Bargaining Theory

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at the Stockholm School of Economics Stockholm 1972

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Foreword

This report, carried out at The Economic Research Institute will shortly be submitted as a doctor's thesis at the Stockholm School of Economics. The author has been entirely free to conduct his research in his own ways as an expression of his own ideas.

The institute is grateful for the financial support which has made this research possible.

Stockholm, November 1972

THE ECONOMIC RESEARCH INSTITUTE
at The Stockholm School of Economics

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This study of bargaining theory is the result of work in this field since 1964 and during this long period of time I have obtained valuable help from a great many persons, for which I would like to express my gratitude.

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This book is an outgrowth of a licentiate thesis presented in 1967, corresponding mainly to Sections 3.2 – 3.3 of this study. The licentiate thesis was written while I was employed at Stockholms Enskilda Bank, where I had the privilege of working with Professor Erik Dahmén. I am grateful for his continuous support of my studies.

During 1967–68 I made a study trip to the United States. My studies in bargaining have benefitted greatly from the discussions I had there with Professors Robert Bishop, Lawrence Fouraker, John Harsanyi, George de Menil, Oskar Morgenstern, Howard Raiffa, Reinhard Selten and Martin Shubik.

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My main source of finance has been the Bank Research Institute, to which I express my sincere gratitude. I also want to thank the Stockholm School of Economics and the Langenskiöld and Söderström Foundations for financing my study trip to the U.S., and the Swedish Council for Social Science Research for financing various costs connected with the publication of this report.

Finally, with deep gratitude I dedicate this book to my wife Monica.

Ingolf Ståhl
Stockholm, November 1972

Contents

Chapter 1
Introductory Summary

1.1 Some Examples of Bargaining Situations

1. Corporation B (the buyer) wants to purchase corporation S (the seller) by issuing new shares in corporation B to the former shareholders of corporation S. S possesses patents, know-how and key personnel which B, with greater financial and managerial resources, is better equipped to utilize than S. During the next decade the merged corporation will generate a synergistic ("2+2=5") merger profit that could not otherwise have been obtained by the separate corporations. This synergistic profit is estimated to be equal to S's present profits of \$ 2 mill. B's present profits are \$ 6 mill. The profits of the two separate corporations as well as the synergistic profit is estimated to grow at the same rate as the market expands, namely 5 per cent a year. While the owners of B have several other investment opportunities with a 25 per cent pay-off, the owners of S have a cost of capital of 15 per cent. With both B and S making a present profit of \$ 1 per share, B proposes an exchange of one share of B for every share of S. But S wants an exchange of three shares of B for two shares of S. Both corporations have the same certified public accountant and they call him in for advice on the terms of exchange.

2. In a small industrialized country there are two producers of a certain building material. They are equally well-located in relation to the main market, which consist of a large number of builders for whom they produce on order. There is no import competition. Variable costs, mainly for raw materials, are virtually the same. The only significant difference in cost is that one producer has a higher cost of capital and hence higher fixed costs. The present plants, each with considerable overcapacity, are expected to last for another decade. After years of competition, during which the prices of the two producers have steadily decreased, the presidents of the two corporations meet in order to discuss their competition. They agree on how the market has behaved and will behave. Since collusion is not prohibited by law they are willing to reach an agreement on market division. The question centers on whether they should split the market into two parts of equal size or whether one party should obtain a larger part.

3. The labor market in a certain country has the following characteristics: Companies negotiate directly with local unions, where the leaders maintain a strong

position vis-à-vis the members. Wage negotiations take place in most major companies at roughly the same time each spring. During a certain year the negotiations are critical. Due to inflationary developments the workers want large wage increases, while the corporations are strictly opposed to cost increases. Strikes are imminent at a number of large corporations. In order to reduce antagonism the government plans to send out mediators to the major corporations. They are to initiate "fact-finding" discussions between the parties about the economic situation during the coming period of agreement, especially in reference to the business cycle, the development of particular markets and the general cost situation. At this time government is also in the process of choosing between increasing the corporate income tax or levying a proportional tax on wages. The decision as to which of these is the most suitable measure is greatly influenced by how these taxes will affect the wage negotiations.

1.2 Introductory Account of Characteristics of the Model

Three examples of different bargaining situations were described in the preceding section. Their common denominator is that they can all be analyzed more closely using a bargaining model that will be developed in this study. Later on the following will be shown to hold for at least some circumstances.[1]

As regards situation 1: An agreement will be reached on the terms desired by the owners of S, i.e. three shares of B for two shares of S.

As regards situation 2: The party with the lower cost of capital will obtain a larger share of the market.

As regards situation 3: An increase in the corporate income tax will *not* have any effect on the outcome of the wage negotiations while the proportional tax on wages will have a considerable effect. The whole wage tax will be shifted backwards to the union.

These examples are intended to bring out some of the main characteristics of our model and to indicate the scope of its ultimate applicability. First of all our bargaining model is not directed solely towards one specific type of situation with respect to the kind of parties involved (producers, middlemen, unions, etc.), nor is it limited to any particular type of bargaining object (wages, price of product, share of capital, etc.). On the other hand the model is far from being general enough to

[1] Pp. 186, 199 and 203, respectively. These examples are on the whole ficticious, but each contains several elements from actual situations known to the author.

be applicable to all kinds of situations. Its direct and immediate applicability is limited primarily to bargaining situations with the following characteristics:

Bargaining takes place between two parties both of whom *think* and act in a *rational* manner. This means among other things that they are logical in their reasoning, that they can state their preferences for possible outcomes[2] and that they consider all consequences of all alternatives of action. Furthermore both parties can estimate the other party's preferences reasonably well as regards the different possible outcomes. Each party knows that the other party is also rational and each party knows what the other party knows about him.

When both parties are rational and can rank all outcomes, the complexity of a negotiation can be reduced to concern only *one single bargaining variable*. This can e.g. be the share of the joint profit that one of the parties receives, the price of the product, the hourly wage rate, etc.

Furthermore each party's present *value of an agreement* on e.g. a certain price or share of the joint profit is assumed to *decrease when the agreement is delayed.* However, this decrease generally takes place at a non-increasing rate. This condition can be fulfilled, for example, if a joint profit is obtained during each period that the contract runs.

We also assume that parties *deliver their bids in turns*.

These are some of the main characteristics of bargaining situations that can be analyzed and in many cases given a solution with the aid of the model to be presented in this book. A solution implies that we can determine what the *terms* (e.g. the price) of the agreement are, *when* the agreement is reached and what each party has *bid* in each period prior to the agreement.

This account might seem difficult to grasp and also imprecise, but all parts of it will be explained in detail in later chapters. This introductory account of the main characteristics of the model is only aimed at emphasizing the following two propositions:

1. The model is a *deductive* one, based on a small number of simple assumptions regarding the rationality of the parties.

2. It is *not* applicable to *all* bargaining *situations*, but only to those that fulfill certain requirements as to how the situation is formed, especially with respect to

[2] In principle the parties can rank the outcomes according to any type of criteria, but our model is probably most applicable to cases where the outcomes can be ranked in terms of their monetary values.

the development over time of the parties' pay-offs or profits from a certain agreement.

1.3 Short Review of the Contents

An overview of the contents of this study is shown in Figure 1 in which the numbers refer to the chapters.

The following short review of the contents will make it easier to understand the various concepts in this figure.

Chapter 2

Chapter 2 is mainly devoted to a discussion of bargaining as a research problem. First bargaining theory is set in relation to other forms of decision theory (2.1). Then the question of why bargaining is an interesting area of study is answered (2.2). The purposes of bargaining models are also discussed in this context (2.3).

In a brief summary (2.4) of the literature review, which is presented as an appendix[3], we note the predominance of models assuming rationality in bargaining. Reasons for this pre-eminence of rationality assumptions are enumerated in 2.5. Against this background the task of our study is formulated (in 2.6), namely to answer the following question: "Is it possible to construct a model of bargaining, based on assumptions of rational behavior that leads to a unique solution for at least some bargaining situations of interest?" Although much of the discussion in bargaining literature revolves around this question, no satisfactory answer has yet been given. We show that answering this question is a necessary first step towards answering the question of whether the construction of a model of bargaining, based on rational behavior, is of interest. It appears fairly likely that a positive answer to the first question will also imply a "yes" to this second question.

Next the main method of study is presented (2.7). The primary emphasis is on using behavioristic assumptions that are as simple as possible. The institutional assumptions concerning the "physical setting" are chosen according to the following principle: While being of interest for some real situations, these assumptions should allow for the determination of a unique solution.

[3] In this literature appendix a taxonomy of bargaining, relating different models to each other is presented first. This is followed by detailed discussions of the bargaining models of: a) Nash b) Zeuthen c) Bishop and Foldes d) Cross and Coddington and e) Hicks.

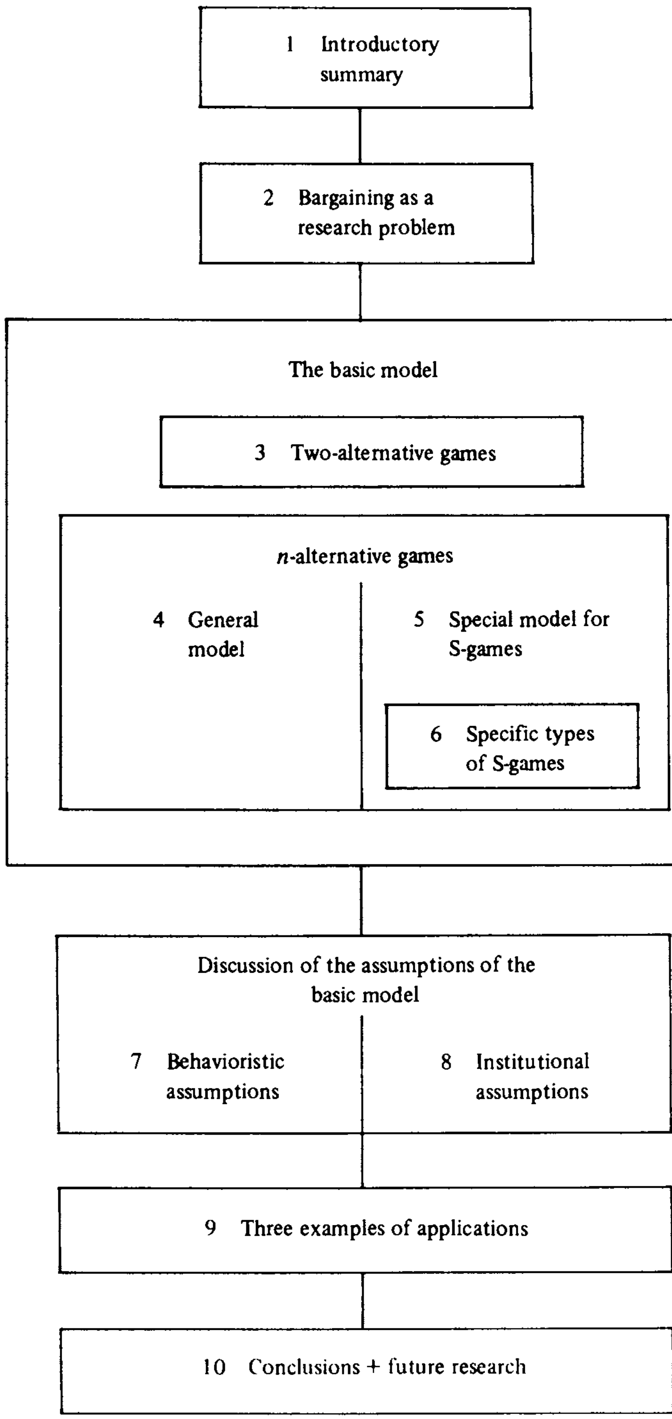

Figure 1 Contents of the study

Chapter 3

Chapters 3–6 are devoted to constructing a basic model of bargaining in which, for a specific set of institutional assumptions, solutions can be deduced on the basis of simple behavioristic assumptions. Due to the strictness of these institutional assumptions the immediate applicability of the basic model is fairly limited. We assume, among other things, the following:

1. Bargaining concerns only *one* variable. This implies that in a sales situation, where both price and quantity have to be determined, the quantity to be transferred is already determined, leading to a joint profit of a fixed size. Bargaining is then said to concern only the *distribution* problem of how this joint profit should be shared. In this connection it should be stressed that a distinction is made in this study between a bargaining game and a negotiation. Negotiation is a broader concept than bargaining game and involves – in addition to the bargaining game – a pre-bargaining phase. In this phase the negotiation is reduced to *one* dimension only, so that the parties have completely opposing interests. The rules of procedure for the bargaining game are also established in this phase.

2. Both parties have *complete information* about each other's pay-offs.

3. The *number of alternatives* to be contemplated is *given*.

These and other institutional assumptions are necessary only for the basic model, and as discussed later in Chapter 8, several of them can be modified without affecting the conclusions of the basic model in any significant way.

We begin to construct this model in Chapter 3 by studying very simple situations with only *two* distribution alternatives (3.2).

It is assumed that the parties alternate bidding, that one bid is delivered in each period and that the money to be divided between the parties decreases the longer an agreement is delayed. Relying only on the behavioristic assumptions of microeconomic theory concerning the rationality of isolated individuals, the following can be determined: A party will accept the distribution alternative proposed by the other party in certain periods designated as *critical*. In such a period a party will obtain more from reaching an agreement on a less favorable alternative (e.g. obtain \$ 6 in each of six periods = \$ 36) than from an agreement one period later on a more favorable alternative (e.g. obtain \$ 7 during five periods = \$ 35).

By assuming that each party knows the other party is rational and that each party in turn knows what the other party knows about him, the solution can step by step be moved backwards in time. In certain situations it can be shown that the party who has the first critical period will accept the other party's terms in his very first bid (3.3).

This holds in one type of bargaining games which we term S-games, characterized by decreasing pay-offs over time and an additional pay-off assumption linked to the concept of a "critical period"[4]. By adding two very weak behavioristic assumptions[5], we show that almost every two-alternative S-game has a unique solution, independent of which party starts bidding (3.4). We do not bother to extend our behavioristic assumptions further and content ourselves with relying on an *ordinal* utility scale (3.5).

Chapter 4

We now proceed to study bargaining involving more than two alternatives. A *general* model – i.e. a general version of our basic model – for using a computer to investigate bargaining games with many alternatives is developed in Chapter 4. First the institutional assumptions of the basic model concerning the number of alternatives are explained (4.2). In order to make bargaining games involving many alternatives and periods solvable by a computer, we introduce the assumption that each agreement on a specific distribution alternative in a certain period is worth the same regardless of what bids were made earlier. "Good-faith" bargaining is also assumed. This implies that if a party has bid a certain alternative in a certain period, the other party can always obtain an agreement on this alternative in any later period (4.3).

The main principle for solving a bargaining game with e.g. 10 alternatives by use of the general method is that the 9 two-alternative games into which this game can be divided are solved first, then the 8 three-alternative games, etc. (4.4).

While discussing (in 4.5) the necessary conditions that must be fulfilled in order for a bargaining game to have a unique solution according to this general method, we note that the study of S-games appears particularly important.

Chapter 5

The importance of S-games gives us reason to develop a *special model* – i.e. a special version of our basic model – for analyzing them. These games can be solved

[4] This assumption – called S_2 – implies *inter alia* that, if a certain period is critical for a party, then all *later* periods are also critical for this party.

[5] These include an assumption of probability dominance and an assumption that it cannot be said with certainty which bid will be chosen, if a party is indifferent between two bids.

in a much more convenient manner by using analytical methods than by using the numerical methods of the general model. First a three-alternative game is solved and then a four-alternative game (5.2 – 5.3). Next, on the basis of these games, theorems are developed for the solution of bargaining games with any number of alternatives. First theorems applying to cases that lead to one party accepting an agreement on his worse alternative (5.4 – 5.5), then theorems applying to cases leading to a "compromise" solution (5.6) are developed. In general, determination of a solution of an n-alternative S-game using these theorems requires us to compare roughly n pairs of pay-offs for each party.

Chapter 6

Chapter 6 contains further development of the special model in Chapter 5. The purposes of Chapter 6 are mainly to obtain an even simpler method for finding the solution and better possibilities of determining which games really lead to a solution and of concretely interpreting the pay-off requirements of the S-games.

First we introduce (in 6.2) S′-games which constitute a subset of all S-games and then (in 6.3) S*-games which constitute a subset of all S′-games.[6] A solution for the n-alternative case in most S′-games and hence also most S*-games can be determined by studying only *two* pairs of pay-offs for each party. In particular we can establish that an agreement will be reached on an alternative x in the very beginning of the bargaining game under the following conditions:

There exists a period such that each party is indifferent between an agreement in this period on alternative x and an agreement in the next period on his closest less favorable alternative.

The solution for S*-games with many periods and alternatives can be approximated with the solution in a hypothetical continuous case having infinitely many alternatives and infinitely short periods.

The pay-off functions of the S*-games can be shown to be of a very simple form. In order to concretize the assumptions on which S*-games are based, three particular pay-off functions fulfilling the requirements of the S*-games are presented (6.4).

Pay-off function 1 implies that the pay-off of an agreement consists of:

> An agreement profit component: the present value of obtaining a share of a certain sum each period, from the agreement time up to the time of the expiration of the contract, which is given,

[6] See Figure 22 on p. 106 for an illustration of the relationship between the different games.

plus a pre-agreement profit component: the present value of all profits, made only prior to the agreement, from time 0 to the agreement time,

minus a bargaining cost component: the present value of all costs of bargaining from time 0 up to the agreement time,

minus an investment component: the present value of the cost of making an investment at the agreement time,

plus a salvage value component: the present value of obtaining a salvage value at the expiration of the contract.

Pay-off functions 2 and 3 refer to cases where the expiration of the contract is *not* given, but depends on when an agreement is reached.[7] In pay-off function 2 the delay in the expiration of the contract is *smaller* than the delay in reaching the agreement. In pay-off function 3 a certain delay in the agreement will lead to a *correspondingly long* delay in the expiration of the contract, i.e. the contract will run over the same length of time regardless of when the agreement is reached. In this instance the model leads to a solution for certain cases when the *periodic* profit varies with the time it takes to reach an agreement.

Analytical methods can be used to draw some more general conclusions about the results of the games. The following can be proved *ceteris paribus* with respect to pay-off function 1:

1. The higher a party's rate of time discount, the smaller his share of the joint profit, provided there are *no* investments.

2. The larger a party's pre-agreement profit, the larger his share.

3. The higher a party's bargaining costs, the smaller his share.

4. The higher a party's interest costs for his *investment*, the larger his share.

Chapter 7

Chapter 7 is devoted to a discussion of the behavioristic assumptions used in the basic model. There is special emphasis on the assumption that each party knows what the other party knows about him, since this is fundamental for our analysis. A set of institutional assumptions can be substituted for this behavioristic assumption.

[7] Since the analysis then becomes more complicated we have limited the function for these two cases to include only *one* component, namely the agreement profit component.

Chapter 8

Next the institutional assumptions are discussed in Chapter 8. First (in 8.2) it is shown that the assumptions of good-faith bargaining and of alternating bidding can be modified, allowing for other types of bidding in a considerable part of the game. The possibility of threats and the influence of liquidity considerations on the outcome of the bargaining are dealt with in 8.3.

Section 8.4 is devoted to the following assumptions of our basic model: The number of alternatives in the bargaining game is given and all these alternatives are such that the parties have diametrically opposing interests. Applying our bargaining model to negotiations concerning e.g. both price and quantity, we show that the parties will only bid alternatives where the quantity exchanged maximizes the joint profit.

The basic model relies on the assumption that the number of alternatives is given prior to the bargaining game. When the number of alternatives is *not* given, the highest share of the joint profit that the party with the higher rate of interest can hope to get can still be established in many cases.

The assumption of *complete* information is discussed in 8.5. It is shown that – at least for S*-games – this assumption can be alleviated considerably. We also show that in several cases of non-complete information our model can determine a fairly small interval within which the solution is to be found.

Chapter 9

Chapter 9 contains some examples of applications. The aim of these applications is to provide a background for discussing the validity of the model. The three examples presented in the beginning of this introductory chapter – a merger case, a duopoly situation and a labor-management negotiation – are all discussed in detail.

As regards mergers: We note that they appear to constitute one of the most promising areas for future application of our model (9.2).

As regards duopoly games: We present cooperative solutions for two well-known games in game theory, the Prisoner's Dilemma Game and the Bertrand Game (9.3).

As regards the labor management case: We note that although several factors make it especially difficult to apply our model to this kind of negotations, it is probable that some aspects might still be of interest for macroeconomic theory (9.4).

Chapter 10

In Chapter 10, the *final chapter* of this monograph, we conclude that it has been possible to show the following: A unique solution for a fairly large group of negotiations which appear to have practical relevance can be deduced on the basis of very simple assumptions of rational behavior.

However, whether or not people really behave or can be made to behave in a rational manner, when bargaining, remains to be investigated. Various forms of laboratory experiments thus appear suitable for continued research in this area. If these experiments would to some extent corroborate our model, then extensive further development of the model would be worthwhile. On the other hand, if our model would be refuted, this would – due to the choice of our behavioristic assumptions – largely imply that any other attempts to construct bargaining models based on rational behavior would be doomed in advance.

Chapter 2
Bargaining as a Research Problem

2.1 What is Bargaining?

2.1.1 Introduction

As the main title of this book indicates the area of study is bargaining. This is such a broad and vague concept that we should begin by defining and specifying our subject area.

The best way our area of study can be described probably involves showing how a bargaining situation is related to other decision situations in economic theory and decision theory. This then enables us to study how some particular bargaining situations such as bilateral monopoly and duopoly fit into the picture.

Figure 2 shows the set of two-person bargaining situations as a subset of other subsets, etc. of more general sets of decision situations or problems, for which theories are formulated.[1]

This figure requires definitions and examples of the different types of situations. It should be stressed that the definitions are considered from the point of view of one specific party, called party 1. It should also be noted that definitions of all the words used frequently in this book with special meaning are listed in alphabetical order on pp. 300 ff.

2.1.2 Definitions of the Situations

1. A decision problem situation: a situation in which party 1 has at least two unequally efficient alternatives for attaining a certain objective.

2. A situation with decision problems that also affect other peoples' pay-offs.

[1] Figure 2 is to be read as follows: Set 1 consists of sets 2 and 2^c, where 2^c is the *complement* of set 2, i.e. the set of all situations that do *not* belong to 2. Set 2 consists in turn of sets 3 and 3^c, etc.

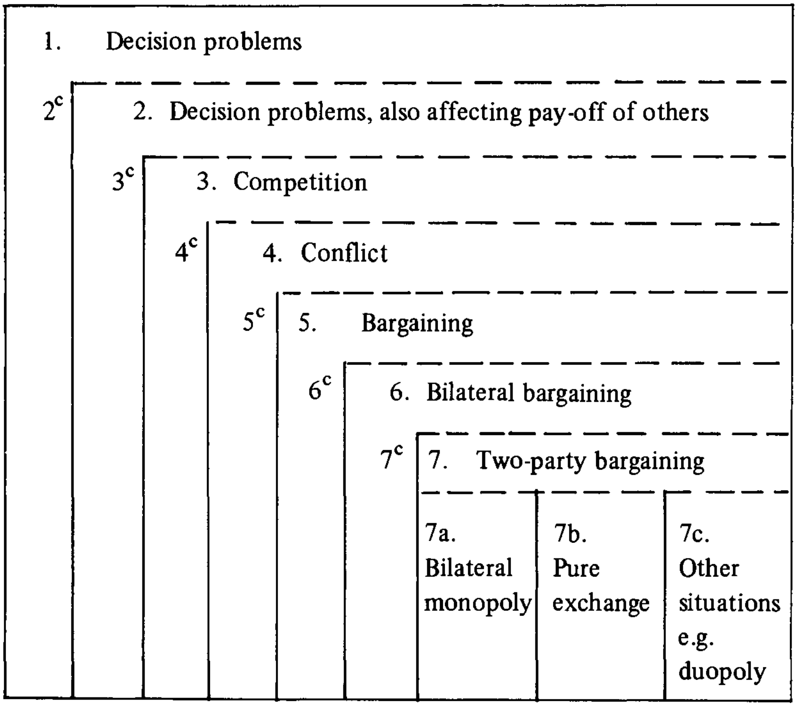

Figure 2 Different types of situations for decision theories

2^c. A situation differing from situation 2 in the sense that the outcome of the decision will not affect any other party's pay-off. This could be a production problem for an individual who wants to produce something as fast as possible using the time saved for leisure.

3. A competitive situation: a situation such that the pay-off of at least one other party is affected in the opposite direction to party 1's pay-off.

3^c. A non-competitive situation such that an increase in party 1's pay-off also leads to an increase in the other parties' pay-offs. This decision situation could be exemplified by a married man contemplating whether he should work harder to earn extra money to share with his wife.

4. A conflict situation: a situation involving competition, where the competitive influence is noticeable and where the parties understand and take into account the fact that they influence each others' pay-offs perceptibly.

4^c. A conflict-free but competitive situation, e.g. the pure competition case of a Swedish wheat farmer competing with a farmer in Nebraska.

5. A bargaining situation: a situation in which there is a potential pay-off that party 1 can obtain only by reaching an *agreement* with some other party and where such an agreement is possible. An agreement is an act by which two or more parties, simultaneously and after exchanging information, irrevocably commit themselves to certain future actions.[2]

5^c. A conflict situation that cannot be solved by bargaining might be exemplified by an oligopoly situation in which price fixing is strictly prohibited.

6. A bilateral bargaining situation: a bargaining situation in which party 1 has to reach an agreement with only *one* other party, party 2, in order to obtain an agreement pay-off, i.e. a pay-off that is *higher* than the one he could have obtained on his own without an agreement.

6^c. A situation in which at least three parties have to reach an agreement in order for party 1 to obtain an agreement pay-off.[3]

7. A two-party bargaining situation: a situation characterized by the assumption that both party 1 and party 2, in terms of certain resources at their disposal, can obtain an agreement pay-off only by making an agreement with each other.[4]

7^c. A bilateral bargaining situation in which either party 1 can obtain an agreement pay-off also by reaching an agreement with someone other than party 2, or party 2 can obtain an agreement pay-off also by an agreement with someone other than party 1.[5] 7^c can be exemplified by a bilateral oligopoly situation.

This study is devoted to two-party bargaining situations. When the word "bargaining" is used below "two-party bargaining" is implied. The set of two-party bargaining situations can, in turn, be divided into three subsets, depending on whether the agreement concerns an exchange of commodities (i.e. goods or services) for money, in which case we refer to bilateral monopoly, whether it means a mutual exchange of commodities (the case of pure exchange) or whether it refers to some other kind of agreement, such as an agreement on limiting competition in a duopoly situation. The theory presented in this study deals with two-party bargaining in general, not limited to any particular one of the three subsets.

[2] Of course, the difference between explicit and implicit agreements is vague but according to our definition an implicit agreement of understanding would be an agreement only if each party is *convinced* that the other party will behave in a certain way in the future.

[3] Using characteristic functions notations – see Luce & Raiffa (1957, p. 182) – we obtain for case 6: $v(\{1,2\})>v(\{1\})+v(\{2\})$, while in case 6^c: $v(\{1,2\}) = v(\{1\})+v(\{2\})$.

[4] In 7: $\forall i>2$: $v(\{1,j\}) = v(\{1\})+v(\{i\})$ and $\forall j>2$: $v(\{2,j\})=v(\{2\})+v(\{j\})$.

[5] In 7^c: $\exists\, i>2$ s.t. $v(\{1,i\})>v(\{1\})+ v(\{i\})$ or $\exists\, j>2$ s.t. $v(\{2,j\})>v(\{2\})+v(\{j\})$.

It should be stressed that the definitions above refer to bargaining situations and not necessarily to bargaining processes. We define a "bargaining process" as a process by which the parties involved exchange a number of bids for the purpose of reaching an agreement. These bids are proposals for reaching an agreement on certain terms at a certain time. A bargaining situation on the other hand is a situation, defined as in point 7 in the list of definitions above that can lead to an agreement in either one of the following ways:

1. A bargaining process of the type mentioned above is carried out.

2. Both parties contemplate different possible bargaining processes. A certain solution then appears to both parties to be the "right" one and an agreement is reached immediately.

3. A mediator, called in by the parties, proposes a solution and succeeds in getting both parties to agree to this proposal, for example by referring to possible bargaining processes.

4. An arbitrator decides the terms of the agreement to which both parties have to abide.

Hence a bargaining process which is carried out in reality is regarded as only one of several ways the parties can reach an agreement in a bargaining situation. As exemplified later in the literature appendix, a considerable part of existing bargaining theory is actually a "bargaining situation" theory rather than merely a "bargaining process" theory. It does not seem suitable to limit the scope of our work to bargaining processes. From now on when we talk about "bargaining theory", a "bargaining situation theory" is implied.

2.2 Why is Bargaining an Interesting Area of Study?

Before proceeding further it seems suitable to state the reasons for choosing bargaining as a subject area. The author's research in this area was originally initiated by the study of a particular problem – a manufacturer's choice between selling through an agent or a subsidiary on a foreign market.[6] Further studies were influenced by the central position of bargaining in economic life and theory. This central position has already been exemplified by the introductory presentation of three concrete examples of areas where bargaining is of great importance. Bargaining is also highly significant in many other areas such as:

[6] Ståhl (1961).

1. Agreements between the producer of a raw material and a processor of it.

2. Determination of intra-company prices settled by negotiations between two departments in a corporation.

3. Agreement on the amount of the share capital in a joint venture to be distributed to each of the two parties initiating this venture.

Most of the situations above and on pp. 1–2 are characterized by one party buying one or several units of some item from the other party. Obviously, all such situations are *not* characterized by bargaining, but there are important elements of bargaining in a fairly large number of these situations. If we look at the way the conditions of transfer, e.g. the prices, are determined in a buyer–seller situation, we can distinguish between two extremes – bilateral monopoly and pure competition. Bilateral monopoly has the following characteristics: one seller is the sole owner of a particular commodity, i.e. a product or a service and the buyer is the only one interested in aquiring it. An example is a seller who has a patent and a buyer who is the only one possessing the technical and financial qualifications necessary for utilizing it. In this type of situation both of these parties, but *only* these two, can directly influence the price. In pure competition, on the other hand, no single buyer or seller can influence the price in any way.

Between these two extremes there are many cases with different degrees of bargaining and market influence on price formation. The following two cases are fairly common in practice.

1. *Bilateral oligopoly*, where a small number of sellers have access to goods which interest only a few buyers. This situation is important, for example, in contemporary marketing of consumer goods, particularly of the everyday type. This is to a large extent characterized by a fairly small number of producers of a certain type of goods and a few large retail chains.

2. *Partial bilateral monopoly* refers mainly to situations with the following characteristics: a. several sellers and buyers, b. the sellers' products are differentiated, c. as compared to the other buyers, a certain buyer has a particularly strong preference for the products of a certain seller over the products of the other sellers. This seller and this buyer are involved in a two-person bargaining situation at least as regards a certain interval on the price scale. An example is a market with several producers, one of which is highly specialized in high quality products. These producers compete for a number of retail stores, one of which in turn has an especially wealthy and quality-minded clientele.

No doubt these situations are of great practical importance. The possibility of developing a bargaining theory to facilitate their analysis contributes to making bargaining theory an interesting area of study.[7]

2.3 Purposes of Bargaining Models

Having noted that bargaining situations constitute an important area of economic activity, our next question concerns the different purposes for which these situations can be studied with the aid of a model. We make our main distinction between the following two aspects:[8]

1. The model is of interest for *research* purposes, such as description and explanation, not directly associated with a decision situation.[9]

2. The model is of interest since – in an extended and improved version – it can *directly* aid decision making. This can be illustrated by the following six examples of possible decision problems in bargaining situations. It should be noted that this enumeration is not exhaustive.

   a. One of the two parties wonders what will happen, especially what the other party will do, if he himself chooses a specific bargaining strategy. We refer to a *simple* (micro) *prediction* aim.

   b. A third party, who e.g. competes with one of the two bargaining parties, wonders if these two will reach an agreement and if so on what terms. We have a *double* (micro) *prediction* aim.

   c. *One* of the bargaining parties wonders which bargaining strategy he should choose. If we want our model to be of use in this situation we can speak of a *normative* aim.

   d. A mediator contemplates which proposal for settlement he should suggest in order to get the two parties to agree. We refer to a *mediation* aim.

[7] It appears reasonable to assume that two-person bargaining is a suitable starting point for studying bargaining in situations such as bilateral oligopoly.

[8] It is difficult to draw a clear line of distinction between these two purposes. Obviously a good explanation or description of how bargaining takes place can be an aid in the development of models for decisions in many cases. However, decision models do not necessarily have to rely on such models. In addition the possibility of improving decision models is *not* regarded as the only *raison d' être* of "pure" research models. As in many other areas of research "pure knowledge" considerations are important.

[9] The theoretical analysis of rational bargaining might also be of interest for "economics regarded as a moral science". The question is then what *would* happen if all economic units pursued their own self-interests in a rational manner.

e. An arbitrator contemplates which proposal for settlement – desirable for society as a whole – he should impose on the two parties. We have an *arbitration* aim.

f. A high government official contemplates – with regard to what will happen during forthcoming labor-management negotiations – whether he should change a certain tax (cf. example 3 on p. 2). This is an example of a *macro prediction* aim.

2.4 Earlier Research on Bargaining

It is natural that a topic as important as bargaining has been the subject of intensive studies. As can be seen from our literature review, bargaining has been studied by a great many social scientists.[10] Among these are some of the best known economists such as Pigou, Wicksell, Schumpeter, Zeuthen, Hicks, von Neumann, Morgenstern, Fellner, etc. Some of the conclusions from this literature survey which are related to our general choice of methodology will be reported in this introductory chapter.

Bargaining theory has been studied mainly in terms of the following widely different approaches:

1. An attempt to generalize on the basis of *empirical* evidence, usually gathered from some specific aspect of bargaining. Most of the literature in this area has not aimed further than establishing some general concepts for describing the bargaining process. These studies contain very few attempts at operationalization and virtually no attempts at establishing any solution.[11] The main purpose of these studies appears to be one of pure research. To the extent that theory discussions take place they are aimed primarily at constructing a system of concepts for describing negotiations.

2. A completely *deductive* approach in which some kind of solution is deduced from a set of assumptions. We can distinguish between two particular types of models with regard to the assumptions:

a) The assumptions generally reflect properties that should intuitively be regarded as "fair" or "equitable". The main purpose of these models is to deduce solutions for *arbitration.*

[10] See pp. 213 ff.

[11] The *solution* of a bargaining game was defined (p. 3) as determination of the bids of each party up to the agreement *and* specification of the agreement, as to the terms and time of the agreement.

b) The assumptions that concern human behavior are intended to conform to some concept of *rationality*, in particular as used in conventional economic theory.[12] Most of these models also assume that each party has access to complete information about his own pay-offs and those of the other party. The purpose of these models is seldom discussed explicitly. But many authors seem to have a "*conditionally predictive*" aim in mind, i.e. to predict how bargainers would behave *if* they were rational. However, in many cases normative, arbitration and mediation purposes as well as pure research purposes cannot be ruled out.

There seems to be very little theory in between these main groups. Hence there are few formal models in which the behavior of the parties is intentionally assumed to be characterized by limited rationality and instead based on psychological theories such as aspiration level and learning.

Another conclusion is that the "rational" theories of group 2b have generally been unsuccessful. There is no consensus on any of these theories and a great deal of the literature is devoted to criticising them.[13] The most severe criticism takes one of the following forms:

1. The model is inconsistent because its outcome does not follow logically from the assumptions of the model *or* some assumptions are inconsistent with other assumptions.

2. Some behavioristic assumptions are *ad hoc*, i.e. applicable only to a specific type of bargaining situation. The question is, in turn, if these *ad hoc* assumptions can be regarded as part of rational behavior.

3. Some models are based on institutional assumptions that correspond so poorly with reality that the models become uninteresting.

Much of the discussion is centered on the question of whether it is possible to construct a model with mutually consistent assumptions about rational behavior leading to a solution for at least some real negotiations of interest. Many economists have answered this negatively.[14] While most of these authors give a solution to the efficiency problem which generally refers to the quantities to be exchanged in a

[12] One difficulty lies in the fact that many authors fail to define what constitutes rational behavior. Rationality will be defined more precisely in Chapter 3. For the time being we define rational behavior as behavior governed by extensive and explicit thought processes of an intelligent and purposive being.

[13] In the literature survey we summarize and add new points of criticism to some of the best known of these theories.

[14] E.g. Menger (1871), Edgeworth (1881), Pareto (1896), Pigou (1908), von Stackelberg (1934), Henderson (1940), von Neumann & Morgenstern (1947), Fellner (1949) and Stigler (1952).

bilateral monopoly situation, they consider the distribution problem – concerning the price at which these quantities are exchanged – to be indeterminate. But they do not give any strict proof for this indeterminateness. Others, however, have felt that a model could be constructed which would lead to a solution of the distribution problem.[15] Thus far there is no concensus.

2.5 Some Reasons for Using Assumptions of Rationality in Bargaining Models

In the preceding section we noted that most formal models of bargaining in the literature rely on the assumption that the parties are rational. Onc is prompted to ask about the reasons why this assumption is so pre-eminent. Although seldom stated explicitly in the literature the following reasons, among others, might have been influential.[16]

1. In accordance with the tradition in much of microeconomic theory assumptions of rational behavior are believed to be the simplest kinds of assumptions that lead to approximately correct *macroeconomic predictions* in many situations.

2. As in many parts of the theory of Operations Research rationality assumptions are considered important to the *normative aim.* Furthermore, if a model is a "good" normative model the number of parties tending to use it might increase. This in turn would make the model able to describe and predict the behavior of more and more parties.[17]

3. In many areas of *game theory*, models of rationality have seemed important as a "*conditionally normative*" tool, i.e. as conditional upon the other party being rational. An example of this is the "minimax model" for the solution of two-person zero-sum games. In the case where the opponent can be regarded as rational, the use of rationality assumptions with regard to both parties appears most suitable for the normative aim. In the case where it is uncertain whether or not the opponent is rational, the use of this "rational model" is not necessarily optimal. It will, however, guarantee the party using the model a certain minimum pay-off. This can be of interest especially against a party whose degree of rationality is unknown.

[15] E.g. Wicksell (1925), Schumpeter (1927), Bowley (1928), Zeuthen (1930), Hicks (1932), Boulding (1950), Harsanyi (1956), Fouraker (1957), Schelling (1960), Bishop (1964), Foldes (1964), Cross (1965), Saraydar (1965) and Coddington (1968).

[16] This belief might to some extent be influenced by the author's own ideas in this area. These are discussed in Chapter 7.

[17] Linear programming, at first regarded only as a normative model for e.g. production choices, can now be used in many cases for forecasting what a company will produce if machine capacities, prices and costs are known.

4. With regard to *mediation*, the assumption of rationality does not seem to be very strong, because the mediator can influence the parties to think rationally by talking to them.

5. The behavior described by the rational model can sometimes be regarded as an *extreme case of "limited rational"* behavior. As decision makers become more educated, obtain better computing and investigating facilities, have more time for decisions on large issues and as bargaining concerns increasingly large sums of money, their behavior might become more consistent with the rational model.

6. The "rational model" might be a suitable *starting point* for constructing a model of limited rationality. Models of rational behavior have generated operational hypotheses which, when tested, have influenced modification of the model towards limited rational behavior.

7. A "rational" theory can be of interest to the extent that it *reveals* phenomena which are not easily observable, such as *thought processes*. Thus it might help the investigator to distinguish more clearly between what seems to happen and what might happen in reality.

8. By predicting the outcomes, a rational theory might be able to *guide empirical research* via indications of which bargaining situations would be interesting to study in terms of situations leading to specific outcomes.

2.6 Statement of Research Problem

In the preceding section we noted several reasons *indicating* why the assumption of rationality should be of interest for bargaining models. These indicators are at least sufficient for justifying the assumption of rationality as a tentative one. However, these indicators do not suffice for *proving* that a model based on the assumption of rational parties can be used *directly* for any of the decision purposes related to some particular type of bargaining situation (cf. p. 17).

In order to prove this we require a positive answer to either of the following two questions as regards some type of bargaining situation:

1. Do the parties really – at least approximately – bargain in the way predicted by a rational model?

2. Can the parties be taught, or influenced in some other way, to bargain in the way prescribed by this type of model?

The following, more fundamental question, has to be answered before these two critical questions can be dealt with: How would the parties behave if they bargained in a rational manner, i.e. according to a model based on assumptions of rational behavior? This is equivalent to the question which is the subject of so much discussion in the literature on bargaining theory (cf. p. 19).

If we do not know what kind of behavior is predicted by such a model, it is impossible to determine whether or not it accurately predicts the behavior of the parties. Nor can we determine whether or not the parties, when given the model, will act in accordance with it. Hence the two questions above can only be answered by testing a rational model *after* it has been constructed. This implies that it is very difficult to pursue any meaningful study of the rationality assumption in bargaining theory without answering the fundamental question raised above. Therefore our study will be devoted to trying to answer this question.

Before proceeding we restate this question more precisely: "Can a model of rational behavior be constructed which would lead to a solution for at least some bargaining situations of interest?"

The requirement that a solution be obtained "for at least some situations of interest" should be explained further. In order to do so we distinguish between two main types of assumptions, behavioristic and institutional. *Behavioristic* assumptions concern the properties of the *parties* – their thought processes and patterns of behavior. *Institutional* assumptions concern the properties of the bargaining *situation*, e.g. how and when bids are exchanged, what the physical result of possible agreements are, what type of communication is possible between the parties and what information is available at the beginning of the negotiation. We regard the assumptions about the physical pay-off of various outcomes as a special kind of institutional assumption.[18] In an experimental reproduction of the bargaining situation, the experimenter will have control over the factors covered by the institutional assumptions, while the parties' behavior cannot be subject to the experimenter's direct control. The requirement that a solution be obtained "for at least some situations of interest" thus implies the following: We are not required to show that the model is applicable to all, or even most sets of institutional assumptions. It is sufficient that the model covers some sets of institutional assumptions of real interest.

Next, we have to define what makes a certain set of institutional assumptions interesting. This is obviously a very subjective matter. It seems most suitable to make this evaluation with reference to those institutional assumptions which more general bargaining theories can be expected to include in the future. A set of

[18] See also the discussion on p. 31, footnote 3 in particular.

institutional assumptions can be regarded as interesting if it appears likely to constitute a subset of the institutional assumptions of such a more general theory. This would make our theory a special case of this more general theory. We do not want to rely on institutional assumptions that are so special that they can be neglected by future research in bargaining theory.[19]

Furthermore, even if our model would be applicable in only a small per cent of all bargaining situations, it would – due to the total importance of bargaining situations – still be of great practical value, meriting considerable research. It also appears reasonable to assume that a model which can be shown to be applicable for some institutional assumptions will also constitute a foundation on which successively more complicated and more realistic versions of the model could possibly be based. The ultimate area of application would then be larger than that of the models to be presented here.[20]

2.7 Establishing a Method of Study

2.7.1 Introduction

The aim of this study was stated above as an attempt to answer the question of whether it is possible to construct a model of rational behavior leading to a solution for at least some institutional assumptions of interest. Before proceeding to try and answer this question we must first determine

1. what to include in our definition of rational behavior, and

2. how to choose the institutional assumptions of the model.

2.7.2 Selecting the Behavioristic Assumptions

We begin by discussing the principles for choosing the assumptions of rational behavior. In a hypothetical choice between several models all leading to solutions for some institutional assumptions, the following model is preferred. It should have as few, simple and general assumptions regarding rational behavior as possible and contain only behavioristic assumptions found in other models. In other words, the set of behavioristic assumptions should constitute a subset of other more complex sets of rationality assumptions. The reasons for this choice are as follows:

[19] For an example of such assumptions see p. 238 in the literature appendix.

[20] This applies particularly to the assumption of complete information. It appears likely that the model can be extended even further, abandoning this assumption more and more (cf. p. 180).

1. The demand for simplicity and the "subset"-principle stem from a desire to obtain reliable feed-back as quickly as possible which would indicate whether the "rational deductive" method is feasible. In order to be certain that the model will be useful for several different purposes it is preferable that both the assumptions and the solution of the model are approximately verified in the experiment.[21] If a model A, based on a certain set of assumptions, cannot be verified in this way, then any model B, relying on – among other assumptions – all the assumptions of model A, can in turn not be verified. If one of A's assumptions is violated, then one of B's assumptions will also be violated.[22]

2. Assumptions of rationality that are general in terms of the situations where they might be applicable have the following advantage over more situation-specific ones, i.e. applicable only to bargaining: They can be utilized for prediction before we know, through a situation-specific analysis, what situation-specific behavioristic assumptions are suitable.

The use of simple assumptions about the parties' behavior seems to have the following disadvantage: The smaller the requirements made on the behavioristic assumptions, the more narrowly one must define the bargaining situation for which a solution can be determined with the aid of the behavioristic assumptions.

Thus we prefer the generality of the behavioristic assumptions to the generality of the bargaining situation. Our aim is not a general bargaining theory, applicable to all institutional assumptions, but rather a theory that would constitute a special case of such a theory if it would ever be constructed (cf. p. 23). Another consideration in favor of simple behavioristic assumptions is the desirability of bringing the general methodology of the model to the "acid test" as soon as possible, e.g. in the form of laboratory experiments. The institutional factors can be almost completely controlled, while no such control is possible with regard to the parties' behavior (cf. p. 22). Hence the behavioristic assumptions are critical for the success of the experiments. It therefore seems suitable to limit their complexity.

[21] Whether it is sufficient for purposes of pure prediction to verify the model only with regard to the outcome is a highly controversial topic (see e.g. Friedman (1953), Machlup (1955) and Melitz (1965)). It appears, however, that more tests are then required to make it probable that the model will lead to correct predictions in situations different from those tested in the experiment. Furthermore, it is sometimes difficult to distinguish clearly between the verification of the assumptions as such and the test of the predictive ability of a model based on the set of assumptions. In many cases the only feasible way of testing the assumption of a model is to test the predictive ability of models based on a proper subset of the model's set of assumptions.

[22] If A e.g. is the set of assumptions necessary for establishing an ordinal utility function and B the set of assumptions necessary for establishing a cardinal utility function (in von Neumann-Morgenstern's sense) then A is a proper subset of B. (See e.g. von Neumann & Morgenstern (1947, p. 26)and Luce & Raiffa, (1957, p. 15)). Then if a party is proved unable to establish an ordinal utility function describing his preferences, he cannot establish a cardinal utility function either.

The question remains, however, as to how many assumptions of rationality have to be included in order to find a solution for some set of interesting institutional assumptions. The only way this can be done seems to involve a *trial-and-error* approach. We start with the simplest and most general behavioristic assumptions to see whether they can be used to find a solution for any kind of interesting institutional assumptions. If this cannot be done, the set of behavioristic assumptions is extended somewhat, in the direction that seems most likely to lead to such a solution. We continue in this manner until a solution for some set of interesting institutional assumptions is found.

2.7.3 Selecting the Institutional Assumptions and Future Testing of the Model

The choice of the institutional assumptions is subordinated to the choice of the behavioristic assumptions. Having chosen a particular set of behavioristic assumptions we try to find some set of institutional assumptions for which the model would lead to a solution. If such a set has been found, we ask whether this set of institutional assumptions can be regarded as interesting in the sense discussed above (p. 23). If not, we continue our search among the possible institutional assumptions. The set of behavioristic assumptions is expanded when we can *not* find any set of interesting institutional assumptions which, for the given set of behavioristic assumptions, leads to a solution.

When choosing between different institutional assumptions that appear equally interesting we are guided by a desire to find a solution without too much computational work. Furthermore it is desirable that the institutional assumptions are such that future testing of the model will not be unduly difficult.

This testing would consist mainly of extensive, rigorous experiments. "Extensive" implies that the model should be tested for several of the different purposes discussed above such as whether parties will act in the manner prescribed by the model, whether they will accept mediation according to this model and whether some party will want to use the model as a normative tool (cf. p. 17). The model should also be tested for different versions of those factors not specifically covered by particular institutional assumptions.[23] The testing should also include different types of persons, i.e. not only students.

It should be stressed that the possibility of parallell "face validity" testing of the model should not be ruled out. This refers to the construction of numerical examples leading to a unique solution, which to some extent would resemble actual bargaining situations. By letting persons familiar with similar real bargaining

[23] E.g. means of communications, different sizes of prizes, etc.

situations discuss these examples, a kind of check is obtained as to whether or not the model is completely inconsistent with reality.

According to this line of reasoning a *basic* bargaining model involving a small set of specific institutional assumptions will be developed in Chapters 3–6. In Chapter 8 we investigate the extent to which these institutional assumptions can be substituted by others, thereby extending the area of application of the model. Some concrete examples of applications of the basic model and its extensions will be discussed in Chapter 9. This will provide a preliminary basis for a discussion of the "face validity" of the model in at least some areas of application.

2.7.4 Reasons for Limiting the Study to the Theoretical Analysis

This study does not contain any attempts to test the model. This restraint is based on the following considerations:

1. A model should not be tested until it has been worked out in reasonable detail and clearly disclosed to the scientific community. This is the only way the development of a theory and attempts to verify it can be separated. Many scientific results evade a proper evaluation, since it is often difficult or impossible to know whether the theory appears as verified, in spite of the fact that it has been modified on the basis of the most recent attempts at verification.

