

# PUBLIC DEBT MANAGEMENT

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EFI RESEARCH REPORT  
May 1992

ISBN NR 91-7258-346-0



A Dissertation for the  
Doctor's Degree in Economics  
Stockholm School of Economics 1992

## Preface

Ever since the day he convinced me to commence graduate studies in economics, my thesis advisor Karl Jungenfelt has always been there to encourage and help me. Without Karl's unique way of guidance, not only characterized by his interest in research but also by his open-mindedness and in particular his interest in the well-being of his students, this thesis would never have been written. Needless to say, my gratitude to Karl is deep.

I am also in great debt to Ragnar Lindgren and Staffan Viotti who have generously spent much time reading and discussing my papers in detail. Their comments and encouragement have been very valuable.

Furthermore, I have benefitted from comments by Peter Högfeltdt, Hans Wijkander and seminar participants at the Stockholm School of Economics. Special thanks are due to Gunnar Dahlfors, Peter Jansson, and Kerstin Niklasson for scrutinizing the papers. With her usual excellence, Kerstin has also assisted me with various administrative details throughout the years. Thanks are due to Rune Castenäs at the Economic Research Institute at the Stockholm School of Economics for his help in arranging financial support for this project. I am also grateful to the faculty, staff and graduate students at the Department of Economics at the Stockholm School of Economics.

During the academic year 1988-89 I had the opportunity to visit the Department of Economics at the University of Rochester. The visit greatly spurred my interest in economics and I wish to thank the faculty, staff and graduate students for their generous hospitality. In particular I remember Lionel McKenzie's classes in general equilibrium theory.

Generous financial support from Finanspolitiska Forskningsinstitutet, the Royal Academy of Sciences, and the Stockholm School of Economics is gratefully acknowledged.

Finally, this thesis is dedicated to my parents, Lars and Märtha, who have encouraged and supported me throughout the 22 years I have spent at school.

Stockholm in May, 1992

*Anders Paalzow*



# Public Debt Management - Introduction and Summary of the Thesis

## 1 Introduction

Public debt management can be defined as open market operations carried out by the government in order to change the composition of the outstanding stock of government-issued debt instruments. Public debt management focuses only on changes in the composition of the outstanding public debt and takes the size of the public debt as given. The composition of public debt is usually characterized by the outstanding debt's maturity structure and public debt management is mainly concerned with changes in the maturity structure. The aim of public debt management can, e.g., be to minimize the cost of public debt or, as a part of the economic policy making, to control aggregate demand. The basic idea behind public debt management as a tool for economic policy making is the following; in order to induce investors to hold the new mix of government-issued debt instruments, changes in the structure of relative asset yields are necessary. These changes might have attendant effects on, e.g., the firms' cost of capital and the agents' consumption-investment decisions, and accordingly on the real economy. The almost permanent budget deficits in many countries during the last two decades have led to a rapid growth in public debt that has stimulated the interest in public debt management. To finance these deficits many governments have been forced to introduce new types of debt instruments and to deregulate financial markets. The growth in debt and the richer menu of debt instruments have increased the scope for using public debt management as a tool of economic policy making<sup>1</sup>.

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<sup>1</sup>For a discussion of public debt management in general, see Dornbusch and Draghi (1990), and Agell and Persson (1992). For a review of public debt management in Sweden,

The early literature on public debt management, e.g., Cohen (1955) and Rolph (1957), mainly discusses minimization of the government's costs for the debt as the objective of public debt management. The rationale behind this objective is that minimization of the costs associated with public debt also implies minimization of the taxes needed to finance the debt and hence minimization of the excess burden.

Musgrave (1959) and in particular Tobin (1963) show that the government also could affect the economic activity through public debt management, i.e., public debt management could be used as an instrument in stabilization policy. The literature following Tobin (1963), e.g., Tobin (1969), Friedman (1978), Roley (1979), and Agell and Persson (1988), focuses mainly on this aspect of public debt management. Although they have focused on the same aspect of public debt management, their approaches have been different. Tobin and Friedman use the same approach as Tobin (1963); an extended version, with respect to the number of assets, of the Keynesian model with two assets (money and capital). In these models the effects of public debt management on economic activity are determined by the substitutability of government-issued bonds *vis-a-vis* capital and money, respectively. In these models the government should, in order to increase the private sectors' investments, reduce the supply of government-issued bonds that are close substitutes to corporate equity and issue bonds that are distant substitutes. This policy increases the demand for corporate equity and hence lowers the firms' costs of capital, which increases (crowds in) private investment. Roley, and Agell and Persson, on the other hand, use the atemporal capital asset pricing model developed by Sharpe (1964) and Lintner (1965) to discuss the effects of public debt management. In these models the effects of public debt management depend on the covariances of the government-issued bonds' returns with the returns on private capital. By changing the supply of government-issued bonds with different covariances, the government can affect the relative asset yields. The changes in relative asset yields affect the valuation of corporate equity and hence the cost of capital, which in turn affects the investments.

However, aspects other than cost minimization and economic stimulation have been discussed in the literature. Chan (1983), for example, discusses the conditions under which the composition of the outstanding public debt

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see Schön (1989).

is irrelevant, i.e., when public debt management is neutral. Gale (1990) discusses the welfare issues of public debt management, in particular the impact of public debt management on the efficiency of risk sharing.

There have also been a number of empirical studies of the effects of public debt management. The empirical studies provide mixed evidence on the effects of public debt management. Modigliani and Sutch (1967) study the effects of Operation Twist, which was carried out in 1961 by the Federal Reserve System in the United States. Operation Twist aimed at shortening the average maturity of the outstanding public debt in order to raise the short-term interest rates and lower the long-term interest rates. Modigliani and Sutch find no or weak evidence of effects on the term structure of interest rates. Agell and Persson (1988) empirically analyze the effects of public debt management on the financial markets using an atemporal capital asset pricing model. They estimate the covariance matrix of asset yields using US data and find that public debt management affects the relative asset yields. However, Agell and Persson find these effects too small to have any impact on the real economy. Friedman (1992) uses US data and a general equilibrium framework where the financial markets are modelled in essentially the same way as in Agell and Persson. He finds that public debt management has effects on the relative asset yields and on the real economy.

## 2 Summary of the Thesis

This thesis consists of three self-contained papers covering different aspects of public debt management. From a methodological point of view they all have in common that results and models from the theory of finance are used to analyze the effects of public debt management.

The first paper, *Neutrality of Public Debt Management*, studies the case when public debt management does not matter, i.e., when it is neutral. Although strong assumptions are needed to ensure neutrality of public debt management it is nevertheless of interest to study it, since an analysis illuminates the mechanisms through which public debt management affects the economy. The paper starts with a discussion of the assumptions that are needed to ensure neutrality in the models used in the literature. The remainder of the paper tries to relax some of these assumptions. The model employed is an intertemporal general equilibrium model. It is shown that if the

agents are identical, public debt is neutral provided the agents pierce the veil of government, and all taxes associated with public debt are lump-sum. It is also shown that if the agents are different but have sufficiently similar utility functions that exhibit hyperbolic absolute risk aversion (i.e., the agents have linear risk tolerance), public debt management is neutral in aggregates, provided the agents pierce the veil of government and all taxes associated with the debt service are lump-sum. This means that public debt management neither affects prices nor aggregate consumption; it might, however, affect the individual agent's consumption. Since the class of utility functions that exhibit hyperbolic absolute risk aversion is widely used in economic analysis, this result has several theoretical and empirical implications. The result also has implications for the choice of model in the third paper.

The second paper, *Objectives of Public Debt Management*, discusses the objectives of public debt management in an atemporal mean-variance framework. The model employed in this paper differs in one important aspect from the ones previously used in the literature; it takes the firms' investment decisions into account and hence endogenizes the supply of assets to some extent. It is shown that if the firms' behavior is introduced, objectives that in the literature have been assumed to stimulate the economic activity do not necessarily have the desired effect. The paper also discusses different objectives aiming at welfare-improvements and economic stimulation. Since the analysis is performed in a unified framework, it is possible to compare the objectives and to discuss their welfare implications. Of particular interest is the welfare aspects of minimization of the costs of public debt. Finally, the paper also discusses the effectiveness of the objectives and it is shown that with one exception, cost minimization, effectiveness declines when the government-issued debt instruments' share of the asset market falls.

The last paper, *Public Debt Management and the Term Structure of Interest Rates*, develops and uses a stochastic overlapping generations model to analyze the impact of public debt management on the term structure of interest rates. In most of the literature public debt management is thought of as changes in the maturity structure of the outstanding public debt. A change in the maturity structure implies that public debt management affects, e.g., future tax liabilities and hedging opportunities. To capture these effects it is necessary to use an intertemporal framework. In contrast to most models in the literature on public debt management, the model in this paper is intertemporal and takes the general equilibrium effects of public debt

management into account, since it integrates the financial and real sectors of the economy. This means that current and future asset prices, as well as investments are affected by public debt management. The analysis suggests that it is not the quantities of long-term and short-term bonds, *per se*, that determine the effects on the term structure of interest rates. What determines these effects is how public debt management affects the hedging opportunities through changes in asset supply, taxes and prices.

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# Neutrality of Public Debt Management

## 1 Introduction

This paper examines neutrality of public management in an intertemporal model with uncertainty. Public debt management is defined as operations made by the government on the financial markets in order to change the composition of the outstanding public debt, for example with respect to the maturity structure. To distinguish between the effects due to public debt management and the effects due to fiscal policy, public debt management has to be self-financed. This means that if the government purchases one type of government-issued bond it simultaneously has to issue other types of bonds in order to finance the purchase without affecting the level of government expenditure. To induce agents to hold the new mix of government-issued bonds, a change in the bond-prices might be necessary. This change might have attendant effects on the whole asset market, and through changes in the agents' asset portfolios the effects of public debt management are transmitted to the real economy. If public debt management does not affect the real economy, then it is neutral. Two types of conditions are crucial for neutrality to emerge. The first condition focuses on the private sector's internalization of the government's budget constraints. Public debt management is neutral if the agents can and also find it optimal to undo the effects of public debt management by changing their consumption and investment decisions. This condition hinges on whether the agents recognize the future effects of public debt management, e.g., on their future tax liabilities. The second condition focuses on the asset markets and how public debt management, via the asset markets, affects the assets' yields, the spanning of the economy's uncertainty, the trading opportunities, and accordingly the set of feasible consumption plans. For public debt management to be neutral it is necessary that the set of feasible consumption plans is not affected.

In order to fulfill these two conditions and hence to ensure neutrality, strong assumptions are required. Throughout the literature on neutrality it has been assumed that the agents pierce the veil of government, i.e., they recognize the relation between their own budget constraints and those of the government. Furthermore to ensure that the set of feasible consumption plans is unaffected, either of the following two assumptions (or versions of them) has been used: the markets are complete or there exist perfect substitutes for government-issued bonds. Although quite strong and unrealistic assumptions are necessary to ensure neutrality, it is nevertheless interesting to analyze neutrality since an analysis illuminates and gives an understanding of the mechanisms of public debt management. The main purpose of this paper is to analyze if it is possible to obtain neutrality without making some of the assumptions associated with the second condition discussed above.

The remainder of the paper is organized in the following way. Section 2 briefly reviews the literature on public debt management. The discussion focuses on the assumptions that are needed to ensure neutrality in the different models used in the literature. The next section presents the model employed; a general equilibrium model with uncertainty and infinitely long-lived agents. In Section 4 public debt management is discussed without making some of the standard assumptions used in the literature. The section begins with the almost trivial case when the agents are identical. This case is used to illuminate the effects of public debt management and to determine the conditions that are necessary to ensure neutrality. The rest of the section is devoted to the case when the agents are different. Of special interest is the case when the agents are different but have utility functions that are sufficiently similar and exhibit hyperbolic absolute risk aversion, i.e., the agents have linear risk tolerance. The last section summarizes the results and discusses their theoretical and empirical implications.

## 2 A Review of the Literature

In the literature public debt management has been analyzed in three different frameworks, atemporal macro-models, atemporal micro-models, and intertemporal micro-models. The atemporal macro-models of public debt management, used by for example Tobin (1963, 1969) and Friedman (1978), are extended versions, with respect to the number of assets, of the Keyne-

sian model with two assets (money and bonds). These analyses focus on the assets' gross substitutability since the effects of public debt management in these models depend on the relative degrees of substitutability of bonds *vis-à-vis* capital and money, respectively. If a certain type of bond is a closer substitute for capital (than it is for money), then an increase in the supply of that type of bond will raise the rate of return on capital as well as the interest rate. On the other hand, if the bond is a closer substitute for money, an increase in the supply of that particular bond will lower the rate of return on capital but still raise the interest rate. Although the question of neutrality of public debt management has not been explicitly addressed in these models, it is obvious that public debt management is neutral if neither the IS-curve nor the LM-curve shifts. It can be shown that if the following two sufficient conditions are fulfilled then public debt management is neutral in this type of models. Firstly, the government-issued bonds whose supply is affected by debt management have to have the same relative asset substitutabilities *vis-à-vis* the returns on all assets in the economy. Secondly, the relative asset substitutabilities of an asset whose supply is not affected by debt management, *vis-à-vis* any of the government-issued bonds whose supply is affected by debt management, should be the same. If the asset demand functions are derived from utility maximization and the agents maximize the expected value of a strictly concave von Neumann-Morgenstern utility function, the conditions for neutrality stated above are fulfilled if the government-issued bonds whose supply is affected by debt management have identical variances and covariances of returns, i.e., if they are perfect substitutes<sup>1</sup>.

In the atemporal micro-models that are used to analyze public debt management the mean-variance model developed by Sharpe (1964) and Lintner (1965) is employed. These models used by for example Roley (1979), Agell and Persson (1988), and Jungenfelt (1989), focus particularly on the impact of public debt management on the relative asset yields. Unlike the macro-models, the demand for assets in the micro-models is explicitly derived from the agents' optimizing behavior. In these models the effects of public debt management are determined by the investors' assessments of the variances and covariances of the future stochastic asset returns that determine the val-

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<sup>1</sup>To derive this result, a result in Blanchard and Plantés (1977) on the relation between the variance-covariance matrix of asset returns and the Jacobian of asset demands with respect to asset returns has been used.

uation and hence the yields of the risky assets. By changing the supply of government-issued bonds, the government not only affects the yields of the government-issued bonds, but also the yields of other assets through changes in the market risk due to public debt management. Public debt management is neutral in these models if it does not affect the asset yields. A sufficient condition for neutrality is that the government-issued bonds whose supply changes through public debt management are perfect substitutes. This condition implies that the bonds (or combination of bonds) whose supply is increased must have a payoff structure identical with that of the bonds (or combination of bonds) whose supply is reduced. If this is the case, the bonds (or combinations of bonds) have identical variances and covariances. Accordingly, public debt management leaves the market risk unaffected, and the asset yields are therefore not affected and public debt management is neutral. A comparison of the sufficient conditions for neutrality in the atemporal macro- and micro-models reveals that the same condition applies in both cases, namely that the government-issued bonds (or combination of bonds) whose supply is changed by public debt management are perfect substitutes. This is hardly surprising since the conditions for neutrality in the two types of atemporal models were derived under the same assumptions regarding the agents' behavior. However, the atemporal models have at least one serious drawback when they are used to analyze neutrality. They only focus on the valuation of the assets and the spanning of the economy's uncertainty, and fail to take the changes in for example future payoffs and taxes due to public debt management into account. Obviously, this is a serious drawback in a model trying to explain the effects of public debt management.

In the literature using intertemporal models to analyze the effects of public debt management, applications of the Modigliani-Miller theorem on the irrelevance of the firm's capital structure [Modigliani and Miller (1958)] have been widely used to show neutrality. To apply the theorem to public debt management it is necessary that the agents pierce the veil of government, that all taxes associated with the debt service are lump-sum, and that the asset markets are complete. The first assumption implies that the agents are able to recognize the relation between private and public budget constraints and accordingly able to take the implicit liabilities associated with public debt into account. The second assumption is necessary in order to exclude distortionary effects of the taxes associated with the debt service. The third

assumption implies that the government-issued bonds are redundant<sup>2</sup>. Redundancy is necessary in order to ensure that public debt management does not affect the set of feasible trading opportunities and that the agents have the opportunity to undo the effects of public debt management. However, this is a very strong assumption and either of the following three somewhat less strong assumptions has been used in the literature in order to ensure neutrality of public debt management when the markets are incomplete. (i) There exist perfect substitutes, issued by the private sector, for the government-issued bonds, and unlimited short sales of the substitutes are allowed. This assumption is used by Chan (1983) as well as Stiglitz (1983). (ii) Unlimited short sales of the government-issued bonds are allowed and the government is neither allowed to create and issue new assets (i.e., assets whose payoff structures cannot be replicated by the existing and outstanding government-issued bonds' payoff structures), nor to withdraw all outstanding bonds of a certain type from the market. Gale (1990) uses this assumption. (iii) The government is only allowed to issue bonds (or linear combinations of bonds) that are perfect substitutes for (i.e., have an identical payoff structure as) the government-issued bonds (or linear combinations of bonds) purchased by the government. This assumption is used by Stiglitz (1988).

Assumptions (i) to (iii) as well as the conditions needed to ensure neutrality in the atemporal models all have in common that they are needed to guarantee that public debt management does not affect the trading opportunities and hence the feasible consumption plans in the economy. The remainder of this paper discusses how far one can get without making these assumptions.

### 3 The Model

To illuminate the ideas behind neutrality of public debt management a model with a minimum of detail is employed. However, the results would be the same if, for example, government consumption and other types of assets (e.g., securities) were introduced in the model. Throughout the analysis it is assumed that the markets are incomplete and that unrestricted short sales are not allowed. Consider an economy with infinitely long-lived agents, where

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<sup>2</sup>An asset is redundant if its cash flows, today and in the future, can be replicated by a linear combination of the other assets' cashflows.

the  $i$ :th agent maximizes a well-behaved utility function:

$$U^i(c^i(t_0), c^i(t_1), c^i(t_2), \dots) \quad (1)$$

where  $c^i(t)$  is the  $i$ :th agent's consumption at time  $t$ . At time  $t_0$  the  $i$ :th agent is endowed with  $\bar{e}^i$  units of the consumption good, there are no endowments in the future periods. The endowment can either be consumed or invested. The agent can either invest in an individual production process or in  $n$  different types of government-issued bonds<sup>3</sup>. Since the markets are assumed to be incomplete the number of non-redundant production processes and bonds is assumed to be less than the number of states. The bonds are traded in perfect capital markets, i.e., there are no transaction costs and the bonds are infinitely divisible. For simplicity but without loss of generality the bonds are assumed to be zero-coupon bonds. The output of the production process as well as the government-issued bonds' redemption values are assumed to be stochastic and depend on the realization of the state variable (or vector of state variables),  $s$ . The redemption of the government-issued bonds is financed through lump-sum taxes, which are stochastic since the redemption prices are stochastic. In period  $t_0$  the tax incidence for all future periods and states is known to the agents. The  $i$ :th agent's budget constraints can now be written as:

$$c^i(t_0) = \bar{e}^i - z^i(t_0) - \sum_{k=1}^n p_k(t_0)q_k(t_0) \quad (2)$$

and for  $t > t_0$ :

$$\begin{aligned} c^i(t, s(t)) &= f^i[z^i(t-1), s(t)] + \sum_{k \in T(t)} \pi_k(t, s(t))q_k^i(t-1) \\ &- \sum_{k=1}^n p_k(t, s(t))[q_k^i(t, s(t)) - q_k^i(t-1)] \\ &- z^i(t, s(t)) - \tau^i(t, s(t)) \end{aligned} \quad (3)$$

where  $s(t)$  is the realized state at time  $t$ ,  $f^i$  the production function of the  $i$ :th agent's production process,  $z^i$  the  $i$ :th agent's investment in the production

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<sup>3</sup>For notational simplicity it is assumed that there is only one type of individual production process, no other types of bonds or securities and no capital accumulation in the model. However, the results in Section 4.1 will still hold if more production processes and/or other types of bonds/securities and/or capital accumulation are introduced in the model.

process,  $\pi_k$  the redemption price of the  $k$ :th bond,  $q_k^i$  the quantity purchased of the  $k$ :th government-issued bond,  $T(t)$  is the set of bonds with maturity at time  $t$ ,  $p_k$  the price of the  $k$ :th government-issued bond, and  $\tau^i$  the lump-sum tax. Hence, the  $i$ :th agent maximizes (1) subject to (2) and (3). The general solution to this problem is characterized by the following sequences of optimal consumption and investment plans:  $\{c^{i*}(t)\}_{t=t_0}^\infty$ ,  $\{z_j^{i*}(t)\}_{t=t_0}^\infty$ , and  $\{q_k^{i*}(t)\}_{t=t_0}^\infty$ .

