuity of price changes, where Z(t) is the price at time t, and s is an arbitrary time lag; however, this Brownian motion model has failed to represent actual financial data well. In reality, the price changes on competitive markets need not be continuous; they are conspicuously discontinuous. Thus, a continuous process, e.g., Brownian motion, cannot account for a phenomenon characterized by extremely sharp discontinuities.

Another example, assuming the continuity of the price changes, is the method of trading to filters as shown in Figure 5.3. In principle, a p% filter is a device that monitors a price continuously, records all the local maxima and minima, issues a buy signal when the price first overshoots the minimum by p%, and issues a sell signal when the price first reaches a local maximum minus p%. This filter assumes that price changes can be treated as continuous through monitoring of the daily highs and lows. In reality, the price jumps take place on many days; a buy signal will be emitted as soon as the price reaches the signal point (p% + minimum); however, the resultant buy price is often significantly higher than that assumed by Alexander due to the presence of a time lag between the signal point and the resultant point.

Besides the discontinuity in the price changes, the non-applicability of a Brownian model is reflected in a large variance of the price changes. This variance of sample data is much larger than that of a Gaussian population as illustrated in Figure 5.4. In chapter 3, we also observed the amount of variance for the sample markets. These histograms demonstrate the departure from normality. In each case, the continuous bell shaped curve represents the Gaussian distributions based on the sample variance; the Gaussian distribution is much lower and flatter than the distribution of the actual data near the midpoint. The tails of the distribution of the price changes are extraordinarily long. Mandelbrot (1982) has explained the large variance as follows:

(1) Values of the variance of the price changes corresponding to different long subsamples often have different orders of magnitude.

(2) As the sample size increases, the variance fails to stabilize, in fact, it tends to increase.

(3) The variance tends to be influenced predominantly by a few contributing squares, i.e., the so-called outliers. When these outliers are eliminated, the estimate of dispersion often changes its order of magnitude.

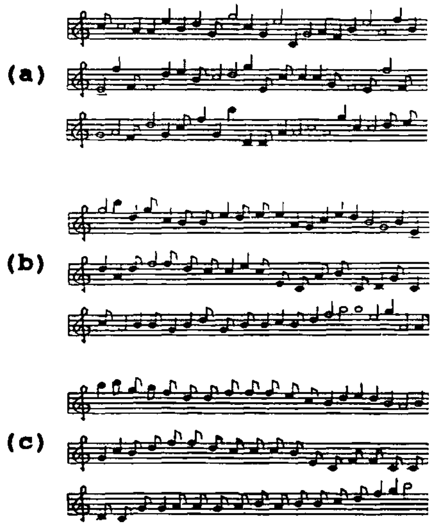



Figure 5.2: Samples of Stochastically Composed Fractal Music Based on the Different Types of Noises Shown in Figure-5.1. (a) "White" music is too random. (b) $1/f$-music is the closest to actual music. (c) "Brown" is too correlated.
(source) Peitgen, et.al.(1988)

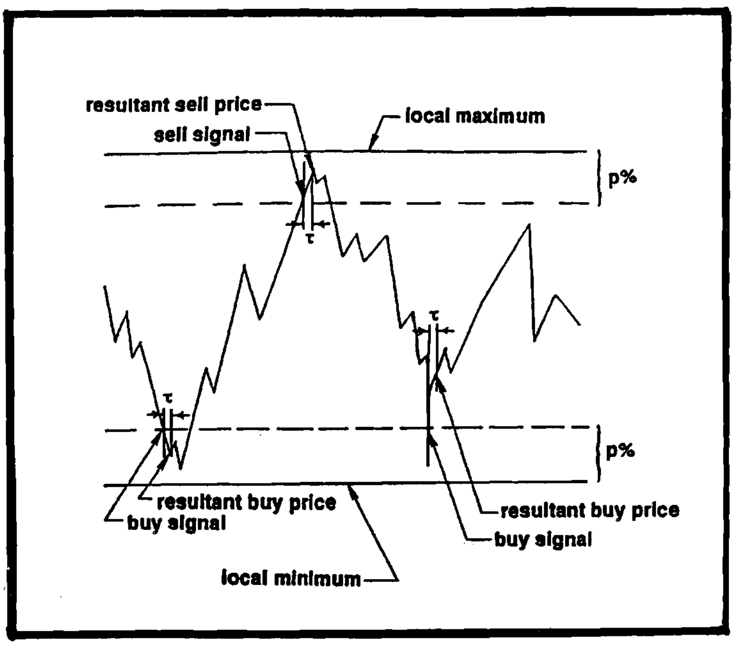


Figure 5.3: p % filter trading
(source) Alexander,S.S.(1961).

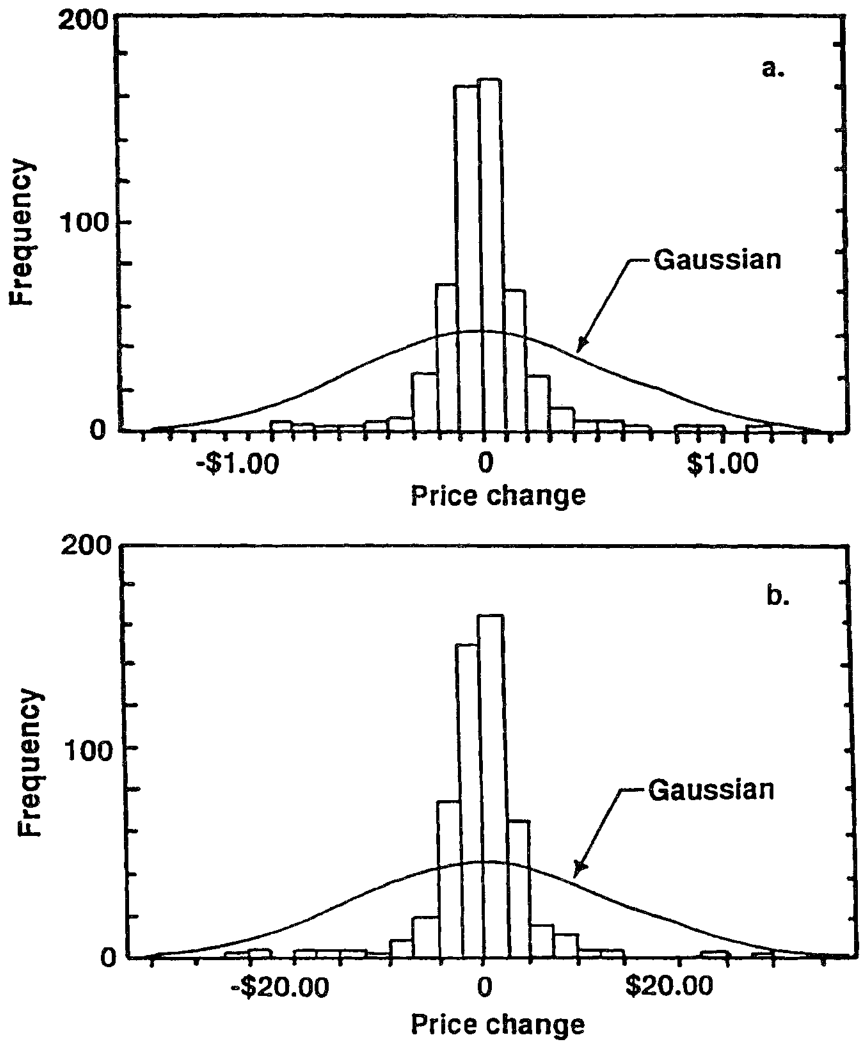


Figure 5.4: Difference between the actual data and the Gaussian distribution:
(a). fifth difference(=5 days)
(b). tenth difference(=10 days).
(source) Mandelbrot,B.B.(1963)

This has led to the rejection of the traditional Brownian hypothesis that price changes are Gaussian. Finally, Mandelbrot found two laws in the randomness of the price changes

(1) Price change in unit time is described by a stable distribution with characteristic exponent $\alpha \fallingdotseq 1.7$.[3)]

(2) The distribution is independent of time unit.

The first law shows a fractal property of the process, since the stable distribution has a long tail characterized by the power exponent α. If we denote the distribution of price change x in a unit time by p(x), then it satisfies[4)]

$$\int_{x}^{\infty} p(x')\,dx' \approx \int_{-\infty}^{-x} p(x')\,dx' \approx x^{-\alpha} \tag{5.7}$$

Therefore there is no characteristic value for the price change.

The lack of a characteristic monetary scale is recognized easily. For a poor man \$1000 is a large amount, but for a rich man it is rather little and perhaps \$1 million is a large amount. The rich man uses \$1000 as easily as the poor man uses \$1. One person may buy and sell in blocks of 1000 stocks and another might buy and sell in blocks of 100 000. Hence, there is no characteristic amount in money and stock price.

The second law indicates that the change of stock price is fractal in time, that is, the graph of stock price change in one day becomes stationary identical to that of the change in a year if we rescale the axes appropriately.

---

3) See Appendix C for the characteristic exponent of stable distribution.

4) See Appendix C for details.

5.3 Power Spectra of Sample Data

The application of the power spectral analysis to stock price sequences or index returns sequences should be straightforward except for two subtle issues. The first is that price may not be a good measure of the value; or it may not be the true quantity we are looking for. The second is that power spectrum is only well defined for stationary series whereas it is typical for economic time series to be non-stationary.

For the first issue, it is argued that the logarithm of the price is a better measure than price itself. This suggestion gains empirical support.

The problem imposed by the non-stationarity has already been discussed by Granger and Hatanaka (1964) in their study of the spectra of economic time series. Basically speaking, the non-stationarity that causes most problems are those that change the spectrum dramatically as more data points are included. On the other hand, some types of non-stationarity only change the total power but do not change the shape of the spectrum when the time span is broadened. Taking random walk, for example, one of the best known non-stationary sequences, the shape of the spectrum is always $1/f^2$, even though the total power diverges in the infinite time limit.

If there is insistence on a meaningful definition of the power spectrum for an infinite number of data points, this is traditionally established by finding out the non-stationary part of the sequence and then removing it from the series. A common practice in analyzing economic time series is to identify the trend, and detrend it. It can be imagined that the detrended sequence will have less power at lower frequencies, since trends are slow-varying functions which contribute most to the lower frequency power spectrum.

Considering these two issues, we have calculated the power spectra for not only the original detrended price sequence but also the log return sequence by the fast Fourier transformation program.[5]

---

5) More details are given to Press, et al. (1988),"Numerical Recipes in Fortran," Cambridge University Press.

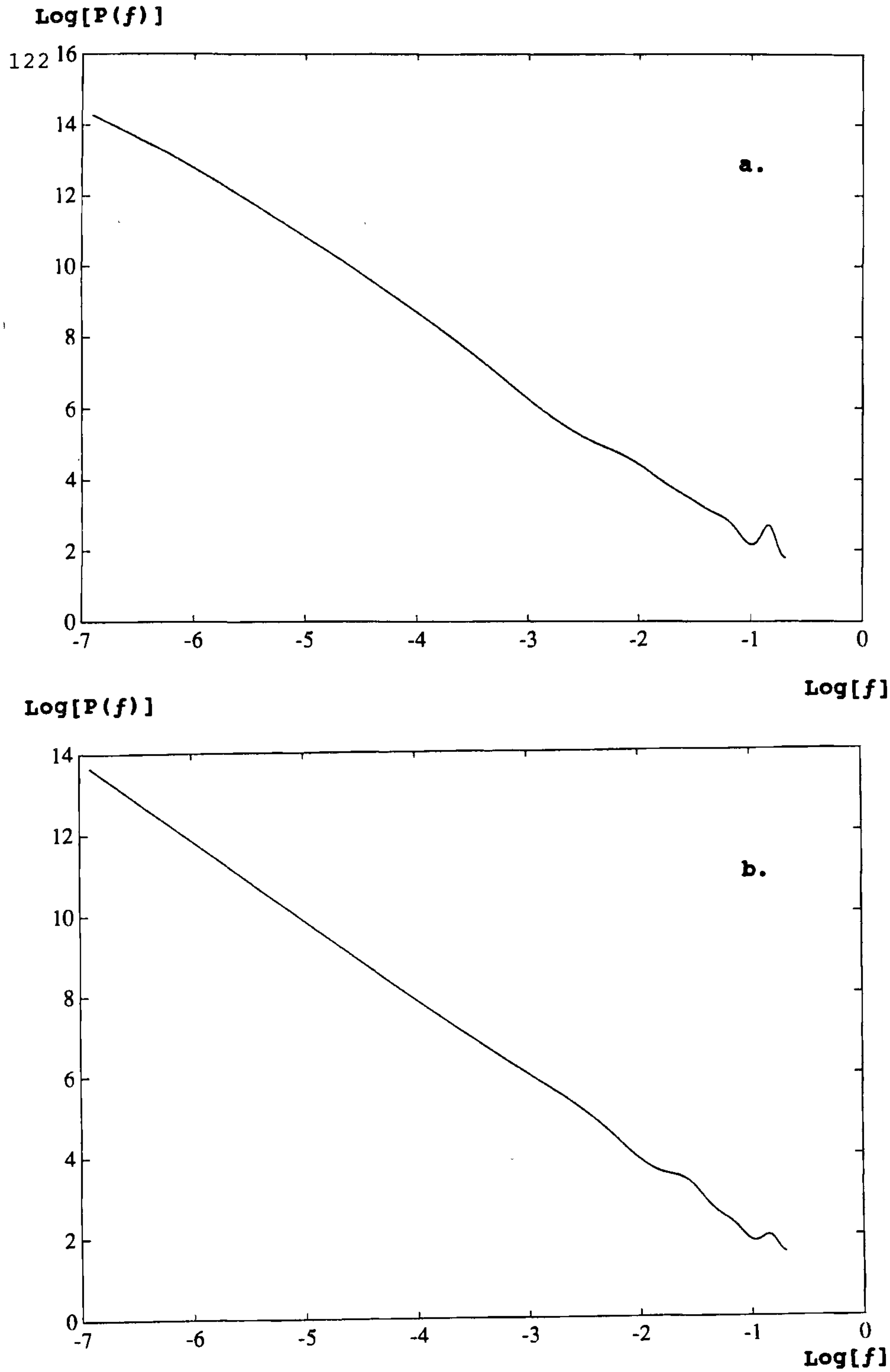


Figure 5.5: The power spectrum $P(f)$ (in log-log scale). The unit of frequency f is cycle/day:
(a) the detrended AFGX sequence
(b) the detrended KCSPI sequence.

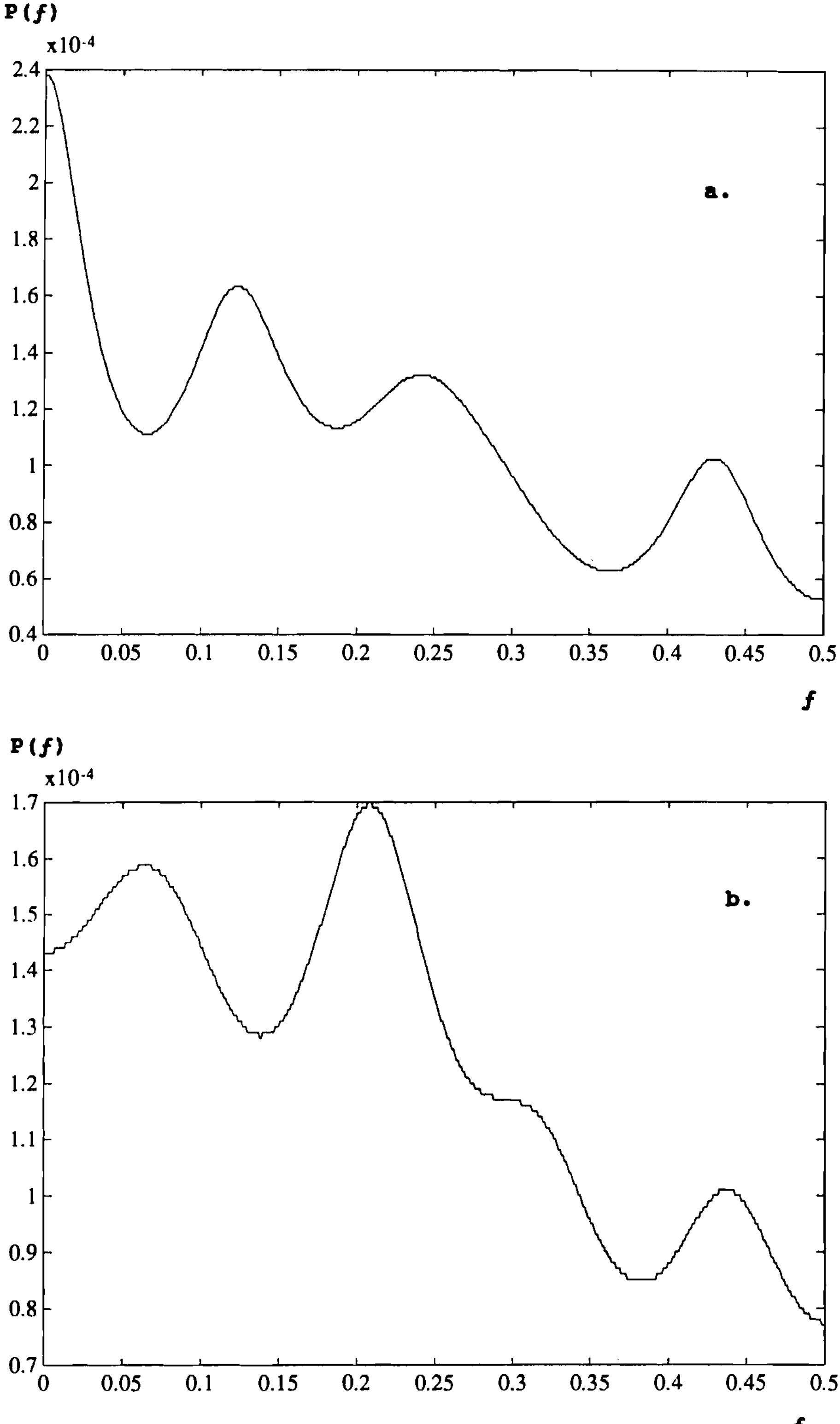


Figure 5.6: The power spectrum P(*f*).
(a) the AFGX return series
(b) the KCSPI return series.

Table 5.1: Some Samples of Power Calculated with Respect to Frequency

Frequency	Power (Sweden)	Power (Rep.Korea)
0.001	1595751.3	845526.9
0.005	90168.53	34357.57
0.01	8647.3	21989.62
0.03	1023.017	1843.414
0.04	890.8388	603.4142
0.05	510.0409	405.9796
0.06	331.2819	294.4909
0.07	238.0843	222.4088
0.08	185.5137	170.9943
0.09	153.6998	132.2881
0.1	132.6579	102.8251
0.15	66.04301	39.9343
0.2	36.03183	30.50044
0.25	23.49419	16.84997
0.3	17.06865	10.97553
0.391	9.596251	6.365663
0.392	9.698253	6.381712
0.4	10.70943	6.559676
0.44	14.02413	7.137749
0.45	11.86991	6.724204
0.46	9.628038	6.169865
0.47	7.905528	5.626709
0.49	6.16225	4.931654
0.5	6.014735	4.864417

Table 5.1 shows a part of calculated values. From the table, we can easily verify that at lower frequencies the sample markets have their peaks. Figure 5.5 shows the power spectra of the detrended original index sequences of the AFGX and the KCSPI (in log-log scale). The unit of the frequency (f) is one cycle per day.

A least square best fit line for the spectra components:

$$P(f) \approx f^{-\alpha}$$

or (5.8)

$$\log[P(f)] \approx b - \alpha \log(f)$$

gives b=0.44 and $\alpha=1.98$ for the AFGX sequence; and b=0.14 and $\alpha=1.94$ for the KCSPI sequence. The fit is good, with an R-squared of 98.8 per cent and a standard error of 0.009 for the AFGX sequence; and with an R-squared of 99.5 per cent and a standard error of 0.005 for the KCSPI sequence. We might smooth the spectrum in order to determine the value of α more accurately. Since fitting the spectrum never gives a more accurate value of the exponent (Dubuc et.al. (1989)), here no attempt is made to polish the original spectrum. We can verify that the sample markets are much closer to $1/f^2$ noise.

Based on our findings by this model, the sample markets can be compared with the USA market. In fact, so far, there have been no reports applied to international markets except the USA market. More recently, Li (1991) has verified the $\alpha=1.79$ for the USA market. He used the daily closing values of the Dow Jones industrial average from October 14, 1974 to February 14, 1991. Compared to Li's results, the sample markets are similar to the USA market, which implies that three stock markets show noise close to $1/f^2$.

In this model, the narrow difference of the exponent between the markets is not important, since the exponent value is more or less dependent on the detrending technique. What is more important is our finding that the sample markets fit power law function: endless sources of self-similarity, even though it is very difficult to believe them to hold the "self-organized criticality", in which the larger events share the same mechanism as the smaller events.

From this power spectral analysis for the detrended index sequences, it is concluded that both markets fluctuate like $1/f^2$ noise(brown noise) rather than $1/f$ noise(pink noise). It is also necessary to check whether the log difference, R(t) =log[x(t+1)] - log[x(t)], is an uncorrelated sequence with a flat power spectrum. Figure 5.6 shows the power spectra (in normal scale) for the sample sequences. The spectra obviously fluctuate around zero and are quite flat on average.

5.4 Summary of Chapter

In the previous chapter, the importance of noise was reviewed and the sample sequences were tested by a time domain methodology - the product process. However, it has been found that only the first lag autocorrelation is significant under the nonlinear generating process. We could not uncover any patterns. This chapter aims at detecting whether the sample markets really lack some patterns and long dependence by a frequency domain methodology - power law function.

We calculated the power spectra for not only the original detrended price sequence but also the log return sequence by the fast Fourier transformation technique. Our results show that the sample sequences fit the power law function well, like the USA's market. Furthermore, it is verified that the power spectra of the sample data follow $P(f) \sim 1/f^{1.9}$, very close to that of the random walk series. Even though there is a little difference in the spectral exponent coefficient, our finding says that the sample markets are similar to the USA market ($\alpha=1.79$) by this model.

Unlike some scientists' and chartists' claim that the reversal of the stock price, being due to the previously inflated price driven by the psychological effects of a big crowd, behaves somehow like an earthquake, the sample markets follow the random walk hypothesis. However, this does not mean that non-trivial long range correlations have not been found on the sample stock markets, since the existence of long-range correlation is necessary but not sufficient for $1/f$ spectra. What is more important is that the sample data fits power laws - endless sources of self-similarity - very well, which implies that there might be some patterns in the sample data. This is discussed in the following chapter.

Chapter 6

Long Term Memory & Fractal Structure in the Capital Markets

A hypothesis of many early theories of the trade and business cycles has been that the economic time series may exhibit long range dependence. Such theories were often motivated by the distinct but non-periodic cyclical patterns that typified plots of economic aggregations over time. In the frequency domain, such time series are said to have power at the low frequencies. In fact, this particular feature of the data was so common that Granger (1966) dubbed it the "typical spectral shape of an economic variable," whereas Mandelbrot and Wallis (1968) used the more colorful term "Joseph Effect."

If a series contains a transitory component that goes away after a while, it will tend to recover to something near its previous levels which we can think of as the trend of a series. Furthermore, if a series has a tendency for large values to be followed by large values of the same sign, it will exhibit trend and irregular cycle. The second type of dependence, called long term memory, is often found in the natural time series such as river flows, rainfall, earthquake frequencies, e.t.c. and is known as the "Hurst phenomenon" in the literature, that is, the heights of the yearly floods were correlated in a way in which several flood years are followed by several arid years. On the other hand, a short-memory process tends to possess frequently offsetting movements such that large positive deviations from a mean value are quickly offset by equally large negative deviations.

In the previous chapter, we have observed that the sample markets hold the power law function of Brownian noise, which implies that the sample markets consist of many more slow (low frequency) than fast (high frequency) fluctuations and their spectral densities are quite steep. It is needed to verify that the much touted randomness of the sample stock markets' returns might mask an underlying fractal structure. A fractal structure means that there exists long term dependence, or memory between observations. That is, the events of one period influence all the periods that follow. So, the presence of long term dependence in stock prices is often viewed as an evidence against the market efficiency in the finance literature.

We are now interested in what kinds of patterns have appeared for the sample markets. We are also interested in the existence of long term memory and patterns on the markets. Hence, this chapter aims at investigating the test statistics, called Rescaled Range(R/S), and the Hurst coefficient (H), which is shown to be asymptotically related with R/S; $R/S(t,s) \sim cs^H$, where c is a constant and s is some time distance.

As mentioned earlier, such as hydrology, meteorology and geophysics, nature's prediction towards long term dependence has been well documented. Furthermore, since there is a certain degree of truth in saying that the ultimate source of economic uncertainty can be natural phenomena like weather or sunspots, we also expect the impact of persistent statistical dependence or long term memory in financial time series.

As a matter of fact, the finding of long memory components in asset returns has important implications for many of the popular paradigms used in modern financial economics. For example, consider the CAPM (Capital Asset Pricing Model) and the APT (Arbitrage Pricing Theory). Since standard methods of statistical inference do not apply to financial time series displaying such persistence, traditional tests of the CAPM and the APT might no longer be valid. Problems can also arise in the portfolio allocation decision, since the decisions may become extremely sensitive to the investment horizon if stock returns are long range dependent. Moreover, we can also identify problems in the pricing of derivative securities such as options and futures via martingale methods, since long term memory is inconsistent with the martingale property. Mandelbrot (1971) has already pointed out this fact and has shown that the random walk and martingale models of speculative prices may not be realizable through arbitrage in the presence of long term memory. More recently, several empirical studies have uncovered anomalous behavior in long horizon stock returns.[1)]

This chapter is organized as follows: The fractional Brownian motion concept is defined and explained in section 6.1. The fractional Brownian motion is a newly expanded concept for the long term memory or dependence. Section 6.2 briefly describes the R/S statistics. The procedure for estimating the Hurst coefficient and the results of the empirical investigations for the sample data are reported in section 6.3. Furthermore, some implications for our results are also discussed in this section. The last section summarizes this chapter.

6.1 Fractional Brownian Motion

In this section, our concern is the extraordinary importance of fractional Brownian motion or fractional random walk process, proposed by Mandelbrot and van Ness (1968) for the long term memory process. It is an extension of the central concept of Brownian motion that has played a significant role in both economics and mathematics. The fractional Brownian motion forms the basis for understanding anomalous diffusion and random walk which can not be reconciled with the classic notion of Brownian motion. Generally, the traces of fractional Brownian motion can be expressed as a single-valued function of one variable, t (usually time), $B_H(t)$. In appearance, it is reminiscent of a mountainous horizon or the fluctuations of an economic variable. Formally, it is the increments of fractional Brownian motion, i.e.,

1) See DeBondt, Werner and Thaler (1985), Fama and French (1988), Jegadeesh (1988) and Poterba and Summers (1988).

$$[B_H(t_2) - B_H(t_1)],$$

that give rise to the noise, and sum of such noises produces the traces in Figure 6.1. The scaling behavior of the different traces in the figure is characterized by a parameter H in the range of $0 < H < 1$. When H is close to zero, the traces are roughest while those with H close to 1 are relatively smooth. H relates the incremental change in B_H,

$$\Delta B_H = B_H(t_2) - B_H(t_1), \tag{6.1}$$

to the time difference,

$$\Delta t = t_2 - t_1,$$

through the simple scaling law

$$\Delta B_H \propto (\Delta t)^H. \tag{6.2}$$

In the classical Brownian motion or random walk, the sum of independent increments or steps leads to a variation that scales as the square root of the number of steps or time increments, as is well known; thus, $H=1/2$ corresponds to the trace of a Brownian motion. On the other hand, the trace follows a fractional Brownian motion when $H \neq 1/2$.[2)]

To differentiate between the classical Brownian motion and fractional Brownian motion, the probability distributions of the incremental jump for both types of motions are presented as follows.

For the Brownian motion, B(t),

$$P[B(t+s) - B(t)] = \frac{1}{\sqrt{4\pi Ds}} \exp\left\{ - \frac{[B(t+s) - B(t)]^2}{4Ds} \right\} \tag{6.3}$$

and for the fractional Brownian motion, $B_H(t)$,

---

2) For details, see Feder, J. (1988) "Fractals", pp. 149-192, Plenum.

$$P\left[B_H(t+s) - B_H(t)\right]$$

$$= \frac{1}{\sqrt{4\pi D_H s}} \, exp\left\{ - \frac{[B_H(t+s) - B_H(t)]^2}{4D_H s} \right\} \tag{6.4}$$

In (6.3) and (6.4), the parameter D, is the diffusivity for classical diffusion and D_H is the anomalous diffusivity for fractal diffusion; D_H is related to D by the following relationship,[3)]

$$D_H = D\, s^{2H-1} \tag{6.5}$$

With this probability distribution, it follows that for the Brownian motion,

$$<B(t+s) - B(t)> = \int_{-\infty}^{\infty} \Delta B P(\Delta B, s) d\Delta B = 0,$$

$$<[B(t+s) = B(t)]^2> = \int_{-\infty}^{\infty} \Delta B^2 \, P(\Delta B, s) d\Delta B = 2ds$$

where ΔB is the increment in position:

$$\Delta B = B(t+s) - B(t).$$

Hence,

$$var\,[B(t+s) - B(t)] = 2Ds \propto s, \tag{6.6}$$

and for the fractional Brownian motion,

$$var\,[B_H(t+s) - B_H(t)] = 2D_H s = 2(Ds^{2H-1})s = 2Ds^{2H} \propto s^{2H}. \tag{6.7}$$

where $\propto$ denotes the proportion operator.

In general, we may say that the increment

---

3) See Feder (1988), p. 177.

$$[B(t+s) - B(t)]$$

of a classical Brownian motion has a Gaussian distribution with a variance proportional to lag s, whereas the increment

$$[B_H(t+s) - B_H(t)]$$

of a fractional Brownian motion also has a Gaussian distribution but its variance is proportional to s^{2H}; for $H=1/2$, this obviously reduces to the result for the classical Brownian motion.

Although $B_H(t)$ is a continuous function, it is nowhere differentiable as in the case of the classical Brownian motion. In other words, whenever we "hit" a point of the process, it is in fact a crest or a trough, with probability of one. It is inconvenient that neither Brownian motion has a derivative. Numerous methods, many of which are not always rigorous, have been evolved in giving meaning to the concept of the "derivative of Brownian motion." The resultant constructs are the so-called "Gaussian white noises." Analogous approaches can be followed with the fractional Brownian motion; they lead to what may be called "fractional Gaussian noises."

The fractional Brownian motion, $B_H(t)$, is said to have a long run correlation; this implies that for any three times, t_1,t, and t_2, such that $t_1 < t < t_2$,

$$B_H(t) - B_H(t_1)$$

is statistically correlated to

$$B_H(t_2) - B_H(t).$$

It is important to realize that fractional Brownian motion has infinitely long run correlations. In particular, past increments are correlated with future increments: Given the increment $B_H(0)$-$B_H(-t)$ from time -t to 0 the probability of having an increment $B_H(t)$-$B_H(0)$ averaged over the distribution of the past increments is

$$< \left[B_H(0) - B_H(-t)\right]\left[B_H(t) - B_H(0)\right] >,$$

where, < • > denotes the expectation operator.

For convenience set $B_H(0)=0$. The correlation function of future increments $B_H(t)$ with past increments $-B_H(-t)$ may be written

$$correl(t) = \frac{< -B_H(-t)\,B_H(t) >}{< B_H(t)^2 >} = 2^{2h-1} - 1. \qquad (6.8)^{4)}$$

First we note that for $H=1/2$ we find that the correlation of past and future increments correl(t) vanished for all t - as is required for an independent random process. However, for $H \neq 1/2$ we have correl(t)$\neq 0$, independent of t. This is a remarkable feature of fractional Brownian motion which leads to persistence or antipersistence. For $H > 1/2$, increments of $B_H(t)$ are positively correlated (persistence), i.e., an increasing trend in the past implies on the average a continued increase in the future; for $H < 1/2$, the increments are negatively correlated (anti-persistence),i.e., an increasing trend in the past implies a decreasing trend in the future. Such correlations extend to arbitrarily long time scales and have a significant effect on visual appearance of the fractional Brownian motion traces, as demonstrated in Figure 6.1.

It should also be noted that the majority of the records in time obtained in the real world are discrete time observations. $B_H(t)$ can be considered to be a tool for

---

4) Since $-B_H(-t)$ and $B_H(t)$ follow $N(0, \sigma^2(ds)^{2H})$,
$-B_H(-t) + B_H(t) \sim N(0, \sigma^2(2ds)^{2H})$.