2. In attempts to verify a model in laboratory experiments involving human subjects there is almost always a risk that the experimenter, more or less unconsciously, will influence the outcome of the experiment. The person who has developed the model probably has a particular desire to see the model verified. Thus his influence on the experiment might be more biased as compared to another experimenter. Hence the "model constructor" is probably *not* the most suitable experimenter from the point of view of reliability. It seems unwise for the model constructor alone to be responsible for the attempts at verification.

3. Finally, a comprehensive attempt at empirical verification of the type described above will be so time-consuming and expensive that it lies outside the scope of this research project.

2.8 Notation Principles

Before proceeding to the construction of our models in the next chapter, we present a brief outline of the system of notations used for assumptions, theorems and variables.

2.8.1 Assumptions

There are three major types of assumptions:

1) behavioristic assumptions

2) general institutional assumptions, i.e. all institutional assumptions except the pay-off assumptions

3) pay-off assumptions

The behavioristic assumptions will be denoted by a B and the general institutional assumptions – henceforth called institutional assumptions – by an I, with the specific number of the assumptions as an index. The behavioristic assumptions will be grouped in sets, denoted by $\boldsymbol{B}$, with an index denoting the number of the set.

As regards the pay-off assumptions we first distinguish between two groups of assumptions – those that are specific to a particular theorem and those that are common to several theorems. The assumptions of the first group do not have any particular notation. As regards the second group of pay-off assumptions, i.e. those assumptions that are common to several theorems, we distinguish between three kinds of assumptions:

General pay-off assumptions that are used for the general model presented in Chapter 4. These assumptions will be denoted by a G.

Special pay-off assumptions. These are assumptions used in the special model presented in Chapters 5 and 6 with the property that – for each party – they concern the pay-offs of *more* than two agreements. An S will be used to denote these pay-off assumptions.

Particular pay-off assumptions. These are assumptions, also used in the special model, but with the properties that they – for each party – refer only to the pay-offs of *two* agreements. These pay-off assumptions are denoted by a P.

The various behavioristic, institutional and pay-off assumptions are introduced as we need them. For easy reference and in order to provide an overview of all the assumptions used in our models, they are all listed again on pp. 289 ff.

2.8.2 Theorems

A number of different theorems are introduced in this book, first at their proper place in the text and second, in a list on pp. 293 ff.[24] Each theorem is denoted by a T followed by a number.

2.8.3 Variables

The various notations for the variables are also introduced for the first time when needed in the text and then for easy reference in a list of symbols on pp. 294 ff.

Some of the main principles for our notations for variables are as follows:

1. The choice of notations is to a large extent guided by a desire for simplicity. Since our model construction is not based on other models, there is less need to adhere to some particular conventions.[25]

2. Our second principle of notations involves designating the parties H(igh) and L(ow) and using different mnemo-techniques for assigning different variables to either one of these parties. In general we apply the principle of using capital letters (i.e. upper-case letters or "*h*igh" letters) for H(igh)'s variables, and small letters (i.e. *l*ower-case letters) for denoting L(ow)'s variables. Hence H's pay-off will generally be denoted V and L's v. As regards the pay-offs of a specific agreement outcome or a specific situation, the principle of setting a "highly" placed "bar" above H(igh)'s alternative and a "lowly" placed one below L(ow)'s alternative is used.

3. In the text the indici will be used mainly to denote the number of a period.

[24] Conclusions that are either of independent importance or referred to repeatedly in various sections of the study will be denoted as theorems.

[25] We use $\bar{y}_j$ to denote the pay-off of one party – party H – from an agreement on alternative y in period j. A more conventional notation for this would have been the far more complicated $U_1(y_j)$.

Chapter 3

Two-alternative Bargaining Games

3.1 Introduction

This chapter deals with the very simplest case of bargaining games, namely those in which each party has only *two* alternatives:

1. *insist* on the terms favorable to himself or
2. *accept* the terms favorable to the other party.

Since the parties have completely opposing interests, we are dealing only with a distribution problem and not an efficiency problem.

The research methodology outlined in Section 2.7 will be used in this chapter. We begin by looking for the simplest possible set of behavioristic assumptions which could contribute to finding a unique solution for some game (i.e. bargaining game).[1] If such a game is found, we ask whether this appears reasonably interesting from a practical point of view. If this requirement is not fulfilled, we proceed by extending the set of behavioristic assumptions.

Following this method we proceed from a very small set of behavioristic assumptions to more complex sets of assumptions. A number of very simple examples of bargaining games will be dealt with in this chapter in order to explain this process. We start with games that are as simple as possible and then develop them into somewhat more complicated ones. Some of the fundamental ideas underlying the bargaining model in this study are presented in conjunction with these examples.

When looking for the simplest and most fundamental behavioristic assumptions, we turn first in Section 3.2 to those dealing exclusively with the isolated behavior of each party. It appears suitable here to adopt the assumptions that are generally used in microeconomic theory. We denote these assumptions $B_1 - B_7$ and assign them to assumption set $\boldsymbol{B}_1$.

[1] From now on, for the sake of simplicity, the term game generally implies a bargaining game.

Next some simple institutional assumptions are introduced. It is e.g. assumed that the parties take turns bidding. A *partial* solution for what appears to be a fairly large class of two-alternative games can be deduced on the basis of $\boldsymbol{B}_1$ and these institutional assumptions. This implies that we can say how some party will behave in at least *some* period, provided we know who starts bidding.

In order to determine behavior in more periods, additional behavioristic assumptions are required that deal with the question of what one party *expects* the other party to do. First we introduce two assumptions, B_8 and B_9, which are later shown to be required in order to solve the efficiency problem. Combined with set $\boldsymbol{B}_1$, these two assumptions constitute assumption set $\boldsymbol{B}_2$ (see Figure 3 below).

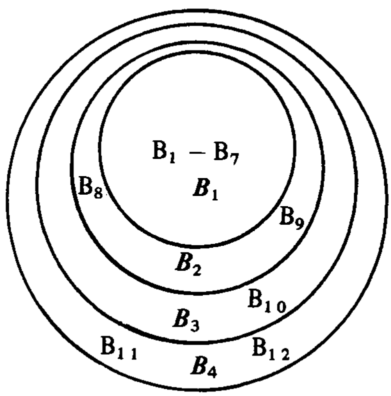

Figure 3 Relation between different behavioristic assumptions and sets of behavioristic assumptions

By also including an institutional assumption that the parties have access to information about each other's pay-offs, the number of periods in which we can determine how the party bidding in this period behaves can now be increased. Assumption B_{10}, which is common to all conflict theory, is also added. As shown in Figure 3, this combined with $\boldsymbol{B}_2$ forms set $\boldsymbol{B}_3$. We can now (in Section 3.3) determine a solution for many games by establishing a choice in *every* period up to the agreement, provided we know who starts bidding.

Next we introduce a *special* type of bargaining games, S-games. These appear to have fairly large practical significance. On the basis of $\boldsymbol{B}_3$ a *unique* solution can be established for most of these games. This is a solution which does *not* rely on the assumption that the party who starts bidding is given (3.4).

Some simple, yet interesting, S-games can still *not* be solved. In order to solve these we add assumptions B_{11} and B_{12} (combined with $\boldsymbol{B}_3$ to form the assumption set

B_4). After this only a few apparently uninteresting two-alternative S-games remain unsolved (3.5).

3.2 Assumption Set B_1 and the Establishment of a Partial Solution

3.2.1 Definition of Assumption Set B_1

As noted above we start with behavioristic assumptions that are as simple as possible when constructing our model. For guidance as to the amount of rational assumptions to include in our first step we turn to traditional microeconomic theory. Almost all rational microeconomic theory relies on the following set of behavioristic assumptions B_1.[2]

B_1 *Preference relations*: On the basis of various factors affecting each outcome[3] a party can define preference relations for pairs of outcomes (a, b). The party can say whether he prefers a to b, or b to a or whether he is indifferent between the two.[4]

If a party's own profit is the only factor taken into account, this assumption implies that out of two outcomes a party will prefer the one with the larger profit. In order to simplify the analysis we assume in the following chapters that a *pay-off* index can be defined. This is an *index* for the combined effect of the factors affecting the outcome such that each party will always prefer an alternative with a higher pay-off to one with a lower pay-off. For the case where the profit is the only factor influencing the preference relations, the pay-off is equivalent to this profit.[5] Our examples below are based on this assumption.

[2] See e.g. Quirk & Saposnick (1968), who however do not list B_6 and B_7 explicitly.

[3] An outcome can be regarded as an ordered set of values of those "objectively measurable" factors which will affect a party's preference for a certain event, e.g. an agreement. ("Objectively measurable" implies that different persons obtain roughly the same result when measuring.) If a party bases his preference on his profit, market share and liquidity, an outcome is a vector with one value of the profit, one value of the market share and one of the liquidity ratio. An outcome is hence the counterpart of the "state of the economy" in traditional economic theory while the "set of outcome factors" corresponds to the "set of commodities". (See e.g. Quirk & Saposnick 1968, p. 8.) As the values of these measurable factors affecting the outcome are determined by institutional assumptions, the actual directions of the preference relations are also determined by these institutional assumptions.

[4] More generally the preference relations could be defined as follows: The party can say whether he does *not* like a less than b and/or if he does not like b less than a. If he does not like a less than b and b not less than a, he is indifferent between them.

[5] In other cases the preference relations might be affected by several factors, such as profit, sales and liquidity. On the basis of each party's preference relations for various possible combinations of these factors, this kind of pay-off index for each outcome can be established. In such a case the assumptions concerning the form of the pay-off function are assumptions of a mixed behavioristic and institutional character; they can in turn be deduced from other behavioristic and institutional assumptions.

B_2 *Completeness:*[6] When comparing any two outcomes a party can *always* say whether he prefers one to the other and if so, which one he prefers, or whether he is indifferent between them.

B_3 *Continuity* in preference relations: If outcome a is *almost* identical to outcome b and a party prefers outcome a to some outcome c, clearly different from a, then the party will not like b less than c.[7]

B_4 *Transitivity*: If a party prefers a to b and b to c he will also prefer a to c. If he is indifferent between a and b and indifferent between b and c, he is also indifferent between a and c.

B_5 *Optimization*: A party will not select an alternative to which an outcome b can be assigned with certainty, if there exists a choice that with certainty leads to an outcome a and he prefers a to b.

B_6 *Information utilization*: Both players utilize all relevant information available to them.

B_7 *Deductive capacity*: Each party is able to carry out complicated logical deductions[8] and use any available computational aid.

Assumptions B_5 – B_7 are often only assumed implicitly, but since they are of fundamental importance they are presented explicitly in this study. It should be noted that as regards the assumption of deductive capacity, B_7, our assumption is much weaker than the assumption usually made – at least implicitly – in economic theory, i.e. an assumption of infinite deductive and computational capacity. Along with the other assumptions this would form what can be called unbounded rationality. Here we refrain from such an extreme assumption.[9]

3.2.2 Example 1

We begin by asking whether there are any bargaining situations that can be given a solution with the aid of only the simple assumptions in set $\boldsymbol{B}_1$.

[6] Assumptions B_1 and B_2 are often combined into a single assumption. For pedagogical reasons we present them as two separate assumptions.

[7] This assumption ensures that an ordinal utility index can be assigned to each outcome. See further Quirk & Saposnick (1968, p. 18). An assumption of continuity over time may also be included in this assumption, implying that the preference relations of the parties do not vary greatly over a very short interval of time.

[8] This assumption implies e.g. that the party is able to carry out the deductions presented in this study.

[9] See furthermore p. 65.

The following case is studied first: Two parties, called L(ow) and H(igh), want to reach an agreement in a bargaining game characterized by the following general institutional assumptions, common to all our bargaining examples:

I_1 The potential bargaining process takes time, i.e. it is distributed over a number of periods. In each period *one* – and only *one* – party delivers a *bid*. This is a written or oral proposal for an agreement on a certain set of terms called an *alternative*.[10]

I_2 An agreement on a certain alternative is reached in any one of the periods, if the party bidding in this period bids the same alternative as the other party has bid in some preceding period. As soon as an agreement has been reached, the game is over. If an agreement is not reached in a period which is not the last one, the game continues into the next period. If it is the last period, the game is discontinued.

I_3 In a certain period each party has information about
(a) the value of the factors (e.g. profits) that can possibly affect his preferences with regard to each future agreement
(b) what he himself has bid in all of his preceding periods[11] and
(c) what the other party has bid in all of his preceding periods.

I_4 The parties alternate bidding.[12] This implies that if H bids in period 1, L bids in period 2 and H again in period 3.

Specifically, it is assumed in example 1 that H and L want to divide \$ 10 between them. They have to divide this sum so that each party gets a number of whole dollars, i.e. no division involving cents is possible. For some reasons, at present not further specified, their demands are only *one* dollar apart when we start to study them. Prior to period 1 L has suggested dividing the \$ 10 so that H gets \$ 6 and L gets \$ 4, while H has proposed a distribution \$ 7 for H and \$ 3 for L.[13]

We turn to the simplest possible version of an example that can be described by the institutional assumptions above, i.e. a bargaining game with only two periods where

[10] For a more precise definition of the alternative concept, see p. 59.
[11] This is called *perfect recall* in game theory.
[12] Combined with part c of I_3, I_4 implies what is called *perfect information* in game theory. The assumption of alternating bidding seems more suitable than the assumption that the parties deliver their bids simultaneously, leading to imperfect information. This choice of assumptions is discussed in greater detail in Chapter 8.
[13] The assumption that L wants a 6,4 and H a 7,3 division is made for the sole purpose of obtaining a very simple example to illustrate the simplest kind of two-alternative games. Later on in this chapter, we investigate examples where the two alternatives involve other divisions of the \$ 10 and in later chapters a great many alternatives are allowed for.

each party bids in one of these periods. Let us also temporarily assume that the order of bidding is *given* and that party H starts bidding. This bargaining situation can be presented in the form of a game tree. It should be emphasized that in our analysis of bargaining situations we work exclusively with games in this form.[14] Thus the bargaining game described by Figure 4 is obtained.

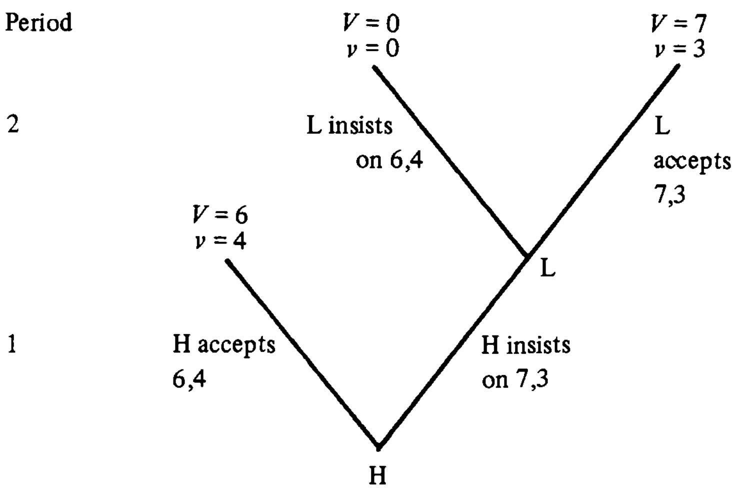

Figure 4 Example 1

V and v at the end points denote H's and L's pay-offs, respectively. It should be stressed that the first period is at the bottom of the figure. We can imagine a vertical time-axis, with the earliest times closest to origo at the bottom of the diagram.

Looking at this figure the following can be determined by the assumptions of B_1: Since L prefers more money to less (cf. p. 31), L will accept the distribution 7,3 in period 2 and obtain $ 3, rather than insist on 6,4 and get nothing. But we can *not* determine how H will bid in period 1, since this would require us to assume that H realizes that L will accept the distribution 7,3 in period 2. This, however, is outside

[14] From now on we work with what in game theory is called the *extensive* form. The alternative is to present the games in the *normal* form, i.e. with game matrices. The result would be the same, if the same behavioristic assumptions hold (see von Neumann & Morgenstern, 1947, p. 82). In the appendix (on p. 253) we also show how a two-alternative game (example 4 below) presented in the extensive form in the text, can be given the same solution using the normal form. The extensive form, however, appears to be much more suitable for our purposes. The analysis in the normal form becomes prohibitively complicated for bargaining games with many alternatives. Furthermore, several important points such as the connections with the behavioristic assumptions seem to become less intuitively clear in the normal form. Finally, as shown on p. 257 in the mathematical appendix, we find no reason to use the concept of equilibrium pairs of strategies. Hence the advantage of the normal form as regards the search for such pairs is without relevance for our analysis.

the scope of assumption set B_1. Hence this simple example cannot be solved by using B_1.

Changing the sum to be divided and the proportions into which it can be divided we realize that the inability of B_1 to determine a solution is *not* affected by a change in the sum that is divided, nor by the proportions into which it is divided.[15] This same indeterminacy prevails as long as the parties are assumed to divide a sum of a constant size.

3.2.3 Example 2

In accordance with the research strategy outlined on p. 25 we proceed to study a case in which the sum to be divided changes over time. Instead of assuming that the parties obtain \$ 10 to divide once and for all, we assume that the parties will receive \$ 10 during *each* of the two periods. The parties are to agree on a division of these \$ 10 that will apply in all remaining periods. In case an agreement has not been reached during a certain period, the \$ 10 of *this* period are forfeited. In case an agreement is reached during the first period, the parties have \$ 20 to divide, while they only have \$ 10 to divide if agreement is delayed until the second period. Retaining all other assumptions from example 1 the following game tree (Figure 5) is now obtained:

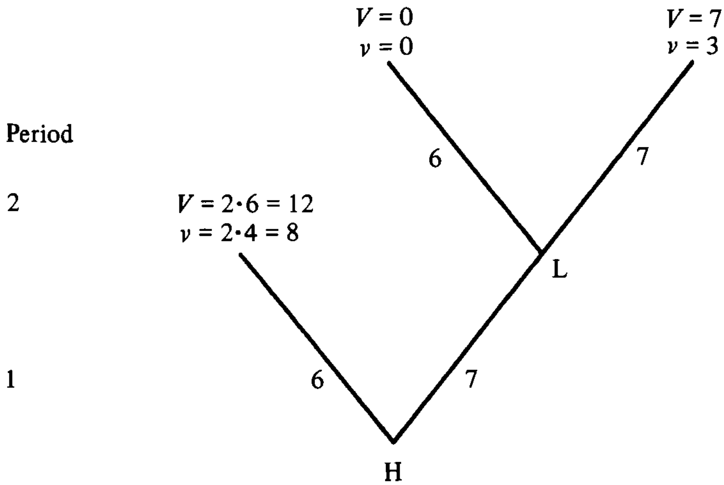

Figure 5 Example 2

Comparing Figure 5 to Figure 4 we see first of all that two notational simplifications have been introduced in Figure 5. By now it should be clear that the

[15] Indeterminacy will prevail as long as H insists on 7,3 in period 1 if L would accept 7,3 in period 2, and H accepts 6,4 in period 1 if L would *not* accept 7,3 in period 2.

letter at the decision nodes denotes which party makes a certain bid and we have only indicated what division is suggested at the branches. Furthermore each proposed distribution alternative is denoted by only *one* figure, i.e. the amount that H gets. This is a sufficient description, since L's pay-off is equal to \$ 10 *minus* H's pay-off. Obviously we could have used L's pay-off instead, but the use of H's pay-off instead of L's has a mnemotechnical advantage. *H* wants the distribution alternative to which we have assigned the *h*igher number while *L* wants the alternative with the *l*ower number.

A second difference between Figure 5 and Figure 4 concerns the pay-off pair resulting from H's acceptance of 6 in period 1. In Figure 5, the pay-off is doubled, since an agreement in period 1 implies that the \$ 10 are obtained twice.

Studying Figure 5, we see that if H insists on alternative 7, he can get either \$ 7 or nothing, while by accepting alternative 6 he will get \$ 12. Hence in period 1 he will accept distribution alternative 6 and an agreement is reached.

Thus in this example with a given order of bidding, we have been able to determine a solution by only looking at H's preferences, i.e. solely by the use of assumption set B_1.

3.2.4 The Concept of a Critical Period

We found an example above where a solution could be determined with regard to a specific order of bidding. The question then arises as to what constitutes the main difference between this example – example 2 – and example 1 which had no solution. In example 1 H would accept 6 in period 1 only if L will *not* accept 7 in period 2. In example 2, on the other hand, H will accept 6 in period 1, regardless of what L does in period 2.

This difference is due to the fact that by accepting L's proposal in period 1, H will get a share of \$ 10 once in example 1, while in example 2 he will get this twice. H accepts L's terms – i.e. alternative 6 – in period 1 in example 2, since it is more favorable for him to get a smaller percentage (60 %) of a larger amount (\$ 20) than a higher percentage (70 %) of a smaller amount (\$ 10). We note that H accepts 6 in period 1 in spite of the conclusion that L would otherwise accept 7 in period 2. Hence period 1 is such that it is better for the party bidding in this period to accept the other party's terms than to wait for the other party to accept his terms in the following period. Any period with this characteristic will be called *critical*.[16] A more precise definition is as follows:

[16] As proved on p. 77, the existence of a critical period is a *necessary* requirement for the existence of a unique solution on the basis of B_4 or any subset of B_4.

In a two-alternative bargaining game characterized by I_1–I_4, a period is *critical* for a party if his pay-off from accepting the other party's terms in this period is *larger* than his pay-off from having the other party accept his terms in the *next* period.

Hence period 1 is critical for H since H's pay-off from an agreement in period 1 on alternative 6 ($ 12) is larger than H's pay-off from an agreement in period 2 on alternative 7 ($ 7).

3.2.5 Example 3

We are obviously not content with studying bargaining games having only two periods. Therefore, we extend example 2 by allowing for more periods – e.g. six periods – in order to see if a solution can still be arrived at using only B_1. With H starting to bid, the game tree in Figure 6 is now obtained.

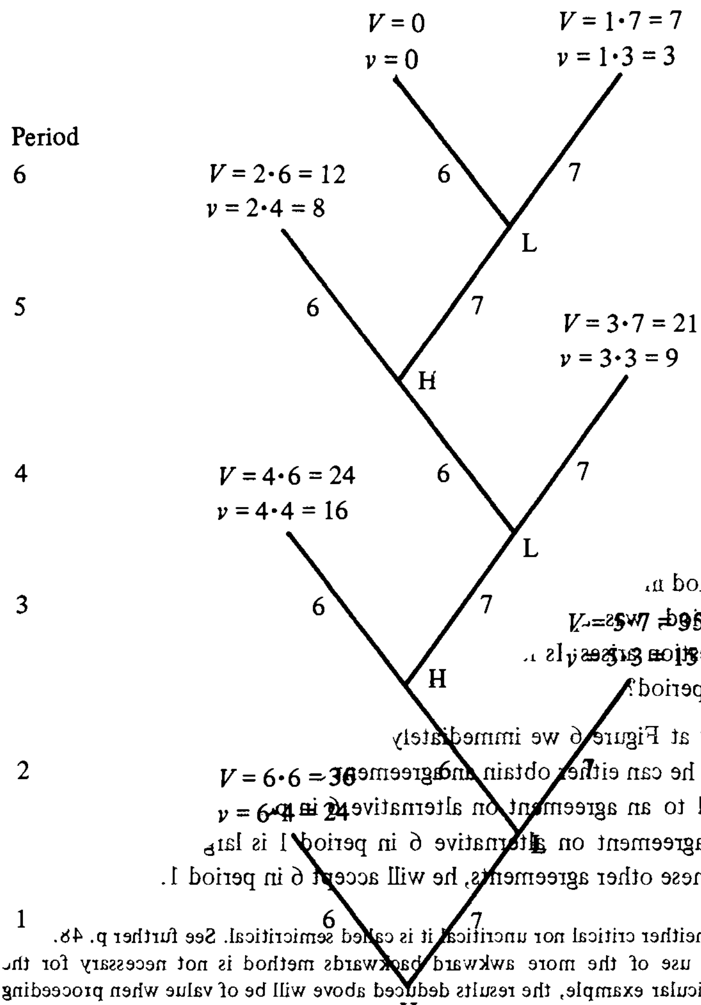

Figure 6 Example 3

To solve this bargaining game we again start with the very last period – in this case period 6. We conclude that L will accept 7 in this period. Next we conclude that H will accept 6 in period 5, since this period is *critical* for H (\$ 12 > \$ 7). Continuing backwards in this manner, we note that period 4 is critical for L (\$ 9>\$ 8) and that L hence accepts 7, while period 3 is critical for H (\$ 24>\$ 21), making H accept 6 in this period.

We can *not* determine L's choice in period 2 since this period is *not* critical for L. In fact it is the very opposite of critical. L's pay-off from accepting 7 in period 2 is *smaller* than the pay-off L gets, if H accepts alternative 6 in period 3 (\$ 15<\$ 16). This kind of period will be called *uncritical* with the following definition:[17]

A period is *uncritical* for a party in a two-alternative game, characterized by $I_1 - I_4$, if his pay-off from accepting the other party's terms in this period is *smaller* than his pay-off from having the other party accept his terms in the *next* period.

Since period 2 is *un*critical, L would insist on 6, *provided* he knew H would accept 6 in period 3. But since $\boldsymbol{B}_1$ does *not* allow us to draw any inferences about what one party expects the other party to do, we must – temporarily – regard L's choice in period 2 as *indeterminate*. This does *not* prevent us from concluding that example 3 does have a solution. Since period 1 is critical for H, H will accept 6 in period 1, regardless of what L does in period 2. Hence an agreement is already reached in period 1. Since the bargaining does *not* continue into period 2, we do not have to bother about the fact that the choice in period 2 is indeterminate.

Thus we have also been able to solve this bargaining game with six periods.

3.2.6 A Simplified Analysis of Example 3

A backwards method involving deduction of the choice in each period, starting with the very last period, was applied in the preceding section. Reflecting on this procedure the question arises: Is it really necessary in this case to start the analysis with the very last period?

If we look closely at Figure 6 we immediately see that H will accept 6 in period 1. If he insists on 7, he can either obtain an agreement on alternative 7 in periods 2, 4 or 6 or be forced to an agreement on alternative 6 in periods 3 and 5. Since H's pay-off from an agreement on alternative 6 in period 1 is larger than his pay-off from any one of these other agreements, he will accept 6 in period 1.[18]

[17] If a period is neither critical nor uncritical it is called semicritical. See further p. 48.

[18] Although the use of the more awkward backwards method is not necessary for the solution of this particular example, the results deduced above will be of value when proceeding to the three-alternative case.

The following question then naturally arises: Is it sufficient to note that period 1 is critical for H in example 3 in order to determine his acceptance of 6 or are further assumptions necessary?

Some new *notations* have to be introduced before this question can be answered.

3.2.7 Notations for Pay-offs

An agreement on alternative y in period j will be denoted y_j. As mentioned earlier we use the general principle of letting the indici of variables refer to the period. Party H's pay-off from agreement y_j is written as $\overline{y}_j$ and L's pay-off from y_j as $\underline{y}_j$. The mnemotechnics are, as before, that the bar is *high* up for H(igh) and *low* down for L(ow).

The assumption that period 1 is critical for H in a game (6,7) is hence equivalent to assuming that $\overline{6}_1 > \overline{7}_2$. But in order to show that H will accept 6 in period 1, we also have to assume that $\overline{6}_1$ is larger than any other pay-off H can possibly get from insisting on 7, i.e. $\overline{6}_3$, $\overline{7}_4$, $\overline{6}_5$ and $\overline{7}_6$. We easily see that it suffices to add the assumption that H has a *decreasing* pay-off over time implying that $\overline{6}_1 > \overline{6}_3 > \overline{6}_5$ and that $\overline{7}_2 > \overline{7}_4 > \overline{7}_6$. In other words, it is sufficient to assume that period 1 is critical for H *and* that H has a decreasing pay-off over time.

3.2.8 Further Generalizations

The next question then is whether this conclusion can be generalized to bargaining games other than example 3. First we ask if example 3 can be changed in some way and still retain the conclusion that H accepts 6 in period 1. It was noted above that the parties' pay-offs decrease over time in example 3. Furthermore, in order to establish that period 1 is critical, we only have to study the pay-offs of periods 1 and 2. This means that the pay-offs in periods 3–6 can be disregarded. The outcome would not be affected, for example, if period 6 were eliminated and period 5 made the last period.

Conclusion: H's acceptance of 6 in period 1 is unaffected by a particular stop rule, terminating the game after a known number of periods.

Any number of periods after period 6 can also be *added* without affecting the solution. Hence, cases where the game in principle continues *indefinitely* can also be investigated. Let us assume that \$ 10 can still be obtained if an agreement has not been reached in the sixth period, but is reached in some later period. If an agreement is reached in period 1 the parties will divide \$ 60; in period 2 \$ 50 and in periods 6, 7, 8, etc. \$ 10. This assumption implies that $\overline{7}_j = 7$ and $\overline{6}_{j+1} = 6$ for $j = 6$, 8, 10, etc. Since $\overline{6}_1 = 36$ will be larger than any of these pay-offs, the conclusion still holds that H accepts 6 in period 1, if he starts bidding.

Conclusion: The total lack of a stop rule does not affect H's acceptance of 6 in period 1.

3.2.9 Theorem T_1

It is obvious from the discussion above that the following holds for any bargaining game, as regards the party bidding in the first period: If the first period is critical for him and he has a decreasing pay-off over time after period 1,[19] he will accept the other party's terms in this period. However, this conclusion is of limited interest. It is not reasonable to assume that the *first* period is critical in most real bargaining games. If we assume e.g. that period 2 is the first period of the game in example 3, the first period would not have been critical.[20] But we can still conclude that a *partial* solution has been obtained.[21] A partial solution implies that – for at least *one* period (e.g. period 3) and for a *particular* bidding order – we have been able to determine what the party bidding in this period would do if the game continued this far. We can now formulate our first general theorem on the basis of this conclusion:

Theorem T_1: On the basis of $\boldsymbol{B}_1$ and I_1–I_4, it can be deduced that *sufficient* conditions for the existence of a *partial solution* in a two-alternative game with a *given* order of bidding are that at least one party

bids in a period that is critical for him and
has a decreasing pay-off function over time after this period.

The partial solution is that this party will accept the other party's terms in the critical period.

It should be noted that T_1 relies only on $\boldsymbol{B}_1$, i.e. the smallest set of behavioristic assumptions involving *no* assumptions of expectations concerning the other party. This follows from the conclusion that the party makes his decision in his critical period solely by looking at his own pay-offs.[22]

[19] The discussion in Section 3.2.8 indicated that we can also allow for the case where the party has a *non-increasing* pay-off after a certain period.

[20] See p. 38.

[21] A *solution* implies as noted that the choice can be determined for *every* period up to the agreement. In the partial solution this choice is *not* determined for *every* period, but at least one.

[22] It should be kept in mind that each party was assumed to prefer an outcome with a high pay-off for himself to one with a lower pay-off, regardless of the other party's pay-off (see p. 31).

3.3 Assumption Sets B_2 and B_3 and the Establishment of a Solution

3.3.1 Example 4

After these generalizations about the existence of a *partial* solution, what further assumptions are needed to establish a solution if the order of bidding is given?

In order to answer this question we extend example 3 by assuming that the \$ 10 are obtained for *eight* periods instead of six. This means that by accepting 6 in period 3, H can obtain \$ 6 during 6 periods, i.e. a total of \$ 36, while if L accepts 7 in period 4, H will get \$ 7 during 5 periods, i.e. a total of \$ 35. Hence period 3 is a critical period for H. Therefore, due to T_1, we can deduce that if L insists on 6 in period 2, this will lead to H accepting 6 in period 3 with an agreement worth \$ 36 to H and \$ 24 to L. The following simple tree is then obtained (Figure 7).

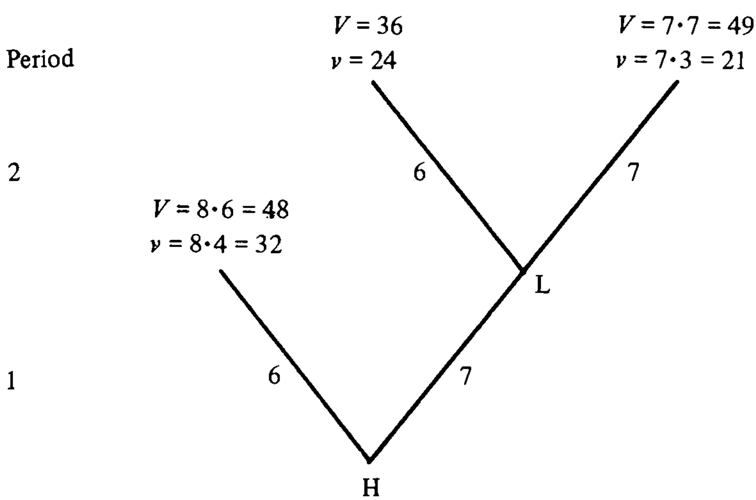

Figure 7 Example 4 (reduced)

3.3.2 Assumptions B_8 and B_9

It should be emphasized, however, that in order for L to assign $V = 36$, $v = 24$ to his bid of insisting on alternative 6, he must *realize* that H will accept 6 in period 3. Among other things this requires L to realize that H acts according to assumption set $\boldsymbol{B}_1$. But this "insight" is *not* included among the assumptions of set $\boldsymbol{B}_1$. Noting that period 2 is uncritical for L and that period 1 is uncritical for H, this simple game can *not* be solved solely on the basis of $\boldsymbol{B}_1$. For obvious reasons games such as examples 1–3 can *not* be regarded as interesting from a practical point of view. In line with our research program outlined on p. 25, we extend our set of behavioristic assumptions.

First some assumptions should be made about what each party believes about the other party. In other words, we require assumptions of rational *expectations*. We introduce the following two assumptions, which combined with set B_1 constitute set B_2:

B_8 *Mutual knowledge of the other party's rationality*: H knows[23] that L is rational (i.e. behaves according to assumption set B_1) and L knows that H is rational.

B_9 *Information about preference relations*: Full information about the other party's preference relations, established according to assumption B_1, is available to each party.[24]

As regards example 4 these two assumptions imply that L realizes that H is rational and that H prefers more money to less. These assumption are of fundamental importance in bargaining theory. Later on (p. 173) they will be shown to be – at least implicitly – assumed by most authors who regard the efficiency problem as solved.

3.3.3 The Assumption of Complete Information

In order for L to understand that H accepts 6 in period 3, an institutional assumption must also be added implying that L knows H's pay-offs from the various outcomes. More generally, we add the following institutional assumption:

I_5 Each party has complete information about the other party's pay-offs, implying that he can assign – to every possible agreement – a correct value of the different outcome factors relevant for the other party's preferences.[25]

This assumption, combined with I_3 (a) implies that both parties have *complete information*. This definition varies in two respects from the more conventional definition of complete information used in game theory, i.e. that both parties know

[23] "Correctly believes" can alternatively be substituted for "knows".

[24] It might be suggested that B_9 is *not* a behavioristic assumption, but rather an institutional one. However, according to the definition of institutional assumptions on p. 22, their fulfillment can be ensured by manipulating the experimental setup. The experiment-leader can very well distribute information to each party saying that the other party's preferences are such and such. The critical question, however, is whether the leader knows the preference relations of the parties and whether the parties believe that he knows them. This cannot generally be assumed to be the case. However, the better the leader can ascertain the true preference relations of the parties, the closer assumption B_9 is to an institutional assumption.

[25] E.g. the size of profits, the size of sales, etc.

the cardinal utility of each outcome for each party.[26] First, as noted, we do *not* assume a cardinal utility function. Secondly, our definition of complete information is an *institutional* one. In other words, this definition does *not* mean that one party knows the other party's preferences for different outcomes. The assumption only implies that he knows the value of all outcome factors that can possibly influence the other party's preferences. Thus, if e.g. assumption B_1 implies that each party's preference relation depends solely on the party's profit, our complete information assumption implies that each party would know the profit the other party obtains from each agreement.[27]

With assumption set B_2 and the institutional assumptions I_1–I_5 at our disposal, we return to the choice situation illustrated by Figure 7. We can now let L assign the pay-off $\nu = 24$ to his bid of insisting on 6 in period 2. This is sufficient for deducing that L will insist on 6 rather than accept 7, which will only give him $\nu = 21$.

3.3.4 Assumption B_{10}

We now turn to the question of how H bids in period 1. Since this period is *uncritical*, H has to determine how L bids in period 2 before making his own choice. To do this H has to realize not only that L is rational, but also that L realizes H is rational. Such an assumption of two steps of insight is *not*, however, included in assumption set B_2. Our set of behavioristic assumptions has to be extended again since we want to analyze games with a great many possible bids. We introduce the following behavioristic assumption, commonly used in game theory.[28]

B_{10} *Mutual knowledge about the other party's knowledge about oneself*: H knows what L knows about H, and L knows what H knows about L.

[26] According to Shubik (1959, p. 5): "Complete information . . . implies that all pay-off values are known." Luce & Raiffa (1957, p. 49) similarily assume that "Each player is fully cognizant of the game in extensive form, i.e. he is fully aware of the rules of the game and the utility function of each of the players."Von Neumann & Morgenstern (1947, p. 30), however, define complete information as: "the assumption that all subjects of the economy under consideration are completely informed about the physical characteristics of the situation in which they operate and are able to perform all statistical, mathematical, etc., operations which this knowledge makes possible". The first part of this assumption is more in line with our definition above.

[27] In an experiment, the complete information assumption would mainly imply that each party is supplied with a table listing the amount the other party obtains from each agreement. Other outcome factors such as time spent on bargaining until a specific agreement would also have to be known. Of course it might be difficult to take all the factors that could affect the preference relations into account, but for practical purposes this distinction between institutional and behavioristic assumptions appears feasible.

[28] See p. 137.

This assumption is sufficient for deducing that H realizes that L realizes that H is rational. The following conclusion can now be drawn: H realizes that if he insists on alternative 7, L will *not* accept this in period 2, but rather carry the game into period 3. Here H will have to accept 6 and obtain \$ 36. H would thus prefer to accept 6 in period 1 and obtain \$ 48.

It should be noted that H accepts 6 in spite of the fact that period 1 is uncritical for him. This implies that H would *not* accept 6 in period 1, if L would accept 7 in period 2. But the assumption that period 1 is uncritical is without significance, since we have determined that L *will* insist on 6 in period 2.

Conclusion: It is *not necessary* that a period be critical for the bidding party in order for the party to accept the other party's terms.

3.3.5 Analysis of Example 4 with Less Specific Pay-off Assumptions

Example 4 contained a game in which we could deduce a solution for a given order of bidding on the basis of a *partial* solution.

The question is now to what extent we can generalize on the basis of this example, particularly as regards the reduced tree presented in Figure 7. We note that the same solution can still be obtained if the pay-off assumptions above, having a specific *numerical* content in Figure 7, are replaced by the following somewhat less specific assumptions:

1. H accepts 6 in period 3
2. Period 2 is uncritical for L
3. $\overline{6}_1 > \overline{6}_3$.

Since H accepts 6 in period 3, assumption 2 is sufficient for deducing that L will *not* accept 7 in period 2. With L preferring more money to less, this follows from the definition of an *un*critical period on p. 38. Assumption 3 is sufficient for deducing that H will accept 6 in period 1 rather than in period 3.

3.3.6 Notations for Two-alternative Games

Before generalizing further, some new notations are introduced for distinguishing between different two-alternative games. A game in which L desires an agreement on alternative 6 and H on alternative 7 is denoted as a game (6,7). More generally we denote a two-alternative game in which L desires an agreement on alternative x and H desires an agreement on alternative y, where $x < y$, as a game (x,y). The *lower* figure to the *left* refers to the alternative desired by L(ow), who definition-wise prefers the alternative with the lower number.[29]

By adding an *index* we indicate the particular *period* of the two-alternative situation we are referring to. This period index refers to the *latest* period in which either x or y was bid. This index also indicates which party bids in the particular period. An index $\bar{j}$ is used if H bids in period j and an index $\underline{j}$, if L bids in period j. As before, we employ the principle of a "highly" placed bar for H(igh) and a "lowly" placed bar for L(ow). Consequently, a situation where H has bid y in period $j-1$ and L has bid x in period j is denoted as $(x,y)_{\underline{j}}$. Likewise $(x,y)_{\bar{j}}$ denotes a situation where L has bid x in period $j-1$ and H has bid y in period j. In a situation $(x,y)_{\bar{j}}$ it is L's turn to bid *next* i.e. in period $j+1$, while in a situation $(x,y)_{\underline{j}}$ it is H's turn to bid. Accordingly, we write $(x,y)_{\bar{0}}$ when L bids in period 1 and $(x,y)_{\underline{0}}$ when H bids in period 1, although no party really makes a bid in period 0.[30]

H's pay-off from e.g. the game $(x,y)_{\bar{j}}$ is denoted as $\overline{(x,y)}_{\bar{j}}$ and L's pay-off from the same game as $\underline{(x,y)}_{\bar{j}}$. We use the same principle as before for $\bar{x}_i$ and $\underline{x}_i$, i.e. with a "highly" placed bar for H(igh) and a "lowly" placed bar for L(ow). The conclusion that H obtains $\bar{x}_i$ in the game $(x,y)_{\bar{j}}$ can now be written as $\overline{(x,y)}_{\bar{j}}=\bar{x}_i$. Likewise, when L's pay-off from $(x,y)_{\bar{j}}$ is $\underline{x}_i$, this is written as $\underline{(x,y)}_{\bar{j}}=\underline{x}_i$. Correspondingly, when the game $(x,y)_{\bar{j}}$ leads to the outcome x we write that $(x,y)_{\bar{j}}=x_i$.[31]

3.3.7 Generalizations on the Basis of Example 4

These new notations can now be used to write the conclusions in Section 3.3.5 above. The assumption that H accepts 6 in period 3, leading to an agreement on 6_3, can be written as $(6,7)_{\underline{2}} = 6_3$. The conclusion that L insists on 6 in period 2, leading to an agreement 6_3, can be written as $(6,7)_{\bar{1}} = 6_3$. Finally, the conclusion that H accepts 6 in period 1, leading to an agreement 6_1, can be written as $(6,7)_{\bar{0}} = 6_1$.

Hence we conclude:

> If period 2 is uncritical and
> $\bar{6}_1 > \bar{6}_3$,
> then $(6,7)_{\underline{2}} = 6_3$ implies that
> $(6,7)_{\bar{1}} = 6_3$ and $(6,7)_{\underline{0}} = 6_1$.

[29] As noted on p. 36 the number of the alternative in the examples in this chapter refers to the amount that H gets. When dividing a fixed sum, L wants H's amount to be as low as possible.

[30] As regards the meaning of $(x,y)_0$ and $(x,y)_j$, with *no* bars for the periods, see pp. 95 and p. 100.

[31] $(x,y)_{\bar{j}}$ can be regarded as an ordered pair consisting of $\overline{(x,y)}_{\bar{j}}$ and $\underline{(x,y)}_{\bar{j}}$ and x_i as another pair, consisting of $\bar{x}_i$ and $\underline{x}_i$. When the elements of each party in the two pairs are equal, the two pairs are also equal. It should be stressed that our notations here differ from those conventionally used in mathematics.

This can be generalized even further: Since our conclusion above does not rely on any alternatives other than those called 6 and 7, we can let 6 represent any alternative x and 7 any alternative $x+1$.[32] Likewise, since no periods other than 0 to 3 are involved, we can let period 3 represent any period j. The conclusion above can then be rewritten in the following way:

> If period $j-1$ is uncritical for L and
> $\bar{x}_{j-2} > \bar{x}_j$
> then $(x,x+1)_{\underline{j-1}} = x_j$ implies that
> $(x,x+1)_{\overline{j-2}} = x_j$ and $(x,x+1)_{\underline{j-3}} = x_{j-2}$

3.3.8 Theorem T_2

Let us now assume that period i is critical for H and that H has a decreasing pay-off over time. Let us study a game where H bids in period i. Using theorem T_1, we can establish that H will accept x. This implies that $(x,x+1)_{\underline{i-1}} = x_i$. Let us also assume that every period prior to i is uncritical for L. The assumption that H has a decreasing pay-off over time implies that $\bar{x}_{j-2} > \bar{x}_j$ holds for every possible j. We can then apply our conclusion at the end of the preceding section. By first setting $j=i$, we deduce that $(x,x+1)_{\overline{i-2}} = x_i$ and $(x,x+1)_{\underline{i-3}} = x_{i-2}$. Next, by setting $j = i-2$, we deduce that $(x,x+1)_{\overline{i-4}} = x_{i-2}$ and $(x,x+1)_{\underline{i-5}} = x_{i-4}$. We proceed backwards in this way and deduce generally that $(x,x+1)_{\underline{j-1}} = x_j$ and $(x,x+1)_{\overline{j-2}} = x_j$ hold for every $j \leqslant i$. It should be noted that this deduction is based on assumption set $\boldsymbol{B}_3$.[33]

Summing up, we present our conclusions in the form of the following theorem:

Theorem T_2: On the basis of $\boldsymbol{B}_3$ and I_1–I_5 it can be deduced that the three assumptions

1. Period i is critical for H, who bids in period i
2. H has a decreasing pay-off over time
3. L has only uncritical periods prior to period i,

imply for every $j \leqslant i$
that $(x,x+1)_{\underline{j-1}} = x_j$ and $(x,x+1)_{\overline{j-2}} = x_j$.

In particular this implies that H will accept x in his very first bid.

[32] From now on we employ the principle of using x to denote an alternative on which an agreement is reached in the different theorems and y in a more general manner to denote any alternative. See also p. 59.

[33] For determining the solution in period 1 we require $i-1$ steps backwards. But since every additional step backwards is covered by assumption B_{10}, no additional behavioristic assumption is required. For a further discussion see p. 137.

It should be emphasized that the real bargaining process in this case is limited to H telling L that he is willing to accept L's terms. Here we encounter a characteristic trait of our model, i.e. that the parties reach an agreement after a very short bargaining process. Our model has this trait in common with other bargaining models based on assumptions of rationality and complete information. The result is a natural consequence of the "insight" assumption B_{10} and the assumption of a decreasing pay-off over time.

3.3.9 Other Bargaining Games with a Solution

Finally, it should be stressed in this context that there are several two-alternative games *not* covered by T_2, for which a solution can be determined, at least for a given order of bidding. A solution can be determined more generally by the following procedure: The pay-offs for *each* period when the party bidding *accepts* the other party's terms can be determined immediately. In order to determine the pay-offs for a period when the bidding party *insists* on his own terms, we have to proceed as follows: Let i be the first period, counted from the start, for which a *partial* solution can be determined. The pay-offs from the outcome in this period, i, can be assigned to the alternative implying that the party in the preceding period, $i-1$, insists on his terms. We can then determine the choice in period $i-1$, provided that the party bidding in period $i-1$ obtains *different* pay-offs from his two alternatives. If so, a pay-off pair can be assigned to the alternative, indicating "insistance", in period $i-2$. In this way, going backwards one period at a time, a solution can be determined, *provided* the party bidding in each period obtains a different pay-off from his two alternatives.

Many games not covered by T_2 will fulfill this requirement.[34] Some of these games seem fairly odd. Furthermore they often appear difficult to solve. However, solutions can be found using the *general* model presented in the next chapter. It seems fairly uninteresting from a practical point of view to pursue a further search for sufficient conditions for the existence of a solution.

[34] As noted on p. 40 we can allow for a constant pay-off over time after period i. Furthermore, every game in which the pay-off of every agreement prior to $i+1$ is different for each party will have a solution, if i has a partial one. Then there is no risk of a party being indifferent between two alternatives. Finally, there are games not covered either by T_2 or by the cases above in this footnote for which a solution for a given bidding order can still be obtained, e.g. example 4 on p. 41 if $\bar{6}_1 = 35$ instead of 48 and $\underline{6}_1 = 21$ instead of 32. A solution is reached in the form of an agreement on 6_3.

3.4 The Establishment of a Unique Solution

3.4.1 The Critical Characteristics of a Period

We shall concentrate on the case given by theorem T_2, in which a solution for a given order of bidding could be found fairly easily. We are obviously interested in finding a solution which is also *independent* of the order of bidding, i.e. what we call a *unique* solution. Before doing so, T_2 is presented in a somewhat revised form, theorem T_3, where the requirements of T_2 are substituted for similar requirements, which can be written in a shorter, more compact manner, without reducing the area of applicability to any significant extent.

Our first step is to introduce some notations that will simplify our characterization of a period as to whether it is critical or uncritical.[35]

Using the pay-off notations introduced on p. 39 the assumption – e.g. made in example 3 – that period 1 is critical for H with respect to the game (6,7) can be written as $\overline{6}_1 > \overline{7}_2$. More generally, the assumption that period j is critical for H as regards the game $(y,y+1)$ can be written as $\overline{y}_j > \overline{y+1}_{j+1}$. The opposite assumption that period j is uncritical for H in $(y,y+1)$ is written as $\overline{y}_j < \overline{y+1}_{j+1}$.