Consider the government, which has inherited a stock of outstanding government-issued zero-coupon bonds of  $n$  different types. The initial stock is given by the vector  $q^g(t_0)$ . The bonds differ with respect to their maturity dates and/or with respect to their state contingent redemption prices. The debt service is the government's only expenditure and it is financed by lump-sum taxes. Hence the government's budget constraint is given by:

$$\tau(t, s(t)) = \sum_{k \in T(t)} \pi_k(t, s(t)) q_k^g(t, s(t)) \quad (4)$$

where  $\tau(t, s(t))$  is the total tax, which is given by:

$$\tau(t, s(t)) = \sum_i \tau^i(t, s(t)) = \sum_i \lambda^i \tau(t, s(t)) \quad (5)$$

where  $\lambda^i$  is the  $i$ :th agent's fraction of the total tax liabilities, and hence  $\sum_i \lambda^i = 1$ . This fraction is known to the agent and the same in all periods and in all states. For the agents to pierce the veil of government it is necessary that they have perfect foresight with respect to (4) and (5).

An equilibrium is defined as a set of stochastic processes  $\{c^{i*}(t)\}_{t=t_0}^\infty$ ,  $\{z_j^{i*}(t)\}_{t=t_0}^\infty$ ,  $\{q_k^{i*}(t)\}_{t=t_0}^\infty$ , and  $\{\tau^{i*}(t)\}_{t=t_0}^\infty$  such that (1) is maximized subject to (2) and (3), the government's budget is balanced and the bond markets clear, i.e.:

$$\sum_i q_k^i(t, s(t)) = q_k^g(t) \quad (6)$$

for all  $k$ , and for all dates and states. Since the purpose of this paper is to show the invariance of the equilibrium under different public debt management policies it is assumed that an initial equilibrium exists and that it is unique.

Finally, it remains to define public debt management. Public debt management is defined as self-financed transactions in government-issued bonds,

carried out by the government, in order to change the composition of the stock of outstanding government-issued bonds. The transactions are assumed to be carried out at the new equilibrium prices. In the general case the government is allowed to issue new types of bonds (i.e., types of bonds that initially do not belong to the stock of outstanding bonds) and not merely restricted to change the composition of the outstanding stock by trading in outstanding and hence already existing types of government-issued bonds. This means that public debt management might change the number of different types of government-issued bonds that are available to the agents. Let  $N$  denote the total number of different types of outstanding and non-outstanding bonds that the government can choose from. Then a feasible public debt management policy must satisfy:

$$\sum_{k=1}^N \hat{p}_k(t, s(t)) [\hat{q}_k^g(t, s(t)) - q_k^g(t)] = 0 \quad (7)$$

where  $q_k^g = 0$  if there are initially no bonds of the  $k$ :th type outstanding and  $\hat{q}_k^g = 0$  if no bonds of the  $k$ :th type are outstanding after public debt management has taken place. Public debt management is assumed to take place at time  $t_0$  and changes the supply of government-issued bonds for all future periods. The agents are assumed to have perfect foresight with respect to the effects of public debt management on the future taxes, i.e., the agents pierce the government veil.

## 4 Neutrality of Public Debt Management

This section studies public debt management without making the ordinary assumptions of the existence of perfect substitutes for the government-issued bonds, complete markets, or unlimited short sales. Its main purpose is to see how far one can get without making these fairly unrealistic assumptions. It also reveals the mechanisms through which public debt management affects the economy. The first part of this section discusses neutrality of public debt management in the general case. Two different cases are discussed, when the agents are identical and when they are not. If the agents are identical, then neutrality is almost trivial. However by using the extremely strong assumption that the agents are identical, it is possible to pinpoint the necessary (although in most cases not sufficient) requirements for neutrality

of public debt management. The analysis also shows how the agents can undo the effects of public debt management. The second part of this section discusses neutrality when the agents are different but have sufficiently similar utility functions that exhibit hyperbolic absolute risk aversion, i.e., when the agents' have linear risk tolerance.

## 4.1 The General Case

To prove neutrality it will first be shown that the initial consumption plan is feasible under the new debt management policy. Then it will be shown that it is optimal for the agents to maintain their initial consumption plans under the new policy<sup>4</sup>.

For simplicity but without loss of generality it is assumed that the stock of government-issued bonds comprises two types of bonds; one-period bonds (denoted  $S$ ) and two-period bonds (denoted  $L$ ). It is also assumed that public debt management changes the supply of these two types of bonds. The changes are given by  $\Delta q_S^g$  and  $\Delta q_L^g$ , respectively.

The feasibility of the agents' old consumption plans under the new policy will be shown by working backwards from the agents' period  $t_2$  budget constraints. Feasibility in the periods beyond  $t_2$  follows from the fact that if the initial consumption plans are feasible in periods  $t_0$  to  $t_2$  then they are feasible in all future periods, since public debt management has no effects in the periods beyond  $t_2$ . Consider the  $i$ :th agent's period  $t_2$  budget constraint, when all agents maintain their pre-public debt management consumption under the new policy:

$$\begin{aligned}
c^i(t_2, s(t_2)) &= f^i[\hat{z}^i(t_1), s(t_2)] + \pi_L(s(t_2))\hat{q}_L^i(t_1) - \hat{z}^i(t_2, s(t_2)) \\
&\quad - \lambda^i \pi_L(s(t_2))\hat{q}_L^g \\
&= f^i[\hat{z}^i(t_1), s(t_2)] + \pi_L(s(t_2))[q_L^i(t_1) + \Delta q_L^i(t_1)] \\
&\quad - \hat{z}^i(t_2, s(t_2)) - \lambda^i \pi_L(s(t_2))[q_L^g + \Delta q_L^g]
\end{aligned} \tag{8}$$

It is easily seen that if the  $i$ :th agent changes his holding of two-period bonds with  $\lambda^i \Delta q_L^g$  units and does not change the investment in the production process, then it is possible to choose the initial consumption path in period  $t_2$  as well as in all future periods provided the choices of  $\hat{z}^i(t_1) = z^i(t_1)$  and

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<sup>4</sup>This approach is similar to the one used in Chan (1983).

$\Delta q_L^i = \lambda^i \Delta q_L^g$  are feasible in the preceding period(s). To see whether this is the case, consider the period  $t_1$  budget constraint when the agent chooses  $\hat{c}^i(t_1) = c^i(t_1)$ ,  $\hat{z}^i(t_1) = z^i(t_1)$ , and  $\Delta q_L^i(t_1) = \lambda^i \Delta q_L^g$  :

$$\begin{aligned}
c^i(t_1, s(t_1)) &= f^i[z^i(t_0), s(t_1)] + \hat{p}_L(s(t_1))[q_L^i(t_0) - \lambda^i \Delta q_L^g - (q_L^i(t_1) \\
&\quad + \lambda^i \Delta q_L^g)] + \pi_S(s(t_1))\hat{q}_S^i(t_0) - \hat{z}^i(t_1, s(t_1)) - \lambda^i \pi_S(s(t_1))\hat{q}_S^g \\
&= f^i[z(t_0), s(t_1)] + \hat{p}_L(s(t_1))[\hat{q}_L^i(t_0) - \hat{q}_L^i(t_1)] \\
&\quad + \pi_S(s(t_1))[q_S^i(t_0) + \Delta q_S^i(t_0)] - \hat{z}^i(t_1, s(t_1)) \\
&\quad - \lambda^i \pi_S(s(t_1))[q_S^g + \Delta q_S^g]
\end{aligned} \tag{9}$$

Hence, if the agent changes his holdings of one-period bonds with  $\lambda^i \Delta q_S^g$  units he can maintain his initial consumption path provided  $\hat{z}(t_0) = z(t_0)$ ,  $\Delta q_S^i(t_0) = \lambda^i \Delta q_S^g$ , and  $\Delta q_L^i(t_0) = \lambda^i \Delta q_L^g$  are feasible and the price of the two-period bonds does not change in period  $t_1$ . The bond markets will clear at the old prices in period  $t_1$  since the change in the supply of bonds is offset by an increase in the demand (since  $\Delta q_L^g = \sum_i \lambda^i \Delta q_L^g$ ), and  $c^i(t_1, s(t_1))$ ,  $c^i(t_2, s(t_2))$ , and  $z^i(t_1, s(t_1))$  are at their initial levels. Finally consider the time  $t_0$  budget constraint:

$$\begin{aligned}
\bar{e}^i - c^i(t_0, s(t_0)) &= z^i(t_0) + \hat{p}_L(t_0)\hat{q}_L^i(t_0) + \hat{p}_S(s(t_0))\hat{q}_S^i(t_0) \\
&= z^i(t_0) + \hat{p}_L(s(t_0))[\hat{q}_L^i(t_0) + \lambda^i \Delta q_L^g] \\
&\quad - \hat{p}_S(s(t_0))[q_S^i(t_0) + \lambda^i \Delta q_S^g]
\end{aligned} \tag{10}$$

Accordingly the old plans  $\{c^i(t, s)\}_{t=t_0}^\infty$  and  $\{z^i(t, s)\}_{t=t_0}^\infty$  are feasible, since the choices of  $c^i$  and  $z^i$  are the same as in the initial equilibrium and the bond markets clear at the new quantities since  $\sum_i \lambda^i \Delta q_S^i = \Delta q_S^g$  and  $\sum_i \lambda^i \Delta q_L^i = \Delta q_L^g$ . Hence, the bond markets clear at the old prices, i.e.,  $\hat{p}_S = p_S$  and  $\hat{p}_L = p_L$ . In other words the agent sells (purchases)  $\lambda^i \Delta q_S^g$  units of the short bond, which will give (cost) him  $p_S(t_0)\lambda^i \Delta q_S^g$ . From the definition of public debt management follows that this is equal to  $p_L(t_0)\lambda^i \Delta q_L^g$ , which is the cost of an investment in (the income from the sale of)  $\lambda^i \Delta q_L^g$  units of the long bond. Accordingly the agent's budget constraint is not affected and it is therefore possible for him to undo the effects of public debt management.

Now it remains to show that it is optimal for them to do so, i.e., that there does not exist a sequence  $\{\hat{c}(t, s)\}_{t=t_0}^\infty$  such that  $U^i(\hat{c}^i(t_0), \hat{c}^i(t_1), \hat{c}^i(t_2), \dots) > U^i(c^i(t_0), c^i(t_1), c^i(t_2), \dots)$ . First consider the case when all agents are identical with respect to endowments and preferences. Furthermore assume that

each agent's fraction of the total tax liabilities is given by  $\lambda^i = \lambda = 1/I$ , where  $I$  is the number of agents. These assumptions make it possible to work with a representative agent and the index  $i$  can therefore be omitted in the analysis. To show that it is optimal for the agents to undo the effects of public debt management a proof by contradiction will be used. Assume that under the new policy there exist sequences  $\{\hat{c}(t, s)\}_{t=t_0}^{\infty}$  and  $\{\hat{z}(t, s)\}_{t=t_0}^{\infty}$  such that  $U(\hat{c}(t_0), \hat{c}(t_1), \hat{c}(t_2), \dots) > U(c(t_0), c(t_1), c(t_2), \dots)$ . Suppose the agent chooses these plans under the initial debt management policy, then his cost in period  $t_0$  is given by:

$$\hat{c}(t_0) + z(t_0) + \tilde{p}_S(t_0)q_S(t_0) + \tilde{p}_L(t_0)q_L(t_0) \quad (11)$$

where a tilde denotes the prices if the agents initially choose  $\hat{c}$ . Let  $\hat{c}(t_0) = c(t_0) + \eta(t_0)$  and  $\hat{z}(t_0) = z(t_0) + \epsilon(t_0)$ . Since the resources available in the economy in period  $t_0$  are given by the agents' endowments it follows that  $\eta(t_0)$  and  $\epsilon(t_0)$  must satisfy:

$$\eta(t_0) + \epsilon(t_0) + [\tilde{p}_S(t_0) - p_S(t_0)]q_S(t_0) + [\tilde{p}_L(t_0) - p_L(t_0)]q_L(t_0) = 0 \quad (12)$$

Use (12) to write (11) as:

$$\begin{aligned} c(t_0) + \eta(t_0) + z(t_0) + \epsilon(t_0) + \tilde{p}_S(t_0)q_S(t_0) + \tilde{p}_L(t_0)q_L(t_0) = \\ c(t_0) + z(t_0) + p_S(t_0)q_S(t_0) + p_L(t_0)q_L(t_0) \end{aligned} \quad (13)$$

where the right hand side is equal to the cost of the initial consumption and investment plan in period  $t_0$ . The same argument can be applied to all periods after  $t_0$ . Accordingly no consumption plan  $\{\hat{c}(t, s)\}_{t=t_0}^{\infty}$  such that  $U(\hat{c}(t_0), \hat{c}(t_1), \hat{c}(t_2), \dots) > U(c(t_0), c(t_1), c(t_2), \dots)$  exists. The following proposition can now be stated.

**Proposition 1** *If all agents are identical with respect to preferences, endowments and fraction paid of the total tax liabilities, then public debt management is neutral provided the taxes are lump-sum and the agents pierce the veil of government, i.e., they internalize the government's budget constraints.*

This result relies upon two facts. Firstly, that public debt management does not create any new resources, i.e., it leaves the resources available for consumption and investment unaffected. Secondly, that when the agents are

similar, there are no gains from trade and hence no trade will occur<sup>5</sup>. Accordingly, the change in the trading opportunities induced by public debt management will not have any effect. Not surprisingly, the analysis of the case when the agents are identical has shown that for public debt management to be neutral it is necessary that the taxes associated with the debt service are lump-sum and that the agents have perfect foresight with respect to these taxes as well as the tax incidence.

Consider the case when the agents are different. If the agents are different, the change in the set of feasible asset trades affects the set of feasible consumption plans, and the arguments used when all agents were identical cannot be used. Assume that the agents' demand functions are continuous, all agents' choices of asset trades are interior, and the government does not create any new assets. If this is the case, public debt management is neutral for marginal changes in the stock of government-issued bonds. This result can be summarized in the following proposition<sup>6</sup>.

**Proposition 2** *If the agents differ, public debt management is neutral for marginal changes in the stock of outstanding government-issued bonds provided the government does not create any new (i.e. non-redundant) assets, the debt service is financed through lump-sum taxation, the agents' demand functions are continuous, their initial asset trades are interior, and they pierce the veil of government.*

This is all that can be said on neutrality of public debt management in the general case when the agents differ. However, in the next section it will be shown that more can be said if the agents' utility functions are sufficiently similar and exhibit hyperbolic absolute risk aversion. Finally, it is important to note that when the agents differ, then in most cases public debt management is not neutral, although it does not change the resources available to the agents. This is due to the fact that public debt management might have redistributive effects, e.g., through changes in taxes, and that it creates new trading opportunities.

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<sup>5</sup>See Lucas (1978).

<sup>6</sup>See Barro (1974) and Chan (1983).

## 4.2 Different Agents with HARA Utility Functions

To discuss neutrality of public debt management when the agents are not identical but have utility functions that exhibit hyperbolic absolute risk aversion (HARA) the following result will be used [see e.g. Rubinstein (1974)]<sup>7</sup>. If all agents have utility functions of the same type that exhibit hyperbolic absolute risk aversion and have homogeneous beliefs and in some cases the same rate of patience, the asset market is effectively complete<sup>8</sup>. This means that the assets are priced as if there was a complete market. It also implies that the asset prices are determined independently of the distribution of initial wealth, i.e., the utility functions satisfy the aggregation property.

The economy considered in this section differs in the following ways from the one discussed in the preceding sections. The agents maximize additive von Neumann-Morgenstern utility functions that exhibit hyperbolic absolute

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<sup>7</sup>If the utility functions exhibit hyperbolic absolute risk aversion then:

$$-\frac{u''(w)}{u'(w)} = \frac{1}{\alpha + \beta w}$$

Since the agent's risk tolerance is defined as the reciprocal of the Arrow-Pratt measure of absolute risk aversion, it is easily seen that hyperbolic absolute risk aversion is equivalent to linear risk tolerance. The utility functions that exhibit linear risk tolerance are of the form:

$$u(w) = \frac{1 - \gamma}{\gamma} \left( a + \frac{bw}{1 - \gamma} \right)^\gamma$$

subject to  $\gamma \neq 1$ ,  $a > 0$ ,  $a + bw/(1 - \gamma) > 0$ , and  $a = 1$  if  $\gamma = -\infty$ . This family of utility functions includes exponential ( $\gamma = -\infty$ ,  $b = 1$  and hence  $\beta = 0$  in the expression for the absolute risk aversion), logarithmic ( $\gamma = 0$  and hence  $\beta = 1$ ) and power utility functions ( $\gamma < 1$  and hence  $\beta \neq 0, 1$ ), i.e.:

$$u(w) \sim -\alpha \exp(-w/\alpha)$$

$$u(w) \sim \ln(\alpha + w)$$

$$u(w) \sim \frac{\beta - 1}{\beta} (\alpha + \beta w)^{(\beta - 1)/\beta}$$

where  $\sim$  means "is equivalent up to an increasing linear transformation to". Hence by a suitable choice of parameters one can have utility functions that exhibit absolute or relative risk aversion that is constant, increasing or decreasing [see e.g. Merton (1971)].

<sup>8</sup>An effectively complete economy exhibits universal portfolio separation, i.e., the optimal portfolio of risky assets is the same for all agents and it is only their holdings of that portfolio that will differ [see Pye (1967) and Rubinstein (1974)].

risk aversion, the agents have access to a risk-free asset in zero net supply, and the agents do not have access to any individual production process (i.e.,  $f^i(\cdot)$  in the previous sections). However, the results are valid in an economy with production where the agents can buy claims (i.e., securities) on the output of the production processes, provided these claims are in zero net supply and the output of the production processes is not affected by the agents' investments in these claims. For simplicity but without loss of generality, consider a two-period effectively complete economy where the  $i$ :th agent maximizes his additive von Neumann-Morgenstern utility function over present consumption and future wealth<sup>9</sup>:

$$u^i(c^i(t_0)) + \rho^i v^i(w^i(t_1, s(t_1))) \quad (14)$$

where  $\rho^i$  is his rate of patience, and  $w^i(t_1, s(t_1))$  his next period wealth after taxes if state  $s$  is realized (this is equal to  $c^i(t_1, s(t_1))$  since a two-period model is considered). The second-period wealth after taxes can be written as:

$$w^i(t_1, s(t_1)) = y^i(t_1, s(t_1)) - \tau^i(t_1, s(t_1)) \quad (15)$$

where  $y^i(t_1, s(t_1))$  is the  $i$ :th agent's second-period pre-tax wealth and where  $\tau^i(t_1, s(t_1))$  is his lump-sum tax. The utility functions  $u^i(\cdot)$  and  $v^i(\cdot)$  are assumed to exhibit hyperbolic absolute risk aversion. All agents' utility functions are assumed to be of the same type (i.e., logarithmic, exponential or generalized power utility functions). However, the agents might differ with respect to endowments, taste, and fraction of the total tax liabilities associated with the debt service<sup>10</sup>. Since the market is effectively complete, the unique state prices can be used to write the  $i$ :th agent's budget constraint as:

$$\bar{e}^i - c^i(t_0) = \sum_s p_s(t_0) y^i(t_1, s(t_1)) \quad (16)$$

The first-order conditions for utility maximization can then be written as<sup>11</sup>:

$$u'_i(c^i(t_0)) = \rho^i p_s(t_0)^{-1} v'_i(w^i(t_1, s(t_1))) \quad (17)$$

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<sup>9</sup>The analysis can also be applied to the case when there are more than two periods.

<sup>10</sup>If the agents' utility functions are negative exponential, heterogeneous beliefs and time preferences are also allowed. The representative agent's expectations and time preferences will then be composite of the agents' expectations and time preferences. See Huang and Litzenberger (1988).