Moreover,
$$E[(-B_H(-t) + B_H(t))^2]$$
$$= E[(-B_H(-t))^2] + E[(B_H(t))^2] + 2\,E[-B_H(-t)B_H(t)]$$
$$= \sigma^2(ds)^{2H} + \sigma^2(ds)^{2H} + 2\,E[-B_H(-t)B_H(t)].$$

Hence,
$$\text{Covariance}(-B_H(-t),B_H(t))$$
$$= E[-B_H(-t)B_H(t)]$$
$$= \sigma^2(ds)^{2H}\,(2^{2H-1} - 1).$$

This leads to $correl(t)=2^{2H-1} - 1$.

interpolating a function of discrete time into a function of continuous time. The sequence of increments of $B_H(t)$, namely, the sequence of values of

$$\Delta B_H(t) = B_H(t+1) - B_H(t)$$

with integer values of t, is called "discrete-time fractional noise." For $H=\frac{1}{2}$, $\Delta B_H(t)$ reduces to a discrete-time Gaussian white noise. Thus, for a record of time starting at time $t=0$ (see Figure 6.2), we get

$$\begin{aligned} X(0) &= 0 \\ X(1) &= B_H(1) - B_H(0) \\ X(2) &= B_H(2) - B_H(1) \\ &\;\;\vdots \\ X(s) &= B_H(s) - B_H(s-1). \end{aligned} \tag{6.9}$$

Both sides of these expressions can be summarized to obtain

$$\sum_{u=1}^{s} x(u) = B_H(s) - B_H(0) \equiv x^*(s) \tag{6.9}$$

where s is the time lag.

Thus, $X^*(s)$ is an increment of fractional Brownian motion and $X(u)$ is a sequence of discrete-time fractional noise. Equation (6.7) and (6.9) imply that

$$var\left[X^*(s)\right] = var\left[B_H(s) - B_H(0)\right] \propto s^{2H}. \tag{6.10}$$

If our recording of a single value is initiated at time t, then the definition of $X^*(s)$, equation (6.9), gives rise to

$$X^*(t+s) - X^*(t) = \sum_{u=1}^{t+s} X(u) - \sum_{u=1}^{t} X(u) = \sum_{u=1}^{s} X(t+u) \tag{6.11}$$

with the variance of this quantity still remaining proportional to s^{2H} as follows:

$$var\left[X^*(t+s) - X^*(t)\right] \propto s^{2H}. \tag{6.12}$$

To understand geometrically the fractional Brownian function $B_H(t)$ and the fractional noise or increments of $B_H(t)$, some examples are simulated as a function of time in Figure 6.3 and Figure 6.4, respectively.

6.2 Rescaled Range(R/S) Analysis

To obtain information about the Hurst coefficient (H) for a given time series, we can resort to the rescaled range analysis (R/S analysis), which was originally proposed by Hurst (1956). R/S analysis is a statistically robust technique that originated in the field of hydrology in the early 1950's as a response to the need to study riverflow and dam overflow. In a series of recent studies, Mandelbrot and Wallis (1969) were the first to apply this method to the determination of the fractal characteristics of a time series: Greene and Fielitz (1977) used the approach to uncover long term dependence in common stock returns and Booth, Kaen and Koveos (1982) also used it for foreign exchange rates. More recently, Haubrich and Lo (1989) made use of this analysis for the stochastic properties of aggregate macroeconomic time series, focusing on the persistence of economic shocks.

Let X(u) be a record containing s readings uniformly spaced from time u=t+1 to time u=t+s; X(u) is a sequence of discrete time fractional noise. Thus, we

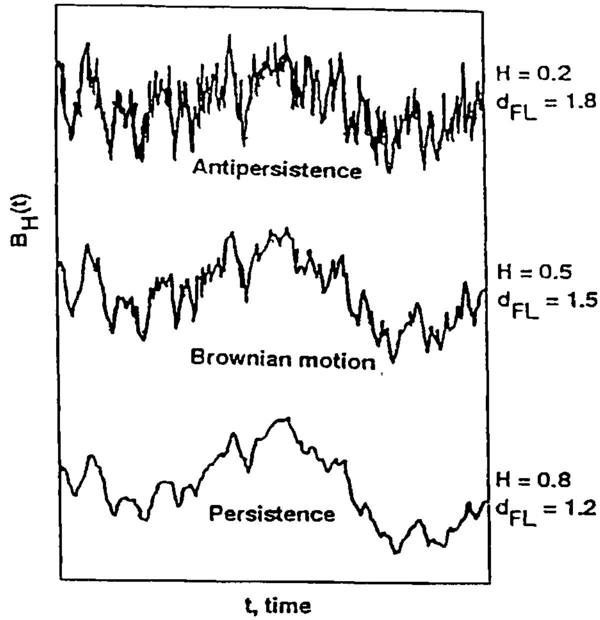


Figure 6.1: Sample plots of the fractional Brownian motion traces, $B_H(t)$, against t for different values of H and D: $\Delta B_H \propto t^H$.
(source) Pietgen,H.O. and Saupe,D.,Eds.(1988)

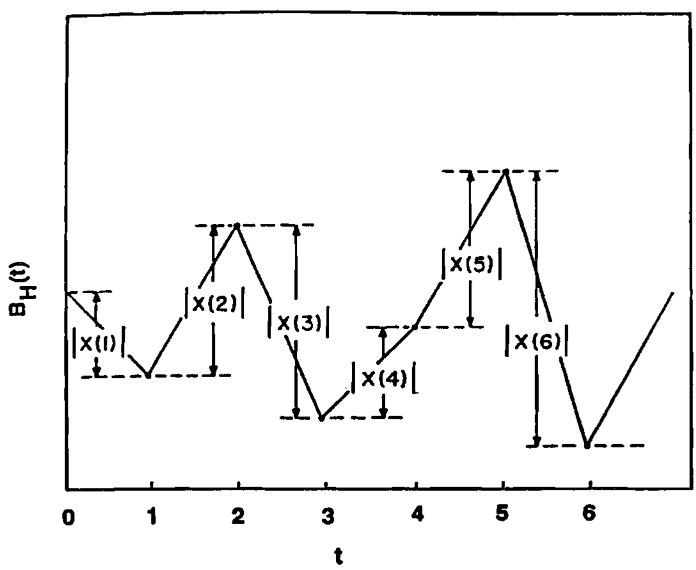


Figure 6.2: Generation of a sequence of "discrete-time fractional noise" from the fractional Brownian motion, $B_H(t)$.

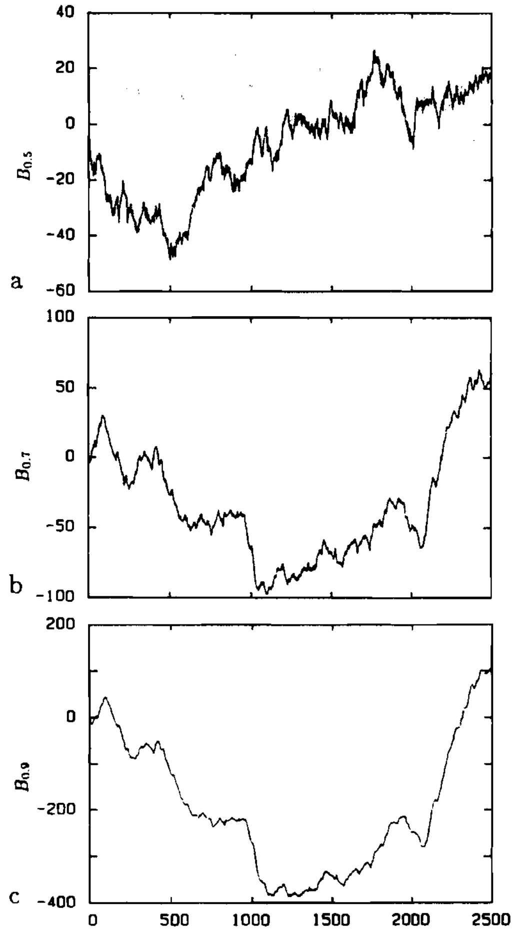


Figure 6.3: Fractional Brownian function B_H simulated with M=700,n=8 and $B_H(0)=0$.
(a) For H=1/2 (b) For H=7/10 (c) For H=9/10.

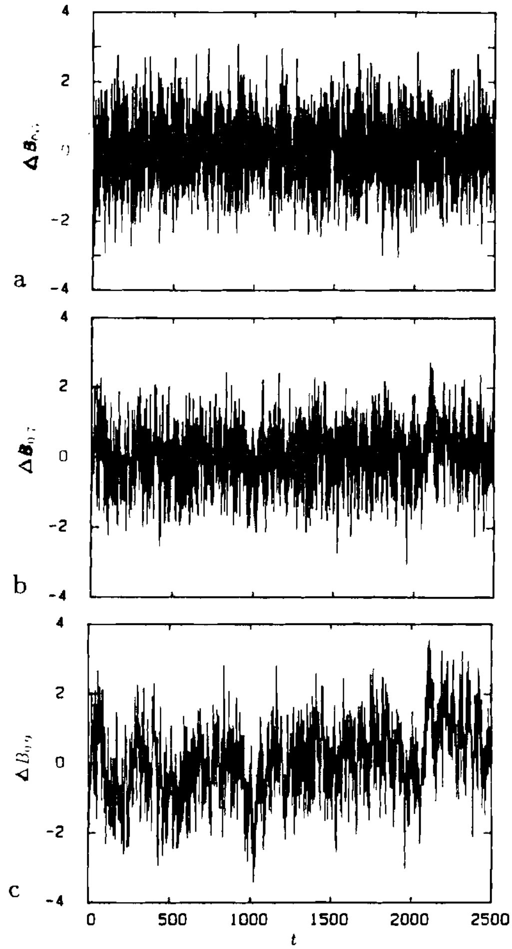


Figure 6.4: Fractional noise or increments of the fractional Brownian function B_H simulated with M=700,n=8. (a) For H=1/2 (b) For H=7/10 (c) For H=9/10.

obtain from equation (6.11),

$$\frac{1}{s}\left[X^*(t+s) - X^*(t)\right] = \frac{1}{s}\sum_{u=1}^{s} X(t+u) \equiv \langle X(t)\rangle_s \tag{6.13}$$

which is the average of readings within the subrecord from time t+1 to time t+s.

Moreover, let c(t,u) denote the cumulative departure of X(t+y) from the mean $<X(t)>_s$, for the subrecord between time t+1 corresponding to y=1 and time t+u corresponding to y=u; note that by definition,

$$c(t,u) = \sum_{y=1}^{u}\left[X(t+y) - \langle X(t)\rangle_s\right] = \sum_{y=1}^{u} X(t+y) - u\,\langle X(t)\rangle_s \tag{6.14}$$

Utilizing (6.11) and (6.13), we obtain

$$c(t,u) = \left[X^*(t+u) - X^*(t)\right] - (u/s)\left[X^*(t+s) - X^*(t)\right] \tag{6.15}$$

The sample sequential range of X(t) for lag s, R(t,s) is defined as (see Figure 6.5)

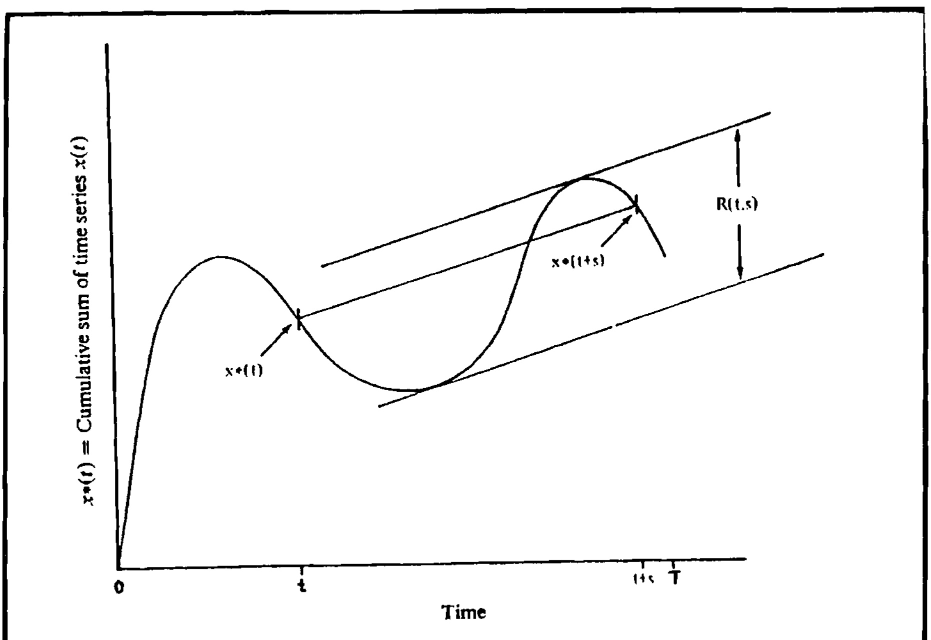


Figure 6.5: Definition of the sample range, R(t,s), of the time series x(t) for starting point t, time interval s, and total sample size T.

$$R(t,s) = \underset{0<u\le s}{Max}\ c(t,u) - \underset{0<u\le s}{Min}\ c(t,u) \tag{6.16}$$

The sample sequential variance of X(t), S^2, is defined by

$$\begin{aligned} S^2(t,s) &= \frac{1}{s}\sum_{u=1}^{s}\left[X(t+u) - \langle X(t)\rangle_s\right]^2 \\ &= \frac{1}{s}\sum_{u=1}^{s}\left\{X(t+u) - \frac{1}{s}\left[X^*(t+s) - X^*(t)\right]\right\}^2 \end{aligned} \tag{6.17}$$

Then, the ratio

$$Q_s = \frac{R(t,s)}{S(t,s)},$$

is called the rescaled range.

Mandelbrot and Wallis (1969) have proposed that R(t,s)/S(t,s) is a random function with a scaling property [5)]

$$Q_s = \frac{R(t,s)}{S(t,s)} \propto s^H \tag{6.18}$$

Thus, the Hurst exponent H can be estimated by performing an ordinary least squares regression between log(R/S) and log(s) for various s.[6)] Furthermore, Feder (1988) has shown that the local fractal dimension-dFl, of the trace of a fractional Brownian motion which is a self-affined curve,[7)] is related to H by

$$dFl = 2 - H, \quad 0 < H < 1.$$

---

5) See Feder (1988), p. 152.

6) $(R/S) = a\, s^H$ (a is constant). This leads to $\log(R/S) = \log(a) + \log(s^H)$.
Hence, $H = (\log(R/S) - \log(a)) / \log(s)$.

7) It is generally believed that neither the classical nor the fractional Brownian motion is self-similar, instead, both are self-affined. Whereas the self-similar shapes repeat statistically or exactly under a magnification, the fractional Brownian traces of Figure 6.1 repeat statistically only when the t and B coordinates are magnified by different degrees. If t is magnified by a factor of r (t becomes rt), then B should be magnified by a factor of r^H (B becomes $r^H B$); for a regular random walk (H=½), we should take four times as many steps to go twice as far. This non-uniform scaling, where the shapes are (statistically) invariant under transformations that scale different coordinates by different degrees, is known as self-affinity.

Thus, H obtained from the R/S analysis yields the fractal dimension of the time series under investigation.

In several seminal papers, Mandelbrot, Taqqu, and Wallis demonstrated the superiority of R/S analysis to more conventional methods of determining long range dependence such as autocorrelation analysis, spectral analysis, and variance ratios. However, Lo (1989) has showed the rescaled range is also sensitive to short range dependence and moreover he has suggested the modified R/S statistic Q'_s.[8)]

Given a sample of observations $X_1, X_2, \ldots, X_s$, the modified rescaled range Q'_s is defined as:

$$\acute{Q}_s(q) \equiv \left(\frac{1}{\hat{Q}_s(q)}\right)\left[Max\sum_{j=1}^{k}(X_j - <X(t)>_s) - Min\sum_{j=1}^{k}(X_j - <X(t)>_s\right. \tag{6.19}$$

where

$$\begin{aligned}\hat{\sigma}_s^2(q) &= \frac{1}{s}\sum_{j=1}^{s}(x_j - <x(t)>_s)^2 \\ &+ \frac{2}{s}\sum_{j=1}^{q}\omega_j(q)\left[\sum_{i=j+1}^{s}(x_i - <x(t)>_s)(x_{i-j} - <x(t)>_s)\right] \\ &= \hat{\sigma}_X^2 + 2\sum_{j=1}^{q}\omega_j(q)\,\hat{\gamma}_j\end{aligned} \tag{6.20}$$

8) Although aware of the effects of short range dependence on the rescaled range, Mandelbrot (1972 and 1975) did not correct for this bias since his focus was the relation of the R/S statistic's logarithm to the logarithm of the sample size as the sample increases without bound. For short range dependent time series such as strong mixing processes the ratio $\log(Q_s)/\log(s)$ approaches 1/2 in the limit, but converges to quantities greater or less than 1/2 according to whether where is positive or negative long range dependence. However, Lo (1989) has claimed that although $H = 1/2$ across general classes of short-range dependent processes, the finite-sample properties of the estimated Hurst coefficient are not invariant to the form of short run dependence.

$$\omega_j(q) \equiv 1 - (j/(q+1)),\ q<s \qquad (6.21)$$
$$1 \le k \le s,$$

and $\hat{\sigma}^2_X$ and $\hat{\gamma}_j$ are the usual sample variance and autocovariance estimators of X.

Q'_s differs from Q_s only in its denominator, which is the square root of a consistent estimator of the partial sum's variance. If $\{X_t\}$ is subject to short range dependence, the variance of the partial sum is not simply the sum of the variance of the individual terms, but also includes the autocovariances. Therefore, the estimator $\hat{\sigma}_s(q)$ involves not only sums of squared deviations of X_j, but also its weighted autocovariances up to lag q. The weights $\omega_j(q)$ are those suggested by Newey and West (1987) and always yield a positive $\hat{\sigma}^2_s(q)$, an estimator of the spectral density function of X_t at frequency zero using a Bartlett window.[9)]

6.3 R/S Results

6.3.1 Results

Two approaches to the selection of lags and starting points are available. These are designed as F Hurst and G Hurst (Wallis and Matalas (1970)). For F Hurst, all possible lags and starting points are used. This procedure requires a large number of calculations. In G Hurst, only certain lags and starting points are used yielding a maximum of 15 values of R/S for each lag.

Theoretically, both F Hurst and G Hurst yield estimates of H which are biased upwards if the true value of H is less than 0.7. The bias and variability of estimates of H obtained with F Hurst are less than those obtained with G Hurst, and they decrease with the sample size under both methods. So, here we use the F Hurst procedure.

9) See Wei (1990), p. 274.

Moreover, R(t,1)/S(t,1) gives the indeterminant form 0/0; s=2 gives the trivial result R(t,2)/S(t,2) = 1. Hence, we consider only non-trivial values of R(t,s)/S(t,s) for $3 \leq s \leq T$. Briefly, 50 subseries ranging in size from s=4 to s=2534 for the AFGX series and from s=3 to s=3374 for the KCSPI series. For each s, all possible samples are obtained, and for each sample, values of R/S are calculated using relations specified in eqs.(6.16) and (6.17). It is the sample mean of the R/S values that is used as the dependent variable in the empirical model.

Table 6.1 shows the data used to estimate the Hurst exponents for the AFGX returns and the KCSPI returns. H was estimated to be 0.639 (standard error of coefficient 0.02) for the AFGX returns; 0.628 (standard error of coefficient 0.02) for the KCSPI returns.
For both sample data, the high R-squared (92.57% for the Swedish market and 93.28% for the Korean market) illustrates the goodness of the fit. Furthermore, the correlation in equation (6.8) was also calculated from the exponent H, which implied that how much stock returns were influenced by the past: the correlation is 21.25% for the AFGX returns and 19.42% for the KCSPI returns.

According to Lo's suggestion, we have checked what is happening in the case of the revised R/S statistic, Q'_s. The results are shown in Table 6.2. We can hardly find the difference between two methodologies in uncovering the type of patterns.

From the tables, it can be noted that the sample markets hold the persistence patterns. This result is similar to the result for the USA market, for which, Peter (1989) has showed H=0.611 under the period 1/50-6/88; Greene and Fielitz (19 77) has showed $0.5 < H < 0.7$ with the probability of 0.82. But the results by Lo's methodology show some interesting remarks. We computed the modified rescaled range with q-values of 30, 90, and 360 lag days , taking short term autocorre lations into account, since we have already captured the importance of short term autocorrelations for the sample stock markets in chapter 3.[10] Using the distri-

10) There still is not known about how best to pick q in finite samples. Here we choose 30, 90 and 360 lag days as done in Lo (1989).

Table 6.2: The Results for Modified R/S.

(Sweden)

lag(q)	*H Coeff.*	*S.E.*	R^2
30	*0.639*	*0.02*	*92.57%*
90	*0.639*	*0.02*	*92.57%*
360	*0.639*	*0.02*	*92.57%*

(Rep.Korea)

(lag)	*H Coeff.*	*S.E.*	R^2
30	*0.629*	*0.02*	*93.22%*
90	*0.628*	*0.02*	*93.28%*
360	*0.628*	*0.02*	*93.28%*

bution calculated by Lo (1989), a test of the null hypothesis of short range dependence was performed at the 95% level of confidence by accepting or rejecting according to whether $V_s(q)$, which is defined by,

$$V_s(q) = (1/\sqrt{s})\,Q_s(q)$$

is or is not contained in the interval [0.809, 1.862].[11]

Table 6.3 shows that while for the Korean stock market the long dependence concept is effective within about three months, for the Swedish stock market there is no evidence of long term dependence.

11) For fractiles of the distribution F(V), see Lo (1989), p. 34.

Table 6.1(a): R/S Analysis for the AFGX Daily Returns.

N	R/S	LOG(R/S)	LOG(N)
4	0.56990278	-0.24419922	0.602059991
6	1.32062554	0.120779692	0.778151250
13	4.81195831	0.682321856	1.113943352
21	9.88184452	0.994838016	1.322219294
34	11.61794472	1.065129305	1.531478917
51	11.61794472	1.065129305	1.707570176
59	11.61794472	1.065129305	1.770852011
62	11.61794472	1.065129305	1.792391689
80	11.61794472	1.065129305	1.903089987
94	11.61794472	1.065129305	1.973127853
101	11.61794472	1.065129305	2.004321373
113	11.61794472	1.065129305	2.053078443
120	12.05052567	1.081005992	2.079181246
142	12.05052567	1.081005992	2.152288344
154	12.41029835	1.093782222	2.187520720
166	16.35616493	1.213681481	2.220108088
173	17.08878708	1.232711238	2.238046103
184	19.274189	1.284976113	2.264817823
197	20.64724922	1.314862199	2.294466226
203	20.64724922	1.314862199	2.307496037
209	20.64724922	1.314862199	2.320146286
229	20.64724922	1.314862199	2.359835482
276	20.64724922	1.314862199	2.440909082
301	21.40671921	1.330550112	2.478566495
312	21.40671921	1.330550112	2.494154594
356	25.55652618	1.407501821	2.551449998
385	32.46869272	1.511464803	2.585460729
412	39.60758972	1.597778414	2.614897216
453	39.60758972	1.597778414	2.656098202
491	39.60758972	1.597778414	2.691081492
526	39.60758972	1.597778414	2.720985744
700	39.60758972	1.597778414	2.84509804
777	54.65084457	1.737596877	2.890421018
829	70.60850525	1.848857017	2.918554530
888	71.10289001	1.851887253	2.948412965
950	72.7906189	1.862075412	2.977723605
994	72.7906189	1.862075412	2.997386384
1063	72.7906189	1.862075412	3.026533264
1212	72.7906189	1.862075412	3.083502619
1307	72.7906189	1.862075412	3.116275587
1500	72.7906189	1.862075412	3.176091259
1623	72.7906189	1.862075412	3.210318519
1954	72.7906189	1.862075412	3.290924559
1992	72.7906189	1.862075412	3.299289334
2113	72.7906189	1.862075412	3.324899497
2199	72.7906189	1.862075412	3.342225229
2233	72.7906189	1.862075412	3.348888723
2314	72.7906189	1.862075412	3.364363354
2457	72.7906189	1.862075412	3.390405156
2534	72.7906189	1.862075412	3.403806610

Regression Output:

Constant	-0.17668864
Std Err of Y Est	0.125990409
R Squared	0.925730620
No. of Observations	50
Degrees of Freedom	48
X Coefficient(s)	0.639545499
Std Err of Coef	0.026146491

Table 6.1(b): R/S Analysis for the KCSPI Daily Returns.

N	R/S	LOG(R/S)	LOG(N)
3	4.48155308	0.651428544	0.477121254
7	5.72888136	0.758069828	0.84509804
24	5.92982626	0.773041969	1.380211241
32	5.92982626	0.773041969	1.505149978
39	5.92982626	0.773041969	1.591064607
47	5.92982626	0.773041969	1.672097857
66	6.00356102	0.778408929	1.819543935
76	6.00356102	0.778408929	1.880813592
82	6.48774719	0.812093918	1.913813852
95	10.67046642	1.028183403	1.977723605
104	11.15799904	1.047586319	2.017033339
117	11.15799904	1.047586319	2.068185861
126	11.15799904	1.047586319	2.100370545
138	11.15799904	1.047586319	2.139879086
169	11.15799904	1.047586319	2.227886704
237	15.92235088	1.202007190	2.374748346
299	31.68833351	1.500899400	2.475671188
345	31.68833351	1.500899400	2.537819095
388	31.68833351	1.500899400	2.588831725
459	44.58275223	1.649166875	2.661812685
477	44.58275223	1.649166875	2.678518379
521	44.58275223	1.649166875	2.716837723
603	44.58275223	1.649166875	2.780317312
653	44.58275223	1.649166875	2.814913181
701	52.4723587	1.719930586	2.845718018
853	52.4723587	1.719930586	2.930949031
923	53.8559761	1.731233901	2.965201701
1089	62.45712662	1.795582000	3.037027879
1132	65.27861786	1.814770950	3.053846426
1511	72.08262634	1.857830601	3.179264464
1623	78.81317139	1.896598803	3.210318519
1700	79.15831757	1.898496555	3.230448921
1761	79.15831757	1.898496555	3.245759356
1827	79.15831757	1.898496555	3.261738547
2000	79.15831757	1.898496555	3.301029995
2030	79.15831757	1.898496555	3.307496037
2122	79.15831757	1.898496555	3.326745379
2301	90.74847412	1.957839331	3.361916618
2399	108.6300354	2.035949921	3.380030248
2456	108.9394226	2.037185069	3.390228362
2577	113.1964569	2.053832833	3.411114418
2832	129.4745483	2.112184404	3.452093249
2888	129.4745483	2.112184404	3.460597188
3011	129.4745483	2.112184404	3.478710755
3065	129.4745483	2.112184404	3.486430478
3121	129.4745483	2.112184404	3.494293768
3199	129.4745483	2.112184404	3.505014240
3251	129.4745483	2.112184404	3.512016969
3300	129.4745483	2.112184404	3.518513939
3374	129.4745483	2.112184404	3.528145078

Regression Output:

Constant		-0.13487726
Std Err of Y Est		0.130599190
R Squared		0.932848025
No. of Observations		50
Degrees of Freedom		48
X Coefficient(s)	0.627977713	
Std Err of Coef.	0.024319103	

6.3.2 Implications

An examination of Table 6.1 and Table 6.2 reveals that the estimated models for both stock indices returns are characterized by very high goodness of fit measures and generally extraordinary low standard errors for the Hurst parameters.

Generally, the higher the Hurst exponent, the stronger the persistence and the less white noise there is in a time series. While the sample stock markets show persistent trends, the relatively low level of H also implies a good deal of noise. Hence, it is expected that attempts to forecast the stock markets over the short term will be difficult, given the level of short term noise. Furthermore, attempts to find "deterministic chaos" in the sample stock markets would be more difficult because of the level of noise in the data.

Table 6.3: Modified R/S Analysis of the AFGX Returns and the KCSPI Returns Using the $V_s(q)$.

(Sweden)

lag (q)	$V_s(q)$
30	***1.589***
90	***1.423***
360	***1.051***

(Rep.Korea)

lag (q)	$V_s(q)$
30	***1.954****
90	***1.869****
360	***1.429***

(Note) Asterisks indicate significance at the 5 per cent level.

A more important finding is that the results show that pure random walk theory does not apply to the sample stock markets; instead the markets would follow a persistently biased (fractional) random walk. What are the underlying reasons for the persistently biased random walk ? We have already reviewed that the Hurst phenomenon indicates that stock market returns are influenced by the past. This influence goes across time scales. That is, for example, one week period influences all subsequent one week periods and two-week period influences all subsequent two-week periods.

We can postulate that this effect is caused by investor bias, or market sentiment. Thus, the correlation derived from the Hurst exponent becomes a measure of the impact of market sentiment, which is generated by past events, upon future returns in the stock markets. Investor sentiment represents investors' interpretation of the events that influence stock markets. This interpretation is not immediately reflected in prices as the efficient market hypothesis asserts. It is instead manifested as a bias in returns. This bias accounts for a significant portion of stock returns. Hence it might be necessary to develop an asset pricing model that takes this pattern into consideration for the sample markets.

The statistical insignificance of the modified R/S statistic in Table 6.3 indicates that the data in the Swedish stock market are consistent with the short memory null hypothesis. This results is similar to Lo's findings for the USA's CRSP index data. However, in the Korean stock market there exists long run dependence within three months.

Considering two methodologies together, we can conclude that persistent patterns surely exist for the sample markets. But there is very weak long term dependence in daily stock returns, beyond three months for the Korean market and one month for the Swedish market.

6.4 Summary of Chapter

The preceding findings concerning the dependence support the contention that patterns in stock index returns do exist. Estimation procedures for long term dependence were discussed in this chapter. First, the extraordinary importance of fractional Brownian random walk process was discussed. This is an extension of the central concept of Brownian motion that has played a significant role in both economics and mathematics. Second, the procedure of estimating parameter H for a given time series was discussed.

The result of this chapter suggests that the sample stock markets have a persistent trend. This result is similar to the USA market. While the result shows persistent trends, the relatively low level of H also implies a good deal of noise. Hence, it can be expected that attempts to forecast the stock markets over the short term will be difficult, given the level of short term noise. Furthermore, attempts to find deterministic chaos in the markets would be more difficult because of the level of noise in the data.

By considering the finite sample size and short run dependence or autocorrelations, we have also checked the robustness of the classical R/S. While the Korean stock market exhibits long range dependence within 90 lag days, the Swedish stock market does not exhibit long term dependence in a really meaningful sense. However, to be sure, a persistent pattern exists for the sample markets.

Chapter 7

Market Efficiency Arising from Nonlinear Dynamical Systems Theory

The stock market crash of 1987, and more recent gyrations, have caused financial economists to doubt more deeply the applicability of the random walk/efficient markets paradigm and have thereby given impetus to the quest for a fresh look at models of how the stock market in particular, and speculative markets in general, work. This calls into question the capital asset pricing model and most option-pricing theories, which are based on normal distributions and finite variances. From the previous chapters, we have also noticed that the sample markets follow a nonlinear generating process between returns; there exist sources of self-similarity for price changes; even though it is not so long, there might be long term dependence (memory) for the Korean stock market. As a matter of fact, these phenomena exist mainly in a nonlinear mechanism. So, more recently, some financial econometricians, before the development of theoretical models and backgrounds on the market crash, have been trying to detect the evidence of nonlinearity, since a market crash (or chaotic dynamics) necessarily requires the nonlinear mechanism.

To detect the evidence of nonlinearity, various algorithms and related statistical tests have been successfully used in models of asset returns. The nonlinear dynamical systems modelling and nonlinear time series analysis have a relatively young history. The statistics community has constructed stochastic nonlinear dependence models since about 1980.[1] They use the concept of autocorrelations.

---

1) For a review see Tong (1990).