There remains the case when $\overline{y}_j = \overline{y+1}_{j+1}$, i.e. when j is neither critical nor uncritical. We then say that j is *semi*critical for H. Likewise, the assumptions that j is critical, uncritical and semicritical for party L are written as $\underline{y+1}_j > \underline{y}_{j+1}$, $\underline{y+1}_j = \underline{y}_{j+1}$, $\underline{y+1}_j < \underline{y}_{j+1}$, respectively.[36]

We now introduce another more compact way of writing the assumptions that a period j is critical, semicritical or uncritical for a party in a two-alternative game $(y,y+1)$. This notation is given in the first column of Table 1 below. The second column contains a full verbal description. The new notations in column 1 are abbreviations of the notations in column 2. We again employ the principle of using capital letters for H and lower-case letters for L. The notations in column 1 are termed the *critical characteristics* of a period. In column 3 the same assumption is given by the notations used earlier.

If a period i is both $C(x)$ *and* $u(x)$ we shall – for the sake of simplicity – write this as $i = Cu(x)$. The principle is that the variable refers to *all* the preceding letters characterizing the period. The notation $i = su(x)$ will also be used in certain cases.

[35] While the advantages of these more compact notations are not very great for two-alternative games, they are considerable for games with more than two alternatives.

[36] It should be remembered that L can accept $y+1$ in period j or wait for H to accept y in period $j+1$.

Since both characteristics refer to the same party and only one can hold, $i = \mathrm{su}(x)$ implies that $i = \mathrm{s}(x)$ *or* $i = \mathrm{u}(x)$.

1	2		3
$j = \mathrm{C}(y)$	j is critical for H	in $(y,y+1)$	$\bar{y}_j > \overline{y+1}_{j+1}$
$j = \mathrm{S}(y)$	j is semicritical for H	" "	$\bar{y}_j = \overline{y+1}_{j+1}$
$j = \mathrm{U}(y)$	j is uncritical for H	" "	$\bar{y}_j < \overline{y+1}_{j+1}$
$j = \mathrm{c}(y)$	j is critical for L	" "	$\underline{y+1}_j > \underline{y}_{j+1}$
$j = \mathrm{s}(y)$	j is semicritical for L	" "	$\underline{y+1}_j = \underline{y}_{j+1}$
$j = \mathrm{u}(y)$	j is uncritical for L	" "	$\underline{y+1}_j < \underline{y}_{j+1}$

Table 1 Definitions of the critical characteristics of a period

3.4.2 Assumption S_1

In proceeding with our revision of the requirements of T_2, we note that T_2 relies on the assumption that party H has a decreasing pay-off over time. Since it appears fairly probable that the pay-off situation is roughly similar for both parties, the following assumption seems appropriate:

S_1: Both parties have a decreasing pay-off over time.

3.4.3 Assumption S_2

Next, we have to deal with the assumption that L will only have uncritical periods, and hence no semi-critical periods, prior to period i. This requirement can be revised with the aid of the following assumption:

S_2: If a period is critical or semicritical for a party in a two-alternative game, then every subsequent period is *critical* for this party and if a period is uncritical or semicritical for him then every preceding period is *uncritical* for him.

Using the critical characteristics, defined above, this can be written as:

For every game $(y,y+1)$ and for $j<j'$ we assume

as regards H: $j = \mathrm{SC}(y) \Rightarrow j' = \mathrm{C}(y)$ and $j' = \mathrm{SU}(y) \Rightarrow j = \mathrm{U}(y)$ and
as regards L: $j = \mathrm{sc}(y) \Rightarrow j' = \mathrm{c}(y)$ and $j' = \mathrm{su}(y) \Rightarrow j = \mathrm{u}(y)$.

We note in this context that examples 3 and 4 above are characterized by S_2.[37]

[37] In example 3 on p. 37 periods 1, 3 and 5 are C(6), while 2 = u(6) and 4 = c(6). Hence example 3 fulfills assumption S_2. We can show that this also holds for example 4.

3.4.4 Definition of S-games

Assumptions S_1 and S_2 will be important in the formulation of our *special* model in Chapter 5. Therefore, a bargaining game in which these two pay-off assumptions hold will be called an S-game.

3.4.5 Theorem T_3

Theorem T_2 can be reformulated for S-games. If we assume that period $i-1$ is *un*critical for L, i.e. that $i-1 = u(x)$ in a game $(x,x+1)$, then according to S_2 *every* one of L's periods *prior* to i is *un*critical. The assumption that H has a decreasing pay-off over time is already covered by assumption S_1. The conclusion of theorem T_2 can therefore be rewritten as the following theorem:

Theorem T_3: In a two-alternative S-game $(x,x+1)$ in which H bids in a period i such that $i = C(x)$ and $i-1 = u(x)$, B_3 and I_1-I_5 imply – for every $j \leqslant i$ – that $(x,x+1)_{\underline{j-1}} = x_j$ and $(x,x+1)_{\overline{j-2}} = x_j$.

Just as for theorem T_2, we deduce that H will accept x in his first period.

3.4.6 Theorem T_4

Theorems T_1-T_3 deal only with the case when the order of bidding has already been determined. It does not appear reasonable to assume that any third party will prescribe the order in which the parties will bid. Instead we introduce the following assumption:

I_6 The parties themselves determine who starts bidding. This is done in a pre-bargaining game before the start of the actual bargaining game. Each party suggests who should start bidding. If both suggest the same one, this party will start.[38]

In order to determine a *unique* solution, i.e. a solution which does *not* rely on a specific bidding order, a solution has to be determined first for each of the two possible bidding orders: 1) H starts first and 2) L starts first. The periods are assigned to the parties for each of these two bidding orders; every other one to each party. Let us, for a specific bidding order, denote the set of all periods in which party H bids as set $\boldsymbol{H}$ and the set of periods in which L bids as set $\boldsymbol{L}$. If H bids in a period j, this can be denoted as $j \in \boldsymbol{H}$, where the symbol "ϵ" is read "belongs to".

[38] If they do not suggest the same party, we can *not* determine who starts bidding.

Let us now assume that for a game $(x,x+1)$ there exists a period i that is $Cu(x)$, i.e. critical for H and uncritical for L. For each bidding order we then prove that H will bid in a period that is critical for him, while the preceding period is uncritical for L.[39] Due to T_3, H will therefore accept x in his first bid for *each* bidding order. This means, due to S_1, that both parties will want H to start bidding. Due to I_6, H starts, immediately accepting x.

Since this conclusion is obviously not dependent on the names of the parties, we can reformulate it as the following more general theorem:

Theorem T_4: In a two-alternative S-game, if there exists a period i, critical for one party and uncritical for the other party, there exists – due to B_3 and I_1–I_6 – a *unique* solution, i.e. that the party for which i is critical will start bidding by accepting the other party's terms in the first period.

Thus in a game $(x,x+1)$, $i = Cu(x)$ leads to an agreement x_1, while $i = cU(x)$ leads to $x+1_1$.

Example 4 can now be analyzed allowing also for the possibility of L bidding first. It was noted above that period 3 is critical for H (p. 41). Period 3 is uncritical for L. If L accepts 7 in period 3, he gets $6 \cdot 3 = 18$, while if H accepts 6 in period 4, L gets $5 \cdot 4 = 20$. With $3 = Cu(6)$, the conclusion according to T_4 is that H starts bidding by accepting 6 immediately. We thus establish a *unique* solution, i.e. a solution that does *not* rely on a given order of bidding.

3.4.7 Effect of Interest Rates – Example 5

In the examples above, where H wanted a 7,3 distribution and L a 6,4 distribution, an agreement was reached on the more "even" distribution 6,4. This result seems to be what most people would anticipate at first glance. It is also the result that can be deduced by some of the bargaining theories referred to on p. 20.[40]

However, already at this early stage, it should be stressed that our model will not always predict that the parties split the total dividable amount into parts that are as even as possible. We can exemplify this by showing that under certain conditions the (6,7)-game will lead to the less even distribution 7,3 instead of the 6,4 distribution. All the assumptions in example 4 are retained except that we assume

[39] 1. $i \epsilon H$: Due to S_2, $i = u(x) \Rightarrow i-1 = u(x)$. We thus have $i = C(x)$ and $i-1 = u(x)$

2. $i+1 \epsilon H$: Due to S_2, $i = C(x) \Rightarrow i+1 = C(x)$. We thus have $i+1 = C(x)$ and $i = u(x)$.

[40] In e.g. the Zeuthen-model the parties will – in the case when both have a linear cardinal utility of money – always agree on the more "even" distribution. See further p. 234 in the literature appendix.

party L has a rate of time preference of 20 per cent per period, i.e. L considers \$ 1 today to be equivalent to \$ 1.20 if received one period from now. In order to keep the analysis simple, the profits of all forthcoming periods are assumed to be paid out at once as soon as an agreement is reached.

Since H's pay-offs are unchanged, we conclude that period 1 is uncritical for H.[41] But period 1 is *critical* for L in this case. If L accepts 7 in period 1, he gets $8 \cdot 3 =$ \$ 24. If L waits for H to accept 6 in period 2, L gets a payment of $7 \cdot 4 =$ \$ 28. However, since \$ 28 obtained in period 2 is worth only \$ 28/1.20 = \$23.33 in period 1, L's pay-off from accepting 7 in period 1 is larger than his pay-off of having H accept 6 in period 2.

With $1 = cU(6)$, theorem T_4 implies that an agreement will be reached on the less even distribution 7,3 in this case when L has a 20 per cent rate of time preference.

3.4.8 Reformulation of Theorem T_1

Finally we note that the notations (introduced on p. 45 and 49) can be used to reformulate theorem T_1 (presented on p. 40), as follows:

Theorem T_1: If H has a decreasing pay-off over time after period i, and $i = C(x)$, then $(x,x+1)_{\underline{i-1}} = x_i$.

Since all S-games have a decreasing pay-off over time, it is sufficient for S-games to assume that $i = C(x)$ in order to deduce that $(x,x+1)_{\underline{i-1}} = x_i$.

3.5 Final Analysis of the Two-alternative S-games

3.5.1 Example 6 and Assumptions $B_{11}-B_{12}$

In Section 3.4 non-trivial results were deduced for some two-alternative games that appear interesting, particularly so-called S-games. Our next question is whether a unique solution for *every* two-alternative S-game can be determined on the basis of assumption set B_3. We can easily show that this is *not* the case. Let us change example 3 above so that L demands an equal split 5,5, while H demands a division 6,4. Studying this case when the \$ 10 are obtained during six periods and when H starts bidding, the following game tree is obtained:

[41] See Figure 7 on p. 41.

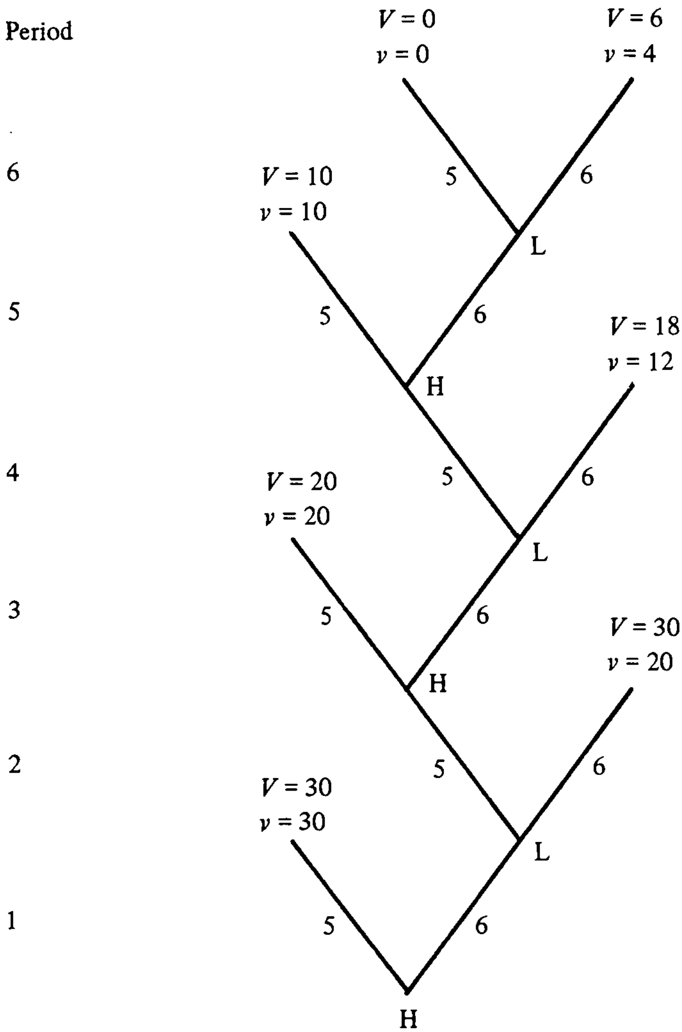

Figure 8 Example 6

First we note that L accepts 6 in period 6, i.e. that $(5,6)_{\bar{5}} = 6_6$, since $4 > 0$. Next we see that periods 5 and 3 are critical for H ($10 > 6$ and $20 > 18$), implying, due to T_1, that $(5,6)_{\underline{4}} = 5_5$ and $(5,6)_{\underline{2}} = 5_3$, while period 4 is critical for L ($12 > 10$), implying that $(5,6)_{\bar{3}} = 6_4$.[42] Since $(5,6)_{\underline{2}} = 5_3$, $\bar{5}_3 = 20$ and $\underline{5}_3 = 20$ are assigned to the branch indicating that L insists on $\bar{5}$ in period 2. We then obtain the following reduced game tree:

[42] These values are deduced for use in a three-alternative game in Chapter 4.

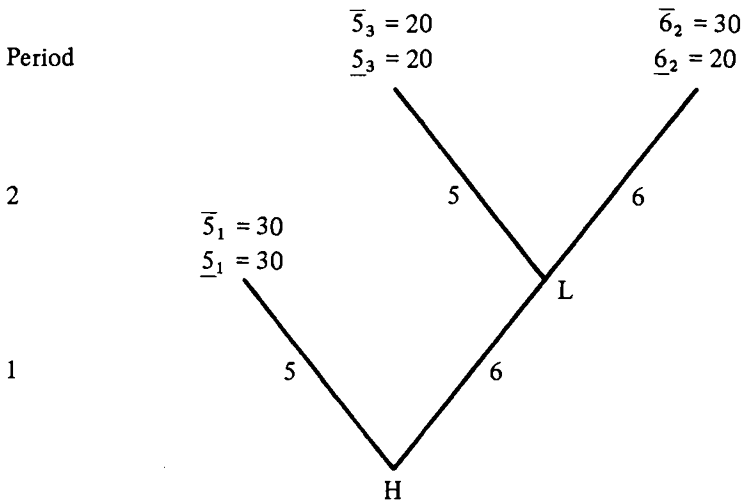

Figure 9 Example 6 (reduced)

We find that 2 = s(5) since L's pay-off $\underline{6}_2$ from accepting 6 in period 2 is *equally large* as his pay-off $\underline{5}_3$ of having H accept 5 in period 3. Since $\boldsymbol{B}_3$ can only generate rules about how a party chooses when he is *not* indifferent, we can *not* determine what L will bid in period 2 on the basis of $\boldsymbol{B}_3$ alone. Furthermore 1 = S(5). Since H would be indifferent between 5 and 6, if L accepts 6 in period 2, we cannot determine what H bids in period 1 either.

Since the solution of this kind of simple games is fundamental to the solution of a large important class of bargaining games involving several alternatives, our bargaining model would be of limited interest if it could not be used to solve this game.[43] In line with our reasoning above (p.25) we extend our set of behavioristic assumptions, but only as much as is absolutely necessary, with the following two assumptions:

B_{11} *Uncertain choice under indifference*: If a party is indifferent between two alternatives, the other party will *not* regard the choice of a specific one of these as certain.

B_{12} *Probability dominance*: If a party prefers an outcome y to an outcome y', the party will prefer receiving y for certain to obtaining a lottery involving y and y' and with some (positive, not extremely small) probability that y' will occur.[44]

[43] Games where the parties have an equal interest rate and where any division alternative is possible cannot be solved. See further p. 111.

[44] This is a special case of Krelle's more general probability dominance axiom. See Krelle (1968, p. 128) and a special case of Luce & Raiffa's monotonicity assumption (1957, p. 28). This assumption is called "the sure thing axiom" by some authors, e.g. Churchman (1961, p.226). Other authors, e.g. Savage (1954, p. 99), use this term, however, in a somewhat different sense, allowing y and y' in turn to be lotteries.

As mentioned earlier, these two assumptions combined with assumption set $\boldsymbol{B}_3$ form assumption set $\boldsymbol{B}_4$.

For the example in Figure 9, B_{11} implies that H does not exclude the possibility of L insisting on 5 in period 2. Hence H cannot be sure of getting $\overline{6}_2$ = \$ 30 by bidding 6. Thus there is a certain risk that L bids 5, which will give H only $\overline{5}_3$ = \$ 20. Since B_{12} implies that H prefers \$ 30 with certainty to a lottery with some chance of getting \$ 30 and some chance of getting less, H assigns a pay-off *smaller* than \$ 30 to his bid of insisting on 6 in period 1, i.e. $(5,6)_{\bar{1}}$ leads to V<30. With H's pay off from $(5,6)_{\bar{1}}$ called $\overline{(5,6)}_{\bar{1}}$, this *conclusion* can also be written as $\overline{(5,6)}_{\bar{1}}<30$, or more generally since $\overline{5}_1 = \overline{6}_2 = 30$ as $\overline{(5,6)}_{\bar{1}}<\overline{5}_1$.

We note that if 2 = u(5) instead of s(5),[45] the conclusion that $\overline{(5,6)}_{\bar{1}}<\overline{5}_1$ still holds.[46] Hence, provided 2 = su(5),[47] H will, when insisting on 6 in period 1, obtain a $V<\overline{5}_1$. Therefore he would rather accept 5 and get $\overline{5}_1$. This implies that $(5,6)_{\underline{0}}=5_1$.[48]

We summarize the conclusions in this section as follows:
1 = S(5) and 2 = su(5) imply that $\overline{(5,6)}_{\bar{1}}<\overline{5}_1$ and $(5,6)_{\underline{0}} = 5_1$.

3.5.2 Theorem T_5

It can easily be proved that 1 = C(5) is also sufficient for deducing that $\overline{(5,6)}_{\bar{1}}<\overline{5}_1$.[49] Furthermore due to theorem T_1 we conclude that 1 = C(5) implies that $(5,6)_{\underline{0}}=5_1$. This conclusion merges with the one at the end of the preceding section into the following theorem:

Theorem T_5: For a two-alternative S-game (5,6), in which either 1 = C(5) holds *or* 1 = S(5) and 2 = su(5) hold, $\boldsymbol{B}_4$ and I_1–I_6 imply
that $\overline{(5,6)}_{\bar{1}}<\overline{5}_1$ and $(5,6)_{\underline{0}} = 5_1$.[50]

[45] This holds e.g. if $\underline{6}_2$ = 19 instead of 20.

[46] 3 = C(5) and 2 = u(5), imply according to T_3 that $(5,6)_{\bar{1}}=5_3$; i.e. since $\overline{5}_1>\overline{5}_3$ that $\overline{(5,6)}_{\bar{1}}<\overline{5}_1$. This conclusion will be of use in Chapter 5.

[47] This implies that 2 = s(5) *or* 2 = u(5); see p. 49.

[48] The case of L bidding first is trivial. 2 = C(5) since 5·5=25>4·6=24, while 1=u(5) since 6·4=24<5·5=25. Hence, due to T_3, an agreement will be reached on 5_2. Since the case of H starting to bid leads to 5_1, this is preferred by both parties and agreed upon, due to I_6.

[49] The best H can get by insisting on 6 is that L accepts this in period 2. Hence, $\overline{(5,6)}_{\bar{1}}\leq\overline{6}_2$. Since 1=C(5) $\Rightarrow \overline{5}_1>\overline{6}_2$, $\overline{(5,6)}_{\bar{1}}<\overline{5}_1$. This conclusion will be used in Chapter 5.

[50] Since we are mainly interested in using theorem T_5 for the analysis of certain *n*-alternative games with 5 as the lowest alternative, we refrain here from writing theorem T_5 in a more general form.

3.5.3 The Question of Extending the Set of Behavioristic Assumptions

Bargaining problems involving the division of \$ 10 obtained during a number of periods, were solved for the cases when bargaining concerned whether H's share should be \$ 5 or \$ 6 and whether it should be \$ 6 or \$ 7. This division problem can be solved in a similar manner for any two-alternative case concerning whether H's share of the \$ 10 shall be \$ y or \$ $y+1$, where $y = 1, 2 \ldots 8$. Theorems $T_1 - T_5$ imply that this solution method is not limited to \$ 10 only, but that the problem of dividing a great many different sums as well as other types of two-alternative bargaining problems can be solved in this way.

It should be stressed, however, that all two-alternative S-games cannot be solved. In order to do so our set of behavioristic assumptions would have to be extended drastically. This is shown in the appendix (p. 257).

First our reliance on ordinal utility would have to be discarded in favor of cardinal utility. The requirement of a cardinal utility function appears less suitable in many cases. It strongly increases the restrictivity of the complete information assumption. The requirement that each party knows the form of the other party's utility function for money appears as a very strong assumption in all cases when it is not roughly linear with money. Another difficulty is that the assumptions behind the cardinal utility concept will in certain cases lead to behavior regarded by many as paradoxical.[51]

Furthermore we would have to commit ourselves to some specific criterion for decisions under genuine uncertainty.

Against this background it is in line with our research policy to refrain from extending our set of behavioristic assumptions further as regards two-alternative bargaining games. With this we conclude our study of two-alternative games and proceed to investigate games with more than two alternatives.

[51] See e.g. Allais (1953, p. 527).

Chapter 4
The General Model

4.1 Introduction

4.1.1 Reasons for Studying the *n*-alternative Case

In the preceding chapter we were able to give a unique solution to what appears to be a large and interesting class of bargaining games in terms of their general pay-off characteristics. But a solution was only determined for the case of two alternatives. Our research methodology (cf. p. 25), is such that we cannot be content with solving games having only two alternatives. One reason is that very few situations in reality are such that a party's only choice lies between accepting the terms most favorable to the other party or insisting on the terms most favorable to himself. There is usually room for some kind of compromise between these alternatives. Furthermore, two-alternative games are often so artifical that it is difficult to test them in an all-round manner even in laboratory situations.[1] For these reasons we proceed to a study of *n*-alternative games.

4.1.2 The Main Principles in Development of the *n*-alternative Model

A basic model relying on a well-defined set of institutional assumptions is presented in Chapters 4, 5 and 6. These institutional assumptions will be investigated in Chapter 8 in order to study the extent to which they can be modified, while still allowing the model to lead to a unique solution.

A more general version of this basic model is studied in this chapter. S-games, i.e. bargaining games fulfilling the special pay-off assumptions S_1 and S_2 are dealt with in Chapter 5. Finally we investigate different subsets of S-games for which a solution can be determined more easily in Chapter 6.

[1] This was shown by some preliminary experiments carried out in the summer of 1967, immediately after the completion of the first study of the two-alternative case (Ståhl, 1967). Proper testing of a two-alternative bargaining game of the type presented in Chapter 3 proved impossible at face-to-face bargaining. The two parties would on paper agree on one of the two "official" alternatives while in reality agreeing on some third alternative, "correcting" the pay-offs afterwards by side-payments. Hence they considered the game as one having more than two alternatives.

4.1.3 Chapter Outline

A general model for investigating games with more than two alternatives will be presented in this chapter. Before proceeding to construct this model we define (in Section 4.2) what we mean by an alternative and introduce some institutional assumptions that define the set of alternatives contemplated in this basic n-alternative model.

Then we proceed to discuss the general principles for investigating an n-alternative game. Since the n-alternative game is so much more complicated than the two-alternative one, additional institutional assumptions and three general pay-off assumptions have to be added in order to analyze these games on a computer, even if n is of moderate size (4.3).

Next we present the principles for solving a three-alternative game in order to exemplify the main principles of the general model. We then extend these principles to the case of an arbitrary number of alternatives (4.4).

The general model – relying on the behavioristic assumptions of B_4 – is intended for use in investigating every game which fulfills the institutional and general pay-off assumptions presented in this chapter. The model determines whether or not the game has a unique solution and, if so, what the solution is. Since the application of this model requires substantial computing time we conclude by asking whether any necessary conditions can be found which have to be fulfilled if the game is to have a solution. It would be meaningless to investigate games that do not fulfill these conditions (4.5).

4.2 Definition of the Alternatives

4.2.1 Definition of the Alternative Concept

First of all, in turning from the two-alternative case to the n-alternative case, we find that it is no longer clear what constitutes an alternative. In Chapter 3 the term alternative was never defined explicitly, but it was noted (on p. 29) that the two alternatives between which a party has to choose were:

1. insist on the terms favorable to him or

2. accept the terms favorable to the other party.

This is insufficient for the n-alternative case. Hence we give the following definition:

An alternative is an ordered set of terms with the following two characteristics:

1. If a party in some period j prefers an agreement on some alternative y to an agreement on some other alternative y', then the party will also prefer an agreement on y to an agreement on y' in any other period.
2. The terms applying if an agreement is reached are determined for each period.[2]

In our analysis below we assume that the pair of pay-offs – V, v – resulting from the terms of agreement for each period can furthermore be determined.[3] Then an alternative can be regarded as an *ordered set of pay-off pairs*, with one pay-off pair for each period. Thus alternative 6 in example 2 on p. 35 could be described as (V = 12, v = 8), (V = 6, v = 4) where the pay-off pair (V = 12, v = 8) is assigned to period 1 and (V = 6, v = 4) to period 2.

Since this is a very awkward way of describing an alternative, we let each alternative be represented by a number, e.g. number 6 in the case above. We use the following principle of representation: Only *one* number is assigned to each alternative and only one alternative to each number. In order to keep our discussions simple, "alternative y" is used to imply "the alternative with the number y." Hence we can continue to refer to the alternative above as alternative 6. The symbol y is used to denote the alternative number regarded as a variable as well as some unspecified alternative. We use y' to denote an alternative different from y and x to denote an alternative which leads to an agreement in a specific situation.

4.2.2 Assumption I_7

In order to further facilitate the study of the basic model of n-alternative bargaining games we introduce – in addition to assumptions I_1–I_6 presented in Chapter 3– three institutional assumptions I_7–I_9 regarding the alternatives that can be bid in the bargaining game. We start with:

I_7 The parties are bound to limit the set of alternatives in the bargaining game to some subset of alternatives with the following characteristics as regards every pair y, y' in the set: If H prefers an agreement on y in some period, then L prefers an agreement on y' in this period.

The effect of this assumption is exemplified in Figure 10.[4]

[2] The terms can e.g. be the price of a product, the quantity exchanged at this price, etc.

[3] The set of *terms*, e.g. prices and quantities, determines the "objectively measurable" *outcome factors* (see p. 31, footnote 3) such as profits, sales, market shares, etc., which in turn determine the *pay-offs*, measured in one dimension, e.g. money.

[4] In Figure 10 set A is characterized by Pareto-optimality. This characteristic does not necessarily follow from I_7.

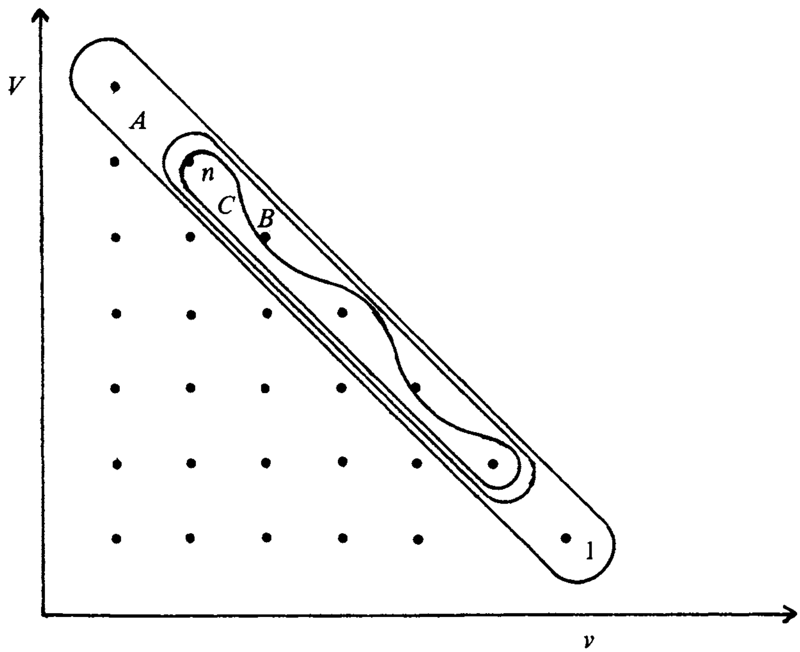

Figure 10 Limitation of the alternative set

In this figure the vertical axis measures H's pay-off V and the horizontal axis L's pay-off v from an agreement in some period on the various alternatives. These are denoted by dots.

The set of alternatives defined by I_7 in this figure could e.g. be the large one denoted by A: The parties' preferences as regards every pair of alternatives in this set are exact opposites. H will prefer the alternative to the left, L will prefer the alternative to the right. Hence I_7 implies that our model at this stage only concerns the pure *distribution* problem, where the two parties have completely opposing preferences. Thus we assume that the efficiency problem, if existent, has already been solved in some way. In Chapter 8 we show how the efficiency problem can actually be solved. In this context it should be noted that there are many negotiations for which the efficiency problem is already solved at the outset. An example is the purchase of a single item, such as a corporation, a patent or a house. Hence the solution of the distribution problem is of practical interest in its own right. The solution of the efficiency problem alone, however, can never be regarded as a complete solution, since a distribution problem will always remain.

Assumption I_7 has the advantage that it reduces the negotiation to *one* dimension only. We can now assign the alternative numbers so that *H(igh)* prefers alternatives with as *high* a number as possible and *L(ow)* prefers alternatives with as *low* a number as possible. For the case when a certain sum is to be divided in each of a number of periods – as in Chapter 3 – the alternatives can e.g. correspond to the number of dollars or the percentage obtained by party H.

4.2.3 Assumption I_8

Next we limit the set of alternatives further by introducing:

I_8 The parties are bound to bid alternatives in the bargaining game such that one given alternative is the most favorable one for L and another given alternative is the most favorable one for H.

In Figure 10 the effect of I_8 is to limit the set of alternatives to set B.

Assumption I_8 implies that an upper and lower limit is placed on the alternatives that the parties will bid in the bargaining game.[5] These extreme values can serve as a basis for numbering the alternatives within the bargaining range. The alternative most favored by L in this range can be called alternative 1 and the one most favored by H, n.[6]

The significance of the establishment of alternatives 1 and n is as follows: Due to B_4, H will realize that if L accepts n in period 1, H cannot hope for anything better. Hence both H and L will want an agreement to be reached on n in this case. Likewise both parties will want an agreement to be reached on alternative 1, if H bids this in period 1. In this case L can be regarded as having bid alternative 1 and H as having bid alternative n prior to period 1.

4.2.4 Assumption I_9

Finally we introduce:

I_9 The alternatives in the subset determined by I_7 and I_8 that can possibly be bid in this game, are determined prior to the bargaining game.

Assumption I_9 implies that the number of alternatives is determined in some way prior to the start of the bargaining game. As shown in Figure 10 this can limit the set of biddable alternatives further, in this case to set C.

In many instances this limitation is determined by purely physical factors. This can occur, for example, in a bargaining situation where a finite number of one

[5] The way these values can be determined is discussed in Chapter 8.

[6] In many cases, however, the alternatives will be numbered in a different way mainly for pedagogical reasons.

commodity, such as axes, is traded against some finite number of another commodity, such as pots, and where the value of an individual item is reduced drastically if it is divided. In other cases long and steady trade usage might determine the number of alternatives. In many labor management negotiations concerning the hourly wages of certain workers, it might appear inappropriate to both parties to reach an agreement on an hourly wage in terms of some fraction of cents. If management initially offers a wage of \$ 3.50 an hour and the union initially demands \$ 4.00 an hour, there would be at most 51 different alternatives.

In other cases the parties are more free to determine the number of alternatives among themselves. I_9 then only implies that this has been done prior to the start of the actual bargaining game[7] – in what we call the pre-bargaining phase of the negotiation – and that we can assume a given number of alternatives for this game.

4.2.5 Assumptions I_{10} and I_{11}

In this context we also want to limit the set of available actions by the following assumption:

I_{10} Prior to the bargaining game the parties agree to limit their bargaining procedure to proposals for an agreement and to adhere to such a procedure until an agreement is reached or bargaining is broken up in a period, called period z, determined prior to the bargaining game.

This assumption implies that the parties cannot threaten to break up the game if the other party does not accept their proposals. It also rules out the possibility of a party breaking up the game in order to reach an agreement with some third party. This rather restrictive assumption is modified in Chapter 8, where threats are discussed.

Finally we rule out the possibility that bargaining is broken up involuntarily prior to period z due to a party making such large losses while the agreement is delayed that he goes into bankruptcy. This situation is avoided by the following institutional assumption:

I_{11} The liquidity of each party is so large in relation to the possible losses from no-agreement that neither party will face bankruptcy, even if no agreement is reached prior to period z.

In the case of a game without a stop rule, i.e. with z infinitely large, this implies that no losses at all are made prior to the agreement.

[7] The way this is determined is discussed in detail in Chapter 8.

4.3 General Principles for Investigating an n-alternative Bargaining Game

4.3.1 An Analysis Based on $I_1 - I_{11}$

Provided a set of alternatives for which the parties have completely opposing preferences has been established and a pair of pay-offs assigned to every period for each alternative, we can proceed to attempt to solve the bargaining game. Just as in the two-alternative games, we rely on the behavioristic assumptions of B_4. As concerns the institutional assumptions, we begin by relying on those presented so far, i.e. $I_1 - I_{11}$.

Our first step is to investigate what further institutional assumptions are necessary in order to find unique solutions for games involving *more* than two alternatives. The question then centers on the differences encountered when leaving the simple two-alternative case.

The first important difference as compared to two-alternative games is that there is no longer an unambiguous answer to which bids will lead to an agreement. From assumption I_2 we can only deduce that an agreement is reached if one party proposes the same alternative as the other party proposed in the preceding period. One very general way of analyzing an n-alternative game is to limit the assumptions concerning which cases will lead to an agreement to this case alone.[8]

Having studied the two-alternative games (5,6) and (6,7) in Chapter 3 we can now study a game (5,7) where there are three alternatives 5, 6 and 7. However, even this simple game – extended for three periods as in Figure 11 *a* – leads to a fairly complex game tree. In period 3 we obtain 12 branches.[9] The number of branches might become tremendous in the general case of n alternatives and j periods. There will be $n(n-1)^{j-1}$ branches in period j.[10] For the case of a sum divided into whole percentages with 99 alternatives[11], there will be $99 \cdot 98^{99}$, i.e. more than 10^{199} branches in period 100.

[8] Other combinations of bids can also be allowed to lead to an agreement, but this is not immediately denoted as an end point. The tree is completed up to period z allowing for all possible bids, but the same outcome is assigned to every end point caused by a play that has passed through the decision node at which an agreement is reached. This procedure could also have been used for the case when a party bids the same alternative as the other party did in the previous period. Then there would be end points only in the last period. This procedure has been suggested by Krelle (1970). According to this procedure a game with three alternatives leads to 27 branches in period 3.

[9] The large number of branches depends *inter alia* on the fact that when H bids 6 and L bids 7, the game still continues *without* an agreement.

[10] Of the branches in period 1, $n-1$ branches lead into period 2 and from each one of these branches $n-1$ lead into period 3, i.e. a total of $(n-1)^2$ branches lead into period . Likewise $(n-1)^{j-1}$ branches lead into period j. With each of these leading to n different branches, there are total of $n(n-1)^{j-1}$ branches in period j.

[11] The parties will find it meaningless to propose an alternative giving the other party 0 per cent. Hence the alternatives consist of proposing 1,2,. . . or 99 per cent of the sum going to H (cf. p. 175).

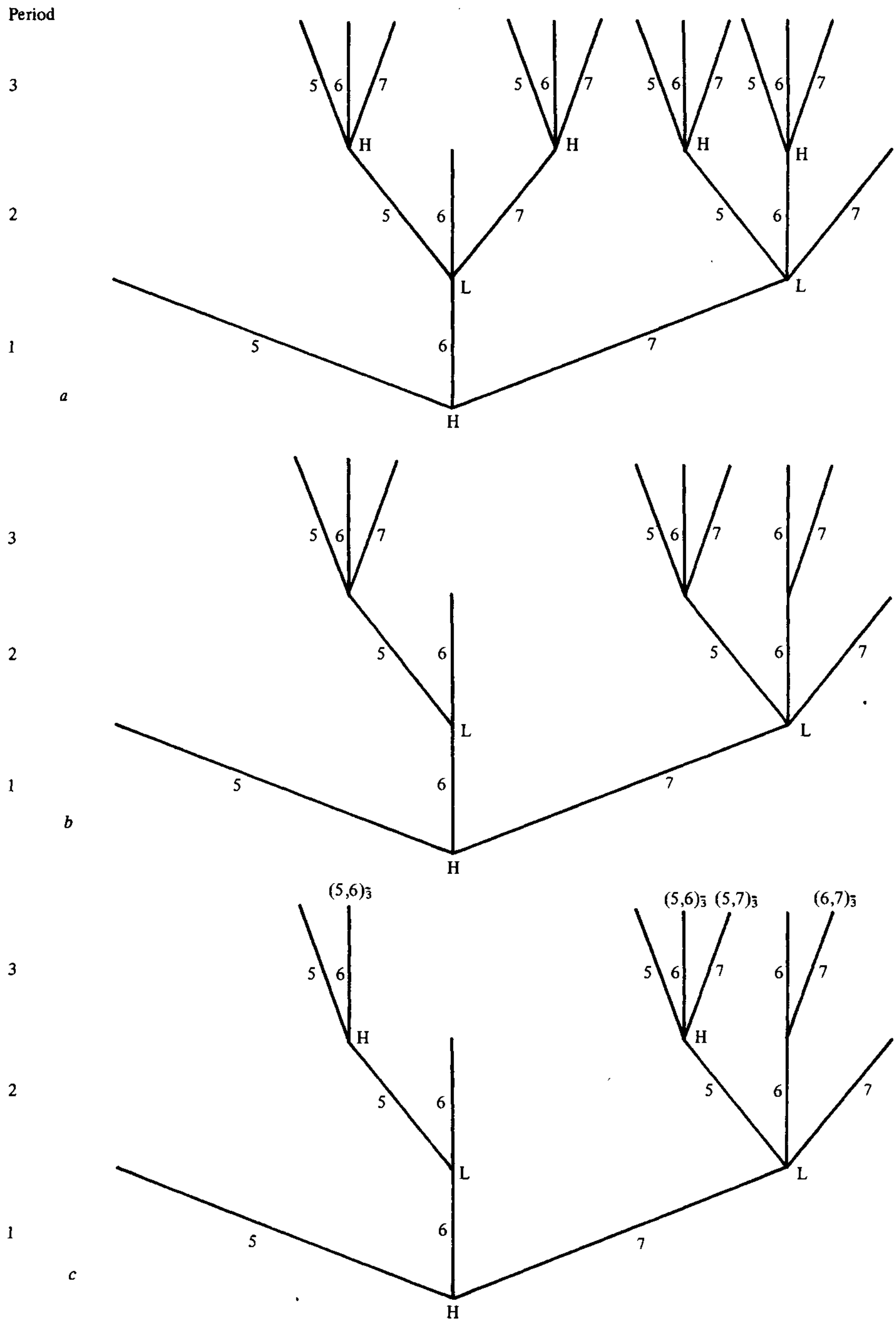

Figure 11 Game trees for three-alternative bargaining games

Run on a single computer, even the fastest possible one that can be foreseen in the future, it will take more than 10^{175} *years*[12] to determine just how the party bidding in period 100 will choose under every possible circumstance.[13] We do not want to rely on the completely unrealistic assumption of unbounded rationality implying unlimited computational ability, but assume instead (see assumption B_7) that the parties can use any existing available computational aid. Hence we must regard the game above as generally *non-solvable* on the basis of our behavioristic assumption for virtually any interesting cases.[14]

4.3.2 Assumption I_{12}

Thus the number of branches has to be reduced in some way, even though this means losing some degree of generality. Probably the most intuitively appealing way is to amplify the institutional assumptions with regard to when an agreement is reached. The following assumption is therefore added to I_2:

I_{12} If one party proposes an alternative y which is *more* favorable for the *other* party than the alternative y' that the other party proposed in the preceding period, then an agreement is reached on y.[15]

I_{12} implies that it is meaningless for a party to bid an alternative less favorable than the alternative proposed earlier by the other party, since he cannot obtain a better outcome anyway. In fact he will get a worse result.[16] We therefore rule out the situation where a party bids alternatives that are worse for him than some alternative his opponent has proposed.

For the three-alternative game, we then obtain case *b* in Figure 11 with 8 branches

[12] According to Poppelbaum (1968) no electronic memory will ever be read from or written into faster than in 10^{-16} seconds, i.e. not faster than in 10^{-24} years.

[13] Unless some *additional* institutional assumptions are introduced, the possibility of all branches leading to a different pair of pay-offs cannot be ruled out entirely, implying that *all* branches have to be taken into consideration.

[14] It should further be stressed that even if we accepted the assumption of unbounded rationality we could still *not* prove that the game would have a solution. For example, we cannot rule out the possibility that several alternatives from the same decision node will lead to the same pay-off for the party bidding in this period, implying a situation similar to the example on p. 257. The assumptions of a cardinal utility function and of equi-probability are required to solve this kind of case (cf. p. 259). Krelle (1970) relies on these two strong assumptions as well as unbounded rationality for the proof that a solution exists.

[15] Hence if L has proposed the distribution 6 to H, 4 to L, then H's proposal 5 to H, 5 to L will lead to an agreement on 5 to H, 5 to L.

[16] If alternative y is more favorable than y' for H, y is, due to I_7, less favorable for L than y'.

in period 3.[17] Obviously this simplification is still far from enough. Determination of the choice in period 100 for the bargaining game concerning percentages will take *far* more than 10^{15} years.[18]

4.3.3 The Good-faith Bargaining Assumption – I_{13}

We have to make further simplifications, limiting the generality of the model somewhat. A game with more than two alternatives cannot be described unambiguously until we have decided whether or not the bargaining is characterized by good-faith bargaining.

Good-faith bargaining implies the following: If a party has bid a specific alternative in a certain period, he may *not* bid an alternative in a later period that is less favorable to the other party.

Although all bargaining is not characterized by good-faith, this property is most likely found in a large part of all bargaining situations in reality. Since the assumption of good-faith also tends to simplify our analysis based on B_4, we adhere to our research methodology outlined on p. 25 and adopt this institutional assumption, regarding it as assumption I_{13}. We write it more precisely as:

I_{13} If party H bids y' in period j', H must bid $y \leqslant y'$ in every period $j > j'$ and if L bids y' in period j' L must bid $y \geqslant y'$ in every period $j > j'$.[19]

For the game in Figure 11 the apparent savings in case *c* compared to case *b* only amount to eliminating H's bid 7 in period 3, when H has bid 6 in period 1. The savings will be much greater in larger games, but still far from enough to make reasonably large games solvable. For example, in the case discussed above with 99 alternatives, determining the choices in period 100 would take more than 10^{15} years.[20]

[17] Compared to case *a*, L will now no longer bid 7 in period 2, if H has bid 6 in period 1. Furthermore H will *not* bid 5 in period 3, if L has bid 6 in period 2. This eliminates 4 of the 12 branches in period 3 in case *a*.

[18] For case *c* (cf. footnote 20 below) we prove that it will take more than 10^{15} years to determine the choice in period 100. Since there will be more branches in case *b*, determination of the choice will take even longer.

[19] Later on in Chapter 8 we prove that the assumption of good-faith bargaining can be relaxed somewhat with regard to certain periods.

[20] Applying common combinatorial reasoning it is proved in the appendix (p. 259) that there are more than $(n-2)^j/j!$ branches in period j, i.e. when n=99 and j=100 more than 10^{39} branches. With less than 10^{24} branches investigated each year, it will then take more than 10^{15} years to determine the choice in period 100 (cf. footnote 12 on p. 65).

4.3.4 The Assumption of Play-independent Pay-offs – G_1

In order to simplify the deduction further we introduce a general assumption concerning the pay-offs, G_1, implying that the parties only value the agreement as such and not the way in which an agreement has been reached. This implies e.g. that the following two agreements are regarded as equivalent:

1. An agreement x_j is reached by H making a series of small concessions until he accepts x in period j.

2. An agreement x_j is reached by H making no concession prior to j and then a large concession in period j so as to accept x.

Using the game-theoretical concept *play* to denote a "sequence of choices, one following the other until the game is terminated"[21], the assumption above is formally written as:

G_1 The pay-offs $\bar{y}_j$ and $\underline{y}_j$, assigned to a specific agreement y_j are independent of the play by which this agreement is reached.

The conditions under which this assumption is a good approximation of reality will be discussed later on (p. 135). For the time being it suffices to note that this assumption appears to be a good approximation in many cases where bargaining concerns large amounts of money and is of a non-recurrent character.

We proceed instead by showing that this assumption leads to a *radical* simplification of all bargaining games of some degree of complexity. Although the savings are not so large for the game in Figure 11 *c* this game can be used to illustrate the advantages of this assumption. Let us assume that this game continues after period 3. We can then indicate what the parties have bid in period 3 at each node implying continued bargaining. There are two nodes at which L has bid 5 and H has bid 6, i.e. two situations $(5,6)_{\overline{3}}$. The assumption that the play up to each agreement is irrelevant for the evaluation of the outcome implies the following: If we can deduce a pair of pay-offs V, v back to one of these two decision nodes, then the same pair of pay-offs can be assigned to the other decision node as well. Hence only the part of the tree following one of these two nodes has to be investigated.

The conclusion reached above also holds for the n-alternative case[22] and for large values of n it implies tremendous simplifications. In our game above where $n = 99$

[21] Luce & Raiffa (1957, p. 39).

[22] Due to the backwards method the assignment of pay-offs to a node representing a situation $(y,y')_j$ is only dependent on that part of the tree that is situated *above* this node, and the trees following each such node are identical due to G_1.

and j = 100 the total number of different decision nodes in period j will be merely 99·49=4851.[23] This means that at most 4851 different pay-off pairs have to be compared for each of the 100 periods. This is easily within the capability of modern computers. Hence, it seems worthwhile to develop an n-alternative model on the basis of the simplifying assumptions presented in this section.

4.3.5 Referring all Pay-offs to period 0

When H makes a choice in period j, he does so in accordance with his evaluation of the outcomes in period j. When making choices among alternatives in some earlier period j', H bases his choice on his evaluations in this earlier period j'. The same outcome y_j might be involved in H's choices in several different periods, but in principle subject to different evaluations.

The most *general* way of solving this problem involves reevaluating all outcomes that the different alternatives lead to in each period. However, this requires a considerable amount of computational work, particularly when the number of periods is large. It is therefore of interest to ask: For which pay-off assumptions is it possible to refer all evaluations to the same period – e.g. period 0 – and still deduce exactly the same choices as would be made if the more general approach above were used?

Due to our assumptions of complete information and rationality we first deduce that each party makes a *correct* estimate of his future discount rate.[24] By adding the following two general pay-off assumptions – G_2 and G_3 – we can ensure that the evaluation of the pay-offs in period 0 can be used to obtain the same solution we would have arrived at by using the evaluation in each specific period of choice.[25]

G_2 The present value of one dollar, discounted from period j back to period 0, is equal to the present value of the same dollar, discounted first back from period j to j' and then from j' to 0.

[23] Calling the highest alternative n, the following situations leading to a continued game are then possible in every period: $(1,n)$, $(2,n)$, $(3,n)$ . . . ,$(n-1,n)$, $(1,n-1)$, $(2,n-1)$, . . . , $(1,2)$. There are hence $n-1+n-2,\ldots,+1=(n-1)n/2$ such situations, all leading to an equivalent outcome according to G_1.

[24] Because of this conclusion and G_2 we avoid the well-known Strotz myopia problem (see Strotz, 1956).

[25] In our theorems presented earlier, it was not assumed explicitly that G_2 and G_3 hold. Reliance on these assumptions can be avoided by regarding the critical characteristics as referring to an evaluation in the first of the two periods referred to by the critical characteristics. This would e.g. imply that i=c(x) refers to an evaluation in period i.

G_3 The discount factor for discounting y dollars back from period j to period $j-1$ is independent of y.[26] We can e.g. allow for $e^{-r(j)}$ or $1/(1+r(j))$, where j can vary with the period number j.

These two assumptions are both very weak and natural.

4.4 Method for Finding the Solution Using the General Model

4.4.1 Introduction

The main points of the general model are presented in this section. It should be kept in mind that the model relies among other things on the following institutional assumptions. For the sake of compactness, they are presented in somewhat different order than before.

1. Prior to the real bargaining game it has been decided
   a) that the parties shall bid only such alternatives for which they have completely opposing preferences (I_7) and
   b) which of these alternatives can be bid (I_8 and I_9).