<sup>11</sup>If there are more than two periods then the Euler equation [see e.g., Sargent (1987)]

For simplicity assume that  $u^i(\cdot)$  and  $v^i(\cdot)$  are generalized power utility functions for all agents, i.e., given by  $[\beta/(1-\beta)][\alpha^i + \beta c^i(t_0)]^{(1-\beta)}$ , and that  $\rho^i = 1$ , then (17) can be written as<sup>12</sup>:

$$[\alpha^i + \beta c^i(t_0)]^{-\frac{1}{\beta}} = p_s(t_0)^{-1}[\alpha^i + \beta w^i(t_1, s(t_1))]^{-\frac{1}{\beta}} \quad (18)$$

or

$$[\alpha^i + \beta c^i(t_0)] = p_s(t_0)^\beta [\alpha^i + \beta w^i(t_1, s(t_1))] \quad (19)$$

Sum over all agents and divide by  $I$ , the number of agents:

$$[\alpha + \beta c(t_0)] = p_s(t_0)^\beta [\alpha + \beta w(t_1, s(t_1))] \quad (20)$$

where  $\alpha = \sum_i \alpha^i/I$ ,  $c(t_0) = \sum_i c^i(t_0)/I$ ,  $w(t_1, s(t_1)) = \sum_i w^i(t_1, s(t_1))/I$ , and where:

$$\bar{e} - c(t_0) = \sum_s p_s(t_0) y(t_1, s(t_1)) \quad (21)$$

where  $\bar{e} = \sum_i \bar{e}^i/I$ , and  $y = \sum_i y^i/I$ . Equation (18) can now be rewritten as:

$$[\alpha + \beta c(t_0)]^{-\frac{1}{\beta}} = p_s(t_0)^{-1} [\alpha + \beta w(t_1, s(t_1))]^{-\frac{1}{\beta}} \quad (22)$$

Equations (21) and (22) imply that  $c(t_0)$  and  $w(t_1, s(t_1))$  are optimal choices for an agent with an endowment of  $\bar{e}$  units of the consumption good.

Now consider the post-public debt management equilibrium. Let a caret denote the post-public debt management variables. From (18) follows that utility maximization implies:

$$[\alpha^i + \beta \hat{c}^i(t_0)]^{-\frac{1}{\beta}} = \hat{p}_s^{-1} [\alpha^i + \beta \hat{w}^i(t_1, s(t_1))]^{-\frac{1}{\beta}} \quad (23)$$

The  $i$ :th agent's post-public debt management optimal choices can be written as:

$$\hat{c}^i(t_0) = c^i(t_0) + \eta^i(t_0) \quad (24)$$

is given by:

$$u'_i(c^i(t)) = \rho^i p_s(t_0)^{-1} v'_i(c^i(t+1), s(t+1))$$

Accordingly the arguments used in the two-period case can be used in the case with more than two periods.

<sup>12</sup>The proofs for the other types of HARA utility functions are similar. See Rubinstein (1974).

$$\hat{w}^i(t_1, s(t_1)) = w^i(t_1, s(t_1)) + \nu_s^i(t_1) - \lambda^i \Delta \tau(t_1, s(t_1)) \quad (25)$$

where  $\Delta \tau(t_1, s(t_1))$  is the change in the tax liabilities due to public debt management. Since public debt management is self-financed by definition it follows that:  $\sum_i \eta_i(t_0) = 0$ . Furthermore, since the debt service is financed by taxes:  $\sum_i \nu^i(t_1) = \Delta \tau(t_1, s(t_1)) = \sum_i \lambda^i \Delta \tau(t_1, s(t_1))$ , where the last equality follows from the fact that  $\sum_i \lambda^i = 1$ . Hence  $\sum_i \hat{c}^i(t_0)/I = \sum_i c^i(t_0)/I = c(t_0)$ , and  $\sum_i \hat{w}^i(t_1, s(t_1))/I = \sum_i w^i(t_1, s(t_1))/I = w(t_1, s(t_1))$ . Substitute this result into (23):

$$[\alpha^i + \beta c^i(t_0)]^{-\frac{1}{\beta}} = p_s(t_0)^{-1} [\alpha^i + \beta w^i(t_1, s(t_1))]^{-\frac{1}{\beta}} \quad (26)$$

Accordingly  $\hat{p}_s(t_0) = p_s(t_0)$  for all  $s$ , and public debt management leaves the asset prices unchanged as well as the aggregate consumption and wealth unaffected. From the proof above follows that this is the case even if the agents' fractions of the total tax liabilities are state dependent, i.e., if  $\lambda^i = \lambda^i(t, s(t))$ . However, inspection of equations (24) and (25) reveals that there is nothing to ensure that public debt management leaves the individual agent's consumption and investment paths unaffected. Hence, public debt management might have (and in most cases it has) redistributive effects. Inspection of (22) reveals that public debt management neither affects the prices, the aggregate consumption nor the aggregate wealth, as long as it does not change the resources available for consumption and investment<sup>13</sup>. The following proposition can therefore be stated.

**Proposition 3** *In an economy where the agents differ but have HARA utility functions of the same type, public debt management is neutral in aggregates (i.e., it does not affect the prices, aggregate consumption or aggregate wealth), provided all assets are in zero net supply, the taxes associated with the debt service are lump-sum, and the agents pierce the veil of government.*

This result is due to the fact that the utility functions satisfy the aggregation property (i.e., the prices are determined independently of the distribution of wealth), and to the general equilibrium properties of the model (i.e., public debt management might affect the distribution of wealth but leaves the

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<sup>13</sup>It is important to note that if the agents had access to assets that are not in zero net supply, e.g., the production processes in the previous sections, the arguments used in this section to prove neutrality cannot be used since changes in the investments in the production processes would affect the resources available.

aggregate wealth unaffected). Finally, this result differs in one important aspect from the previous results on neutrality. In this case public debt management is neutral (in aggregates) although the agents do not undo its effects through changes in their investments.

## 5 Conclusions

The analysis above has shown that for public debt management to be neutral it is necessary, although in most cases not sufficient, that the agents pierce the veil of government and that the taxes associated with the debt service are lump-sum. It has also been shown that public debt management, besides the redistributive effects it might have through changes in taxes and in agents' portfolios, also affects the feasible consumption plans through changes in asset trading opportunities. To avoid the latter effects two approaches are possible. The first approach, used in most of the literature on public debt management, is to ensure that public debt management does not affect the set of feasible asset trades. This is done by assuming that the markets are complete, or that there exist perfect substitutes for the government-issued bonds, and that unlimited short sales are allowed. The other approach, discussed in this paper, is to consider the cases when public debt management changes the set of feasible asset trades, but when this change does not have any effect on the prices, or on the consumption and investment plans. The analysis above shows that this might be the case if either the agents are identical, the agents differ but the change in the stock of government-issued bonds is marginal, or when the agents are different but have utility functions that exhibit hyperbolic absolute risk aversion. Of special interest is the case when the agents' utility functions exhibit hyperbolic absolute risk aversion since members of the HARA family are widely used in empirical and theoretical studies. From a methodological point of view, the findings in this paper imply that the intertemporal models, such as the models in Samuelson (1969), Merton (1969, 1971), and Cox, Ingersoll and Ross (1985), where the agents' demand for risky assets is explicitly derived cannot be used to study the effects of public debt management since public debt management is neutral in these models. This is due to the fact that Samuelson and Merton use HARA utility functions to solve for asset demands, and Cox, Ingersoll and Ross use a representative agent framework, and that it is not possible to solve

these models if the agents do not take the future tax liabilities into account. The findings also suggest that when public debt management is analyzed in a mean-variance framework [e.g., in Roley (1979), and Agell and Persson (1988)] it is necessary that the agents do not pierce the veil of government, since the results are derived under the assumption that the agents' utility functions exhibit hyperbolic risk aversion. It is worth noting that the neutrality proposition derived for HARA utility functions does not merely apply to public debt management, but also to other cases, e.g., fiscal policy, as long as the aggregate resources are not affected. Finally, this paper shows that if the agents do not pierce the government veil or if the taxes associated with the debt service are not lump-sum, then public debt management cannot be neutral, i.e., public debt management matters.

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# Objectives of Public Debt Management

## 1 Introduction

The purpose of this paper is to analyze different objectives of public debt management within a unified framework. Most of the objectives analyzed in this paper have previously been discussed in the literature. They have, however not been analyzed within a unified analytical framework. The use of a unified framework allows comparisons of the different objectives as well as a discussion of their welfare implications. The model employed is an atemporal mean-variance model.

Public debt management is defined as self-financed operations on the financial markets, carried out by the government in order to change the composition of the outstanding public debt. An example is substitution of short-term government-issued bonds for long-term government-issued bonds. The general objective of public debt management is usually considered to be maximization of the social welfare. Since this is a rather abstract objective, other less abstract objectives have been discussed in the literature. The early literature on public debt management, for example Cohen (1955), Rolph (1957), and Brownlee and Scott (1963), focused mainly on minimization of the cost of the outstanding public debt. The rationale behind this objective is that minimization of the cost of public debt implies minimization of the taxes needed to finance the debt service and hence a reduction in the excess burden [see Rolph (1957)]. Musgrave (1959) regarded public debt management as an instrument of stabilization policy. This approach is further developed in Tobin (1963, 1969). In his seminal paper Tobin (1963) states the following criterion for an optimal debt management policy; minimum cost for required economic impact. The basic argument is that the government, by changing the composition of the outstanding public debt, can affect the valuation of private (corporate) capital through changes in Tobin's  $q$ , and thereby create

incentives for the corporate sector to invest in new capital. Most of the later literature on public debt management, for example Friedman (1978) and Roley (1979) have followed the tradition of Tobin, and considered public debt management as a part of the stabilization policy aiming at controlling the corporate sector's investment. The analysis of public debt management as an instrument of stabilization policy has primarily concentrated on the covariance between the existing (i.e., installed) corporate capital's returns and the government-issued bonds' returns in order to assess the effects on the corporate sector's investments<sup>1</sup>. Less attention has been paid to the welfare issues of public debt management<sup>2</sup>. When the impact of public debt management on welfare has been discussed in the literature, other objectives, for example cost minimization and economic stimulation, have usually been taken as surrogates for welfare improving policies. The lack of attention to the welfare issues of public debt management is probably due to the fact that in many models used to analyze public debt management the demand for assets has not been derived from the agents' optimizing behavior, and the agents' utility functions have therefore not been introduced in the models<sup>3</sup>.

In this paper public debt management is analyzed within the context of a single-period mean-variance equilibrium model of asset pricing. As in most other studies of the objectives of public debt management, the economy considered is closed, i.e., the domestic capital market is not integrated with its foreign counterparts and hence there is no capital mobility<sup>4</sup>. To analyze the effects of public debt management, the fact that the government can affect the valuation of government-issued bonds as well as the valuation of other assets by changing the composition of the outstanding public debt, is used.

The following characteristics of the approach taken in this paper are worth

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<sup>1</sup>Tobin (1969) discusses the effects of public debt management in terms of gross substitutability. However, Blanchard and Plantés (1977) have shown that a necessary (although not sufficient) condition for gross substitution is a positive covariance between the assets' returns.

<sup>2</sup>One exception is Gale (1990) who analyzes the impact of public debt management on the efficiency of risk sharing in an intertemporal model.

<sup>3</sup>In Roley (1979), and Agell and Persson (1988) the demand for assets is derived from the agents' maximizing behavior. However, they do not mainly focus on the welfare aspects of public debt management.

<sup>4</sup>One exception is Boothe and Reid (1992) who analyze minimization of the cost of public debt in an open economy using a numerical simulation model and Canadian data.

emphasizing. Firstly, the demand for assets is derived from the agents' optimizing behavior, and the corporate sectors' demand for new investment is derived from the firms' optimizing behavior. Hence both the agents' and the firms' responses to public debt management are taken into consideration. The introduction of the firms' behavior implies that the supply of securities to some extent is endogenized. Secondly, it allows a discussion of the welfare issues of public debt management, particularly how public debt management affects the welfare through changes in risk and changes in consumption opportunities. Thirdly, since the analysis is performed in a unified framework, comparisons of different objectives discussed in the literature are possible.

The remainder of the paper is organized in the following way. In Section 2 the mean-variance model is formulated and the equilibrium valuation model is derived. Furthermore, the firms' demand for new investment is derived under the assumption that each firm tries to maximize its market value. In Section 3 public debt management is defined and its impact on the capital market equilibrium is discussed. Moreover, the objectives of public debt management are analyzed. The objectives considered are: welfare maximization, maximization of the agents' wealth, minimization of the interest cost of public debt, maximization of Tobin's  $q$ , and maximization of the firms' investments. The objectives are examined under two different types of debt management policies; when the government changes the composition of the outstanding debt without introducing any new types of bonds, and when the government is allowed to introduce new types of bonds. Section 3 ends with a comparison of the different objectives. In this part it is shown that objectives that intuitively might be thought of as welfare-improving do not necessarily improve the agents' welfare. It is also shown that the conclusions in Tobin (1969), Friedman (1978), and Roley (1979) on the effects of public debt management, based on the characteristics of the private sector's installed capital (e.g., the existing capital's covariances), do not hold when the firms' explicit investment behavior is taken into account. Section 4 contains the conclusions and a summary of the results.

## 2 The Model and the Equilibrium

To value the assets in the economy, the atemporal capital asset pricing model developed by Sharpe (1964) and Lintner (1965) will be used. However, at

least three caveats are necessary when public debt management is analyzed in a capital asset pricing model<sup>5</sup>. Firstly, in the capital asset pricing model agents can diversify all diversifiable risk. Hence there is not any role for public debt management since all the diversifiable risk has been diversified away and only the market (non-diversifiable) risk remains. For public debt management to have effects it is therefore necessary that the agents do not pierce the government veil. If this is the case, the agents do not recognize the relation between their budget constraints and the government's, i.e., they do not realize that the change in payoff structure associated with the government-issued bonds is completely offset by a change in the taxes. Hence, for public debt management to have effects it is assumed in the forthcoming analysis that the agents do not pierce the veil of government, i.e., they act myopically. Secondly, the capital asset pricing model is only a partial equilibrium model of the financial market. It takes the agents' first period consumption as well as the size of their investments (savings) as given. However, the consumption and investment problems are in general not independent. Accordingly, if public debt management changes the (perceived) set of feasible second period consumption opportunities, the effects of changes in the agents' consumption-savings decisions are not captured in the model. This means that the analysis merely captures the first-order effects of public debt management, i.e., the changes in the market structure that might induce changes in the consumption-savings decision, and not the effects due to changes in the agents' consumption-savings decision. Finally, in most of the literature public debt management has been thought of as changes in the maturity structure of the outstanding public debt. However, the capital asset pricing model is atemporal and accordingly it does not capture the effects on the hedging opportunities due to changes in the maturity structure.

## 2.1 The Agents

All agents are assumed to be risk-averse expected utility maximizers, who maximize  $U_i(c_i, e_i, v_i)$  where  $c_i$  is the level of current (i.e., at the beginning of the period) consumption and  $e_i$  and  $v_i$  are the expected value and variance, respectively, of the agent's end-of-period wealth (consumption). It is assumed

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<sup>5</sup>These caveats also apply to other studies that analyze public debt management in a mean-variance framework, e.g., Roley (1979), and Agell and Persson (1988).

that each investor already has decided how much of his total initial wealth he wishes to use for current consumption and how much of his total initial wealth he wishes to invest, i.e., the agents have already made their consumption-savings decisions. These fractions of his total initial wealth are taken as given and therefore not affected by changes in the supply of risky assets. Furthermore it is assumed that  $U_c > 0$ ,  $U_{cc} < 0$ ,  $U_e > 0$ ,  $U_v < 0$ , that all assets are infinitely divisible, and that transaction costs are zero. The agents enter the market with homogeneous expectations of the end-of-period cash flow per bond ( $\bar{d}_j$ ), and their variances and covariances ( $\sigma_j^2$  and  $\sigma_{ij}$ ). These expectations are independent of today's market clearing prices. The bond's returns are therefore endogenously determined. All agents can borrow or lend at the riskfree interest rate  $i$ . To simplify the notation let  $r = 1 + i$ . The agent's budget constraint is:

$$\sum_i p_i q_i = w + b \quad (1)$$

where  $p_i$  is the price of the  $i$ :th risky asset,  $q_i$  the quantity of the  $i$ :th risky asset,  $w$  the amount of initial wealth used for investment (henceforth this is called the agent's wealth), and  $b$  the agent's borrowing (lending if  $b$  is negative)<sup>6</sup>. To simplify the analysis it is assumed that all agents' utility functions exhibit constant absolute risk aversion and that the cash flows are joint normally distributed. As shown in Section 2.2, these two assumptions imply that the market price per unit of risk is constant. There are two types of utility functions that exhibit constant absolute risk aversion; linear and exponential utility functions<sup>7</sup>. In the following it is assumed that the agents' utility functions are exponential. Exponential utility functions also exhibit linear risk tolerance, which implies that the agents' utility functions satisfy the aggregation property, i.e., the equilibrium prices are determined independently of the distribution of the agents' wealth [see Rubinstein (1974)]<sup>8</sup>. The aggregation property also implies that the agents are so similar that there is no demand for endogenous assets (i.e., assets issued by the agents).

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<sup>6</sup>Throughout the analysis the definition of wealth does not include non-marketable assets, e.g., human capital, which might constitute a large part of the agents' wealth. However, Mayers (1972) has shown that it is possible to introduce non-marketable assets in the model.

<sup>7</sup>See, e.g., Krouse (1986).

<sup>8</sup>Risk tolerance is defined as the reciprocal of the Arrow-Pratt measure of absolute risk aversion.

The  $k$ :th agent's exponential utility function is given by:

$$U_k(D_k) = -\exp(-a_k D_k) \quad (2)$$

where  $D_k$  is the  $k$ :th investor's end-of-period wealth and  $a_k$  is his absolute risk aversion. The  $k$ :th agent's problem is therefore equivalent to:

$$\max E[-\exp(-a_k \tilde{D}_k)] \quad (3)$$

subject to his budget constraint. To rewrite (3) use the characteristic function of a normal distribution with the same mean and variance as the agent's portfolio and use the fact that the utility function is ordinal:

$$\max[a_k(\bar{D}_k - a_k \frac{\sigma_{wk}^2}{2})] \quad (4)$$

where  $\bar{D}_k$  is the  $k$ :th agent's expected end-of-period wealth and  $\sigma_{wk}^2$  is the variance of his portfolio<sup>9</sup>. Hence, the agent's optimal investment position is the one that maximizes the certainty equivalent of his end-of-period wealth,  $Q_k$ , which is given by:

$$Q_k = \bar{D}_k - a_k \frac{\sigma_{wk}^2}{2} \quad (5)$$

Given the assumptions above, it will be optimal for each agent to hold a fraction of the market portfolio in equilibrium [see, e.g., Mossin (1966)]. In other words, the agents' portfolios differ only with respect to scale (i.e., the amount invested in the market portfolio).

## 2.2 The Representative Agent's Utility Function and the Social Welfare Function

In this section the representative agent's utility function is derived. In the following analysis it will be used as a social welfare function. The section ends with a discussion of the representative agent's utility function as a social welfare function.

A distinctive feature of the representative agent's utility function is that the heterogeneous agents' behavior can be replaced by a single representative agent's behavior. Lintner (1969) shows that if all agents' utility functions

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<sup>9</sup>For a derivation of (4) see Ingersoll (1987).

are negative exponential, as assumed in this paper, then the representative agent's utility function will be a composite of the individual agents' preferences. In this case the representative agent's utility function is exponential and given by:

$$U_m(D_m) = -\exp(-a_m D_m) \quad (6)$$

where  $D_m$  is the end-of-period value of the market portfolio and  $a_m$  is the representative agent's risk aversion, which is defined as  $a_m = [\sum_k a_k^{-1}]^{-1} = H_{a_k}/N$ , where  $H_{a_k}$  is the harmonic mean of the investors' risk aversions, and  $N$  is the number of investors. Since the individual agents' utility functions exhibit constant absolute risk aversion, it follows that the representative agent's risk aversion is constant. Furthermore  $a_m^{-1}$  is the representative agent's risk tolerance, which is equal to the sum of the individual agents' risk tolerances. Lintner (1970) shows that  $a_m$  is equal to the market price of risk, i.e.,  $a_m = \lambda$ . Hence, constant absolute risk aversion implies a constant market price of risk. The market price of risk (and hence the representative agent's risk aversion) measures the change in expected return, required by the market, per dollar change in the market portfolio variance, i.e., the marginal rate of substitution between expected return and variance in the market.