In the previous chapters, we also used this concept. However, the autocorrelation function of many economic time series decays rather slowly to zero,i.e., the spectrum has a peak at the origin and displays a downward slope as frequency increases. For example, returns volatility series appear to have autocorrelation function which dies off with roughly power law tails in contrast to the one which dies off exponentially and exists typically in ergodic stationary processes. Furthermore, the autocorrelation function is of little use in nonlinear structure. As shown by Sakai and Tokumaru (1980) most trajectories generated by

$$x_{t+1} = \frac{1 - |2x_t - 1|}{1 - |2a - 1|} \quad 0 \le x_t \le 1 \; and \; 0 < a < 1$$

yield the same autocorrelation function as the AR(1) process

$$x_{t+1} = (2a - 1)x_t + \epsilon_t$$

where ϵ_t is a white noise error term. Hence, if a is equal to 1/2, $\{x_t\}$ generated by the above equation appears to be a pure random process to a researcher calculating its autocorrelation function. So, some natural scientists have constructed deterministic nonlinear models independently, motivated by the phenomenon of chaos since about 1983.[2)] They use the concept of correlation integral, instead of the autocorrelations. Since economic time series appear "random" and are difficult to predict, it is natural for financial economists to take a look at nonlinear science (chaos theory) to see if this concept can be applied to macroeconomics and finance. Therefore, chaos has now captured the fancy of some macroeconomists and financial economists.

The purpose of this chapter is to discuss some of the methodological issues concerned with the detection of chaotic and nonlinear behavior in financial economics and to apply them to the sample stock markets. Section 7.1 reviews what chaos theory means to financial econometricians. Section 7.2 discusses some of issues developed to detect and to describe nonlinear structure in macroeconomic and financial time series. In section 7.3, the nature and notations

---

2) See Crutchfield and McNamara (1987), and Farmer and Sidorowich (1988) and references therein.

of dynamical systems are presented. The purpose of this section is solely to provide background material on the nature of a dynamical system in such a way that the measures on which the statistical tests and estimates are based on, can be understood. Section 7.4 details descriptions for testing independence under nonlinear dynamics. This section begins with a discussion of correlation integral and its connection to U-statistics. The discussion then turns to the BDS test and its power. In section 7.5, we explain how the geometric and dynamic properties of a stock market can be identified, by using the correlation integral. In particular, we will focus on determining the correlation dimension by the correlation integral. In section 7.6, we calculate the correlation dimension of phase space and discuss our findings for the sample markets. This chapter is summarized in section 7.7.

7.1 What Does Chaos Theory Mean to Financial Econometricians?

The idea that a simple equation, algorithm or process can generate complex results has been the cornerstone of chaos theory. The name 'chaos' derives from this. Now, chaos theory has induced applied financial econometricians to develop tests which reveal whether financial time series are generated by deterministic chaos or stochastic processes. This is definitely not an easy task since it is a characteristic of chaotic time series that the variables look random and pass usual tests of randomness. "A host of examples shows that traditional linear statistical methods are often useless in this respect" (Grandmont & Malgrange, 1986, p. 6).

In the natural sciences, the method used to locate chaos has been to represent the data in various forms of phase space and by using sufficient amounts of data to observe regularities or bounds in the data.[3] As an example, if data is generated by the logistic equation $x_t = A\ x_{t-1}(1-x_{t-1})$, where A is between 0 and 4, when mapped onto the dimensions of itself in the current and next time period, the

3) Phase space describes a graph which portrays dynamic behavior along the dimensions of free variables in the system without time as a dimension.

variable would yield a hill-shaped curve. The curve itself rises with the parameter A. This phenomenon is outlined in an excellent review by Baumol and Benhabib (1989). The problem, however, with such an approach in financial econometrics, is the same that has plagued the subject since its beginnings. Economic phenomena are rarely reducible to laboratory conditions where the number of degrees of freedom are limited. Thus, even if a time series is actually generated by a simple equation such as the logistic curve, how do we know that the parameter will remain fixed over time? It is probably affected by other forces in the economy and these may also be nonlinear. Because of the number of free variables in the economy, it is hard to believe that the nonlinear equation defining the economy will be anything but complex. That factor makes it virtually impossible to discover regularities hidden in the data even if they actually exist.

This is not to say that it is impossible that an economic time series might be generated by a simple nonlinear system of a low dimension. In fact, if it is possible to come up with a measure for the complexity or dimension of the forces underlying the data, we can reasonably assert that if the dimension is low, the time series will more likely be generated by deterministic chaos than stochastic processes. It has not been easy, unfortunately, to come up with a notion of dimension that is easy to calculate and gives reliable results (Brock and Malliaris (1989)).[4] This is a possible direction for applied work but in reality, econometricians still doubt whether any models can be generated in a purely deterministic sense. There is always some noise present (Brock and Malliaris (1989)). Also, intuitively, it is difficult to believe that low dimensional chaos really exists. Take the example of asset prices. "There are many participants in a financial market with many complex sets of human relationships, motivations, and reactions" (Savit (1988)). It could hardly be expected that these would

---

4) Paul Davies (1987) has surveyed attempts to come up with measures for complexity. These include measuring systems by the number of equations needed to specify them, a concept called 'logical depth' based on heuristics, forms of interaction, and information content. However, "organization and complexity," in spite of their powerful intuitive meanings, lack generally agreed definitions in the rigorous mathematical sense.

converge to a simple model of any sort. Thus, it is not surprising that the theoretical results computed with data have been inconclusive.[5)]

If we can investigate how much of the irregularity in an aperiodic time series is due to low dimensional chaotic dynamics, as opposed to stochastic or high dimensional dynamics, this problem can be easily solved. However, to detect high dimensional (stochastic) chaotic dynamics, we need a vast amount of data, which is impossible in reality. Furthermore, there has not been developed a way to distinguish nonlinear deterministic systems from time-varying linear stochastic systems. So, the goal of financial econometricians in this area is to reject the null hypothesis that the original time series is just linearly correlated noise. Rejecting a null hypothesis is less ambitious than estimating a dimension, which was originally proposed as a way to distinguish systems, but it is a statistically more well-posed problem, and in general can be done more reliably.

7.2 *Recent Developments and Issues*

7.2.1 *The Recent Developments*

Arising out of the nonlinear dynamical system literature, a new class of statistical tests and estimates has recently been developed to detect and describe nonlinearity in macroeconomic and financial data. Here, many of the recent developments are applied to address a variety of issues in finance. The hot issues fall under the general category regarding the detection and description of serial dependence in

5) It is interesting to speculate on what deterministic chaos means for the rational expectations hypothesis. According to that hypothesis agents should be able to learn the underlying laws of movements in economic variables from observation of the data and hence not make systematic prediction errors. But what if the underlying model is generated by a simple chaotic system? The question is, of course, empirical. A test for experimental economists could be as follows: program a computer with the simple logistic equation described above. Inform that participants of this equation save for the value of the fixed parameter. Then allow the subjects to put data into the computer and observe the response. If the rational expectations hypothesis holds under conditions of deterministic chaos then, eventually, the subject should be able to discover the value of the parameter. It is left for the interests to image and verify the plausibility of the rational expectations hypothesis given this simple test of its validity at the micro-micro level.

time series, since conventional tests are plagued by the inability to detect nonlinear dependence. The focus of this section centers on that problem.

Regarding the new tests and estimates, there have been two research tracks of the literature. In both tracks, the research relies on the discovery by Grassberger and Procaccia (1983) of the connection between the correlation integral and the principal geometric and dynamic properties of dynamical systems.

One track has focused on the connection to design a new class of statistical tests for independence. In a variety of Monte Carlo studies (Brock, Dechert and Scheinkman (1987), Baek and Brock (1988), and Hsieh and LeBaron (1988)), the new tests are shown to have power against a wide variety of nonlinearly dependent alternatives. The power property, coupled with Brock's (1987) demonstration of the "nuisance-parameter-free" character of the new tests, has led financial econometricians to view the new tests as promising diagnostics for model specification in cases where a specific alternative to the null is left unspecified (Brock and Dechert (1988)). As such, one of the tests — called the BDS test after its authors: W.A.Brock, W.D.Dechert, and J.A.Scheinkman — is now frequently employed. For example, Hsieh (1989), Scheinkman and LeBaron (1989), and Willey (1992) have used the test to reject the random walk and certain autoregressive conditional heteroskedasticity (ARCH) models of aggregate stock prices and foreign exchange rates.

The other track has relied on the new tests and estimates to address issues related to the more general category of detecting and describing nonlinear structure in macroeconomic and financial data. The second track has criticized the conventional methodology approach to macroeconomic and financial theory. The conventional approach to macroeconomic theory assumes (after appropriate changes of units and detrending) that a system of economic variables converges to a steady state in the absence of exogenous stochastic shocks. Moreover, if the exogenous shocks are relatively small, and if the underlying dynamics of the system are smooth, then the dynamics can be approximated by a linear model. However, much of the criticism of the approach is directed at the equilibrium business cycle literature. Equilibrium business cycle methodology models the macroeconomy as a globally asymptotically stable system which is subjected to exogenous shocks on tastes, technology, or resource endowments. The approach can be criticized by the second track on the grounds that if the dynamics of the system are not smooth, or if it does not converge to a steady state in the absence of exogenous stochastic shocks, then a linear-quadratic approximation of its solution can be an entirely misleading representation of the macroeconomy and financial economy.

Instead the second track has suggested the alternative approach relies on "complex" dynamics to describe a wide range of macroeconomic and financial issues. The alternative approach allows for dynamics which even in the absence of stochastic shocks, do not converge to a steady state (Grandmont (1985), Brock (1988), and Baumol and Benhabib (1989)). In contrast to the conventional approach, the dynamics generated from such models are endogenous. The dynamics are also fundamentally nonlinear.

Furthermore, the second track has tried to describe the kind of nonlinearity by the principal geometric and dynamic measures of dynamical systems. In fact, one problem which has plagued the search for complex dynamics in macroeconomic and financial time series is the lack of a firm econometric foundation for the estimation techniques which are used in the search. Therefore, the development of econometric foundations for estimates of empirical approximations of the principal dynamic and geometric properties has been important for the second track.

Advances from the two research tracks are used to address broad issues. Two of the issues fall in the realm of the new tests as diagnostics of model specification. The other regards the econometric development and uses of the new estimates of the geometry and dynamics of dynamical systems. These will be discussed in the following.

7.2.2 *The Importance of Low Power Problems of Standard Tests*

One issue regards the practical importance of low power problems associated with standard tests of serial dependence. In particular, the issue is addressed in the context of weak-form tests of capital market efficiency. As mentioned earlier, one requirement of capital market efficiency is that, conditioned on any known information, the expected value of a market forecast error is zero (Fama (1976) and Abel and Mishkin (1983)). Weak-form tests focus on whether the conditional mean of a market forecast error is independent of the forecast errors which are known at the time when the forecast is made. The conventional tests of weak-form market efficiency are based on autocorrelations (Fama (1970), Hoffman, Low, and Schlagenhauf (1984), Mishkin (1978), and Summers (1986)). One problem with such tests is that they have low power against nonlinearity dependent alternatives (Brock, Dechert, and Scheinkman (1987), Baek and Brock (1988), inter alia). As a result, it is indeed possible that the standard tests incorrectly accept the efficient market hypothesis simply since they cannot detect the presence of nonlinearity in time series of market forecast errors. The first of the broad issues considered here addresses the practical importance of that possibility.

The issue is addressed by subjecting market forecast errors of several models of market expectations to the standard tests, to the new tests, as well as to other tests which are designed to detect particular kinds of dependence. Specifically, market forecast errors are generated from the models of individual stock returns, the random walk model of aggregate stock returns, and the constant expected returns model of real bond returns.In many instances, the new tests reject a time series of forecast errors which is accepted as serially independent by the standard tests. Such results suggest that in practice the possibility that tests of autocorrelations incorrectly accept independence simply on account of their inability to detect nonlinear dependence is of some importance.

Another finding which arises from addressing the issue of the practical importance of the low power problem of autocorrelation tests regards the power of the BDS test. The consensus view on Hsieh and LeBaron's (1988) and Hsieh (1991) studies of the power of the BDS test is that their study demonstrates that the test has high power compared with a wide variety of nonlinearly dependent alternatives (Frank, Sayers, Stengos (1989), Hsieh and LeBaron (1988), and Sayers (1989)).

7.2.3 Identification of the Source of Model Mis-specification

We are concerned with the possibility, implications, and the sources of model mis-specification. It, too, is addressed in the context of weak-form tests of capital market efficiency. Tests of capital market efficiency which are based on market forecast errors are actually joint tests of efficiency and the model of market expectations. As a result, any rejection of the joint hypothesis could stem from an inefficiency, a mis-specification of market expectations, or from both sources. For the forecast errors associated with the models of stock returns, an attempt can be made to isolate the source of a detection of dependence by the new tests. With regard to the models of individual stock returns, about one-half of the forecast errors display ARCH dependence and about one-tenth show signs of AR dependence. Non-linear dependence, however, does not appear to plague the forecast errors.

Identification of the source of model mis-specification is useful for two purposes. First, the source can be used as "stylized" fact to be accounted for by a better model. The second purpose specifically regards the models of individual stock returns. The two models (i.e., the market model and an empirical version of the two-factor capital asset pricing model) serve as standard models of the stock market in event studies. Such studies attempt to assess the effect of specific information on asset prices. The approach compares the average of the market forecast errors before the information becomes known with the average after the information is known. The statistical tests used to compare the two averages depend on assumptions made about forecast errors under the null hypothesis, that specific information has no effect on the market. Moreover, the implications surrounding event study methodology of nonlinear dependence in the null market forecast errors have only recently been considered (Diebold, Im, and Lee (1988)). As a result, identification of nonlinear dependence in the forecast errors associated with the two models have important implications regarding the validity of many existing event studies which generally only account for the possibility of AR dependence. Identification of the source could also be used to suggest efficient designs of future event studies which are based on the two models.

7.2.4 *The Detection and Description of Nonlinear Structure*

The final issue is related to the research track surrounding the general area of detecting and describing nonlinear structure in macroeconomic and financial data. The two principal measures of the geometry and dynamics of dynamical systems — the correlation dimension and Kolmogorov entropy — are often discussed in relation to the fields of causality and forecasting. The correlation dimension yields a lower bound on the number of variables which evolve in a dynamic system. The Kolmogorov entropy measures the average per-period rate at which information gained from the observation of a system is rendered useless for forecasting by subsequent observations of the system. In principle, estimates of empirical approximations of the dimension and entropy measures can be used in the fields of causality and forecasting. With regard to causality, estimates of the dimension of two time series can be used to determine whether the two series are generated by the same system. In regard to forecasting, estimates of the Kolmogorov entropy can be used to determine the period over which information embedded in the observed values of a time series can be used to predict future values.

7.3 *The Nature and Notations of Dynamical Systems*

In subsequent sections, a variety of statistical tests and estimates based on measures of geometric and dynamic properties of a time series are described and applied to the sample markets. The purpose of this section is solely to provide background material on the nature of a dynamic system in such a way that the measures on which the statistical tests and estimates are based can, to some degree, be understood. More detailed descriptions can be found in Brock (1986), Brock and Dechert (1988), Eckmann and Ruelle (1985), Farmer (1982), Shaw (1981), and in Schuster (1984).

A dynamical system can be described as a collection of variables which evolve according to a set of equations of motion. For the reason of mathematical and empirical tractability, a number of restrictions are placed on the nature of dynamical systems used to describe observed phenomena. To begin, consider a collection of,say, N variables. The focus here is on discrete, as opposed to continuous, systems. Suppose, then, that each of the N variables is observed at discrete time intervals, $t=1,2,\ldots..T$. The collection of variables is said to evolve in a phase space of Euclidean dimension N. Now, denote the N variables by x^1, $x^2, \ldots..,x^N$, and denote the value of the i-th variable in the t-th period by x_{it}. The vector of observations of the variables in period t is said to be the state of the collection. Let x_t denote the state. That is, let

$$x_t \equiv (x_{1t}, x_{2t}, \ldots\ldots, x_{Nt}) .$$

The time series of states, $\{x_t\}^T{}_{t=1}$, is said to be the trajectory of the states of the collection. Finally, let the vector value function $F:R^N \rightarrow R^N$ denote the equation of motion which describes the trajectory of the states. Operationally, it is then said that the collection of variables comprises a dynamical system if the states can be described as

$$x_t = F(x_{t-1}),\ t = 1,2,\ldots..T;\ x_0\ given . \qquad (7.1)$$

That is, the collection of variables comprises a dynamical system if the states can be expressed solely in terms of the history of the collection.

The framework under which applied work using dynamical systems theory is conducted generally presupposes that the equation of motion, F, is not known and that only one of the variables in the system is known and observed. Indeed, typically the research goal is to uncover properties of a dynamical system simply on the basis of using observations of the only known variable in the systems. As a result, an "observer" function is generally appended to the equations of motion of a system. For example, for the system $x_t=F(x_{t-1})$ an observer function $h:R^N \rightarrow R$ would be

$$a_t = h(x_t), \tag{7.2}$$

where a_t simply denotes the observation in period t of the only observed variable in the system.

As described by Brock (1986), restrictions are placed on the nature of the dynamical system so that it is possible to determine, from the time series of observations of the single variable known to evolve in the system, i.e. $\{a_t\}^T_{t=1}$, some geometric and dynamic properties of the system. One of the restrictions requires that the equation of motion possesses an attractor. Intuitively, an attractor is a bounded region of phase space into which all possible trajectories of the system evolve. The restriction imposes global, but not necessarily local stability on the system. As will be explained shortly, it must also be assumed that F is dense on its attractor. The reason is straightforward. In applied work, there is frequently only one observed trajectory with which to study a system. That is, unlike the experimental sciences, frequently one cannot repeatedly "restart" a system to generate a sample of observed trajectories. For instance, because the macroeconomy cannot be restarted, to study macroeconomic systems one must make do with only a single set of historical observations. Consequently, it should be assumed that the dynamical system which generates a time series of states is such that single time series of states can be used to recover certain properties of the system. A dynamic system which generates for any initial state, x_0, a trajectory which asymptotically covers all regions of the attractor, will satisfy the denseness assumption. Mathematically, the restriction that the trajectory $\{x_t\}^{\infty}_{t=0}$ covers the attractor for any x_0, requires that F be dense on its attractor. Finally, some of the properties which a time series of observations $\{a_t\}$ can reveal about F are spatial in nature. One example is the probabilistic properties of locating a state or a sequence of states in some region. In order to ensure that the time series of observations $\{a_t\}$ can be used to measure spatial properties, it must be assumed that F possesses a unique invariant ergodic measure. Essentially, the assumption ensures that there is a connection between long-run time averages and long-run spatial averages. It also ensures that the spatial averages determined from the time averages are in some sense meaningful.

Given that the system is dense on its attractor and possesses a unique invariant ergodic measure, under certain conditions it is possible to recover information about the system simply on the basis of a single time series of one variable in the system. The method by which such information is recovered entails the use of m-histories. An m-history of the observation a_t is denoted by $a^m{}_t$ and is defined by the expression

$$a_t^m \equiv (a_t, a_{t+1}, \ldots, a_{t+m-1}), \; t = 1, 2, \ldots, T - m + 1, \qquad (7.3)$$

where m denotes any positive integer less than T+1. As can be seen in its definition, the m-history of a_t is the sequence of observations of the known variable in the system from period t to period t+m-1.

We can illustrate how it is possible to recover information about F from $a^m{}_t$ through the use of the observer function h. In terms of the dynamical system described in equations (7.1) and (7.2), namely,

$$x_t = F(x_{t-1}), \;\; a_t = h(x_t), \;\; t = 1, 2, \ldots, T, \; x_0 \; given,$$

note that the m-history $a^m{}_t$ can be expressed as

$$\begin{aligned} a_t^m &\equiv (a_t, a_{t+1}, \ldots, a_{t+m-1}) \\ &= (h(x_t), h(x_{t+1}), \ldots, h(x_{t+m-1})) \\ &= (h(F(x_{t-1})), h(F(x_t)), \ldots, h(F(x_{t+m-2}))) \end{aligned} \qquad (7.4)$$

Now denote the i-th iterate of x_t by $F^i(x_t) = x_{t+i}$. For example, the third iterate of x_t, x_{t+3} is simply

$$F^3(x_t) = F(F(F(x_t))) = F(F(x_{t+1})) = F(x_{t+2}) = x_{t+3}.$$

Using F^i to denote the iterates of x_t, the m-history of a_t, $a^m{}_t$, can be written as

$$a_t^m = \left(h\left(F^0(x_t)\right), h\left(F^1(x_t)\right), \ldots\ldots, h\left(F^{m-1}(x_t)\right)\right) \quad (7.5)$$

Finally, let $J_m : R^m \rightarrow R^m$ denote the relation in equation (7.5) which expresses the connection between the observable m-history to the unknown function F and to the unknown collection of variables x. That is, let

$$\begin{aligned} J_m(x_t) &\equiv \left(h\left(F^0(x_t)\right), h\left(F^1(x_t)\right), \ldots\ldots, h\left(F^{m-1}(x_t)\right)\right) \\ &= a_t^m . \end{aligned} \quad (7.6)$$

If the inverse of J_m, say $J^{-1}{}_m$, exists, then x_t can be expressed from equation (7.6) as

$$x_t = J_m^{-1}(a_t^m) \quad (7.7)$$

Letting "$\circ$" denote the composition of functions, note that one can use J_m and its inverse to express the dynamical system in terms of the m-history $a^m{}_t$ instead of the state x_t. That is, from equations (7.6) and (7.7), x_t can be written as

$$x_t = F(x_{t-1}) = F\left(J_m^{-1}(a_{t-1}^m)\right) = F \; o \; J_m^{-1}(a_{t-1}^m)$$

As a result, $a^m{}_t$ can be expressed as

$$a_t^m = J_m \; o \; F \; o \; J_m^{-1}(a_{t-1}^m) \quad (7.8)$$

Equation (7.8) is called an equivalent dynamical system for dynamical system $x_t = F(x_{t-1})$. Note that the equivalent system in (7.8) is expressed solely in terms of observable m-histories.

Equation (7.8) displays mathematically the connection between observations of a single component of a system and the entire system itself. As demonstrated by Takens (1981), under certain conditions, some properties of F can be recovered from the time series of observations

$$\{ a_t^m \}_{t=1}^{T-m+1}$$

The equivalent dynamical system shown in equation (7.8) displays the sort of properties of F which can be recovered from $\{a^m{}_t\}$. Specifically, note in equation (7.8) that the only kinds of properties that can be recovered are those which are invariant under the conjugacy transformation from F to $J_m \circ F \circ J^{-1}{}_m$. A few geometric and dynamic properties of F can indeed be recovered from time series of m-histories, provided that $m \geq 2N+1$. If F is assumed to be a smooth function with at least continuous first and second derivatives, if the inverse of F exists, and if the attractor of F is a smooth manifold, a variety of geometric and dynamic properties of F can be recovered by $\{a^m{}_t\}$. In other words, under such conditions the dynamical system $x_t = F(x_{t-1})$ and its equivalent dynamical system share many of the same geometric and dynamic properties. Such properties are said to be conjugacy invariants.

7.4 Test for Independence under Nonlinear Dynamics

This chapter has a twofold purpose. One is solely to establish, by reviewing the relevant literature, the concept of the correlation integral. To understand this concept is very important since the principal measures can be expressed in terms of a correlation integral. What is more, statistical theory for the correlation integral is established for time series which are completely random as well as for some time series which are serially dependent. As a result, it is possible to use the correlation integral to distinguish time series of observations which are generated by a dependent process from those which are generated by a completely random process.

The other is to describe the statistical tests of estimates related to the principal measures of dynamical systems. The statistical tests are based on the fact that correlation integral are U-statistics. The tests also rely on the well-understood asymptotic properties of U-statistics for independent and weakly dependent time series. The section begins with a discussion of U-statistics and their connection to correlation integral. The discussion then turns to independence tests based on estimates of the principal measures and to how those tests can be used as diagnostic tests for independence and for model mis-specification. Also included is a discussion of the power of the new tests against dependent alternatives.

It should be pointed out that the material presented here is devoted to describing, in depth, what is known in the literature about the nature and properties of the new statistical tests.

7.4.1 The Correlation Integral

The correlation integral can be used to determine the fraction of pairs of states, trajectories, or m-histories of observations which are within a certain distance of each other. Focusing first on the states of a system, consider the time series of states $\{x_t\}^T_{t=1}$ $(x \in R^N)$. Let $\{x_t\}$ be strictly stationary stochastic process with distribution function F. We shall call $(x_t, x_{t+1}, \ldots, x_{t+m-1})$ an m-history, and denote it by x^m_t like (7.3). Its distribution function is denoted by F_m. When the $\{x_t\}$ are independent, then

$$F_m(u_1, u_2, \ldots, u_m) = \prod_{k=1}^{m} F(u_k).$$

Let $\Re^j_i$ be σ-algebra generated by $\{X_t | i \leq t < j\}$ $(1 \leq i \leq j \leq \infty)$. The stochastic process $\{x_t\}$ is absolutely regular if

$$\beta_k = \sup_{n \geq 1} \left\{ E[\sup\{\, | P(A \mid \Re^n_1) - P(A) \| A \in \Re^\infty_{n+k}\}] \right\} \tag{7.9}$$

converges to zero (see e.g. Denker and Keller (1983)).

For $x \in R^m$ we shall use the maximum norm,

$$\| x \| = \max_{1 \leq k < m} \{ |x_k| \}.$$

When it is important to emphasize the dimension of the underlying space we shall use the notation $\| \bullet \|_m$ for this norm. Let χ_A be the characteristic function of the set A. For the special case that $A = [0, \epsilon)$ we denote its characteristic (or indicator) function by χ_ϵ. This function determines whether a pair of states (x_t, x_s) are "within" the scale parameter ϵ of each other. Since the norm $\| \bullet \|$ be the maximum norm, for $x_t \equiv (x_{1t}, x_{2t}, \ldots, x_{Nt})$,

$$\| x_t - x_s \| = \max \{ |x_{1t} - x_{1s}|, |x_{2t} - x_{2s}|, \ldots, |x_{Nt} - x_{Ns}| \} \tag{7.10}$$

As a result, $\chi_\epsilon(x_t, x_s)$ can be expressed as

$$\chi_\epsilon(x_t, x_s) = \begin{cases} 1, & if \ \| x_t - x_s \| < \epsilon \\ 0, & otherwise \end{cases} \tag{7.11}$$

The number of distinct (x_t, x_s) pairs associated with the time series $\{X_\tau\}^T_{\tau=1}$ which are within ϵ of each other is simply

$$\sum_{t=1}^{T-1} \sum_{s=t+1}^{T} \chi_\epsilon(x_t, x_s),$$

or, for short

$$\sum_{1 \leq t < s \leq T} \chi_\epsilon(x_t, x_s).$$

Consequently, the correlation integral, $C_T(\epsilon)$, which determines the fraction of distinct (x_t, x_s) pairs which are within ϵ of each other can be expressed as

$$C_T(\varepsilon) \equiv \frac{2}{T(T-1)} \sum_{1 \leq t < s \leq T} \chi_\epsilon(x_t, x_s). \tag{7.12}$$

(The number of distinct (x_t, x_s) pairs associated with $\{X_t\}^T_{t=1}$ is $T(T-1)/2$.)

A generalization of the correlation integral can be used to compute the fraction of pairs of trajectories of length m which are within the scale parameter ϵ of each other. Consider the pair of m-length trajectories $(\{X_\tau\}_{\tau=t}^{t+m-1}, \{X_\tau\}_{\tau=s}^{s+m-1})$ generated by the time series of states $\{X_t\}^T_{t=1}$. Again $\chi_\epsilon(\{X_\tau\}_{\tau=t}^{t+m-1}, \{X_\tau\}_{\tau=s}^{s+m-1}$ denote the indicator function which determines whether the two m-length trajectories are within ϵ of each other. The function is defined as

$$\chi_\varepsilon\left(\{X_\tau\}_{\tau=t}^{t+m-1}, \{X_\tau\}_{\tau=s}^{s+m-1}\right) \equiv \prod_{\tau=0}^{m-1} \chi_\varepsilon(x_{t+\tau}, x_{s+\tau}) \tag{7.13}$$

Finally, let n denote the number of distinct (t,s) pairs of m-length trajectories which can be generated from the time series of states $\{X_t\}^T_{t=1}$. That is, $n=T-m+1$. The fraction of pairs of m-length trajectories, $C_{m,n}(\epsilon)$, is then given by the relation

$$C_{m,n}(\varepsilon) \equiv \frac{2}{n(n-1)} \sum_{1\le t<s\le n} \left[\prod_{\tau=0}^{m-1} \chi_\varepsilon(x_{t+\tau}, x_{s+\tau})\right] \tag{7.14}$$

The generalized correlation integral $C_{m,n}(\epsilon)$ can also be used to determine the fraction of pairs of m-histories $(a^m{}_t, a^m{}_s)$ associated with the time series of observations $\{a_t\}$ which are within the scale parameter ϵ of each other. As discussed in section 7.3, the scalar a_t corresponds to the observation of a single variable in a system. As before, the m-history of a_t is defined as $a^m{}_t \equiv (a_t, a_{t+1}, \ldots, a_{t+m-1})$. In an analogous way to the generalized correlation integral for m-length trajectories of states, the correlation states, the correlation integral associated with the time series of m-histories $\{a_t^m\}_{t=1}^{n=T-m+1}$ can be written as

$$C_{m,n}(\varepsilon) \equiv \frac{2}{n(n-1)} \sum_{1\le t<s\le n} \left[\prod_{\tau=0}^{m-1} \chi_\varepsilon(a_{t+\tau}, a_{s+\tau})\right] \tag{7.15}$$

If the data is generated by a strict stationary stochastic process which is absolutely regular, a large sample property of the correlation integral is satisfied. Brock and Dechert (1988) have shown that for any system or process which possesses a unique ergodic measure, the limit as $T\to\infty$ of the correlation integral associated with the trajectory of states $\{X_t\}^T_{t=1}$ exists. That is, the authors showed that

$$C(\varepsilon) = \lim_{T \to \infty} C_T(\varepsilon) \tag{7.16}$$

exists. Moreover, $C(\epsilon)$ and its analog $C_m(\epsilon)$ defined by the expression

$$C_m(\varepsilon) \equiv \lim_{n \to \infty} C_{m,n}(\varepsilon) \tag{7.17}$$

are related to the probability of locating states and trajectories in the same region of phase space (see Grassberger and Procaccia (1983)).

7.4.2 *Statistical Tests Based on the Correlation Integral*

7.4.2.1 U-Statistics

When one estimates an object such as, for example, a correlation integral, the error bar (the standard error for statisticians) depends on the nature of the underlying process. A precise theory of error bars can be developed with the concept of U-statistic theory for weakly dependent processes. Furthermore this concept can be applied for tests for noisy low dimensional chaos in limited data sets. We will follow Sen (1972), Serfling (1980) and Denker and Keller (1983).

Consider a sample of independent observations $\{X_t\}^n_{t=1}$ ($x^m_t \in R^m$) generated by a probability distribution $F(x^m_t)$. Let $h(x^m_t, x^m_s)$ denote a symmetric real valued function, so called a "kernel", where x^m_t, x^m_s are in R^m. And let θ denote the expected value of $h(x^m_t, x^m_s)$, i.e.,

$$\theta \equiv E\left[h(x_t^m, x_s^m) \right]. \tag{7.18}$$

Note that a measurable function $h: \Omega^n \rightarrow R$ is called a kernel for $\theta = E[h]$ if it is symmetric in its n arguments.

Under the IID assumption for $\{X_t^m\}$, a minimum variance estimator for θ in the class of unbiased estimators is the U-statistic, U_n, given by

$$U_n \equiv \frac{2}{n(n-1)} \sum_{i \le t < s \le n} h(x_t^m, x_s^m) \qquad (7.19)$$

(See Serfling (1980)). Any statistic which can be written in the (7.19), where the kernel" is symmetric in its arguments, is called a U-statistic. It is easy to see

hat the correlation integral is a U-statistic. In the context of a sample of independent m-histories $\{a^m_t\}^n_{t=1}$ for which, under the null, the sample is generated from a distribution $F(x^m_t)$, $x^m_t \in R^m$, let the kernel $h(x^m_t, x^m_s)$ be the indicator function. That is, let

$$h(x_t^m, x_s^m) = \chi_\epsilon(x_t^m, x_s^m) = \begin{cases} 1, & if \ \| x_t^m - x_s^m \| < \epsilon \\ 0, & otherwise \end{cases}$$

The expected value of the indicator kernel is simply

$$C_m(\epsilon) \equiv \int\int_R \chi_\epsilon(x_t^m, x_s^m)\, dF(x_t^m)\, dF(x_s^m) = E[\chi_\epsilon(a_t^m, a_s^m)] = \theta,$$

and the U-statistic for $C_m(\epsilon)$ is the correlation integral:

$$U_n = \frac{2}{n(n-1)} \sum_{1 \le t < s \le n} \chi_\varepsilon(a_t^m, a_s^m) = C_{m,n}(\varepsilon).$$

Asymptotic theory for U-statistic is based on the so-called "projection" of U_n. This is a device to reduce a complicated looking U-statistic to a simple average so that standard central limit theorems may be applied. Define the conditional expectation, $h_1(x^m_t)$ as

$$h_1(x_t^m) \equiv E\left[h(x_t^m, x_s^m) \mid x_t^m\right] \qquad (7.20)$$

Then the projection of U_n, say $\hat{U}_n$, is given in terms of $h_1(\bullet)$ by the projection method of Hoeffding (1948):

$$\hat{U}_n \equiv \theta + \frac{2}{n}\sum_{t=1}^{n}\left[h_1(x_t^m) - \theta\right] + R_n(x_1, x_2, \ldots, x_n) \qquad (7.21)$$

where R_n is a remainder that goes to zero in distribution when multiplied by $\sqrt{n}$ as $n \to \infty$. As shown by Hoeffding(1948) and as described in Serfling (1980), the decomposition

$$U_n = \hat{U}_n + R_n, \qquad (7.22)$$

where $R_n \to 0$ as $n \to \infty$ yields many results regarding the asymptotic theory for the U-statistic U_n.