2. The parties determine among themselves, immediately prior to the bargaining game, who shall start bidding (I_6).

3. The parties deliver their bids alternatingly (I_4), one party bidding in each period (I_1)[27]. When bidding in a period the parties are aware of all earlier bids and know both their own and their opponent's pay-off (I_3 and I_5).

4. An agreement is reached if one party proposes an alternative that the other party proposed in some earlier period of the game (I_2) or if he proposes an alternative even more favorable for the other party (I_{12}).

5. Bargaining is characterized by "good faith", implying that a party can always obtain an agreement on an alternative that the other party has proposed in an earlier period (I_{13}).

6. The bargaining game will continue either until an agreement has been reached or up to period z when the game is broken up (I_{10} and I_{11}).

[26] It appears that we can also allow r to vary with y in certain ways, e.g. decrease with y. It does not, however, appear interesting to carry this analysis any further.

[27] As concerns the general model our only requirement as to the length of a period is that it shall be sufficiently long for one, and only *one* party to deliver a proposal in this period, i.e. *one* decision node is assigned to each period.

We continue to rely on the behavioristic assumptions of set $\boldsymbol{B}_4$. We also assume that the game is characterized by the general pay-off assumptions $G_1 - G_3$. We begin by looking at games with *three* alternatives.

4.4.2 A Three-alternative Game – Example 7

In order to introduce the main points of the following discussion in a simple manner we present a game with three alternatives similar to examples 3 and 6 in Chapter 3. We assume that H desires alternative 7, implying a distribution 7 to H, 3 to L of the \$ 10 received each period, while L desires alternative 5, implying a 5, 5 distribution. In the two-alternative case the parties only had the choice of either accepting the demands of the other party or insisting on their own most desired alternative. In the three-alternative case both parties have the possibility of making a "compromise" bid by suggesting an alternative lying in-between these two alternatives. Still assuming that the agreement is made on a number of whole dollars, the parties in this specific game can also suggest alternative 6, implying that H gets \$ 6 and L \$ 4 each period.

We examine a game where the \$ 10 are obtained during 6 periods.[28] We limit our study to the case when H starts bidding[29], and assume that both parties have a 0 per cent interest rate. This game can be described by the tree in Figure 12.

At each node in this figure, where the game has been reduced to one involving only two alternatives, we have noted which two alternatives remain. The index denotes the period in which the parties have come this close to each other for the first time (cf. p. 45). E.g.we get $(5,6)_{\bar{3}}$ if L insists on 5 in period 2 and H bids 6 in period 3. We get $(6,7)_{\underline{2}}$, if H insists on 7 in period 1 and L bids 6 in period 2.

As regards the game (6,7) we noted that periods 3 and 5 were critical for H and that H will hence accept 6 in these periods, if the bargaining continues that far (see p. 38). This implies that we assign $\overline{6}_5$=12 and $\underline{6}_5$=8 to $(6,7)_{\underline{4}}$ and $\overline{6}_3$=24 and $\underline{6}_3$=16 to $(6,7)_{\underline{2}}$.

As regards the game (5,6) we noted that $(5,6)_{\bar{5}} = 6_6$ and $(5,6)_{\bar{3}} = 6_4$ (cf. p. 53). Hence we assign $\overline{6}_6$=6 and $\underline{6}_6$=4 to $(5,6)_{\bar{5}}$ and furthermore $\overline{6}_4$=18 and $\underline{6}_4$=12 to $(5,6)_{\bar{3}}$. Finally we noted that $\overline{(5,6)_{\bar{1}}}<30$ and that $(5,6)_{\bar{1}} = 20$ (see p. 55).

We also see that L will accept 7 in period 6. H's bid 7 in period 5 will therefore lead to the profit distribution \$ 7 to H, \$ 3 to L.

Figure 12 can thus be reconstructed as shown by Figure 13.

[28] Hence the game is broken up in period 7, i.e. z=7.
[29] The case of L bidding first can be analyzed in a similar manner. See further footnote 31 on p. 87.

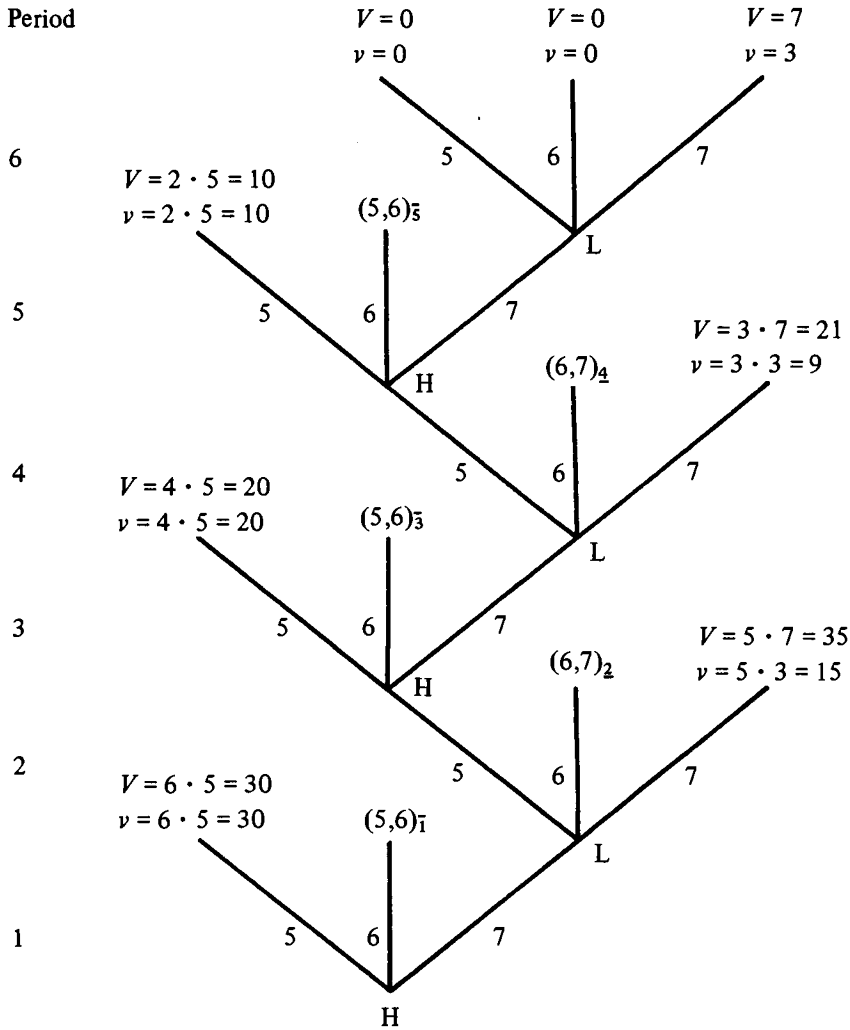

Figure 12 Example 7

Using the "backwards" method we note that H bids 5 in period 5, since 6 and 7 give him less than $ 10. Hence 6 and 7 are eliminated.

L bids 5 in period 4, since we have now determined that this will give him $ 10, rather than bidding 6 or 7, which only gives him $ 8 or $ 9. Hence alternatives 6 and 7 in period 4 are crossed out.

H chooses alternative 5 in period 3, giving him $ 20, rather than alternative 6, giving him $ 18 or alternative 7, giving him $ 10. Hence 6 and 7 are eliminated also for this period.

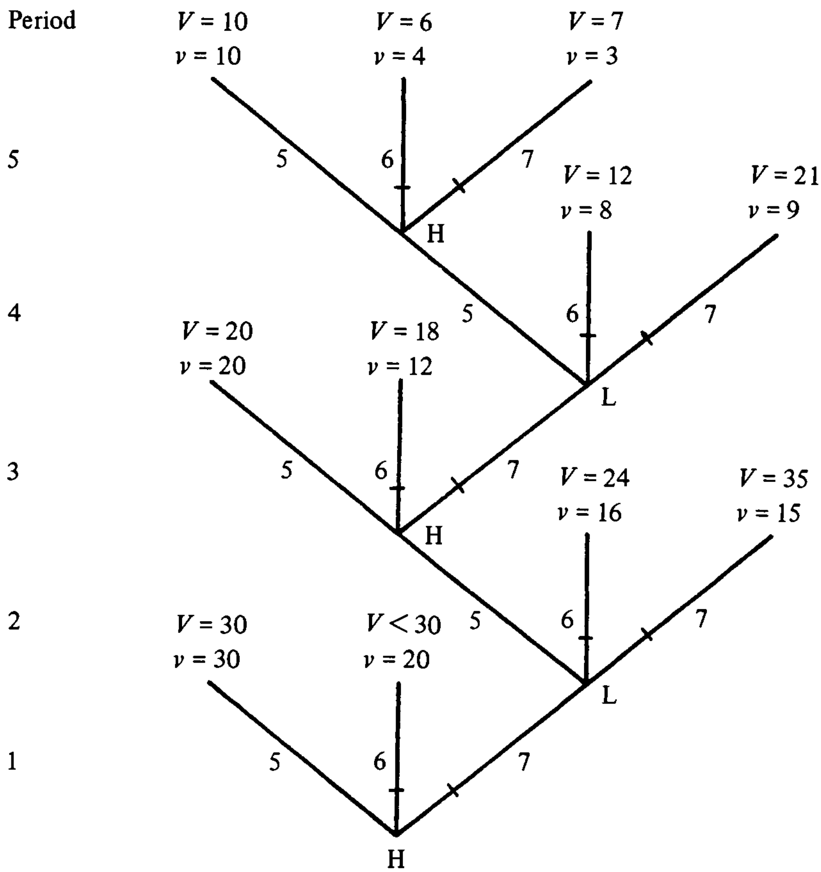

Figure 13 Example 7 (first five periods)

L bids 5 in period 2 giving him \$ 20 rather than 6 or 7, which would give him \$ 16 and \$ 15, respectively. After the elimination of alternatives 6 and 7 in period 2, H's bid 7 in period 1 will give H only \$ 20. Since alternative 5 gives H \$ 30 and H assigns a value V smaller than \$ 30 to alternative 6, H will bid 5 in period 1.

Conclusion: In this three-alternative game with H starting to bid, an agreement is reached immediately on the division of the \$ 10 into two equal shares during all periods.

It can also be shown that regardless of how many periods precede the period designated here as period 1, the solution will remain a distribution 5,5.[30] It can

[30] We content ourselves with adding only one period in the beginning – called period 0 – i.e. we explicitly study 7 periods. If L insists on alternative 5 in period 0, this will lead to a game $(5,7)_0$, in which L gets \$ 30. If L bids 6 in period 0, we obtain a game $(6,7)_0$ giving L \$ 24 (see Figure 6 on p. 37). Finally, if L accepts 7, he will obtain 7 · 3=\$ 21. Hence L will bid 5 in period 0, which H will accept in the next period.

also be shown that the same 5,5 distribution is agreed upon when L starts the bidding, but in another period.[31]

4.4.3 Main Principles for Solving a Three-alternative Game

The main principles for using the general backwards method to find a solution to games with three alternatives can now be summarized.

Each bidding order is studied separately.

For each period, when the parties have bid their most preferred alternatives in their preceding periods (e.g. in the example above 5 for L and 7 for H), the three bids in this period can be distinguished between in the following way:

1. An *acceptance* bid: a bid in which a party accepts the alternative that the other party likes the most. In the game above, alternative 5 is H's acceptance bid.

2. A *compromise* bid: a bid in which a party offers an agreement on an alternative which is not the best – nor the worst – for either party. An example of this is H's bid 6 in the game above.

3. An *insistance* bid: a bid in which the party insists on the alternative most favorable for him. An example of this is H's bid 7 in the game above.

First, for a given bidding order, the pay-offs of each party associated with the acceptance bid can be directly established for each period.

Next the pay-offs associated with the compromise bid can be determined for each period. This compromise bid, combined with the insistance bid in the preceding period, forms a two-alternative game. We refer to this as a subgame of the original three-alternative game. The methodology developed (cf. p. 47) for two-alternative games can thus be used to determine the pay-offs of the compromise bid (cf. p. 70).[32]

After this the only item remaining is determination of the pair of pay-offs for each period from the insistance bid. An insistance bid in one period can be answered by

[31] In a similar manner we can deduce that in this game involving 7 periods, with H bidding in period 0 and L in period 1, an agreement is reached on alternative 5 in period 0. Thus a unique solution can be obtained for this three-alternative game with the aid of the general "backwards" method. This can also be proved by using theorem T_{11} on p. 95.

[32] In some cases, when a party is indifferent between two alternatives, one has to be content with determining an upper limit below which the pay-off that other party gets from the compromise bid will lie. An example of this is given on p. 54. But, as shown by theorem T_5 this does not in general make it impossible to determine a unique solution.

three different bids in the next period; an acceptance bid, a compromise and a *new* insistance bid. For the last period of bidding, period $z-1$ (in the example above = 6), the insistance bid will cause the bargaining to be broken up in period z, giving each party his "break-up pay-off", generally zero. Since all three alternatives in period $z-1$ have determined pay-offs, a choice can be determined[33] in period $z-1$ and a pay-off pair can be assigned to the insistance bid in period $z-2$.

Going backwards, period by period, this procedure can be employed for each period j to determine which of these three bids is preferred by the party bidding in this period. The pair of pay-offs previously assigned to the bid thus preferred in period j, is then in turn assigned to the insistance bid of the preceding period $j-1$. In this way a pair of pay-offs is assigned to each insistance bid and the choice in each period can be determined.

Finally, the results of the two different bidding orders are compared. If an agreement has been reached on the same alternative for both bidding orders, a *unique* solution for the game has been found due to I_6.[34]

4.4.4 General Method for Solving Games with Many Alternatives

On the basis of the method presented above for investigating three-alternative games we outline more generally how games involving a great many alternatives can be investigated. This is done in the text for the simplest case, when the game has a last period for which a pay-off can be assigned to every bid and when a party is *not* indifferent between two alternatives in any period. A procedure for investigating games in which there is not any known last period and/or games involving cases of indifference is outlined in the appendix (pp. 260 ff.).

When proceeding to games with *more* than three alternatives, the only new difficulty encountered is that there are *several compromise* bids. In a four-alternative game there are two such bids, in a five-alternative game there are three and in an n-alternative game there are $n-2$ compromise bids. Otherwise, the procedure is the same as in the three-alternative case.

Let us concretize this by looking at a four-alternative game (5,8), where the alternatives are called 5, 6, 7 and 8. Then the compromise bids are alternatives 6 and 7.

[33] Provided that no indifference occurs. This problem is dealt with in the appendix (p. 261).
[34] If a particular bidding order leads to an earlier agreement, both parties will prefer this bidding order. If an agreement is reached in the same period for both bidding orders it is irrelevant who starts bidding. This can then be determined e.g. by tossing a coin.

If L bids 6 this can lead to either a three-alternative subgame (6,8), a two-alternative subgame (6,7) or an agreement on 6, all depending on what H bid in the previous period.

If L bids 7 this can lead to either a two-alternative subgame (7,8) or an agreement on 7.

Likewise, H's compromise bids in this four-alternative game can lead to a three-alternative subgame (5,7), to either one of the two-alternative subgames (5,6) and (6,7) or to an agreement.

All of these subgames have to be solved if we want to determine the pay-offs of the compromise bids of the original game (5,8). The three-alternative subgames (5,7) and (6,8) can be solved using the method in the previous section. This requires solving the two-alternative subgames into which these can be divided, namely (5,6), (6,7) and (7,8).

We *conclude* that the pay-offs of the compromise bids in a four-alternative game are determined by first analyzing three two-alternative subgames and then two three-alternative subgames. Likewise, the pay-offs of the $n-2$ compromise bids of an n-alternative game are determined by first solving $n-1$ two-alternative subgames, then $n-2$ three-alternative subgames, etc. and finally two $n-1$ alternative subgames.

The method presented above obviously requires fairly extensive calculations. The author has therefore written a computer program (in FORTRAN IV) on the basis of this method.[35] Every possible game fulfilling $I_1 - I_{13}$ and $G_1 - G_3$ can be investigated using this program to see whether it can be given a unique solution using B_4. Since any more detailed description of this computer model, beyond that presented above and in the appendix, would be of too technical a nature for this study, it has been presented elsewhere.[36]

[35] Supplemented by the procedure for investigating games in which there is not any known last period and/or games involving cases of indifference. See pp. 260 ff.

[36] See Ståhl (1972). This computer program can also be obtained on cards from the author. It should be mentioned that the computer model differs from the model outlined above and in the appendix in one respect. In order to decrease the requirement for memory space, the computer program first solves all games – i.e. both the subgames and the original game – for period $z-1$, then all games for the preceding period $z-2$, etc. back to period 1. Hence this program resembles some programs for dynamic programming (see e.g. Bellman & Dreyfus, 1962). This difference, however, has no effect on the solution.

4.5 Necessary Requirements for the Existence of a Solution

4.5.1 Introduction

The general model is intended for investigating virtually any bargaining game to see whether or not it has a unique solution. Since its application requires substantial computing time, it is of interest to find at least some necessary conditions which have to be fulfilled if a solution is to be found using the general method.[37] If a game does *not* fulfill these requirements it is meaningless to analyze it. Thus it is possible to save oneself the trouble and cost of analyzing a certain game using the computer program, by first finding out whether or not the game fulfills these necessary requirements.

It should be kept in mind that the concept of a unique solution (see p. 48) implies two things:

1. For each specified bidding order a solution, i.e. a choice in each period up to the agreement, can be determined.
2. The same alternative is agreed upon for ***both*** bidding orders, since otherwise the parties will disagree as to who shall start the bidding.

In order to find the necessary requirements for such a unique solution, we begin by looking at some requirements necessary for obtaining a solution for each bidding order (4.5.2). We then look at some further requirements necessary for obtaining a unique solution (4.5.3).

4.5.2 Requirements for a Solution when the Bidding Order is Given

The solution of an n-alternative game ultimately relies on solving the various two-alternative subgames into which the n-alternative game can be divided (cf. p. 75). Thus the assumptions necessary for obtaining a solution for the two-alternative game are also necessary for establishing a solution in the n-alternative game. Furthermore, in order to obtain a solution – i.e. to determine the choice in all bids up to the agreement – it is necessary to be able to determine the choice in at least one period, i.e. to establish a *partial* solution.

It can now be proved that a necessary condition for the existence of a partial solution of the two-alternative game for a given order of bidding and thus also for a solution of an n-alternative game is as follows:

> The game is over after a known and finite number of periods or there exists a critical period for one party for the given bidding order.

[37] The alternative of presenting assumptions that are both necessary and sufficient has to be ruled out since this task appears enormously complicated. This is very difficult even in the two-alternative case. Some sufficient, but definitely not necessary, requirements will be presented in Chapter 5.

Let us study a two-alternative game with *no* critical period. First we look at a period j in which H bids. Since j is not critical H has to determine how L will bid in $j+1$ in order to determine how he himself should bid in j. But L's choice in period $j+1$, which is not critical for him, has to be based on how H in turn bids in period $j+2$, etc. Unless there is a last period, we can continue indefinitely without being able to determine how H or L will choose in any period.

4.5.3 Requirements for a Unique Solution

Next we proceed to the requirement that there be an agreement on the same alternative for both bidding orders. We then note the following theorem:

Theorem T_6: No unique solution can be established on the basis of B_4 in a game with the following characteristics:

a. The game is broken up in period z, which is known. Each party obtains a smaller pay-off from this than from an agreement on any of the possible alternatives.

b. All periods are *un*critical for both parties for all two-alternative subgames.

c. $I_1 - I_{13}$ and S_1 hold.

Proof: We first look at the case when H bids in period $z-1$. When bidding in this period, H will accept whatever alternative L proposed in period $z-2$ rather than have the game broken up. This implies that L knows that by bidding alternative 1 in period $z-2$, he will get an agreement on 1 in $z-1$. Furthermore, since period $z-2$ is uncritical for L as regards every two-alternative subgame and hence also for the game (1,2), L will insist on alternative 1 rather than reach an agreement on alternative 2 or any higher numbered alternative in period $z-2$. This implies that if H bids $2 \ldots n$ in period $z-3$, L will *not* accept this. Hence H will have the alternative of either accepting alternative 1 in period $z-3$ or – by bidding something else – be forced to accept alternative 1 in period $z-1$. Due to S_1, H will accept 1 in period $z-3$. Going backwards in this manner two periods at a time we deduce that H will accept 1 in his first bid. Similarly, for the case of L bidding in period $z-1$, we can prove that L will accept alternative n in his first bid. This means that both parties will want the other party to bid in period $z-1$ with the result that *no* unique solution can be determined.

Theorem T_6 is important for the following reasons. It covers a case that we are particularly interested in, i.e. when there are decreasing pay-offs over time. This theorem implies that a unique solution in this case requires one party to have at least one critical (or semicritical) period as regards some two-alternative subgame. Combined with the conclusion in 4.5.2 this implies that the solution of a game based on S_1 and I_1–I_{13} requires the existence of at least one critical or semicritical period. The importance of these critical characteristics makes the study of S-games, with an emphasis on these characteristics, appear particularly important.

Chapter 5
The Special Model

5.1 Introduction

5.1.1 Reasons for Studying S-games

We hypothesized that S-games form an important subset of those games which can be given a unique solution using our general model (cf. p. 78). n-alternative S-games will be analyzed more thoroughly in this chapter, especially in order to see whether these games can be solved using a *simpler* model than the general model. This seems desirable particularly because the general model has the following disadvantages:[1]

1. As the number of alternatives and periods grows, it becomes increasingly time-consuming and hence increasingly expensive to run the general model on a computer.[2] The general model cannot be used to obtain an exact solution when there are very many alternatives and periods. It can still be used to find an approximate solution[3], but the higher the precision requirements, the more difficult it becomes to use the general model.

2. The game cannot be assumed to have a solution simply because it conforms to the few necessary, but not sufficient, assumptions mentioned in the preceding chapter.

3. Games without a stop rule can be analyzed by starting the backwards deduction from a more or less arbitrarily chosen period, j' (p. 261). If no solution is found for a certain value of j', it is not possible to determine whether the game really lacks a solution or whether j' was merely too small.

Two-alternative S-games have the great advantage that a solution can be obtained by merely looking for a period which is critical for one party and uncritical for the

[1] It should be stressed that the general model also has advantages over the special model. Besides being more general it avoids the problem of having to establish whether S_2 holds. Furthermore, the general model is probably more easy to understand than the special model.

[2] As shown in Ståhl (1972) the number of alternatives n is the most critical factor for the time needed to solve a game using the general model.

[3] We might then only be able to say what the approximate solution would be, *if* one exists. See furthermore p. 112.

other one. The following questions now arise: Under what circumstances can a unique solution be expected with respect to n-alternative S-games? To what extent can these solutions be found by examining the critical characteristics of the periods?

5.1.2 Chapter Outline

We begin by analyzing a *three*-alternative game which is a somewhat generalized version of example 7, solved in the preceding chapter by use of the general model.

Gradually extending the complexity of our examples, a *four*-alternative game where H starts bidding is studied. On the basis of the analysis of this game and the conclusions from two- and three-alternative games, an important theorem – Theorem T_{11} – can be developed step by step. This theorem can then be applied to S-games with a particular set of pay-off assumptions $\boldsymbol{P}$ in order to determine a unique solution: H accepts x in period 1 of a game where L prefers an alternative x and H prefers some higher alternative called n.

Likewise we can deduce a unique solution for a similar set of pay-off assumptions $\boldsymbol{P}'$, i.e. L accepts x in period 1 of a game where L prefers 1 and H prefers x.

Finally, on the basis of both $\boldsymbol{P}$ and $\boldsymbol{P}'$ a unique solution can be deduced for an n-alternative game, where L prefers 1 and H n. An agreement is reached on x in period 2. A numerical method for finding the solution x is provided.

5.1.3 Foundations for the Analysis

Six different theorems, T_8–T_{13}, will be presented in this chapter. Since this chapter is devoted exclusively to S-games, all games covered by these theorems will be assumed to be characterized by assumptions S_1 and S_2 which can be written as follows.[4]

S_1: $\bar{y}_j > \bar{y}_{j+1}$ and $\underline{y}_j > \underline{y}_{j+1}$; i.e. the parties have decreasing pay-offs over time.

S_2 consists of the following assumptions concerning any two periods j and j' such that $j < j'$:

S_{2A}: $j = SC(y) \Rightarrow j' = C(y)$ and S_{2B}: $j = sc(y) \Rightarrow j' = c(y)$,
implying that if a period is critical or semicritical for a party as regards the two-alternative game $(y, y+1)$, then every subsequent period is critical in this respect.

[4] We assume that j is an integer such that $0 < j < z-1$.

$S_{2B}: j' = SU(y) \Rightarrow j = U(y)$ and $S_{2b}: j' = su(y) \Rightarrow j = u(y)$
implying that if a period is uncritical or semicritical for a party as regards the game $(y,y+1)$, then every preceding period is uncritical in this respect.

Furthermore, all theorems will be based on the institutional and general pay-off assumptions presented in Chapters 3 and 4, i.e. $I_1 - I_{13}$ and $G_1 - G_3$.[5]

Finally, the deduction in all of these theorems will rely solely on the behavioristic assumptions of set $\boldsymbol{B}_4$.[6]

5.2 Analysis of a Three-alternative Bargaining Game

A generalized version of the three-alternative game (5,7), studied with the aid of the general model in Chapter 4, will be dealt with in this section. The concept of a generalized version means two things:

1. We are interested in a game with any number of periods, not just six.

2. Pay-off assumptions are used without a specific numerical content. Hence we are no longer bound by the particular situation (e.g. the division of \$ 10 in each of 6 periods) of example 7 (p. 70) but only by the following particular pay-off assumptions, numbered according to the alternative involved.[7]

P_5^0: Period 1 = Cu(5)[8] allowing for 1 = S(5), if 2 = su(5).

P_6^0: Period 2 = Cu(6).

The assumption that 1 = Cu(5) refers to the characteristics of subgame (5,6), implying that $\overline{5}_1 > \overline{6}_2$ and that $\underline{5}_2 > \underline{6}_1$.[9] In order to be able to include cases like example 6 (p. 53), we allow for period 1 to be S(5), i.e. for $\overline{5}_1 = \overline{6}_2$ to hold instead of 1 = C(5), *provided* period 2 is su(5), i.e. that $\underline{5}_3 \geqslant \underline{6}_2$ holds.

Assumption P_6^0 that 2 = Cu(6) likewise implies that $\overline{6}_2 > \overline{7}_3$ and $\underline{6}_3 > \underline{7}_2$.

Two of the reasons for making these particular pay-off assumptions are as follows:[10]

[5] $I_1 - I_{13}$ are also listed together on pp. 290–291 and $G_1 - G_3$ on pp. 291–292.
[6] See p. 289.
[7] The notation P^0 instead of P is used to stress that these pay-off assumptions are less general than those presented later, for which the notation P is used.
[8] 1 = Cu(5) implies that 1 is C(5) *and* that 1 = u(5) (see p. 48).
[9] See Table 1 on p. 49.
[10] Furthermore, it appears difficult to find a solution if we assume instead that 2 = Cu(5) and 1 = Cu(6), unless the very strong assumption that $\overline{5}_1 > \overline{7}_2$ is made.

1. They hold for example 7 in Chapter 4.[11]

2. P^0_5 holds for theorem T_5.

The case where party H bids in period 1, like in example 7, will be studied here.[12] Careful study of Figure 13 (p. 72) shows that all periods of this game do *not* necessarily have to be analyzed. In example 7 it is sufficient to study the choices in periods 1 and 2 only. Fig. 13 indicates that L's bid 5 in period 2 will give H less than $ 30, regardless of what is bid in subsequent periods. Against this background our study of this more general case is also limited to the first two periods, whereby the following figure is obtained (Fig. 14).

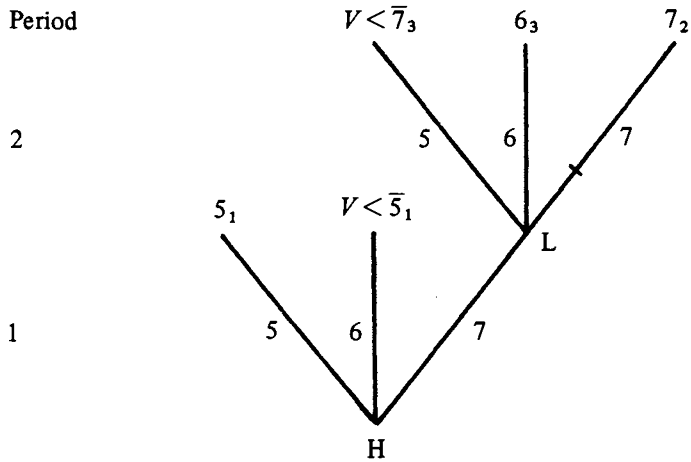

Figure 14 A reduced (5,7) game

Regardless of what he bids in period 3, H will obtain a less favorable outcome than 7_3, i.e. an agreement in this period on his most preferred alternative. Hence a $V<\bar{7}_3$ can be assigned to L's bid 5 in period 2.

L's bid 6 in period 2 will lead to a situation $(6,7)_{\underline{2}}$. Since $P^0_6 \Rightarrow 3 = C(6)$[13] and this in turn implies, due to T_1, that $(6,7)_{\underline{2}} = 6_3$, L's bid 6 leads to 6_3.

L's acceptance of 7 in period 2 obviously leads to 7_2.

In *period 1*, H's acceptance of 5 leads to 5_1, while H's bid 6 leads to $\overline{(5,6)_1} < \bar{5}_1$, due to P^0_5 and theorem T_5.[14]

[11] In example 7, 1 = Su(5), 2 = s(5)Cu(6).

[12] The case where L is also allowed to start the bidding will not be analyzed until the more general theorem T_{11} has been constructed (p. 95).

[13] $P^0_6 \Rightarrow 2 = C(6) \Rightarrow 3 = C(6)$, due to S_2.

[14] The pay-off assumptions of theorem T_5 follow from P^0_5.

In Figure 14, L's bid 7 in period 2 can be eliminated since $P_6^0 \Rightarrow 2 = u(6) \Rightarrow \underline{6}_3 > \underline{7}_2$. After this elimination H's bid 7 leads to $V < \overline{7}_3$ or $V = \overline{6}_3$. Since $\overline{5}_1 \geqslant \overline{6}_2 > \overline{7}_3 > \overline{6}_3$ we conclude that $\overline{(5,7)}_{\overline{1}} < \overline{5}_1$. Since H's bid 6 also gives him less that $\overline{5}_1$, H will accept 5 in period 1. These conclusions can be formulated as:

Theorem T_7. P_5^0 and P_6^0 imply that $\overline{(5,7)}_{\overline{1}} < \overline{5}_1$ and $(5,7)_{\underline{0}} = 5_1$.[15]

5.3 Analysis of a Four-alternative Bargaining Game

We proceed towards the study of more complex games by examining a game with four alternatives. Our aim is to use the analysis of this game to develop a theorem for analyzing games with more than four alternatives in Section 5.4.

This four-alternative game is very similar to the three-alternative game presented in Section 5.2 above. It is again assumed that H bids in period 1 and that L's most desired alternative is alternative 5. In contrast to the earlier example we now assume, however, that H's most desired alternative is alternative 8.

Pay-off assumptions P_5^0 and P_6^0 are retained implying that 1=Cu(5), allowing for 1 = S(5), if 2=su(5), and that 2=Cu(6).

Regarding assumptions of this type as suitable[16], we continue to use this pattern of 1=Cu(5), 2=Cu(6) by assuming:

P_7^0: 3 = Cu(7).

Theorem T_8 is deduced in four steps.

Step 1. If H bids 6 in period 1, a game $(5,6)_{\overline{1}}$ is obtained. Due to theorem T_5, P_5^0 implies that $\overline{(5,6)}_{\overline{1}} < \overline{5}_1$.

Step 2. If H bids 7 in period 1, a game $(5,7)_{\overline{1}}$ is obtained. Due to theorem T_7, P_5^0 and P_6^0 imply that $\overline{(5,7)}_{\overline{1}} < \overline{5}_1$.

Step 3. When H bids 8 in period 1, we obtain the following game as illustrated by Figure 15.

[15] A more generalized version of theorem T_7 (with x instead of 5 and i instead of 1) is presented in the list of theorems on p. 293.

[16] For reasons similar to those stated on p. 82.

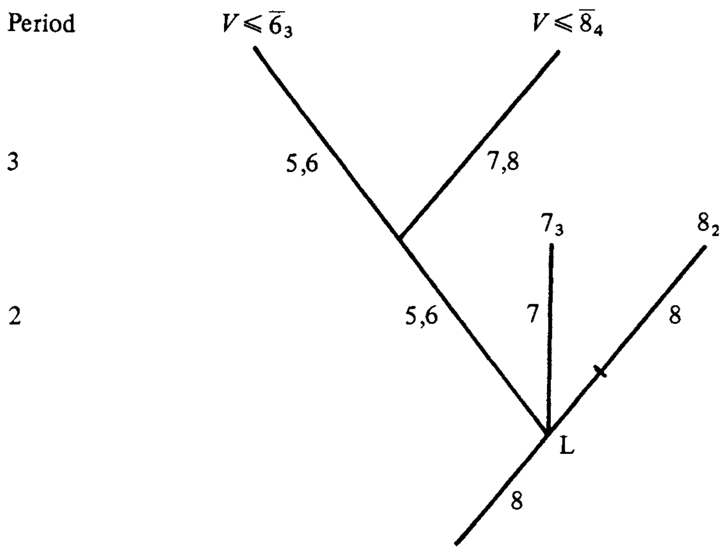

Figure 15 The choice situation in step 3 of theorem T_8

The pay-offs of the tree can be commented on as follows:

By bidding 5 or 6 in period 3, H can at best get an agreement on 6 in this period.

By bidding 7 or 8 in period 3, H can at best get an agreement on 8 in the next period, i.e. H gets a $V \leqslant \bar{8}_4$.

If L bids 7 in period 2, we get a game $(7,8)_{\underline{2}}$. Due to T_1 and P_7^0, implying that 3 = C(7), we deduce that $(7,8)_{\underline{2}} = 7_3$.[17]

If L accepts 8 in period 2, L obtains $\underline{8}_2$.

Since $\underline{7}_3 > \underline{8}_2$[18], L will bid 7 rather than 8 in period 2. After eliminating L's bid 8, we can, since $\bar{7}_3 > \bar{6}_3$ and $\bar{7}_3 > \bar{8}_4$[19], conclude that by bidding 8 in period 1 H will get a $V \leqslant \bar{7}_3$. Since $\bar{5}_1 \geqslant \bar{6}_2$[20] and $\bar{6}_2 > \bar{7}_3$[21] imply that $\bar{5}_1 > \bar{7}_3$, we conclude that H will get less than $\bar{5}_1$, i.e. $\overline{(5,8)}_1 < \bar{5}_1$.

[17] See p. 52.

[18] $P_7^0 \Rightarrow 3 = u(7)$, implying, due to S_2, that 2 = u(7), i.e. that $\underline{8}_2 < \underline{7}_3$.

[19] $P_7^0 \Rightarrow 3 = C(7) \Rightarrow \bar{7}_3 > \bar{8}_4$.

[20] $P_5^0 \Rightarrow 1 = SC(5) \Rightarrow \bar{5}_1 \geqslant \bar{6}_2$

[21] $P_6^0 \Rightarrow 2 = C(6) = \bar{6}_2 > \bar{7}_3$.

Step 4. We noted in steps 1–3 that H's bids 6, 7 or 8 in period 1 give H less than $\bar{5}_1$. Since H's acceptance of alternative 5 gives him $\bar{5}_1$, 5 is his best bid. Hence $(5,8)_{\underline{0}}=5_1$.

On the basis of the conclusions in steps 3 and 4 we obtain:

Theorem T_8: P_5^0, P_6^0 and P_7^0 imply that $\overline{(5,8)}_{\bar{1}}<\bar{5}_1$, and that $(5,8)_{\underline{0}}=5_1$.[22]

5.4 Determination of a Partial Solution for an *n*-alternative Game

5.4.1 Section Outline

On the basis of our theorems for establishing partial solutions for games with two, three and four alternatives, we deduce in Section 5.4 a partial solution for one type of bargaining games with an arbitrarily large number of alternatives. Since this is the most central part of the deduction of our special model, the whole deduction is given step by step in the text, even though it is fairly complex.

After presenting a notation system in 5.4.2, the four-alternative game is extended upwards (in 5.4.3) by studying a game $(5,8+k)$. Through theorem T_9 this game can be given a partial solution, *provided* we give a partial solution to a game $(7,7+k)$.

We next extend theorems T_5 and $T_7 - T_9$ in 5.4.4, allowing L also to bid a lower alternative than alternative 5.

Combining theorems T_5, T_7, T_8, and T_9 and their extensions a game with any number of alternatives can be solved[23] (in 5.4.5), *provided* we can solve a game which has two alternatives less.

Then, on the basis of solutions of some 2- and 3-alternative games (presented in 5.4.6) 4- and 5-alternative games can be solved. The process can be repeated in order to solve 6- and 7-alternative games and so on. In this way, after some further generalizations, we deduce (in 5.4.7–8) a solution for the $(n+1-x)$-alternative game (x,n) of theorem T_{10}.

5.4.2 Notational System for Analyzing *n*-alternative Games

Before developing our model further, we introduce some additional notations which will be used frequently in the remainder of Chapter 5.

[22] A more generalized version of theorem T_8 (with x instead of 5 and i instead of 1) is presented in the list of theorems on p. 293.

[23] "Solve" in this context implies "give a partial solution to".

x was used above to denote a specific alternative, generally one with the characteristics that period i was SC(x). The notation $x+k$ will now be used to denote any alternative with a *higher* number than x; i.e. we assume that $k \geqslant 1$.

We can also establish *upper* limits to k. Since $x+k$ in a bargaining game cannot exceed the highest alternative n, k in connection with x, used in the notation for a *game*, e.g. $(x,x+k)$, will take any integer value $\leqslant n-x$. Likewise, when k is used in a game in connection with some other alternative, e.g. 5, k will be such that $5+k \leqslant n$. On the other hand, when k is used in a *critical characteristic*, e.g. j=C($x+k$), then k is such that $x+k \leqslant n-1$.[24]

Likewise we shall use the notation $x-m$, where m, used in the connection with a game, e.g. $(x-m,x)$, is an integer 1 . . . $x-1$ and used in a critical characteristic, e.g. j=c($x-m-1$), is an integer 1 . . . $x-2$.

Furthermore, we shall use the notation $x-m'$ in games, where m' is an integer 0 . . . $x-1$, i.e. allowing $x-m'$ also to take the value x.

5.4.3 Extending the Four-alternative Game Upwards – Theorem T_9

We proceed to analyze games with *more* than four alternatives. First we ask what would happen if H – in a situation similar to that in Section 5.3 – is allowed to bid alternatives higher than alternative 8 in period 1. This means studying the case when H bids some alternative 8+k in period 1.

Let us introduce as *independent assumptions* that $(7,7+k)_{\underline{2}} = 7_3$ and that $\overline{(7-m', 7+k)}_3 < \overline{7}_3$. The following game tree is then obtained (Fig. 16).

The pay-offs of the tree can be commented on as follows:

When L has bid 5 or 6 in *period 3*, H can at best get an immediate agreement on 6 by accepting it, i.e. 5 and 6 lead to a $V \leqslant \overline{6}_3$. By bidding 7 H can at best get an agreement on this in the next period – period 4 – i.e. H gets a $V \leqslant \overline{7}_4$. By bidding $7+k$ (i.e. 8, 9, 10, etc.) H will get a $V < \overline{7}_3$ due to the independent assumption that $\overline{(7-m',7+k)}_3 < \overline{7}_3$, implying for m'=1 and 2 that $\overline{(5,7+k)}_3$ and $\overline{(6,7+k)}_3 < \overline{7}_3$.

In *period 2*, L's bid 7 will lead to a situation $(7,8+k)_{\underline{2}}$ which in turn leads to 7_3, due to the independent assumption that $(7,7+k)_{\underline{2}} = 7_3$.

[24] j = C($n-1$) implies that j is critical as regards the game $(n-1,n)$, i.e. as regards the highest relevant two-alternative subgame.

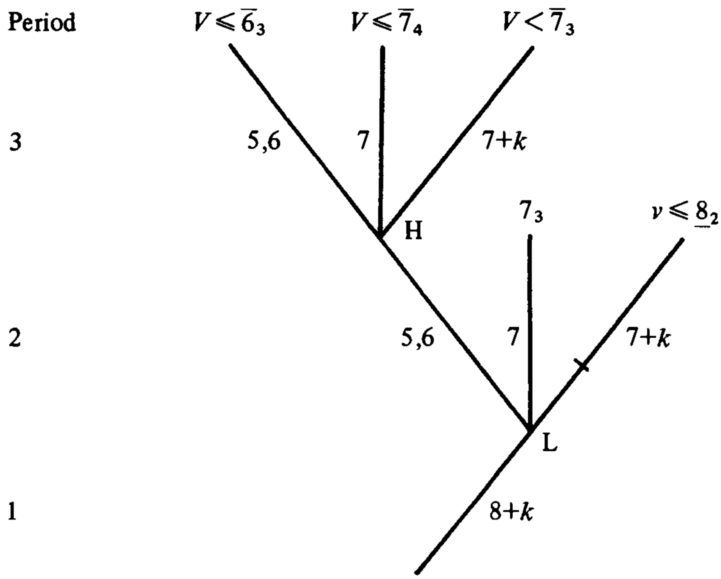

Figure 16 The choice situation of theorem T_9

L's bid $7+k$ can at best lead to an immediate agreement on 8, i.e. to a $\nu \leqslant \underline{8}_2$. Since $\underline{7}_3 > \underline{8}_2$[25], L will bid 7 rather than $7+k$ in period 2.

After crossing out L's bid $7+k$, H's bid $8+k$ in *period 1* will give H at most $\overline{7}_3$ since $\overline{7}_3 > \overline{6}_3$ and $\overline{7}_3 > \overline{7}_4$. Hence $\overline{(5,8+k)}_{\overline{1}} \leqslant \overline{7}_3$. Since $\overline{5}_1 > \overline{7}_3$[26], we conclude that $\overline{(5,8+k)}_{\overline{1}} < \overline{5}_1$. This implies in turn that $(5,8+k)_{\underline{0}} = 5_1$, since every alternative larger than 5 will then give H less than 5_1[27]. We write these conclusions as

Theorem T_9: P^0_5–P^0_7, $\overline{(7-m',7+k)}_{\overline{3}} < \overline{7}_3$ and $(7,7+k)_{\underline{2}} = 7_3$ imply that $\overline{(5,8+k)}_{\overline{1}} < \overline{5}_1$ and $(5,8+k)_{\underline{0}} = 5_1$.

5.4.4 Extensions of Theorems T_5 and $T_7 - T_9$

In theorems T_5, T_7, T_8 and T_9, alternative 5 was the lowest alternative bid by L. We extend these four theorems here, allowing L to bid also some alternative lower than 5. In particular we study the situation when H bids 6, 7, 8 or $8+k$, respectively, in period 1 and L has bid $5-m'$, where $m' = 0, \ldots 4$, prior to period 1.[28] In theorems T_5, T_7, T_8 and T_9 we deal with this situation only for the case when $m' = 0$. We call the extended theorems dealing also with the case, when $m' > 0$, T'_5, T'_7, T'_8 and T'_9. We shall now prove that the same conclusions are

[25] See footnote 18 on p. 84.
[26] See step 3 on p. 84.
[27] H's bids 6, 7 and 8 in period 1 lead to a $V < \overline{5}_1$ (cf. step 4 on p. 85).
[28] m', an integer $\geqslant 0$, should be distinguished from m, an integer $\geqslant 1$ (cf. p. 86).

obtained in these extended theorems with $m' \geqslant 0$ as in the original theorems with $m'=0$.

Theorem T_5': $P_5^0 \Rightarrow \overline{(5-m',6)}_{\bar{1}} < \bar{5}_1$

We first note that $\overline{(5-m,6)}_{\underline{2}} \leqslant \max\,[\bar{4}_3, \bar{6}_4]$, i.e. that H in period 3 can at best get an agreement on 4 and in period 4 on 6. Since $\bar{5}_1 \geqslant \bar{6}_2 > \bar{6}_4$ and $\bar{5}_1 > \bar{4}_1 > \bar{4}_3$, we conclude that $\overline{(5-m,6)}_{\underline{2}} < \bar{5}_1$.

Furthermore, $\overline{(5-m',6)}_{\bar{1}} \leqslant \max\,[\overline{(5-m,6)}_{\underline{2}}, \overline{(5,6)}_{\bar{1}}]$, i.e. $\overline{(5-m',6)}_{\bar{1}}$ is smaller than or equal to the highest of $\overline{(5-m,6)}_2$ and $\overline{(5,6)}_{\bar{1}}$.[29] Since $\overline{(5,6)}_{\bar{1}} < \bar{5}_1$, due to T_5, this implies in turn that $\overline{(5-m',6)}_{\bar{1}} < \bar{5}_1$.

Theorem T_7': $P_5^0 + P_6^0 \Rightarrow \overline{(5-m',7)}_{\bar{1}} < \bar{5}_1$.

We first note that $\overline{(5-m,7)}_{\underline{2}} \leqslant \max[\bar{4}_3, \bar{7}_4] < \bar{5}_1$, since $\bar{5}_1 > \bar{7}_3 > \bar{7}_4$. Since $\overline{(5,7)}_{\bar{1}} < \bar{5}_1$ and $\overline{(5-m',7)}_{\bar{1}} \leqslant \max[\overline{(5-m,7)}_{\underline{2}}, \overline{(5,7)}_{\bar{1}}]$ we conclude that $\overline{(5-m',7)}_{\bar{1}} < \bar{5}_1$.

Theorem T_8': $P_5^0 - P_7^0 \Rightarrow \overline{(5-m',8)}_{\bar{1}} < \bar{5}_1$.

$\overline{(5-m,8)}_{\underline{2}} \leqslant \max[\bar{4}_3, \bar{8}_4] < \bar{5}_1$, since $\bar{5}_1 > \bar{7}_3$ and $\bar{7}_3 > \bar{8}_4$[30] imply that $\bar{5}_1 > \bar{8}_4$. Since $\overline{(5,8)}_{\bar{1}} < \bar{5}_1$ we hence conclude that $\overline{(5-m',8)}_{\bar{1}} < \bar{5}_1$.

Theorem T_9': $P_5^0 - P_7^0$, $\overline{(7-m', 7+k)}_{\bar{3}} < \bar{7}_3$ and $(7,7+k)_{\underline{2}} = 7_3$ imply that $\overline{(5-m',8+k)}_{\bar{1}} < \bar{5}_1$.

To prove this theorem we substitute $5-m'$ for 5 in Figure 16 on p. 87. We then note that all the pay-offs at the end-nodes are unchanged.

If H bids $5-m'$ or 6 in period 3 he will obviously not get more than $\bar{6}_3$; by insisting on 7 he will *not* get more than $\bar{7}_4$. Finally, by bidding $7+k$ a situation $(5-m', 7+k)_{\bar{3}}$ is obtained. Since the independent assumption $\overline{(7-m', 7+k)}_{\bar{3}} < \bar{7}_3$ implies that $\overline{(5-m', 7+k)}_{\bar{3}} < \bar{7}_3$, H will still get a $V < \bar{7}_3$.

With all the pay-offs unchanged, the same conclusion as in T_9 is obtained, i.e. that by bidding $8+k$ in period 1 H will get a $V < \bar{5}_1$.

[29] $\overline{(5,6)}_{\bar{1}}$ represents H's pay-off from L's choice between 5 and 6 in period 2. It should be stressed that L's choice between 5 and 6 in period 2 is completely unaffected by what L has bid in period 0, since each pay-off from a bid in period 2 is uniquely determined by the choices in periods 1 and 2.

[30] See footnote 19, p. 84.

5.4.5 Joint Effects of the Theorems Regarding S-games

The conclusions from the various theorems regarding S-games can now be combined. All conclusions rest on some or all of the pay-off assumptions P^0_5–P^0_7 and, with respect to theorems T_9 and T'_9, also on the two independent assumptions: (1) $\overline{(7-m',7+k)}_{\bar{3}} < \bar{7}_3$ and (2) $(7,7+k)_{\underline{2}} = 7_3$.

Due to theorem T'_5: $\overline{(5-m',6)}_{\bar{1}} < \bar{5}_1$

Due to theorem T'_7: $\overline{(5-m',7)}_{\bar{1}} < \bar{5}_1$

Due to theorem T'_8: $\overline{(5-m',8)}_{\bar{1}} < \bar{5}_1$

Due to theorem T'_9: $\overline{(5-m',8+k)}_{\bar{1}} < \bar{5}_1$

Combining these conclusions we obtain (3) $\overline{(5-m',5+k)}_{\bar{1}} < \bar{5}_1$.

Furthermore:

Due to theorem T_5: $(5,6)_{\underline{0}} = 5_1$
Due to theorem T_7: $(5,7)_{\underline{0}} = 5_1$
Due to theorem T_8: $(5,8)_{\underline{0}} = 5_1$
Due to theorem T_9: $(5,8+k)_{\underline{0}} = 5_1$

Combining these conclusions we obtain (4) $(5,5+k)_{\underline{0}} = 5_1$.