Let  $Q_m$  denote the certainty equivalent of the market portfolio's end-of-period value and  $\bar{D}_m$  its expected end-of-period value, then:

$$Q_m = \bar{D}_m - a_m \frac{\sigma_m^2}{2} \quad (7)$$

By using the arguments in the previous section it can be shown that maximization of the representative agent's expected utility implies maximization of the certainty equivalent of the market portfolio's end-of-period value. The representative agent's problem can therefore be stated as:

$$\max \bar{D}_m - \lambda \frac{\sigma_m^2}{2} \quad (8)$$

The representative agent's utility function in (8) will be used as a social welfare function in the forthcoming analysis. This implies that the market price of risk measures the social marginal rate of substitution between net expected return and variance. If the individuals differ in risk aversion, the harmonic mean of the individual risk aversions is less than the arithmetic mean. Due to the properties of harmonic means, agents with a low risk

aversion will have greater influence on the social marginal rate of substitution (i.e., the market price of risk), than agents with a high risk aversion. This implies that used as a social welfare function, the representative agent's utility function will not satisfy the weak Pareto principle, which states that allocation  $x$  is socially preferred to allocation  $y$  if  $x$  is unanimously preferred to  $y$  by all agents in the economy<sup>10</sup>.

### 2.3 The Valuation Model

Lintner (1965) shows that the assumptions made in the previous sections imply that the equilibrium value of the total stock of the  $j$ :th uncertain asset is given by:

$$V_j = \frac{1}{r}(\bar{D}_j - \lambda\sigma_{jm}) \quad (9)$$

where  $\bar{D}_j = q_j\bar{d}_j$  is the expected end-of-period cash flow of the total stock of the  $j$ :th asset,  $r$  is the risk-free rate of interest,  $\sigma_{jm}$  is the covariance of the cash flow of the total stock of the  $j$ :th asset ( $\tilde{D}_j$ ) with the cash flow of the market portfolio ( $\tilde{D}_m$ ),  $\lambda = (\bar{D}_m - rV_m)/\sigma_m^2$  is the market price per unit of risk (which is constant),  $\bar{D}_m = \sum_{j=1}^n \bar{D}_j$  is the expected cash flow of the market portfolio,  $V_m = \sum_{j=1}^n \bar{V}_j$  is the market portfolio's value, and  $\sigma_m^2$  is the variance of the market portfolio's cash flow (i.e., of  $\tilde{D}_m$ ).

### 2.4 The Assets

There are two categories of assets in the economy, riskfree and risky. The riskfree asset is riskfree borrowing and lending at the riskfree rate of interest  $i$  and it is in zero net supply. The risky assets are of two types, government-issued bonds and securities issued by the firms. Furthermore since the utility functions are assumed to be exponential, there will not be any endogenously issued assets (i.e., assets issued by the agents and sold to other agents), since there is no demand for them. The asset market is assumed to be incomplete, i.e., the number of assets with linearly independent payoffs is less than the number of states. For simplicity it is assumed that there exist two types of government-issued bonds (S and L) and one type of security (P). Their equilibrium values defined in (9) are given by  $V_S$ ,  $V_L$  and  $V_P$ , respectively.

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<sup>10</sup>See, e.g., Boadway and Bruce (1984).

Public debt management (to be defined in the next section) changes the supply of S and L. Since these three types of assets are the only ones in the economy, the total market value is given by:

$$V_m = V_S + V_L + V_P \quad (10)$$

or:

$$V_m = q_S v_S + q_L v_L + q_P v_P \quad (11)$$

where  $q_j$  is the quantity of the  $j$ :th asset, and  $v_j$  is the value per unit of the  $j$ :th asset. Let  $\sigma_j^2$  denote the variance per unit of the  $j$ :th asset, then:

$$\sigma_m^2 = q_S^2 \sigma_S^2 + q_L^2 \sigma_L^2 + q_P^2 \sigma_P^2 + 2q_S q_L \sigma_{SL} + 2q_S q_P \sigma_{SP} + 2q_L q_P \sigma_{LP} \quad (12)$$

$$\sigma_{jm} = \sum_{k=1}^n q_j q_k \sigma_{jk} \quad (13)$$

Use equation (12) to rewrite the expression for  $V_L$  in (9) as (the expressions for  $V_S$  and  $V_P$  are similar):

$$V_L = \frac{1}{r} [q_L \bar{d}_L - \lambda (q_L q_S \sigma_{LS} + q_L^2 \sigma_L^2 + q_L q_P \sigma_{LP})] \quad (14)$$

Hence the value of the total stock of an asset is not only affected by changes in its own quantities and variances. It is also affected by changes in the quantities of the other assets as well as changes in its covariances with other assets.

## 2.5 The Firm

To simplify the analysis it is assumed that there is one representative firm. For new investments the firm has access to a stochastic constant returns to scale technology. It is assumed that some, although not necessarily all, of the firm's existing investments are in this technology. This technology yields a random dollar return of  $\tilde{\rho}$  per dollar invested, with  $E(\tilde{\rho}) = \bar{\rho}$ ,  $var(\tilde{\rho}) = \sigma_\rho^2$ , and  $cov(\tilde{D}_P, \tilde{\rho}) = \sigma_{P\rho}$ . Since the output of the firm's existing production processes is equal to the payoff on the securities issued by the firm, it follows that  $cov(\tilde{D}_P, \tilde{\rho})$  can be interpreted as the covariance of the output of the

firm's existing investment with the output of the firm's new investment<sup>11</sup>. The firm finances the investment by issuing new securities. It is assumed that the firm's objective is to maximize its market value,  $V_P$ , which is given by<sup>12</sup>:

$$V_P = \frac{1}{r}(\bar{D}_P - \lambda\sigma_{Pm}) \quad (15)$$

Let a prime denote the post-investment variables, then:

$$V'_P = \frac{1}{r}(\bar{D}'_P - \lambda\sigma_{P'm}) \quad (16)$$

where the expected post-investment output is given by:

$$\bar{D}'_P = \bar{D}_P + I\bar{\rho} \quad (17)$$

and where the covariance of the post-investment return on the securities issued by the firm (or output of the firm) with the return of the market portfolio is given by<sup>13</sup>:

$$\sigma'_{P'm} = \sigma_{Pm} + I\sigma_{\rho m} + I\sigma_{P\rho} + I^2\sigma_\rho^2 \quad (18)$$

Let  $\Delta V_P(I)$  denote the change in the firm's market value brought about by an investment of  $I$  dollars, then the net increase in the value of the firm after an investment of  $I$  dollars is:

$$\Delta V_P(I) - I = V'_P - V_P - I = \frac{I}{r}[(\bar{\rho} - r) - \lambda(\sigma_{\rho m} + \sigma_{P\rho} + I\sigma_\rho^2)] \quad (19)$$

To maximize its market value the firm will continue to invest as long as a dollar invested gives a positive net change in the firm's market value. Hence, the firm's first-order condition for value maximization is given by:

$$\frac{d[\Delta V_P(I) - I]}{dI} = \frac{1}{r}[(\bar{\rho} - r) - \lambda(\sigma_{\rho m} + \sigma_{P\rho} + I\sigma_\rho^2)] - \frac{I}{r}\lambda\sigma_\rho^2 = 0 \quad (20)$$

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<sup>11</sup>Since all of the firm's existing (i.e., initial) investments are not necessarily of the same type as the technology presently available to the firm, a situation where  $\sigma_{P\rho} \neq \sigma_P^2$  is possible.

<sup>12</sup>For a discussion of value maximization as an objective see, Leland (1974).

<sup>13</sup>The expression follows from (13).

To determine the optimal level of new investment (i.e., the firm's demand for new investment),  $I^*$ , use the first-order condition and solve for  $I^*$ :

$$I^* = \frac{1}{2\lambda\sigma_\rho^2}[(\bar{\rho} - r) - \lambda(\sigma_{\rho m} + \sigma_{\rho P})] \quad (21)$$

In the initial (pre-public debt management) equilibrium it is assumed that the firm has made all investments it wants. Hence its demand for investment in the new technology is equal to zero in the initial equilibrium.

The firm finances new investments through sale of additional securities. These securities as well as the previously issued securities entitle the owner to a fraction of the firm's total end-of-period output. To finance the purchase of these securities the agents increase their borrowing at the riskfree interest rate. Unless  $\sigma_\rho^2 = \sigma_{\rho P}$  the new investment changes the risk of all the securities issued by the firm. Hence the new investment creates a new type of security and the agents' risk-sharing opportunities are therefore affected. Due to the difference in risk, the pre-investment security and the post-investment security are two completely different assets. The spanning of the state space is therefore changed by the firm's new investment.

### 3 Public Debt Management and Its Objectives

The objectives of public debt management obviously depend on the government's economic objectives. In this section two different types of objectives are considered, social objectives and objectives aiming at economic stimulation<sup>14</sup>. The following five objectives are analyzed: (i) maximization of the representative agent's welfare, (ii) maximization of end-of-period private wealth, (iii) minimization of the cost of public debt, (iv) maximization of Tobin's  $q$ , and (v) maximization of the representative firm's investment. The first three are social objectives and the last two aim at economic stimulation. In the following analysis the objectives are characterized by their respective objective functions. This section ends with a comparison of the objectives discussed.

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<sup>14</sup>The problems analyzed as well as the methods used in this section are somewhat similar to those in the literature on corporate investment and Pareto optimality in the capital market. See, e.g., Jensen and Long (1972), and Long (1972).

### 3.1 Public Debt Management

In this section public debt management is defined and its effects on the valuation of the assets and on the agents' end-of-period consumption (wealth) are discussed. Public debt management is defined as self-financed operations by the government on the financial markets in order to change the composition of the outstanding public debt. The government is by assumption not allowed to create and issue new types of bonds, i.e., such bonds that have a different risk than the already existing (i.e., outstanding) government-issued bonds. This assumption is relaxed in Section 3.7. The transactions associated with public debt management are assumed to be carried out at the new equilibrium prices. Let the variables with carets denote the new equilibrium values. The definition of public debt management can then be written as<sup>15</sup>:

$$\hat{v}_S(\hat{q}_S - q_S) + \hat{v}_L(\hat{q}_L - q_L) = 0 \quad (22)$$

where:

$$\hat{v}_L = \hat{V}_L / \hat{q}_L = \frac{1}{r} [\bar{d}_L - \lambda(\hat{q}_S \sigma_{LS} + \hat{q}_L \sigma_L^2 + q_P \sigma_{LP})] \quad (23)$$

$$\hat{v}_S = \hat{V}_S / \hat{q}_S = \frac{1}{r} [\bar{d}_S - \lambda(\hat{q}_S \sigma_S^2 + \hat{q}_L \sigma_{SL} + q_P \sigma_{SP})] \quad (24)$$

The definition of public debt management implies that the government is free to choose the quantity of one of the bonds, since the quantity of the other follows from the definition of public debt management. From equations (12) and (13) it is obvious that, although the assets' expected returns and covariances are exogenously given,  $\sigma_m^2$  as well as  $\sigma_{jm}$  are affected by the composition of the outstanding government debt. Hence, public debt management also affects the value of the outstanding securities,  $V_P$ . Furthermore, the change in market risk due to public debt management also affects  $\sigma_{\rho m}$ , i.e., the new project's (or investment's) covariance with the market. From equation (21) follows that the firm's equilibrium condition is not fulfilled if  $\sigma_{\rho m}$  is changed. Hence, the firm invests and a new (i.e., post-public debt management) equilibrium is established. In the forthcoming analysis it is assumed that the government takes the firm's behavior into consideration, i.e., the government knows that the firm's demand for new investment is given by equation (21).

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<sup>15</sup>In principle it is possible to solve for  $\hat{q}_L$  as a function of  $\hat{q}_S$  or vice versa using the definition of public debt management. However, the expressions are analytically too difficult to work with.

However, the results are essentially the same if the government does not take the firm's behavior into account.

Finally, consider the effects of public debt management on the agents' end-of-period consumption. Let  $W^s$  denote the aggregate end-of-period state contingent wealth (i.e., state contingent consumption) if state  $s$  is realized, i.e.,  $W^s$  is defined as<sup>16</sup>:

$$W^s = q_S d_S^s + q_L d_L^s + q_P d_P^s \quad (25)$$

Inspection of (25) reveals that  $W^s$  depends on the number of different assets in the market ( $q_S, q_L, q_P$ ), i.e., on the market structure. Hence, public debt management changes the market structure, which might induce a change in  $W$ , and the change in  $W$  affects the agents' state contingent end-of-period consumption opportunities.

### 3.2 Welfare Maximization

To analyze welfare maximization it is assumed that the agents' welfare can be described by a social welfare function. It is assumed that the social welfare function is equal to the representative (composite) agent's expected utility function. In Section 2.2 it was shown that maximization of the representative agent's expected utility is equivalent to maximization of the certainty equivalent of the market portfolio's end-of-period value. However, when public debt management is considered, it is necessary to take the effects of the firm's investment behavior into account. The effect on the expected value of the market portfolio is captured in the term  $\bar{D}_m$  and the effect on the risk is captured in  $\sigma_m^2$ . The public debt management policy that maximizes the welfare is therefore the policy that maximizes the certainty equivalent of the market's end-of-period wealth. Consequently, the government's problem is:

$$\max_{\hat{q}_L, \hat{q}_S} \bar{D}_m - \lambda \frac{\sigma_m^2}{2} \quad (26)$$

subject to equation (22) (the definition of public debt management).

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<sup>16</sup>From the definition of  $W^s$  it follows that the effect of the riskfree borrowing/lending on the state contingent end-of-period consumption is not captured in  $W^s$ .

### 3.3 Wealth Maximization

Since each agent holds a fraction of the market portfolio in equilibrium, maximization of the expected end-of-period private wealth is equivalent to maximization of the market portfolio's expected end-of-period value (i.e.,  $\bar{D}_m$ ). In other words, the government's problem is:

$$\max_{\hat{q}_L, \hat{q}_S} \bar{D}_m \quad (27)$$

subject to (22).

### 3.4 Minimization of the Cost of Public Debt

To minimize the expected cost of public debt, the government should choose a public debt management policy that minimizes the expected redemption value of the outstanding government-issued bonds<sup>17</sup>. The government's objective function is therefore given by:

$$\min_{\hat{q}_L, \hat{q}_S} \bar{D}_S + \bar{D}_L \quad (28)$$

subject to (22). In the literature [see, e.g., Rolph (1957) and Musgrave (1959)] minimization of the cost of public debt for a given level of social welfare (utility) has been suggested as an objective of public debt management. This implies minimization of (28) subject to (22) and  $U(\cdot) \geq \bar{U}(\cdot)$ , where  $\bar{U}(\cdot)$  is the desired level of social welfare.

To illuminate the mechanisms behind minimization of the expected cost of public debt, rewrite  $\bar{D}_j$  as:

$$\bar{D}_j = q_j \bar{d}_j \quad (29)$$

It is easily seen that to minimize the expected cost of public debt the government should issue bonds with a low  $\bar{d}_j$ . The quantities,  $q_j$  are endogenously determined in (22) (the definition of public debt management). From (22)

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<sup>17</sup>The analysis does not take the effects of taxation of the yields on the government-issued bonds into account, when minimization of the cost of public debt is discussed. See Agell and Persson (1988) for an analysis that takes the effects of taxation into account.

follows that a low  $q_j$  requires a high  $v_j$ , i.e., a high value (price) of the  $j$ :th bond government-issued bond. Consider the expression for  $v_j$ :

$$v_j = \frac{1}{r}[\bar{d}_j - \lambda q_j^{-1} \sigma_{jm}] \quad (30)$$

From (30) follows that  $v_j$  is high if  $\bar{d}_j$  is high. This implies that the effects of issuing bonds with a low  $\bar{d}_j$  on the cost of public debt are ambiguous, since a low  $\bar{d}_j$  implies a high  $q_j$  and *vice versa*. However, inspection of (30) also reveals that a bond has a high value if its covariance with the market,  $\sigma_{jm}$ , is low or negative. To analyze the case when the government issues bonds with a low covariance with the market, suppose the  $j$ :th bond's return is negatively correlated with the return on the market portfolio. Inspection of (30) reveals that this bond will have a high price. This means that the investors are willing to accept a lower expected rate of return on the bond since it is a good hedge against bad states. Furthermore, since the price of the bond is high, the government has to issue a relatively small quantity of bonds to finance a given purchase of outstanding bonds. Hence to minimize the cost of public debt, the government could issue bonds that are negatively correlated with the return on the market portfolio, and buy bonds that are positively correlated with the return on the market portfolio. If there does not exist a government-issued bond whose return is negatively correlated with the market return, the government should issue bonds of the type that has the smallest covariance with the market. However, suppose the government issues a large amount of a bond whose return initially is negatively correlated with the return on the market portfolio. Then, if the bond becomes a sufficiently large part of the market portfolio, its post-public debt management return becomes positively correlated with the return on the market portfolio.

### 3.5 Maximization of Tobin's $q$

Tobin's  $q$  [see Tobin (1969)] is defined as the ratio of the market value of the firm's assets to the replacement cost of its assets. The rationale behind the  $q$  theory of investment is that the firm will have an incentive to invest when  $q$  is greater than one, and it will stop investing when  $q$  is equal to one<sup>18</sup>. In

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<sup>18</sup>It is important to keep in mind that these results for the  $q$  theory of investment do not necessarily hold if, as assumed in this paper, the firm's objective is to maximize its market value.

the following analysis the market value of the firm's assets is equated with the firm's market value ( $V_P$ ), and the replacement cost of the firm's assets is assumed to be constant. Hence, maximization of Tobin's  $q$  implies maximization of the firm's pre-investment market value, since the denominator in the definition of Tobin's  $q$  is constant by assumption. The argument for maximization of Tobin's  $q$  as an objective of public debt management, is that an increase in  $q$  creates incentives to invest and hence economic stimulation provided public debt management raises  $q$  above one [see, e.g., Roley (1979)]. Consider the expression for the firm's post-public debt management but pre-investment market value:

$$\hat{V}_P = \frac{1}{r}(\bar{D}_P - \lambda\hat{\sigma}_{Pm}) \quad (31)$$

Since  $\lambda$  is constant, all terms on the right hand side of (31) except  $\hat{\sigma}_{Pm}$  are unaffected by public debt management. Hence, maximization of the firm's market value implies minimization of  $\hat{\sigma}_{Pm}$ , i.e., the post-public debt management covariance of the returns on the firm's existing (initial) investment with the return on the market portfolio. To maximize Tobin's  $q$  the government's problem is therefore to:

$$\min_{\hat{q}_L, \hat{q}_S} \hat{\sigma}_{Pm} \quad (32)$$

subject to equation (22) (i.e., the definition of public debt management). In other words, to maximize Tobin's  $q$ , the government should increase the supply of the bond with the lowest covariance with the firm's existing capital.

### 3.6 Maximization of the Firm's Investment

To obtain maximal economic stimulation, the government should choose a public debt management policy that maximizes the firm's post-public debt management investment ( $I^*$ ). Consider the expression for the optimal level of post-public debt management investment:

$$I^* = \frac{1}{2\lambda\sigma_\rho^2}[(\bar{\rho} - r) - \lambda(\hat{\sigma}_{\rho m} + \sigma_{\rho P})] \quad (33)$$

The only term on the right hand side of (33) that is affected by public debt management is  $\hat{\sigma}_{\rho m}$ <sup>19</sup>. Since  $I^*$  is inversely related to the post-public debt management covariance of the return of the new investment opportunity with the return on the market portfolio (i.e.,  $\hat{\sigma}_{\rho m}$ ), maximization of the post-public debt management investment level implies minimization of  $\hat{\sigma}_{\rho m}$ . The public debt management policy for maximum economic stimulation is therefore a solution to the following problem:

$$\min_{\hat{q}_L, \hat{q}_S} \hat{\sigma}_{\rho m} \quad (34)$$

subject to equation (22). Hence, in order to maximize the firm's investment the government should increase the supply of the bond whose return has the lowest covariance with the new investment opportunity.

### 3.7 Public Debt Management when the Government Issues New Bonds

In this section the assumption that the government is allowed to change the composition of the outstanding debt and not allowed to issue new assets is relaxed, i.e., the government is allowed to change the set of assets in the economy. By new assets is meant assets whose payoffs are linearly independent of the existing assets' payoffs. If the market is incomplete, the possibility to create new assets means that through public debt management the government can affect the spanning of the state space directly, and not indirectly via changes in the firm's investment caused by public debt management. Hence, the possibility to issue new assets gives the government access to additional policy instruments. Consider the objectives discussed above. The objective functions as well as the constraints are the same, but there are more instruments. Accordingly, in most cases it will be possible for the government to find policies that are better than the policies that were feasible when the government was restricted to changes in the composition of the outstanding debt.