An important result regarding asymptotic theory for U-statistics is by Denker and Keller (1983). By replacing U_n with its projection, the authors have shown that U_n is distributed asymptotically normal provided that the sample $\{X^m{}_t\}$ satisfies certain regularity conditions concerning the rate of decay of the dependence between $x^m{}_t$ and $x^m{}_{t+k}$ as k increases. Specifically, the authors have shown that under their regularity conditions

$$\sqrt{n}\, U_n \to N(\theta,\, 4\sigma^2),\ approximately, \qquad (7.23)$$

where

$$\sigma^2 = E\left[\left(h_1(x_1^m) - \theta\right)^2 + 2\sum_{t=2}^{n}\left(h_1(x_1^m) - \theta\right)\left(h_1(x_t^m) - \theta\right)\right]. \qquad (7.24)$$

Provided that the sum in equation (7.24) converges absolutely, the asymptotic variance can be expressed as

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} E\sum_{t=1}^{n}\left(h_1(x_t^m) - \theta\right)^2. \qquad (7.25)$$

The result of Denker and Keller can easily be applied to the correlation integral. As before, consider the time series of m-histories $\{a^m{}_t\}$ for which each is assumed to be generated from the distribution $F(x^m{}_t)$. Likewise, let the kernel be the indicator function, and note then that

$$h_1(a_t^m) = E\left[h(a_t^m, x_s^m)\right] = \int_R \chi_\epsilon(a_t^m, x_s^m)\, dF(x_s^m)$$

Let $h_1(a^m{}_t,\epsilon) \equiv h_1(a^m{}_t)$, so that the dependence on ϵ of $h_1(a^m{}_t)$ is made explicit. Denker and Keller's result can then be expressed for the correlation integral as

$$9\sqrt{n}\, C_{m,n}(\epsilon) \sim N\left(C_m(\epsilon),\ 4\sigma^2(m,\epsilon)\right),\ approximately, \qquad (7.26)$$

where

$$\begin{aligned}\sigma^2(m,\varepsilon) = E\,[\,(h_1(a_1^m,\varepsilon) - C_m(\varepsilon))^2 \\ + 2\sum_{t=1}^{n}(h_1(a_1^m,\varepsilon) - C_m(\varepsilon))\,(h_1(a_t^m,\varepsilon) - C_m(\varepsilon)\,)\,]\end{aligned} \qquad (7.27)$$

Theorem 1 *(Denker and Keller)*

Provided $\sigma^2 > 0$ and for some $\delta > 0$,

i) $\Sigma\, \beta_n^{\delta/(2+\delta)} < \infty$

ii) $\sup_{i<j}\ E[h(x^m{}_i, x^m{}_j)^{2+\delta}] < \infty$

then

$$\sqrt{n}\ \frac{U_n - \theta}{2\,\sigma_n/\sqrt{n}} \ \to\ N(0,1),\ as\ n \to \infty.$$

So far, we assume that $\{x_t\}$ are IID. However, for general dependent processes, i.e., the case when $\{x_t\}$ are not IID, U-statistic central limit theory is also available (Brock, Dechert and Scheinkman (1987), Baek and Brock (1988), inter alia). U-statistics are surely interesting because (1) they have many of the desirable properties that averages of IID variable possess, including central limit theorems and law of large numbers, (2) they are minimum variance estimators of θ in the class of all unbiased estimators of θ (Serfling (1980)), (3) they converge rapidly to normality (Serfling (1980), p. 193 Theorem B).

7.4.2.2 The BDS Test

Now it turns out that the correlation integral can be written as functions of U-statistics. Moreover, because the asymptotic distribution of the correlation integral is well understood for weakly dependent and independent time series, a variety of tests based on the correlation integral can be devised. The Brock, Dechert and Scheinkman's test (hereafter BDS test) is one of the tests based on the correlation integral.

The BDS test is based on the dimension of IID time series. Brock, Dechert and Scheinkman (1987) have shown that under the assumption that a time series $\{X_t\}^T_{t=1}$ ($x_t \in R^1$) is IID that

$$\lim_{n \to \infty} C_{m,n}(\varepsilon) = C_m(\varepsilon) = C_1^m(\varepsilon) \tag{7.28}$$

That is, for a scalar IID time series, the correlation integral for its m-histories is the same as the correlation integral for the scalar series raised to the power of m. We interpret the result from a geometrical point of view. Since x_t is independent, the probability of finding two m-histories within ϵ of each other is equal to the product ofcanthe probabilities of finding m consecutive (x_t, x_s) pairs of observations within ϵ of each other.

On the basis of the asymptotic theory for U-statistics, the result of Brock, Dechert, and Scheinkman can be converted into a statistical test. Rearranging equation (7.28) suggests that we can use the U-statistics $C_{m,n}(\epsilon)$ and $C_{1,T}(\epsilon)$ to test for independence based on the result that

$$C_m(\epsilon) - C_1^m(\epsilon) = 0.$$

As before, assume that the sample of scalar observations, say, $\{a_t\}^T_{t=1}$, is generated by an IID process $\{x_t\}$. The aim is to generate a test for independence based on the asymptotic behavior of the estimates for $C_m(\epsilon)$ and $C_1(\epsilon)$, namely, $C_{m,n}(\epsilon)$ and $C_{1,T}(\epsilon)$.

Firstly, it should be noted that any test for independence based on the asymptotic behavior of correlation integral of the same scalar parameter ϵ can be generated from the principal result in Brock, Dechert and Scheinkman (1987). That result can be stated in the following way. Consider a vector of correlation integral of the variety

$$(C_{m1,n}(\epsilon), C_{m2,n}(\epsilon), \ldots\ldots, C_{mq,n}\epsilon)),$$

where q denotes any positive integer. The asymptotic distribution of that vector was shown by the authors to be given by

$$\sqrt{n}\,(C_{m1,n}(\epsilon),\ldots,C_{mq,n}(\epsilon)) \sim N(C^{m1}(\epsilon),\ldots,C^{mq}(\epsilon)),\Sigma(\bullet), \qquad (7.29)$$

approximately

where the (i-j) element of the variance-covariance matrix $\Sigma(m_1, m_2, \ldots.. m_q)$ is given by the expressions

$$(\Sigma(m_1,m_2,\ldots,m_q))_{i,j} = (½)[\xi(m_i,m_j,1,1) - \xi(1,m_i,0,1)-\xi(1,m_j,0,1)],$$

$$\xi(m_i,m_j,\lambda_1,\lambda_2) = 4[\xi_0(m_i,m_j,\lambda_1,\lambda_2) + 2\sum_{k=1}^{m_j-1} \xi_k(m_i,m_j,\lambda_1,\lambda_2)],$$

$$\xi_0(m_i,m_j,\lambda_1,\lambda_2) = \lambda^2{}_1K^{mi}(\epsilon) + \lambda^2{}_2k^{mj}(\epsilon) + 2\lambda_1\lambda_2k^{mi}(\epsilon)C^{mj-mi}(\epsilon)$$

$$- (\lambda_1C^{mi}(\epsilon) + \lambda_2C^{mj}(\epsilon))^2.$$

$$k(\epsilon) = \int\int\int_R \chi_\epsilon(x_t,x_s)\chi_\epsilon(x_s,x_r)dF(x_t)dF(x_s)dF(x_r),$$

and (omitting the explicit reference to ϵ in $k(\epsilon)$ and $C(\epsilon)$),

$$\xi_k(m_i,m_j,\lambda_1,\lambda_2) = \lambda^2{}_1k^{(mi-k)}C^{2k} + \lambda^2{}_2k^{(mj-k)}C^{2k} + \lambda_1\lambda_2k^{mi-k}C^{mj-mi+2k}$$

$$-(\lambda_1C^{mi}+\lambda_2C^{mj})\lambda_1C^{mi} - (\lambda_1C^{mi}+\lambda_2C^{mj})\lambda_2C^{mj}$$

$$+ \lambda_1\lambda_2k^{\min(mi,mj-k)}C^{(mi+mj-2\min(mi,mj-k))}$$

if $1 \leq k \leq m_1$, otherwise

$$\xi_k(m_i,m_j\lambda_1,\lambda_2) = \lambda^2{}_1C^{2mi} + \lambda_2{}^2k^{mi-k}C^{2k} + \lambda_1\lambda_2C^{mi+mj}$$

$$- (\lambda_1C^{mi}+\lambda_2C^{mj})\lambda_1C^{mi} - (\lambda_1C^{mi}+\lambda_2C^{mj})\lambda_2C^{mj}$$

$$+ \lambda_1\lambda_2k^{\min(mi,mj-k)}C^{(mi+mj-2\min(mi,mj-k))}.$$

With the aid of the delta method (Serfling (1980)), the BDS test satisfies the conditions for the authors' main result. In the context of the vector of correlation integral $(C_{m1,n}(\epsilon),\ldots\ldots, C_{mq,n}(\epsilon))$, the delta method yields the result that for any C^1 function of the vector, say

$$g(C_{m1,n}(\epsilon), C_{m2,n}(\epsilon), \ldots\ldots, C_{mq,n}(\epsilon)),$$

that

$$g(C_{m1,n}(\epsilon),\ldots\ldots,C_{mq,n}(\epsilon)) \sim N(g(C^{m1}(\epsilon),\ldots,C^{mq}(\epsilon)), D\Sigma(\bullet)D^T), \quad (7.30)$$

approximately,

where the j-th element of D, say, D_j, is given by the expression

$$D_j = \frac{\partial g(\bullet)}{\partial C_{mj,n}(\epsilon)} \Big|(C^{m1}(\epsilon),\ldots\ldots,C^{mq}(\epsilon)).$$

Given the results of Brock, Dechert, and Scheinkman (1987) and the delta method, it is easy to see that

$$\sqrt{n}(C_{m,n}(\epsilon) - C^m{}_{1,n}(\epsilon)) \sim N(C_m(\epsilon) - C^m{}_1(\epsilon), D\Sigma(m,1)D^T).$$

where $D = (1, - mC_1{}^{m-1}(\epsilon))$.

However since $C_m(\epsilon)\text{-}C_1{}^m(\epsilon) = 0$ under the IID assumption, the asymptotic distribution can be expressed simply

$$\sqrt{n}(C_{m,n}(\epsilon)-C^{m}{}_{1}(\epsilon)) \sim N(0,(1,-mC_{1}{}^{m-1}(\epsilon))\Sigma(m,1)(1,-mC_{1}{}^{m-1}(\epsilon))^{T}).$$

Moreover, consistent estimates under the null for $C_1(\epsilon)$ and $k(\epsilon)$ as described in Brock, Dechert and Scheinkman (1987) and in Hsieh and LeBaron (1988) are $C_{1,T}(\epsilon)$ and $k_n(\epsilon)$, where

$$k_n(\varepsilon) \equiv \frac{6}{n(n-1)(n-2)} \sum_{1\le t<s<r\le n} \chi_\varepsilon(a_t, a_s)\,\chi_\varepsilon(a_s, ar).$$

Combining the results yields the simplified form of the BDS test:

$$\sqrt{n}(C_{m,n}(\epsilon) - C^{m}{}_{1,n}(\epsilon)) \sim N(0, \sigma^2{}_{BDS}(m,\epsilon)), \tag{7.31}$$

where a consistent estimate for the variance $\sigma^2{}_{BDS}(m,e)$[6)]

$$\begin{aligned}\hat{\sigma}^2_{BDS}(m,\varepsilon) = 4[&m(m-2)C_{1,n}^{2m-2}(\varepsilon)(k_n(\varepsilon) - C_{1,n}^2(\varepsilon)) + k_n^m(\varepsilon)\\ &- C_{1,n}^{2m}(\varepsilon) + 2\sum_{j=1}^{m-1}[C_{1,n}^{2j}(\varepsilon)(K_n^{m-j}(\varepsilon)\\ &- C_{1,n}^{2m-2j}(\varepsilon)) - mC_{1,n}^{2m-2}(\varepsilon)(K_n(\varepsilon) - C_{1,n}^2(\varepsilon))]]\end{aligned} \tag{7.32}$$

Theorem 2 (Brock, Dechert and Scheinkman)

Let $\{X_t\}$ be IID. Then for $m \ge 2$

$$\sqrt{n}\;\frac{C_{m,n}(\epsilon) - C_{1,n}^{m}(\epsilon)}{\sigma_{BDS}(m,\epsilon)} \tag{7.33}$$

converges in distribution to N(0,1), where $\sigma_{BDS}(m,\epsilon)$ satisfies equation (7.32).

---

6) The proof is given by Brock, Dechert, Scheinkman, and LeBaron (1991).

So far, our discussion is focused on the case of the IID. Our interest is how we can conduct statistical tests based on the correlation integral when it is assumed that a time series is weakly dependent. To begin, recall (7.26) and (7.27) . Now suppose for example that we wanted to compute the BDS test statistic under an alternative hypothesis of some form of weak dependence. That is, suppose that we wanted to look at the statistic

$$C_{m,n}(\epsilon) - C_m(\epsilon) - \left(C_{1,n}(\epsilon) - C_1(\epsilon)\right),$$

under an hypothesis about $\{a_t\}$ other than IID. Along with lines of Hsieh and LeBaron (1988), the procedure under which such a test could be devised can be described as follows. From the result of Denker and Keller shown in equations (7.26) and (7.27), the asymptotic distributions of $C_{m,n}(\epsilon)$ and $C_{1,n}(\epsilon)$ are known. The trick is to find the covariance between $C_{m,n}(\epsilon)$ and $C_{1,n}(\epsilon)$ and then apply the delta method to get the asymptotic distribution of the BDS statistic under an alternative hypothesis. The covariance, say $\sigma_{1,m}(\epsilon)$, between the above two U-statistics is simply

$$\sigma_{1,m}(\varepsilon) = E\left[\left(\sum_{t=1}^{n}\left(h_1(a_t^m,\varepsilon) - C_m(\varepsilon)\right)^2\right)\left(\sum_{t=1}^{n}\left(h_1(a_t,\varepsilon) - C(\varepsilon)\right)^2\right)\right] \tag{7.34}$$

On the basis of the delta method, then, the BDS statistic under an alternative hypothesis of weak dependence is

$$\sqrt{n}(C_{m,n}(\epsilon) - C^m{}_{1,n}(\epsilon)) \sim N((C_m(\epsilon) - C_1{}^m(\epsilon)), D\Sigma(\bullet)D^T), \tag{7.35}$$

approximately

where $D=(1,-mC_1{}^{m-1}(\epsilon))$, and

$$\Sigma = \begin{bmatrix} 4\sigma^2(m,\epsilon) & 4\sigma_{1,m}(\epsilon) \\ 4\sigma_{1m}(\epsilon) & 4\sigma^2(1,\epsilon) \end{bmatrix}$$

Denote the variance $D\Sigma(\bullet)D^T$ by $\sigma^2_{BDSA}(m,\epsilon)$.

To use equation (7.35) as a test for some form of weak dependence requires that consistent estimates of the mean $C_m(\epsilon)$-$C^m{}_1(\epsilon)$ and the variance $\sigma^2_{BDSA}(m,\epsilon)$ are computed. One straightforward way to generate estimates of the mean and variance is by Monte Carlo simulation. Such a method requires that the statistic $C_{m,n}(\epsilon)$-$C^m{}_{1,T}(\epsilon)$ should be computed for a large number of replications under the alternative hypothesis. The mean value of the replications is used as an estimate of $C_m(\epsilon)$-$C^m{}_1(\epsilon)$, and the square of the standard deviation is used as an estimate of $\sigma^2_{BDSA}(m,\epsilon)$.

Brock (1987) has shown that the BDS test is nuisance-parameter-free in a relatively wide sense. Borrowing the notation in Brock and Dechert (1988) to describe the property, let I_t denote a regressor set of information used to describe the behavior of, say, y_t. Let the nature of the relation between y_t and I_t be of the general functional form

$$y_t = g_1(I_t) + g_2(I_t)\,U_t,\; t=1,2,\ldots,T,$$

where U_t is assumed to be IID with mean zero and finite variance. Further, let $\hat{g}_1(\bullet)$ and $\hat{g}_2(\bullet)$ denote $T^{1/2}$ consistent estimates of $g_1(\bullet)$ and $g_2(\bullet)$. Finally, let the estimate of U_t based on the consistent estimates of $g_1(\bullet)$ and $g_2(\bullet)$ be defined as

$$\hat{U}_t(T) \equiv \frac{y_t - \hat{g}_1(I_t)}{\hat{g}_2(I_t)}$$

Brock has shown that one can use $\{\hat{U}_t(T)\}$ in (7.31) to test whether the unobservable errors $\{U_t\}$ are IID. That is, Brock has verified that the test statistic $C_{m,n}(\epsilon)$-$C^m{}_{1,T}(\epsilon)$ converges to the same asymptotic distribution when one evaluates the statistic with $\{\hat{U}_t(T)\}$ as when one evaluates the statistic with $\{U_t\}$.

Furthermore, Brock and Dechert (1988) have shown that the BDS test, as well as any other nuisance-parameter-free test for IID, can be used in some cases as a test for model specification error. They have verified that if the residuals associated with the differences between a null model and the actual process which describes the behavior of the dependent variable are IID, then one can conclude that the null model is indeed the actual process which generates the dependent variable.

7.4.3 The Power of the BDS Test

In fact, the BDS statistic (7.33) is a distribution-free statistic. Thus, it has the advantage that no distributional assumptions need to be made in using it as a test statistic for IID random variables. On the other hand, it has the disadvantage that it will not be the most powerful statistic to use in testing a parametric hypothesis against a parametric alternative. Furthermore, in practical applications we rarely find infinite series. So, we must make sure that the finite sample distribution is actually well approximated by the asymptotic distribution. For the BDS statistic, since the BDS test is computed for a given imbedding dimension m, and a given distance ϵ (in number of standard deviation of the data), there are more complications.

We know intuitively what will happen in a finite sample.

(1) If embedding dimension m is too large relative to the sample size, the BDS statistic will be very ill-behaved, since there are too few independent points.

(2) If ϵ is too small or too large, the BDS statistic will also be ill-behaved, since there are too few or too many points "close" to any given m-vector.

(3) The shape to the data distribution also matters, since it affects how far the points are spread apart.

Brock, Hsieh and LeBaron (1987), Hsieh and LeBaron (1988) and Ramsey and Yuan (1989) accessed the finite distribution of the BDS statistic under the null hypothesis of IID. While Ramsey and Yuan generated three distributions of random variables: uniform, normal and gamma, Hsieh and LeBaron(1988) generated pseudo random numbers for three sample sizes (100, 500, 1000) and six distributions: first order autoregression(AR(1)); first order moving average(MA(1)); the tend map; non-linear moving average threshold (NMA); threshold autoregression (TAR); and the autoregressive conditional heteroskedasticity (ARCH) model. They addressed two issues:

(1) What region of ϵ yields the BDS statistic that are well approximated by the asymptotic distribution, and whether that region expands as the sample size increases.

(2) What the effect of increasing the dimension m is.

Hsieh and LeBaron's (1988) results are interesting. For each of the alternatives, the authors compared the BDS statistic for values of the scale parameter ϵ equal to 1/4, 1/2, 1, 3/2 and 2 times the sample standard deviation in the dependentvariable for various sample sizes T and embedding dimensions m. Specifically, for each of the scale factors ϵ, the BDS test statistic was determined for the following sample size-embedding dimension paris (T,m): (200,2),(500,2),(1000,2),(500,5),(1000,10).

On the basis of 2000 replications for each Monte Carlo experiment, the authors found that the power of the BDS test to detect the dependence in the time series was good. For example, in all but six cases, Hsieh and LeBaron determined that at 5% significance the power of the BDS test against the alternatives is greater than 95% for values of ϵ equal to 0.5 and 1.0 times the standard deviation in the dependent variable when $T/m \geq 100$. The six exemptions are: The MA(1) when $(T,m,\epsilon)=(1000,10,0.5\sigma)$ where the power is 69%; the TAR when $(T,m,\epsilon)=(500,2,1\sigma)$ when $(500,5,0.5\sigma)$ and when $(T,m,\epsilon)=(500,5,1\sigma)$ where the respective power estimates are 93%, 84% and 80%; the NMA when $(T,m,\epsilon)=(1000\ ,10,0.5\sigma)$ where the power is 86%; and the ARCH(1) model when $(T,m,e)=(1000,10,0.5\sigma)$ where the power is 85%. An important conclusion to be drawn from Hsieh and LeBaron's study is that the BDS test has good power against nonlinearity dependent alternatives which are uncorrelated.

Most researchers of nonlinear dynamics who have applied the BDS test to macroeconomic and financial time series commonly interpret the Hsieh and LeBaron's study as indicating that the test has high power against a wide range of alternatives to the nulling of IID (see, for example, Frank, Sayers and Stengos (1989)). As a result, the test is viewed as a useful diagnostic for model specification in cases where there are many relevant alternatives or in class where an explicit alternative to the nulling is left unspecified. However, such a conclusion might be premature and erroneous. According to Baek and Brock's (1988) Monte Carlo study, against the MA(1) and AR(1), the power of the BDS test fails considerably in relative and absolute terms when ρ is decreased from 0.5 to 0.1. Note that they also claimed that the BDS test indeed did well in an absolute and relative sense in detecting the nonlinear dependence in the ARCH(1), NMA(1) ,TAR and the tent map alternatives. One conclusion is that the BDS test performs relatively poorly against time series possessing low-level autocorrelations.

Given the results of many researchers, the common claim for the BDS test as a promising diagnostic test for model specification must be qualified. However, it appears that the BDS test indeed has high power against nonlinear dependence alternatives.

7.5 Dimension Measures and the Geometry of a Stock Market

To uncover the geometric and dynamic properties of a stock market is another evidence against the market efficient hypothesis. Throughout this section, the discussion will focus on measures which describe the geometric and dynamic properties of a dynamical system. It is assumed the equation of motion F possesses an attractor for which F is dense. It is also assumed that F possesses a unique invariant measure. Again, in this section, much of the material is taken from Schuster (1984). In particular, among dimension measures we will focus on only the correlation dimension.

7.5.1 Preliminaries

It is convenient to begin by considering some of the probabilistic properties of F. Moreover, a convenient way of expressing the properties relies on partitioning the phase space which contains the attractor into N-dimensional cubes. Let the cubes have sides of edge length ϵ. Figure 7.1 displays such a partitioning for a system embedded in a 2-dimensional phase space. Next, denote and number by i those cubes which contain a part of the attractor. Let i run from 1 to $M(\epsilon)$. As shown in Figure 7.1, the number of cubes of edge length ϵ which contain a part of the attractor is 12. Note as well that the number of cubes of edge length 2ϵ is 4. Clearly the number of cubes which contain a part of the attractor, M, depends on the edge length.

Now, let $\mu_i(t)$ denote the relative frequency up to period t of locating a state in cube i. That is, define $\mu_i(t)$ as

$$\mu_i(t) \equiv \frac{1}{t} \sum_{\tau=1}^{t} I_i(\tau),$$

where

$$I_i(\tau) \equiv \begin{cases} 1, & if\ x_t \in cube\ i \\ 0, & otherwise \end{cases}$$

The probability $P_\epsilon(i)$, then, of locating a state in cube i can be expressed in terms of the relative frequency as

$$P_\epsilon(i) = \lim_{T \to \infty} \mu_i(T) \tag{7.36}$$

The probability $P_\epsilon(i)$ is meaningful because of the assumptions made regarding the nature of F. The assumption that F possesses a unique invariant ergodic measures that the system has a unique limiting density function as $T \to \infty$. Moreover, the assumption that F is dense ensures that $P_\epsilon(i)$ can be calculated simply on the basis of a single trajectory.

7.5.2 *The Correlation Dimension*

There is a family of dimension measures which are functions of the probability of locating a state in some region on the attractor. As will become evident, the dimensions describe a few geometrical properties of a dynamical system. The family of dimensions are special cases of the general function of Renyi (1970),

$$D_f \equiv \lim_{\varepsilon \to 0} \frac{1}{f-1} \frac{\ln \sum_{i=1}^{M(\varepsilon)} P_\varepsilon^f(i)}{\ln \varepsilon}, \quad f \in R^+ \tag{7.37}$$

where, $P^f_\epsilon(i)$ denotes the probability raised to the power f. In particular, when the dimension is given by

$$D_2 = \lim_{\varepsilon \to \infty} \frac{\ln \sum_{i=1}^{M(\varepsilon)} P_\varepsilon^2(i)}{\ln \varepsilon} \tag{7.38}$$

D_2 is called the correlation dimension.

The reason it gets its name is related to the fact that the term $\Sigma P^2_\epsilon(i)$ is simply the probability of locating any two observations of the state of the system in the same region. A simple example illustrates the claim. Consider the 2-dimensional system displayed in Figure 7.2 which is partitioned into 4 cubes. As shown in the figure, the probability of finding two states in cube 1, $P^2_\epsilon(1)$ is 0.25, in cube 2 the probability is 0.01, and in cube 3 and 4 the probabilities are each 0.04. The probability, then, of locating two states in the same cube is

$$\sum_{i=1}^{4} P_\varepsilon^2(i) = 0.25 + 0.01 + 0.04 + 0.04 = 0.34 .$$

Since $\Sigma P^2_\epsilon(i)$ is the probability of locating two states in the same region, a geometrical interpretation of the correlation dimension is evident in equation (7.38): D_2 measures the proportional rate of change in the probability of locating two states in the same region as the size of the region as indicated by the length ϵ is changed.

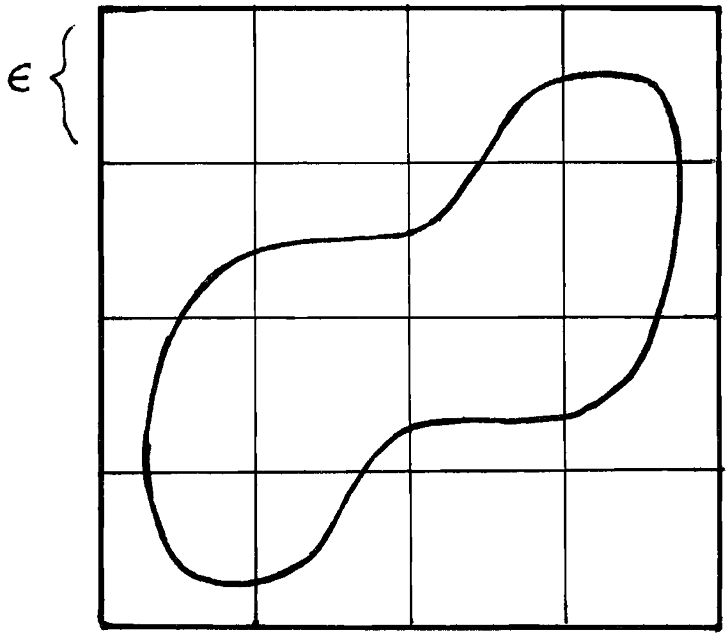


Figure 7.1: The Attractor of a 2-Dimensional System
Partitioned into Cubes of Edge Length ϵ

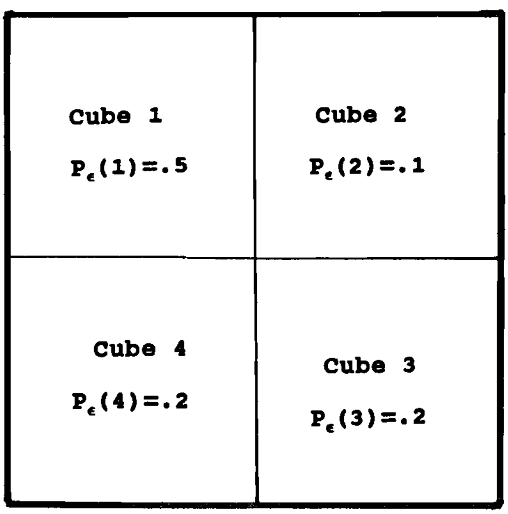


Figure 7.2: Probabilities for a 2-Dimensional System
Partitioned into 4 Cubes

The more interesting is that the geometrical and dynamical properties which the correlation integral can be used to measure are the correlation dimension. Recall that the probability of locating any two states in the same partition is given by

$$\sum_{i=1}^{M(\epsilon)} P_\epsilon^2(i),$$

where, as before, $M(\epsilon)$ denotes the number of cubes which contain a part of the attractor and $P_\epsilon(i)$ denotes the probability of locating a state in the i-th cube. Recall as well that the joint probability of locating state x_t in cube i_t, state x_{t+1} in $i_{t+1}, \ldots,$ and state x_{t+m-1} in cube i_{t+m-1} is denoted by $P_\epsilon(\{i_\tau\}_{\tau=t}^{t+m-1})$. The probability, then of finding two m-length trajectories which follow the same path (i.e., which remain in the same cubes) can be written as

$$\sum_{\substack{\text{all possible}\\ \text{trajectories}}} P_\epsilon^2\left(\{i_\tau\}_{\tau=t}^{t+m-1}\right).$$

Grassberger and Procaccia (1983) show that a relevant limiting correlation integral is proportional to the probability of locating two states which are within ϵ of each other. The authors also show that another limiting correlation integral is proportional to the probability of observing two trajectories which follow the same path through the partitioned phase space. That is, they showed that

$$C_m(\epsilon) \approx \sum_{\substack{\text{all possible}\\ \text{trajectories}}} P_\epsilon^2(\{i_\tau\}_{\tau=t}^{t+m-1}): \qquad (7.39)$$

Note for m=1, equation (7.39) becomes

$$C_1(\epsilon) = C(\epsilon) \approx \sum_{\text{all possible } (i_t)\text{'s}} P_\epsilon^2(i_t) = \sum_{i=1}^{M(\epsilon)} P_\epsilon^2(i) \qquad (7.39)'$$

The connection between the correlation integral and the correlation dimension is readily apparent from equation (7.39) and the definition of the correlation dimension given in (7.38). Clearly, since $C(\epsilon)$ is proportional to the probability of locating two states in the same region, D_2 can be expressed as

$$D_2 = \lim_{\varepsilon \to 0} \frac{\ln(KC(\varepsilon))}{\ln \varepsilon}, \tag{7.40}$$

where k denotes the constant of proportionality between $C(\epsilon)$ and the probability of locating two states in the same cube. So far, the discussion has ignored the fact that in practice the components of a system are not known. As a result, the discussion has ignored the fact how the states and thereby the trajectories are rarely observable. This complication severely restricts the kinds of systems to which the measures presented in the preceding sections can be applied. Brock and Dechert (1988) showed that the correlation dimension was a conjugacy invariant. That is, the authors showed that the correlation dimension for a system which evolves on an N-dimensional smooth manifold is N, and that the time series of observations of a single component of the system $\{a_t\}$ can be used to determine the dimension.