These two new conclusions (3) and (4) are similar to the two independent assumptions (1) and (2) of the extension upwards.

Defining more generally

$A_1(y)$ as $\overline{(v-m',y+k)}_{\overline{-4+y}} < \bar{y}_{-4+y}$ and

$A_2(y)$ as $(y,y+k)_{\underline{-5+y}} = y_{-4+y}$

the two independent assumptions (1) and (2) can be written as $A_1(7)$ and $A_2(7)$ and the two conclusions (3) and (4) as $A_1(5)$ and $A_2(5)$. A partial solution has thus been given to a game with any number of alternatives, *provided* we can give a partial solution to a game which has two alternatives less.

We now *conclude*:

If P^0_5: 1 = Cu(5), allowing for 1 = S(5), if 2 = su(5),
P^0_6: 2 = Cu(6) and P^0_7: 3 = Cu(7) hold,
then $A_1(7)$ and $A_2(7)$ lead to $A_1(5)$ and $A_2(5)$.

Defining P^0_y as $-4+y$=Cu(y), allowing, when y=5, for 1=S(5), if 2=su(5), and then substituting period y for period 5 in the conclusion above, we can more generally deduce:[31]

P^0_y, P^0_{y+1} and P^0_{y+2} imply that $A_1(y+2)$ and $A_2(y+2)$ lead to $A_1(y)$ and $A_2(y)$.

If P^0_y holds for every y from 5 to $n-1$, then $P^0_y - P^0_{y+2}$ hold for every y from 5 to $n-3$. We can then deduce that the implication

$A_1(y+2)$ and $A_2(y+2) \Rightarrow A_1(y)$ and $A_2(y)$
is true for every y from 5 to $n-3$.

5.4.6 Deduction of $A_1(n-1)$, $A_2(n-1)$, $A_1(n-2)$ and $A_2(n-2)$

1. Due to theorems T_5 and T'_5 : $P^0_5 \Rightarrow \overline{(5-m',6)}_{\bar{1}} < \bar{5}_1$ and $(5,6)_{\underline{0}} = 5_1$.

With k = 1[32], we can write this as:

$P^0_5 \Rightarrow \overline{(5-m',5+k)}_{\bar{1}} < \bar{5}_1$ and $(5+k)_{\underline{0}} = 5_1$.

Substituting n for 6, i.e. $n-1$ for 5 and period $-4+n-1$ for period 1 we write this in turn as:

$P^0_{n-1} \Rightarrow \overline{(n-1-m',n-1+k)}_{\overline{-4+n-1}} < \overline{n-1}_{-4+n-1}$ and

$(n-1,n-1+k)_{\underline{-5+n-1}} = n-1_{-4+n-1}$, i.e. that $A_1(n-1)$ and $A_2(n-1)$ hold.

2. Due to theorem T_5 and T'_5: $P^0_5 \Rightarrow \overline{(5-m',6)}_{\bar{1}} < \bar{5}_1$ and $(5,6)_{\underline{0}} = 5_1$

Due to theorem T_7 and T'_7: $P^0_5 + P^0_6 \Rightarrow \overline{(5-m',7)}_{\bar{1}} < \bar{5}_1$ and $(5,7)_{\underline{0}} = 5_1$

Combining these conclusions we obtain:

$P^0_5 + P^0_7 \Rightarrow \overline{(5-m',5+k)}_{\bar{1}} < \bar{5}_1$ and $(5,5+k)_{\underline{0}} = 5_1$, where k goes from 1 to 7−5 = 2.

Substituting n for 7, i.e. $n-2$ for 5 and hence period $-4+n-2$ for period 1, we obtain:

[31] Since the alternative number 5 can be regarded as arbitrary, any other number we choose can be substituted for it, provided this is done consistently throughout all assumptions and conclusions.

[32] In this case k = 1 ... $n-5$ according to the definition on p. 86. With the highest alternative n = 6, k can only take the value 1.

$P^0_{n-2}+P^0_{n-1} \Rightarrow \overline{(n-2-m',n-2+k)}_{\overline{-4+n-2}}<\overline{n-2}_{-4+n-2}$ and

$(n-2,n-2+k)_{\underline{-5+n-2}}=n-2_{-4+n-2}$, i.e. that $A_1(n-2)$ and $A_2(n-2)$ hold.

5.4.7 Deduction of a Partial Solution for a Specific *n*-alternative Game

We deduced that P^0_y, holding for $y = 5 \ldots n-1$, implies that $A_1(y+2)$ and $A_2(y+2)$ lead to $A_1(y)$ and $A_2(y)$ for every $y \ldots n-3$ (cf. p. 90) and that $A_1(n-1)$, $A_2(n-1)$, $A_1(n-2)$ and $A_2(n-2)$ hold.

For the case where $n-5$ is odd we set $n-5=2k+1$, i.e. $5 = n-1-2k$. Then setting $y+2=n-1$, $A_1(n-3)$ can be deduced from $A_1(n-1)$. Next setting $y+2=n-3$ we deduce $A_1(n-5)$. By continuing in this way we finally deduce $A_1(n-1-2k)=A_1(5)$. Likewise $A_1(n-2k) = A_1(6)$ can be deduced step by step from $A_1(n-2)$.

In the case where $n-5$ is even $A_1(6)$ can be deduced from $A_1(n-1)$ and $A_1(5)$ from $A_1(n-2)$.

Hence, regardless of whether $n-5$ is odd or even, we deduce that $A_1(y)$ holds for every $y=5 \ldots n-1$. Substituting A_2 for A_1 in the proof above we likewise deduce that $A_2(y)$ holds for every $y=5 \ldots n-1$.

These important conclusions can be summed up as follows:

If P^0_y: $-4+y = \text{Cu}(y)$, allowing (when $y = 5$) for $1=\text{S}(5)$, if $2 = \text{su}(5)$, holds for every y from 5 to $n-1$, we can deduce that

$A_1(y)$: $\overline{(y-m',y+k)}_{\overline{-4+y}} < \overline{y}_{-4+y}$ and

$A_2(y)$: $(y,y+k)_{\underline{-5+y}}=y_{-4+y}$

likewise hold for every $y = 5 \ldots n-1$.

5.4.8 Theorem T_{10}

The generality of the conclusions above might appear limited by the use of alternative 5 as the lowest numbered alternative and –4 as the "base" from which the other periods are numbered. Since the use of these numbers was motivated only by a desire to keep the deductions notationally simple, any other alternative number can be substituted for 5 and any other period number for –4, provided this is done consistently throughout all assumptions and conclusions. Hence alternative

x will be substituted for alternative 5 and period i for period 1, i.e. period $i-x$ for period $-4=1-5$.

P^0_y: $1-5+y = \mathrm{Cu}(y)$, allowing (when $y = 5$) for $1 = \mathrm{S}(5)$, if $2 = \mathrm{su}(5)$,
is rewritten as the more general pay-off assumption:
P_y: $i-x+y = \mathrm{Cu}(y)$, allowing (when $y = x$) for $i = \mathrm{S}(x)$, if $i+1 = \mathrm{su}(x)$.

The specific assumption that P^0_y holds for $y=5 \ldots n-1$ can be written more generally such that P_y holds for $y=x \ldots n-1$. The set of all pay-off assumptions P_y such that $y=x \ldots. n-1$ will be called set $\boldsymbol{P}$.

Next the conclusions above that
$\overline{(y-m',y+k)_{\overline{-4+y}}} < \bar{y}_{-4+y}$ and $(y,y+k)_{\underline{-4+y-1}} = y_{-4+y}$,

are rewritten more generally as
$\overline{(y-m',y+k)_{\overline{i-x+y}}} < \bar{y}_{i-x+y}$ and $(y,y+k)_{\underline{i-x+y-1}} = y_{i-x+y}$.

In particular when $y = x$ and $m' = 0$, we obtain

$\overline{(x,x+k)_{\bar{i}}} < \bar{x}_i$ and $(x,x+k)_{\underline{i-1}} = x_i$.

These conclusions are formalized into the following theorem:

Theorem T_{10}: $\boldsymbol{P}$, consisting of
P_x: $i = \mathrm{Cu}(x)$ allowing for $i = \mathrm{S}(x)$, if $i+1 = \mathrm{su}(x)$
P_{x+k}: $i+k = \mathrm{Cu}(x+k)$
implies that
1. $\overline{(x,x+k)_{\bar{i}}} < \bar{x}_i$
2. $(x,x+k)_{\underline{i-1}} = x_i$

Thus a partial solution can be deduced for a game which could have a very large number of alternatives. We note that theorems T_5, T_7 and T_8 constitute special cases of T_{10} with $k = 1, 2$ and 3, respectively, $x = 5$ and $i = 1$.

5.4.9 Example 8

We now return to the case of dividing \$ 10 as in example 7. A somewhat more general version of this example – called example 8 – can now be investigated. Just as in example 7 we assume that the \$ 10 are divided in each of a number of periods and that each party has a 0 per cent interest rate. The difference is that in example

8 we assume the \$ 10 are obtained during $z-1$ periods where $z \geqslant 7$ and all alternatives $y = 1 \ \ldots \ 9$ are available as bids.

For the case when the game is broken up in period 7, i.e. $z = 7$, as in example 7, we proved that $1 = \mathrm{Su}(5)$ and $2 = \mathrm{s}(5)\mathrm{Cu}(6)$. It can also be proved that $3 = \mathrm{Cu}(7)$ and $4 = \mathrm{Cu}(8)$ in this case.[33] Due to theorem T_{10} this implies that $(5,9)_{\underline{0}} = 5_1$.

5.5 Determination of a Unique Solution where One Party Accepts the Other Party's Terms

5.5.1 Determining a Partial Solution when L Bids in Period *i*

In our search to determine a *unique* solution we proceed as follows. Having established a partial solution for the case when $i \epsilon H$, a partial solution can now be established for the case when $i \epsilon L$. After this the solution will be carried backwards (in 5.5.2 and 5.5.3).

It has already been proved that P leads to: $(x,x+k)_{i-1}=x_i$. This implies that if L bids x in period $i-1$, then H will accept this in period i. Turning to the case where L bids in period i, and hence H in period $i-1$, we deduce the following: If $k=1$, T_3 implies, when $i=\mathrm{Cu}(x)$ that $(x,x+k)_{i-2}=x_{i-1}$, i.e. that H will accept x in period $i-1$. Since $P \Rightarrow i=\mathrm{Cu}(x)$ it appears reasonable to hypothesize that H will accept x in period $i-1$, also when $k>1$, i.e. that $(x,x+k)_{i-2}=x_{i-1}$ holds for *every* $k \geqslant 1$. A method similar to the one used to deduce theorem T_{10} above can be used to prove that this hypothesis is true. The proof is given on p. 262 in the appendix.

5.5.2 Carrying the Solution Backwards Two Periods

Next we proceed to the determination of a unique solution, allowing for any number of periods prior to period i.

This can be done by carrying the solution backwards two periods at a time with the aid of Figure 17 where period $j-1$ is prior to period i.

[33] $\bar{7}_3=4 \cdot 7=28>\bar{8}_4=3 \cdot 8=24 \Rightarrow 3 = \mathrm{C}(7)$

$\underline{7}_4=3 \cdot 3=9>\underline{8}_3=4 \cdot 2=8 \Rightarrow 3 = \mathrm{u}(7)$

$\bar{8}_4=3 \cdot 8=24>\bar{9}_5=2 \cdot 9=18 \Rightarrow 4 = \mathrm{C}(8)$

$\underline{8}_5=2 \cdot 2=4>\underline{9}_4=3 \cdot 1=3 \Rightarrow 4 = \mathrm{u}(8)$

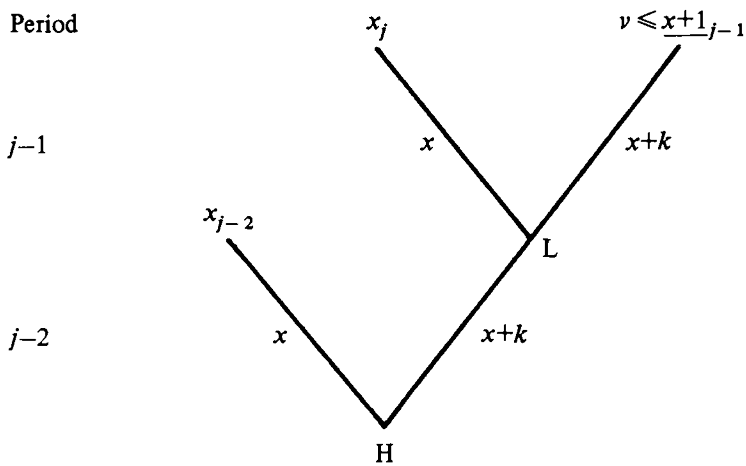

Figure 17 Tree for carrying the solution backwards two periods

Assuming that $(x,x+k)_{\overline{j-1}}=x_j$, L's bid x in period $j-1$ gives him $\underline{x}_j$, while L's bid $x+k$ can at best give L $\underline{\overline{x+1}}_{j-1}$. Since $P_x \Rightarrow \underline{x}_j > \underline{x+1}_{j-1}$,[34] L bids x rather than $x+k$ in period $j-1$, implying that $(x,x+k)_{\overline{j-2}}=x_j$. After eliminating L's bid $x+k$, we see that H's bid $x+k$ in period $j-2$ will give him $\overline{x}_j$. Since $\overline{x}_{j-2}>\overline{x}_j$, H will bid x in period $j-2$, implying that $(x,x+k)_{\underline{j-3}}=x_{j-2}$.

Conclusion: For $j \leqslant i$: P_x and $(x,x+k)_{\underline{j-1}}=x_j$ imply that $(x,x+k)_{\overline{j-2}}=x_j$ and $(x,x+k)_{\underline{j-3}}=x_{j-2}$.

5.5.3 Carrying the Solution All the Way Back – Theorem T_{11}

According to theorem T_{10} $(x,x+k)_{\overline{i-1}}=x_i$. Setting $j=i$, the conclusion in the preceding section can be used to deduce that $(x,x+k)_{\overline{i-2}}=x_i$ and $(x,x+k)_{i-3}=x_{i-2}$. By iteratively setting $j=i-2$, $i-4$, etc. and merging all the conclusions including the original one, we deduce, for $j=i$, $i-2$, $i-4$, etc. that $(x,x+k)_{\underline{j-1}}=x_j$ and $(x,x+k)_{\overline{j-2}}=x_j$.

Next we employ the conclusion in Section 5.5.1 that $(x,x+k)_{\underline{i-2}}=x_{i-1}$. By iteratively setting $j=i-1$, $i-3$, etc. we can deduce, for $j=i-1$, $\overline{i-3}$, etc. that $(x,x+k)_{\overline{j-2}}=x_{j-2}$ and $(x,x+k)_{\underline{j-1}}=x_j$.

Combining these conclusions the following theorem can be formulated:

[34] $P_x \Rightarrow i = u(x) \Rightarrow j-1 = u(x)$ (due to S_2 since $j < i$) $\Rightarrow \underline{x}_j > \underline{x+1}_{j-1}$.

Theorem T_{11} : $\boldsymbol{P}$ implies that, for every j such that $j \leqslant i-1$[35]
$(x,x+k)_{\overline{j-2}} = x_j$ and $(x,x+k)_{\underline{j-1}} = x_j$.

For $1 \epsilon \boldsymbol{H}$ we deduce that $(x,x+k)_{\underline{0}}=x_1$ and for $1 \epsilon \boldsymbol{L}$ that $(x,x+k)_{\overline{0}}=x_2$. With both parties better off when $1 \epsilon \boldsymbol{H}$, they will agree – due to I_6 – on H starting to bid. This means we can deduce that $(x,x+k)_0=x_1$[36] and in particular that $(x,n)_0=x_1$.[37]

Thus, we have been able to find a unique solution for a bargaining game with a great many possible alternatives and a large number of periods.

As regards example 8 (p. 92) we deduce that if L offers an even division of \$ 5 to each (in period $j \leqslant i-1$), then H will accept this division.

5.5.4 Solutions Implying that L Accepts H's Terms

All our examples and theorems have thus far always included assumptions such that H is the party who will accept the least favorable terms. As regards example 8 it was proved that if L proposed an even division 5,5 he could obtain an agreement on this proposal. The question then arises as to whether H, when bidding 5, can also obtain an agreement on an even division by getting L to accept 5 in the next period. It appears quite reasonable to hypothesize that this is the case.

That this hypothesis is true can easily be proved by letting H and L exchange names in each of the assumptions and conclusions. On the basis of theorem T_{11}, a new theorem – theorem T_{12} – can be deduced whereby L will accept the most favorable terms for H. A strict proof of how theorem T_{12} is obtained from theorem T_{11} is given in the mathematical appendix (pp. 263 ff.). One very simple example of this exchange of names will be discussed here in the text, namely the conclusion of theorem T_1 where i=C(5) implies that $(5,6)_{\underline{i-1}} = 5_i$.

We deal first with the assumption i = C(5). In the case of dividing \$ 10 in each period, i=C(5), i.e. $\overline{5}_i > \overline{6}_{i+1}$, reads "H prefers an agreement in period i on a division 5 to H, 5 to L to an agreement in period i+1 on a division 6 to H, 4 to L". If we now let the two parties exchange names, this can be written as: "L prefers an agreement in period i on a division of 5 to H, 5 to L to an agreement in period i+1 on a division 4 to H, 6 to L." Retaining the principle of denoting the alternatives

[35] $j \leqslant i-1$ is required since, for the case of undetermined bidding order, i can belong to L so that no solution can be determined for $j = i$.

[36] $(x,x+k)_0$ with no bar above or below the period index refers to a situation where it has not yet been determined who will start the bidding.

[37] We can also deduce unique solutions for the three- and four-alternative games and deduce that P_5^0–P_7^0 imply that H bids first and that $(5,7)_0 = 5_1$ and $(5,8)_0 = 5_1$.

according to the number of dollars that H obtains, this can be written as $\underline{5}_i > \underline{4}_{i+1}$. This in turn can be written as i=c(4), since the alternative in a critical characteristic refers to the alternative with the *lower* number. Hence the "mirror picture" of i=C(5) is i=c(4).

In a similar – though somewhat more complicated – way, we deduce generally that the "mirror picture" of i=Cu(x) is i=cU($x-1$) and that the "mirror picture" of $i+k$=Cu($x+k$) is $i+m$=cU($x-1-m$); see p. 265.

Thus, the "mirror picture" of set $\boldsymbol{P}$ consisting of

P_x: i=Cu(x), allowing for i=S(x), if $i+1$=su(x)
P_{x+k}: $i+k$=Cu($x+k$)

is the set $\boldsymbol{P'}$ consisting of

P'_x: i=cU($x-1$), allowing for i=s($x-1$), if $i+1$=SU($x-1$)
P'_{x+m}: $i+m$=cU($x-m-1$), where $m = 1 \ldots x-2$[38].

Next the "mirror picture" of the conclusion is established, to be exemplified in the text by the conclusion that $(5,6)_{i-1}=5_i$. This conclusion reads "If H suggests a division 6 to H, 4 to L in period $\overline{i-2}$ and L suggests a division 5 to H, 5 to L in period $i-1$, then an agreement is reached in period i on a division 5 to H, 5 to L". Letting the parties exchange names, this can be written as "If L suggests a division 4 to H, 6 to L in period $i-2$ and H then suggests a division 5 to H, 5 to L, an agreement is reached in period i on a division 5 to H, 5 to L". Using our ordinary notations this is written as $(4,5)_{\overline{i-1}}=5_i$. In a similar but more complicated way, we deduce more generally that the mirror picture of "$(x,x+k)_{\underline{j-1}}=x_j$, where $k = 1 \ldots n-x$" is "$(x-m,x)_{\overline{j-1}} = x_j$, where $m = 1 \ldots x-1$"[38] and that of "$(x,x+k)_{\overline{j-2}}=x_j$" is "$(x-m,x)_{\underline{j-2}}=x_j$" (see p. 266).

The following theorem can then be deduced from theorem T_{11}:

Theorem T_{12}: $\boldsymbol{P'}$ implies that
for every j, such that $j \leqslant i-1$ and
for every m such that $m=1 \ldots x-1$
$(x-m,x)_{\underline{j-2}}=x_j$ and $(x-m,x)_{\overline{j-1}}=x_j$.

Setting $m = x-1$ we obtain $(1,x)_{\bar{0}} = x_1$ and $(1,x)_{\underline{0}} = x_2$. Due to I_6, the parties will agree on L starting to bid. Therefore $(1,x)_0 = x_1$.

[38] m is defined as an integer $\geqslant 1$, such that $m \leqslant x-2$, when used in a critical characteristic and $m \leqslant x-1$, when used in the notation for a game (see p. 86).

Thus a theorem has been obtained according to which L will also accept H's best terms in the first period of the bargaining game. This theorem implies that in a game $(1,n)$ for which P' holds H can *ensure* an agreement on x by bidding x. With respect to example 8 this theorem implies that H can also ensure an agreement on dividing the \$ 10 into two equal parts.[39]

5.6 Determination of a Compromise Solution

5.6.1 Introduction

We just proved that either party can obtain an agreement in example 8 on a 5,5 split by making this bid. The question now is whether any one of the parties *would* really bid alternative 5 in a game like this. In other words, do assumptions P and P', which are sufficient for deducing that $(x-m,x)$ and $(x,x+k)$ lead to x, also imply that $(x-m,x+k)$ leads to an agreement on x?

This can be answered by using the same procedure as above, starting by deducing a partial solution, first for the case where $i \in H$ (in 5.6.2), then for the case where $i \in L$ (in 5.6.3) and finally by carrying the solution backwards, to deduce that P and P' lead to a unique solution (in 5.6.4).

Then (in 5.6.5) we discuss the question of whether there can be several alternative numbers y fulfilling P and P', leading to the unique solution x. It is shown that there can only be *one* such value.

Finally (in 5.6.6 and 5.6.7) we deal with the question of numerical methods for finding the value of y which – if it exists – is the solution x.

5.6.2 Establishing a Partial Solution When H Bids in Period i

A partial solution on the basis of P and P' is deduced in two steps.

Step 1: We study L's choice in period $i-1$ when H has insisted on $x+k$ in period $i-2$. This is illustrated by Figure 18.

[39] In a manner similar to the procedure for the (5,9) game, we can compute that i = sU(4), $i+1$ = S(4) cU(3), $i+2$ = cU(2), $i+3$ = cU(1), where $l=z-6$, implying, due to T_{12}, that $(1,5)_{\overline{j-1}}$ $= 5_j$ for $j \leqslant i-1$.

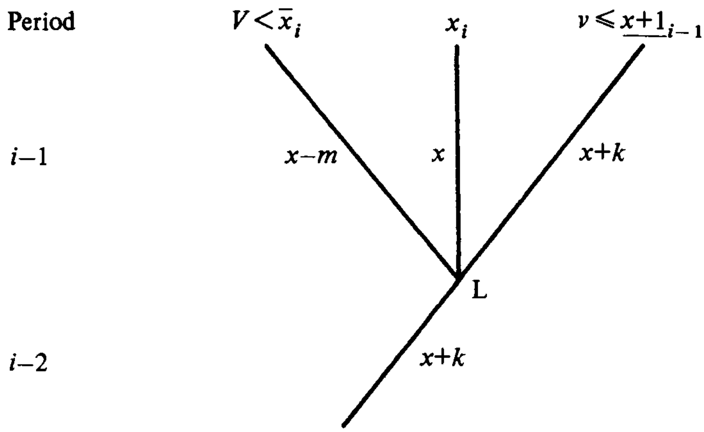

Figure 18 L's choice in period $i-1$

If L bids x in period $i-1$, leading to $(x,x+1)_{i-1}$, then there is agreement on x_i, since $\boldsymbol{P}$ implies that $(x,x+k)_{\underline{i-1}}=x_i$ (Theorem T_{10}).

If L makes a bid that is less favorable for H, i.e. $x-m$, it is reasonable to hypothesize that H obtains a less favorable agreement than x_i. That this hypothesis is in fact true can be proved rigorously. Since the proof is somewhat complicated and the hypothesis reasonable the proof is given in the appendix (p. 266).

L's bid $x+k$ can for L at best lead to an agreement in this period on $x+1$, i.e. to a $v\leqslant \underline{x+1}_{i-1}$.

Since $\boldsymbol{P}$ implies that $\underline{x}_i>\underline{x+1}_{i-1}$[40], L will bid x rather than $x+k$ in period $i-1$. Hence H's bid $x+k$ in period $i-2$ leads for H at best to x_i, i.e. to a $V\leqslant \bar{x}_i$.

Step 2. We study H's choice in period $i-2$, when L has insisted on $x-m$ in period $i-3$. This situation is illustrated by Figure 19.

H's bid $x-m$ can for H at best lead to an agreement on $x-1$ in this period, i.e. it leads to $V\leqslant \overline{x-1}_{i-2}$.

H's bid x leads to a game $(x-m,x)_{\overline{i-2}}$ which, according to Theorem T_{12}, leads to x_{i-1}.

H's bid $x+k$ leads, as noted in step 1, to a $V\leqslant \bar{x}_i$.

Since $\boldsymbol{P}'$ implies that $\bar{x}_{i-1}>\overline{x-1}_{i-2}$[41] and since $\bar{x}_{i-1}>\bar{x}_i$, H will bid x leading to x_{i-1}.

Conclusion: $\boldsymbol{P}$ and $\boldsymbol{P}'$ imply that $(x-m,x+k)_{\underline{i-3}}=x_{i-1}$.

[40] $P\Rightarrow P_x\Rightarrow i=u(x)\Rightarrow$, due to S_2, $i-1=u(x)\Rightarrow \underline{x+1}_{i-1}<\underline{x}_i$.
[41] $P'\Rightarrow i=U(x-1)\Rightarrow$, due to S_2, that $i-1=U(x-1)\Rightarrow \overline{x-1}_{i-2}<\bar{x}_{i-1}$.

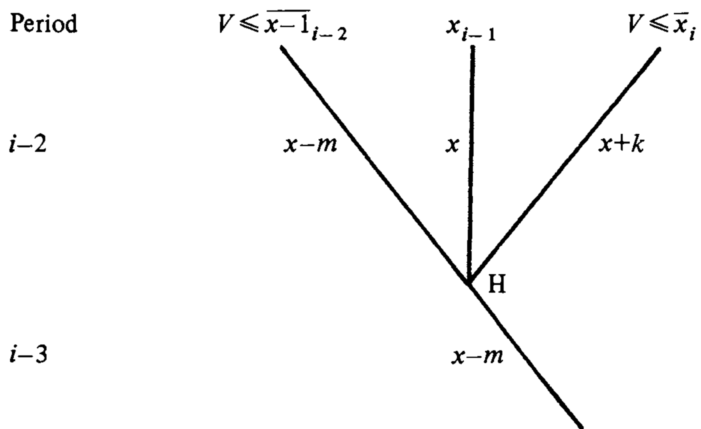

Figure 19 H's choice in period $i-2$

5.6.3 Establishing a Partial Solution When L Bids in Period i

We now turn to the problem of establishing a partial solution for the case when party L bids in period i. This is done by establishing a mirror-picture of the conclusion above. P' is the mirror-picture of P (see p. 266) and since the process of establishing a mirror-picture is completely reversible[42], P is in turn the mirror-picture of P'. Using a method similar to the one referred to in Section 5.5.4, we deduce that the mirror-picture of the conclusion $(x-m,x+k)_{\underline{i-3}} = \underline{x_{i-1}}$ is that $(x-m,x+k)_{\overline{i-3}} = x_{i-1}$.[43]

Conclusion: P and P' imply that $(x-m,x+k)_{\overline{i-3}} = x_{i-1}$.

5.6.4 Determination of a Unique Solution – Theorem T_{13}

The solution can now be carried backwards.

If $(x-m,x+k)_{\overline{j}} = x_{j+2}$ holds for some period $j \leqslant i-2$, then H's choice in period j can be described by Figure 20.

H's bid $x-m$ will at best for him lead to $x-1_j$, i.e. to a $V \leqslant \overline{x-1}_j$.

H's bid x leads to $(x-m,x)_{\overline{j}} = x_{j+1}$, due to Theorem T_{12}.

H's bid $x+k$ leads to $(x-m,x+k)_{\overline{j}} = x_{j+2}$, according to the assumption above.

[42] See p. 266 in the appendix.
[43] See p. 266 in the appendix.

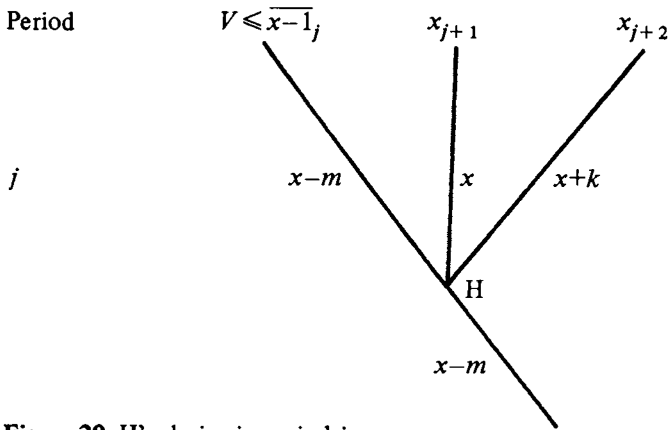

Figure 20 H's choice in period j

Since $j < i$, $P' \Rightarrow \bar{x}_{j+1} > \overline{x-1}_j$[44] and since $\bar{x}_{j+1} > \bar{x}_{j+2}$, H will bid x leading to x_{j+1}.

Thus we have deduced that $(x-m,x+k)_{\bar{j}} = x_{j+2}$ implies that $(x-m,x+k)_{\underline{j-1}} = x_{j+1}$. The "mirror-picture method" (p. 96) can then be used to deduce that $(x-m,x+k)_{\underline{j}} = x_{j+2} \Rightarrow (x-m,x+k)_{\overline{j-1}} = x_{j+1}$. Hence the same conclusion is obtained regardless of the bidding order. If we let $(x-m,x+k)_j$, with no bar above or below the period index, denote the case when the bidding order is unimportant, we can combine the two conclusions above into:
$(x-m,x+k)_j = x_{j+2} \Rightarrow (x-m,x+k)_{j-1} = x_{j+1}$ holds for $j \leqslant i-2$

Likewise the conclusions in 5.6.2 and 5.6.3 combined imply: $(x-m,x+k)_{i-3} = x_{i-1}$.

By iteratively setting $j = i-3, i-4, i-5$, etc. the solution can be deduced backwards one period at a time whereby the following theorem is obtained.

Theorem T_{13}: P+P' imply, for every $j \leqslant i-2$, that $(x-m,x+k)_{j-1} = x_{j+1}$.

Specifically (for $j = 1$, $m=x-1$ and $k=n-x$) we deduce that $(1,n)_0 = x_2$, provided i is finite and $\geqslant 3$.[45]

Thus a unique *compromise* solution for an n-alternative game with P and P' has been obtained. In this context we note that in games with a compromise solution an agreement will be reached in the *second* period and *not* in the first, as in the earlier games where one party accepted the other party's best alternative. The

[44] $P' \Rightarrow i = U(x-1) \Rightarrow j = U(x-1) \Rightarrow \overline{x-1}_j < \bar{x}_{j+1}$.

[45] In order to set $j = 1$ we require that $1 = j \leqslant i-2$, i.e. that $i \geqslant 3$.

reason for this is as follows: Assume that no outcome has been reached prior to the start of the game and that L's and H's initial requirements are alternatives 1 and n. Then no agreement can be reached in period 1 on alternative x, since an agreement requires a party either to bid the same alternative that the other party has bid in a preceding period *or* to accept the other party's initial position. We also note that the order of bidding can be disregarded. This can very well be decided by tossing a coin.

5.6.5 Number of Values of y Fulfilling P and P'

The following solution for an n-alternative S-game, characterized by I_1–I_{13} and G_1–G_3 has been deduced: If there is some value x fulfilling the requirements of assumption sets P and P', the parties will reach an agreement on this alternative x.

The question then arises as to whether there can possibly be more than one value of y that will fulfill the requirements of assumption sets P and P'. We attempt to answer this question with the aid of Figure 21.

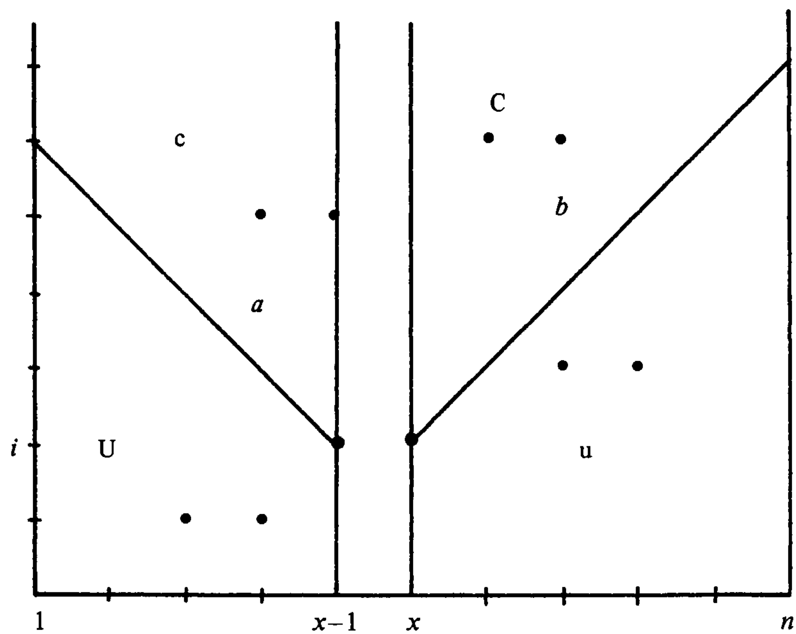

Figure 21 Critical characteristics of different points (y,j)

The number of the alternative is on the horizontal and the number of the period is on the vertical axis. For the sake of simplicity, the differences between each period and each alternative have been made equally large.[46] Let us now assume that there

[46] This, however, is not necessary for the proof.

is a period i and an alternative x such that P_x and P'_x both hold. This is denoted in Figure 21 by the pair of boldface points consisting of one point $(x-1,i)$ fulfilling P'_x and one point (x,i) fulfilling P_x. We also assume that P_{x+k} and P'_{x+m} hold, implying that $i+k$ =Cu$(x+k)$ and $i+m$=cU$(x-m-1)$.

A straight line – line a – can now be drawn from point $(x-1,i)$ upwards to the left. This line has the characteristic that every point $(x-1-m,i+m)$ lies on it and since $i+m$ = cU$(x-1-m)$ all points on this line will be cU. Due to S_2, every point above this line will be c and every period below this line will be U.

Likewise, a straight line – line b – can be drawn from point (i,x) upwards to the right. Every point $(x+k,i+k)$ lies on it and all points on this line will be Cu. Due to S_2, every point above it will be C and every point below it will be u.

Thus every possible point (y,j) is located in one of four fields C, c, U, u, where every point (y,j) in field C is such that j = C(y). It can now be proved that there can *not* exist any pair (y,j) with $y \neq x$ such that P_y and P'_y hold.[47]. This means that if a value of y can be found which fulfills *P* and *P'* then this is the unique solution, x.

5.6.6 Methods for Determining the Solution

We next outline – on the basis of theorem T_{13} – a method for finding the unique solution in an n-alternative S-game, if such a solution exists.

This method consists of two steps.

Step 1. Finding a period j and an alternative y such that j = cU$(y-1)$Cu(y), allowing for j = S(y), if $j+1$ = su(y), and j = s$(y-1)$, if $j+1$ = SU$(y-1)$.

If such a pair of values (y,j) is found, then $y = x$ and $j = i$ of P_x and P'_x.

This search is carried out for every alternative y from 2 to $n-1$ and for each alternative for every period j from 1 to z^*, where z^* (as explained further in the next section) is a period such that no period after z^* can affect the solution of an S-game.

In order to make the search more efficient it is asked, as regards a given pair (y,j), *first*, if j = U$(y-1)$u(y). If this is not true then, due to S_2, j' = U$(y-1)$u(y) cannot

[47] Every such pair (y,j) would require a pair of points in the diagram where the left point would be cs$(y-1)$ and U$(y-1)$ (in order for P'_y to hold) and the right point CS(y) and u(y) (in order for P_y to hold). This is obviously impossible. In field C, the left point cannot be U$(y-1)$; in field c, the right point cannot be u(y); in field U, the right point cannot be CS(y) and in field u the left point cannot be cs$(y-1)$.

hold for any later period j'. In other words, this y cannot be the x of P_x and P'_x. Hence if j is *not* $U(y-1)u(y)$, we proceed directly to the next *alternative.*

Step 2. If P_x and P'_x are fulfilled by step 1, we investigate whether P_{x+k} and P'_{x+m} hold, i.e. whether $i+k = Cu(x+k)$, for $k = 1 \ldots n-x-1$ and $i+m = cU(x-m-1)$ for $m = 1 \ldots x-2$. If this also holds, then a unique solution x has been found, provided i is finite and $\geqslant 3$. Ståhl (1972) contains computer programs for carrying out these procedures.

It might, however, be difficult to ascertain whether a particular bargaining game is an S-game or not. It is not always easy to establish whether or not requirement S_2 is fulfilled by merely looking at the pay-off functions.[48] But we can determine that S_2 holds, if prior to z^*[49] there is no pair (y,j) such that $j = SC(y)$ *and* $j+1 = SU(y)$ or such that $j = sc(y)$ *and* $j+1 = su(y)$.[50]

5.6.7 Establishing a Last Interesting Period

A period z^* with the following characteristics was introduced in the preceding section: The exclusion of all periods after z^* will not affect the analysis, provided all periods are characterized by S_1 and all periods $1, \ldots, z^*$ are characterized by S_2.[51] We call z^* "the last interesting period". This can be regarded as an abbreviation of "the last period that can *possibly* be interesting".[52] Thus the period after this, i.e. period z^*+1, can be called uninteresting. Our method of establishing z^* consists of finding an *uninteresting* period and setting it as z^*+1. This is done in the following manner:

First, we note that T_{13} relies solely on $\boldsymbol{P}$ and $\boldsymbol{P'}$. $\boldsymbol{P}$ only involves periods such that $i+k = Cu(x+k)$ with $k = 1 \ldots n-x-1$. The last period that $\boldsymbol{P}$ involves is hence $i+n-x-1$, where $i+n-x-1 = Cu(n-1)$. Likewise the last period that $\boldsymbol{P'}$ involves is $i+x-2$, where $i+x-2 = cU(1)$.

In other words, z^*+1 has to come after both $i+x-2$ and $i+n-x-1$. Since $i+x-2 = U(1)$, every period *prior* to $i+x-2$ must, due to S_2, be $U(1)$. Likewise, every period

[48] It is generally easy to determine whether or not the requirement S_1 of a decreasing pay-off over time holds by merely looking at the pay-off functions.

[49] S_2 does not have to hold for any period after z^*, since the solution does not rely on any pay-off assumptions whatsoever as regards periods after z^*, except that S_1 holds. This is seen by studying the deductions of theorems T_1-T_{13} closely.

[50] Since S_2 implies that $j=SC(y) \Rightarrow j+1 = C(y)$ and $j+1 = SU(y) \Rightarrow j = U(y)$, $j = SC(y)$ *and* $j+1= SU(y)$ violate S_2.

[51] See footnote 49.

[52] z^* can alternatively be called a "sufficient economic horizon". See O. Langholm (1964, p. 485).

prior to $i+n-x-1$ is u($n-1$). If a period j such that j = C(1)c($n-1$) can be found, then we know that j must come after both $i+x-2$ and $i+n-x-1$.

In order to lessen our search in 5.6.6 we want to minimize z^*. z^*+1 is set as the *first* period, counted from the start, that is C(1)c($n-1$). The first period that is C(1) and the first period that is c($n-1$) are determined and z^*+1 is then set as the last of these two periods.

Chapter 6

Specific Types of S-games

6.1 Introduction

6.1.1 Reasons for Studying Specific Types of S-games

A method for determining a unique solution for S-games in a far less complicated way than by the general method was presented in the preceding chapter. In addition to not being general, however, this method has the following drawbacks:

1. The computations required for finding the solution might be considerable if n is a large number (cf. p. 103).

2. Without further specification of the pay-off assumptions, a fairly large amount of computational work might be required to establish the fact that S_2 holds.

3. We have *not* been able to establish which S-games really have a solution.

Due to these drawbacks we intend to look for some assumptions that can reduce the amount of computation required without severely limiting the applicability of the model. A simultaneous aim is to acquire more knowledge about which S-games have a solution.

6.1.2 Chapter Outline

We begin by introducing three assumptions S_3', S_4' and S_5' which can be used to deduce that a unique solution can be determined by studying the individual assumptions P_x and P_x' instead of the whole sets of assumptions $\boldsymbol{P}$ and $\boldsymbol{P}'$.

Along with S_1' and S_2', equivalent to S_1 and S_2, these three new assumptions characterize what we call S′-games, for which a unique solution can be determined in a fairly simple manner.

Next we proceed to a subset of these S′-games for which both the solution and the truth of assumptions S_1'–S_5' can be determined in an even simpler fashion. These games are called S*-games, characterized by requirements $S_1^* - S_4^*$. Each period is assumed to last the same length of time. It is also assumed that H's pay-off can be written as $AyF(T)+B$, where A and B are parameters and $F(T)$ is a function of the time of agreement T only.

Finally, three concrete examples of pay-off functions that fulfill the requirements of the S*-games are given. A solution can often be found for these examples very easily and the exact conditions under which these pay-off functions lead to a unique solution can also be deduced.

The following picture (Figure 22) shows the relationship between the different types of games dealt with in this study.

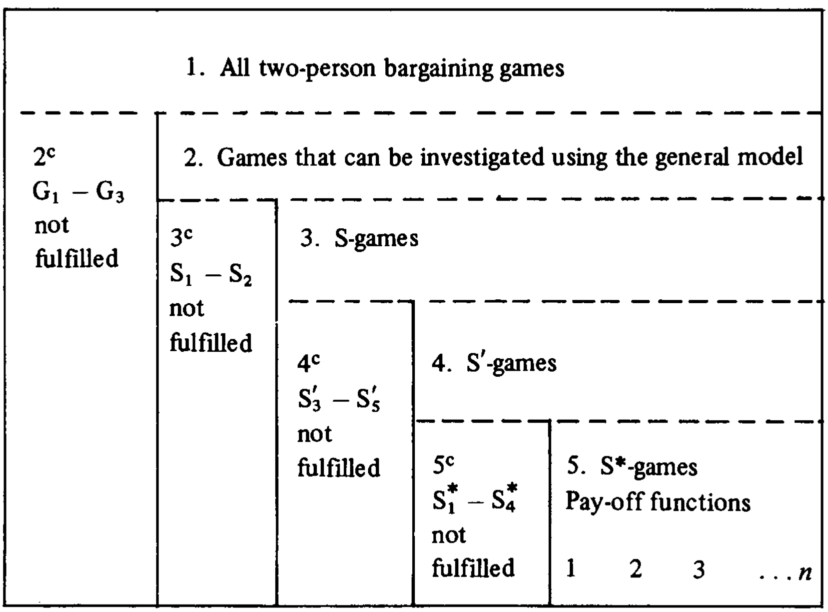

Figure 22: Different types of games

6.2 S′-games

6.2.1 Introduction

Solution of the S-games was shown above to involve a fairly large amount of computational work if n is large. We therefore proceed to see if a solution can be found using simpler methods for at least some subset of all S-games. A particular type of S-games called S′-games are studied first. These are characterized by five assumptions $S_1' - S_5'$.

S_1', which is equivalent to S_1, can be written as: $\bar{y}_j > \bar{y}_{j+1}$ and $\underline{y}_j > \underline{y}_{j+1}$.

S_2' is virtually equivalent to S_2, since S_2' consists of the following assumptions concerning two periods j and j' such that $j < j'$:[1]

S'_{2A}: $j = SC(y) \Rightarrow j' = C(y)$ and S'_{2a}: $j = sc(y) \Rightarrow j' = c(y)$

S'_{2B}: $j' = SU(y) \Rightarrow j = U(y)$ and S'_{2b}: $j' = su(y) \Rightarrow j = u(y)$.

The only difference compared to S_2 is that j and j' are no longer required to be integers.[2]

S_3', which like S_2' concerns the critical characteristics, is discussed in 6.2.2, while S_4' and S_5', related to the rate of change of the pay-off, are discussed in 6.2.3.

6.2.2 Assumption S_3' and its Implications

Assumption S_3' is similar to assumption S_2', because they both concern the critical characteristics of periods. While S_2' is related to the question of how these characteristics, for a given game $(y,y+1)$, vary with the period number j, S_3' concerns how these characteristics, for a given j, vary with the alternative number y of the game $(y,y+1)$. With respect to H, S_3' can be written as follows:

If a period is semicritical or critical as regards the two-alternative game $(y,y+1)$, then this period is critical as regards $(y',y'+1)$ where $y'>y$. If it is semicritical or uncritical as regards $(y',y'+1)$, it is uncritical as regards $(y,y+1)$.

Using our shorter notations, S_3' consists of the following assumptions about y and y', not necessarily integers, such that $y' > y$.

S'_{3A}: $j = SC(y) \Rightarrow j = C(y')$ and S'_{3a}: $j = sc(y') \Rightarrow j = c(y)$

S'_{3B}: $j = SU(y') \Rightarrow j = U(y)$ and S'_{3b}: $j = su(y) \Rightarrow j = u(y')$.

Assumption S_3' can now be combined with assumption S_2':

For H, S_2' implies that i=$SC(x) \Rightarrow i+1 = C(x)$[3].

S_3' can be written as $i+1 = SC(x) \Rightarrow i+1 = C(x+1)$.[4]

[1] Just as in the case of S_1 and S_2 we assume that $0<j<z-1$ with regard to S_1' and S_2'.
[2] In certain approximations of the solution j is allowed to take any real value (see p. 111).
[3] This is obtained by setting j=i and j'=$i+1$.
[4] This is obtained by setting j=$i+1$ and y=x and y'=$x+1$.

Together they imply that $i = \mathrm{SC}(x) \Rightarrow i+1 = \mathrm{C}(x+1)$. By substituting $i+1$ for i and $x+1$ for x, we can likewise deduce from these assumptions that $i+1 = \mathrm{SC}(x+1) \Rightarrow i+2 = \mathrm{C}(x+2)$. Combining these conclusions we conclude that $i = \mathrm{SC}(x) \Rightarrow i+2 = \mathrm{C}(x+2)$. By continuing in this manner we can generally deduce that S_2' and S_3' imply that $i = \mathrm{SC}(x) \Rightarrow i+k = \mathrm{C}(x+k)$.

Similarly we can deduce – with respect to L – that S_2 and S_3 combined imply that $i = \mathrm{sc}(x-1) \Rightarrow i+m = \mathrm{c}(x-m-1)$.

6.2.3 Assumptions Concerning the Rate of Change in a Party's Pay-off

Assumption S_4' implies that each party's pay-off falls at a decreasing or constant rate over time. For party H this can be written as $\overline{y}_{j+2}-\overline{y}_{j+1} \geqslant \overline{y}_{j+1}-\overline{y}_j$ and for L as $\underline{y}_{j+2}-\underline{y}_{j+1} \geqslant \underline{y}_{j+1}-\underline{y}_j$.[5] This assumption is fulfilled e.g. in example 8 (p. 92), since the rate of change over time is a constant, $-y$.[6]

Assumption S_5' concerns the rate of change of the pay-off when the alternative number is changed. As regards H, for whom the pay-off increases when this number *increases*, we assume that the rate of growth in the pay-off does *not* increase as the alternative number increases. This implies that $\overline{y+2}_j-\overline{y+1}_j \leqslant \overline{y+1}_j-\overline{y}_j$. As regards L, for whom the pay-off increases when the alternative number *decreases*, we assume that the rate of growth of the pay-off does *not* increase as the alternative number decreases, i.e. $\underline{y-2}_j-\underline{y-1}_j \leqslant \underline{y-1}_j-\underline{y}_j$. This assumption is also fulfilled by example 8, since this rate of change is a constant, $z-j$.[7]

Next we prove that assumptions S_4' and S_5' combined are sufficient for deducing that $i=\mathrm{U}(x-1)$ implies that $i+m=\mathrm{U}(x-m-1)$.[8]

[5] For H the decrease between periods $j+1$ and j is $-(\overline{y}_{j+1}-\overline{y}_j)$. A non-increasing rate of decrease hence implies that $-(\overline{y}_{j+2}-\overline{y}_{j+1}) \leqslant -(\overline{y}_{j+1}-\overline{y}_j)$.

[6] In example 8 H's pay-off can be written as $y(z-j)$. Then for every j: $\overline{y}_{j+1}-\overline{y}_j=y(z-j-1-z+j)=-y$.

[7] With $\overline{y}_j=y(z-j)$: $\overline{y+1}_j-\overline{y}_j=(y+1-y)(z-j)=z-j$.