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<sup>19</sup>The covariance  $\hat{\sigma}_{\rho m}$  is defined as:

$$\hat{\sigma}_{\rho m} = \sigma_{\rho}^2 + \hat{q}_L \sigma_{\rho L} + \hat{q}_S \sigma_{\rho S} + q_P \sigma_{\rho P}$$

Of particular interest is the effects on welfare when the government is allowed to create new assets<sup>20</sup>. To analyze these effects it is useful to distinguish between two cases; when public debt management affects the aggregate state contingent wealth,  $W$  [i.e.,  $W = \{W^s\}$  where  $W^s$  is defined in (25)], and when it does not. First consider the case when public debt management leaves the set of aggregate end-of-period consumption,  $W$ , unaffected. If the market is already complete, then any new asset issued by the government is redundant. Hence public debt management does not affect the welfare. In an incomplete market, creation of new assets through public debt management will generally affect the welfare. If the government completes the market by issuing new bonds there will be a Pareto improvement [see, e.g., Hart (1975)]. However, from the theory of second best it follows that if the government issues new assets, but does not complete the market, the effect on welfare is ambiguous; it might even be the case that everybody is worse off [see Hart (1975)]. To analyze the effects when the government issues new bonds but does not complete the market, let  $F_W$  denote the set of feasible allocations of a given aggregate end-of-period wealth,  $W$ . Unless the market is complete, the set  $F_W$  is obviously affected by changes in the market structure, i.e., by changes in  $(q_S, q_L, q_P)$ . Let  $\hat{F}_W$  denote the post-public debt management set of feasible allocations. There are three possible changes in  $F_W$  due to changes in the market structure<sup>21</sup>: (i) feasible preserving, i.e.,  $F_W = \hat{F}_W$ , (ii) feasible expanding,  $F_W \subset \hat{F}_W$ , and (iii) feasible altering changes,  $F_W \cap \hat{F}_W^c \neq \emptyset$ , where the  $\hat{F}_W^c$  denotes the complement to  $\hat{F}_W$ . In Hakansson (1982) the following results are shown. (i) Feasible preserving changes either yield Pareto equivalence or Pareto redistributions. (ii) Feasible expanding changes in the market structure in general imply unpredictable changes in welfare. However, a sufficient condition for a change to be Pareto equivalent or Pareto improving is that the change does not affect the value of each agent's wealth. (iii) The effect on welfare of feasible altering changes in the market structure is unpredictable. The implication of these results for public debt management is that by issuing new types of assets/opening new markets the government can in principle make everybody better off. However, it is not the case that whatever issuance of new assets/opening of new markets is welfare increasing

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<sup>20</sup>The following analysis of welfare effects of public debt management is in terms of Pareto improvements and not in terms of the social welfare function defined in (7). The results are therefore valid for arbitrary utility functions.

<sup>21</sup>The terminology is due to Hakansson (1982).

(Pareto improving).

Finally consider the case when public debt management affects  $W$ . This means that the government can affect the spanning of the states, and in most cases it will therefore be possible to find policies that Pareto dominate the ones available when  $W$  is constant.

### 3.8 A Comparison of the Objectives

This section analyzes the divergence between the different objectives of public debt management discussed above. The analysis is based on comparisons of the different objective functions, i.e., equations (26), (27), (28), (32), and (34). In the comparisons welfare maximization will serve as a benchmark. However, when the welfare aspects of public debt management are discussed, one has to keep in mind that the representative agent's utility function does not satisfy all the desired properties of a social welfare function, e.g., it does not take the redistributive effects of public debt management into account.

Firstly consider the three social objectives. A comparison of the objective functions for welfare maximization and wealth maximization [i.e., equations (26) and (27)] reveals the obvious fact that wealth maximization could only be welfare maximizing if the agents are risk neutral. If the agents are risk averse, wealth maximization would lead to a too risky portfolio. Now consider minimization of the cost of public debt. As shown in Section 3.4, one way for the government to lower the cost of public debt is to issue bonds whose returns have a negative or low covariance with the market. In most cases this will lower the overall market risk ( $\sigma_m^2$ ). Examination of (26) shows that reduction in the market risk will have a positive impact on the welfare in terms of the representative agent's utility as long as:  $\lambda/2 |d\sigma_m^2| > |d\bar{D}_m|$ , provided  $d\sigma_m^2$  and  $d\bar{D}_m$  have the same sign, and where  $d$  is the differential operator. Hence, if the initial composition of the outstanding public debt is not welfare maximizing, cost minimization might be welfare improving, at least in terms of the representative agent's utility. This result is not immediately obvious, since the positive effects (from the agents' point of view) of cost minimization, e.g., through a reduction in the future taxes needed to finance the cost of public debt, are not captured in this model. However, an introduction of taxes for example would probably strengthen these results, since minimization of the cost of public debt and hence minimization of the taxes associated with public debt, also implies that the excess burden is reduced,

which might have a positive impact on welfare.

Secondly, consider the objectives aiming at economic stimulation. Comparison of (32) and (34) reveals that the criteria for maximization of Tobin's  $q$  and maximization of the firm's investment in general are inconsistent. The discrepancy between the two objectives aiming at economic stimulation is due to the fact that one (maximization of Tobin's  $q$ ) focuses on the installed capital's risk characteristics, while the other (maximization of the investments) focuses on the risk characteristics of the new technology. An implication of this is that the results in, e.g., Roley (1979) on economic stimulation, stating that the debt management policy for maximum economic stimulation is the policy that maximizes the aggregate value of the private securities (i.e., that maximizes Tobin's  $q$ ), do not hold in a model where the firm's optimizing behavior is introduced. The magnitude of the discrepancy between the two objectives depends on the resemblance between the risk characteristics of the installed capital and the new technology, respectively. If their risk characteristics are fairly similar, the discrepancy between the objectives is small, and *vice versa*. The impact on welfare of the objectives aiming at economic stimulation is ambiguous and depends on how the new investment affects the market portfolio's expected end-of-period value and variance. However, the model is not suitable to analyze the welfare effects of economic stimulation, since it does not take the effects of increased capital accumulation and hence output into consideration.

To discuss the effectiveness of the different objectives consider their respective objective functions. Inspection of the objective functions for welfare maximization, wealth maximization, maximization of Tobin's  $q$ , and maximization of the firm's investment reveals that they involve  $\bar{D}_m$ ,  $\sigma_m^2$ ,  $\sigma_{Pm}$ , and  $\sigma_{pm}$ , respectively. From the definitions of these variables it follows that the government's ability to affect these variables is, *ceteris paribus*, related to the government-issued bonds' fraction of the total asset market. The larger fraction of the asset market, the greater impact of public debt management on welfare, wealth and economic activity. On the other hand, the effectiveness of cost minimization is not affected by the government-issued bonds' fraction of the total asset market. This is due to the fact that cost minimization does not hinge upon the government's ability to influence market variables, but merely on its ability to choose a mix of bonds that minimizes the cost of public debt.

To summarize, the analysis has shown that an objective that intuitively

might be thought of as welfare increasing, not necessarily increases the social welfare, and that an objective that intuitively might be thought of as stimulative, not necessarily increases the economic activity.

## 4 Conclusions

In the preceding analysis the firms' behavior was introduced in order to capture the effects of public debt management on the firms' investments and to some extent endogenize the supply of assets. The analysis has shown that debt management policies aiming at economic stimulation through changes in Tobin's  $q$  might not have the desired effects when the firms' behavior is explicitly taken into account.

It has also been shown that the effectiveness of policies aiming at welfare maximization in terms of the representative agent's utility, wealth maximization and economic stimulation is positively correlated with the government-issued bonds' relative share of the asset market. However, this is not the case for a policy aiming at minimization of the cost of public debt. On the contrary it might be the case that the effectiveness of a cost minimizing policy is enhanced when the government-issued bonds' share of the total asset market falls. In the analysis above it has been assumed that the economy considered is closed, i.e., an economy where the domestic capital market is not integrated with the foreign capital markets. Consider the polar case, a small open economy where the domestic capital market is fully integrated with its foreign counterparts. In this case the government-issued bonds of the small open country only represent a small fraction of the market portfolio of domestic and foreign assets. The government's ability to affect welfare, wealth and economic activity through public debt management is therefore limited. However, there is still scope for a debt management policy that minimizes the costs of public debt.

Of particular interest is minimization of the costs of public debt, not only because it is the most clear-cut of the objectives discussed. It is also the operative objective used by debt managing institutions, e.g., the Swedish National Debt Office. The analysis suggests that to minimize the cost of public debt the government should issue bonds that have a payoff structure that is negatively correlated (or have a weak positive correlation) with the market portfolio's return, since these bonds can be issued at a higher price than

bonds with a strong positive correlation with the market portfolio. However, this means that the cost of debt service and accordingly the budget deficits and taxes increase in states when the market portfolio's return is low, e.g., in recessions, and this might have negative impact on welfare. On the other hand, minimization of the cost of public debt implies minimization of the taxes associated with public debt, which in turn reduces the excess burden. Minimization of the costs of public debt may also reduce the market risk, which may have a positive effect on the representative agent's utility. The discussion above has only focused on the direct costs associated with public debt. Minimization of the direct costs of public debt may also have indirect effects, e.g., through crowding out of private investment. However, these indirect costs are of less importance if the government-issued bonds' fraction of the market portfolio is small. Throughout the analysis it has been assumed that the risk characteristics of the government-issued bonds are exogenous. However, part of the risk is due to the functioning of the second-hand markets for government-issued bonds, which, e.g., affect the bonds' liquidity. Accordingly, the government can affect the risk and hence the cost of public debt by improving the functioning of the second-hand markets for government-issued bonds.

The analysis above has discussed the objectives of public debt management as if there were a single well-defined objective. However, in the real world there are many objectives, and as shown in the analysis they are in most cases inconsistent. This means that there is a trade-off between different objectives, e.g., cost minimization and economic stimulation. The potential conflict between different objectives is even more evident in an open economy where the central bank tries to control the capital flows through changes in the interest rates induced by changes in the supply of government-issued bonds, while the debt managing institution tries to minimize the cost of public debt.

It has also been shown that, under certain conditions, introduction of new types of government-issued bonds can lead to Pareto improvements. However, one might ask why the private sector has not already created these new types of assets, since an introduction of them leads to Pareto improvements. The explanation for this might be that the public sector has some sort of advantage over the private sector in supplying certain types of assets.

In the preceding analysis it has been assumed that there are no costs (e.g., transaction costs) associated with public debt management. However,

these costs might be considerable, particularly when the government issues new types of bonds, since the issuance of new types of bonds involves opening and operation of new asset markets. Hence, the non-existence of some types of government-issued bonds might be an indication that the cost for the government to issue these types of bonds outweighs the benefits in terms of, e.g., welfare improvements or cost reductions.

Finally a comment on the assumptions is necessary. Throughout the analysis it has been assumed that the agents' preferences are such that the market price of risk is constant. This assumption has simplified the analysis considerably. Nevertheless, there is reason to believe that the results are qualitatively the same for other types of well-behaved preferences.

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# Public Debt Management and the Term Structure of Interest Rates

## 1 Introduction

The purpose of this paper is to analyze the effects of public debt management on the term structure of interest rates. Public debt management means that the government changes the composition of the outstanding public debt through self-financed operations in the financial markets. For example, the government issues short term bonds using the proceeds from the issuance to simultaneously purchase long term bonds, i.e., the policy increases the stock of short term government issued bonds and reduces the stock of long term government issued bonds. To define the term structure of interest rates, consider default free bonds that differ only with respect to their terms to maturity. The relationship between the yields of these bonds is measured by the term structure of interest rates. Public debt management can affect the term structure of interest rates since it changes the supply of financial assets, which in turn might alter the yields on other assets.

Public debt management and its effects on the term structure of interest rates are discussed in Musgrave (1959) and Tobin (1963). The discussion focuses on public debt management as an instrument of stabilization policy. The basic argument is that public debt management affects the economic activity through changes in the availability of capital and the cost of capital. These changes in turn affect the level of the private sector's investments. One example of such a debt management policy is Operation Twist in the United States, where the Federal Reserve, in 1961, via public debt management changed the maturity structure of the outstanding public debt in order to simultaneously raise the short term interest rates while lowering the long

term interest rates<sup>1</sup>. Most of the literature after Musgrave and Tobin e.g., Friedman (1978), Roley (1979), and Agell and Persson (1988), has analyzed the impact of public debt management within the context of atemporal models. This approach means that the effects of public debt management on the term structure of interest rates cannot be explicitly analyzed, since an intertemporal model is necessary in order to distinguish between short term and long term interest rates.

In this paper the impact of public debt management is analyzed in a general equilibrium framework, using a stochastic version of Samuelson's (1958) overlapping generations model (OLG-model) populated by two-period-lived agents. The choice of model is due to the fact that in most cases public debt management is neutral in models with homogeneous agents, and accordingly public debt management has no impact on the term structure of interest rates<sup>2</sup>. An OLG-model permits heterogeneous participation in the asset market since the young generation in every period acts as buyer of the long-lived assets and the old generation always acts as seller of the long-lived assets. This means that there is transfer of resources from the young to the old through the asset sale. Accordingly the multiperiod government-issued bonds can be interpreted as a state contingent intergenerational transfer scheme. Public debt management can therefore be interpreted as changes in the state contingent intergenerational transfer scheme. This means that the government can affect the welfare through public debt management. However, the welfare aspects of public debt management are not discussed in this paper<sup>3</sup>.

The basic concepts used in this paper to model the term structure of interest rates are due to Benninga and Protopapadakis (1986). In the model of Benninga and Protopapadakis the term structure of interest rates is derived in a binomial model where the agents' behavior is characterized by a single three-period-lived representative agent. To determine the term structure of interest rates, Benninga and Protopapadakis use a complete market framework, and the fact that in a complete market the unique state prices determine all asset prices and hence the term structure of interest rates. To

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<sup>1</sup>See Modigliani and Sutch (1967) for a discussion of Operation Twist.

<sup>2</sup>See Paalzow (1990) for a discussion of neutrality of public debt management in models with homogeneous agents.

<sup>3</sup>Gale (1990) discusses the welfare aspects of public debt management. However, he uses a framework that differs from the one in this paper.

employ their approach in an OLG framework where the agents are heterogeneous it is necessary to modify their model. The first modification is that asset trade between the agents is introduced. This modification is necessary since a complete market requires that there exist state prices for all possible states at all future dates. Although the agents in an OLG-model live for just two periods, state prices for all future dates may exist since all generations are linked to each other through intergenerational asset trading. The second modification is that an additional source of uncertainty is introduced into the model. In an OLG-model, this modification is necessary in order to make the future asset prices uncertain. In the model developed below this additional source of uncertainty is uncertainty about the future generations' endowments.

The approach taken in this paper has three features that are worth emphasizing. Firstly, it models the term structure of interest rates in an OLG-model with two-period-lived agents. Secondly, unlike previous studies in this field the approach allows public debt management to be analyzed in an intertemporal framework where the asset demands are derived from the agents' optimizing behavior. Thirdly, the analysis is carried out in a general equilibrium framework in order to capture the real effects of public debt management and not only the effects on the financial markets.

The remainder of the paper is organized as follows. Section 2 introduces the model and the assumptions, and describes the way uncertainty is modelled. In Section 3 public debt management is defined. The term structure of interest rates is derived in Section 4. This section also discusses the properties of the term structure of interest rates in an OLG-model. Section 5, the core of the paper, explores some properties of the model as well as the impact of public debt management on the term structure of interest rates. The concluding section summarizes and discusses the results. It also relates the results in the paper to some empirical studies of the effects of public debt management on the term structure of interest rates.

## 2 The Model

This section introduces the stochastic OLG-model. It starts with a description of the agents and their behavior as well as a description of the asset markets. Then the government is introduced and the equilibrium conditions

are derived. It is shown that public debt can be interpreted as an intergenerational transfer. Furthermore the economy's state prices are determined. These will be crucial when the term structure of interest rates is derived. The remaining part of this section discusses the binomial model of uncertainty that is used.

## 2.1 The Agents

The model employed is a stochastic overlapping generations model with two-period-lived agents. Each generation consists of a single, representative agent. All generations are identical with respect to their preferences, which are given by an intertemporally additive state independent utility function:

$$u(c_1, c_2) = U(c_1) + V(c_2) \quad (1)$$

where  $c_1$  denotes the consumption of the agent when he is young and  $c_2$  the consumption when he is old.  $U(\cdot)$  as well as  $V(\cdot)$  are well-behaved concave utility functions that are assumed to exhibit decreasing absolute risk aversion. The function  $V(\cdot)$  is given by:

$$V(c_2) = \sum_s \pi_s v(c_2, s) \quad (2)$$

where  $\pi_s$  denotes the probability of occurrence of state  $s$  and  $v(c_2, s)$  is a well-behaved concave utility function. There is a single good and each generation receives an endowment of the good at birth. The size of the endowment depends on the state in which the generation is born. There is no endowment when the agents are old. The endowment can either be consumed, invested in complex government-issued bonds, or used as an input in complex production processes (or risky storage technologies)<sup>4</sup>. The bonds are traded in a perfect market, i.e., there are no transaction costs and all bonds are infinitely divisible. It is assumed that the bond market and the production processes are complete, i.e., the number of bonds with linearly independent payoffs is equal to the number of states of nature, and the number of production processes with linearly independent outputs is equal to the number of states

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<sup>4</sup>A complex security is a security with a non-zero cash flow in more than one state of the world. Hence, a complex security is a bundle of Arrow-Debreu securities (i.e., securities with a non-zero cash flow in only one state of the world.)

of nature<sup>5</sup>. The original economy with complex bonds and production processes can therefore - and in the following analysis it will - be replicated by an economy with Arrow-Debreu government-issued bonds and Arrow-Debreu production processes in order to simplify the analysis<sup>6</sup>. Furthermore, since the original bond market and the production processes by assumption are complete, the number of different types of government-issued Arrow-Debreu bonds in the replicating economy as well as the number of Arrow-Debreu production processes in the replicating economy is equal to the number of states. Without loss of generality it is assumed that each government-issued Arrow-Debreu bond pays one unit of the consumption good if a particular state of the world occurs. To study the agents' budget constraints, let  $s$  denote the present state (known to the generation alive),  $s \in S$ ,  $S = (s_1, s_2, \dots, s_n)$ , and let  $\tilde{s}$  denote the state in the next period,  $\tilde{s} \in S$ . Furthermore, let  $t$  denote the date,  $t \in T$ ,  $T = (t_0, t_1, \dots, t_m)$ . Moreover, let  $e(t, s)$  denote the endowment received by generation  $t$  if they are born in state  $s$  and let  $z(t, s, \tilde{s})$  denote the investment of generation  $t$  born in state  $s$  in the Arrow-Debreu production process with a non-zero output in the next period if state  $\tilde{s}$  occurs.

The production function is given by  $\alpha(\tilde{s})f[z(s, \tilde{s})]$ , where  $\alpha(\tilde{s})$  is a stochastic (state contingent) factor of multiplicative efficiency<sup>7</sup>. The function  $f(z)$  is assumed to have the following properties:  $f'(z) > 0$  and  $f''(z) < 0$ . The production function is also assumed to satisfy the Inada conditions, i.e.,  $f(0) = 0$ ,  $\lim_{z \rightarrow 0} f'(z) = \infty$ , and  $\lim_{z \rightarrow \infty} f'(z) = 0$ . The representative agent of generation  $t$  born in state  $s$  maximizes his lifetime expected utility  $u(c_1, c_2)$  subject to the budget constraints<sup>8</sup>:

$$c_1(t, s) = e(t, s) - \sum_{\tilde{s} \in S} p(t, s, \tilde{s})q(t, s, \tilde{s}) - \sum_{\tilde{s} \in S} z(t, s, \tilde{s}) - \tau(t, s) \quad (3)$$

$$c_2(t, s, \tilde{s}) = q(t, s, \tilde{s}) + \alpha(t + 1, \tilde{s})f(z(t, s, \tilde{s})) \quad (4)$$

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<sup>5</sup>The bonds and the production processes are not redundant since by assumption the production processes exhibit decreasing returns to scale and the supply of bonds is fixed (see below), and since there are no short sales in an OLG-model where all agents of the same generation are identical.

<sup>6</sup>An Arrow-Debreu (or primitive or state-contingent) security is an asset with a non-zero cash flow in only one state of the world. See Arrow (1964) and Debreu (1959).