More generally, as shown by Grassberger and Procaccia (1983), the correlation dimension, $D_2(\{x_t\})$, associated with the process which produces the time series $\{x_t\}$ can be expressed as

$$D_2(\{x_t\}) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\ln(C_{m,n}(\varepsilon))}{\ln \varepsilon}. \tag{7.41}$$

or

$$D_2(\{x_t^m\}) = \lim_{\varepsilon \to 0} \frac{\ln C_m(\varepsilon)}{\ln \varepsilon}. \tag{7.42}$$

Note that the dimension is the limiting slope of the relation between the log of the probability $C_m(\epsilon)$ and the log of the distance ϵ. As a result, an empirical approximation of the dimension is

$$D_2(\{x_t^m\}, \varepsilon, \Delta\varepsilon) \equiv \frac{lnC_m(\varepsilon + \Delta\varepsilon) - lnC_m(\varepsilon)}{ln(\varepsilon + \Delta\varepsilon) - ln\varepsilon}. \tag{7.43}$$

Moreover, an estimate of the approximation can be expressed in terms of correlation integral:

$$\hat{D}_2(\{a_t^m\}, n, \varepsilon, \Delta\varepsilon) \equiv \frac{lnC_m(n, \varepsilon + \Delta\varepsilon) - lnC_{m,n}(\varepsilon)}{ln(\varepsilon + \Delta\varepsilon) - ln(\varepsilon)}. \tag{7.44}$$

However, some care should be taken in regard to the interpretation of such correlation dimension estimates. As demonstrated by Ramsey and Yuan (1989) and by Ramsey, Sayers, and Rothman (1988), the standard error of the estimate of the dimension is considerably influenced by such factors as the embedding dimension, m, and the sample size, T.

7.6 Correlation Dimension Results

First we calculate the correlation dimension of phase space for the sample markets. The correlation dimension of the phase space is a little different from the dimension of a time series. The phase space includes all of the variables in the system. Its is dimensionality is dependent on the complexity of the system being studied. The more important thing is that correlation dimension gives us important information about the underlying system. More precisely, the next higher integer above the dimension tells us the minimum number of variables we need to model the dynamics of the system. This also gives us a lower bound on the number of degrees of freedom in the whole system.

A pure random process, such as white noise, fills whatever space it is plotted in. Since its elements are uncorrelated and independent, white noise assumes the dimension of whatever space we place it in.

Recall that Grassberger and Procaccia (1983) have found that if we increase the value of scale parameter ϵ, $C_{m,n}(\epsilon)$, should increase at the rate of R^D, where D denotes "correlation dimension":

$$ln(C_{m,n}(\epsilon)) = D*ln(\epsilon) + constant. \qquad (7.41)'$$

By calculating the correlation integral, $C_{m,n}(\epsilon)$, for various embedding dimension,m, we can estimate D as the slope of a log/log plot of $C_{m,n}(\epsilon)$ and ϵ. Grassberger and Procaccia (1983) have also shown that as m increases, D will eventually converge to its true value. It is believed that the Grassberger and Procaccia method offers a reliable, relatively simple method for estimating correlation dimensions when only one dynamic observable variable is known. This method is data-intensive and requires a fair amount of computer time. However the results are reliable.

Figure 7.3 represents the plots of $\log[C_{m,n}(\epsilon)]$ against $\log(\epsilon)$ for embedding dimensions 2,3,5,8,12, and 20 for the detrended Swedish daily returns and for the detrended Korean daily returns.[7)] The figures show the linear regions where the regression can be performed.

Figure 7.4 shows the results of the estimates of correlation dimension (D) for various embedding dimension (m). This analysis indicates that the correlation dimension of the Swedish daily returns series exists in approximately between 5 and 6. The case is same for the Korean daily series. This means that the dynamics of the sample stock markets can be defined with a minimum of six dynamic variables. Figure 7.5 represents the comparison of convergence of correlation dimensions for the sample markets with those of pure random walk. We can easily verify that changes in returns for the sample markets are not governed by a completely random process. If they were, the correlation dimension would roughly equal the embedding dimension associated with the estimate.

We can make an international comparison regarding our findings. Concerning the market efficiency test arising out of the dynamical systems theory, a few results for the international markets have been reported. Since our studies are done only

---

7) To reduce the possible effects of serial correlations on the test results, we test to the residual series of a linear model for the returns series.

for the daily returns, here we confine the results for the daily returns. As a matter of fact, only the USA's market has been studied, based on the daily returns. At first, Scheinkman and LeBaron (1989) researched the USA's market for 5,200 + daily returns on the value-weighted portfolio of the CRSP. They have shown that patterns of past residuals help predict future ones. Also implicitly they have shown the correlation dimension lies between 5 and 7. Yang (1989) used the S&P 500 daily index covering January 1979 to December 1987. He has concluded that the hypothesis of deterministic chaos is not supported by the empirical results. However, he claimed that individual stocks had lower dimensions. Reviewing his calculated table, we might guess that the dimension of S&P daily index is above 5. Hiemstra (1990) has shown more concretely that the USA stock market has her correlation dimension between 6 and 7. Compared to these results, the sample markets' dimensions are not higher than that of the USA market. As a matter of fact, the exact estimate of correlation dimension is more or less difficult in empirical research, because it depends on the sample size and embedding dimension. Empirically more interesting thing is when we apply this model to weekly or monthly data, the dimension might be reduced.[8] This implies that the changes for weekly or monthly prices are somewhat less complex and hence their movements can be explained and forecasted more simply rather than daily movements.

Our finding is good news, since our findings mean that we could model the dynamics of the sample markets with a few variables. Moreover, they constitute low dimensional systems.[9] Of course, our findings do not tell us what the

8) Correlation Dimensions for Monthly Equity Indices

Index	Correlation Dimension
S&P 500	2.33
MSCI Japan	3.05
MSCI Germany	2.41
2MSCI U.K.	2.94

(Note) MSCI (Morgan Stanley Capital International)
(Source) Peter (1992)

9) When we speak of "low dimensional chaos," we usually think of their correlation dimensions to be substantially lower than 10, perhaps 5 or 6.

variables are. This problem will be solved in the field of neural networks (see Trippi (1992)).

To detect whether the stock markets follow really deterministic chaotic systems, we took Brock's residual test. As a first task, we removed any linear structure in the raw data. This filtering was done by fitting an autoregressive model to the returns series. We took the AR(1) model. One reason is that the AR(1) suggests the optimal lag structure for our sample series, based on the criterion of "the law of iterated logarithms(LIL)", which is reported to Table 7.1. Another reason is that Brock and Sayers (1988) have shown that the power of the BDS statistic declines as higher-order AR(p) models are applied, leading to a need of set p as low as possible. The AR(1) structures for the sample series are as follows:

For the AFGX series,

$$x_t = \underset{(9.00)}{0.176}\, x_{t-1} + \delta x_t;$$

R^2=0.028, SSR=0.277, NOBS=2534, DW=1.99, SER=0.010

$K_{\delta x}$=11.003, $SK_{\delta x}$=-0.528, $\sigma_{\delta x}$=0.010.

For the KCSPI series,

$$x_t = \underset{(8.02)}{0.136}\, x_{t-1} + \delta x_t;$$

R^2=0.018, SSR=0.412, NOBS=3374, DW=1.98, SER=0.011

$K_{\delta x}$=4.616, $SK_{\delta x}$=0.118, $\sigma_{\delta x}$=0.011.

The number in parentheses are t-statistics, SSR is the sum of squared residuals, SER is the standard error of the regression, NOBS is the number of observations and DW is the Durbin-Watson statistic. The symbols $\sigma_{\delta x}$, $K_{\delta x}$, and $SK_{\delta x}$ represent the standard deviation of δx, the kurtosis of δx, and the skewness of δx, where δx is the residual of whitened series.

Table 7.1: Information Criteria(LIL) for AR(p) Processes Modelling Daily Returns on the AFGX and the KSCPI.

lag	AFGX	XCSPI
1	6.818	9.013
2	6.820	9.018
3	6.821	9.018
4	6.824	9.020
5	6.825	9.021
6	6.827	9.024
7	6.837	9.024
8	6.838	9.026
9	6.842	9.027
10	6.845	9.028

Table 7.2 contains the results of the autocorrelation tests for the unwhitened and whitened series. All linear structure has been removed by the AR(1) process for the AFGX series, except that at lag 16, and for the KCSPI series except those at lags 2 and 25.

Figure 7.6(a) and (b), and Table 7.3 show that the estimated correlation dimensions for each series for embedding dimensions from 2 to 20. Since there is more or less apparent difference between the correlation dimensions of the original series and the AR(1) white noise residuals, we may say that the sample markets are stochastic, not chaotic. However the results of this diagnostic should be interpreted with caution. A bias has usually been shown, when the relatively smaller (less than 10,000 observations) data set is applied to this model. This bias

Table 7.2: ***Autocorrelations of the Raw and Whitened Series***

lag	index(1) (Sweden)	index(2) (Sweden)	index(1) (Rep. Korea)	index(2) (Rep. Korea)
1	0.169*	-0.001	0.134*	0.006
2	-0.000	-0.031	-0.040*	-0.058*
3	-0.002	-0.009	-0.020	-0.019
4	0.033	0.031	0.030*	0.029
5	0.024	0.022	0.018	0.018
6	-0.014	-0.034	-0.021	-0.021
7	0.080*	0.039	-0.024	-0.017
8	0.030	0.008	-0.038*	-0.029
9	0.050*	0.035	0.015	0.021
10	0.013	-0.000	0.002	-0.002
11	0.029	0.031	0.021	0.018
12	-0.010	-0.018	0.022	0.015
13	0.010	0.015	0.038*	0.029
14	-0.012	-0.018	0.039*	0.029
15	0.013	0.003	0.032*	0.028
16	0.071*	0.075*	-0.021	-0.027
17	-0.006	-0.019	0.005	0.004
18	-0.003	0.001	0.027	0.026
19	-0.021	-0.019	0.009	0.004
20	-0.012	-0.006	0.011	0.011
21	-0.014	-0.018	-0.001	-0.004
22	0.031	0.031	0.006	0.007
23	0.027	0.021	-0.000	-0.001
24	0.009	0.001	-0.000	0.005
25	0.018	0.019	-0.040*	-0.037*
26	-0.010	-0.017	-0.031*	-0.024
27	0.015	0.020	-0.014	-0.012
28	-0.009	-0.016	0.013	0.016
29	0.017	0.023	-0.006	-0.004
30	-0.017	-0.023	-0.026	-0.024

(Note): Index(1)= logged first difference of the closing value of stock index; index(2)= linear structure removed from (1) using AR(1) process; the lags are significant if the reported values are greater than or equal to the critical value. The asterisked values are significant at 5% level. The critical values are 0.04 (for Sweden) and 0.03 (for Korea).

results in estimation errors in the dimension estimates that lead to rejection of deterministic chaos, even when it is present. (Brock (1988); Hsieh (1989); Ramsey et al. (1988)).

The added audit of the BDS test for IID conditions in the data is applied to each of the four series. Generally it is believed that BDS test has its power between ϵ=0.5s.d. and ϵ=1.5s.d, where ϵ denotes the scale parameter and s.d. denotes the standard deviation of the series. Thus, we checked the BDS statistics within these bounds. Table 7.4 shows the result in case of ϵ=1.0s.d.. In summary, the conclusion is to reject the IID null hypothesis for the sample series. Also, we can see that there is considerable evidence of nonlinear structure in the residuals of the linear specification. Of course if we consider the cases of the embedding dimensions above 20, there exists high possibility of the acceptance of the null hypothesis of IID. Unfortunately at the present time, no statistical test has been developed to determine the optimal embedding dimension. Empirically embedding dimensions below 10 or 20 are usually considered for the BDS test.

7.7 Summary of Chapter

In this chapter, we dealt with the market efficiency arising from nonlinear dynamical systems theory. Since conventional statistical tests are plagued by the inability to detect nonlinear dependence, the new tests and estimates based on the nonlinear dynamical systems have been developed rapidly, and have been applied more intensively to the fields of macroeconomics and finance.

First, the recent developments and issues concerning this area were reviewed. It has been pointed out that research on the applications of nonlinear dynamical systems theory to empirical issues in macroeconomics and finance follows along two broad tracks: One track centers on recently devised tests for statistical independence which are used as diagnostics for model specification. The other track addresses issues related to the more general category of detecting and describing nonlinear structure in macroeconomic and financial data. Moreover, we classified the advances from two tracks into 3 issues:

(1) the importance of low power problems of standard (conventional) tests;

(2) identification of the source of model misspecification;

(3) detection and description of nonlinear structure.

Second, we discussed a new concept "correlation integral" which is used in the dynamical systems theory, in place of the autocorrelation concept. In particular, we focused on the statistical properties of the correlation integral- U statistics, and introduced the BDS test for detecting the nonlinear dependence.

Third, we detailed the correlation dimension measure which describes the geometric properties of a dynamical systems. It has also been explained how to find out the dimensions underlying the system for the price changes.

Finally, we have applied this model to the sample series. We have found the sample markets are not governed by a completely random process. The correlation dimensions of the sample series have been estimated. The result says that the sample markets can be described on the phase space of R^6 or R^7. As a matter of fact, this result is not greatly different from that of the USA market. To detect whether the stock markets follow really deterministic chaotic systems, we also took the Brock's residual test. The linear structure from original data was removed and then the residuals were applied to this model. We have found that the sample markets are stochastic, not chaotic. Considering the BDS statistics, we can claim that our series are surely not IID, and moreover they constitute low dimensional systems.