[8] This is deduced as follows:

Substituting $y+1$ for y S_4' is $\overline{y+1}_{j+2}-\overline{y+1}_{j+1} \geqslant \overline{y+1}_{j+1}-\overline{y+1}_j$, i.e. $\overline{y+1}_{j+2}+\overline{y+1}_j \geqslant 2(\overline{y+1}_{j+1})$

Substituting $j+1$ for j S_5' is $\overline{y+2}_{j+1}-\overline{y+1}_{j+1} \leqslant \overline{y+1}_{j+1}-\overline{y}_{j+1}$, i.e. $\overline{y+2}_{j+1}+\overline{y}_{j+1} \leqslant 2(\overline{y+1}_{j+1})$

Combined they imply $\overline{y+1}_{j+2}+\overline{y+1}_j \geqslant \overline{y+2}_{j+1}+\overline{y}_{j+1}$, i.e. $\overline{y+1}_{j+2}-\overline{y}_{j+1} \geqslant \overline{y+2}_{j+1}-\overline{y+1}_j$

Hence $\overline{y+2}_{j+1}-\overline{y+1}_j>0 \Rightarrow \overline{y+1}_{j+2}-\overline{y}_{j+1}>0$, i.e. $\overline{y+1}_j<\overline{y+2}_{j+1} \Rightarrow \overline{y}_{j+1}<\overline{y+1}_{j+2}$,

i.e. $j=\mathrm{U}(y+1) \Rightarrow j+1 = \mathrm{U}(y)$.

After replacing $y+1$ by $x-1$ and then by $x-2$, etc., this implies

that $i=\mathrm{U}(x-1) \Rightarrow i+1=\mathrm{U}(x-2)$ and that $i+1=\mathrm{U}(x-2) \Rightarrow i+2=\mathrm{U}(x-3)$, etc.

Thus, from $i = \mathrm{U}(x-1)$, we can deduce step by step that $i+m = \mathrm{U}(x-m-1)$.

Similarly with respect to L, assumptions S'_4 and S'_5 are sufficient for deducing that i=u(x) implies that $i+k$=u($x+k$).[9]

6.2.4 Presentation of Theorem T_{14}

For S′-games, we have deduced that:

1. $i=SC(x) \Rightarrow i+k = C(x+k)$

2. $i=u(x) \Rightarrow i+k = u(x+k)$

3. $i=sc(x-1) \Rightarrow i+m = c(x-m-1)$

4. $i=U(x-1) \Rightarrow i+m = U(x-m-1)$.

1 and 2 imply that
P_x: $i=Cu(x)$, allowing for $i=S(x)$, if $i+1=su(x)$, is sufficient for deducing P_{x+k}: $i+k=Cu(x+k)$.

3 and 4 imply that
P'_x: $i = cU(x-1)$, allowing for $i = s(x-1)$, if $i+1 = SU(x-1)$, is sufficient for deducing
P'_{x+m}: $i+m = cU(x-m-1)$.

Hence for an S′-game, assumptions P_x and P'_x are sufficient for deducing both P+P' and in turn deducing the results of theorem T_{13}.

This conclusion is formalized into the following theorem:

Theorem T_{14}: In an S′-game P_x and P'_x imply for every $j \leqslant i-1$ that $(x-m,x+k)_{j-2} = x_j$.

Theorem T_{14} enables us to find the solution of S′-games without having to resort to the procedure in step 2 on p. 103. This reduces the computational effort considerably.

[9] This can also be proved by the mirror-picture of footnote 8 above.

6.2.5 Conditions for the Existence of a Solution for an S′-game

It was noted in Chapter 5 (p. 103) that P and P' alone are *not* sufficient for establishing the solution x in the S-game $(1,n)_0$. We also required i to be finite and ≥ 3.

We can now prove that it is sufficient to assume the existence of an alternative y such that $3=\mathrm{su}(y-1)\mathrm{SU}(y)$ in order to be certain that $i \geq 3$.[10]

The requirement that i be finite can be deduced from the following assumption: There exists a period j within finite time, such that $j=\mathrm{C}(1)\mathrm{c}(n-1)$.[11]

Combining these conclusions we obtain

Theorem T_{15}: Every S′-game $(1,n)_0$ in which

1. there exists a y such that $3 = \mathrm{su}(y-1)\mathrm{SU}(y)$
2. there exists within finite time a period j such that $j = \mathrm{C}(1)\mathrm{c}(n-1)$
3. P_x and P'_x hold

has a unique solution x.

Theorems T_{14} and T_{15} can now be applied to example 8 as it can be proved that these examples fulfill $\mathrm{S}'_1-\mathrm{S}'_5$.[12] Since for $x = 5$ period $z-6$ is $\mathrm{s}(x-1)\mathrm{S}(x)$[13] and $z-5=\mathrm{s}(x)\mathrm{S}(x-1)$[14], it can be deduced directly that an agreement will be reached on a 5,5-split in the second period of the bargaining game, provided $z-6 \geq 3$, i.e. that $z \geq 9$.

6.2.6 Solution when $i=\mathrm{s}(x-1)\mathrm{S}(x)$

In certain cases a solution can be deduced very easily on the basis of theorem T_{14}. First we note that S'_4 and S'_5 combined imply that $i = \mathrm{S}(x) \Rightarrow i+1 = \mathrm{SU}(x-1)$[15] and

[10] Due to S'_3, $3=\mathrm{su}(y'-1)\mathrm{SU}(y')$ implies that $3=\mathrm{su}(y-1)$ for every $y > y'$ and that $3 = \mathrm{SU}(y)$ for every $y \leq y'$. Hence for *every* y either $3=\mathrm{SU}(y)$ *or* $3=\mathrm{su}(y-1)$. If there then exists a period i such that $i=\mathrm{sc}(x-1)\mathrm{SC}(x)$, we can, due to S'_2, be assured that $i \geq 3$.

[11] Due to S'_3 this implies that $j=\mathrm{C}(y)\mathrm{c}(y-1)$ for every integer value of y. Hence, if there exists a period such that $i=\mathrm{U}(x-1)\mathrm{u}(x)$, it can, due to S'_2, not come later than j and hence i must also be finite.

[12] For a formal proof see p. 271.

[13] Both parties are indifferent between obtaining \$5 during 6 periods and \$6 during 5 periods, i.e. $z-6=\mathrm{s}(4)\mathrm{S}(5)$.

[14] Both parties are indifferent between obtaining \$4 during 5 periods and \$5 during 4 periods, i.e. $z-5=\mathrm{s}(5)\mathrm{S}(4)$.

[15] The conclusion in footnote 8 on p. 108 that $\overline{y+1}_{j+2}-\bar{y}_{j+1} > \overline{y+2}_{j+1}-\overline{y+1}_j$ implies $\overline{y+2}_{j+1}-\overline{y+1}_j = 0 \Rightarrow \overline{y+1}_{j+2}-\bar{y}_{j+1} > 0$, i.e. $j = \mathrm{S}(y+1) \Rightarrow j+1 = \mathrm{SU}(y)$. Setting $x = y+1$ and $i = j$ this is written as $i = \mathrm{S}(x) \Rightarrow i+1 = \mathrm{SU}(x-1)$.

that $i = s(x-1) \Rightarrow i+1 = su(x)$. In other words the assumption $i = s(x-1)S(x)$ leads to fulfillment of P_x and P'_x.

The following verbal interpretation can be given to i=s$(x-1)$S(x): Agreement x is such that each party, in some period i, will be *indifferent* between an agreement on x in this period and an agreement on his closest better alternative one period later.

The question now is how to find out whether there are values of i and x such that i=s$(x-1)$S(x), i.e. such that $\underline{x}_i = \underline{x-1}_{i+1}$ and $\overline{x}_i = \overline{x+1}_{i+1}$. This can easily be answered if j and y are allowed to vary continuously – at least hypothetically – and the pay-offs $\overline{y}_j$ and $\underline{y}_j$ to vary continuously as y and j change.[16] We can then set up the equation system 1. $\overline{y}_j = \overline{y+1}_{j+1}$ and 2. $\underline{y}_j = \underline{y-1}_{j+1}$. In accordance with our discussions on p. 101, we know that this equation system can have only *one* solution. The solution values, called y^* and j^*, are most likely to be found by using standard computer library programs.[17] If the solution values y^*, j^* of this equation system are *both* integers, then the true solution to our problem has been found and we can set $i=j^*$ and $x=y^*$.

The equation system above can now be used to find the solution to example 8 in a simple manner. With $\overline{y}_j = y(z-j)$ and $\underline{y}_j = (10-y)(z-j)$, the equation system above is 1. $y(z-j) = (y+1)(z-j-1)$ and 2. $(10-y)(z-j) = (10-y+1)(z-j-1)$, implying that $y = 10-y$, i.e. that $y^* = x = 10/2 = 5$. This is the same solution obtained earlier by more cumbersome methods.

Writing this more generally and substituting N for 10, we obtain $\overline{y}_j$ as $N(z-j)$ and $\underline{y}_j$ as $(N-y)(z-j)$, where $N=n+1$ is the total sum to be divided.[18] The solution $x=N/2$ is then deduced, provided $N/2$ is an integer.[19]

6.2.7 Approximating the Solution x with y^*

An equation system was provided for computing a pair of values j^*, y^* which, if both were integers, constituted the solution values of theorem T_{14}: i and x. The

[16] In example 8 where $\overline{y}_j$ can be written as $y(z-j)$, we can let y and j hypothetically also take any real values, i.e. y involving fractions of dollars and j involving fractions of periods. j can then be either proportional to real time, as we shall assume for S*-games, or related to real time in some other way, allowing for periods of different lengths.

[17] In most cases Newton's iterative method can be used to solve simultaneous non-linear equation systems of the type $f(x,y)=0$; $g(x,y)=0$. Such computer programs are described in Ben-Israel (1966) and SAPSYMD (1970). Furthermore, for some special cases exemplified by the S*–games, the two equations can be directly reduced to *one* equation with j as the only unknown variable. A great number of different methods are available for solving this equation. See also footnote 30, p. 116.

[18] While n (in example 8:9) is the number of *alternatives*, $N=n+1$ can be regarded as the total number of *parts* involved in the division.

[19] Since the solution relies on i=s$(x-1)$S(x), the deduction requires not only assumption set B_3 but also set B_4. See furthermore p. 54.

following question then arises: If j^* and y^* are *not* both integers, will there still exist a unique solution x, and if yes, to what extent does y^* approximate this solution value x?

We shall try to answer this question by investigating the relation between x and y^* more closely. A function INT(y) is used, which gives the truncated value of a real number, i.e. the largest *integer* that is smaller than or equal to the real number; e.g. INT(5.5)=5; INT(5.9)=5.

Next x' is defined as INT(y^*) and i' as INT(j^*). Then $x' \leqslant y^* < x'+1$ and $i' \leqslant j^* < i'+1$.

On the basis of S_2' and S_3', j^*=s(y^*-1)S(y^*) can now be shown to imply that $i'+1$=cU($x'-1$)Cu($x'+1$).[20] This conclusion allows x to be either x' or $x'+1$.[21]

Conclusion: If a compromise solution x does exist, x is such that $y^*-1 < x \leqslant y^*+1$.[22]

If a *unique compromise* solution does *not* exist, one of the following two cases can occur.

1. There exists a unique "capitulation" solution, i.e. one party is willing to accept an agreement on his worst alternative. This can happen even when $x'<1$ or $x'>n-1$.[23] If $x'<1$, we can deduce, due to S_3, that $i'+1$=Cu(1). Since $P_x \nRightarrow P$ for S′-games (see p. 109), we can with x=1, due to theorem T_{11}, deduce that $(1,n)_0=1_1$. If $x'>n-1$, we can deduce, due to S_3, that $i'+1$=cU($n-1$) and then, due to theorem T_{12}, that $(1,n)_0=n_1$.

2. L can enforce x' and H can enforce $x'+1$. $i'+1$=cU($x'-1$) implies, due to theorem T_{12}, that $(1,x')_{\overline{j-1}}=x_j'$ for every $j<i'$ and $i+1$ = Cu($x'+1$) implies, due to theorem T_{11}, that $(x'+1,n)_{j-1}=x'+1_j$ for $j<i'$. Hence H can with certainty enforce an agreement on x', while $\overline{\text{L}}$ can enforce an agreement on $x'+1$. In games with many alternatives and with x' and $x'+1$ close to each other, both parties can enforce roughly the same agreement. Although a solution cannot be deduced rigorously in this case, we can deduce with reasonable precision what the agreement will be, *if* it is reached prior to period i'. It appears justified in these cases to *hypothesize* that a fairly early agreement is reached on x' or $x'+1$.

[20] The proof is given on p. 267.

[21] x can *not* be smaller than x'. If we set $x=x'-1$, P_x would imply that i=Cu($x'-1$) or Su($x'-1$). Due to S_2, this is contrary to our conclusion that $i+1$ = cU($x'-1$). Likewise it can be proved that x can *not* be larger than $x+1$.

[22] This follows from the conclusions that $y^*+1 \geqslant x'+1, y^*-1 < x'$ and $x' \leqslant x \leqslant x'+1$.

[23] If e.g. y^* is such that the \$ 10 in our example should be divided giving \$ 8.10 to H and \$ 1.90 to L, but L can ensure \$ 3 on his own and hence n=7 (cf. p. 175), we obtain $x'=8>n-1=6$. In this case H can enforce an agreement on n=7.

To find out whether the special model leads to a unique solution, or whether it merely establishes that H can enforce x' and L $x'+1$, we can test whether x' or $x'+1$ fulfills assumptions P_x and P'_x. This can e.g. be done using the algorithmic routine in step 1 on p. 102, by letting y take the values x' and $x'+1$ and assigning different values to j. Since only two alternatives are possible, the search is easy even if *no* limit is established with regard to the periods involved in the investigation other than stopping at the last interesting period. But it appears probable that i is close to j^*. It should be noted that even if there is no unique solution according to this analysis, a unique solution might still be found by use of the general model.

6.2.8 Games without a Unique Solution

The question then arises as to which games have a unique solution and which do not.

The following is a simple example of a game *without* a unique solution. The two parties want to divide \$ 11, obtained during a large number of periods. Both parties are assumed to have zero per cent interest rate. In the continuous case where any kind of division is allowed, an agreement is reached on an equal split, i.e. on a division \$ 5.50 to each party.[24] In a discontinuous case where the agreement has to be in whole dollars, this solution is no longer possible. It can be shown that if H suggests a division of \$ 6 to L and \$ 5 to H, he can get L to agree on this and that if L suggests a division \$ 5 to L, \$ 6 to H, he can also get H to agree.[25] But we can *not* establish that either one of the parties will voluntarily suggest an agreement more favorable to the other party. We have to be content with deducing that either party can assure \$ 5 for himself. This limitation should not be regarded as a serious flaw in our model. In fact, the model would appear strange if it assigned \$ 6 to one of the parties and \$ 5 to the other party in this *completely* symmetric situation.

Although we cannot predict whether the outcome of this bargaining game is 5 or 6, we hypothesize that an agreement is likely. But the possibility of a stalemate cannot be completely ruled out.

In many cases where the parties themselves can establish the alternatives in the bargaining game, problems of this type will not be severe.[26] In the example above the parties could have agreed – prior to the bargaining game – to allow agreements involving e.g. whole half-dollars. Then \$ 5.50 would be an alternative and an agreement would be reached on this.

[24] $N=11, y^*=N/2=5.5$ (See p. 111).

[25] With $y^*=5.5, x'=5$ (See p. 112).

[26] This applies in particular to cases when the parties have equal interest rates. See the discussion on p. 176.

6.3 S*-games

6.3.1 Characteristics of S*-games

Our next step aimed at increasing the possibility of judging the applicability of the basic model, is to introduce a particular set of S′-games which we call S*-games. As shown below, S*-games are very simple to work with. They also resemble bargaining situations of practical relevance. This makes S*-games suitable for exemplifying how the basic model works. S*-games are characterized by four pay-off assumptions, $S_1^* - S_4^*$:

S_1^*: $\bar{y}_j = AyF(T) + B$ and
$\underline{y}_j = a(N-y)f(T) + b$, where

A, a, B and b are constants, and $N = n+1$, i.e. N is an integer dependent on the total number of alternatives.

$F(T)$ and $f(T)$ are continuous functions of real time, measured in years, (with continuous derivatives of the first and second order), where T denotes the time of agreement. Each period is assumed to be equally long, Δt, implying that $j\Delta t = T$.[27]

S_2^*: $F' = F'(T)$ and $f' = f'(T) < 0$, where $F'(T) = dF(T)/dT$

S_3^*: $F'' = F''(T)$ and $f'' = f''(T) \geqslant 0$, where $F''(T) = d^2F(T)/dT^2$.

S_4^*: dF^*/dT and $df^*/dT < 0$, where $F^* = -F/F'$ and $f^* = -f/f'$,
implying that $d^2(\log f(T))/dT^2$ and $d^2(\log F(T))/dT^2 < 0$.[28]

It can now be proved that $S_1^* - S_4^*$ combined are sufficient for deducing $S_1' - S_5'$ and hence that every S*-game is an S′-game and therefore also an S-game. Since the proof is fairly complicated it is given in the mathematical appendix (pp. 268 ff.).

6.3.2 Graphical Representations of S*-games

The question naturally arises as to the concrete implications of these four requirements, $S_1^* - S_4^*$.

[27] As shown in Section 6.4, t will be used to denote time in general, with T reserved for the contemplated time of agreement. Hence Δt appears more suitable for denoting the length of a period than ΔT.

[28] $df^*/dT = d(-f/f')dT < 0 \Rightarrow d(f/f')/dT > 0 \Rightarrow f(T+\Delta t)/f'(T+\Delta t) > f(T)/f'(T) \Rightarrow f'(T)/f(T) > f'(T+\Delta t)/f(T+\Delta t) \Rightarrow 0 > d(f'/f)/dT = d(d(\log f(T))/dT)/dT = d^2(\log f(T))/dT^2$

Three particular pay-off functions fulfilling these four requirements are presented in the following section (6.4). First, however, the meaning of the four requirements S_1^*–S_4^* should be exemplified. Figure 23 depicts three functions, *a*, *b*, and *c*, all fulfilling these requirements, and two functions, *d* and *e*, which do *not* fulfill these requirements.[29]

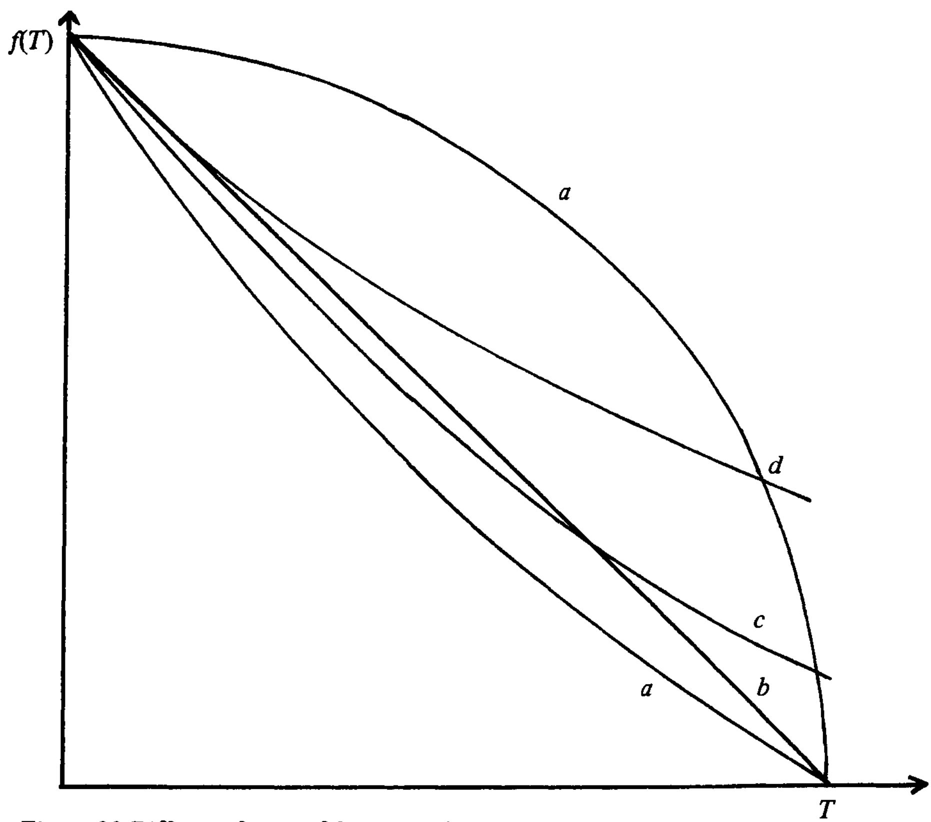

Figure 23 Different forms of function $f(T)$

6.3.3 Establishing a Solution

One great advantage of S*-games is that the equation system on p. 111 can be solved fairly easily.

[29] $f(T)$ is given by $1.59 \int_T^{10} e^{-0.1t}dt$ for function *a*; $(10-T)$ for *b*; by $10e^{-(0.1+0.01T)T}$ for *c*; by $10e^{-0.1T}$ for *d* and by $\sqrt{100-T^2}$ for *e*. That *a*, *b* and *c* fulfill the requirements is proved on pp. 270, 271 and 273 respectively. *d* does not fulfill requirements S_4^*, since $df^*/dT = 0$ (see p. 126) and *e* does not fulfill S_3^*, since $f'' < 0$. The difference between functions *c* (fulfilling S_4^*) and *d* (not fulfilling S_4^*) would be more evident if we had log $f(T)$, instead of $f(T)$ on the vertical axis. Then *d* would be a straight line, while *c* would be a parabola.

With $j=T/\Delta t$ and $j+1=(T+\Delta t)/\Delta t$, this system can now be written as

1. $AyF(T)+B=A(y+1)F(T+\Delta t)+B$

2. $a(N-y)f(T)+b=a(N-y+1)f(T+\Delta t)+b$.

Equation 1 can next be simplified into $yF(T)=(y+1)F(T+\Delta t)$, implying that $-y(F(T+\Delta t)-F(T))=F(T+\Delta t)$, i.e. that $y=-F(T+\Delta t)/(F(T+\Delta t)-F(T))$.

Likewise, $N-y =-f(T+\Delta t)/(f(T+\Delta t)-f(T))$.

The solution value of T, called T^*, can thus be determined by the equation

$$N+\frac{F(T+\Delta t)}{F(T+\Delta t)-F(t)}+\frac{f(T+\Delta t)}{f(T+\Delta t)-f(t)}=0.$$

This equation can be solved using one of the several iterative methods available as computer library subprograms.[30]

y^* is then $-F(T^*+\Delta t)/(F(T^*+\Delta t)-F(T^*))$.

For very short periods, $-F(T+\Delta t)/(F(T+\Delta t)-F(T))$ can be approximated by

$$\lim_{\Delta t\to 0} -\frac{F(T+\Delta t)}{\frac{F(T+\Delta t)-F(T)}{\Delta t}\Delta t}=-\frac{F(T)}{\frac{dF(T)}{dT}\Delta t}=-\frac{F}{F'\Delta t}=\frac{F^*}{\Delta t}.$$

In this case $-f(t+\Delta t)/(f(t+\Delta t)-f(t))$ can likewise be approximated by $f^*/\Delta t$.

T^* is then determined by $F^*(T)+f^*(T)=N\Delta t=\mu$.[31] This equation can be solved analytically for several cases.[32] Iterative methods are available and easy to use for the other cases.[33] y^* is then finally established as $F^*(T^*)/\Delta t$.

It should be noted in this connection that the solution y^* is completely independent of the constants A, a, B and b.

[30] When the first derivatives of the terms of the equation are easy to compute, the Newton-Raphson method appears most suitable. See e.g. IBM (1970, p. 119). When the first derivatives are more difficult to compute, the Wegstein method seems preferable. See Wegstein (1960) and IBM (1970, p. 116).

[31] Since $N\Delta t$ will often be used below, we shall represent this by the simple symbol μ. We note in this connection that our bargaining game has been transformed into a continuous *differential game;* see Isaacs (1965) and Intriligator (1971). The relationship between the various methods of differential game theory and our bargaining theory must be left for future research.

[32] See e.g. pp. 272, 274 and 287.

[33] The Newton-Raphson method appears suitable. See footnote 30.

6.3.4 Conditions for the Existence of a Solution

Theorem T_{15} contained three conditions which implied that the S′-game had a unique solution (p. 110).

Requirement 1, 3 = su($y-1$)SU(y), for at least same value of y, can be written as the two equations 1. $\overline{y}_3 \leqslant \overline{y+1}_4$ and 2. $\underline{y}_3 \leqslant \underline{y-1}_4$. When $T = j\Delta t$, period 3 can be written as $3\Delta t$ and period 4 as $4\Delta t$.

Applying the procedure used in the preceding section, we deduce that this implies the following:

1. $y \leqslant -F(4\Delta t)/(F(4\Delta t)-F(3\Delta t))$
2. $N+f(4\Delta t)/(f(4\Delta t)-f(3\Delta t) \leqslant y$

hold for at least some value of y.

It is then both necessary *and* sufficient that

$$N+\frac{f(4\Delta t)}{f(4\Delta t)-f(3\Delta t)} \leqslant -\frac{F(4\Delta t)}{F(4\Delta t)-F(3\Delta t)}\ ^{34}.$$

For very short periods, i.e. when $\Delta t \to 0$:

$-F(4\Delta t)/(F(4\Delta t)-F(3\Delta t)) \to F^*(0)/\Delta t$ so that $N-f^*(0)/\Delta t \leqslant F^*(0)/\Delta t$ is obtained, i.e. $f^*(0)+F^*(0) \geqslant N\Delta t = \mu$.

Requirement 2, there exists a period j within finite time such that j=C(1)c($n-1$), is easy to analyse. j=C(1) is for S*-games written as: There is a period T such that $AF(T)+B>A2F(T+\Delta t)+B$, i.e. such that $F(T)>2F(T+\Delta t)$. Likewise j=c($n-1$) is fulfilled by $f(T)>2f(T+\Delta t)$.

Requirement 3, P_x and P'_x hold, can be simplified only in the continuous case, i.e. if we let Δt approach 0. Then P_x and P'_x hold, if there is a pair of values y and T, called y^* and T^*, such that $y=F^*(T)/\Delta t$ and $N-y=f^*(T)/\Delta t$ (cf. p. 116). Since $F(T)$ is defined as a continuous function of a real number T, with a continuous first derivative, requirements 1 and 2 above, which imply that T^* will assume a positive, finite value, are then sufficient for establishing this requirement.

[34] That the requirement is necessary follows directly. That it is sufficient follows from the conclusion that we can set

$$y = -\frac{F(4\Delta t)}{F(4\Delta t)-F(3\Delta t)} = N + \frac{f(4\Delta t)}{f(4\Delta t)-f(3\Delta t)}$$ and make the equation system hold.

6.4 Three Examples of S*-games

6.4.1 Introduction

Three examples of pay-off functions that fulfill the four requirements $S_1^*-S_4^*$ will now be studied in order to concretize the meaning of the requirements behind the S*-games. These pay-off functions also exemplify how the outcomes of various bargaining games depend on different parameters. Some special cases of these pay-off functions, where analytical solutions are fairly easy to deduce, will also be studied. Finally, these three specific pay-off functions will be helpful in investigating particular bargaining situations in order to ascertain the extent to which our model is applicable (Chapter 9).

It should be stressed that these functions constitute only three of many examples of possible pay-off functions which would fulfill $S_1^*-S_4^*$. The three functions presented in this chapter have been chosen with the following requirements in mind:

1. The pay-off functions should be representative of other pay-off functions. The functions below appear to contain characteristics common to several other possible pay-off functions.

2. The pay-off functions should be of interest as regards the application of either these functions directly, or of similar functions, to practical problems.

3. It should be fairly simple to investigate whether or not the functions fulfill $S_1^*-S_4^*$ and to find the solution of the game.

6.4.2 Pay-off Function 1

6.4.2.1 Presentation of the Function

Pay-off function 1 implies that the pay-off of an agreement consists of:

An *agreement* profit component

plus a *pre-agreement* profit component
minus a *bargaining cost* component
minus an *investment* component
plus a *salvage value* component.

These five components will be commented on separately. We limit ourselves to presenting L's pay-off function. H's pay-off function is similar, the only difference being that all variables are assigned capital letters.

1. The *agreement profit* component consists of the present value of obtaining a certain share of a sum in *each* period from the time of agreement T up to the time of the expiration of the contract Z. It has the following form:

$$s\,\pi \int_T^Z e^{-rt}dt, \text{ where}$$

π – a *constant* – is the total profit to be divided each year;

s – the bargaining variable – is the share of this annual profit which the agreement will give to party L. Thus $s\pi$ is the profit, computed at an annual rate, that L receives from an agreement on s;

e^{-rt} is L's present value – i.e. his value at the start of the bargaining game in period 0 – of \$ 1.00 paid out at time t and discounted by a constant annual interest rate of 100 r per cent;[35]

$\int_T^Z e^{-rt}dt$ is the *sum* of L's present values of an income flow, amounting to \$ 1.00 annually, accumulated from the time of agreement T up to the time of the expiration of the contract, called Z[36] and

$s\pi \int_T^Z e^{-rt}dt$, finally, is the *sum* of the present values of obtaining $s\pi$ annually from the time of agreement T up to Z, when the contract expires.

The first thing to be noted in this context is that the total time during which the agreement is in effect decreases in *exact proportion* to the time that an agreement is delayed.

It should also be noted that we assume payments are made and interests computed *continuously*. This implies that all periods are in principle assumed to be infinitely short. This is not true in reality (cf. p. 151). But it is reasonable to assume that the continuous function provides a fairly good approximation of the function that would have been obtained in the discontinuous case. The smaller the real period length Δt, the better the approximation. As Δt approaches 0, the two functions coincide. Even if Δt is relatively large, i.e. there are few periods each year, the continuous case appears to approximate the discontinuous case with insignificant errors.[37]

[35] e is the base of the natural logarithmic system.

[36] $Z = z\Delta t$ or $(z+1)\Delta t$ (see p. 273).

[37] If the real periodic interest rate is $r\Delta t$, and the periodic payment $\pi\Delta t$, then discontinuous payments and compounding lead to an agreement pay-off component

$$s\pi\Delta t \sum_{j=T/\Delta t}^{Z/\Delta t} (1+r\Delta t)^{-j}.$$ For e.g. r=0.1, Δt=0.1, π=1, T=1 and Z=5,

we obtain $10\,s(1.01^{-10}-1.01^{-50})=2.973s$ to be compared with the continuous case leading to $s\int_1^5 e^{-0.1t}dt=2.983s$. The difference set in relation to either value is less than 0.004.

2. The *pre-agreement profit* component, which is added, is the sum of the present values of all pre-agreement profits obtained from the start of the bargaining game at time 0 up to the time of agreement T. This component can be written as $w\int_0^T e^{-rt}dt$, where w is the profit paid annually to party L prior to an agreement, i.e. paid provided that *no* agreement has been reached in this period.

3. The *bargaining cost* component, which is subtracted, is the sum of the values of all annual costs of continuing the bargaining from the start of the game at time 0 up to the time of agreement. This component is written as $c\int_0^T e^{-rt}dt$, where c is L's cost of continuing the bargaining during one year.

4. The *investment* component, which is subtracted, is the present value of the cost of making an investment at the time of agreement T. It is written as $k_T e^{-rT}$, where k_T is the cost of investment evaluated at the time of agreement T.[38]

5. The *salvage value* component, which is subtracted, is the present value of obtaining a salvage value at time Z when the contract expires. It is written as $k_Z e^{-rZ}$, where k_Z is the salvage value evaluated at time Z. For the sake of simplicity it is assumed that k_Z is *not* affected by use, i.e. that it is independent of when an agreement is reached.

Combining the pre-agreement and bargaining cost components, the total pay-off function 1 can be written as:

$$s\pi\int_T^Z e^{-rt}dt+(w-c)\int_0^T e^{-rt}dt-k_T e^{-rT}+k_Z e^{-rZ}$$

We can now prove that pay-off function 1 fulfills all four assumptions S_1^*–S_4^*. Since the proof is somewhat complicated, it is given in the mathematical appendix (pp. 270 ff.).

6.4.2.2 General Conclusions Regarding the Outcome

The next question has to do with what kind of agreement will be reached in a game characterized by pay-off function 1. First, some general conclusions are drawn which are independent of the exact relationship between H's annual rate of interest

[38] This does *not* mean that the investment is necessarily made exactly at the time of the agreement. The investment can be made at any time (or times, as k_T can be the discounted value of several smaller investments) after the agreement. The only requirement is that the interval of time between the time of investment and the time of agreement T is the same for every value of T.

R and L's annual rate of interest r. The case when $r = R$ is then analyzed in 6.4.2.3 and the case when $r = 2R$ in 6.4.2.4.

The following can be proved, *ceteris paribus*[39], with respect to pay-off function 1:[40]

1. The higher a party's rate of time discount (r or R), the smaller his share of the joint profit, provided there are no investment costs (k_T or K_T).

2. The larger a party's pre-agreement profit (w or W), the larger his share.

3. The higher a party's bargaining cost (c or C), the smaller his share.

4. The higher a party's interest costs for his *investment* (rk_T or RK_T), the larger his share.

Conclusion 1 can be given the following verbal interpretation: If the party's rate of interest is high the agreement profits received in the immediately subsequent periods will be important in relation to the agreement profits in later periods. Then the party will be more anxious to reach an early agreement, even on less favorable terms than he might have received at a later agreement.

Conclusion 2 can be explained as follows. A party will feel less eager to obtain an early agreement when the difference between agreement and no-agreement in a period is small than when it is large.

Conclusion 3 can be explained in a similar manner. The higher a party's bargaining costs, the more he experiences the *disadvantage* of having the agreement delayed. This means he is more pressed to reach an early settlement.

Conclusion 4 is explained as follows: As shown in Chapters 3 and 5, the solution of an S-game is *inter alia* determined by comparing the pay-off from an agreement in a certain period j and an agreement one period later, i.e. in $j+1$. If an agreement is reached in period j instead of $j+1$, the investment expenditure is made one period earlier and the party will pay *interest* on the investment during one extra period. The higher the interest cost of the investment, the more the party saves by *postponing* the agreement one period and the less he is pressed to reach an early settlement.

It should be stressed that in a situation *with* an investment component, conclusions 1 and 4 above lead to *conflicting* influences with regard to the effect of e.g. R on the share S^* that the agreement gives H. In such a situation, it can *not* generally be said whether this share will increase or decrease, when R increases.[41]

[39] If e.g. we let r vary, the values of all other parameters (R, w, W, c or C) are assumed *given*.
[40] The proofs of the conclusions are given on p. 272.
[41] A numerical example of these two conflicting effects is given on p. 123.

6.4.2.3 Solution when $r = R$

Next we look more carefully at the case when the parties have equal interest rates, i.e. where $r = R$. The following solution is deduced: H's share of the annual profit $S^*\pi$ is $(\pi+(W-w)-(C-c)+R(K_T-k_T))/2$, where π is the annual profit, W and w are H's and L's annual pre-agreement profits, C and c their respective annual bargaining costs and K_T and k_T their investment costs.[42] This means that the three general conclusions 2–4 above hold, i.e. that S^* increases when W and RK_T increase and C decreases.

When the parties have the same periodic pre-agreement profits, the same bargaining costs and the same investment costs, i.e. when the situation is *completely* symmetric, the parties will divide the annual profit into two equally large shares. This appears very natural. It is indeed difficult to understand why the parties should obtain unequal shares in a *completely* symmetrical situation.

The determination of S^* when $r = R$ can be illustrated by a numerical example:

Let us assume that each year during a number of years [43] two parties shall divide \$ 100,000 of a joint project; that H can assure himself of \$ 20,000 annually if no agreement is reached and that L can assure himself of \$ 10,000. Furthermore H is assumed to have a bargaining cost of \$ 1,000 computed on an annual basis, while L has an equivalent cost of \$ 5,000. The joint project will require an investment of \$ 150,000 from H and \$ 50,000 from L. Both parties have an annual interest rate of 10 per cent.

With every amount in thousands of dollars, $W = 20$, $w = 10$, $C = 1$, $c = 5$, $\pi = 100$, $R = 0.1$, $K_T = 150$ and $k_T = 50$, $S^*\pi = (100+(20-10)-(1-5)+0.1(150-50))/2=62$, i.e. H will obtain \$ 62,000 and L \$ 38,000.

If both parties instead have 0 per cent interest rates or no investment costs, i.e. if $rk_T = RK_T = 0$, $S^*\pi = (100+10+4)/2=57$ and H will obtain only \$ 57,000. This difference is due to the fact that when $r = R = 0$, H will *not* make a larger interest saving than L by *delaying* the investment one period.

6.4.2.4 Solutions when $r = 2R$

Although it is generally difficult to determine a solution analytically for the case where $r \neq R$, an analytical solution can easily be found, when $r = 2R$. We then

[42] This is deduced on p. 272.

[43] In the mathematical appendix we prove that $\int_0^Z (e^{-rt}+e^{-Rt})dt > \mu$ is sufficient in order to ensure the existence of a solution for pay-off function 1. When $r \leqslant 0.2$, $R \leqslant 0.1$ and $\mu \leqslant 2$, it is quite sufficient to assume that $Z \geqslant 2$ (cf. p. 272).

deduce the following solution in the continuous case, i.e. when Δt approaches 0:[44] H's share of the annual profit $S^*\pi$ is

$((1-\sqrt{1-\alpha})/\alpha)(\pi-W+C-Rk_T-w+c-rk_T)+W-C+RK_T$, where $\alpha=R\mu/2$.[45]

As mentioned earlier $\mu=N\Delta t=(n+1)\Delta t$, where n is the total number of alternatives and Δt is the length of each period measured in years. Therefore the solution depends not only on the relationship between r and R, c and C and k_T and K_T, but *also* on the number of alternatives n and the length of each period Δt.

This does not cause any additional problems in our basic model where the total number of alternatives and periods are assumed *given* prior to the start of the bargaining game. But this does prove to be somewhat problematic when the institutional assumptions are further developed in Chapter 8 and the parties in certain cases are allowed to determine the length of the periods and the number of alternatives. However, the *highest* share party L can hope to get can be established (see p. 178). For many games this share appears likely to be agreed upon.

The determination of $S^*\pi$ when $r=2R$ can be illustrated by the following numerical example: Let us again study the division of \$ 100,000 obtained annually during a number of years.[46] Let us assume that $\mu=2$[47]. Furthermore assuming that $R=0.1$ and $r=0.2$, $\alpha=0.1\cdot 2/2=0.1$. We retain our assumptions from the first example (cf. p. 122), i.e. that $W = 20$, $C = 1$, $K_T = 150$, $w = 10$, $c = 5$ and $k_T = 50$, all measured in thousand of dollars.

Then $S^*\pi$ is $10(1-\sqrt{1-0.1})\cdot(100-20+1-15-10+5-10)+20-1+15=0.513\cdot 51+34=60.172$. In this case H will obtain around \$ 60,000 and L around \$ 40,000.[48]

We note that L obtains *more*, if his annual rate of interest increases from 10 per cent (in which case he obtains \$ 38,000, as noted on p. 122) to 20 per cent. This is solely due to the large investment components. If the parties' investments k_T and K_T were both 0, L obtains around \$ 42,000, when $r=0.2$ and $R=0.1$.[49] This should

[44] The proof is given in the mathematical appendix (p. 272).

[45] α can more generally be written as $(r-R)\mu/2$ (see p. 127). In this case with $r=2R$, $r-R=R$.

[46] See footnote 43.

[47] $\mu=2$ can be obtained in several ways, e.g. the following:

a. Only whole thousands of dollars or percentages can be involved in the agreement, implying that $N = 100$. The bargaining parties are corporations with a board of directors, deciding on the bargaining bids. Every week one of these boards meets, implying that $\Delta t = 0.02$.

b. Only whole hundreds of dollars or per mills can be involved in the agreement, implying that $N = 1000$. Each party can deliver one bid every week-day, implying that there are two periods each week-day, i.e. that $\Delta t=0.002$.

[48] In accordance with the conclusions on p. 112, when e.g. $N = 1000$ (cf. case b in footnote 47), L can ensure himself of \$ 39,800 and H can ensure himself of \$ 60,100.

[49] $s\pi=100-S^*\pi=100-(0.513\cdot 76+19)=42.004$

be compared to the case with no investments and $r=R=0.1$, giving L \$ 43,000 (cf. p. 122). If *no* investments are involved, L obtains a *smaller* share when he has a higher rate of interest.

6.4.3 Pay-off Function 2

In the preceding section we noted that the agreement pay-off for pay-off function 1 was only received up to a time Z, when the contract expires. Every delay in reaching an agreement resulted in a *corresponding* reduction in the total number of periods under which the agreement pay-off was obtained. We will now show two pay-off functions in which the agreement profit can still be obtained after a fixed date, when the contract expires.

In pay-off function 2, the delay in the expiration of the contract is *smaller* than the delay in reaching the agreement. Just as in the case of pay-off function 1 the agreement will start at the agreement time T, but – in contrast to function 1 – it will not run only to a fixed time Z, but instead to a possibly variable time $Z+\theta\ T$, where $1>\theta\geqslant 0$.[50]

The agreement will run to a fixed time-point only if $\theta = 0$ and only then will any delay of the agreement imply a corresponding reduction in the contract time. In the case of $1>\theta>0$, a delay in reaching the agreement will cause a reduction in the contract time that is smaller than the delay. It should be stressed that we exclude the case where $\theta=1$, when a delay in reaching a contract will *not* affect the total period of time during which the contract will run. This case will be covered by pay-off function 3.

Since the analysis of this function is already fairly complicated with regard to the agreement pay-off component and the addition of other components would not yield very much extra insight compared to pay-off function 1, we shall only include an agreement pay-off component.

In this case the annual profit π can also be allowed to increase over time. First of all we can allow it to increase at a rate of 100 β per cent each year, until an agreement has been reached at time T.

We also allow for another growth rate, taking into account the growth which takes

[50] In general it seems difficult to establish the value of θ, if it is not 1 or 0. However, in many cases when an agreement is delayed – e.g. owing to a strike – the parties are increasingly prone to settle for a *shorter* contract just to get the conflict over with. If θ is interpreted as the *expected* prolongment of the contract, the assumption of the contract running between T and $Z+\theta T$ might be a rough *approximation* of reality in several instances.

place *after* the agreement at time T. The profit π is assumed to increase annually by 100 γ per cent *between* time T and time t. We assume that $r > \gamma$ and that $r > \beta$.

Hence for a time t, *after* the agreement time T, L's profit – computed at an annual rate – is $s\pi e^{\beta T} e^{\gamma(t-T)}$[51]. Since this expression can also be written as $s\pi e^{\beta t} e^{(\gamma-\beta)(t-T)}$, we can alternatively distinguish between a *normal* growth rate β holding for the entire time and an *extra* growth rate $\gamma-\beta$ holding only for the time *after* the agreement time T. For example, the normal growth rate β can be due to market expansion, while the extra growth rate $\gamma-\beta$ can be dependent on the rate of market penetration or learning *after* the agreement has been reached on e.g. the establishment of a joint venture or the launching of a new product.

With the agreement running from T to $Z+\theta T$, the sum of the present values of all these profits can be written as $s\pi e^{\beta T} e^{-\gamma T} \int_T^{Z+\theta T} e^{\gamma t} e^{-rt} dt$.

It can now be proved that for every $0 \leqslant \theta < 1$, pay-off function 2 as stated above will fulfill $S_1^*-S_4^*$.[52]

It seems most suitable to use an iterative method for computing the solution. For the specific case where $\gamma=\beta$ and $\theta=0$ the deductions for pay-off function 1 can be used, substituting $r-\gamma$ for r and $R-\gamma$ for R.

The following case can then be analyzed:

1. $r=0.25, R=0.15$.

2. The parties divide a periodic profit, initially \$ 100,000, which increases by 5 per cent each year (i.e. $\beta=\gamma=0.05$).

3. $\mu=2$ (cf. footnote 47 on p. 123).

4. There are no pre-agreement profits, bargaining costs or investments.

We can deduce that $S^*=0.513$[53], implying that H will obtain around 51 per cent of the joint profits.[54]

[51] There is no indication that exponential growth rates are *necessary* for the applicability of the special model. We could probably assume many different types of functions for the variation of profit over time. There are two main reasons for using exponential growth functions:

a. They appear to constitute a fairly good approximation of how profits vary over time in many cases,
b. They are very convenient to manipulate, especially when used in combination with a continuous method of discounting.

[52] The proof is given on p. 273 for the case when $r > 0$.

[53] When $r-\beta=0.2$, $R-\beta=0.1$ and W,w,C,c,k_T and $K_T=0$, we can deduce that $S^*=0.513$ (see p. 123).

[54] Cf. footnote 48 on p. 123.

6.4.4 Pay-off Function 3

6.4.4.1 Presentation of the Function

Just as in the case of pay-off function 2 there is only one component – the agreement profit component – and the last period of the contract is a variable. In contrast to pay-off function 2, the delay in reaching an agreement is assumed to lead to a *corresponding* delay in the last date of the contract. Hence the contract in pay-off function 3 runs over a constant amount of time, regardless of *when* an agreement has been reached. This is equivalent to assuming that $\theta = 1$ in pay-off function 2. This means that the last period of the contract can be written as $T+Z$. Allowing, as in pay-off function 2, the joint periodic profit to increase annually by 100 β per cent *prior* to the agreement at time T and by 100 γ per cent *after* this agreement, L's pay-off function can then be written as $s\pi \int_T^{Z+T} e^{\beta T} e^{\gamma(t-T)} e^{-rt} dt$.

Provided γ and r are constants, this expression can also be written as $s\pi e^{-(r-\beta)T} \int_0^Z e^{-(r-\gamma)t} dt = s\pi k e^{-(r-\beta)T}$, where $k = \int_0^Z e^{-(r-\gamma)t} dt$, a constant[55]. Likewise H's pay-off function is written as $S\pi K e^{-(R-\beta)T}$.

It should be noted that the same expression for the pay-off functions is obtained in the following two cases:

1. A one-shot payment $ke^{\beta T}$ is obtained at the agreement time T.

2. The contract runs infinitely far into the future. We then obtain L's pay-off as

$$s\pi \int_T^{\infty} e^{\beta T} e^{\gamma(t-T)} e^{-rt} dt = s\pi e^{-(r-\beta)T} \int_0^{\infty} e^{-(r-\gamma)t} dt = s\pi k e^{-(r-\beta)T}.$$

The first thing we notice when starting to analyze this pay-off function is that no solution can be established if β is also a constant.[56] But in many cases it appears natural to assume that β varies over time; in particular that β decreases over time, implying that the *increase* in the annual profit *decreases* as the agreement is delayed. The simplest form of β fulfilling this requirement is that β *decreases* at a constant rate from the growth rate β_0 at time 0.[57] $\beta = \beta(T)$ is then written as $\beta_0 - \beta' T$, where β_0 and β' are constants.

[55] $\int_T^{Z+T} e^{\beta T} e^{\gamma(t-T)} e^{-rt} dt = e^{\beta T} e^{-\gamma T} \int_T^{Z+T} e^{\gamma t} e^{-rt} dt = e^{\beta T} e^{-\gamma T} e^{\gamma T} e^{-rT} \int_0^Z e^{-(r-\gamma)t} dt =$
$e^{-(r-\beta)T} \int_0^Z e^{-(r-\gamma)t} dt.$

[56] Then $f^* = -f/f' = 1/(r-\beta)$ and $df^*/dT = 0$, i.e. S_4^* is violated.

[57] As shown on p.274 in the mathematical appendix, other – more complicated – forms for the variation of β will also fulfill $S_1^* - S_4^*$.

It can now be proved that $S_1^*-S_4^*$ hold, provided $\beta'>0$ and that both $(r-\beta_0)^2$ and $(R-\beta_0)^2 \geqslant 2\beta'$ (cf. the mathematical appendix, p. 273).

6.4.4.2 *Establishing a Solution*

It can now be deduced that

$S^* = 1/(1-\alpha+\sqrt{\alpha^2+1})$, where $\alpha=(r-R)\mu/2$.[58]

We note that S^* is completely independent of the *size* of γ, β_0 and β'. This does not, however, allow us to set $\beta'=0$, since we require that $\beta'>0$ in order for S_4^* to hold (cf. p. 274). We furthermore note that $\partial S^*/\partial R<0$ and that $\partial S^*/\partial\mu>0$, when $r>R$.

As for the existence of a solution, it is sufficient to furthermore assume that $1/(r-\beta_0)+1/(R-\beta_0) \geqslant \mu$ (cf. p. 000).

6.4.4.3 *A Numerical Example*

A numerical example will illustrate the effect of our model in terms of pay-off function 3. Just as in the example on p. 125 let us assume that $r=0.25$, $R=0.15$ and $\beta_0=0.05$ and that $\mu=2$. We then obtain $S^*=0.525$.[59]

We notice that the effect of R being smaller than r in this example without a fixed last date for the expiration of the contract is stronger than in the example on p. 125 where the end of the contract is fixed. However, an investigation of the extent to which this conclusion holds and of the reasons for this must be left to future research.

[58] See p. 275 in the appendix.

[59] When $R-r = 0.10$ and $\mu = 2$: $\alpha = 0.1 \cdot 2/2 = 0.1$. Since $1/(r-\beta_0)+1/(R-\beta_0) =$ $= 1/0.20+1/0.10 = 15>\mu = 2$ and $\beta' > 0$, the existence of a solution is assured in the continuous case.