<sup>7</sup>This specification of the production function is due to Diamond (1967). It can be used to represent production functions that can be ordered by their marginal product, i.e., if the original production functions, e.g.,  $g_1$  and  $g_2$ , are such that  $g'_1(z) > g'_2(z)$  for all  $z$ .

<sup>8</sup>The budget constraints for the other generations are similar.

where  $c_1(t, s)$  is generation  $t$ 's first-period consumption if they are born in state  $s$ ,  $c_2(t, s, \tilde{s})$  is generation  $t$ 's (born in state  $s$ ) second-period consumption if state  $\tilde{s}$  is realized when they are old,  $p(t, s, \tilde{s})$  is the issue price at time  $t$  in state  $s$  of a government-issued Arrow-Debreu bond with a payoff of one unit of the consumption good in the next period if state  $\tilde{s}$  is realized,  $q(t, s, \tilde{s})$  is the quantity purchased of that bond, and  $\tau(t, s)$  is the tax on the young generation born in state  $s$  at time  $t$ . The first-order conditions for expected utility maximization with respect to  $z$  and  $q$  are given by:

$$-U'(c_1(t, s)) + \pi_s V'(c_2(t, s, \tilde{s}))\alpha(t+1, \tilde{s})f'(z(t, s, \tilde{s})) = 0 \quad (5)$$

$$-U'(c_1(t, s))p(t, s, \tilde{s}) + \pi_s V'(c_2(t, s, \tilde{s})) = 0 \quad (6)$$

for  $s, \tilde{s} \in S$ .

## 2.2 The Government

At time  $t_0$  (i.e., when the analysis starts) there is an outstanding stock of government-issued bonds, which the government has inherited. Since the bond market by assumption is complete, the inherited stock of government-issued bonds will be replicated by a bundle of Arrow-Debreu bonds in order to simplify the analysis. The Arrow-Debreu government-issued bonds are one-period bonds, and each type has a redemption value of one unit of the consumption good if a particular state of the world occurs and a zero redemption value otherwise. In the absence of public debt management (to be defined later) the stock and hence the composition of the outstanding public debt is the same at all dates and in all states, and therefore given by the composition of the inherited stock of government-issued bonds<sup>9</sup>. Let  $\bar{q}(t, s, \tilde{s})$  denote the supply at time  $t$  in state  $s$  of an Arrow-Debreu bond that pays one unit of the consumption good in state  $\tilde{s}$ , then this condition can be written as:

$$\bar{q}(t, s, \tilde{s}) = \bar{q}(t', s', \tilde{s}) = \bar{q}(\tilde{s}) \quad (7)$$

for all  $s, s', \tilde{s} \in S$  and for all  $t, t' \in T$ . Accordingly the supply of government-issued Arrow-Debreu bonds is independent of time as well as of the realized states and can therefore be written as  $\bar{q}(\tilde{s})$ . The payments associated with

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<sup>9</sup>Consider the original economy with complex bonds, then this assumption means that at every date and in every state the stock of, e.g., two-period bonds is the same.

the redemption of the Arrow-Debreu bonds are the government's only expenditure, and the government primarily finances the redemption by issuing new bonds and selling them to the young generation. However the quantity of bonds of different types that the government can (must) issue is determined by relation (7), and a tax might therefore be necessary in order to balance the government's budget. The tax is a lump-sum tax. In order to avoid intergenerational transfers due to asymmetric tax treatment of the generations, it is assumed that all taxes are imposed on the young generation. To determine the size of the tax on the young generation born in state  $s$ , i.e.,  $\tau(s)$ , write the government's budget constraint as:

$$\bar{q}(s) - \sum_{\tilde{s} \in S} p(t, s, \tilde{s}) \bar{q}(\tilde{s}) = \tau(s) \quad (8)$$

### 2.3 The Equilibrium

An equilibrium is characterized by an array  $[e, q, z, p]$  such that (1) is maximized subject to (3) and (4), the government's budget balances and the bond markets clear. The market clearing condition for the bond markets is given by:

$$q(t, s, \tilde{s}) = \bar{q}(\tilde{s}) \quad (9)$$

for all  $s, \tilde{s} \in S$  and  $t \in T$ . Consider the equilibrium price vector  $p(t, s, \tilde{s})$ . Let the price of today's consumption good serve as a numeraire, then  $p(t, s, \tilde{s})$  is a  $(n + 1)$ -vector since there are  $n$  different government-issued Arrow-Debreu bonds, i.e.,  $p(t, s, \tilde{s}) = [1, p(t, s, \tilde{s}_1), \dots, p(t, s, \tilde{s}_n)]$ , where the first element is the price of the numeraire, i.e., the consumption good. The  $p$ 's are the economy's unique Arrow-Debreu state prices<sup>10</sup>. To determine the one-period state prices, rewrite the first-order conditions in (5) and (6):

$$\frac{\pi_{\tilde{s}} V'(c_2(t, s, \tilde{s}))}{U'(c_1(t, s))} = \frac{1}{\alpha_{\tilde{s}} f'(z(t, s, \tilde{s}))} = p(t, s, \tilde{s}) \quad (10)$$

for  $s, \tilde{s} \in S$  and  $t \in T$ . Hence the economy's state prices are determined by the agents' marginal rate of substitution between consumption today and

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<sup>10</sup>The state price  $p(t, s, \tilde{s})$  is the price in state  $s$  at time  $t$  of an asset that pays off one unit of consumption good if and only if state  $\tilde{s}$  is realized in the future.

consumption in the different states of the world in the next period. Furthermore, in equilibrium, this is equal to the marginal rate of transformation<sup>11</sup>.

The assumption made in Section 2.2 that the supply of government-issued bonds is constant over time, implies that the agents' budget constraints as well as the equilibrium are independent of time and only depend on the realized state. Therefore, the equilibrium depends only on the current state and the state realized in the preceding period. However, the asset prices are solely determined by the young generation's investments. Hence, the equilibrium asset prices as well as the size of the tax - and accordingly the young generation's investment and consumption decisions - only depend on the current state and not on the preceding one<sup>12</sup>. Use this steady state result and the equilibrium condition for the bond market in (9) and the definition of the tax on the young generation in equation (8) to rewrite the agents' budget constraints (3) and (4):

$$c_1(s) = e(s) - \sum_{\tilde{s} \in \mathcal{S}} z(s, \tilde{s}) - \bar{q}(s) \quad (11)$$

$$c_2(s, \tilde{s}) = \bar{q}(\tilde{s}) + \alpha(\tilde{s})f(z(s, \tilde{s})) \quad (12)$$

Inspection of the steady state budget constraints reveals that public debt can be interpreted as a state contingent intergenerational transfer, i.e.,  $\bar{q}(\tilde{s})$  is the transfer from the young to the old if state  $\tilde{s}$  is realized. This result is due to the fact that the old generation is always the one that holds and sells bonds and the young generation always the one that purchases bonds. The payments associated with the government-issued bonds can therefore be replicated by a state contingent intergenerational transfer scheme. Since public debt management changes the supply of  $\bar{q}(\tilde{s})$ , it is equivalent to changes in the state contingent intergenerational transfer scheme<sup>13</sup>. Accordingly public debt management affects the welfare, and the government might improve the welfare for at least some generation(s) through public debt management.

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<sup>11</sup>It is worth noting that (10) implies that every equilibrium in this OLG-model is Pareto optimal.

<sup>12</sup>The steady state property of the model follows from the facts that the stock of outstanding government-issued bonds is independent of state and time and that there is not any capital accumulation in the model.

<sup>13</sup>See Gale (1990) for a discussion of public debt management and intergenerational transfers.

## 2.4 The State Space

To keep the analysis as simple as possible, uncertainty is modelled by a binomial structure, i.e., a structure where two possible states follow each state. Denote these two states by  $a$  and  $b$ , respectively, i.e.,  $S = (a, b)$  for all  $t \in T$ . The stochastic environment is by assumption stationary, i.e., each state has the same probability of occurrence at each date. Each state is fully characterized by the following triplet of exogenous stochastic variables:  $[e(\tilde{s}), \bar{q}(\tilde{s}), \alpha(\tilde{s})]$ . The  $a$ - and  $b$ -states are chosen in such a way that  $c_2(a) > c_2(b)$ , i.e., the old generation's consumption is always higher if state  $a$  is realized and accordingly state  $a$  is preferred to state  $b$  by the old generation. For simplicity, the probability of occurrence of each state is 0.5 conditional upon the occurrence of its predecessor (i.e.,  $\pi_s = 0.5$  for all  $s \in S$ ).

The state at time  $t_0$  is realized and known when the analysis starts. The binomial structure of the model implies that at time  $t_1$  two possible states of the world can occur, and that each of these two states has two possible successor states at time  $t_2$ , i.e., the uncertainty follows a tree structure<sup>14</sup>. Accordingly there are six uncertain states in this two-period economy. To ease the notation in the coming analysis, let states 1 and 2 be the two possible period  $t_1$  states, let states 3 and 4 be the two period  $t_2$  states following state 1, and let states 5 and 6 be the two  $t_2$  states following state 2. Since there are two possible states in each period and the realizations are independent it follows that:  $[e(1), \bar{q}(1), \alpha(1)] = [e(3), \bar{q}(3), \alpha(3)] = [e(5), \bar{q}(5), \alpha(5)]$  and  $[e(2), \bar{q}(2), \alpha(2)] = [e(4), \bar{q}(4), \alpha(4)] = [e(6), \bar{q}(6), \alpha(6)]$ . Let states 1, 3, and 5 be  $a$ -states, i.e., characterized by  $[e(a), \bar{q}(a), \alpha(a)]$ , and let states 2, 4, and 6 be  $b$ -states, i.e., characterized by  $[e(b), \bar{q}(b), \alpha(b)]$ . Although the states are characterized by the same triplet, their history is not the same, e.g., state 3 does not have the same history as state 5. This implies that the old agent in period  $t_2$  is not indifferent between the two states (e.g., state 3 and state 5), since his state contingent endowment is not the same in state 1 (the state preceding state 3) as in state 2 (the state preceding state 5). However, the young agent in period  $t_2$  is indifferent between the two states since his endowment is the same in state 3 as in state 5 (since both are  $a$ -states).

Before closing this section a few comments on the generality of the binomial model of uncertainty are necessary. Cox and Ross (1976) show that a binomial representation of uncertainty can be used to model relatively

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<sup>14</sup>See the appendix for a diagram of the state space.

complex problems under uncertainty. Furthermore the results in this paper, obtained by using a binomial structure, can be generalized to the case when there are more than two states following each state [see Benninga and Protopapadakis (1986)].

To summarize, after each state two states,  $a$  and  $b$ , are possible. Accordingly, in the three-period economy studied there are six possible states, numbered 1 to 6, where odd numbers denote  $a$ -states and even numbers  $b$ -states.

### 3 Public Debt Management

Public debt management is defined as self-financed operations by the government in the financial markets in order to change the composition of the outstanding public debt. The requirement that public debt management should be self-financed is necessary in order to isolate the impact of the changes in the composition of the outstanding public debt from the impact of changes in the fiscal policy. Public debt management changes the supply of government-issued bonds at all future dates and in all future states, and accordingly it affects the equilibrium prices and returns. The transactions associated with public debt management are assumed to be carried out at the new equilibrium prices. Let the variables with primes denote the new equilibrium values. Then the general definition of public debt management, when the government issues Arrow-Debreu bonds, can be written as:

$$\sum_{\tilde{s} \in S} p'(s, \tilde{s}) [\bar{q}'(\tilde{s}) - \bar{q}(\tilde{s})] = 0 \quad (13)$$

However, when there are only two possible states in the next period and accordingly only two types of government-issued Arrow-Debreu bonds, the definition above reduces to:

$$p'(t, s, a) d\bar{q}(t, s, a) + p'(t, s, b) d\bar{q}(t, s, b) = 0 \quad (14)$$

for  $s = (a, b)$ . Since public debt management by definition is self-financed, it does not affect the agents' budget constraints for the period when public debt management takes place. However, through changes in the state contingent intergenerational transfers, public debt management affects the agents' budget constraints in all future periods, and accordingly every generation, except

the old one alive at the date when public debt management takes place. Public debt management therefore cannot be neutral in this model. Finally it is worth noting that the set of feasible public debt management policies is state dependent. Let  $Q(s)$  denote the set of feasible future transfers if public debt management takes place in state  $s$ , then:

$$Q(s) = \{\bar{q}'(s) \mid \sum_{s \in S} p'(s)[\bar{q}'(s) - \bar{q}(s)] = 0\} \quad (15)$$

From the definition of the set  $Q(s)$  it is easily seen that the set of possible future intergenerational transfers that can be obtained through public debt management depends on the initial composition of the outstanding public debt as well as the state in which public debt management takes place.

Finally, it is assumed that public debt management does not affect the agents' expectations. For notational simplicity, the date at which public debt management takes place is denoted  $t_0$  in the forthcoming analysis.

## 4 The Term Structure of Interest Rates and Its Properties

The first part of this section derives the term structure of interest rates. In the second part, properties of the term structure of interest rates in an OLG-model are discussed and the sign of the term structure premium is determined.

### 4.1 The Term Structure of Interest Rates

The general equilibrium properties of the term structure of interest rates in a model with a three-period-lived representative agent have been analyzed by Benninga and Protopapadakis (1986). They use a three-date binomial model with uncertain production, where the agents have access to a complete set of Arrow-Debreu production technologies. By using no arbitrage arguments, they show that the term structure of interest rates for bonds with certain (i.e., non-stochastic) redemption values is determined by the economy's state prices.

In this section, the term structure of interest rates is modelled within the overlapping generations framework discussed above, using the arguments in

Benninga and Protopapadakis. However, the OLG-model discussed above differs from the model of Benninga and Protopapadakis in two important ways. Firstly, it allows for heterogeneous agents since two generations are alive at the same time. In contrast to the representative agent framework in Benninga and Protopapadakis, asset trading between the agents will occur. This means that the different generations will be linked to each other through intergenerational asset trading<sup>15</sup>. Secondly, while Benninga and Protopapadakis just have one source of uncertainty, uncertainty about the outcomes of the production processes, the model employed in this paper has two sources of uncertainty; uncertainty about the outcomes of the production processes and uncertainty about the future generations' endowments. If the next generation's endowment is not uncertain at time  $t_0$ , the time  $t_1$  state prices are not uncertain since they are determined by generation  $t_1$ 's consumption and investment decisions which are certain at time  $t_0$  if generation  $t_1$ 's endowment is certain. Hence to obtain uncertainty about the future state prices it is necessary that the future generations' endowments are stochastic.

Within the framework of the previous sections, the term structure of interest rates for two hypothetical bonds is examined. The bonds are hypothetical in the sense that the bonds' equilibrium prices are determined without introducing them in the economy. Two types of default-free bonds are considered, a one-period bond and a two-period zero-coupon bond. Both pay a certain (i.e., independent of the state of the world) payoff of one unit of the consumption good at their respective redemption dates<sup>16</sup>. To derive the term structure of interest rates the economy's state prices will be used<sup>17</sup>.

To determine the term structure of interest rates in this two-period economy, six period  $t_0$  state prices are needed. The two date  $t_0$  one-period state

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<sup>15</sup>It is worth emphasizing that the asset prices in an OLG-model with intergenerational asset trading are solely determined by the young generation's preferences, since the old generation's supply of bonds is inelastic.

<sup>16</sup>Throughout the analysis it is important to bear in mind that these hypothetical bonds are not the same as the government-issued Arrow-Debreu bonds.

<sup>17</sup>The natural way to analyze the effects of public debt management on the term structure of interest rates would have been to study the effects of public debt management when the government-issued bonds are bonds with different maturity dates and not (as modelled in this paper) one-period Arrow-Debreu bonds. However, when the bond market is complete the two approaches are equivalent, since the payoff structure associated with any mix of outstanding government-issued bonds of any type can be replicated by an appropriate mix of the two government-issued one-period Arrow-Debreu bonds.

prices  $p_1(t_0)$  and  $p_2(t_0)$ , and the four date  $t_0$  two-period state prices  $p_3(t_0)$ ,  $p_4(t_0)$ ,  $p_5(t_0)$ , and  $p_6(t_0)$ . The state prices  $p_1(t_0)$  and  $p_2(t_0)$  are the time  $t_0$  prices of one unit of the consumption good in state 1 and state 2, respectively, and the state prices  $p_3(t_0)$ ,  $p_4(t_0)$ ,  $p_5(t_0)$ , and  $p_6(t_0)$ , the time  $t_0$  prices of one unit of the consumption good two periods ahead in state 3, 4, 5 and 6, respectively. Let  $r(t_0, s)$  denote the interest rate, at  $t_0$  given the realization of state  $s$ , of a one-period hypothetical bond with a certain payoff of one unit of the consumption good in period  $t_1$ , and  $R(t_0, s)$  the interest rate, at  $t_0$  given the realization of state  $s$ , of a two-period hypothetical bond with a certain payoff of one unit of the consumption good in period  $t_2$ . Accordingly the prices of the two bonds are  $[1 + r(t_0, s)]^{-1}$  and  $[1 + R(t_0, s)]^{-1}$ , respectively. In equilibrium the absence of arbitrage implies that these two prices are given by:

$$[1 + r(t_0, s)]^{-1} = p_1(t_0) + p_2(t_0) \quad (16)$$

$$[1 + R(t_0, s)]^{-1} = p_3(t_0) + p_4(t_0) + p_5(t_0) + p_6(t_0) \quad (17)$$

To rewrite (17) in terms of one-period state prices, let  $p_3(t_1)$  and  $p_4(t_1)$  denote the time  $t_1$  one-period state prices provided state 1 (the  $a$ -state) is realized at time  $t_1$ , and  $p_5(t_1)$  and  $p_6(t_1)$  denote the time  $t_1$  one-period state prices provided state 2 (the  $b$ -state) is realized at time  $t_2$ . Then the price in state  $s$  at time  $t_1$  of a one-period bond can be written in terms of the time  $t_1$  one-period state prices:

$$[1 + r(t_1, s)]^{-1} = p_k(t_1) + p_m(t_1) \quad (18)$$

for  $k = 3, m = 4$  if  $s = 1$ , or  $k = 5, m = 6$  if  $s = 2$ . These period  $t_1$  state prices can be interpreted as the period  $t_0$  forward one-period state prices, and by no arbitrage they are given by:

$$p_i(t_1) = \frac{p_i(t_0)}{p_j(t_0)} \quad (19)$$

for  $i = 3, 4$  if  $j = 1$  and for  $i = 5, 6$  if  $j = 2$ . Now use (18) and (19) to rewrite the time  $t_0$  price of a two-period bond given in equation (17) as:

$$\begin{aligned} [1 + R(t_0, s)]^{-1} &= p_1(t_0)[p_3(t_1) + p_4(t_1)] + p_2(t_0)[p_5(t_1) + p_6(t_1)] \\ &= \pi_1 \frac{p_1(t_0)}{\pi_1} [p_3(t_1) + p_4(t_1)] + \pi_2 \frac{p_2(t_0)}{\pi_2} [p_5(t_1) + p_6(t_1)] \\ &= E_{t_0} \left[ \left( \frac{p(t_0)}{\pi} \right)' \left( \frac{1}{1 + r(t_1, \tilde{s})} \right) \right] \end{aligned} \quad (20)$$

where  $[p(t_0)/\pi]' = [p_1(t_0)/\pi_1, p_2(t_0)/\pi_2]$  and  $[1/(1 + r(t_1, \tilde{s}))]' = [p_3(t_1) + p_4(t_1), p_5(t_1) + p_6(t_1)]$ . Use the definition of the expected value of a product to rewrite the expression for  $[1 + R(t_0, s)]^{-1}$  as:

$$E_{t_0} \left[ \left( \frac{p(t_0)}{\pi} \right)' \left( \frac{1}{1 + r(t_1, \tilde{s})} \right) \right] = \frac{1}{1 + r(t_0, s)} E_{t_0} \left( \frac{1}{1 + r(t_1, \tilde{s})} \right) + \text{cov} \left( \frac{p(t_0)}{\pi}, \frac{1}{1 + r(t_1, \tilde{s})} \right) \quad (21)$$

where the fact that  $E_{t_0}[p(t_0)/\pi]' = p_1(t_0) + p_2(t_0) = 1/[1 + r(t_0, s)]$  has been used to rewrite the first factor in the first term. To rewrite the covariance term in (21) in terms of state prices, use equations (18) and (20) and the assumption that  $\pi_a = \pi_b = 0.5$ .

$$\text{cov}(\cdot) = 0.5[p_1(t_0) - p_2(t_0)][p_3(t_1) + p_4(t_1) - p_5(t_1) - p_6(t_1)] \quad (22)$$

In order to interpret the covariance term above, the term structure premium has to be defined. The term structure premium, measured in bond prices, is defined as the difference between the expected cost today of a portfolio of period  $t_0$  and period  $t_1$  one-period bonds such that the portfolio has a certain payoff of one unit of the consumption good two periods ahead, and the price of a two-period bond that delivers one unit of the consumption good with certainty two periods ahead, i.e.:

$$\frac{1}{1 + r(t_0, s)} E_{t_0} \left( \frac{1}{1 + r(t_1, \tilde{s})} \right) - \frac{1}{1 + R(t_0, s)} = -\text{cov}(\cdot) \quad (23)$$

where the right hand side follows from expression (21). From the definition of the term structure premium, it is easily seen that the term structure premium is positive (negative) if the price of the long bond is lower (higher) than the expected price of the portfolio of one-period bonds. Accordingly, the term structure premium is positive (negative) when the long-term rate is higher (lower) than the product of the expected short-term rates. The term structure premium can be interpreted as the period  $t_0$  price of the systematic risk that an agent born in period  $t_0$  takes on by owning a two-period bond.