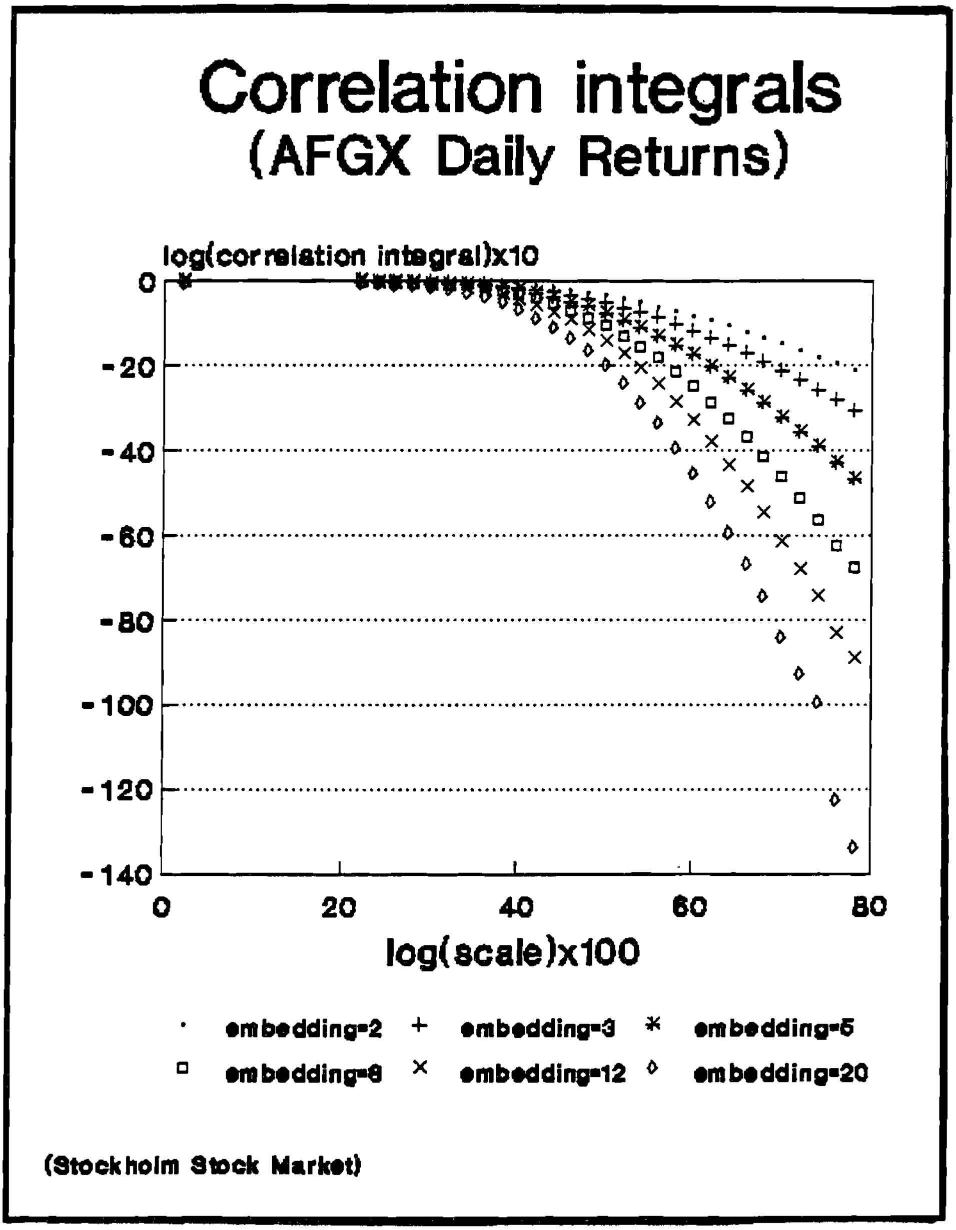


Figure 7.3(a): Correlation Integrals for the AFGX Returns

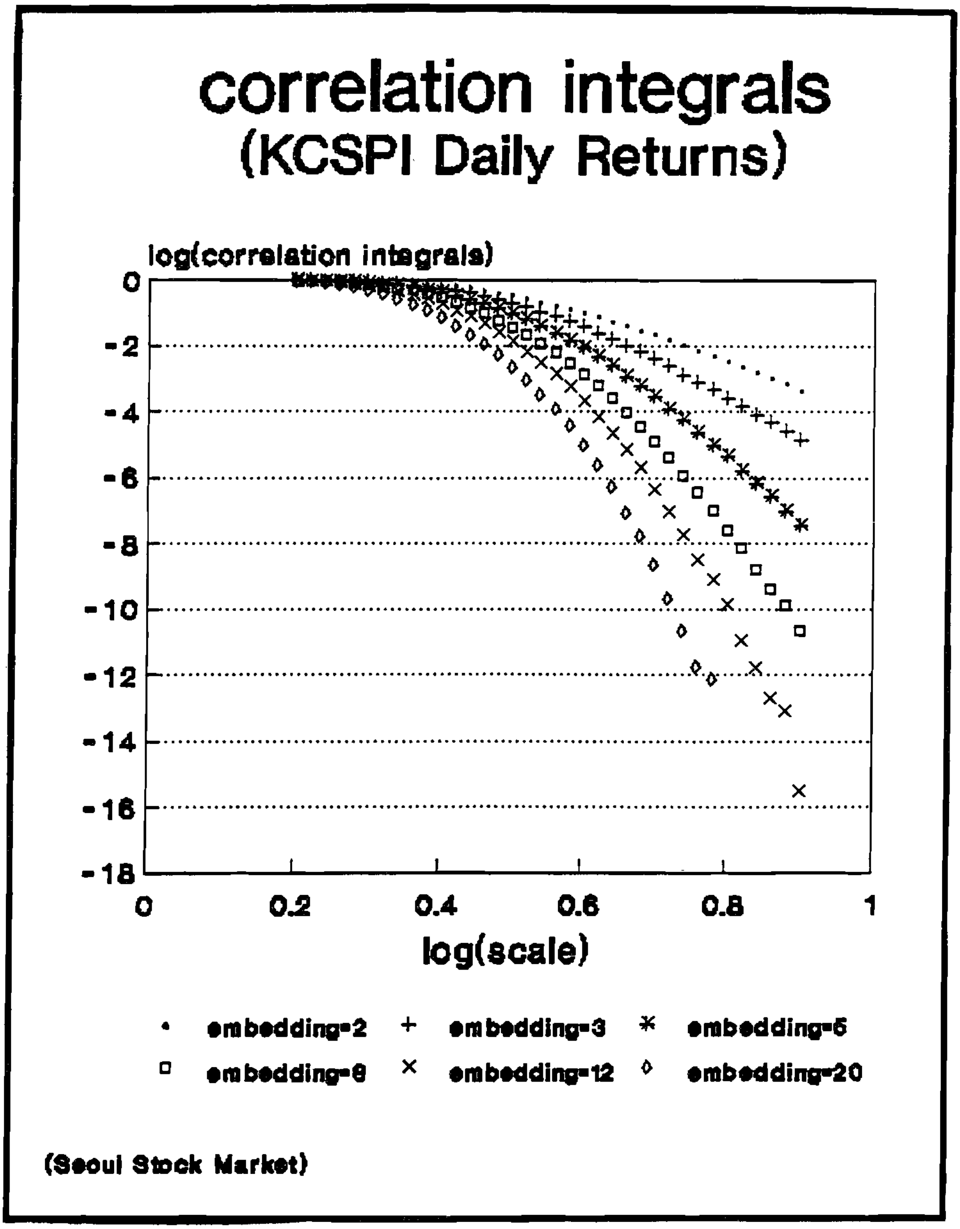


Figure 7.3(b): Correlation Integrals for the KCSPI Returns

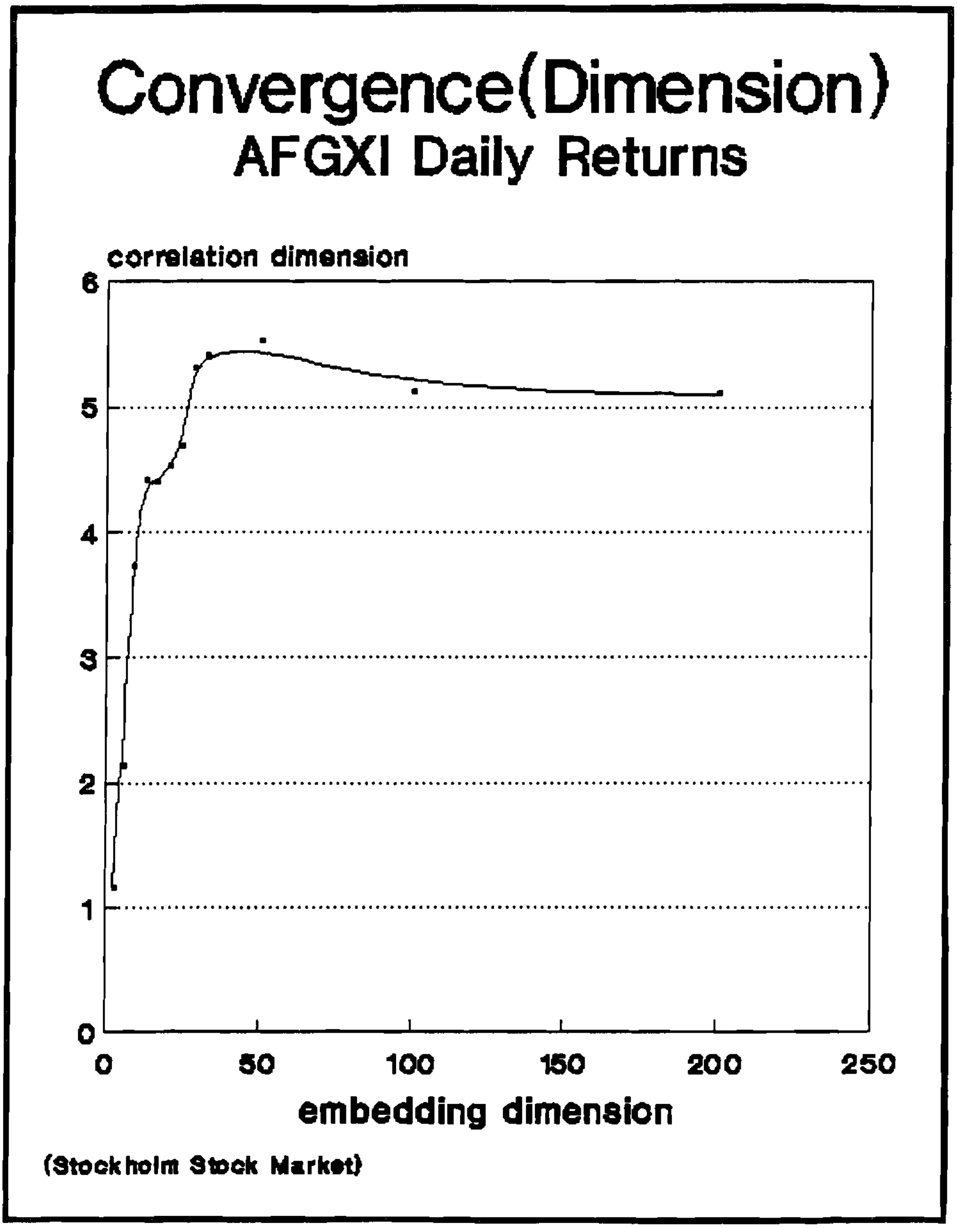


Figure 7.4(a): Convergence of Dimension for the AFGX Returns

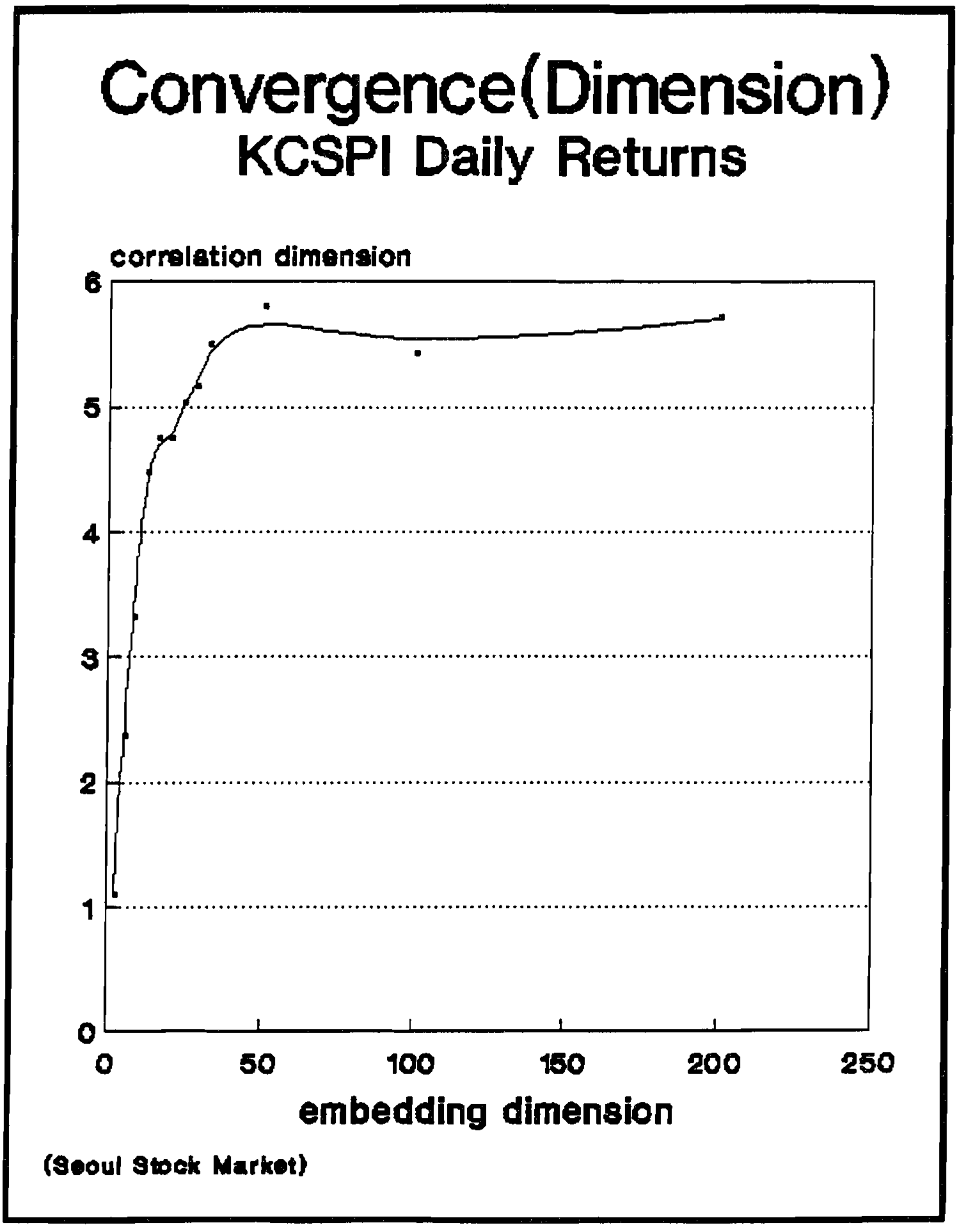


Figure 7.4(b): Convergence of Dimension for the KCSPI Returns

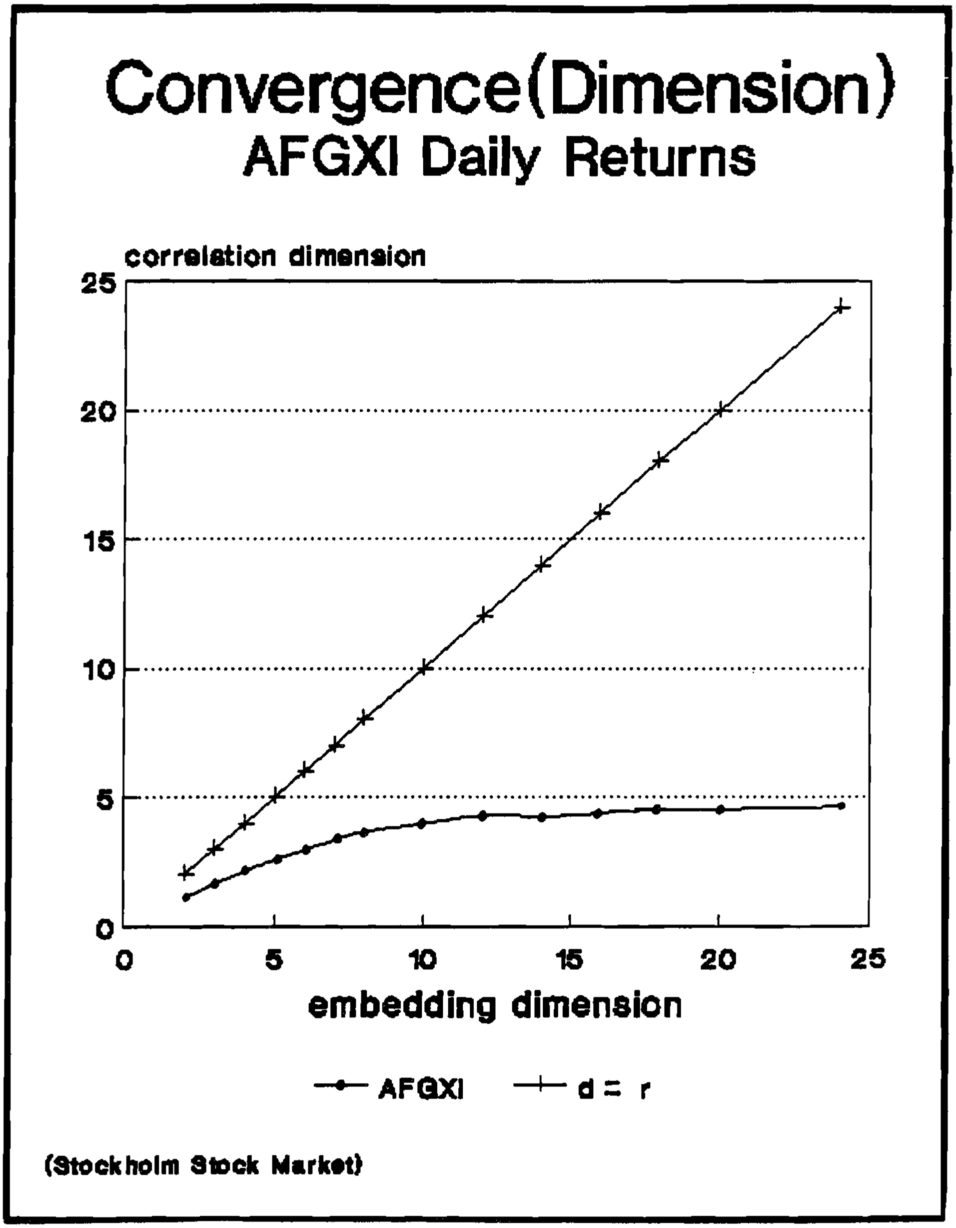


Figure 7.5(a): Comparision with Pure Random Walks in Dimension

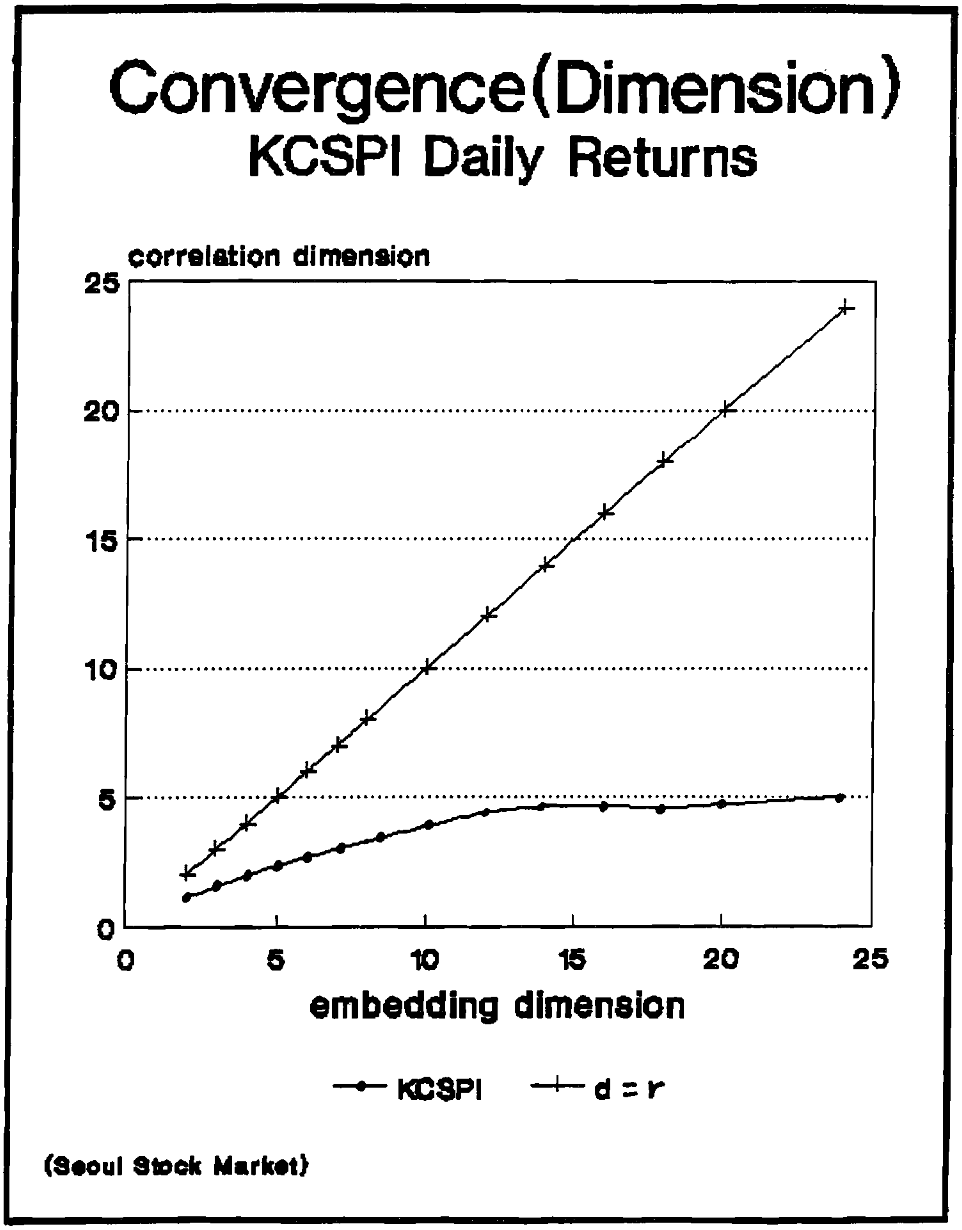


Figure 7.5(b): Comparision with Pure Random Walks in Dimension

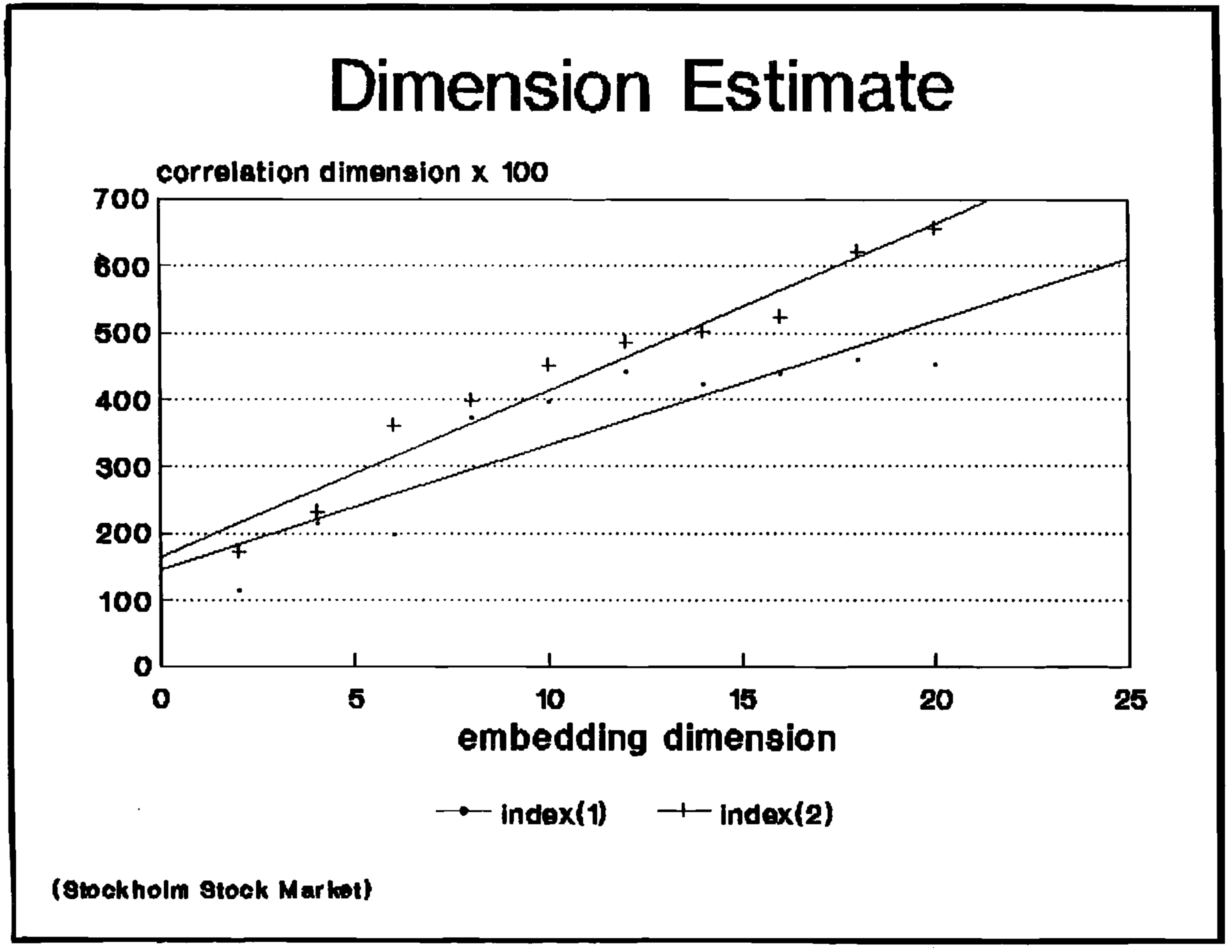


Figure 7.6(a): Comparision with Raw Series and Whitened Series in Correlation Dimension

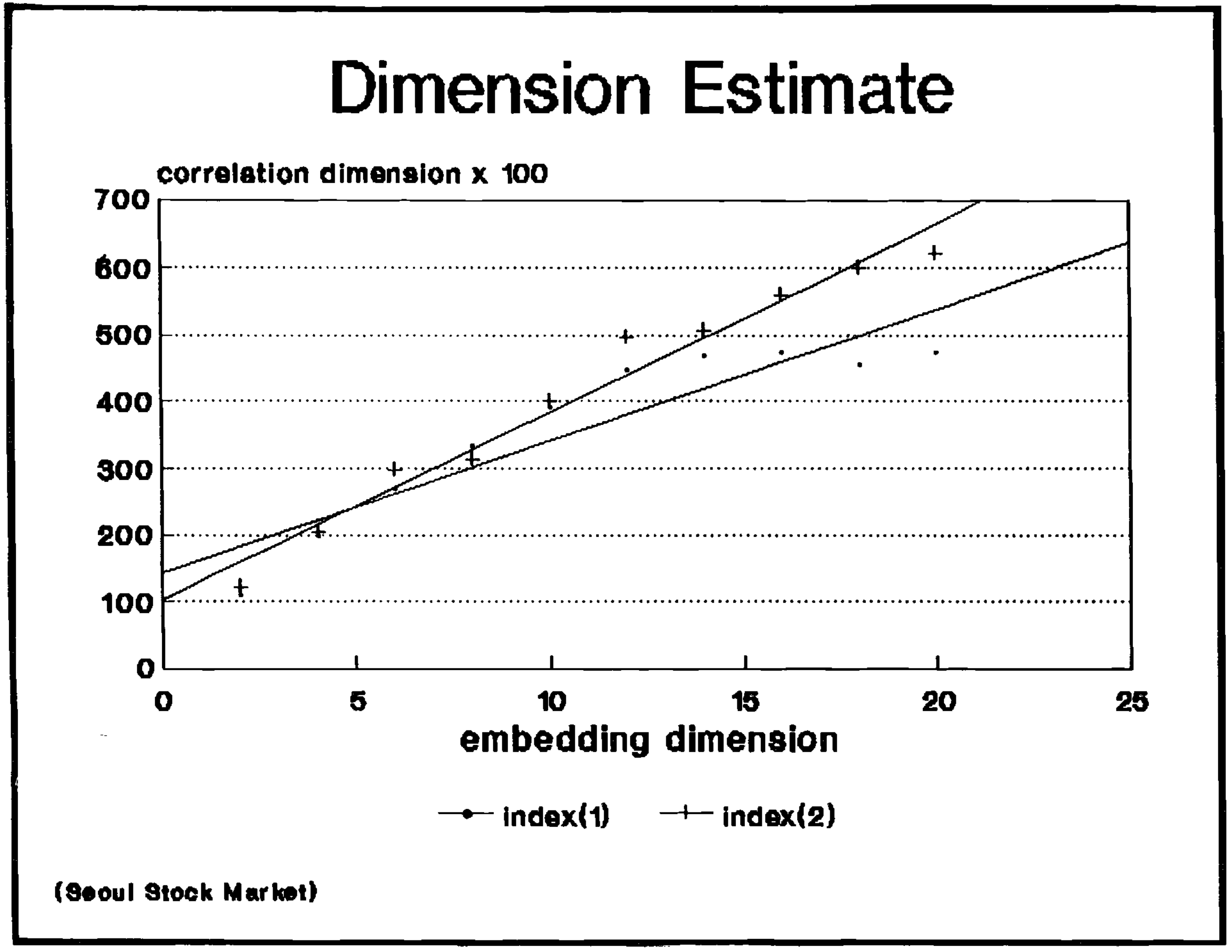


Figure 7.6(b): Comparision with Raw Series and Whitened Series in Correlation Dimension

Table 7.3: Correlation Dimension Estimates

m	AFGX series		KCSPI series	
	index(1)	index(2)	index(1)	index(2)
2	1.14	1.72	1.09	1.21
4	2.13	2.31	1.97	2.03
6	2.96	3.61	2.69	2.97
8	3.71	3.98	3.31	3.14
10	3.96	4.51	3.90	4.02
12	4.41	4.87	4.46	4.98
14	4.21	5.01	4.67	5.08
16	4.39	5.23	4.73	5.61
18	4.59	6.21	4.53	6.01
20	4.52	6.56	4.73	6.21

Table 7.4: BDS Statistics for IID of Residuals - Neighboring (Scale) Parameter - Neighboring (Scale) Parameter ϵ=1.0s.d.

Embedding Dimension (m)	AFGX		KCSPI	
	Index(1)	Index(2)	Index(1)	Index(2)
2	17.29	13.21	15.18	13.47
3	22.09	18.72	23.61	17.45
5	42.83	28.56	53.70	31.49
10	4.72	8.32	4.46	3.14
15	2.26	2.14	1.98	1.02
20	1.12	1.62	0.09	0.01

(Note) The neighbouring parameter is measured in standard deviations of the relevant series. We report absolute values of the BDS statistics. The critical value is 1.96 at the five percent level.

Chapter 8

Conclusions

8.1 Summary of Problem Statement and Procedures

The proposition that securities markets are efficient forms the basis for most research in financial economics. The efficient market hypothesis [EMH], as the cornerstone of modern finance theory, has been one of the most intensely researched topics in the field of financial economics. Even though, so far, there has appeared stronger evidence against the EMH, it has still been central to fundamental polices of investment practice and corporate financial management. Can we really believe that financial markets are efficient?

Generally, it is believed that stock index returns do not follow a Gaussian distribution. They follow a dependent and nonlinear generating process between returns. Consequently, most financial models and assumptions as well as tests, assuming an independent and identical distribution for price changes and a linear generating process between them, can mislead the results. In particular, the concept of nonlinear dependence structure in the second moment has appeared to be more important in testing the EMH in the late of 1980s.

Concerning the nonlinear dependence between returns, some dynamic models were introduced for the EMH tests. They have focused on detecting a new view of the EMH, and on verifying the nonlinear structure for the process, criticizing the conventional statistical approaches which usually assume a linear and independent process. For example, Taylor (1986) has proposed the product process model, which is effective in testing the random walk hypothesis by the nonlinear process. Bollerslev (1987) has tried to verify the nonlinear structure of generating processes for price changes by an extended GARCH model. Even some researchers (e.g., Lo (1989); Peter (1992)) have introduced a new concept taken from fractal geometry to detect a pattern and long term memory. If it exists, it is also evidence against the EMH. In fact, this kind of evidence against the EMH can be detected only under the nonlinear structure. More recently, some scientists (Li (1991); Mandelbrot (1983)) as well as some financial econometricians (Brock (1991); Scheinkman and LeBaron (1989); Hsieh (1991); Baek and Brock (1991)) have applied a new class of statistical tests and estimates arising out of the dynamical systems theory to the EMH-tests.

As far as stock markets are concerned, these new models based on the nonlinear structure have been applied to some thick markets: the USA, the UK and Japan. Most of their findings are evidence against the EMH. Our major contribution in this study is to apply these new models to two relatively small thin markets: the Swedish stock market and the Korean stock market. We propose to find answers to the following statements:

1. Are there any differences between the previous results analyzed under the linear and independent generating process and our findings?

2. Are there any differences between the results for some thick markets and our findings for the sample markets?

3. Can we really believe that the sample markets are efficient under the nonlinear and dependent generating process? If we can not, "why"?

4. If the sample markets are not efficient, can volatilities be predictable?

5. What are the suggestions for future study in this area?

There are many alternative models available. The author classified them into 4 categories, according to the utilized concept, and applied 4 models, which are taken from each category to the sample markets: the product process model (time domain and autocorrelation concept); the power law function model (frequency domain and Fourier transformation of autocorrelation concept); the R/S model (Fractals and rescaled range concept); Dynamical systems model (Chaos and correlation integral concept).

The initial data set consist of 2534 daily AFGX series for the period of January 2, 1980 to March 6, 1990 for the Swedish stock market and 3374 daily KCSPI series covering January 4,1980 to July 16, 1991 for the Korean stock market. The stock index returns are calculated as the difference in natural logarithm of the index value for two consecutive days.

8.2 Conclusions Based on Empirical Findings

Chapter 3 aims at verifying whether nonlinear dependence between returns exists for the sample series. In the basic statistical analysis, the following was observed:

(1) As expected, the average (mean) value is very small, compared to the standard deviation. This result can be one of reasons why many investors are very interested in the variation of the standard deviation than in the variation of the mean.

(2) Somewhat surprisingly to the author, the standard deviation is not much larger than on thicker markets.

(3) Under the null hypothesis that the sample returns processes $\{x_t\}$ are IID with zero mean, the t-values of the mean are significant. Hence, the average is not zero. However, the variation of the average is small, compared to the variation of the deviations.

(4) Significant excess kurtosis appeared for both markets.

(5) By the Anderson and Walker's theorem and the Lindberg-Feller Central Limit theorem, we have uncovered that the sample series show the dependence between index returns.

(6) Under the assumption that the relative efficiency of a simple predictor compared with the best predictor is not large, the sample series showed the nonlinear generating process between index returns.

In chapter 4, the efficient market hypothesis was tested under the product process model, which copes well with a nonlinear generating process. We applied this model as a tool of time domain methodologies. The basic concept for this model is still the autocorrelation coefficient. Following the theoretical discussion, we have derived the rescaled data and verified that raw index returns are not of use in testing the EMH under the nonlinear generating process (see Figure 4.1). The null hypothesis of IID was tested by some powerful tests: the first autocorrelation test, Box-Pierce test, price-trend test, and excessive response test. The test results are follows:

(1) The first autocorrelation is significant.

(2) Box-Pierce test statistic Q_{10} as well as Q_{30} rejected the null hypothesis of IID for both markets. Compared to the first lag autocorrelation coefficient, higer order autocorrelations seem to be negligible. So, we conclude that the Box-Pierce test results are largely dependent on the first autocorrelation.

(3) The price trend test also rejected the null hypothesis. However, considering the values of adjusted trend test (the price trend test which removes the first autocorrelation's effect), we can infer that there exists no really meaningful price trend alternative. The acceptance of a price trend alternative for the sample markets is due to the largeness of the first autocorrelation, not due to the market's own patterns. This observation is strengthened by (1).

(4) Concerning the excessive response alternative, the similarity to the price trend alternative can be inferred. That is, there exists no really meaningful excessive response alternative due to the market's own patterns.

In summary, this model establishes that the sample markets do not follow the IID process, even under the product process model which is characterized by the dependence and the nonlinear generation between returns. As a matter of fact, this conclusion has already been derived by the linear generating process, as shown by the previous studies (see Table 4.5). The author believes that the time domain methodologies using the autocorrelation function fail to detect long term memory and patterns, since all the test statistics are very sensitive to the relative greatness of the first lag autocorrelation, which is one of characteristics for stock price changes.

In chapter 5, we applied a frequency domain model to the sample series. First, we discussed a concept "power law function." We have focused on detecting whether slow-varying and non-periodic components exist in the sample time series. From the calculations of the power spectra, it has been found that at lower frequencies, the sample series have their peaks, which implies that the power law function model fits well to the sample markets (see Figure 5.5). Furthermore, the power exponent was calculated for the detrended samples: $\alpha=1.98$ for the Swedish stock market, and $\alpha=1.94$ for the Korean stock market. These values say that the Swedish stock market as well as the Korean stock market has noise similar to the USA market ($\alpha=1.79$).

In summary, the manifest non-trivial long range correlations could not be identified. However, an endless source of self-similarity has been found, which implies that there exist some patterns. Since the power spectrum is the Fourier transformation of the autocorrelation function, we applied this model as a tool of frequency methodologies using the autocorrelation concept.

In chapter 6, we tried to uncover the types of patterns (memory) by R/S analysis, which is powerful in detecting the long term dependence (memory). As a matter of fact, the finding of long memory components in asset returns has very important implications, since many of the popular paradigms in modern financial economics assume that there exist no patterns or memory. Before applying the Hurst's rescaled range (R/S) analysis, we detailed a basic concept, fractional Brownian motion, and furthermore showed some simulation results on the fractional Brownian noise to understand this concept geometrically. It has been verified that the sample series have persistent patterns. This result says that an increasing trend in the past implies on the average a continued increase in the future. This result is also similar to the USA market. To check the sensitivity of short lags autocorrelations' effects, we re-applied Lo's modified R/S model to the samples. It has been found that the long dependence model is effective within only three months for the Korean market, while there exists no evidence of long term dependence for the Swedish stock market. Considering calculated H's coefficients, it would be difficult to attempt to find "deterministic chaos" and to forecast the stock markets over the short term for the sample series, since there are a good deal of noise. To be sure, there exist persistent patterns in both markets and the patterns hold for a longer time on the Korean stock market as compared to the Swedish stock market.

In chapter 7, we presented the EMH arising from nonlinear dynamical systems theory. In fact, the above mentioned models, excluding the R/S analysis, are based on the autocorrelation concept. As shown by Sakai and Tokumaru (1980), the autocorrelation concept sometimes misleads the results. Furthermore, many economic time series decay rather slowly to zero, which has already proved to be true for the samples in chapter 5. Hence, we need to specify the nonlinear structure for the sample markets in detail. First, we discussed the weakness of the conventional (standard) statistical tests based on the autocorrelation distribution:

(1) They have low power in detecting the nonlinear dependence between returns.

(2) They sometimes mislead the identification of model specifications.

(3) They cannot detect and describe the nonlinear structure between returns.

Second, we detailed a new concept "correlation integral." In particular, we focused on the statistical properties of the correlation integral-U statistics and its connection to the dimension measures. Then, the BDS test was introduced, which is of great use in testing the IID hypothesis under the nonlinear process. By applying this model to the sample series, we have uncovered the following:

(1) The sample markets have a low dimension structure. The sample markets' movements for daily price changes can be described with only six or seven independent variables.

(2) The sample series are not chaotic, but stochastic. This result is consistent with the findings in chapter 6.

Results of the present research provide an important addition to knowledge about the market efficiency and price behavior in stock markets. Now, we are ready to answer the statement of problems.

1. Are there any differences between the previous results analyzed under the linear and independent generating process and our findings?

   (Yes) The previous results analyzed under the linear and independent process for the sample markets indicate only two remarks: significant first lag autocorrelation, and dependence over daily intervals. However, even though we observed the same results under a time domain, we verified persistent patterns and low dimension structure between returns under a frequency domain and Fractals. These were not found under the linear process.

2. Are there any differences between the results for some thick markets and our findings for the samples?

(No) Confining the discussion to the above-mentioned models, we can see that there might be no deep difference for daily stock price changes between the sample markets and the USA market. They have, in common, the same persistent patterns; they have the significant first lag autocorrelation; the series constitute low dimensional structure around 6 or 7; furthermore the series seem to be stochastic, not chaotic.

3. Can we really believe that the sample markets are efficient under the nonlinear and dependent generating process? If we can not, "Why"?

(No) Assuming that the rejection of the random walk hypothesis or uncovering patterns and memory indicates market inefficiency, we can say that the sample markets are not efficient. The reasons are as follows: there exist significant lag autocorrelations; there exist persistent patterns (memory); the hypothesis of IID was rejected below 10 embedding dimensions by the BDS test. Note that our findings can be treated as indications of the failure of models specifying equilibrium returns, rather than as evidence against the efficient market hypothesis.

4. If the sample markets are not efficient, can volatilities be predictable?

(Yes) We have noticed that the sample series fit the power law function well and that the sample series follow the $1/f^2$-noise, not the $1/f^2$-noise, not $1/f$-noise. As shown in Figure 5.2, $1/f^2$-noise is more correlated. Furthermore, we have verified that the sample stock markets follow persistent patterns, and also observed by the correlation dimension that the sample markets constitute relatively low dimensional structure. We can describe the movements of price changes in the space of six or seven dimensions. This is relatively simple space compared to the space for purely random processes. Hence, by the technique of neural networks or self-organized regression models, we can establish a forecasting model for the volatilities.

5. What are the suggestions for future study in this area?

While answering the above statements, we have uncovered the serial dependence, nonlinear dependence, long memory and patterns, time-varying variance, and low dimensional structure, compared to the pure random process for the stock markets. Serial dependence in daily stock price changes immediately invalidates the random walk model that assumes IID price changes. Nonlinear dependence also rejects the martingale efficient markets model since past daily returns help to predict future ones. Moreover, evidence of the power law function and the low dimension structure implies that the variability in price changes is changing over time with some patterns and that it can be predictable. Thus, as a future study we propose the following:

(i) If we want to keep supporting the market efficiency, we should develop the concept of market efficiency accounting for this nonlinearity

(ii) Technical trading systems need be designed to exploit this nonlinear dependence and low dimensional structure.

(iii) The finding of the time-varying variance for the stock markets implies that a stock index option pricing model should allow the variance to change stochastically as in the study of Hull and White (1987). However, such stochastic option pricing models could not account for conditional non-normality. Moreover, evidence of conditional leptokurtosis and long memory and patterns implies the inapplicability of the Gauss-Wiener process assumed in deriving the pricing model. A correct financial pricing model should cope with the changing variances and the non-normality as well as the patterns. It remains as a future research.

(iv) Parametric tests for deterministic chaos are needed. The nonparametric residual test and the shuffling diagnostics were quite arbitrary. Even though the BDS test also has power against a nonlinear deterministic system, a more powerful parametric test is essential to confirm the validity of the stochastic approach to determine the returns generating process in speculative markets.

(v) Lower dimensionality simply implies that the process can be described by a few nonlinear factors. Identifying those factors among many state variables in an economy is not easy. Unless we can identify the process, tests for chaos may not be worthwhile. Thus, much further research should be focused on modelling a chaotic process by the technique of neural networks or self-organized regression models, and on studying the econometric properties of the chaotic process.

Bibliography

Abel, A., "Stock Prices Under Time-Varying Dividend Risk: An Exact Solution in an Infinite-Horizon General Equilibrium Model," Journal of Monetary Economics 22(1988), 375-394.

Abel, A. and F. Mishkin, "An Integrated View of Tests of Rationality, Market Efficiency and the Short-Run Neutrality of Monetary Policy," Journal of Monetary Economics 11(1983), 3-24.

Akgiray, V., "Conditional Heteroskedasticity in Time Series of Stock Returns," Journal of Business 62(1989), 55-80.

Akgiray, V. and G. Booth, "Stock Price Processes with Discontinuous Time Path: an Empirical Examination," The Financial Review 21(1986), 163-184.

Akgiray, V. and G. Booth, "Mixed Diffusion-Jump Process Modeling of Exchange Rate Movements," The Review of Economics and Statistics Feb.(1988), 631-637.

Alexander, S.S., "Price Movements in Speculative Markets: Trends or Random Walks?," Industrial Management Review of MIT, II(1961), 7-26.

Ali, M.M. and C. Giaccotto, "The Identical Distribution Hypothesis for Stock Market Prices," Journal of American Statistical Association 77(1982), 19-28.

Anderson, R.W., "Some Determinants of the Volatility of Futures Prices," Journal of Futures Markets 5(1985), 332-348.

Anderson, T.W., The Statistical Analysis of Time-Series, Wiley, New York, 1971.

Anderson, T.W. and A.M. Walker, "On the Asymptotic Distribution of the Autocorrelations of a Sample from a Linear Stochastic Process," Annals of Mathematical Statistics 35(1964), 1296-303.

Attanasio, O.P., "Risk Time Varying Second Moments and Market Efficiency," Review of Economic Studies 58(1991), 479-494.

Bachelier, L., "Theorie de la Speculation," in P.Cootner (ed.) Random Character of Stock Market Prices, MIT, Cambridge, 1964.

Baek, E. and W. Brock, "A Nonparametric Test for Temporal Dependence in a Vector of Time Series," mimeo(1988), University of Wisconsin.

Baek, E. and W. Brock, "Some Theory for Statistical Inference for Nonlinear Science," mimeo(1991), Santa Fe Institute.

Bak, P., C. Tang and K. Wiesenfeld, "Self-Organized Criticality," Physical Review A. 38(1988), 364-374.

Barnett, W. and P. Chen, "The Aggregation-Theoretic Monetary Aggregates are Chaotic and Have Strange Attractors," in (ed.) Barnett,W. Dynamic Econometric Modelling, Cambridge University Press, Cambridge, 1987.

Baumol, W.J. and J. Benhabib, "Chaos: significance, mechanism and economic applications," Journal of Economic Perspectives 3(1989), 77-105.

Benhabib, J and R.H. Day, "Characterization of Erratic Dynamics in the Overlapping Generations Model," Journal of Economic Dynamics and Control 4(1982), 37-55.

Bera, A. and M.L. Higgins, "A Test for Conditional Heteroskedasticity in Time Series Models," mimeo(1991), University of Illinois, Champaign.

Blanchard, O.J., "Speculative Bubbles, Crashes and Rational Expectations," Economic Letters (3) 1979, 387-389.

Blattberg, R.G. and N.J. Gonedes, "A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices," Journal of Business 47(1974), 244-280.

Bollerslev, T., "Generalized Autoregressive Conditional Heteroskedasticity," Journal of Econometrics 31(1986), 307-327.

Bollerslev, T., "A Conditionally Heteroskedastic Time Series Model for Speculative Prices and Rates of Returns," The Review of Economics and Statistics 69(1987), 542-547.

Booth, G., F. Kaen, and F. Koveos, "R/S Analysis of Foreign Exchange Rates under Two International Monetary Regimes," Journal of Monetary Economics 10(1982), 407-415.

Box, G.E. and J.M. Jenkins, Time Series Analysis: Forecasting and Control, Holden-Day, San Francisco, 1976.

Brillinger, D. and M. Rosenblatt, "Asymptotic Theory of Kth Order Spectra," in B . Harris, (ed.) Spectral Analysis of Time Series, John Wiley, New york, 1967, 153-188.

Brock, W.A., "Distinguishing Random and Deterministic Systems: Abridged Version," Journal of Economic Theory 40(1986), 168-195.

Brock, W.A.,"Notes on Nuisance Parameter Problems in BDS Type Tests for IID," mimeo(1987), University of Wisconsin.

Brock, W.A.,"Nonlinearity and Complex Dynamics in Economics and Finance," in Anderson, P., K. Arrow, and D. Pines (eds.), The Economy as a Evolving Complex System, Addison-Wesley 1988.

Brock, W.A., "Understanding Macroeconomic Time Series Using Complex Systems Theory," Structural Change and Economic Dynamics 2(1991), 119-141.

Brock, W.A. and W.D. Dechert, "Theorems on Distinguishing Deterministic from Random Systems," in Barnett, W.A., E.R. Bernd and H. White (eds.) Dynamic Econometric Modeling, Cambridge University Press, Cambridge, 1988.

Brock, W.A., W.D. Dechert, and J.A. Scheinkman, "A Test for Independence Based on the Correlation Dimension," mimeo(1987), University of Wisconsin.

Brock, W.A., W.D. Dechert, J.A. Scheinkman and B. LeBaron, "A Test for Independence Based on the Correlation Dimension," mimeo(1991), University of Wisconsin.

Brock, W.A., D.A. Hsieh, and B. LeBaron, Nonlinear Dynamics, Chaos, and Instability, MIT press, Cambridge, 1991.

Brock, W.A. and A.G. Malliaris, Differential Equations, Stability and Chaos in Dynamic Economics, North-Holland, Amsterdam, 1989.

Brock, W.A. and C. Sayers, "Is the Business Cycle Characterized by Deterministic Chaos?," Journal of Monetary Economics 22(1988), 71-90.

Brorsen, B.W. and S.H. Irwin, "Futures Funds and Price Volatility," Review of Futures Markets 6(1987), 118-138.

Camerer, C., "Bubbles and Fads in Asset Prices: A Review of Theory and Evidence," mimeo(1987), University of Pennsylvania.

Campbell, J.Y. and R. Shiller, "Cointegration and Tests of Present Value Models," Journal of Political Economy 95(1987), 1062-88.

Campbell, J.Y. and R. Shiller, "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors," Review Financial Studies 1(1988), 195-228.

Cargill, T.F. and G. Rausser, "Temporal Behavior in Commodity Futures Markets," Journal of Finance 30(1975), 1043-1053.

Chang, E.and C.W. Kim, "Day of the Week Effects and Commodity Price Changes," Journal of Futures Markets (1988), 229-241.

Chang, E. and J.M. Pinegar, "Seasonal Fluctuations in Industrial Production and Stock Market Seasonals," Journal of Financial and Quantitative Analysis 26(1988), 229-241.

Chiang, R.C. and T.C. Tapley, "Day-of-the-Week Effects and the Futures Market," Journal of Futures Markets 2(1983), 356-410.

Claesson, K., Effektiviteten på Stockholms Fondbörs, Ph.D. dissertation, Stockholm School of Economics, 1987.

Clark, P.K., "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices," Econometrica 41(1973), 135-155.

Corhay, A. and A.T. Rad, "Conditional Heteroskedasticity in Stock Returns: International Evidence," mimeo(1991), Limburg University.

Copeland, T.E. and D. Mayers, "The Value Line Enigma [1965-1978]: A Case Study of Performance Evaluation Issues," Journal of Financial Economics 10(1982), 289-321.

Cross, F., "The Behavior of Stock Prices on Fridays and Mondays," Financial Analysts Journal 29(1973), 67-69.

Crutchfield, J.P. and B.P. McNamara, "Equations of Motion from a Data Series," Complex Systems 1(1987), 417-452.

Davidson, L. and R.T. Froyen, "Monetary Policy and Stock Returns: Are Stock Markets Efficient?," Review, Federal Reserve Bank of St.Louis, March(1982), 3-12.