Chapter 7

Behavioristic Assumptions

7.1 Introduction

7.1.1 Introductory Remarks about Chapters 7 and 8

Having established a basic model for the analysis of many different bargaining situations in the preceding four chapters, the assumptions behind this model will now be analyzed in this and the next chapter. The special pay-off assumptions (S_1–S_2, S'_1–S'_5 and S_1^* – S_4^*) and the particular pay-off assumptions (of sets $\boldsymbol{P}$ and $\boldsymbol{P}'$) have already been described rather extensively and will be referred to again in connection with application of the bargaining models in Chapter 9. No particular discussion of the general pay-off assumptions G_1–G_3 seems necessary. G_1 is closely related to B_{10} and does not require any individual attention. G_2 and G_3 are so weak that no further discussion seems required. Thus our analysis in the following two chapters is devoted to the behavioristic assumptions of $\boldsymbol{B}_4$ (to be discussed in this chapter) and the institutional assumptions I_1–I_{13} (to be discussed in Chapter 8).

Besides dealing with the validity of these assumptions, we shall investigate whether alternative and possibly less restrictive assumptions could be substituted for these assumptions of the basic model without significantly affecting the conclusions of this model[1].

7.1.2 Outline of Chapter 7

This chapter deals with the behavioristic assumptions used in the basic model, i.e. those of set $\boldsymbol{B}_4$. The assumptions generally used in microeconomic theory and referring to only one party's isolated behavior, i.e. the assumptions of set $\boldsymbol{B}_1$, are treated in 7.2. Section 7.3 deals with assumptions B_8 and B_9 (combined with $\boldsymbol{B}_1$ to form set $\boldsymbol{B}_2$) concerning each party's expectations about the other party's *behavior*[2]. Section 7.4 deals with B_{10} (combined with $\boldsymbol{B}_2$ to form $\boldsymbol{B}_3$) concerning

[1] Assumptions are substituted for each assumption separately, with all the other assumptions remaining in force. The substitution of combinations of assumptions has to await future research.

[2] As defined by $\boldsymbol{B}_1$.

each party's expectations about the other party's *expectations*. The remaining assumptions of set B_4 are discussed in 7.5. In the final section 7.6 we sum up our discussions of the various assumptions and relate them to the different purposes for studying bargaining discussed in Chapter 2.

7.2 Assumptions of Rationality in Microeconomic Theory

The assumptions of set B_1 are fundamental not only to our model but to virtually all other microeconomic models. The validity of these assumptions has therefore been the subject of a great deal of debate in economic literature[3]. We do not intend to replicate this debate, but only very briefly summarize *some* of the main points of the criticism directed against these assumptions.

B_1 and B_2 – the assumptions of complete preference ordering, are criticized on the following grounds: People are often unable or unwilling to state or explicitly contemplate preferences over two widely different outcomes. Outcomes have to be measured in several dimensions[4].

B_3 – the assumption of continuity, is criticized on the grounds that some people, such as drug addicts, have an extreme preference for certain outcomes.

B_4 – the assumption of transitivity, is criticized on the grounds that transitivity is often violated, particularly when decisions are made jointly by several persons, e.g. through majority rule.

B_5 – the assumption of optimization, is confronted by the principle of satisficing. This principle in its simplest form implies the following kind of search procedure: One alternative is investigated at a time; an alternative is accepted if it is regarded as satisfactory; otherwise the search continues until a satisfactory alternative is found. The satisficing principle is often defended with reference to high search costs. The assumption of optimizing is also confronted by "organizational slack", implying that possibilities sometimes exist for improving the result when pressure is applied.

[3] See e.g. Hall & Hitch (1939), Lester (1946), Machlup (1946), Simon (1947), Alchian (1950), Arrow (1951), Friedman (1953), Earley (1956), Margolis (1958) and Cyert & March (1963).

[4] It should be stressed in this context that assumption B_1 in principle allows for virtually *any* kind of preference relations. The only restriction is due to B_3, which rules out lexiographic utility orderings, implying that a person prefers a small quantity of some good to large quantities of any other good. Hence profit maximization is *not* required. On the other hand, the more complex the preference relations of B_1 are, the less likely it is that B_9 holds. This is discussed further on pp. 132 ff.

B_6 – the assumption of full information utilization, is criticized with reference to the high costs of processing available information[5].

B_7 – the assumption of deductive capacity, is confronted by the extensive use of very simple rules of thumb in actual decision-making as well as empirical evidence indicating fairly low average knowledge of e.g. mathematics among decision-makers.

The extensive debate in the literature makes a *general* discussion of the relevance of this criticism superfluous[6]. Instead, we should direct our attention to the *specific* relevance of these assumptions to bargaining situations. The following hypotheses then appear reasonable:

1. The fewer the persons involved in the bargaining and the more central their position in the organizations on whose behalf the bargaining is carried out, the more probable it is that the decisions concerning bargaining strategy are made in accordance with principles of logic. There would *ceteris paribus* probably also be fewer intransitivities and greater possibilities for taking different outcome factors into account simultaneously, improving the possibility of measuring the outcome in *one* dimension.

2. A party's behavior in a bargaining situation depends to a large extent on the party's (i.e. the bargaining agent's) intelligence, memory, formal schooling, experience from earlier bargaining situations, etc. The more intelligent and educated the party, the more likely he is to act in accordance with the assumptions of B_1.

3. The ability to make complicated deductions and computations and to utilize available information is highly dependent on the availability of cheap computer capacity. The cheaper computers become in terms of capacity, the more probable it is that bargaining decisions will be characterized by rationality. The trend in this respect is strongly in favor of rationality.

4. The higher the monetary rewards for "good" decisions, the higher the relevance of rationality assumptions. Several studies have indicated that parties in an experimental situation acted much more in accordance with rationality assumptions when there were monetary prizes than when the only prize was the honor of

[5] The B_1-assumptions cannot be criticized on the grounds that they fail to incorporate considerations of risk and uncertainty. The assumptions of B_1 refer only to how the parties act in situations characterized by institutional assumptions I_3 and I_5, implying that each party has complete information. Instead, criticism should perhaps be directed towards the use of assumptions I_3 and I_5 in a specific bargaining situation.

[6] Much of this criticism has been directed towards application of assumptions B_1–B_7 to the theory of consumer behavior.

winning[7]. High rewards for good decisions are also more likely to make the extensive use of computers economically justifiable. High monetary rewards will also motivate considerable search activity. Hence, if the bargaining concerns large amounts in relation to potential search and computation costs, an important reason for satisficing instead of optimizing disappears.

5. The longer the time available for making a decision, the greater the number of decision alternatives which can be investigated and the greater the care that can be taken in analyzing the consequences of these alternatives. Lack of time pressure will allow the decision-maker to specify the purpose of his decisions more carefully, which e.g. should counteract intransitive choices.

6. The more closely the real information situation approximates that of complete information, the smaller the cost of the information search. Hence I_3 and I_5 imply that one reason for satisficing instead of optimizing decreases in importance.

On the basis of these hypotheses we believe that the assumptions of economic rationality are more justified for bargaining situations in which the following conditions hold, *ceteris paribus*.

The bargaining
1. concerns large amounts of money;
2. can take place over a long span of time;
3. involves parties with intelligent, well-educated and skilled personnel and with computing capacity;
4. involves the bargaining organization's top management with clear preferences as regards possible outcomes of the bargaining;
5. takes place closer to the year 2000 than to 1972.

7.3 Assumptions Concerning Expectations about the Other Party's Behavior

7.3.1 Introduction

The following two assumptions will now be discussed:

B_8 *Mutual knowledge of the other party's rationality*: H knows that L is rational (i.e. behaves according to assumption set $\boldsymbol{B}_1$) and L knows that H is rational.

[7] See e.g. Siegel (1964).

B_9 *Information about preference relations*: Full information concerning the other party's preference relations, established according to B_1, is available to each party.

These two assumptions are fundamental to any kind of interesting bargaining theory based on rational behavior. Later on it is proved that they are necessary for the solution of the efficiency problem (see Section 8.4). Hence all bargaining theories which involve some kind of rational behavior and which regard the *efficiency* problem as solved in the sense that the contract will be Pareto-optimal are at least implicitly based on B_8 and B_9.[8]

7.3.2 Assumption B_8

If the extent to which a party behaves according to $\boldsymbol{B}_1$ is a function of the various factors discussed in 7.2 such as degree of centralization, computer capacity, intelligence, etc., then the validity of B_8 hinges mainly on whether or not these factors in reference to one party are known to the other party. This means that the validity of assumption B_8 depends on whether or not certain institutional factors such as the organizational structure or the computational capacity of the other party are known.

Furthermore, a party – e.g. H – can determine whether the other party – L – is rational or not by looking at L's decisions and actions in other areas. Since $\boldsymbol{B}_1$ covers a great many decision areas, H can analyze many of L's decisions in order to find out whether or not L acts according to $\boldsymbol{B}_1$.

Conclusion: It appears reasonable to assume that there are many circumstances when a party's rationality can be recognized by the other party and that B_8 holds, provided $\boldsymbol{B}_1$ holds.

7.3.3 Assumption B_9

7.3.3.1 Introduction

The validity of assumption B_9, implying e.g. that H knows L's preference relations over various outcomes factors,[9] is highly dependent on the nature of the parties' preference relations. We distinguish mainly between the following two cases[10]:

[8] See p. 173.
[9] See footnote 3, p. 31.
[10] There are obviously other possible cases. For example, a party's goal might be the maximization of some index, determined in turn by several non-prestige factors such as profits and sales. These cases, however, seem to be of less interest.

1. The preferences are such that B_1 implies that the parties have a maximization goal, i.e. they seek to *maximize* one outcome factor such as profits or sales, possibly under various restrictions.

2. The preferences are such that the parties also take other factors into account which are difficult to measure objectively, mainly those factors commonly referred to as *prestige* factors.

Case 2 can be exemplified by a party who prefers obtaining an agreement on his terms, even if reached after a long strike, to an early agreement on a compromise giving him a higher total profit, just because he wants to "win" the bargaining game. Likewise a party in a two-alternative game might *not* accept the other party's proposal, even though it would give him a profit larger than the one he could possibly obtain by insisting on his own alternative. Hence a period that would be critical if the pay-offs were influenced only by outcome factors such as profits and sales might become uncritical if prestige factors were introduced.

These prestige factors can be related to either the *outcome*, i.e. the terms and time of agreement, or the *play*, i.e. the actual bargaining process by which an agreement is reached. If prestige is dependent on the play, then G_1 will not hold (see p. 67). An example is that a party might regard it as "humiliating" to make *one* big concession instead of several small concessions.

We can also distinguish between indirect and direct prestige factors. The term *indirect* prestige factors refers to cases when prestige is regarded only as the *means* to an end, namely to obtain a higher value of some outcome factor such as profits in *other* negotiations (either later negotiations with the same party or negotiations with some other party). In other words, if H makes large concessions in his present negotiation with L, he might be expected to make large concessions in the other negotiations. Direct prestige factors on the other hand, refer to cases when prestige is regarded as an important *ultimate* goal.

Against the background of these distinctions, the validity of B_9 will be discussed first for maximization goals (7.3.3.2), then for indirect prestige factors (7.3.3.3) and finally for direct prestige factors (7.3.3.4).

7.3.3.2 Maximization Goals

If we adhere to traditional microeconomic theory and assume e.g. that the two parties are firms seeking to maximize their profits, then B_9 implies that each party recognizes that the other party is a profit-maximizing firm. The problem is not augmented to any great extent if we assume that one party is a labor union seeking to maximize the total amount of wages obtained by its members.

We next proceed to cases of revised economic theory involving suggestions of other maximization goals for a firm such as the maximization of sales or market share under the restriction that profits should not fall below a certain level. We can very well allow for such goals without having to make any significant changes in our basic model. B_9 then implies that it be known that the other party is a sales maximizer. This might e.g. be evident from his earlier actions. If the parties agree to solve the efficiency problem prior to the actual bargaining game, this solution will probably make it evident whether a specific party is a profit or a sales maximizer. The quantity at which the joint profit is maximized will generally differ from the quantity at which the joint sales are maximized, under restrictions of minimum profits. If these quantities are equal it might very well be that, with respect to solving the distribution problem, it will not matter whether a party believes that the other party is a sales or a profit maximizer[11].

7.3.3.3 Indirect Prestige Factors

The first question to be dealt with here is whether it is realistic to assume that a party – e.g. H – is able to make a rough estimate of the size of the indirect prestige factors which influence his opponent's – L's – preferences. Factors of importance in this case are whether H knows which future negotiations L is likely to enter into and the pay-off functions of the different parties in these future negotiations.

A special case in this instance is one where indirect prestige factors refer only to the parties' future negotiations with each other. This kind of situation can be handled by applying our bargaining model to a consecutive *series* of negotiations. This is exemplified on p. 205.

It is much more difficult to evaluate indirect prestige factors with respect to their influence on negotiations with *other* parties. The evaluation problem could be solved in principle by applying the bargaining model to these other negotations. But this would *inter alia* require the parties to know the pay-off functions involved in all of these other negotiations. Another requirement would be that B_9 and I_5 also concern parties other than H and L. It suffices for our purposes here to note

[11] If the parties have the same time preference for sales as for profits, L's pay-off function in the sales maximization case can sometimes be written as $sq \int_T^Z e^{-rt}dt$, where q is the maximum joint sales, and in the profit maximization case as $s\pi \int_T^Z e^{-rt}dt$. If H's functions are similar, the pay-offs will only differ in terms of a constant. The size of this constant will not affect the solution (cf. p. 184).

that the influence from future negotiations with other parties is very small and that indirect prestige factors can be disregarded as long as the negotiation studied has one or several of the following characteristics:

1. The parties will not be involved in negotiations with other parties. This might be true e.g. for negotiations between a corporation and a company union.

2. If H and L would be involved in other negotiations, these are of such a different nature that the other parties involved will not base their judgements about H or L on the outcome of the present negotiation between H and L. For example, the way the management of a corporation behaves towards its trade union might have very little influence on the way it is believed to act vis-à-vis other corporations, e.g. in merger negotiations.

3. Outsiders do not know the exact nature of the outcome and/or the starting positions of the parties well enough to be able to judge the extent of each party's actual concessions.

It should be stressed that under the three conditions stated above G_1 will not be violated, at least not due to indirect prestige factors. Point 3 becomes somewhat different and much more important with respect to the validity of G_1. The requirement that outsiders do not receive information about the bargaining *process* can then be added to point 3. This process is never revealed for many negotiations, e.g. between two corporations, and hence G_1 is not affected. In other cases, such as many labor-management negotiations, something *called* the bargaining process is revealed to the public. However, what is revealed to the public is sometimes a *sham* bargaining process and the true bargaining process is kept secret.

7.3.3.4 Direct Prestige Factors

It is probably exceedingly difficult for a party to estimate the other party's evaluation of direct prestige factors. This means that if direct prestige factors are very important, B_9 will most probably *not* hold. But it does appear reasonable to assume that there are a great many negotiations where direct prestige factors will play an insignificant role. This is based on the following hypothesis:

The prestige involved in a certain agreement does *not* rise in proportion to the monetary pay-off of this agreement. Instead it is either constant or increases at a slower rate.

It is quite conceivable, for instance, that party H in example 3 on p. 37 would insist on his terms, i.e. alternative 7, in period 1 in order to get \$ 35 rather than accept L's terms, alternative 6, and get \$ 36. The prestige of having "won" might very well be worth more than one dollar. But it is much less likely that party H would rather have the prestige of winning, if the choice were between \$ 36,000 and \$ 35,000 and it is even less likely that he would rather take \$ 35 million instead of \$ 36 million[12]. The hypothesis that *direct* prestige factors decrease in relative importance as the monetary pay-off increases is a natural extension of assumption B_3, the continuity assumption. If prestige always outweighed monetary pay-off, the party would behave contrary to B_3[13].

It should be emphasized that if a highest possible value of the direct prestige factors can be determined, then we might be able to make a reasonably good approximation of the solution even though the solution cannot be determined exactly[14].

The last question in this context is whether or not direct prestige factors invalidate assumption G_1. In general the prestige difference between different *plays* leading to the same outcome seems smaller than the prestige difference between different *outcomes*. In a three-alternative game (5,7) for example, H's prestige difference between conceding first to 6 then to 5 and accepting 5 immediately is most likely much *smaller* than H's prestige difference between accepting 5 and getting L to accept 7. This makes it reasonable to hypothesize that the validity of G_1 will generally not be affected significantly by direct prestige factors in negotiations involving considerable amounts of money.

7.3.3.5 Summary

B_9 seems more likely to hold for bargaining games with the following characteristics

1. The games concern large amounts of money.
2. The bargaining process and the initial positions of the parties are kept secret from outsiders.

[12] It should be noted that direct prestige factors are likely to carry more weight in ordinary bargaining experiments, involving at most a few dollars, than in real negotiations involving much larger sums.

[13] He would then have a lexiographic utility ordering, not allowed according to B_3.

[14] This can be exemplified by a division of \$ 100,000 into whole 1,000 dollars during each of 100 periods. We assume that the maximum loss of prestige in reaching an agreement on a less favorable alternative in a two-alternative subgame is worth less than a monetary loss of \$ 1,000. Then, for e.g. the case when $r=R=0$, we can deduce that the solution must give either party between 49,000 and 51,000 each period. This is deduced on the basis of theorem T_{11}. Since $50\cdot 51-1>49\cdot 52$, while $50\cdot 48<49\cdot 49$; $50=Cu(51)$ holds even when a prestige loss of \$ 1,000 is deducted from H's acceptance pay-off. Likewise we deduce that $50=cU(48)$. Hence H can enforce an agreement on 49 and L can enforce an agreement on 51.

3. The bargaining parties are not likely to carry out negotiations with other, similar parties in the near future.

G_1 is also more likely to hold under these circumstances than otherwise.

7.4 The Insight Assumption B_{10}

7.4.1 Introduction

We now turn to the insight assumption:

B_{10} *Mutual knowledge about the other party's knowledge about oneself*: H knows what L knows about H, and L knows what H knows about L.

As noted earlier, this assumption combined with $\boldsymbol{B}_2$ forms the set $\boldsymbol{B}_3$, which in turn is sufficient for deducing a unique solution for some games (see e.g. theorem T_4). B_{10} is a crucial assumption (cf. Chapter 3), since it is required in a two-alternative game to bring the solution backwards all the way from the first critical period i to the very first period of the game. The choice in period i is determined by $\boldsymbol{B}_1$ and the choice in period $i-1$ by $\boldsymbol{B}_2$, but the choice in every period prior to $i-1$ is determined by the use of B_{10}. If $i \in \boldsymbol{H}$ and i=Cu(x) then the choice in period i is determined by H being a rational pay-off maximizer. The choice in period $i-1$ is determined *inter alia* by L perceiving – due to B_8 and B_9 – that H is a rational pay-off maximizer. The choice in $i-2$ depends in turn on the use of B_{10}, implying that H knows what L knows about H, i.e. that H knows that L knows that H is a rational maximizer. Next the choice is determined in $i-3$, since B_{10} then implies that L knows that H knows that L knows that H is a rational maximizer. By iteratively applying B_{10}, we can deduce the choice one period backwards at a time. If B_{10} is applied many times, we can deduce the solution in a game with any number of periods prior to the first critical period i.

7.4.2 Replacing B_{10} with Five New Institutional Assumptions

Just as the validity of B_8–B_9 is critical for the rational solution of the efficiency problem, B_{10} is fundamental for many solutions in game theory[15]. As regards the

[15] The method of iterative elimination of dominated strategies relies on this assumption (cf. p. 254). Many methods for computing the solution according to the minimax principle of zero-sum games are based on the assumption that all dominated strategies have to be eliminated prior to computation of the probabilities for various strategies; see e.g. Allen (1959, p. 523), Williams (1954, p. 112) and Repoport (1966, p. 87). When there are several dominated strategies, this implies that an iterative elimination of dominated strategies must take place, requiring assumption B_{10}. Furthermore, Zermelo's theorem (cf. p. 257) relies implicitly on B_{10} (see McKinsey, 1952, p. 130).

validity of B_{10}, we limit ourselves to showing that B_{10} can be replaced by five assumptions of an *institutional* nature, called $I'_1 - I'_5$. An examination of alternative *behavioristic* assumptions appears to be fairly complicated and will be left for future research[16].

The purpose of replacing B_{10} with $I'_1 - I'_5$ is twofold:

It can be shown that even *if* some parties would regard B_{10} as an unacceptable assumption, a unique solution can still be obtained provided B_2 holds for a set of institutional assumptions, namely $I_1 - I_{13}$ *and* $I'_1 - I'_5$.

In situations where $I'_1 - I'_5$ only hold to some extent, increased credibility can be assigned to assumption B_{10}.

In order to limit our discussion to B_{10} alone we analyze a bargaining game for which $B_1 - B_9$ and hence B_2 hold unconditionally. If B_1 implies that each party seeks to maximize his profits, we assume that each party realizes the other party is a rational profit maximizer. If H knows with certainty that L is a rational maximizer, it is also reasonable to assume that he has some informational grounds for this belief, such as evidence in the form of L's statements, L's earlier actions or orders from L's principals[17]. We therefore introduce

I'_1 Each party can prove that he has the information he really possesses about the other party's preferences and about the outcome factors relevant to the other party.

This assumption also implies that a party can prove he has complete information, i.e. that I_5 holds.

We also introduce

I'_2 The parties have complete facilities for exchanging information.

[16] A project for future research could consist of comparing a two-alternative bargaining game and an iterated Prisoner's Dilemma game (see p. 190 for definition) with the aid of a simulation model for cases where $I'_1 - I'_5$ do not hold unconditionally. The model should allow the parties to *learn* the behavioristic assumptions according to which the other party behaves. Very preliminary attempts have indicated that behavior in the bargaining game will be similar to that proposed by our bargaining model, based completely on B'_4, while the behavior in the P(risoner's) D(ilemma) game will be different from that of traditional game theory analysis based on B_4. According to the simulation model there will be cooperative choices in all but the last few periods in the P. D. game as opposed to the analysis based on B_4, according to which the iterated P. D. game with a known and finite number of periods should be played non-cooperatively in every period.

[17] E.g. in the form of a budget, the fulfillment of which requires profit maximization.

I'_3 Each party has committed himself (e.g. in the pre-bargaining phase of the negotiation) to the norm of *not* persisting in lying if the other party proves that a statement is untrue.

I'_4 Each party has committed himself to the norm of accepting the other party's statement if he cannot bring forth any evidence indicating that it is incorrect.

I'_5 Each party has committed himself to the norm of answering the other party's questions.

Our next step is to outline the main ideas of the proof that assumptions I'_1–I'_5 are sufficient to replace B_{10}[18]. The only proof given here is for the simple two-alternative game (6,7) in example 4, where 3 = Cu(6) (cf. p. 41). The deduction consists of the following steps:

1. L first asks H "Is 3=C(6)?" or in other words "Do you prefer $\overline{6}_3$ or $\overline{7}_4$"? Due to assumption I'_5 H must answer. Since B_2 holds and L knows and – due to I'_1 – can prove that 3=C(6), H must – due to I'_3 – admit that 3=C(6).

2. L can next – due to I'_3 and I'_5 – make H admit that H will accept 6 in period 3.

3. L tells H that L will insist on 6 in period 2. When L claims that 2=u(6), H must – due to I'_4 – admit that this is true and that he, H, will only get $\overline{6}_3$ from insisting on 7 in period 1.

4. L asks H how H will choose in period 1. Since L can prove that $\overline{6}_1 > \overline{6}_3$ and H has admitted that 7 leads to $\overline{6}_3$, H must admit that he will accept 6 in period 1.

Proceeding backwards in this manner, one period at a time, we deduce more generally that H accepts x in his first bid in any game $(x,x+1)$ where i=Cu(x). Similarly, B_{10} can be replaced by I'_1–I'_5 in any of our other deductions.

7.4.3 The Validity of I'_1–I'_5

We now proceed to discuss briefly the validity of these five assumptions.

The validity of I'_1 has already been dealt with above. We only reiterate here that it appears dubious to assume that B_2 would hold *unconditionally* if each party could

[18] Whether they are necessary or not is a more complicated question. Some assumptions of this type, however, appear necessary in order to completely rule out bluffing. Bluffing would occur e.g. in example 4 if H says in period 1 that he will insist on 7 in period 3, in spite of the fact that he actually plans to accept 6.

not prove his belief related to $\boldsymbol{B}_2$. The effect of I_1' can probably also be obtained, in a less rigorous manner, by alternative assumptions. The important thing is for H to *convince* L that H has the knowledge implied by assumption B_9, making it impossible – due to I_3' – for L to make statements contrary to this information.

Assumption I_2' of complete facilities for informational exchange can be regarded as fairly natural in many situations. Such facilities are available, for example, in face-to-face bargaining between two parties speaking the same language. Even if I_1', I_3'–I_5' do not hold, this assumption alone is likely to increase the probability that B_{10} holds.

Assumption I_3' also appears fairly natural. It is reasonable to assume that most parties would agree to a norm of prohibiting consistent lying against evidence. Alternatively one party can force the other party to accept this norm, e.g. in the pre-bargaining phase of the negotiation. If a party does not agree to refrain from lying consistently against evidence, the other party can threaten to break up the negotiation. A threat in this situation appears fairly credible. This is discussed further on p. 169.

The arguments for assumptions I_4' are similar to those for I_3'. If H claims that L is a liar although H is unable to present any form of evidence, L's threat of breaking up the negotiation appears credible.

Assumption I_5' that each party will answer the other party's questions might be more doubtful. The norm of answering all of the other party's questions does not appear equally accepted as e.g. the norm of not lying against evidence. Therefore threats of breaking up the game in case of refusal to answer questions seems less credible than withdrawal due to obvious and consistent lying. However, it is plausible to hypothesize that roughly the same psychological effect can be obtained without forcing the other party to answer. Instead of forcing H to admit that he will accept 6 in period 3, when 3=C(6), (see steps 1 and 2 on p. 139), L can say: "I know that $\overline{6}_3 > \overline{7}_4$ and that you want to maximize your pay-offs. You cannot deny this, can you? If you refuse to answer, I regard this as your consent." Since H cannot deny this without departing from the norm of not lying he will have to remain silent or consent. It then appears reasonable that H will assume, with a very high probability, that L will regard this as H having consented to accept 6 in period 3. This in turn implies that H's subjective probability of L insisting on 6 in period 2 will be very close or equal to 1.

7.5 The Remaining Behavioristic Assumptions

In addition to the assumptions of $\boldsymbol{B}_3$, assumption set $\boldsymbol{B}_4$ consists of

B_{11} *Uncertain choice under indifference*: If a party is indifferent between two alternatives, the other party will *not* regard the choice of a specific one of these as certain.

B_{12} *Probability dominance*: If a party prefers an outcome y to an outcome y', the party will prefer y for certain to a lottery involving y and y' with some (positive, not extremely small) probability that y' will occur.

It should be kept in mind that these two assumptions are only necessary for the solution of S-games when $i = S(x)$ and/or $i = s(x-1)$. Hence, many bargaining games can be solved without the use of B_{11} and B_{12}[19].

Furthermore these assumptions are very weak. As concerns B_{11}, "H is indifferent between y and y' in period j" can be defined as "prior to period j, H cannot determine whether he will choose y or y' in period j". Then it is reasonable to assume that L has no way of determining with certainty whether H will choose y or y' in period j.

B_{12} has been criticized on the grounds that some people prefer an act involving a considerable risk of death – e.g. when mountain climbing – to an act with no significant risk of death – e.g. staying at home – although life is preferred to death[20]. The question, however, is whether this criticism really concerns B_{12}. Mountain climbing also involves rewards such as prestige, beautiful scenery and fresh air, not obtained by staying at home. Furthermore it should be stressed that we only require B_{12} to be applicable to bargaining situations where "mountain climbers" and their counterparts probably constitute a minute minority.

7.6 Relations between the Purpose of the Bargaining Model and the Behavioristic Assumptions

7.6.1 Introduction

Various purposes for which a bargaining model could be used were discussed in Chapter 2. Now that all of the behavioristic assumptions have been discussed group by group, we turn to a brief study of the relations between some of these purposes and the behavioristic assumptions as a whole.

[19] See p. 111.
[20] See e.g. Luce & Raiffa (1957, p. 28).

We first note that if our purpose is to provide an *explanation* of the behavior of both parties on the micro level, then all of the behavioristic assumptions have to hold – at least approximately – for *both* parties.

There are two schools of thought with respect to *predictions*[21]. One says that all the behavioristic assumptions should hold if the predictions are to be reliable. According to the other school, the validity of the behavioristic assumptions is less essential. It suffices for the model to have shown predictive ability in other similar situations. However, spokesmen of both schools would probably agree on the following judgement: The requirement regarding the precision with which the behavioristic assumptions approximate real behavior is weaker when the predictions are related to macroeconomic studies involving many bargaining parties than when they refer to a single pair of bargainers on the micro level.

Generally speaking with regard to the predictive purpose, it is *not* possible to establish that some *particular* behavioristic assumptions among those of B_3 are more essential than other assumptions. On the other hand, as indicated by the discussion in the following three sections, it suffices to rely on specified subsets of B_3 as concerns the normative purpose, mediation and to some extent also arbitration[22].

7.6.2 The Normative Purpose

7.6.2.1 Definitions

The normative purpose can be given the following operational definition:

A model fulfills the normative purpose with regard to *one* party in a decision situation, characterized by the institutional assumptions of the model, if the following is true: After being thoroughly informed about the characteristics of the model and the mode of behavior recommended by the model, the party will want to follow this recommended mode of behavior – at least to a significant extent – and, after being informed about the result, will continue to use it.

A normative model is then a model aimed at fulfilling this kind of purpose. When judging whether a model can be regarded as fulfilling the normative purpose, we ask whether one party will want to use it or not[23]. In answering this question, it is of particular importance to distinguish between the following two cases:

[21] See e.g. Friedman (1953), Machlup (1955) and Melitz (1965).

[22] B_4, instead of B_3, is required mainly when both parties have a 0 per cent interest rate (see p. 141). For the sake of simplicity, this case will be disregarded in this section.

[23] This establishes a basis for a later test of our model's value as regards the normative purpose (cf. p. 210).

1. It is known with reasonable certainty that the other party behaves rationally[24]

2. The behavioristic assumptions according to which the other party will act cannot be determined.

7.6.2.2 Study of a Particular Situation

Both of these two cases will be investigated in terms of a situation with the following characteristics:

1. B_1–B_4 and B_9 hold for both parties. This implies that each party can establish an ordinal utility for each outcome on the basis of his preferences and that each party is aware of the other party's preferences. We can e.g. assume that each party prefers more money to less and that each party is aware of this also being true for the other party.

2. B_5–B_7 also hold with certainty for L. Combined with point 1 this implies that $\boldsymbol{B}_1$ holds in full for L, whom we therefore regard as rational. B_5–B_7 might *not* hold for H, so we regard H as *possibly* less rational[25].

3. I_1–I_{13} hold.

4. I'_1–I'_5 hold (see p. 138).

5. The bargaining game is an S-game for which $\boldsymbol{P}$ holds with i=C(x)[26].

It can now be proved that L *can* ensure himself of an immediate agreement on x, i.e. the solution to be obtained, if $\boldsymbol{B}_3$ would hold and both parties were fully rational. The proof is as follows: According to point 1 above, B_1–B_4 hold for H and L is aware of this. L can make B_5–B_7 hold for H, since L, who is rational, *can* supply H with computations, deductions, information processing,etc [27]. Hence L can make $\boldsymbol{B}_1$ hold also for H[28]. Since L knows that he can make H follow $\boldsymbol{B}_1$, B_8 holds for L. Furthermore by showing his deductional capacity, etc. to H, L can make H realize that $\boldsymbol{B}_1$ holds for L and hence make B_8 hold also for H. Since it is assumed in point 1 that B_9 holds for both parties, $\boldsymbol{B}_2$ holds for both parties. $\boldsymbol{B}_2$ and

[24] This implies here that he acts according to B_1 and that B_9 holds.
[25] However, we have to assume that H understands simple statements when spoken to.
[26] Thus we avoid the case of i = S(x) requiring B_4.
[27] Due to B_1–B_4, a party will choose the optimal alternative when it is presented to him. Violation of B_5, due to lack of search capability, is hence avoided.
[28] I'_1–I'_5 ensure that H obtains the information L wants to give him.

$I'_1 - I'_5$ combined imply that B_3 holds (see p. 139). Hence, L can – due to Theorem T_{11} – enforce an agreement on alternative x.

7.6.2.3 Bargaining against a Rational Opponent

Against the background of the analysis in the preceding section, we turn to a study of the case where a party is in a position to use the model as a normative one when bargaining against a party who, with certainty, is rational[29]. We let H be the party who uses the normative model and L be his rational opponent[30]. For a situation characterized by the five points in 7.6.2.2 we can then deduce that L can ensure himself of an immediate agreement on x, i.e. the same alternative on which an agreement would be reached if H were also fully rational. Thus we conclude that if H does not use our model in this situation, he cannot get an agreement more favorable than the immediate agreement on x which he will get by using our model. Instead there would most likely be at least some risk of H obtaining a less favorable agreement.

Conclusion: In a situation with the above-mentioned characteristics, the choice of our model is optimal for a party playing against a rational opponent.

7.6.2.4 Bargaining against a Possibly Irrational Opponent

Next we assume that L is the party for whom our model is to be regarded as a normative model and that H is his opponent, whose state of rationality is unknown to L. The deduction in 7.6.2.2 can then be used to conclude that in a situation with the characteristics of p. 143, L can assure himself of an agreement on x by using our model. However, we cannot rule out the possibility that L could get an even more favorable agreement if H is irrational. On the other hand, we cannot determine that this is the case.

Use of our model is similar to the use of the minimax strategy in zero-sum games. With the aid of our model, a party can *guarantee* himself a certain minimum pay-off. But it is generally not known whether this is the best attainable result, if the opponent is *not* rational.

7.6.3 Mediation

The mediation purpose can be regarded as a double normative purpose. The mediator must be able to convince both parties that they can *not* obtain a better

[29] As defined in footnote 24.

[30] For example, H might have a consultant who supplies him with advice according to the normative model.

agreement by using any other bargaining strategy than the one proposed by the mediator.

We study a situation with the following characteristics:

1. $B_1 - B_4$ hold for each party, i.e. both parties can determine an ordinal utility for the various outcomes. This is true e.g. when both parties seek to maximize their profits.

2. The mediator knows the preference relations of each of the parties and has some evidence of this knowledge (cf. p. 138).

3. $B_5 - B_7$ hold for the mediator, i.e. he has computational and deductive capacity.

4. The mediator has the same information which we earlier assumed each of the parties possessed according to I_5, i.e. complete information about all outcome factors of relevance to each party's preferences. The mediator is also able to prove that he has this information. If the parties are corporations seeking to maximize their profits, this implies that the mediator has information about the forecasted revenues and costs associated with each outcome. This can e.g. be true when the mediator is a certified public accountant working for both corporations and with access to the accounting and budgeting data of each.

5. The bargaining game is characterized by $I_1 - I_4$ and $I_6 - I_{13}$[31].

6. There are good facilities for exchanging information.

7. The parties are committed to answering all the mediator's questions, not lying to him in face of evidence and not claiming that he is lying. This seems to be quite a plausible assumption as regards mediation, since a mediator would appear credible when threatening to break up his mission if these conditions are violated.

8. The game is an S-game for which $\boldsymbol{P}$ and $\boldsymbol{P}'$ hold with i=c$(x-1)$C(x)[32].

By supplying each party with capacity for computation, deduction, information processing, etc. the mediator can make $\boldsymbol{B}_1$ hold for each of them. Furthermore, by supplying information about the opponent's preferences and by indicating that he has fulfilled $\boldsymbol{B}_1$ as regards the other party, he can make $\boldsymbol{B}_2$ hold for each party. The mediator can also tell each party that he has given the other party information

[31] Assumption I_5 is no longer needed as regards the two parties, i.e. we do not have to assume that each party has complete information about the other party's pay-off.

[32] This is assumed in order to be able to rely solely on B_3.

about the party's preference relations. This would make I_1' hold. Points 6 and 7 above imply that the mediator can make I_2'–I_5' hold. We can then deduce that B_3 holds for each party (see p. 139). This is sufficient for deducing that an immediate agreement will be reached on x.

More concretely, one way the mediator can enforce an agreement on x is as follows: The mediator lets the parties play a series of very simple experimental games. If the parties behave approximately according to our model[33] after a few games needed to perceive the main ideas, the mediator discusses the model with the parties. If the parties accept the assumptions of the model for the simpler games, they might be willing to accept these principles for more complicated games. However, if one of the parties does *not* accept the mediator's proposal – which is the solution x according to our model – the mediator can tell him: "If you do not accept x, I will advise your opponent to use a strategy which will ensure him of an agreement on x. So if you do not accept x immediately, you will only delay an agreement on x. This delay is clearly unfavorable to you."

7.6.4 Arbitration

Arbitration is defined as the case where a third party *imposes* a solution on the two bargainers. Our model appears to be of little use if this arbitrated solution is aimed at fulfilling some social welfare criteria. But if the arbitrator is mainly interested in ensuring that the parties are satisfied with the imposed solution (e.g. so that he or other arbitrators will have good chances of being called in to settle other disputes, the conclusions with respect to use of our model for mediation are of interest. The arbitrator will then want to explain to each party that the imposed solution gives him at least as good a result as the one he could have obtained on his own, if the arbitrator had helped the other party.

7.6.5 The Solution as a Focal Point

It also seems possible to fulfill several of the purposes of our model in another way. This has to do with the idea of a *focal point*, which is a concept originally introduced by Schelling. According to Schelling the parties are likely to agree on some point with some outstanding characteristics, such as "prominence, uniqueness, simplicity, precedent, or some rationale that makes [it] qualitatively differentiable from the continuum of possible alternatives."[34] There are a few such focal points in a game concerning the division of a given sum. These include the

[33] Whether or not this is true is one of the main questions to be investigated by future laboratory testing of the model. See also p. 209.

[34] Schelling (1960, p. 70).

equal split, the Nash point and the solution according to our model[35]. The equal split has the characteristics of being very simple and appearing "fair" on an *ad hoc* basis. The Nash point has the characteristics of being the solution of two of the best-known bargaining theories and of being based on axioms of "fairness" in Nash's original version[36]. Our model has the special characteristic of being based on a small set of simple, rational behavioristic assumptions generally used within economic theory.

The use of the Nash model is problematical since it relies on cardinal utility, implying that the solution is very difficult to establish except in the case when both parties have a linear utility for money. The equal split solution is problematical since it does not establish exactly what will be evenly divided. Suppose $ 10 are to be divided and H can ensure $ 3 for himself and L $ 1 for himself, if no agreement is reached. Will the equal split principle then assign $ 5 to each or will the parties split the "agreement" profit $ 6 evenly so that H gets $ 6 and L $ 4? Many authors imply the first outcome, although the establishment of such an outcome would not take considerations of strategic positions into account.[37] Finally, both the equal split and the Nash solutions are independent of the fact that the parties are affected differently by a delay in reaching an agreement, if the parties have different time preferences[38].

If the parties regard it as a natural principle that the outcome *should* be affected by strategic aspects of the bargaining situation, such as what can be obtained if no agreement is reached and the effect of a delay in reaching an agreement, then the only acceptable one of these three focal points is that provided by our model.

[35] The other theories mentioned in the literature appendix would be less likely to provide focal points, e.g. due to the *ad hoc* character of their behavioristic assumptions.
[36] See pp. 222 and 234 in the literature appendix.
[37] See e.g. Grubbström (1972).
[38] See p. 227 for a proof that the Nash model does *not* take this into consideration.

Chapter 8
Institutional Assumptions

8.1 Introduction

The most important institutional assumptions made in the basic model will be dealt with in this chapter. This discussion will be carried out along the following lines:

1. We shall try to discuss the validity of the assumptions as such.

2. We shall investigate whether some of these assumptions can be replaced by other, possibly less demanding assumptions.

In particular, we examine how reasonable it is to assume that the parties agree freely among themselves, in the pre-bargaining phase of the negotiation, to play the bargaining game according to specific institutional assumptions. Since an exhaustive analysis would be very extensive, this discussion will be brief and scetchy. We concentrate to a great extent on presenting various hypotheses without any rigorous proofs, which will be left for future research.

The institutional assumptions of the basic model presented in Chapters 3–4 were numbered according to the order in which they were introduced. We no longer adhere to this enumeration in this chapter. Instead the assumptions are divided into four groups of similar assumptions and discussed groupwise.

A. Assumptions concerning the *bidding procedure* (discussed in 8.2):

   1. The assumption that only one bid is delivered each period (I_1).

   2. The assumption of alternating bidding (I_4).

   3. The assumption of good-faith bargaining (I_{13}).

   4. Other assumptions concerning the bargaining procedure (I_2, I_6 and I_{12}).

B. Assumptions concerning the *break up* of the bargaining game (discussed in 8.3):

1. The assumption that the parties cannot willfully break up the game prior to period z (I_{10}). The ruling out of threats is discussed in particular.

2. The assumption of sufficient liquidity (I_{11}).

C. Assumptions regarding determination of the *alternatives* (discussed in 8.4):

1. The assumption that the efficiency problem is solved (I_7). We prove here that assumption set B_2 is fundamental for solving this problem.

2. The assumption regarding determination of each party's most preferred alternative (I_8).

3. The assumption that the total number of alternatives in the bargaining game is given (I_9).

D. Assumptions concerning the *information* available to the parties (discussed in 8.5):

1. The assumption of complete information about the opponent's pay-off (I_5).

2. Other information assumptions (covered by I_3).

8.2 Bidding Procedure

8.2.1 The Assumption of One Bid in Each Period

8.2.1.1 Introduction

We defined a period as the amont of time during which a particular bid is made (see p. 33). Since the periods were not required to be of equal length, the period concept used for the basic model had a very broad application.[1] Looking back at Chapters 4–6, however, we notice that our model requires us to be able to assign a pair of pay-offs to each possible agreement in each period. Since this pay-off most probably depends on the amount of real time elapsed, the length of each period in real time has to be given and known by both parties. This gives rise to a fairly strong informational problem, unless the periods are of equal length. Even if the periods are of equal length, the problem of how to determine the length of each

[1] It thus corresponds to the original game theory concept of a *move*. But since there has been a certain amount of ambiguity with respect to the use of this concept and the moves follow upon each other in time, the period concept has appeared more suitable. Cf. von Neumann & Morgenstern (1947, p. 49) and Kuhn (1953, p. 194).

period still remains. The way in which the length of each period can be determined in certain cases will be discussed in 8.2.1.2. Then, in 8.2.1.3 we examine the circumstances under which determination of the solution will not be affected by whether the exact length of each period is given.

In many games where the size of Δt is important to the solution, we can rely on a period concept other than the "bidding period" of assumption I_1. The concept of a "pay-off period" is introduced in 8.2.1.4. This kind of period is linked to the distribution of certain pay-offs, e.g. profits, to the parties. This period is often much easier to determine than the "bidding period". Under certain sets of natural institutional assumptions, it appears likely that the pay-off period can be used instead of the bidding period and that a unique solution can then be obtained.

8.2.1.2 Determination of the Length of the Periods

We have already dealt with some cases where determination of the length of each period in real time was fairly clear.[2] An example of this is the case where each new bid has to be confirmed by some group of people, such as a board of directors, convening only at certain intervals of time. More concretely, we could imagine a bargaining agent acting on behalf of a board of directors, that has given him strict instructions as to the worst possible alternative on which he is allowed to reach an agreement. This means that the bargaining agent cannot reach an agreement on some less favorable alternative until the board has convened again. The period concept can be modified towards some degree of greater realism, if we assume that the board of directors keeps the bargaining agent under very strict reins, limiting the agent's choice so that he would have to propose an agreement at the limit set by the board soon after the board has met. The board of directors' minimum limit can then be regarded as the real bid. If the board of directors of each party convenes at known intervals, the length of the periods can be determined – provided the negotiation is *not* of such vital interest to the board that it can convene on extraordinary occasions just to set the limits for the bargaining agent.

In other negotiations, however, the bargaining agent is independent enough to reach an agreement on his own. This implies that the party's bids can be delivered in very short intervals of time through the bargaining agent, who might be involved in face to face bargaining with the other party. The question in this case is what constitutes a period. The proposals actually *stated* will be given at very unequal intervals of time in many negotiations. A person's silence can also be regarded as a bid in this instance, implying that he proposes the same alternative he gave the last time he spoke. The important thing then is how often a party *can* potentially deliver a bid.

[2] See footnote 47 on p. 123.

A period in this case is thus regarded as the amount of time during which a party *can* deliver one bid. This period is very short – probably only a few seconds – in face to face bargaining where the parties e.g. state a price, such as "ten thousand". However it should be stressed that the period is not infinitely short, since a second is probably the shortest possible period length.[3] Furthermore it will be very difficult to say whether a period consists of one, two or perhaps more seconds. The question of whether this really matters to the solution will be discussed below.

8.2.1.3 Importance of Determining the Length of the Periods

The discussion in this section is devoted to the extent to which a good approximation of the length of the periods in a bargaining game is required in order to approximate the solution of the game reasonably well. In this context we distinguish between the case where the solution is not at all dependent on Δt and the case where it does depend on Δt.

As regards pay-off functions 1 and 3 the solution is completely *independent* of Δt, if $r=R$. (See pp. 122 and 127)[4] For pay-off function 1 we then obtain $S^*\pi = (\pi+W-w-(C-c)+RK_T-rk_T)/2$, i.e. $S^* = 1/2$ when W, w, C, c, K_T and $k_T = 0$. For pay-off function 3 we obtain $S^* = 1/2$. Our study of the same functions indicated that when $r{\neq}R$, the solution might depend on $\mu{=}N\Delta t$. Our next step is to distinguish between the following three cases:

1. μ is of moderate size
2. μ is very small (e.g.$\leqslant 0.001$)
3. μ is very large (e.g. > 100).

1. *μ of moderate size*

When μ is of a moderate size, we can only conclude that both N and Δt probably have to be known.[5]

2. *μ very small*

μ cannot be very small unless Δt is very small, since N is an integer ($\geqslant 3$). Our problem is less complex in this case, since it appears likely that S will approach a

[3] At least when bargaining takes place between human beings. In the future, when bargaining might take place between computers, it might be reduced to a few nano-seconds.

[4] For pay-off function 3 $r{=}R$ implies that $\alpha = 0$.

[5] When the pay-off period is equal to the bidding period. The case where this is not true is dealt with in 8.2.1.4 below.

certain value asymptotically, as μ goes towards 0, with a very small difference for all values of $\mu \leqslant 0.001$. We hypothesize that the solution will be the same as the one obtained when the parties have the *same* interest rate, i.e. $r=R$. It suffices for our purposes here to prove this hypothesis for pay-off function 3 in general and pay-off function 1 for the case where $r=2R$.

According to pay-off function 3, $S^*=1/(1-\alpha+\sqrt{\alpha^2+1})$. When $\mu \to 0$, $\alpha=(r-R)\mu/2$ will also go towards 0[6] and S^* towards $1/(1+ \quad 1)=1/2$. We note that for $\mu \leqslant 0.001$ and $r-R \leqslant 1, S^*-1/2 \leqslant 0.0001$.

According to pay-off function 1, S^* will also go towards 1/2, when W, w, C, c, K_T and $k_T = 0$.[7] For $\mu \leqslant 0.001$ and $R \leqslant 1$, we obtain $S^*-1/2 \leqslant 0.0001$. When $\mu \to 0$ and W, w, C, c, K and $k \neq 0$ we obtain $S^*\pi = (\pi+W-w-(C-c)+RK_T-rk_T)/2$, i.e. the same expression as when $r = R$.

3. μ *very large*

μ cannot be very large unless N is very large, since there will most probably be several bids each year, implying that $\Delta t<1$. When μ becomes very large we see that, at least for noticeable differences between r and R, no solution will exist. E.g. we require that $1/(r-\beta_0) + 1/(R-\beta_0) \geqslant \mu$ for pay-off function 3. It is necessary, though not sufficient, that $1/r+1/R \geqslant \mu$ for pay-off function 1.[8] If interest rates are at least ten per cent a year for both parties, there would be no solution e.g. if $\mu > 20$. The question is then under what circumstances μ can be expected to be this large. This question will be answered in 8.4.3 where determination of the number of alternative is discussed.

8.2.1.4 The Relationship between Bidding Periods and Pay-off Periods

A new period concept, the pay-off period, is introduced in this section. This is followed by a brief discussion of the extent to which the pay-off period concept can be substituted for the bidding period concept.

[6] $r-R$ is finite, in fact most probably <1.

[7] We write $(1-\sqrt{1-\alpha})/\alpha = f(\alpha)/g(\alpha)$, where $f(\alpha) = 1-\sqrt{1-\alpha}$ (cf. 123). Thus $f'(\alpha) = 1/2\sqrt{1-\alpha}$ and $g'(\alpha) = 1$. Hence $f'(\alpha)/g'(\alpha) \to 1/2$, when $\alpha \to 0$. Due to L' Hospital's rule (described in any standard calculus textbook) $S^* = f(\alpha)/g(\alpha) \to 1/2$, when $\alpha \to 0$.