The right hand side of expression (23) shows that the term structure premium is equal to the negative of the covariance term in (21), which gives the covariance between the pairs  $p_1(t_0)/\pi_1, p_3(t_1) + p_4(t_1)$ , and  $p_2(t_0)/\pi_2, p_5(t_1) + p_6(t_1)$ .

$p_6(t_1)$ . This is the covariance between the probability-normalized state prices in period  $t_0$ , and the price of a one-period bond in period  $t_1$ . From (10) it follows that:

$$\frac{p_i(t)}{\pi_i} = \frac{V'(c_2(i))}{U'(c_1)} \quad (24)$$

for  $i = 1, 2$ . Consider the case when the covariance term is negative, i.e., a high probability-adjusted state-price in period  $t_0$  is accompanied by a low price of a one-period bond in period  $t_1$  (and *vice versa*). A high probability adjusted state price means that the marginal utility of second period consumption is high [follows from equation (24)]. A low price of a one-period bond in period  $t_1$  means that the price in period  $t_1$  of a two-period bond issued in period  $t_0$  is low (since no arbitrage implies that the two prices are equal). The two-period bond is therefore a poor hedge against low consumption (low utility) in period  $t_1$ . A low price and hence a high interest rate as well as a positive term structure premium is therefore necessary in order to induce agents to hold the long-term bond if the covariance term is negative. The same argument applies, *mutatis mutandis*, to the case when the covariance term is positive.

## 4.2 Properties of the Term Structure Premium

First consider the case when the production technology is linear i.e., when the production technology exhibits constant returns to scale. Benninga and Protopapadakis (1986) have shown the following.

**Proposition 1** *If the production technology is linear, i.e., if  $f''(\cdot) = 0$ , then the term structure premium is zero.*

The intuition for this result is as follows. If the technology is linear, the marginal product of investment is constant and hence the state prices are the same for all states. The two-period bonds are therefore not risky since their price is not stochastic. Accordingly the covariance term is zero and hence the term structure premium is zero.

Secondly, consider the case when the production technology exhibits decreasing returns to scale. To determine the sign of the term structure premium when the production technology is concave, recall from Section 2.4 that the states are numbered in such a way that the agents have higher second period consumption in the  $a$ -states (i.e., states 1, 3, and 5) than in the

$b$ -states (i.e., states 2, 4, and 6), i.e.:

$$c_2(i) > c_2(j) \quad \text{for } (i, j) = (1, 2), (3, 4), (5, 6) \quad (25)$$

This assumption together with the first-order conditions in (5) implies that:

$$\alpha_i f'(z_i) > \alpha_j f'(z_j) \quad \text{for } (i, j) = (1, 2), (3, 4), (5, 6) \quad (26)$$

Now, consider the case when  $c_2(3) > c_2(5)$ , and  $c_2(4) > c_2(6)$ , i.e., the agents of generation  $t_1$  have a higher second-period consumption if they are born in state 1 than if they are born in state 2. This implies that  $c_1(1) > c_1(2)$ . The result follows from the fact that it cannot be optimal for the agents to consume more in the states following a bad state (i.e., a state with low consumption) than in the states following a good state, when the utility function is state-independent and the production possibilities are state-independent and symmetric (i.e., they are the same regardless of which state has occurred). Furthermore, by using the same argument it can be shown that it is not optimal for an agent born in the bad state to invest more in the production processes than an agent born in the good state. Hence,  $f'(z_3) < f'(z_5)$ , and  $f'(z_4) < f'(z_6)$ . Combining these results with the agents' budget constraints in (11) shows that a necessary and sufficient condition for  $c_1(a) > c_1(b)$  [or  $c_1(1) > c_1(2)$ ] is that  $e(a) - \bar{q}(a) > e(b) - \bar{q}(b)$ . This means that the young agent consumes more (less) in state  $a$  than in state  $b$  if and only if the resources available to him when he is born are greater (less) in state  $a$  than in state  $b$ . Finally, before the sign of the term structure premium can be determined, rewrite the term structure premium as the negative of the right hand side of the expression for the covariance term in (22) and substitute the state prices by the marginal rates of transformation [follows from the definition in (10)], i.e.:

$$\left( \frac{1}{\alpha_2 f'(z_2)} - \frac{1}{\alpha_1 f'(z_1)} \right) \times \left( \frac{1}{\alpha_3 f'(z_3)} + \frac{1}{\alpha_4 f'(z_4)} - \frac{1}{\alpha_5 f'(z_5)} - \frac{1}{\alpha_6 f'(z_6)} \right) \quad (27)$$

Inspection of (27) reveals that the results above imply a positive term structure premium, since the first factor as well as the second factor in (27) is positive. Hence, the term structure premium is positive if  $c_2(3) > c_2(5)$ ,

and  $c_2(4) > c_2(6)$ . Conversely, if  $c_2(3) < c_2(5)$ , and  $c_2(4) < c_2(6)$ , then  $c_1(1) < c_1(2)$ ,  $f'(z_3) > f'(z_5)$ ,  $f'(z_4) > f'(z_6)$ , and  $e(a) - \bar{q}(a) < e(b) - \bar{q}(b)$ . Moreover if this is the case, the term structure premium is negative since the first factor in equation (27) is positive and the second is negative<sup>18</sup>. The following proposition can now be stated.

**Proposition 2** *If  $e(a) - \bar{q}(a) > e(b) - \bar{q}(b)$  then the term structure premium is positive, and if  $e(a) - \bar{q}(a) < e(b) - \bar{q}(b)$  then the term structure premium is negative.*

The intuition for this proposition is straightforward. First consider the case when  $e(a) - \bar{q}(a) > e(b) - \bar{q}(b)$ , i.e., when the resources available to the young agent are greater in state 1 than in state 2. Generation  $t_1$ 's investment in the two production processes is therefore greater in state 1 than in state 2. Hence,  $\alpha_3 f'(z_3) < \alpha_5 f'(z_5)$  and  $\alpha_4 f'(z_4) < \alpha_6 f'(z_6)$ . From the definition of the state prices in equation (10) follows that  $p_3(t_1) + p_4(t_1) > p_5(t_1) + p_6(t_1)$ . The period  $t_1$  price of a two-period bond issued in period  $t_0$  (i.e., the price paid by generation  $t_1$  when they buy the two-period bonds from generation  $t_0$ ) is higher in state 1 than in state 2. This implies that, for generation  $t_0$ , the two-period bond is a poor hedge against low second-period consumption, since  $c_2(1) > c_2(2)$  by definition. The two-period bond's price is therefore low in period  $t_0$  (the date of issuance). Accordingly its interest rate,  $R(t_0, s)$ , is high, and from equation (23) it is easily seen that a high  $R(t_0, s)$  means a high term structure premium. Secondly, by using the arguments above it can be shown that if  $e(a) - \bar{q}(a) < e(b) - \bar{q}(b)$ , the two-period bond is a good hedge against low second-period consumption in  $t_1$  and its period  $t_0$  price is high. The two-period bond's interest rate is therefore low and the term structure premium is negative.

## 5 The Effects of Public Debt Management

In order to determine the effects on the term structure premium the analysis in this section uses the fact that public debt management affects the agents'

<sup>18</sup>A situation where  $c_2(3) < c_2(5)$ , and  $c_2(4) > c_2(6)$  or vice versa cannot be optimal (and hence cannot occur) when the utility function and the production possibilities are state-independent.

investments in the Arrow-Debreu production processes, i.e., it affects the  $z$ :s. To do this the definition of the term structure premium written in terms of marginal rates of transformation (i.e., in terms of  $z$ :s) in equation (27) is used. In other words, the general equilibrium effects on the term structure premium are determined through the changes in the  $z$ :s due to public debt management. The first part of this section analyzes the impact of public debt management on the  $z$ :s. The main results of the analysis of the comparative statics are summarized in three propositions. In the second part these results are used to discuss the effects on the term structure premium.

## 5.1 Comparative Statics

To determine the general equilibrium effects on the  $z$ :s of public debt management rewrite the first-order conditions with respect to  $z$  and define the functions  $F_i$ :

$$F_i = -U'(c_1) + \pi_i V'(c_2, i) \alpha(i) f'(z(i)) = 0 \quad i = 1, 2, \dots, 6 \quad (28)$$

Let  $dF_z$  denote the matrix of partial derivatives of  $F$  with respect to  $z$ , and  $dF_{\bar{q}}$  the vector of partial derivatives with respect to  $\bar{q}$ . Then the effect on  $z$  is given by the following system of equations (in matrix form):

$$dF_z dz = -dF_{\bar{q}} d\bar{q} \quad (29)$$

To ease the notation define:

$$d\bar{q}_a = d\bar{q}(1) = d\bar{q}(3) = d\bar{q}(5) \quad (30)$$

$$d\bar{q}_b = d\bar{q}(2) = d\bar{q}(4) = d\bar{q}(6) \quad (31)$$

$$a_{ii} = U''(c_1) + \pi_i V''(c_2, i) [\alpha(i) f'(z(i))]^2 + \pi_i V'(c_2, i) \alpha(i) f''(z(i)) \quad (32)$$

$$a_{ij} = U''(c_1) \quad i \neq j \quad (33)$$

Concavity of the utility function implies:

$$a_{ii} < a_{ij} < 0 \quad i \neq j \quad (34)$$

Furthermore let:

$$d_{ij} = a_{ii} a_{jj} - a_{ij} a_{ji} > 0 \quad (35)$$

Let  $\bar{A}_i$  denote the Arrow-Pratt measure of generation  $t_0$ 's absolute risk aversion for second period consumption in state  $i$ , i.e.:

$$\bar{A}_i = -\frac{V''(c_2(i))}{V'(c_2(i))} > 0 \quad i = 1, 2 \quad (36)$$

and let  $A_i$ , for  $i = 1, 2$ , denote generation  $t_1$ 's absolute risk aversion for first period consumption in state  $i$ , and let  $A_i$  for  $i = 3, 4, 5, 6$ , denote their absolute risk aversion for second period consumption in state  $i$ . Then the solution to the system of equations in (29), i.e.,  $dz$ , can be written as<sup>19</sup>:

$$\begin{aligned} dz_1 &= \frac{1}{d_{12}} \pi_2 \bar{A}_2 V'(c_2(2)) \alpha_2 f'(z_2) a_{12} d\bar{q}_b \\ &\quad - \frac{1}{d_{12}} \pi_1 \bar{A}_1 V'(c_2(1)) \alpha_1 f'(z_1) a_{22} d\bar{q}_a \end{aligned} \quad (37)$$

$$\begin{aligned} dz_2 &= \frac{1}{d_{12}} \pi_1 \bar{A}_1 V'(c_2(1)) \alpha_1 f'(z_1) a_{21} d\bar{q}_a \\ &\quad - \frac{1}{d_{12}} \pi_2 \bar{A}_2 V'(c_2(2)) \alpha_2 f'(z_2) a_{11} d\bar{q}_b \end{aligned} \quad (38)$$

$$\begin{aligned} dz_3 &= \frac{1}{d_{34}} A_1 U'(c_1(1)) (a_{34} - a_{44}) d\bar{q}_a \\ &\quad - \frac{1}{d_{34}} \pi_4 A_4 V'(c_2(4)) a_{34} d\bar{q}_b \\ &\quad + \frac{1}{d_{34}} \pi_3 A_3 V'(c_2(3)) a_{44} d\bar{q}_a \end{aligned} \quad (39)$$

$$\begin{aligned} dz_4 &= \frac{1}{d_{34}} A_1 U'(c_1(1)) (a_{43} - a_{33}) d\bar{q}_a \\ &\quad + \frac{1}{d_{34}} \pi_4 A_4 V'(c_2(4)) a_{33} d\bar{q}_b \\ &\quad - \frac{1}{d_{34}} \pi_3 A_3 V'(c_2(3)) a_{43} d\bar{q}_a \end{aligned} \quad (40)$$

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<sup>19</sup>The equation system is solved in the appendix.

$$\begin{aligned}
dz_5 &= \frac{1}{d_{56}} A_2 U'(c_1(2))(a_{56} - a_{66}) d\bar{q}_b \\
&+ \frac{1}{d_{56}} \pi_5 A_5 V'(c_2(5)) a_{66} d\bar{q}_a \\
&- \frac{1}{d_{56}} \pi_6 A_6 V'(c_2(6)) a_{56} d\bar{q}_b
\end{aligned} \tag{41}$$

$$\begin{aligned}
dz_6 &= \frac{1}{d_{56}} A_2 U'(c_1(2))(a_{65} - a_{55}) d\bar{q}_b \\
&- \frac{1}{d_{56}} \pi_5 A_5 V'(c_2(5)) a_{65} d\bar{q}_a \\
&+ \frac{1}{d_{56}} \pi_6 A_6 V'(c_2(6)) a_{55} d\bar{q}_b
\end{aligned} \tag{42}$$

The first term in the expressions for  $dz_i$  ( $i = 3, 4, 5, 6$ ) mainly captures the effects public debt management has, through changes in taxes, on the resources available to generation  $t_1$  when they are young<sup>20</sup>. The second and third terms capture the effects on the resources available to them in their second period of life, i.e., the effects through changes in the transfers received when they are old.

Since there are only two types of government-issued bonds it is reasonable to assume that  $\text{sgn } d\bar{q}_a = -\text{sgn } d\bar{q}_b$ , i.e., if public debt management increases the supply of one type of bond, it reduces the supply of the other type. As one would expect, inspection of the expressions for  $dz_1$  and  $dz_2$  reveals that  $\text{sgn } dz_1 = -\text{sgn } d\bar{q}_a$ , and that  $\text{sgn } dz_2 = -\text{sgn } d\bar{q}_b$ , and accordingly  $\text{sgn } dz_1 = -\text{sgn } dz_2$ . Furthermore if  $d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$  then  $dz_4 < 0, dz_5 > 0$ , and if  $d\bar{q}_a > 0$  and  $d\bar{q}_b < 0$  then  $dz_4 > 0, dz_5 < 0$ . To see this, use the fact that  $a_{43} - a_{33} > 0$  and  $a_{56} - a_{66} > 0$ . To determine the sign of  $dz_3$  and  $dz_6$  is more troublesome. Let the public debt management policy be such that  $d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$ . Consider generation  $t_1$  born in state 1. From the discussion above follows that  $dz_4 < 0$ . Substitute the agents' post-public debt management budget constraints into the first-order condition for

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<sup>20</sup>It is important to keep in mind that a debt management policy that makes the second period  $a$ -state worse and the second period  $b$ -state better has the opposite effect to the agents when they are young, since it reduces the tax in the  $a$ -state and increases it in the  $b$ -state (i.e., makes it better to be born in state  $a$  and makes it worse to be born in state  $b$ ).

utility maximization with respect to investment in the production process with output in state 3 [i.e., equation (5)]:

$$U'[c_1(1) - d\bar{q}_a - dz_3 - dz_4] = V'[\bar{q}_a + d\bar{q}_a + \alpha_3 f(z_3 + dz_3)] \alpha_3 f'(z_3 + dz_3) \quad (43)$$

where  $c_1(1)$  is the pre-public debt management consumption level. To determine the sign of  $dz_3$  a proof by contradiction will be used. First let  $dz_3 < 0$ . Inspection of (43) reveals that if  $dz_3 < 0$  the value of the term on the left hand side falls, while the value of the term on the right hand side increases relatively to the pre-public debt management situation when the two sides were identical. A situation where  $dz_3 < 0$  cannot therefore be optimal. If  $dz_3 = 0$  then the value of the term on the left hand side falls and the value of the term on the right hand side increases, i.e.,  $dz_3 = 0$  cannot be optimal. Accordingly  $dz_3 > 0$  must be the case. By using the same argument, *mutatis mutandis*, it can be shown that if  $d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$ , then  $dz_6 < 0$ , and if  $d\bar{q}_a > 0$  and  $d\bar{q}_b < 0$ , then  $dz_3 < 0$  and  $dz_6 > 0$ . The effects of public debt management on the  $z$ s are summarized in the following proposition.

**Proposition 3** *If  $d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$  then  $dz_a > 0$  and  $dz_b < 0$ . If  $d\bar{q}_a > 0$  and  $d\bar{q}_b < 0$  then  $dz_a < 0$  and  $dz_b > 0$ .*

The proposition says that a reduction (increase) in the state contingent intergenerational transfer received in state  $i$  when the agent is old increases (reduces) the investment in the state contingent production process with output in state  $i$ . For the original economy, i.e., the economy with complex bonds and complex production processes, the proposition can be interpreted in the following way. If the complex bond is a close substitute for the production process, an increase in the bond's supply is likely to crowd out part of the investment in the production process.

Equations (37) and (38) also reveal that, *ceteris paribus*, the higher generation  $t_0$ 's absolute risk aversion for second-period consumption, i.e.,  $\bar{A}_i$ ; ( $i = 1, 2$ ), the greater the effect on  $dz_i$  ( $i = 1, 2$ ). The explanation is straightforward. Since the second period utility function,  $V(\cdot)$ , exhibits decreasing absolute risk aversion, a high  $\bar{A}_i$  means low utility in the second period. Concavity of the utility function and decreasing absolute risk aversion imply that the lower the second period utility is, the greater is the impact on  $dz_i$ .

( $i = 1, 2$ ) of a change in the transfer that generation  $t_0$  receives when it is old. From equations (39) to (42) it follows that, *ceteris paribus*,  $|dz_3|$  is decreasing in  $A_1$ ,  $|dz_4|$  is increasing in  $A_1$ ,  $|dz_5|$  is increasing in  $A_2$ , and  $|dz_6|$  is decreasing in  $A_2$ , where  $A_i$  is the first period risk aversion of generation  $t_1$  born in state  $i$  ( $i = 1, 2$ ). To interpret these results, use the fact that the change in resources available to generation  $t_1$  in state 1 (i.e.,  $-d\bar{q}_a$ ) is perfectly negatively correlated with the change in resources available to them in state 3 (i.e.,  $d\bar{q}_a$ ) and positively correlated with the change in resources available to them in state 4 (since  $\text{sgn } -d\bar{q}_a = \text{sgn } d\bar{q}_b$ ). From concavity of the utility function follows that the higher (lower) utility in the first period, the greater the tendency to smooth the consumption over the states. Accordingly, if utility is high (low) the change is large (small) in the investment in the production process with output in the second period state whose change in resources is negatively correlated with the change in resources in the first period, and the change will be small (large) in the investment in the production process with output in the state whose change in resources is positively correlated with the change in resources in the first period. The same argument can be applied to  $dz_5$  and  $dz_6$ . However, in this case the change in resources available to generation  $t_1$  in state 2 (i.e.,  $-d\bar{q}_b$ ) is perfectly negatively correlated with the change in resources available to them in state 6 (i.e.,  $d\bar{q}_b$ ), while it is positively correlated with the change in resources available to them in state 5 (i.e.,  $d\bar{q}_a$ ). The following proposition can now be stated<sup>21</sup>.