Davies, P. The Cosmic Blueprint, Urwin, London, 1987.

Day, R.H., "Irregular Growth Cycles," The American Economic Review, June(1982), 407-414.

DeBondt, W.F., F.M. Werner, and R. Thaler, "Does the Stock Market Overreact?," Journal of Finance 40(1985) 793-808.

DeBondt, W.F., F.M. Werner, and R. Thaler, "Further Evidence on Investor Overaction and Stock Market Seasonality," Journal of Finance 42(1987), 557-581.

De Long, J.B., A. Schleifer, L. Summers and R. Waldman, "The Economic Consequences of Noise Traders," mimeo(1987), NBER.

Denker, M. and G. Keller, "On U-Statistics and Von.Mises Statistics for Weakly Dependent Processes," Z. Wahrscheinlichkeitstheorie Verw. Gebiete 64(1983), 505-522.

Diebold, F., J. Im, and C.J. Lee, "Conditional Heteroskedasticity in the Market Model," mimeo(1988), Federal Reserve Board.

Dubuc, B., et.al., "Evaluating the Fractal Dimension of Profiles," Physical Review A 39(1989), 1500-1512.

Dusak, K., "Futures Trading and Investor Returns," Journal of Political Economy (1973) 1387-1406.

Eckmann, J.B. and D. Ruelle, "Ergodic Theory of Chaos and Strange Attractors," Reviews of Modern Physics 57(1985), 617-656.

Engle, Robert F., "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation," Econometrica 50(1982), 987-1007.

Engle, R.F. and C.W. Granger, "Co-integration and Error Correction," Econometrica 55(1987), 251-276.

Engle, R.F. and C. Mustafa, "Implied ARCH Models from Options Prices," Journal of Econometrics 52(1992), 289-311.

Epps, T.W. and M.L. Epps, "The Stochastic Dependence of Security Price Changes and Transaction Volumes," Econometrica 44(1976), 305-322.

Fabozzi, F.J. and J. Francis, "Beta As a Random Coefficients," Journal of Financial and Quantitative Analysis (1978) 101-116.

Fama, E.F., "The Behavior of Stock Market Prices," Journal of Business 38(1965), 34-105.

Fama, E.F., "Efficient Capital Markets: A Review of Theory and Empirical Work," Journal of Finance 25(1970), 383-417.

Fama, E.F., Foundations of Finance, Basic Books, New York, 1976.

Fama, E.F., "Term-Structure Forecasts of Interest Rates, Inflation, and Real Returns," Journal of Monetary Economics 25(1990), 59-76.

Fama, E.F., "Efficient Capital Markets: II," Journal of Finance 46(1991), 1575-1617.

Fama, E.F. and K. French, "Permanent and Temporary Components of Stock Prices," Journal of Political Economy 96(1988a), 246-73.

Fama, E.F. and K. French, "Divident Yield and Expected Stock Returns," Journal of Financial Economics 22(1988b), 3-26.

Fama, E.F. and K. French, "Business Conditions and Expected Returns on Stocks and Bonds," Journal of Financial Economics 25(1989), 23-49.

Farmer, J.D., "Chaotic Attractors of an Infinite Dimensional Dynamical Systems," Physica D. 4(1982), 366-393.

Farmer, J.D. and J. Sidorowich, "Exploiting Chaos to Predict the Future and Reduce Noise," mimeo(1988), Los Alamos.

Feder, J., Fractals, Plenum, New York, 1988.

Feller, W., An Introduction to Probability Theory and Its Applications, Wiley, New York, 1966.

Fischer, S. and R. Merton, "Macroeconomics and Finance," Carnegie-Rochester Conference Series on Public Policy, Vol.21, 1984.

Flannery, M.J. and A. Protopapadakis, "From T-Bills to Common Stocks," Journal of Finance (1988), 431-450.

Frank, M.Z. and T. Stengos, "Some Evidence Concerning Macroeconomic Chaos,"Journal of Monetary Economics 22(1988), 423-438.

Frank, M.Z., C. Sayers, and T. Stengos, "Evidence Concerning Nonlinear Structure in Provincial Unemployment Rates," mimeo(1989), University of British Columbia.

Frank, M.Z. and T. Stengos, "Measuring the Strangeness of Gold and Silver Rates of Returns," Review of Economic Studies 56(1989), 553-568.

French, K., "Stock Returns and the Weekend Effect," Journal of Financial Economics 8(1980), 55-69.

French, K., G. Schwert, and R. Stambaugh, "Expected Stock Returns and Volatility," Journal of Financial Economics 9(1987), 3-29.

Friedman, D. and S. Vandersteel, "Short-Run Fluctuations in Foreign Exchange Rates," Journal of Int. Economics 13(1982), 171-186.

Garber, P., "Tulipmania," Journal of Political Economy 97(1989), 535-560.

Gibbons, M and P. Hess, "Day of the Week Effects and Asset Returns," Journal of Business 54(1981), 579-596.

Grandmont, J., "On Endogenous Business Cycles," Econometrica 53(1985), 995-1045.

Grandmont, J. and P. Malgrange, "Nonlinear Economic Dynamics: Introduction," Journal of Economic Theory 40(1986), 3-12.

Granger, C.W.J., "The Typical Spectral Shape of an Economic Variable," Econometrica 34(1966), 150-161.

Granger, C.W.J. and P. Anderson, An Introduction to Bilinear Time Series Models, Göttingen, 1978.

Granger, C.W.J. and M. Hatanaka, Spectral Analysis of Economic Time Series, Princeton University, Princeton, 1964.

Granger, C.W.J. and O. Morgenstern, "Predictability of Stock Market Prices," Lexington, Massachusetts, 1970.

Granger, C.W.J. and P. Newbold, "Forecasting Transformed Series," Journal of Royal Statistical Society 38B(1976), 189-203.

Grassberger, P and I. Procaccia, "Measuring the Strangeness of Strange Attractors," Physica D. 9(1983), 189-208.

Greene, M. and D. Fielitz, "Long Term Dependence in Common Stock Returns," Journal of Financial Economics 4(1977), 339-349.

Hamilton, J.D., "On Testing for Self-Fullfilling Speculative Price Bubbles, "International Economic Review 27(1986), 545-552.

Harvey, A.C., The Economic Analysis of Time Series, Wiley, Oxford, 1981.

Harvey, A.C., Forecasting, Structural Time Series Models and the Kalman Filter, Cambridge University Press, Cambridge, 1989.

Haubrich, J. and A. Lo, "The Sources and Nature of Long Term Dependence in the Business Cycle," mimeo(1989), NBER.

Hayek, F., Monetary Theory of the Trade, Harcourt, New York, 1933.

Helms, B.P., F.Kaen, and R. Rosenman, "Memory in Commodity Futures Contracts," Journal of Futures Markets 4 (1984), 559-567.

Hiemstra, C., "Chaos, Correlation, and Complex Dynamics," mimeo (1990), University of Maryland, College Park.

Hinich, M. and D. Patterson, "Linear Versus Nonlinear Macroeconomies: A Statistical Test," International Economic Review 30(1989), 685-704.

Hinich, M. and D. Patterson, "A New Diagnostic Test of Model Inadequacy Which Uses the Martingale Difference Criterion," Journal of Time Series Analysis 13(1992), 13-30.

Hoeffding, W., "A Class of Statistics with Asymptotically Normal Distribution," Journal of the American Statistical Association 58(1948), 13-30.

Hoffman, D., S. Low, and D. Schlagenhauf, "Tests of Rationality, Neutrality, and Market Efficiency," Journal of Monetary Economics 14(1984), 339-363.

Holloway, C., "A Note on Testing an Aggressive Investment Strategy Using Value Line Ranks," Journal of Finance June(1981), 711-719.

Homa, K. and D. Jaffee, "The Supply of Money and Common Stock Prices," Journal of Finance December(1971), 1056-1066.

Hsieh, D., "The Statistical Properties of Daily Foreign Exchange Rates," Journal of International Economics 24(1989), 339-368.

Hsieh, D., "Testing for Nonlinear Dynamics," Journal of Finance (1991), 1839-1877.

Hsieh, D. and B. LeBaron, "Finite Sample Properties of the BDS Statistics," mimeo(1988), University of Chicago.

Hull, J.and A. White, "The Pricing of Options on Assets with Stochastic Volatilities," Journal of Finance 62(1987), 281-30.

Hurst, H., "Methods of Using Long Term Storage in Reservoirs," Proceedings of the Institute of Civil Engineers 1(1956), 519-543.

Jegadeesh, N, "Evidence of Predictable Behavior of Security Returns," mimeo(1988), UCLA.

Jennergren, L.P.,"Filter Tests of Swedish Share Prices," in Elton, E. and M. Gruber (eds.) International Capital Markets, North Holland, Amsterdam, 1975.

Jensen, M., "Some Anomalous Evidence Regading Market Efficiency," Journal of Financial Economics 6(1978), 95-101.

Kariya, T., Price Change Movements in the Japanese Stock Market, (in japanese), Toyang Press, Tokyo, 1989.

Keim, D., "Size-Related Anomalies and Stock Return Seasonality," Journal of Financial Economics June(1983), 13-22.

Keim, D. and R. Stambaugh, "A Further Investigation of the Weekend Effect in Stock Returns," Journal of Finance 39(1984), 819-835.

Kenyon, D.K., J. Kling, W. Jordan, and N. McCabe, "Factors Affecting Agricultural Futures Price Variance," Journal of Futures Markets (1987), 73-91.

Keynes, J., The General Theory of Employment, Interest, and Money, Macmillan, London, 1936.

Kim, M., C. Nelson, and R. Startz, "Mean Reversion in Stock Prices?" Review of Economic Studies 58(1991), 515-528.

Kleidon, A.W., "Variance Bounds Tests and Stock Price Valuation Models," Journal of Political Economy Oct.(1986), 953-1001.

Kon, S., "Models of Stock Returns," Journal of Finance 39(1984), 147-165.

Krasker, W., "The Peso Problem in Testing the Efficiency of Forward Exchange Markets," Journal of Monetary Economics 6(1980), 269-276.

Landry, L. and Y. Lepage, "Empirical Behavior of Some Tests for Normality," Commun. Statist.-Simula. 21(1992), 971-999.

Lee, S., Test of Weak Form Stock Market Efficiency on the Korea Stock Exchange, Ph.D. dissertation, Kent University, 1989.

Lehmann, B., "Fads, Martingales and Market Efficiency," mimeo (1987), Columbia University.

LeRoy, S. and R. Porter, "The Present Value Relation: Tests Based on Implied Variance Bounds," Econometrica 45(1981), 555-74.

Leuthold, R.M., "Random Walks and Price Trends," Journal of Finance 27(1972), 879-889.

Li, W., "Absence of $1/f$ spectra in Dow Jones Daily Average," mimeo(1991), Rockefeller University.

Lo, A., "A Long Memory in Stock Market Prices," mimeo(1988), University of Pennsylvania.

Lo, A., "The Size and Power of the Variance Ratio Test in Finite Samples,"Journal of Econometrics 40(1989), 203-238.

Lo, A. and A. McKinley, "Stock Market Prices Do Not Follow Random Walks," mimeo(1987), University of Pennsylvania.

Lukac, L.P., B. Brorsen, and S. Irwin, "A Test of Futures Market Disequilibrium Using Twelve Different Technical Trading Systems," Applied Economics 20(1988), 623-639.

Mandelbrot, B., "The Variation of Certain Speculative Prices," Journal of Business 36(1963), 394-419.

Mandelbrot, B., "When Can Price Be Arbitraged Efficiently?" Review of Economics and Statistics 53(1971), 225-236.

Mandelbrot, B., "Statistical Methodology for Non-Periodic Cycles: From the Covariance to R/S Analysis," Annals of Economic and Social Measurement 1(1972), 259-290.

Mandelbrot, B., "Limit Theorems on the Self-Normalized Range for Weakly and Strongly Dependent Process," Z. Wahrscheinlichkeitstheorie Verw. Gebiete 31(1975), 271-285.

Mandelbrot, B., The Fractal Geometry of Nature, Freeman, New York, 1983.

Mandelbrot, B. and J. Van Ness, "Fractional Brownian Motion, Fractional Noises and Applications," SIAM Review 10(1968), 422-437.

Mandelbrot, B. and J. Wallis, "Noah, Joseph and Operational Hydrology," Water Resource Research 4(1968), 909-918.

Mandelbrot, B. and J. Wallis, "Computer Experiments with Fractional Gaussian Noises," Water Resource Research 5(1969), 228-267.

Marsh, T. and R. Merton, "Dividend Variability and Variance Bounds Tests for the Rationality of Stock Prices," The American Economic Review June(1986), 483-498.

McCulloch, J., "Interest-Risk Sensitive Deposit Insurance Premia Stable ARCH Estimates," Journal of Banking and Finance 9(1985), 137-156.

McFarland, J., R. Pettit and S. Sung, "The Distribution of Foreign Exchange Price Changes," Journal of Finance 37(1982), 693-715.

McQueen, G., "Long-Horizon Mean-Reverting Stock Prices Revisited," Journal of Financial and Quantitative Analysis 27(1992), 1-18.

Merton, R.,"Option Pricing When Underlying Stock Returns are Discontinuous," Journal of Financial Economics 3(1976), 125-44.

Milonas,N., "Price Variability and the Maturity Effect in Futures Markets," Journal of Futures Markets (1986), 443-460.

Mishkin, F., "Efficient Market Theory: Implications for Monetary Policy," Brookings Papers on Economic Activity 3(1978), 707-52.

Montroll, E.W. and M. Schlesinger, "On the Wonderful World of Random Walks," in Lebowl, J. and E. Montroll (eds.) Nonequilibrium Phenomena II, North Holland, Amsterdam, 1984.

Newey, W. K and K.D. West, "A Simple Positive Definite Heteroskedasticity and Autocorrelation Consistent Covariance Matrix," Econometrica 55(1987), 703-705.

Näslund, B., "Deterministic Chaos and Market Volatility," mimeo (1988), Stockholm School of Economics.

Officer, R., "The Distribution of Stock Returns," Journal of the American Statistical Association 67(1972), 807-12.

Oldfield, G., R. Rogalski and R.A. Jarrow, "An Autoregressive Jump Process for Common Stock Returns," Journal of Financial Economics 5(1977), 211-221.

Pearce, D.K., "Challenges to the Concept of Stock Market Efficiency," Economic Review 72(1987), 16-23.

Peitgen, H.O. and D. Saupe, The Science of Fractal Images, Springer-Verlag, New York, 1988.

Perry, P., "More Evidence on the Nature of the Distribution of Security Returns," Journal of Financial and Quantitative Analysis 18(1983), 211-221.

Peter, E.E., "Fractal Structure in the Capital Markets," Financial Analysts Journal July(1989), 32-37.

Peter, E.E., Chaos and Order in the Capital Markets, Wiley, New York, 1992.

Poterba, J. and L. Summers, "Mean Reversion in Stock Prices," Journal of Finance 43(1988), 27-59.

Praetz, P.D., "The Distribution of Share Price Changes," Journal of Business 45(1972), 49-55.

Praetz, P.D., "On the Methodology of Testing for Independence in Futures Prices," Journal of Finance 31(1976), 977-979.

Praetz, P.D., "Testing for a Flat Spectrum on Efficient Market Price Data," Journal of Finance 34(1979), 645-658.

Priestiley, M.B., Spectral Analysis of Time Series, 2 vols., Academic Press, New York, 1981.

Ramsey, J.B., C.L. Sayers, and P. Rothman, "The Statistical Properties of Dimension Calculations Using Small Data Sets," mimeo (1988), New York University.

Ramsey, J.B. and H.Yuan, "The Statistical Properties of Dimension Calculations Using Small Data Sets," mimeo (1989), New York University.

Reinganum, M., "The Anomalous Stock Market Behavior of Small Firms in January," Journal of Financial Economics 12(1983), 89-104.

Renyi, A., Probability Theory, North Holland, Amsterdam, 1970.

Richardson, M., "Temporary Component of Stock Prices: A Skeptic's View," mimeo (1989), University of Pennsylvania.

Richardson, M. and J. Stock, "Drawing Inferences from Statistics Based on Multiyear Asset Returns," Journal of Financial Economics 25(1989), 323-348.

Rocca, L.H., "Time Series Analysis of Commodity Futures Prices," Ph.D. dissertation (1969), UC Berkeley.

Rozeff, M., "Money and Stock Prices," Journal of Financial Economics 13(1984), 65-89.

Rozeff, M. and W. Kinney, "Capital Market Seasonality," Journal of Financial Economics 3(1976), 379-402.

Sakai, H. and H. Tokumaru, "Autocorrelations of a Certain Chaos," IEEE Transactions on Acoustics, Speech and Signal Processing 1(1980), 588-590.

Savit, R., "When Random is Not Random: An Introduction to Chaos in Market Prices," Journal of Futures Markets 8(1988), 271-289.

Sayers, C.L., "Work Stoppages: Exploring the Nonlinear Dynamics," mimeo (1989), University of Houston.

Scheinkman, J. and B. LeBaron, "Nonlinear Dynamics and Stock Returns," Journal of Business 62(1989), 311-337.

Schumpeter, J.A., Business Cycle: A Theoretical, Historical, and Statistical Analysis of the Capitalist Process, McGraw-Hill, New York, 1939.

Schuster, H.G., Deterministic Chaos, Verlag-Physik, New York, 1984.

Sen, P.K., "Limiting Behaviour of Regular Functionals of Empirical Distributions for Stationary-Mixing Process," Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 25(1972), 71-82.

Serfling, R., Approximation Theorems of Mathematical Statistics, Wiley, New York, 1980.

Shaw, R., "Strange Attractors, Chaotic Behavior and Information Flow," Z. Naturforsch A. 36(1981), 80-112.

Shiller, R.J., "Do Stock Prices Move Too Much To Be Justified by Subsequent Changes in Dividends?" The American Review 71(1981), 421-436.

Shiller, R.J., "Reply," American Economic Review 73(1983), 236-237.

Simonds, R., L. Lamotte and A. McWhorter, "Testing for Nonstationarity of Market Risk," Journal of Financial and Quantitative Analysis 21(1986), 207-220.

Singleton, C. and J. Wingerder, "Skewness Persistence in Common Stock Returns," Journal of Financial and Quantitative Analysis 21(1986), 335-342.

Smirlock, M. and L. Starks, "Day-of-the-Week and Intraday Effects in Stock Returns," Journal of Financial Economics 17(1986), 197-210.

So, J.C., "The Sub-Gaussian Distribution of Currency Futures," Review of Economics and Statistics 69(1987), 100-107.

Sprinkel, B., Money and Stock Prices, Irwin, Homewood, Ill., 1964.

Stevenson, R.A. and R. Bear, "Commodity Futures: Trend or Random Walks," Journal of Finance 25(1970), 65-81.

Stickel, S.E., "The Effects of Value Line Investment Survey Rank Changes on Common Stock Prices," Journal of Financial Economics 14(1985), 121-144.

Stutzer, M.J., "Chaotic Dynamics and Bifurcation in a Macro Model," Journal of Economic and Dynamics and Control 2(1980), 353-376.

Subba, R. and M. Gabr, "A Test for Linearity of Stationary Time Series Analysis," Journal of Time Series Analysis 1(1980), 145-158.

Summers, L.H., "Does the Stock Market Rationally Reflect Fundamental Values," Journal of Finance 41(1986), 591-601.

Takens, F., "Detecting Strange Attractors in Turbulence," mimeo (1981), University of Groningen.

Tauchen, G.E. and M. Pitts, "The Price Variability-Volume Relationship on Speculative Markets," Econometrica 51(1983), 485-505.

Taylor, S.J., "Tests of the Random Walk Hypothesis Against A Price Trend Hypothesis," Journal of Financial and Quantitative Analysis 17(1982), 37-61.

Taylor, S.J., "The Behavior of Futures Prices Over Time," Applied Economics 17(1985), 713-734.

Taylor, S.J., Modelling Financial Time Series, Wiley, New York, 1986.

Tinic, S.M. and R. West, "Risk and Return: January vs the Rest of the Year," Journal of Financial Economics 13(1984), 561-574.

Tirole, J., "On the Possibility of Speculation Under Rational Expectations," Econometrica 50(1982), 1163-1181.

Tirole, J., "Asset Bubbles and Overlapping Generations," Econometrica 53(1986), 1071-1100.

Tong, H., Nonlinear Time Series, Oxford University Press, Oxford, 1990.

Trippi, R. and D. Desieno, "Trading Equity Index Futures with a Neural Network," Journal of Portfolio Management Fall(1992), 27-33.

Tucker, A.L. and L. Pond, "The Probability Distribution of Foreign Exchange Changes," Review of Economics and Statistics Feb.(1988), 638-647.

Wallis, J.R. and N.C. Matalas, "Small Sample Properties of H and K-Estimators of the Hurst Coefficient h," Water Resources Research 6(1970), 1583-1594.

Wei, W.W.S., Time Series Analysis, Addison-Wesley, 1990.

West, K.D., "Bubbles, Fads, and Stock Price Volatility Tests," Journal of Finance 43(1988), 639-660.

Willey, T., "Testing for Nonlinear Dependence in Daily Stock Indices," Journal of Economics and Business 44(1992), 63-74.

Yang, S.R., The Distribution of Speculative Price Changes, Ph.D. dissertation, University of Purdue, 1989.

Appendix A: Summary of Estimated Autocorrelations for Index Returns($R_\tau(x)$), Absolute Returns($R_\tau(|x|)$), Squared Returns($R_\tau(x^2)$), Adjusted Squared Returns($R_\tau(S=(x-\mu)^2)$), Rescaled Returns($R_\tau(y)$), Conditional Variance($R_\tau(v)$), and Squared Variance($R_\tau(v^2)$).

(Sweden)—$R_\tau(X)$

lag	estimate	standard error	lag	estimate	standard error
1	.16902	.01987	2	-.00060	.02042
3	-.00297	.02042	4	.03302	.02043
5	.02452	.02045	6	-.01439	.02046
7	.08012	.02046	8	.03037	.02059
9	.05050	.02060	10	.01342	.02065
11	.02996	.02066	12	-.01039	.02067
13	.01072	.02067	14	-.01293	.02068
15	.01349	.02068	16	.07175	.02068
17	-.00695	.02078	18	-.00314	.02078
19	-.02114	.02078	20	-.01214	.02079
21	-.01449	.02079	22	.03177	.02080
23	.02767	.02082	24	.00938	.02083
25	.01852	.02083	26	-.01051	.02084
27	.01593	.02084	28	-.00985	.02085
29	.01709	.02085	30	-.01758	.02085

(Sweden)—$R_\tau(|x|)$

lag	estimate	standard error	lag	estimate	standard error
1	.33806	.01987	2	.25395	.02202
3	.26797	.02315	4	.26763	.02434
5	.24751	.02546	6	.23132	.02641
7	.28120	.02719	8	.21775	.02832
9	.17274	.02897	10	.21491	.02938
11	.23917	.02999	12	.21193	.03073
13	.17455	.03130	14	.19526	.03169
15	.17697	.03216	16	.17699	.03254
17	.17181	.03292	18	.18673	.03327
19	.13366	.03368	20	.12107	.03389
21	.12204	.03406	22	.10378	.03423
23	.13252	.03436	24	.13353	.03456
25	.10839	.03476	26	.06653	.03489
27	.10058	.03494	28	.08758	.03506
29	.08501	.03514	30	.10887	.03522

(Continued)

(Sweden)—$R_\tau(X^2)$

lag	estimate	standard error	lag	estimate	standard error
1	.045342	.01987	2	.21677	.02360
3	.28366	.02437	4	.29362	.02564
5	.19978	.02694	6	.23963	.02751
7	.41854	.02833	8	.31413	.03067
9	.19708	.03191	10	.24712	.03239
11	.26268	.03313	12	.16153	.03394
13	.10937	.03424	14	.15384	.03438
15	.15214	.03465	16	.11688	.03491
17	.12155	.03507	18	.14559	.03523
19	.07202	.03547	20	.05083	.03553
21	.06584	.03566	22	.07431	.03560
23	.06572	.03566	24	.04723	.03571
25	.05195	.03574	26	.00499	.03577
27	.02630	.03577	28	.02544	.03577
29	.03585	.03578	30	.04682	.03579

(Sweden)—$R_\tau(S)$

lag	estimate	standard error	lag	estimate	standard error
1	.44822	.01987	2	.21757	.02378
3	.28645	.02430	4	.29857	.02528
5	.20191	.02694	6	.23727	.02714
7	.42200	.02832	8	.31991	.03069
9	.19602	.03199	10	.24069	.03283
11	.25884	.03316	12	.15814	.03384
13	.10667	.03424	14	.15338	.03494
15	.15541	.03464	16	.11529	.03479
17	.11408	.03506	18	.14244	.03572
19	.06981	.03544	20	.04936	.03524
21	.06611	.03552	22	.07566	.03599
23	.06579	.03563	24	.04584	.03513
25	.04961	.03570	26	.00183	.03541
27	.02475	.03573	28	.02416	.03585
29	.03465	.03574	30	.04686	.03538

(Continued)

(Sweden)—$R_\varepsilon(Y)$

lag	estimate	standard error	lag	estimate	standard error
1	.20936	.01994	2	.02680	.02067
3	-.01061	.02081	4	.03865	.02086
5	.03250	.02084	6	.01694	.02048
7	.00853	.02087	8	-.01022	.02003
9	.04702	.02078	10	.02028	.02036
11	.02445	.02092	12	-.01127	.02078
13	.02585	.02093	14	.01974	.02021
15	.02510	.02095	16	.01947	.02095
17	-.02059	.02097	18	-.01268	.02049
19	.03090	.02098	20	.00625	.02153
21	.00443	.02101	22	.01846	.02152
23	-.00637	.02101	24	.01580	.02139
25	.02169	.02101	26	-.00983	.02171
27	-.00775	.02102	28	-.00774	.02135
29	.00815	.02102	30	.00172	.02139

(Sweden)—$R_\varepsilon(v)$

lag	estimate	standard error	lag	estimate	standard error
1	.98216	.01994	2	.96102	.03413
3	.94012	.04358	4	.91875	.05101
5	.89638	.05721	6	.87380	.06255
7	.85159	.06723	8	.82671	.07139
9	.80193	.07510	10	.77915	.07843
11	.75620	.08145	12	.73184	.08419
13	.70717	.08669	14	.68354	.08895
15	.65968	.09101	16	.63599	.09290
17	.61237	.09461	18	.58902	.09617
19	.56447	.09760	20	.54122	.09889
21	.51956	.10006	22	.49897	.10113
23	.48006	.10210	24	.46116	.10299
25	.44222	.10524	26	.42401	.10456
27	.40852	.10524	28	.39395	.10587
29	.38048	.10645	30	.36833	.10699

(Continued)

(Sweden)—$R_t(v^2)$

lag	estimate	standard error	lag	estimate	standard error
1	.98035	.01994	2	.95566	.03400
3	.93474	.04345	4	.91152	.05082
5	.88299	.05695	6	.85855	.06216
7	.83481	.06671	8	.79895	.07074
9	.76284	.07424	10	.72919	.07729
11	.69230	.07998	12	.64862	.08233
13	.60638	.08434	14	.56900	.08505
15	.53167	.08754	16	.49589	.08881
17	.45966	.08991	18	.42518	.09084
19	.38895	.09162	20	.35746	.09228
21	.32931	.09283	22	.30332	.09329
23	.27928	.09368	24	.25572	.09401
25	.23493	.09429	26	.21536	.09452
27	.20050	.09472	28	.18705	.09488
29	.17515	.09503	30	.16513	.09516

(Korea)—$R_t(X)$

lag	estimate	standard error	lag	estimate	standard error
1	.13472	.01722	2	-.04050	.01753
3	-.02028	.01755	4	.03061	.01756
5	.01869	.01758	6	-.02135	.01756
7	-.02469	.01759	8	-.03855	.01760
9	.01525	.01762	10	.00201	.01763
11	.02100	.01763	12	.02289	.01769
13	.03822	.01764	14	.03945	.01767
15	.03230	.01774	16	-.02102	.01772
17	.00567	.01773	18	.02788	.01771
19	.00956	.01773	20	.01187	.01774
21	-.00191	.01774	22	.00689	.01774
23	-.00076	.01774	24	-.00066	.01774
25	-.04058	.01776	26	-.03124	.01775
27	-.01432	.01778	28	.01357	.01779
29	-.00611	.01779	30	-.02656	.01779

(Continued)

(Korea)—$R_\tau(|X|)$

lag	estimate	standard error	lag	estimate	standard error
1	.28757	.01722	2	.26567	.01859
3	.24896	.01968	4	.19366	.02059
5	.19285	.02112	6	.17391	.02164
7	.14591	.02205	8	.13569	.02233
9	.10666	.02258	10	.11905	.02272
11	.12843	.02291	12	.10269	.02312
13	.10055	.02326	14	.06195	.02338
15	.08039	.02343	16	.09884	.02351
17	.07975	.02364	18	.08053	.02372
19	.10177	.02380	20	.05805	.02393
21	.07608	.02397	22	.10015	.02404
23	.08533	.02416	24	.09449	.02425
25	.07143	.02436	26	.10655	.02442
27	.09170	.02474	28	.10655	.02460
29	.07397	.02474	30	.06917	.02480

(Korea)—$R_\tau(X^2)$

lag	estimate	standard error	lag	estimate	standard error
1	.30139	.01722	2	.18988	.01871
3	.16509	.01928	4	.10995	.01969
5	.11524	.01987	6	.10613	.02007
7	.09706	.02024	8	.10005	.02037
9	.04557	.02052	10	.05212	.02055
11	.06009	.02059	12	.04639	.02066
13	.04304	.02069	14	.01560	.02071
15	.02623	.02072	16	.02500	.02073
17	.02435	.02074	18	.01991	.02074
19	.02919	.02075	20	-.00687	.02076
21	.00937	.02076	22	.02557	.02076
23	.02447	.02077	24	.03993	.02078
25	.01432	.02080	26	.03115	.02081
27	.02584	.02082	28	.05331	.02083
29	.02366	.02087	30	.02178	.02088

(Continued)

(Korea)—$R_t(S)$

lag	estimate	standard error	lag	estimate	standard error
1	.30096	.01722	2	.18997	.01872
3	.16661	.01927	4	.10938	.01986
5	.11449	.01987	6	.10647	.02041
7	.09859	.02024	8	.10003	.02011
9	.04623	.02052	10	.05266	.02089
11	.06866	.02059	12	.04761	.02049
13	.04401	.02069	14	.01525	.02033
15	.02566	.02073	16	.02441	.02087
17	.02445	.02076	18	.01977	.02024
19	.02955	.02076	20	-.00599	.02071
21	.00994	.02077	22	.02481	.02019
23	.02273	.02078	24	.03750	.02097
25	.01284	.02081	26	.03029	.02067
27	.02558	.02082	28	.05164	.02069
29	.02368	.02087	30	.02164	.02038

(Korea)—$R_t(Y)$

lag	estimate	standard error	lag	estimate	standard error
1	.10734	.01726	2	.00467	.01783
3	.02005	.01746	4	.02643	.01762
5	.02552	.01748	6	-.01074	.01745
7	-.01061	.01769	8	-.03030	.01726
9	-.00029	.01751	10	.00684	.01758
11	.00917	.01751	12	.02292	.01761
13	.03882	.01752	14	.02367	.01731
15	.01271	.01751	16	.00339	.01711
17	.01193	.01756	18	.02645	.01734
19	.00162	.01758	20	.01795	.01721
21	-.00359	.01758	22	.00890	.01782
23	-.01183	.01749	24	-.01531	.01723
25	-.02875	.01759	26	-.01850	.01749
27	-.00671	.01761	28	.00987	.01767
29	-.00020	.01761	30	-.02859	.01763

(Continued)

(Korea)—$R_\tau(v)$

lag	estimate	standard error	lag	estimate	standard error
1	.94088	.01726	2	.88068	.02874
3	.81962	.03589	4	.75738	.04109
5	.70093	.04506	6	.64730	.04820
7	.59676	.05073	8	.55100	.05278
9	.50938	.05447	10	.47488	.05587
11	.44405	.05706	12	.41378	.05808
13	.38668	.05895	14	.36162	.05970
15	.34467	.06035	16	.33210	.06094
17	.32042	.06147	18	.31165	.06197
19	.30500	.06243	20	.29671	.06288
21	.29379	.06329	22	.29297	.06370
23	.28995	.06410	24	.28718	.06523
25	.28304	.06487	26	.28120	.06523
27	.28045	.06559	28	.27836	.06595
29	.27239	.06630	30	.26744	.06663

(Korea)—$R_\tau(v^2)$

lag	estimate	standard error	lag	estimate	standard error
1	.92555	.01726	2	.84549	.02844
3	.76284	.03514	4	.68239	.03977
5	.61690	.04312	6	.55160	.04568
7	.49047	.04762	8	.43580	.04910
9	.38940	.05024	10	.35163	.05113
11	.31748	.05185	12	.35163	.05113
13	.24923	.05288	14	.21922	.05323
15	.19631	.05349	16	.17729	.05271
17	.16005	.05388	18	.14691	.05402
19	.13681	.05414	20	.12758	.05425
21	.12554	.05434	22	.12615	.05442
23	.12567	.05451	24	.12666	.05460
25	.12588	.05468	26	.12852	.05477
27	.13059	.05486	28	.13205	.05495
29	.12629	.05505	30	.12130	.05513

Appendix B: Derivation of Test Statistic for Excessive Response Hypothesis

Assume that the $\{X_t\}$ is a linear process and it is satisfied by the theorem in 4.2.2. Then as $n \to \infty$,

$$\sqrt{n}\,(R - \rho) \to N(0, \Omega_k). \tag{B.1}$$

Also assuming that $\Omega_k \approx I_k$, similar to the price trend hypothesis, we can obtain that for large n,

$$V_\tau = (\sqrt{n}\,R_\tau)^2 \tag{B.2}$$

is approximated by a non-central χ^2-distribution with degree of freedom 1 and noncentrality parameter

$$\lambda_\tau = \frac{N}{2} D^2 \phi^{2\pi} \tag{B.3}$$

where distribution of V_τ is asymptotically independent for $\tau(\tau = 1,2.....,k)$ and the p.d.f of V_τ is given to

$$f(V_\tau) = \sum_{j=0}^{\infty} \frac{e^{-\lambda_\tau}\lambda^{j}_{\tau}}{j!} \frac{V_\tau^{1/2+j-1} e^{-1/2V_\tau}}{2^{1/2+j}\Gamma(1/2+j)} \tag{B.4}$$

For the fixed D and ϕ, the likelihood ratio is

$$L = \prod_{\tau=1}^{k} \left[\frac{\displaystyle\sum_{j=0}^{\infty} \frac{e^{-\lambda_\tau}\lambda^{j}_{\tau}}{j!} \frac{V_\tau^{1/2+j-1} e^{-1/2V_\tau}}{2^{1/2+j}\Gamma(1/2+j)}}{\displaystyle\frac{V_\tau^{1/2-1} e^{-1/2V_\tau}}{2^{1/2}\Gamma 2(1/2)}} \right] \tag{B.5}$$

$$= \prod_{\tau=1}^{k} \left[\sum_{j=0}^{\infty} \frac{e^{-\lambda_\tau}(\lambda_\tau V_\tau)^j}{2^j j!} \frac{\Gamma(1/2)}{\Gamma(1/2+j)} \right]$$

Hence, the log likelihood ratio is

$$
\begin{aligned}
logL &= \sum_{\tau=1}^{k}\left\{-\lambda_\tau + log\left[\sum_{j=0}^{\infty}\frac{(n/2D^2\phi^{2\tau}V_\tau)^j}{2^j j!}\frac{\Gamma(1/2)}{\Gamma(1/2+j)}\right]\right\} \\
&= constant + \sum_{\tau=1}^{k}\frac{n}{2}D^2\phi^{2\tau}V_\tau + O_p(\frac{n}{2}D^2\phi^{2\tau})
\end{aligned}
\tag{B.6}
$$

For any positive real number a, a test statistic E^* defined by

$$
E^* = a\sum_{\tau=1}^{\infty}\phi^{2\tau}V_\tau = an\sum_{\tau=1}^{k}\phi^{2\tau}R_\tau^2(x) \tag{B.7}
$$

is approximately equivalent to the likelihood ratio test as $n\to\infty$.
In fact, the Neyman-Pearson lemma indicates that this is the most powerful asymptotic test.