**Proposition 4** *The change in generation  $t_0$ 's investment in the production processes,  $|dz_i|$  ( $i = 1, 2$ ), is increasing in  $\bar{A}_i$ , i.e., in generation  $t_0$ 's absolute risk aversion for second-period consumption in state  $i$ . Furthermore,  $|dz_3|$  is decreasing in  $A_1$ ,  $|dz_4|$  is increasing in  $A_1$ ,  $|dz_5|$  is increasing in  $A_2$ , and  $|dz_6|$  is decreasing in  $A_2$ , where  $A_1$  and  $A_2$  denote generation  $t_1$ 's risk aversion for first-period consumption in state 1 and 2, respectively.*

To determine the size of  $|dz_3|$  relatively to the size of  $|dz_5|$  and the size of  $|dz_4|$  relatively to the size of  $|dz_6|$ , consider the case when the term structure premium initially is positive. Let the debt management policy be such that

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<sup>21</sup>Throughout the analysis the impact of public debt management is discussed in terms of generation  $t_1$ 's first-period risk aversion. It is important to remember that from the first-order conditions it follows that the first-period risk aversion is high (low) when the second-period risk aversion is high (low). Inspection of the expressions for  $dz_i$  shows that this implies that the effects on  $dz_3$  and  $dz_6$  might be ambiguous.

$d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$ . From Proposition 3 follows that  $dz_3, dz_5 > 0$  and  $dz_4, dz_6 < 0$ . Suppose that  $dz_5 > dz_3$ . In Section 4.2 it was shown that a positive term structure premium implies that  $f'(z_3) < f'(z_5)$  and  $f'(z_4) < f'(z_6)$ . Accordingly, due to changes in the investment in the production processes, the increase in consumption in state 5 is greater than the increase in consumption in state 3 if  $dz_5 > dz_3$ . By using the same argument it can be shown that if  $dz_6 < dz_4$ , the reduction in consumption in state 6 is greater than the reduction in state 4. These results imply that the changes in consumption are greater in the bad states, since a positive term structure premium implies that  $c_2(3) > c_2(5)$  and  $c_2(4) > c_2(6)$ . From concavity of the utility function follows that this cannot be optimal, i.e.,  $dz_5 > dz_3$  and  $dz_6 < dz_4$  cannot be optimal if the term structure premium initially is positive. This argument also applies to the cases when  $dz_3 = dz_5$  and  $dz_4 = dz_6$ , and when either  $dz_5 \geq dz_3$  or  $dz_6 \leq dz_4$ . The argument also applies, *mutatis mutandis*, to the cases when  $d\bar{q}_a > 0$ ,  $d\bar{q}_b < 0$ , and when the term structure initially is positive. The results are summarized in the following proposition.

**Proposition 5** *If the term structure premium initially is positive it follows that  $|dz_3| > |dz_5|$  and  $|dz_4| > |dz_6|$ . If the term structure premium initially is negative it follows that  $|dz_3| < |dz_5|$  and  $|dz_4| < |dz_6|$ .*

## 5.2 The Effects on the Term Structure Premium

In Section 4 it was shown that the term structure premium could be written as:

$$- cov \left( \frac{V'(c_2)}{U'(c_1)}, \frac{1}{1+r(t_1)} \right) \quad (44)$$

i.e., the negative of the covariance between the pairs  $V'(c_2(1))/U'(c_1), p_3(t_1) + p_4(t_1)$ , and  $V'(c_2(2))/U'(c_1), p_5(t_1) + p_6(t_1)$ . Loosely spoken, the first factor in the covariance term measures the direct effects of public debt management on the utility of generation  $t_0$ . The second factor measures the indirect effects public debt management has on the utility of generation  $t_0$ , through changes in the next generation's budget constraints and hence on the time  $t_1$  state prices that determine the price of the two-period bond in the next period.

To analyze the effects of public debt management on the term structure premium consider the expression for the term structure premium in equa-

tion (27), where the term structure premium is written in terms of marginal rates of transformation. Differentiate it with respect to  $z$ :

$$\begin{aligned}
& \left( \frac{1}{\alpha_1 [f''(z_1)]^2} dz_1 - \frac{1}{\alpha_2 [f''(z_2)]^2} dz_2 \right) \\
& \times \left( \frac{1}{\alpha_3 f'(z_3)} + \frac{1}{\alpha_4 f'(z_4)} - \frac{1}{\alpha_5 f'(z_5)} - \frac{1}{\alpha_6 f'(z_6)} \right) \\
& + \left( \frac{1}{\alpha_5 [f''(z_5)]^2} dz_5 + \frac{1}{\alpha_6 [f''(z_6)]^2} dz_6 - \frac{1}{\alpha_3 [f''(z_3)]^2} dz_3 - \frac{1}{\alpha_4 [f''(z_4)]^2} dz_4 \right) \\
& \times \left( \frac{1}{\alpha_2 f'(z_2)} - \frac{1}{\alpha_1 f'(z_1)} \right) \quad (45)
\end{aligned}$$

where the first term captures the effects of public debt management on the time  $t_0$  term structure premium through changes, induced by public debt management, in the behavior of generation  $t_0$  (direct effects), and where the second term captures the effects due to changes, induced by public debt management, in generation  $t_1$ 's behavior (indirect effects). By combining (26) and (27) it is easily seen that the second factor in the first term in (45) is positive (negative) when the term structure premium is positive (negative). Furthermore, from relation (26) follows that the second factor in the second term always is positive.

To determine the effects of public debt management on the term structure premium, the case when the term structure initially is positive [i.e.,  $e(a) - \bar{q}(a) > e(b) - \bar{q}(b)$ ], and  $d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$ , is analyzed in detail in this section. The arguments used to discuss this case will then be applied to the other cases.

Firstly, consider the effects on the first term in (45). From Proposition 3 it follows that  $dz_1 > 0$  and  $dz_2 < 0$ . Consequently, the first factor in the first term is positive and the debt management policy considered brings about an increase in the first term and hence an increase in the term structure premium, *ceteris paribus*. The intuition for the result is the following. The increase in  $\bar{q}_b$ , the intergenerational transfer to the old generation in the bad state, and the reduction in  $\bar{q}_a$ , the intergenerational transfer to the old in the good state, reduces the risk and thus the demand for a hedge against low second period consumption falls. Accordingly the price of the long bond,  $[1/(1 + R(t_0))]$ , falls and the term structure premium increases. If  $d\bar{q}_a < 0$ ,  $d\bar{q}_b > 0$ , and the term structure premium initially is positive,

it follows from Proposition 4 that the higher  $\bar{A}_i$  ( $i = 1, 2$ ), i.e., the lower utility in state  $i$ , the higher value of the first term and the more likely it is that the change in the term structure premium is positive. By applying the same arguments, *mutatis mutandis*, the following proposition, giving the relation between generation  $t_0$ 's utility and the sign of the change in the term structure premium, can be proven.

**Proposition 6** *If the term structure premium initially is positive, then a debt management policy that, from generation  $t_0$ 's point of view, reduces (increases) the difference between the good and the bad second period states, i.e., a policy such that  $d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$  ( $d\bar{q}_a > 0$  and  $d\bar{q}_b < 0$ ), is more likely to raise the term structure premium the lower (higher) second period utility generation  $t_0$  has. If the term structure premium initially is negative, then a debt management policy that, from generation  $t_0$ 's point of view, reduces (increases) the difference between the good and bad second period states, i.e., a policy such that  $d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$  ( $d\bar{q}_a > 0$  and  $d\bar{q}_b < 0$ ), is more likely to raise the term structure premium the higher (lower) second period utility generation  $t_0$  has.*

Secondly, consider the impact of public debt management on the second term in (45). This term captures the effects of public debt management on the state prices in period  $t_1$ , which determine the period  $t_1$  price of the hypothetical two-period bond issued in period  $t_0$ . For the change in the term structure premium to be positive it is necessary that the second term in (45) either is positive or not "too negative" (since the first term is positive when  $d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$  and the term structure premium is positive, and the second factor in the second term always is positive). From Proposition 3 follows that when  $d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$  then  $dz_3, dz_5 > 0$  and  $dz_4, dz_6 < 0$ . By using this result, it is easily seen that the effects on the second term in (45) are ambiguous. However, inspection of the first factor in the second term reveals that it is positive or not too negative if the sum  $\alpha_5^{-1}[f''(z_5)]^{-2}dz_5 + \alpha_6^{-1}[f''(z_6)]^{-2}dz_6$  is sufficiently large compared with the sum  $\alpha_3^{-1}[f''(z_3)]^{-2}dz_3 + \alpha_4^{-1}[f''(z_4)]^{-2}dz_4$ , i.e., the larger  $dz_5$  and the smaller  $|dz_6|$ , *ceteris paribus*, and the smaller  $dz_3$  and the larger  $|dz_4|$ , *ceteris paribus*, the higher the probability that the change in the term structure premium is positive. In terms of state prices, this condition says that  $p_5(t_1) + p_6(t_1)$  should increase relatively to  $p_3(t_1) + p_4(t_1)$ , i.e., the two-period bond's period  $t_1$  price in the bad state (from generation  $t_0$ 's point of view) should increase

relatively to its price in the good state. The explanation for this somewhat counter-intuitive result is the following. For the term structure premium to increase, it is necessary that the covariance in (44) falls (i.e., “becomes more negative” since the term structure premium is assumed to be positive and hence the covariance is negative). Consider the effect on generation  $t_0$ 's marginal rate of substitution of consumption in the second period for consumption in the first period. Since  $dz_1 > 0$  and  $dz_2 < 0$ , inspection of (10) reveals that the marginal rate of second-period consumption in state 1 (the good state) for consumption in the first period increases, while the marginal rate of substitution of second period consumption in state 2 (the bad state) for first period consumption falls. If the period  $t_1$  prices of a two-period bond issued at  $t_0$ , i.e.,  $p_3(t_1) + p_4(t_1)$  and  $p_5(t_1) + p_6(t_1)$ , are held constant, the covariance increases and the term structure premium falls. Hence, for the covariance to fall and the term structure premium to increase it is necessary that  $p_3(t_1) + p_4(t_1)$  falls and  $p_5(t_1) + p_6(t_1)$  increases. From Proposition 4 follows that  $dz_3$  is decreasing in  $A_1$ ,  $|dz_4|$  is increasing in  $A_1$ ,  $dz_5$  is increasing in  $A_2$ , and  $|dz_6|$  is decreasing in  $A_2$ . Accordingly, the higher  $A_1$  and  $A_2$ , i.e., the lower utility generation  $t_1$  has, the higher probability that the change in the term structure premium is positive if the term structure premium initially is positive and the debt management policy is such that  $d\bar{q}_a < 0$  and  $d\bar{q}_b > 0$ . This argument also applies to the cases when  $d\bar{q}_a > 0$ ,  $d\bar{q}_b < 0$ , and/or when the term structure premium initially is negative. The following proposition can therefore be stated.

**Proposition 7** *The lower (higher) first period utility generation  $t_1$  has, the higher is the probability that the change in the term structure premium is positive (negative).*

This result is independent of the sign of the initial term structure premium as well as of the debt management policy. The reason for this result is that a policy that from generation  $t_0$ 's point of view makes their second period  $a$ -state better and  $b$ -state worse, makes it worse for generation  $t_1$  to be born in state  $a$  and better to be born in state  $b$ , i.e., the effects on the two generations' utilities are negatively correlated. From concavity of the utility function follows that a change of a given size in the transfers paid by generation  $t_1$  will have greater impact the lower utility generation  $t_1$  has. The negative impact on the covariance (positive impact on the term structure premium) will therefore, *ceteris paribus*, be greater the lower utility generation  $t_1$  has.

In the preceding analysis it has been assumed that the probabilities of occurrence are identical, i.e.,  $\pi_i = 0.5$ . However the results above also apply to the case when the probabilities are not identical. If this is the case, then the good (bad) state is defined as the state which has the highest (lowest) “probability adjusted” consumption.

## 6 Conclusions

The examination above of the term structure of interest rates in an OLG model has shown that the sign on the term structure premium depends on the correlation between the young generation’s and the old generation’s consumption. For example, if the good state (i.e., the state with high consumption) for the old coincides with the good state for the young generation, the term structure premium is positive. This result is due to the fact that in an OLG model the old generation’s supply of bonds is inelastic. The bond prices are thus solely determined by the young generation’s investments, which in turn are determined by the resources available to them over the life cycle. Since public debt serves as an intergenerational transfer from the young to the old, the effects of public debt management on the resources available to the young generation through changes in the intergenerational transfers are crucial in determining the effects on the term structure of interest rates. Accordingly a partial equilibrium analysis of the financial markets is not adequate in order to determine the effects on the term structure of interest rates. For the analysis of the impact of public debt management on the term structure, this means that a debt management policy which intuitively might be thought of as reducing the demand for a hedge against low second-period consumption and hence reduce the term structure premium, might have the opposite effects (i.e., increase the demand for a hedge and accordingly reduce the term structure premium).

The analysis above has also shown that although the agents have access to perfect and complete capital markets, public debt management matters, i.e., it affects the term structure of interest rates. There are two assumptions that are crucial for the emergence of this result. Firstly, there are heterogeneous agents in an OLG model and accordingly public debt management has redistributive effects. Secondly, the production processes exhibit decreasing returns to scale, which implies that the economy’s state prices and hence

interest rates are not the same at different levels of investment. Furthermore, decreasing returns to scale imply that the agents cannot undo the effects of public debt management through changes in their investments in the production processes.

The analysis in this paper suggests that it is the covariances of the bonds' payoff structures, and not the change in the supply of long term and short term bonds, *per se*, that determine the effect of public debt management on the term structure of interest rates. In other words, what determines the effects on the term structure is how public debt management affects the hedging opportunities through changes in asset supply, taxes, and prices.

Modigliani and Sutch (1966, 1967) have empirically analyzed the effects of Operation Twist on the term structure of interest rates. By increasing the supply of short term bonds and reducing the supply of long term bonds, the Federal Reserve tried, in 1961, to increase the short term interest rates and lower the long term interest rates in the United States. Modigliani and Sutch find weak or no evidence of any effects of Operation Twist on the term structure of interest rates. This paper offers a possible explanation for this result; if the long term and short term bonds were close substitutes, the effect on the term structure of interest rates should be small.

Finally, some comments are necessary on the assumptions made and the generality of the results in this paper. It has been assumed that the stock of outstanding government-issued bonds is constant over time, e.g., that the number of one-period and two-period bonds, respectively, is constant. However the analysis also applies to the case when the stock of outstanding government-issued changes, e.g., through redemption of mature bonds. The assumption that there is one production process for each state is less restrictive than it might appear since any technology can, in a complete market, be represented by an Arrow-Debreu technology. The binomial modelling of uncertainty is also less restrictive than it might appear; the results obtained in a binomial model of uncertainty can be generalized to the case when there are more than two states following each state [see Benninga and Protopapadakis (1986)]. However, as shown in Cox and Ross (1976), the binomial representation of uncertainty can be used to model relatively complex problems under uncertainty.

## Appendix

Consider the equation system in (29) i.e.,  $dF_z dz = -dF_{\bar{q}} d\bar{q}$ . The matrix  $dF_z$  is given by:

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{pmatrix} \quad (46)$$

where:

$$a_{ii} = U''(c_1) + \pi_i V''(c_2, i) [\alpha(i) f'(z(i))]^2 + \pi_i V'(c_2, i) \alpha(i) f''(z(i)) \quad (47)$$

$$a_{ij} = U''(c_1) \quad i \neq j \quad (48)$$

$$a_{ii} < a_{ij} < 0 \quad i \neq j \quad (49)$$

The determinant of  $dF_z$  is given by:

$$|dF_z| = (a_{11}a_{22} - a_{12}a_{21})(a_{33}a_{44} - a_{34}a_{43})(a_{55}a_{66} - a_{56}a_{65}) > 0 \quad (50)$$

where the inequality follows from (49). The matrix  $dF_{\bar{q}}$  is given by:

$$\begin{pmatrix} b_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 & 0 \\ b_{31} & 0 & b_{33} & 0 & 0 & 0 \\ b_{41} & 0 & 0 & b_{44} & 0 & 0 \\ 0 & b_{52} & 0 & 0 & b_{55} & 0 \\ 0 & b_{62} & 0 & 0 & 0 & b_{66} \end{pmatrix} \quad (51)$$

where:

$$b_{ii} = \pi_i V''(c_2, i) \alpha(i) f'(z(i)) \quad (52)$$

$$b_{ij} = U''(c_1) \quad i \neq j \quad (53)$$

where  $b_{ii}, b_{ij} < 0$ . Define the vector  $\theta$ :

$$\theta = -dF_{\bar{q}} d\bar{q} \quad (54)$$

whose elements are given by:

$$\theta_1 = -\pi_1 V''(c_2(1))\alpha_1 f'(z_1)d\bar{q}_a \quad (55)$$

$$\theta_2 = -\pi_2 V''(c_2(2))\alpha_2 f'(z_2)d\bar{q}_b \quad (56)$$

$$\theta_3 = [U''(c_1(1)) - \pi_3 V''(c_2(3))\alpha_3 f'(z_3)]d\bar{q}_a \quad (57)$$

$$\theta_4 = U''(c_1(1))d\bar{q}_a - \pi_4 V''(c_2(4))\alpha_4 f'(z_4)d\bar{q}_b \quad (58)$$

$$\theta_5 = U''(c_1(2))d\bar{q}_b - \pi_5 V''(c_2(5))\alpha_5 f'(z_5)d\bar{q}_a \quad (59)$$

$$\theta_6 = [U''(c_1(2)) - \pi_6 V''(c_2(6))\alpha_6 f'(z_6)]d\bar{q}_b \quad (60)$$

Define:

$$d_{ij} = a_{ii}a_{jj} - a_{ij}a_{ji} > 0 \quad (61)$$

for  $i = 1, j = 2$ , or  $i = 3, j = 4$ , or  $i = 5, j = 6$ , and where the inequality follows from (49). Then the solution of the equation system in (29) can be written as:

$$dz_1 = \frac{1}{|dF_z|}(\theta_1 a_{22} - \theta_2 a_{12})d_{34}d_{56} \quad (62)$$

$$dz_2 = \frac{1}{|dF_z|}(\theta_2 a_{11} - \theta_1 a_{21})d_{34}d_{56} \quad (63)$$

$$dz_3 = \frac{1}{|dF_z|}(\theta_3 a_{44} - \theta_4 a_{34})d_{12}d_{56} \quad (64)$$

$$dz_4 = \frac{1}{|dF_z|}(\theta_4 a_{33} - \theta_3 a_{43})d_{12}d_{56} \quad (65)$$

$$dz_5 = \frac{1}{|dF_z|}(\theta_5 a_{66} - \theta_6 a_{56})d_{12}d_{34} \quad (66)$$

$$dz_6 = \frac{1}{|dF_z|}(\theta_6 a_{55} - \theta_5 a_{65})d_{12}d_{34} \quad (67)$$

Finally, in order to obtain equations (37) to (42) substitute for  $\theta_i$  from equations (55) to (60), use the fact that  $|dF_z| = d_{12}d_{34}d_{56}$ , the first order conditions in (5) and the definition of absolute risk aversion in (36).

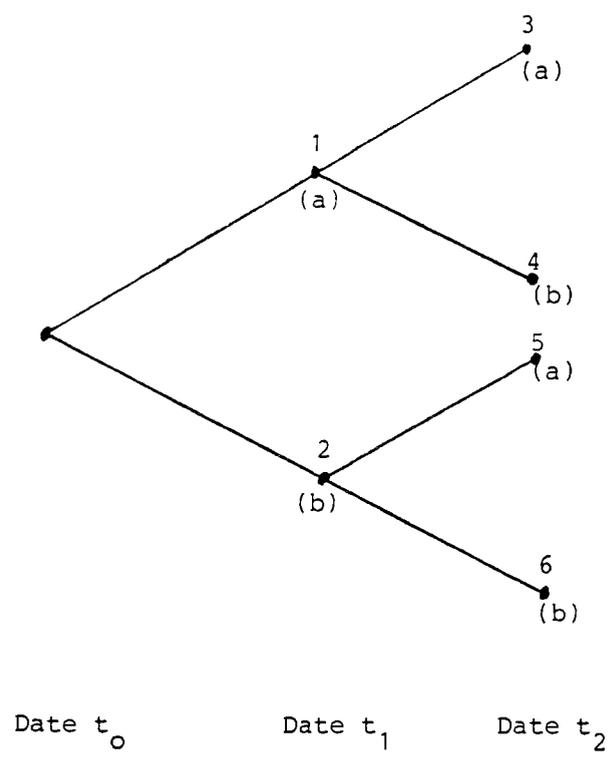


Diagram of the state space.

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