The critical region can be set up for an arbitrary c:

$$
E^* > c .
$$

To compute c, two parameters β and υ are determined by approximating E^*'s distribution to be a $\beta\chi^2_\upsilon$, where β is an integer. From the fact that under the H_0 the distribution of V_τ can be approximated to χ^2_1,

$$
E(E^*) = a\sum_{\tau=1}^{k}\phi^{2\tau} \approx \frac{a\phi^2}{1-\phi^2} = \beta\nu \tag{B.8}
$$

$$
\begin{aligned}
var(E^*) &= a^2\sum_{\tau=1}^{k}var(V_\tau)\phi^{4\tau} \\
&\approx \frac{2a^2\phi^4}{1-\phi^4} \\
&\approx 2\beta^2\nu
\end{aligned}
\tag{B.9}
$$

Furthermore these two equations lead to

$$\nu = \frac{1 - \phi^4}{(1 - \phi^2)^2} \tag{B.10}$$

and

$$\beta = \frac{a\phi^2}{\nu(1 - \phi^2)} \tag{B.11}$$

Assume that a=0.4274 and ϕ=0.92 as in the price trend model. Then, υ=12.021 and β=0.196 are calculated.

At the 5 percent significance level, c=4.120 is obtained from

$$P\,(\chi^2_{12} > c/\beta) = 0.05. \tag{B.12}$$

As in the price trend hypothesis, a revised test statistic disregarding the autocorrelation coefficient at lag 1 is made and it is defined as

$$E^{**} = b \sum_{\tau=2}^{k} \phi^{2\tau} V_\tau . \tag{B.13}$$

By approximating E^{**}'s distribution to be a $\beta\chi^2_\upsilon$,

$$E(E^{**}) \simeq \frac{a\phi^4}{1 - \phi^2} = \beta\nu , \tag{B.14}$$

$$var(E^{**}) \simeq \frac{a^2\phi^8}{1 - \phi^4} = \beta^2\nu . \tag{B.15}$$

are given. Furthermore,

$$\nu = \frac{1 + \phi^2}{1 - \phi^2}, \tag{B.16}$$

and

$$\beta = \frac{a\phi^4}{\nu(1 - \phi^2)} . \tag{B.17}$$

Also assume that b=0.4649 and ϕ=0.92 as in the price trend model. Then, υ=12.0208 and β=0.1804 are calculated.

Also at 5 per cent significance level, c=3.793 is obtained from

$$P = (\chi^2_{12} > c/\beta) = 0.05. \tag{B.18}$$

Appendix C: Stable Distribution

Here we present a more detailed explanation on the stable distribution since it is very closely related to fractals.

The notion of stable distribution was invented by P.Lévy in 1925. Nonetheless, this distribution did not gain much attention in the fields of natural science and social science because of the fact that in general the family of this distribution possesses infinite second moments and sometimes also infinite means. Only a special singular case of the family of this distribution, the Gaussian distribution, having both finite mean and variance, has became extremely popular. He studied summation of random variables and found that there were some distributions which reproduced themselves up to a linear change of variable under the operation of convolution.

The importance of stable distributions was reiterated by one of Lévy's students, Mandelbrot (1963), in connection with his work on commodity price change. Mandelbrot in his article on commodity price change has asserted that the empirical distribution of price change is too "peaked" to be a Gaussian and that it can be better described by a stable distribution. Furthermore Montroll and Schlesinger(1984) have also suggested that stable distributions are often needed for describing a fractal stochastic process.

A definition of stable distribution is given by Feller(1966) as follows:

Let $X, X_1, X_2, \ldots\ldots, X_n$ be independent random variables with a common distribution R. The distribution R is stable if and only if for $Y_n \equiv X_1 + X_2 + \ldots + X_n$ there exist constants c_n and γ_n such that

$$Y_n \simeq_{(d)} c_n X + \gamma_n \qquad (C.1)$$

where $\simeq_{(d)}$ indicates that the random variable of both sides have the same distribution.

In general, the sum of random variables with a common distribution becomes a random variable with a distribution of different form. However, for random variables with a stable distribution, an appropriate linear transformation makes the sum of random variables obey the same distribution. In other words, the distribution of the sum of many random variables is similar to that of one variable. This may be regarded as a kind of self-similarity.

Using the characteristic function of a distribution F(x)

$$\phi(t) \equiv E[e^{itx}] = \int_{-\infty}^{\infty} exp(itx)\,dF(x) \tag{C.2}$$

the relation (C.1) is transformed into

$$\phi^n(t) = \phi(c_n t) \cdot exp(i\gamma_n t). \tag{C.3}$$

This functional equation can be solved completely and the solution becomes:

$$\phi(t) = exp\left[i\delta t - \gamma\,|t|^{\alpha} \cdot \{1 + i\beta(t/|t|)\,\omega(t,\alpha)\}\right] \tag{C.4}$$

where

$$\omega(t,\alpha) \equiv \begin{cases} tan(\pi\alpha/2), & \alpha \neq 1, \\ (2/\pi)\log|t|, & \alpha = 1. \end{cases} \tag{C.5}$$

Here, α,β,γ and δ are parameters which satisfy $0<\alpha\leq 2$, $-1<\beta<1$ and $\gamma>0$. The most important of the four is α, the characteristic exponent. This is an index of "peakedness." It has been proved that the normalization factor c_n in (C.1) must be $n^{1/\alpha}$. The case $\alpha=2$ corresponds to the Gaussian distribution. For $\alpha>2$, the distribution would have negative probability, which is inconsistent. The parameter β governs symmetry of the distribution; for example, $\beta=0$ corresponds to a symmetric distribution. δ is the parameter which translates the distribution and γ dominates the scale of X. The latter two parameters are not essential since they do not change the shape of distributions. If we disregard such parameters, characteristic functions of the stable distributions, except in the case $\alpha=1$, can be transformed into the following simple form with two parameters:

$$\phi(t) = exp\left\{-|t|^{\alpha} \cdot e^{\pm i\pi\theta/2}\right\} \tag{C.6}$$

where the symbol $\pm$ takes the sign of t. Here θ is the symmetry parameter instead of β and its domain is restricted in the following region:

$$|\theta| \leq \begin{cases} \alpha, & 0 < \alpha < 1, \\ 2-\alpha, & 1 < \alpha < 2. \end{cases} \tag{C.7}$$

(See Figure C.1.)

The probability density p(X;α,θ) is the Fourier transform of (C.6) and is given by

$$p(X;\alpha,\theta) = \frac{1}{\pi} Re\int_0^{\infty} dt\ exp\left[iXt - t^{\alpha} \cdot e^{i\pi/2\theta}\right] \tag{C.8}$$

where Re denotes the real part. From this expression it is obvious that

$$p(X;\alpha,\theta) = p(-X;\alpha,-\theta), \tag{C.9}$$

and for θ=0,

$$p(X;\alpha,0) = p(-X;\alpha,0). \tag{C.10}$$

That is, the distribution is symmetric.

Expanding the right-hand side of (C.6) in powers of X and integrating each term with respect to t, we obtain the following expansion formulae.

For X>0, 0<α<1,

$$p(X;\alpha,\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n\,\Gamma(n\alpha+1)}{n!\,X^{n\alpha+1}}\, sin\frac{n\pi}{2}(\theta-\alpha) \tag{C.11}$$

For X>0, 1<α<2,

$$p(X;\alpha,\theta) = \frac{1}{\pi X}\sum_{n=1}^{\infty} \frac{(-X)^n\,\Gamma(n\alpha^{-1}+1)}{n!}\, sin\,\frac{n\pi}{2\alpha}(\theta-\alpha) \tag{C.12}$$

Similar expressions for X<0 are easily found by using (C.9). When 0<α<1 and θ=-α, we can see from (C.9) and (C.11) that for X<0,

$$p(X;\alpha,-\alpha) = 0. \tag{C.13}$$

This means that the distribution is one-sided, that is, the random variable never takes negative values.

In this way, stable distributions are generally expressed in the integral form (C.8) or in the form of power series expansion as in (C.11) and (C.12). However, only four cases have been found to be expressed by elementary functions: $\alpha=2$, $\theta=0$ corresponds to the Gaussian distribution

$$p(X;2,0) = (1/\sqrt{\pi})\, exp(-X^2) \tag{C.14}$$

$\alpha=1$, $\theta=0$ corresponds to the Lorentzian (or Cauchy) distribution.

$$p(X;1,0) = (1/\pi)\,(1/1+X^2) \tag{C.15.}$$

$\alpha=½$, $\theta=-½$ corresponds to a one-sided distribution:

$$p(X;\frac{1}{2},-\frac{1}{2}) = \begin{cases} 0, & X \le 0, \\ (\frac{1}{2\pi})\, exp(-\frac{1}{2}X)\cdot X^{-3/2}, & X>0. \end{cases} \tag{C.16}$$

For $\alpha \approx 0$, it is known that for not too small and not too large X, the distribution of X is approximated by a log-normal distribution:

$$p(X;\alpha,-\alpha) \approx (1/X)\, exp\{-1/2\,\alpha^2 (log X)^2\}. \tag{C.17}$$

The following two relations are sometimes useful in applications of stable distributions:

(1) For $X>0$ and $½<\alpha<1$,

$$\frac{1}{X^{\alpha+1}} \cdot p\left(\frac{1}{X^{\alpha}};\frac{1}{\alpha},\gamma\right) = P(X;\alpha,\alpha(\gamma+1)-1) \tag{c.18}$$

(2) Let p be a variable with a stable positive distribution with index α(i.e.p(Y;α,-α)) and let X be a variable with a stable distribution with index β. Then $Z \equiv X \bullet Y^{1/\alpha}$ becomes a stable distribution with index $\alpha\beta$. For example, when we take Y as p(Y;½,-½) and X as Gaussian, then $Z=X \bullet Y^2$ becomes Lorentzian.

The most important peculiarity of stable distributions is that, as long as $\alpha \neq 2$, they have long tails of power type for large $|X|$, such as

$$p(X;\alpha,\theta) \sim X^{\alpha^{-1}}, \text{ approximately.} \tag{C.19}$$

Note that, for asymmetric distributions, the longer tail obeys this relation but the other side decays faster than this. The existence of such tails is expected from the fact that the characteristic function given by (C.6) is singular at the origin. Since the characteristic function is the Fourier transform of its distribution function, its singularity at the origin corresponds to the singularity of the distribution function at infinity. Using the cumulative distribution function, the power law distribution (C.19) is written as

$$P(X;\alpha,\theta) \equiv \int_X^\infty p(X';\alpha,\theta)\,dx' \propto X^{-\alpha}. \tag{C.20}$$

Therefore, in the case that X is a quantity expressing some kind of length, the characteristic exponent α can be regarded as the fractal dimension. At this point, stable distributions cannot be separated from fractals.

If the distribution has a power-type long tail as in (C.20), the qth-order absolute moment $<|X|^q>$ diverges. Hence, for $\alpha \neq 2$, the variance is always divergent. Among stable distributions, the Gaussian distribution is very exceptional and its moment is finite for any order.

Historically, stable distributions other than the Gaussian have not attracted much of the attention of physicists. This seems to be due to the divergence of variance. Many scientists are apt to think divergent variance is not natural. However, even if the variance of a population of some distribution is infinite, this does not mean that infinity should be observable. For a finite number of samples, the observed variance is finite with probability unity just as in the case with finite variance. However, as the number of samples is increased, the variance becomes larger and larger showing a tendency to diverge. This is not unphysical at all-in fact, there are many experimental examples. The well-known central limit theorem states that a quantity made of a number of stochastic variables follows a Gaussian distribution. As a result of this theorem, the Gaussian distribution is treated specially in every field of natural science as well as social science and is frequently applied. However, the fact that this theorem has a limitation is not so well known. For a counter-example, any sum of random variables of a stable distribution follows, by definition, a stable distribution with the same characteristic exponent. Hence, if its characteristic exponent is less than 2, the sum never approaches the Gaussian. The following generalized central limit theorem clarifies the conditions for the ordinary central limit theorem to hold.

For random variables $X_1, X_2, \ldots\ldots, X_n$, let $Y_n = \Sigma X_i$. If the distribution of Y_n, with an appropriate normalization, converges to some distribution R in the limit $n \to \infty$, then the distribution R is stable. In particular, when its variance is finite, R is the stable distribution with characteristic exponent 2, that is, the Gaussian distribution.

It is clear from this theorem that the ordinary central limit theorem is only applicable to random variables with finite variance and that stable distributions other than the Gaussian are fundamental for stochastic phenomena with divergent variance.

While a number of phenomena with fractal distribution have been found recently, no one has succeeded in explaining from a general point of view why their distributions follow the power law. The above generalized central limit theorem is expected to play on important role in this problem. According to this theorem, a sum of many random variables with divergent variances is likely to follow a stable distribution which has a power tail. So,if we can decompose a phenomenon into superposition of elementary random processes whose variances are divergent, then we may expect a power tail of a stable distribution to be observed.

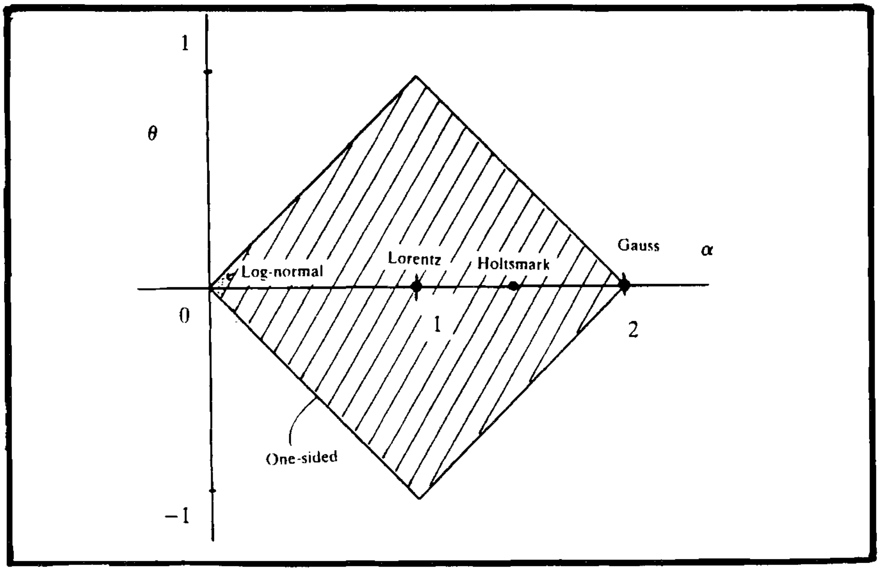


Figure C.1: The Parameter Space (α,θ) for Stable Distribution.

EFI - reports since 1987

Published in the language indicated by the title

Andersson, T., Ternström, B., Private Foreign Investments and Welfare Effects: A comparative study of five countries in southeast Asia. Research report

Andersson, T., Ternström, B., External Capital and Social Welfare in South-East Asia. Research report

Benndorf, H., Marknadsföringsplanering och samordning mellan företag i industriella system. EFI/MTC

Bergman, L., Mäler, K-G., Ståhl, I., Överlåtelsebara utsläppsrätter. En studie av kolväteutsläpp i Göteborg. Research report

Björkegren, D., Mot en kognitiv organisationsteori. Research report

Claesson, K., Effektiviteten på Stockholms Fondbörs

Davidsson, P. Growth Willingness in Small Firms. Entrepreneurship - and after? Research report

Engshagen, I., Finansiella nyckeltal för koncern versus koncernbolag. En studie om finansiella måltal. Research report

Fredriksson, O., Holmlöv, PG., Julander, C-R., Distribution av varor och tjänster i informationssamhället.

Hagstedt, P., Sponsring - mer än marknadsföring. EFI/MTC

Järnhäll, B., On the Formulation and Estimation of Models of Open Economies. Research report

Kylén, B., Digitalkartans ekonomi. Samhällsekonomiska modeller för strategiska val. Delrapport 2 i "Nordisk Kvantif"

Lundgren, S., Elpriser: Principer och praktik. Research report

Schwarz, B., Hederstierna, A., Socialbidragens utveckling - orsaker eller samvariationer? Research report

Swartz, E., Begreppet federativ organisation belyst i ett organisationsteoretiskt, juridiskt och empiriskt perpektiv. Research report.

Westlund, A.H. Öhlén, S., Business Cycle Forecasting in Sweden; A problem analysis. Research report

1988

Andréasson, I-M., Costs of Controls on Farmers' Use of Nitrogen. A study applied to Gotland.

Björkegren, D., Från sluten till öppen kunskapsproduktion. Delrapport 3 i forskningsprojektet Lärande och tänkande i organisationer. Research report

Björkegren, D., Från tavistock till Human-relation. Delrapport 4 i forskningsprojektet Lärande och tänkande i organisationer. Research report

Björklund, L., Internationell projektförsäljning. En studie av ad hoc-samverkan mellan företag vid internationella projekt. Research report

Bojö, J., Mäler, K-G., Unemo, L., Economic Analysis of Environmental Consequences of Development Projects. Research report

Brynell, K., Davidsson, P., Det nya småföretagandet? - en empirisk jämförelse mellan små high-tech företag och konventionella småföretag. Research report

Dahlgren, G., Witt, P., Ledning av fusionsförlopp. En analys av bildandet av Ericsson Information Systems AB.

Forsell, A., Från traditionell till modern sparbank. Idé organisation och verksamhet i omvandling. Research report

Hirdman, V., The Stability and the Interest Sensitivity of Swedish Short Term Capital Flows. Research report

Hultén, S., Vad bestämmer de svenska exportmarknadsandelarnas utveckling?

Häckner, J., Biobränslenas konkurrenskraft i ett framtida perspektiv. Research report

Jennergren, P., Näslund, B., If the Supreme Court had Known Option Theory: A Reconsideration of the Gimo Case. Research report

Jennergren, P., Näslund, B., The Gimo Corporation Revisited: An Exercise in Finance Theory. Research report

Jennergren, P., Näslund, B., Valuation of Debt and Equity Through One- and Two-State Contingent Claims Models - with an Application to a Swedish Court Case. Research report

Jonung, L., Laidler, D., Are Perceptions of Inflation Rational? Some Evidence for Sweden. Research report.

Lagerstam, C., Business Intelligence. Teori samt empirisk studie av elektronik- och bioteknikbranschen i Japan. Research report

Liljegren, G., Interdependens och dynamik i långsiktiga kundrelationer. Industriell försäljning i nätverksperspektiv.

Näslund, B., Några synpunkter på börsfallet i oktober 1987. Research report

Olsson, C., The Cost-Effectiveness of Different Strategies Aimed at Reducing the Amount of Sulphur Deposition in Europe. Research report

Philips, Å., Eldsjälar. En studie av aktörsskap i arbetsorganisatoriskt utvecklingsarbete.

Sellstedt, B., Produktionsstrategier. En diskussion med utgångspunkt från litteraturen. Research report.

Skogsvik, K., Prognos av finansiell kris med redovisningsmått. En jämförelse mellan traditionell och inflationsjusterad redovisning.

Wahlund, R., Skatteomläggningen 1983 - 1985. En studie av några konsekvenser. Research report

Wahlund, R., Varför och hur olika svenska hushåll sparar. Research report

Vredin, A., Macroeconomic Policies and the Balance of Payments.

Åkerman, J., Economic Valuation of Risk Reduction: The Case of In-Door Radon. Research report.

1989

Andersson, T., Foreign Direct Investment in Competing Host Countries. A Study of Taxation and Nationalization.

Björkegren, D., Skönhetens uppfinnnare. Research report

Björkegren, D., Hur organisationer lär. Studentlitteratur.

Blomström, M., Transnational Corporations and Manufactoring Exports from Developing Countries. Research report.

Carlsson, A., Estimates of the Costs of Emission control in the Swedish Energy Sector. Research report

Blomström, M., Foreign Investment and Spillovers. London; Routledge

Bordo, M.D., Jonung, L., The Long-Run Behavior of Velocity: The Institutional Approach Revisted. Research report

Davidsson, P., Continued Entrepreneurship and Small Firm Growth.

DeJuan, A., Fiscal Attitudes and Behavior. A study of 16 - 35 years old Swedish citizens. Research report

Edlund, P-O., Preliminary Estimation of Transfer Function Weights. A Two-Step Regression Approach.

Fridman, B., Östman, L., eds. Accounting Development - some perspectives. In honour of Sven-Erik Johansson.

Gadde, L-E., Håkansson, H., Öberg, M., Stability and Change in Automobile Distribution. Research report

Glader, M., Datorer i småföretag. Teldok

Jakobsson, B., Konsten att reagera. Intressen, institutioner och näringspolitik. Carlssons Bokförlag

Jonung, L., The Economics of Private Money. Private Bank Notes in Sweden 1831 - 1902. Research report

Jonung, L., Batchelor, R.A., Confidence about Inflation Forecasts: Tests of Variance Rationality. Research report.

Kylén, B., Hur företagschefer beslutar innan de blir överraskade. Ett försök till förklaring av svarsmönster i svagsignalsitutationer.

Lagerstam, C., Jämförelse av skydd med samlad valuta/aktieoption och skydd med separata aktie- och valutaoptioner samt härledning av optionspriser.
Research report

Lagerstam, C., On the Pricing and Valuation of Forwards and Options on Futures and their Relationsship. Research report.

Larsson, B., Koncernföretaget. Ägarorganisationer eller organisation för ägare?

Lindberg, C., Holmlöv, PG., Wärneryd K-E., Telefaxen och användarna. Teldok

Löwstedt, J., Föreställningar, ny teknik och förändring. Tre organisationsprocesser ur ett kognitivt aktörsperspektiv. Doxa Förlag

Löwstedt, J., (red) Organisation och teknikförändring. Studentlitteratur.

Tollgerdt-Andersson, I., Ledarskapsteorier, företagsklimat och bedömningsmetoder.

Schuster, W., Ägandeformens betydelse för ett företag - en studie av ICA-rörelsen.

Schwartz, B., Företaget som medborgare. Samhällskontakter och reklam som legitimeringsinstrument. Research report

Spets, G., Vägen till nej. Anade och oanade konsekvenser av en OS-satsning. Research report.

Stymne, B., Information Technology and Competence Formation in the Swedish Service Sector. IMIT/EFI

1990

Björkegren, D., Litteraturproduktion - en fallstudie. Delrapport 2 i forskningsprojektet Det skapande företaget: Management of Narrativation.
Research Report.

Bojö, J., Economic Analysis of Agricultural Development Projects. A Case Study from Lesotho. Research Report.

Brunsson, N., Forsell, A., Winberg, H., Reform som tradition. Administrativa reformer i Statens Järnvägar.

Drottz-Sjöberg, B-M., Interests in Humanities, Social Science and Natural Science. A Study of High School Students.

Eklöf, J.A., Macro Ecoonomic PLanning with Quantitative Techniques - and Effects of Varying Data Quality. Research Report.

Flink, T., Hultén, S., Det svenska snabbtågsprojektet - de första 20 åren. Research Report.

Hesselman, A., Ett lokalt ekonomisystem. En fallstudie inom byggbranschen. Research Report.

Jennergren, L.P., Näslund, B., Models for the Valuation of International Convertible Bonds. Research Report.

Jonung, L., Introduction and Summary to The Stockholm School of Economics Revisited. Research Report.

Lagerstam, C., Hedging of Contracts, Anticipated Positions, and Tender Offers.

Lindkvist, H., Kapitalemigration

Normark, P., Strategiska förändringar i kooperationer fallet Lantmännen.
Research Report.

Patrickson, A., Essays in the Latin American Fertilizer Industry.

Sjöstrand, S-E., Den dubbla rationaliteten. Research Report.

Steiner, L., Ledningsfunktionen i tillväxtföretag. Ledningsteamens sammansättning och funktion i tillverkande företag inom informationsteknologiindustrin.

Warne, A., Vector Autoregressions and Common Trends in Macro and Financial Economics

Wärneryd, K., Economic Conventions. Essays in Institutional Evolution.

1991

Bergholm, M., Hemdistribution med hjälp av datorkommunikation. Research Report.

Bojö, J., The Economics of Land Degradation. Theory and Applications to Lesotho.

Brytting, T., Organizing in the Small Growing Firm - A grounded theory approach

Edlund, P-O., Soegaard, H., Business Cycle Forecasting: Tracking Time - Varying Transfer Functions. Research Report.

Ericson, M., Iggesundsaffären - Rationaliteter i en strategisk förvärvsprocess

Horn av Rantzien, M., Endogenous Fertility and Old-Age Security. Research Report.

Jennergren, L.P., Näslund, B., Options with Stochastic Lives. Research Report.

Jonung, L., Gunnarsson, E., Economics the Swedish Way 1889-1989. Research Report.

Jungenfelt, K., An Analysis of Pay as Yo Go Pension Systems as Dynastic Clubs. Research Report.

Lundgren, A., Technological Innovation and Industrial Evolution - The Emergence of Industrial Networks.

Nilsson, A. G., Anskaffning av standardsystem för att utveckla verksamheter. Utveckling och prövning av SIV-metoden.

Nilsson, J., Ekonomisk styrning i ett multinationellt företag. Research Report.

Normark, P., Swartz, E., Bolagisering av ekonomiska föreningar. Research Report.

Rutström, E., The Political Economy of Protectionism in Indonesia A Computable General Equilibrium Analysis.

Sjöstrand, S-E., Institution as Infrastructures of Human Interaction. Research Report.

Söderqvist, T., Measuring the Value of Reduced Health Risks: The Hedonic Price Technique Applied on the Case of Radon Radiation. Research Report.

Wahlund, R., Skatter och ekonomiska beteenden. En studie i ekonomisk psykologi om främst skattefusk och sparande utifrån 1982 års skatteomläggning.

Wahlund, R., Studenternas betalningsvilja för studier vid Handelshögskolan i Stockholm. Research Report.

Westelius, A., Westelius, A-S., Decentraliserade informationssystem. Två fallstudier inom ekonomistyrning. Research Report.

Westlund, K., Affärsetik. En studie av etiska bedömningar. Research Report.

Wirsäll, N-E., Julander, C-R., Den lättrörliga detaljhandeln.

1992

Charpentier, C., Ekonomisk styrning av statliga affärsverk.

Edlund, P-O., Karlsson, S., Forecasting the Swedish Unempolyment Rate: VAR vs. Transfer Function Modelling.

Eklöf, J., Varying Data Quality and Effects in Economic Analysis and Planning.

Eliasson, M., Julander, C-R., Productivity in Swedish Grocery Retailing. - changes over time and a causal model.

Ewing, P., Ekonomisk styrning av enheter med inbördes verksamhetssamband.

Fredriksson, O., Datorkommunikation i Distributionssystem.

Erfarenheter och effekter vid införandet av två multilaterala interorganisatoriska informationssystem - exemplet BASCET Infolink AB.

Fredriksson, T., Policies Towards Small and Medium Enterprises in Japan and Sweden. Research Report.

Gerdtham, U., Jönsson, B., Sjukvårdskostnader i de nordiska länderna. Research Report.

Hedvall, M., Produktkvalitet som konkurrensmedel i producentvaruproducerande industri. Research Report.

Holmberg, C., Effects of Feature Advertising and Other Promotions Research Report.

Jansson, D., Spelet kring investeringskalkyler. Norstedts

Jennergren, L.P., Näslund, B., Valuation of Executive Stock Options. Research Report.

Ljung, A., Intressentstrategier - En longitudinell studie av utvecklingen i två svenska företag.

Kokko, A., Foreign Direct Investment, Host Country Characteristics and Spillovers.

Mårtensson, P., Mähring, M., Information Support From Staff to Executives. - An Explorative Study. Research Report

Paalzow, A., Public Debt Management. Research Report.

Persson, P-G., Basket Analysis. A New Way of Studying Short Term Effects of Promotions in Grocery Retailing.

Sjöstrand, S-E., On Institutions and Institutional Change. Research Report.

Södergren, B., Decentralisering. Förändring i företag och arbetsliv.

Rabiee, M., A Review of Demand for Telecommunication: An Analytical Approach. Research Report.

Thodenius, B., Användningen av ledningsinformationssystem i Sverige: Lägesbild 1991. Research Report.

Tollgerdt-Andersson, I., Sjöberg, L., Intresse och kreativitet inom tjänsteproducerande företag. Research Report.

Wahl, A., Kvinnliga civilekonomers och civilingenjörers karriärutveckling.

1993

Ekvall, N., Studies in Complex Financial Instruments and their Valuation.

Söderlund, M., Omvärldsmodeller hos beslutsfattare i industriföretag - en studie av svenska leverantörer till fordonsindustrin.

Whitelegg, J., Hultén, S., Flink, T., High Speed Trains. Fast tracks to the future.