[8] $1/r+1/R= \int_0^\infty e^{-rt}dt+ \int_0^\infty e^{-Rt}dt> \int_0^Z (e^{-rt}+e^{-Rt})dt$. On p. 122, (footnote 43) we required that $\int_0^Z (e^{-rt}+e^{-Rt})dt \geqslant \mu$.

First we define a "pay-off period". Let us call the start of a certain period, time t_1, the end of the period, time t_2 and the start of the next period, time t_3. Times t_2 and t_3 are very close, while t_1 and t_2 are assumed to have some interval between them. The period studied will constitute a pay-off period if

1. the difference in pay-offs between t_2 and t_3[9] is significant[10] and

2. this difference in pay-offs (between t_2 and t_3) is larger than the difference in pay-offs between t_1 and t_2.

If there are two pay-off periods of different lengths, partly covering the same span of time[11] and fulfilling the requirement above, then only the *shorter* of the periods is regarded as a pay-off period.

In order to concretize this definition, let us go back to one of the simple examples in Chapter 3: example 3. In this example it was assumed that the parties obtained \$ 10 in each of six periods. Each period constituted a bidding period, since one bid and one bid only was delivered, *and* a pay-off period, since the pay-off remained constant during the period but decreased significantly (for H by \$ 6 or \$ 7), when we passed from one period to the next.

The same conclusion holds if we allow for a (virtually) continuous calculation of interest. Let us e.g. assume that the interest is calculated continuously but that the profit to be divided decreases by \$ 10 at the end of each month. Then, if H's pay-off is \$ 36 from accepting alternative 6 at t_1, it might be $36\,e^{-R/12}$ at time t_2 and $30\,e^{-R/12}$ at time t_3.[12] Then, for resonable values of R, the time between t_1 and t_2 can be regarded as a pay-off period.

A further analysis shows that the pay-off period and the bidding period might very well be of different lengths. In example 3, if several bids could be delivered before the joint profit decreased by \$ 10, then the bidding period and pay-off periods would no longer coincide. There will generally be a difference between the bidding

[9] I.e. the difference between the pay-off obtained from an agreement at time t_2 and the pay-off obtained from an agreement at time t_3.

[10] In order for the difference in pay-offs between t_2 and t_3 to be regarded as significant, it has to be so large that at least some pay-off period of this length of time is *critical* for at least one party as regards every game $(y, y+1)$.

[11] E.g. one period covers t_1-t_4, while the other covers t_3-t_4.

[12] If interest is calculated daily, the pay-offs at t_2 and t_3 can be written e.g. as $36(1+R/360)^{-30}$ and $30(1+R/360)^{-31}$; $e^{-R/12}$, however, is a very good approximation of both $(1+R/360)^{-30}$ and $(1+R/360)^{-31}$, for reasonable values of R. Furthermore, the pay-off will decrease from 36 to $36/(1+R/360)$ between the first two days in the first month. With $R \leqslant 0.30$ the decrease is smaller than one tenth of a per cent and hence probably negligible at least for moderate values of n (see footnote 10). In other words, the procedure of regarding each day as a pay-off period is not warranted in this case.

period and the pay-off period when bids can be delivered at fairly short intervals, e.g. at face to face bargaining, but the pay-off decreases significantly at certain, more sparsely distributed, intervals of time.

This type of situation is undoubtedly very common in reality. One good example is a labor-management negotiation where a delay in reaching an agreement implies prolongment of a strike. If an agreement is reached before a certain hour, e.g. 1.00 p.m., work can start the following day. But if an agreement is reached after this hour, work cannot start until another day has passed.[13] Similar effects also occur in other types of bargaining situations. Joint ventures or mergers might have to be started on a certain day of a week or a certain day of a month, in order to avoid excessive administrative difficulties.[14] In duopoly negotiations, e.g. to limit price competition, prices might have to be set at certain dates so as to remain unchanged during a certain span of time. Then the joint collusion profit cannot be obtained in smaller parts than e.g. monthly pay-offs.

In other words, the bidding period is frequently shorter than the pay-off period. The question then is how this will affect the use of our model. We introduce the following institutional assumptions:

I'_6 Each pay-off period consists of a number of bidding periods, such that one party and one party only bids in each bidding period.

I'_7 Each pay-off period contains an *odd* number of bidding periods.

I'_7 implies that if one party bids in the *last* bidding period of one pay-off period, then his opponent will bid in the *last* bidding period of the following pay-off period. It also implies that if one party bids in the *first* bidding period of one pay-off period, then his opponent will bid in the *first* bidding period of the following pay-off period.

Assumption I'_7 does not appear absolutely necessary for the hypotheses about to be introduced[15], but it simplifies our deductions. It also seems to be a fairly natural assumption. The original assumption of alternating bidding I_2 can be regarded as a special case of assumption I'_7, namely when the odd number in I'_7 is 1. In any case I'_7 is more natural than the assumption of an even number of bidding periods in

[13] This is probably most clear in the newspaper industry, but it is significantly less profitable to work a fraction of a day in many other industrial sectors as well.

[14] Joint projects are often begun by an advertising campaign. Then the publication dates of the most important media might determine the length of the pay-off periods.

[15] The possibility of letting the number of bidding periods in each pay-off period be a random variable, where each pay-off period has even chances of containing an even or an odd number of bidding periods is discussed briefly on p. 277.

each pay-off period, since this would imply that the same party will bid in the last bidding period of each pay-off period.

With respect to most S-games, we now *hypothesize* that a solution identical to that of our basic model can be obtained on the basis of I_1–I_{13} *and* I'_6–I'_7. In line with our discussion on p. 148, we refrain from giving a general proof of this hypothesis. As shown in the appendix (p. 276) it holds at least for any two-alternative S-game with i=Cu(x) and also many three-alternative S-games.

If the above hypothesis is true in a more general sense, our model's area of applicability would be greatly increased. In many cases we would get rid of the problem of having to determine the *length* of the bidding period. It then suffices to determine the *pay-off* period, which is often given institutionally as in the examples on p. 154. The parties can then be allowed to deliver an arbitrarily large number of bids without affecting the solution.

8.2.2 The Assumption of Alternating Bidding

8.2.2.1 Introduction

Our basic model relies on the assumption of alternating bidding, I_4, implying that the parties take turns bidding. Our model differs in this respect from some of the best known bargaining models, which are based on the assumption of simultaneous bidding.[16] Simultaneous bidding means that the parties deliver their bids at the same time in each period or "round" without knowing what the other party bids in this period.

Thus the relevance of the a.b. (alternating bidding) assumption has to be judged in relation to the alternative of assuming s.b. (simultaneous bidding). This involves the following two questions:

1. To what extent can the assumption of s.b. be substituted for that of a.b. so that the same result is obtained?

2. When are the parties likely to settle on a.b. rather than s.b. in the pre-bargaining phase of the negotiation?

Before answering these two questions, we examine some general characteristics of the s.b. assumption.

[16] See p. 219.

8.2.2.2 Characteristics of Simultaneous Bidding

In order to exemplify how the s.b. assumption works, let us study a period j of a finite bargaining game – similar to example 1 (p. 34) – involving two alternatives, called 6 and 7. Each party will deliver a bid in this period. The situation in this period is illustrated by the following tree (Figure 24).

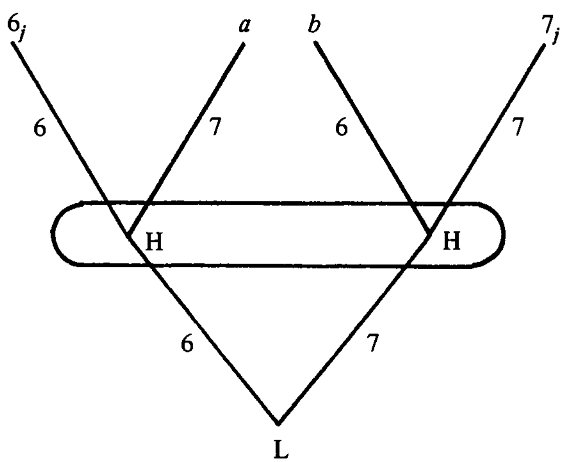

Figure 24 A game with simultaneous bidding

The "ellipse" around the two nodes, denoting H's decisions, indicates the following: When H delivers his bid, he does not know whether he is at one node or the other, since he does not know which of the two alternatives L has proposed in period j.

The following can be noted with respect to the pay-offs at the end points: If both parties bid 6, an agreement is reached on 6_j. If both bid 7, an agreement is reached on 7_j. If L insists on 6 and H insists on 7 we reach the node denoted a. If $j = z-1$, the last period of bidding, node a leads to the break-up outcome. This is denoted 0_z, since it often gives both parties zero pay-off. If $j < z-1$, node a only implies that bargaining continues. Finally, if L accepts 7 and H accepts 6, node b is reached. The subsequent outcome must be subject to some rule made in the pre-bargaining phase of the negotiation. The following rules are possible:

1. Bargaining continues without agreement. This rule would be highly inefficient, e.g. if $j=z-1$, since *both* parties would be better off by reaching any agreement in this period.

2. Each party obtains the pay-off of his least preferred alternative; i.e. b gives H $\overline{6}_j$ and L $\underline{7}_j$. This assumption is used in the Zeuthen model.[17] It is also inefficient.[18]

[17] See p. 231.
[18] See furthermore p. 235.

In a division of \$ 10, this implies that H would get \$ 6 and L \$ 3. Who would then get the remaining \$ 1?

3. An agreement is reached on either 6 or 7 in period j. The question of which of these pay-offs is agreed upon is decided by some random event, e.g. a coin toss, leading to *even* chances for an agreement on 6 or 7.[19] This is an efficient solution because one of the parties obtains his best possible agreement. Since this is the only efficient decision rule of the three, we assume in our analysis of the simultaneous case below that the parties agree to adhere to this rule.

8.2.2.3 Use of Simultaneous Bidding Throughout the Bargaining Game

The first thing we note is that the assumption of s.b. cannot be substituted for that of a.b. with regard to *every* possible period. This is most clearly seen in the case of a two-alternative bargaining game with a stop rule.

Let us study the choice in period $z-1$. If H insists on 7 and L insists on 6, the outcome 0_z is obtained. If H accepts 6 and L accepts 7 there will – according to the rule above – be a lottery with 1/2 chance that H will get $\bar{7}_{z-1}$ and 1/2 chance that he will get $\bar{6}_{z-1}$. Due to B_{12} this is worth less to H than $\bar{7}_{z-1}$ for certain. Hence the following simple game matrix, with H's pay-offs in the cells, is obtained.

		L	
		insists on 6	accepts 7
H	accepts 6	$\bar{6}_{z-1}$	$V<\bar{7}_{z-1}$
	insists on 7	$\bar{0}_z$	$\bar{7}_{z-1}$

Table 2 H's pay-off in a simultaneous bidding game

If L insists on 6, H would accept 6, since $\bar{6}_{z-1} > \bar{0}_z$, while if L accepts 7, H would insist on 7. Hence H can *not* determine how he should bid without determining the probability of L accepting 7. Behavioristic assumptions, probably of a very particular nature, are required in addition to those of B_4 in order to determine this probability.[20] Furthermore, computation of an expected value requires the establishment of cardinal utility, e.g. of the von Neumann-Morgenstern type. This is also outside the scope of B_4. Since we only have B_4 at our disposal, the choice of

[19] This rule can also be used in the Zeuthen model (see p. 235).
[20] See e.g. Harsanyi's assumptions with regard to the Zeuthen-model on p. 232.

the parties in this simple situation cannot be determined. And if we cannot determine how the parties bid in the last period, then we cannot determine how they will bid in earlier periods either. In these periods we also have the problem of determining the outcome at node a.

8.2.2.4 Use of Simultaneous Bidding in Part of the Bargaining Game

The conclusion above that s.b. can *not* be used during the *whole* bargaining game in our model does not rule out use of this type of bidding in *part* of the bargaining game. It is proved in the mathematical appendix (p. 278) that s.b. can be allowed for in S-games for every period prior to period i–2, where i is the period determined by P and P'. Since the special model does not investigate any periods after the last interesting period z^*, we can also allow for s.b. after this period. In many games the majority of periods will be prior to i–2 or after z^*, implying that the majority of periods can be played simultaneously without affecting our solution. The smaller n is in an S-game, the earlier z^* arises in relation to i and the smaller the number of periods that have to be played with a.b.

It should be noted that a solution can be obtained in the very *first* period by letting the first period be characterized by s.b. This differs from the case where *every* period is characterized by a.b. In this case the solution will generally be reached in the second period, since one party bids x in the first period and the other party cannot accept this until the second period.[21] This difference does not matter in the continuous case, but in the discontinuous case with fairly few periods, it might be advantageous for both parties to agree on playing at least the first period simultaneously.

8.2.2.5 Reasons for Selecting Alternating Bidding

We now turn to the conditions under which the parties are more likely to agree on a.b. than s.b. at least as regards periods i–2 to z^*. We try to answer this question by giving some reasons why the parties should select a.b.

1. At the point in the pre-bargaining phase when the form of the bargaining process is decided, each party might very well believe that he will be better off under a.b. than under s.b. It might appear probable that an agreement will be reached *earlier* under a.b. than under s.b. We know that an agreement is reached in the very beginning of the bargaining game in the a.b. case.[22] But, on the basis of B_4, it

[21] An exception is the case where one party accepts the other party's most desired alternative.

[22] An agreement can be reached in period 1 of the bargaining game, if the "alternating bidding case" is interpreted to imply that alternating bidding takes place in periods $i - 2 \ldots z^*$ or if $x = 1$ or n. Otherwise it is reached in period 2.

cannot be determined with certainty whether this is also true for the s.b.-case. Much more demanding behavioristic assumptions are needed to establish an agreement in the s.b. case and the risk that an immediate agreement cannot be reached is therefore likely to be higher.

2. The pre-bargaining phase of the negotiation is likely to be longer, and hence the agreement later, in the s.b.-case. In many instances it seems more suitable for the parties to have a fact-finding session prior to the bargaining game in which they try to agree on each other's pay-off functions (see p. 181). As noted above, *any* logically deduced solution based on rational behavior in the s.b. case would have to rest on assumptions of cardinal utility. Thus establishment of the relevant pay-off functions in the s.b. case would require the parties to agree upon each other's cardinal utility functions for money. Since it is usually very difficult to determine another person's cardinal utility function, this fact-finding session would constitute a far greater problem and probably be longer in the s.b. than in the a.b. case.

3. There appear to be difficult decisions in the s.b. case involving the specific form of the bargaining process, which might prolong the pre-bargaining phase and hence delay the start of the bargaining game even further.[23]

4. Even if the possibility of a different time of agreement is disregarded, both parties might perceive s.b. as leading to a lower expected utility. For example, if H lacks complete information about L's utility function for money, H may believe that he gets a lower expected utility under s.b. than under a.b., although he would not do so with complete information. The conclusion is that in a situation where the monetary pay-offs are reasonably well-known to both parties, but their cardinal utility functions for money are not, a.b. might very well appear best for both parties.

5. One party's possibilities of enforcing a.b. during periods $i-2$ to z^* appear to be considerably larger than the other party's possibilities of enforcing s.b. during these periods. L might want these periods to be played under a.b. because he wants these periods to be *critical* for H as regards certain two-alternative games. L then wants H to be certain that L will *not* deliver any new bid during such a period. L also wants H to understand that if H does not accept L's terms in one of these periods, there will not be any agreement in this period. Hence the "a.b.-effect" during a certain period is obtained if L can convince H that L will *not* make a new bid in this period. For instance, the following procedure could produce this kind of effect: Party L is represented by a bargaining agent who will have to get each *new* bargaining proposal approved by his board before reaching an agreement on these

[23] As discussed on p. 238, the logical deduction of Zeuthen's solution for the n-alternative case appears to require a very particular bargaining procedure. This procedure might have to be agreed upon prior to the start of the bargaining game.

terms. If the board cannot convene for decisions during the period in question, this will ensure that the bargaining agent will not be able to deliver any new bids. H's possibilities of forcing the board to convene or getting it to empower the bargaining agent to decide on new proposals on his own are probably small.

6. Finally, but not least important, it appears reasonable to assume that a.b. is more common in reality. To the extent that the parties are guided in their choice by common business practice, they would more likely agree on a.b.

8.2.3 The Assumption of Good-faith Bargaining

Assumption I_{13} of good-faith bargaining (abbreviated as g.b.) was introduced in Chapter 4. This assumption implies that a party cannot bid an alternative which is less advantageous to the other party than an alternative he has bid in an earlier period. This assumption will now be compared with its opposite, i.e. an assumption implying that each party can bid independently of his previous bids. Lacking a better name, we call this "bad-faith bargaining" (b.b.).

As in our comparison of simultaneous and alternating bidding, it can be shown that the assumption of g.b. only has to refer to a limited number of periods. First of all, nothing has to be assumed about the periods after the last interesting period z^*. The reasons are the same as in the case of alternating bidding. Furthermore, as regards the early parts of the bargaining game, we can prove that $\boldsymbol{P}$ and $\boldsymbol{P'}$ imply that $(1,n)_0=x_2$ for every S-game, even if every period *prior* to $i-2$ is played with b.b.[24] The proof of this is given in the mathematical appendix (p. 281). Hence we can very well allow the parties to withdraw their previous bids during the early part of the potential bargaining process.

General conclusions regarding the question of whether we can also allow for b.b. in periods $i-2$ to z^* are left for future research. But it can easily be proved that the conclusion for the *three*-alternative game (5,7) is completely unaffected by allowing for b.b. in *every* period of the game.[25] Figure 11 on p. 64 shows that the only difference between b.b. – case b – and g.b. – case c – for the first three periods of this three-alternative game is that in the b.b. case, H can bid 7 in period 3, *even* if

[24] It should be noted that b.b. is substituted for g.b. under the assumption of alternating bidding. The question of whether we can substitute simultaneous bidding for alternating bidding and b.b. for g.b. at the same time must be left for future research.

[25] It should be mentioned that there are at least *some* games which lead to a different result depending on whether g.b. or b.b. is assumed. An example is a three-alternative game with 4 periods, $1 \in H$ and $\overline{1}_1 = 10$; $\underline{1}_1 = 12$; $\overline{2}_2 = 11$; $\underline{2}_2 = 6$; $\overline{3}_2 = 12$; $\underline{3}_2 = 3$; $\overline{1}_3 = 7$; $\underline{1}_3 = 7$; $\overline{2}_3 = 9$; $2_3 = 5$; $\overline{2}_4 = 5$; $\underline{2}_4 = 4$; $\overline{3}_4 = 8$; $\underline{3}_4 = 2$. This is not an S-game. The solution, using the general model of Chapter 4, is 1_1 for the g.b. case but 2_2 for the b.b. case.

he bids 6 in period 1. Since H's bid 7 in period 3 will give H less than $\overline{7}_3$ and since $\overline{5}_1 > \overline{7}_3$, H will accept 5 in period 1, i.e. $(5,7)_{\underline{0}} = 5_1$.

Substituting alternative x for alternative 5 and period i for period 1, we deduce more generally that $\boldsymbol{P}$ implies $(x,x+2)_{\underline{i-1}} = x_i$, even if period i and later periods are played with b.b. Since we can deduce that $\boldsymbol{P}$ implies that $(x,x+k)_j = x_{j+1}$ holds for every $j < i$, provided $(x,x+k)_{\underline{i-1}} = x_i$[26], we can deduce that $(x,x\mp 2)_{\underline{j-1}} = x_j$ holds for every $j < i$, even if *every single* period of the bargaining game is played with b.b.

As a mirror picture of this conclusion we deduce that $\boldsymbol{P}'$ implies that $(x-2,x)_{\overline{j-1}} = x_j$ holds for every $j < i$ also in the case of b.b. in every period. These two conclusions combined imply that in the five-alternative game $(x-2,x+2)$ played with b.b. in every period, both H and L can assure themselves of an agreement on x. If this game has a solution it must be x. We hypothesize that similar conclusions hold for S-games with more than five alternatives. However, investigation of this hypothesis has to be left for future research.

If different outcomes would be determined for the g.b. case and the b.b. case, the parties are fairly likely to play the game with g.b. This is mainly because the g.b. idea undoubtedly has a fairly strong ethical weight. It therefore appears reasonable to assume that a party's threat in the pre-bargaining phase of the negotiation to abandon the negotiation completely, unless the other party agrees to the g.b. norm, is fairly credible. This is discussed further on p. 169.

8.3 Break up of the Bargaining Game

8.3.1 Introduction

In the basic model it was assumed that the bargaining game would *not* be broken up prior to period z without an agreement. We ruled out situations where the parties themselves could voluntarily break up the game and where some party would have to break up the game involuntarily by going bankrupt, due to liquidity problems. This section deals with the validity of this assumption.

The first question is why a party would *want* to break up the game. The most obvious reason is that in certain cases he can only obtain an agreement which would be *worse* for him than the one he could obtain on his own after the break up of the game. This reason and some conditions under which it might apply will be discussed in 8.3.2. The effect of this kind of break up on the outcome of the bargaining game will be dealt with in 8.3.3.

[26] The proof is similar to that on pp. 281–282 with the difference that $x-m$ is not available and the deduction backwards starts with $(x,x+2)_{\underline{i-1}} = x_j$.

Another reason why a party might break up the game involves carrying out a *threat*. The role of threats and the conditions under which it appears reasonable to assume that threats are *not* credible and hence less likely to be used, are discussed in 8.3.4.

Finally, some ways in which we can allow for liquidity considerations and use at least the general version of our basic model without major modifications are treated briefly in 8.3.5.

8.3.2 Reasons for Breaking up a Bargaining Game without Threats

In this section we study the circumstances under which a party might find himself forced to break up the bargaining game permanently because he cannot obtain any agreement which gives him a pay-off higher than the one he can obtain on his own from the resources involved in the bargaining. This can be determined by considering the party's break up pay-off, i.e. the pay-off he would obtain on his own if he broke up the bargaining permanently. Let us call the break up pay-offs H and L can obtain at time j, $\overline{0}_j$ and $\underline{0}_j$, respectively. Next we can determine that in period j L will with certainty break up the game, if he cannot obtain a higher agreement pay-off than the break up pay-off of this period $\underline{0}_j$. This implies, due to S_1, that L will break up the game if $\underline{y}_j < \underline{0}_j$ holds for every y. The condition that $\underline{y}_j < \underline{0}_j$ can be written as $v(S,T) < v(0,T)$ for S*-games.

Let us concretize this condition by looking at pay-off function 1 in Section 6.4. First, the break up pay-off, $v(0,T)$, has to be determined. It contains the same pay-off components as $v(S,T)$ with respect to the time prior to T, i.e. the pre-agreement and bargaining cost components. As regards the pay-off *after* the break up of the game at time T, we limit our study to assuming that an income stream w^*, computed at an annual rate, is obtained up to time Z.[27] Thus $v(0,T) = (w-c)\int_0^T e^{-rt}dt + w^* \int_T^Z e^{-rt}dt$. This means that L will break up the bargaining, if $(w-c)\int_0^T e^{-rt}dt + s\pi \int_T^Z e^{-rt}dt - k_T e^{-rT} + k_Z e^{-rZ} < (w-c)\int_0^T e^{-rt}dt +$ $+ w^* \int_T^Z e^{-rt}dt$, i. e. if $(s\pi - w^*)\int_T^Z e^{-rt}dt < k_T e^{-rT} - k_Z e^{-rZ}$.

For the case where the periodic break up pay-off w^* is 0 we note that a break up can occur only if $k_T > 0$, i.e. if there is an investment component. In the case where there are no investment and salvage value components, the condition above narrows down to $s\pi - w^* < 0$, i.e. $s < w^*/\pi$.

[27] This could be true in duopoly games, where w^* could be the profit obtained if both parties play the periodic sub-game non-cooperatively (cf. p. 187), or in a merger game where w^* would be the profit obtained without an agreement. In many cases w^* would be equivalent to the pre-agreement profit w.

8.3.3 Effect of Break up on the Determination of the Solution

A solution for various games was rigorously deduced in Chapters 4–6 under the assumption that it was *not* possible to break up the bargaining game. This meant we did not have to take special notice of the possibility that the comparisons involved agreements such that $\bar{y}_j < \bar{0}_j$ or $\underline{y}_j < \underline{0}_j$. The question then is whether this deduction can still be relied on if we allow the parties to break up the bargaining game.

We can now – at least for S′-games with many alternatives – prove that in period $j-1$ L will not bid an alternative that would give H a profit smaller than or equal to $\bar{0}_j$.[28] Hence we can deduce that L will not bid any alternative y such that $\bar{y}_j \leq \bar{0}_j$. Likewise H will not bid any alternative y such that $\underline{y}_j \leq \underline{0}_j$. This implies that only such alternatives, which give the other party a *better* pay-off than the one he can obtain on his own, will be bid in the bargaining game. In the example above of pay-off function 1 with no investments the worst share for L and H would thus be $w^*/\pi+\epsilon$ and $W^*/\pi+\epsilon$, respectively. Thus our deduction of the basic model, made without considering whether the parties might bid alternatives which give the other party such a negative pay-off, still stands.

There is, however, one important *limitation* to this conclusion: We must *not* rule out so many alternatives that the solution given by the basic model cannot be deduced. As noted in Chapter 5, deduction of the solution of S-games relies on the assumption that there is a period i and an alternative x such that i=sc$(x-1)$SC(x), making it possible to deduce a *choice* in period i. In order to rely on this deduction we must assume that $\bar{x}_i > \bar{0}_i$ and/or $\underline{x}_i > \underline{0}_i$.

In the case of S*-games, these conditions are written as $V(S^*,T^*) > V(0,T^*)$ and/or $v(S^*,T^*) > v(0,T^*)$. For pay-off function 1 as regards L, this implies that $(s^*\pi-w^*)\int_{T^*}^{Z} e^{-rt}dt > k_T e^{-rT^*} - k_Z e^{-rZ}$, where $s^* = 1-S^*$. In the particular case with no investment component it is sufficient to assume that $s^* > w^*/\pi$ and that $S^* > W^*/\pi$.

Provided that these conditions hold, our basic model can be used. If they do *not* hold, a solution may or may not exist. An exhaustive investigation must be left for future research. For the time being, we take a brief look at the case of pay-off function 1 with no investment component, but with an annual break up pay-off of w^*. If e.g. $s^* < w^*/\pi$, the solution will be different from that of the no-break up case of the basic model. The worst that can happen to L is having to agree to w^*/π. However, as noted above, H is unwilling to risk a break up and thus will not bid w^*/π, but $w^*/\pi+\epsilon$. This means that an agreement will be reached on $w^*/\pi+\epsilon$, i.e. the lower limit of the bargaining game.

[28] The proof is given on p. 283.

Summing up we conclude that the break up pay-offs will *not* affect the solution of S*-games provided s^* and S^* lie within the limits established by these break up pay-offs. Otherwise the solution might be equal to one of these limits.

8.3.4 Threats

8.3.4.1 Introduction

Assumption I_{11} implies that the parties will *not* use any threats in the bargaining game.[29] In this regard our model differs greatly from those of some other authors such as Schelling[30], in which use of threats is essential. Various conditions under which the exclusion of threats appears reasonable will be discussed in this section.

In order to simplify our discussion below, the threatening party is called H and the threatened party L.

8.3.4.2 Definition of the Threat Concept

The threat concept is defined as follows: A threat, made by party H to party L, consists of H's announcement that unless party L fulfills certain conditions, H will be committed to playing a strategy which he would not play without having made this announcement.

A *threat* can be regarded as consisting of *three* parts:

1. the threat *action* – an action which the threatening party H might take against the threatened party L.[31]

2. the threat *condition* – the condition that L has to fulfill in order to make H refrain from the threat action.

3. the threat *announcement* – H declares that he will carry out the threat action if L does not fulfill the threat condition.

[29] This does not rule out the use of threats in the pre-bargaining phase of the negotiation in order to enforce certain norms. See p. 169.

[30] See p. 221 in the literature appendix.

[31] The following two types of threat actions are possible in a bargaining game:

a. The threat action eliminates some of H's strategies such as those implying an early agreement. The pay-off of all agreement outcomes and the pay-off from the break up of bargaining remain unchanged.
b. The threat action adds *new* strategies to H's total set of strategies and will thereby affect L's break up pay-off.

This definition elucidates some of the most important aspects of a threat:

1. the carrying out of the threat action is *conditional* upon the other party's action,

2. the threat action will *not* be carried out, if the threat condition is fulfilled and

3. the threat action will *not* be carried out, if no threat announcement has been made.

According to this definition of a threat, certain acts which one might call *commitments*, can be regarded as threats. An example: H, who has earlier bid y', says that he commits himself to not bidding $y<y'$, in period j to k. This is really a *threat*, since it can be rewritten as H saying that he is committed to *not* reaching an agreement in periods j to k, *unless* L agrees to alternative y'.[32]

On the other hand, H's statement that he will break up the bargaining permanently if L will *not* bid an alternative that gives H a higher pay-off than his *break up pay-off*, does *not* constitute a threat. This is because H would break up the game if the "threat condition" is not fulfilled, regardless of whether or not the threat had been announced.

Our definition of a threat also includes promises that are conditional on other acts. In everyday language, the difference between a threat and a promise is that in a threat, a person commits himself conditionally to an act that is *unfavorable* to the other party, while a promise implies that a person commits himself conditionally to an act that is *favorable* to the other party. Since it does *not* matter in our analysis whether the act is favorable or unfavorable to the other party, we do not need to retain this distinction between threats and promises. For the sake of simplicity the term "threats" as used below also covers promises.

8.3.4.3 Factors Making Threats Credible

Before studying the circumstances under which our disregard of the use of threats in the bargaining game appears reasonable, we discuss why a party would regard the use of threats as inefficient and refrain from it. The very core of the efficiency of threat strategies lies in the question of whether they can be made *credible*, i.e.

[32] A pure commitment *without* any threat ingredient would exist if H says that he is committed to not reaching an agreement in periods i to k, regardless of what L does, i.e. even if L agrees to y'.

whether the threatened party L believes that the threatening party H will really carry out the threat action if L does not fulfill the threat condition. The credibility of a threat depends in turn on L's expectations of a) H's cost of carrying out the threat and b) H's cost of *not* carrying out the threat. It should be stressed that "H's cost" implies "H's subjective cost", i.e. H's estimates of his costs or disadvantages from this act. The question then becomes: Which factors will tend to make a threat credible? The following five factors will be dealt with:

1. Institutions enforcing commitments. This refers to the idea of institutions which specialize in making threats enforcable.[33] By having the power to make failure to carry out a threat very costly, these institutions can force H to carry out his threat.

2. Considerations of one's principals and similar groups. This refers to H's attempts to commit himself by e.g. raising the outcome expectations of his principals (e.g. the workers in a union, etc.), thereby increasing the cost of losing face if he fails to carry out his threat.

3. Considerations of future negotiations with *other* parties. If party H fails to carry out a threat in his negotiation with party L, and some other party perceives this, H will have difficulty making a threat against this other party credible in the future.

4. Considerations of future negotiations with the *same* party. If H fails to carry out a threat in a negotiation with party L, then L will be more reluctant to regard H's threats as credible in future negotiations with H.

5. Considerations of later stages of the negotiation in progress. A series of threats are presented during the negotiation, where the early threats are made credible by the consideration that they must be carried out in order to make later threats during the negotiation credible. These early threats constitute a restricted type of threats having the following characteristics:

a) The threat action does not refer to all remaining periods, i.e. we rule out the threat of a complete break up and the threat that the party will refrain from a certain bid in every period.

b) The threat condition is not *complete* acceptance of the terms of the threatening party H. The threat condition can be either 1) a lowering of L's demand or 2) a particular action in some game, outside of the pure bargaining game, but related to it. An example is duopoly games.

[33] See Schelling (1960, p. 25).

8.3.4.4 Reasons for Excluding Threats in our Model

The main reason for excluding threats is that the set of negotiations in which rational parties[34] would regard any threat as non-credible (or slightly credible) appears to be large – sufficiently large, according to our discussion on p. 23 – to make our model interesting. This hypothesis is based on the following brief study of the reasons presented above with respect to the credibility of a threat. This study also indicates the conditions under which the exclusion of threats appears reasonable.

1. Very few institutions which enforce commitments appear to exist in reality.

2. The credibility factors concerning commitment appear to lose much of their efficiency if the parties have roughly equal commitment possibilities. There seems to be a high risk that *both* parties will commit themselves to a threat action, and hence obtain a worse outcome than they otherwise would have obtained if they had made an explicit agreement to rule out threats at the very beginning of the bargaining.[35] Since it is important to be the first party to commit oneself, there would be a rush to commit oneself if no such agreement had been made.[36]

As concerns considerations of one's principals, commitment possibilities will theoretically depend on the assumption that the principals have limited rationality. If the principals have the same B_4-rationality assumed for the bargaining party and can be given full information – e.g. by the other party – no commitment can be obtained in this respect. This might be a rough approximation for some cases where the principal consists of a board of directors. If the principal consists of a great many union members or voters, our rationality and information assumptions can hardly be considered appropriate. Hence commitment possibilities can very well exist for a union.[37] Therefore there are probably more unequal commitment possibilities in labor-management negotiations than in negotiations between corporations.

3. Considerations of future negotiations will make a threat credible only if the other parties can perceive whether or not a threat is carried out. Hence, if the

[34] As defined e.g. by set B_4.

[35] It should be noted that considerations of other negotiating parties and future negotiations with the same party are factors which tend to enforce this agreement. If someone cheats on a no-threat agreement, he might have great difficulty reaching such an agreement later on.

[36] Cf. Schelling (1960, Chapter 5).

[37] However, it should be stressed that it is often very difficult to commit oneself, also against less rational principals. Commitment might require time, since a threat might have to be repeated or delivered by different people close to the bargaining agent. Furthermore there are difficulties in ensuring that a failure to carry out a threat would necessarily be costly to oneself.

bargaining process is kept secret to outsiders, credibility factor 3 will not be effective. This appears to be true for many types of business negotiations, particularly those which concern limitation of competition, such as the duopoly games presented in Section 9.3.

Furthermore, there are roughly equal commitment possibilities if *both* parties can be foreseen to enter into several future negotiations, but it cannot be said whether any one of the two parties will engage in considerably more of such activity than the other party. This situation is probably quite common. Then the same kind of reasoning used for factor 2 applies.

Credibility factor 3 might also be weak in situations with unequal commitment possibilities. If H – with several future negotiations ahead of him – would enforce a threat against party L, e.g. because L has no future negotiations with other parties, H would run the risk of having his future negotiating parties become reluctant to negotiate with him and turn to other parties instead. Thus credibility factor 3 would tend to apply mainly to cases when H's future negotiating parties do *not* have other parties to turn to, i.e. we require H to have strict bilateral monopoly relations with his future negotiating parties.

4. Credibility factor 4, i.e. considerations of future negotiations with the same party, is inefficient not only in all negotiations of a non-recurrent type, e.g. merger negotiations, but also if we assume that the parties can only foresee a limited number of future negotiations with each other. This is due to our insight assumption B_{10}.[38] Assume that the parties foresee n future negotiations and that they also foresee they will understand that a certain negotiation is the last one – or at least one of the few last ones[39] – when they come to it. Then threats will not be credible due to factor 4 in negotiation n. Since the threats in period n will not be credible in this respect, threats in negotiation $n-1$ cannot be made credible by referring to the argument that the threats will be carried out just to make the threat in negotiation n credible. We can proceed backwards in this way, one negotiation at a time, and finally prove that there is no reason to carry out the threat in the present negotiation.

5. Likewise, as regards consideration of future smaller threats in the same negotiation, assumption B_{10} will imply that a threat will not be credible if the

[38] It should be noted that $I_1'-I_5'$ can be substituted for B_{10} (cf. Chapter 7). Cf. also footnote 16 on p. 138.

[39] The incentive to carry out a threat would be considerably smaller if a few, rather than a great many future negotiations, might remain. The backwards deduction also holds if a party, when coming to a negotiation n, believes that either this or $n+1$ is the last negotiation. He will regard threats in negotiation n as non-credible also if negotiation $n+1$ is the last negotiation.

game will be broken up in a period z within finite time.[40] A threat in period $z-1$ cannot be made credible by referring to credibility factor 5 and hence a threat in period $z-2$ will not be perceived as credible either. Proceeding backwards in this fashion, we deduce that threats will *not* be credible in any period.

8.3.4.5 Threats in the Pre-bargaining Phase of the Negotiation

It should be stressed that we have only ruled out threats in the bargaining game, not in the pre-bargaining phase of the negotiation.[41] Earlier (cf. pp. 140 and 161) we discussed the use of threats in the pre-bargaining phase in order to enforce certain norms with an ethical content, namely the following:

1. A norm ruling out persistent lying against evidence.

2. A norm ruling out that a party claims the other party is lying without presenting any evidence of this.

3. A norm establishing good-faith bargaining.

The main reasons for regarding these threats as considerably more credible than threats in the bargaining game are:

1. We can allow for equal commitment possibilities, since the use of threats will not be necessary if both parties want to enforce one of these norms. This also implies that the order in which the parties make their threats is unimportant.
2. Due to the strong ethical content of these norms, a party might experience the other party's refusal to accept these norms as so unpleasant[42] or as leading to such a loss of face that he might actually prefer carrying out his threat of withdrawing from the negotiation, if these norms are *not* accepted.

8.3.5 Liquidity

According to I_{11}, we assumed that liquidity is so large in relation to the possible losses from no-agreement, that neither party will go bankrupt even if no agreement

[40] In principle the same kind of reasoning could also be used against credibility factor 3, regard to future negotiations with other parties, e.g. when a party negotiates with a known number of future parties. However, this application involves very extreme information requirements.

[41] It should be stressed that we do not have to make a chronological distinction between the bargaining game and the pre-bargaining phase. The bargaining game concerns proposals for agreement, the pre-bargaining phase concerns the establishment of rules for the bargaining game.

[42] This is probably particularly true of norms 1 and 2.

is reached prior to period z. The implications of assumption I_{11} can be made more concrete by looking at its application to pay-off function 1. In this instance it would imply that total initial liquidity is large enough to cover the bargaining costs and possible pre-agreement losses (if w is negative). Assuming that the initial liquidity for party L – called k_0 – is invested at interest rate r, this initial liquidity k_0 has to be larger than $\int_0^Z (c-w)e^{-rt}dt$.[43]

The question then is what would happen if I_{11} were violated. Does it become impossible to use our basic model?

First of all, we note that with regard to S-games, the requirement can be made less restrictive by demanding that no bankruptcy occurs prior to the last interesting period z^*, instead of prior to z.

The problem becomes more difficult if this modified requirement is also violated. In order to investigate this problem in a simple manner, we refrain from assuming that there is a fixed amount of liquidity available to the parties. It is more convenient and reasonable to assume that a party, e.g. a corporation, has considerable possibilities of obtaining additional capital. However, the cost of capital (e.g. the interest rate) can increase very drastically as the amount of capital required increases. One can imagine a corporation obtaining loans from more and more dubious sources or raising additional money through stock issues. In the latter case, the increasing cost to old share-holders of raising new capital can e.g. be reflected by rapidly falling share prices. In principle, a complete liquidity crisis can be represented by letting the cost of capital go towards infinity.

The advantage of looking at the bargaining game in this way is that the pay-off functions can still be regarded as continuous. Any bargaining games having such a function can be investigated using our general model presented in Chapter 4. In this context, however, we refrain from any investigation of such pay-off functions, leaving this to future research.

8.4. The Alternatives in the Bargaining Game

8.4.1 Solving the Efficiency Problem

8.4.1.1 Introduction

In our basic model we were concerned only with the solution of the distribution problem. It was assumed in I_7 that the two parties in the bargaining game had

[43] For the sake of a simple deduction, the cost $c-w$ can be regarded as covered by loans, carrying a loan rate r, while k_0 is invested at rate r. We then require that at time Z, k_0 plus accumulated interests are larger than the accumulated debts, i.e. that
$k_0 e^{rZ} > \int_0^Z (c-w)e^{r(Z-t)}dt$, implying that $k_0 > \int_0^Z (c-w)e^{-rt}dt$.

completely opposing interests with respect to the different alternatives. We now turn to the question of the relevancy of this assumption. It obviously holds for negotiations in which no efficiency problem exists at all, such as the purchase of a given unit. Our question here is whether the applicability can also be extended to other situations. In particular we are interested in bilateral monopoly (cf. p. 16). The agreement then concerns two dimensions: the *number of units* bought by the monopsonist from the monopolist and the *price* which the monopsonist pays the monopolist for each unit.

In dealing with situations such as this we begin (in 8.4.1.2) by defining a *contract curve.* Next we investigate (in 8.4.1.3) the behavioristic assumptions required in order to place every final agreement on the contract curve. We find that B_1 is *not* sufficient, and that B_2 is also required. Finally, we show (in 8.4.1.4) that for many S*-games, B_4, $G_1 - G_3$ and $I_1 - I_{13}$ can be used to deduce that the parties will only deliver bids along this contract curve.

8.4.1.2 Establishing the Contract Curve

Let us look at a bargaining situation characterized by bilateral monopoly. A buyer and a seller, e.g. an agent and a manufacturer, try to reach an agreement on the number of units, q, to be transferred annually and the transfer price P which the buyer pays the seller for each unit. The seller wants the price P to be as *high* as possible for a given quantity. In accordance with our previous method of notations the seller will be called H(igh) and the buyer L(ow). Each party will have preference relations between possible agreements on a specific quantity q and transfer price P. A set of indifference curves can be determined for each party on the basis of these relations. Each indifference curve of a party will consist of all those agreements between which the party is indifferent. For the sake of simplicity let us start with the case commonly examined in economic theory, namely the static case, when the pay-off of an agreement on a certain alternative does not vary over time.

For the particular case where we assume that each party's utility is dependent only on the monetary value of the agreement and each party prefers more money to less, the indifference curves will consist of all agreements which lead to the same profit. The seller H's annual profit, valled W, can be written as $q(P-C)$, where C is H's average cost per unit, possibly varying with q. Assuming that buyer L in turn resells the product on a market at market price p, L's annual profit w can be written as $q(p-P)$.[44] Assuming in particular that $p=8-q$ and $C=4$, we obtain the diagram shown in Figure 25.[45] This figure can be used to determine what is called the

[44] For the sake of simplicity we assume that L's sales costs are insignificant.
[45] The line dd' can be disregarded in this context. It is used in the literature appendix.

contract curve. This curve is defined as the locus of all points of tangencies between the two parties' indifference curves. It is represented in the figure by the vertical line with $q = 2$. The contract curve has the following property: For every point outside the contract curve (e.g. *a*) there is at least one point *on* the contract curve (any point between *b* and *c*) which is such that *both* parties can obtain a better agreement than at the original point *not* on the contract curve. In other words, all solutions on the contract curve are Pareto-optimal.

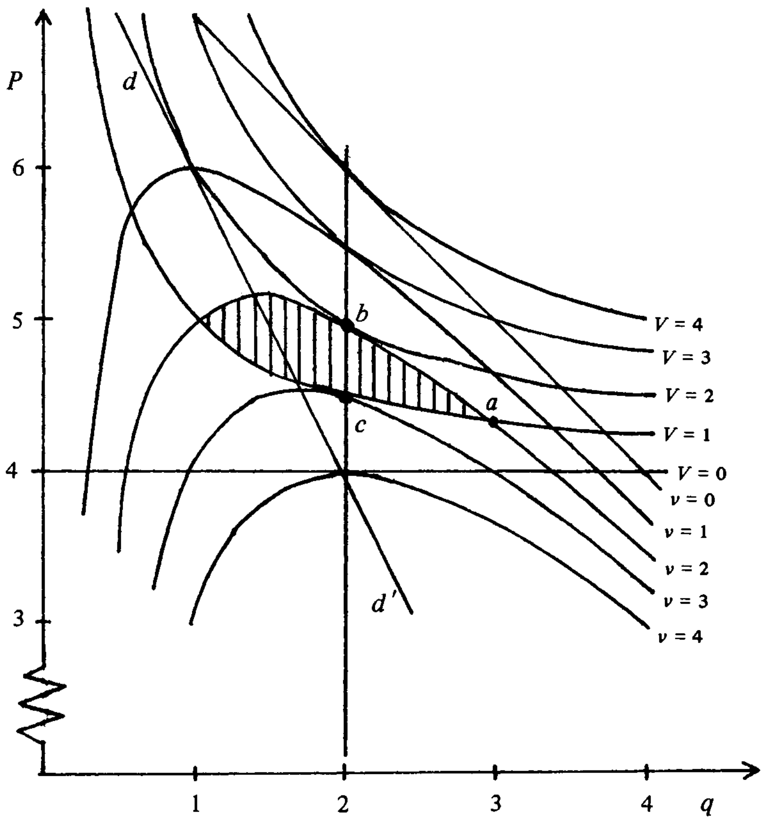

Figure 25 A bilateral monopoly situation

8.4.1.3 Establishing that the Solution will Lie on the Contract Curve

As shown in the literature appendix (p. 220) most authors regard the efficiency problem in bilateral monopoly situations as solved in the sense that the agreement will lie on the contract curve. In order to ascertain the behavioristic assumptions implicitly made by these authors, we ask: Which behavioristic assumptions are necessary for arriving at this conclusion?

The first question is whether assumption set $\boldsymbol{B}_1$ alone suffices for establishing that the final solution must lie on the contract curve. Let us assume that both parties are about to reach an agreement on a point a outside the contract curve. Is it then self-evident that the parties would rather reach an agreement on the contract curve? Let us e.g. localize point a at the intersection of the curves implying that v=2 and V=1, respectively (see Figure 25). L is then willing to switch to any point below the curve implying that v=2, but not to any point above this curve. If only $\boldsymbol{B}_1$ holds, H does not necessarily know either L's preferences or that L is rational and H does not necessarily realize that L will only accept points below this curve. Hence, relying only on B_1, we cannot be certain that the parties will limit themselves to the lined area between these curves.[46]

Hence, assumption set $\boldsymbol{B}_1$ is *insufficient* for deducing that the solution will lie on the contract curve. The assumptions of $\boldsymbol{B}_2$ also have to be included, which in addition to $\boldsymbol{B}_1$ imply that each party realizes that the other party acts according to $\boldsymbol{B}_1$, and that each party knows the other party's preference relations. It can now be shown that if the behavioristic assumptions of $\boldsymbol{B}_2$ are also included, the efficiency problem for bilateral monopoly situations can be solved for at least some set of institutional assumptions. In addition to I_3 and I_5, implying that each party has information about his own and the other party's pay-offs, let us rely on the following institutional assumption:

I_8': If an agreement has been reached in one period, a party e.g. L can propose a *new* agreement. If H rejects this new agreement in the next period, a final agreemcnt is reached on the terms of the original agreement. On the other hand, if H accepts this new agreement, H can in turn make a new proposal.

In this way the parties can successively reach new agreements until some party rejects the other's proposal for a new agreement. At this point a final agreement is reached on the terms of the latest accepted agreement.

It can be proved that every such *final* agreement will lie *on* the contract curve. Let us assume that an agreement is reached with the pay-off pair V,v outside the contract curve. Then if e.g. L proposes an agreement with V',v' such that $V' > V$ and $v' > v$, H will accept this agreement. If this agreement is not on the contract curve, H can in turn suggest an agreement with V'',v'' such that $V'' > V'$ and $v'' > v'$. In this way, the parties can move towards more and more favorable agreements, until they reach the contract curve.

46 This conclusion obviously holds if the parties can make only a few proposals for a new agreement. It also holds when the parties are able to make many proposals for a new agreement, since it cannot be ruled out that H will bid so as to influence L's perception of H's preference relations in a false direction.

A similar analysis applies to the case where the value of an agreement on a certain q and P varies over time. We refrain from analyzing this case here, since more extensive conclusions with respect to S*-games will be established in the next section.

8.4.1.4 Establishing that All Bids will be on the Contract Curve

The conclusion that *every* final solution will ultimately lie on the contract curve is insufficient for deducing that the parties will only suggest proposals for agreement which lie on the contract curve. On the basis of B_2 alone, we can *not* rule out the possibility that some party might believe he can obtain a more favorable final agreement, if the parties first reach a preliminary agreement outside the contract curve. However, it can be proved for at least all S*-games for which the distribution problem could be solved, that no bids outside the contract curve will be contemplated.

In the bilateral monopoly situation under study, $W=q(P-C)$ and $w=q(p-P)$. Hence $W+w=q(p-C)=\pi$. Setting $S=(P-C)/(p-C)$, we can write W as $S\pi$ and w as $(1-S)\pi$.[47]

Let us next assume either

1) that the contemplated contract will run to a fixed time Z, or

2) that the contract will run to a time $Z+\theta T$, where $\theta<1$ and T is the time of agreement or

3) that the sales at the time of agreement, computed on an annual basis, will increase $(A-BT)$ 100 per cent as the agreement is delayed[48] and that the contract will run to a time $Z+T$.

These three conditions correspond to pay-off functions 1, 2 and 3, respectively in Section 6.4, implying that we can write $V(S,T) = S\pi F(T)$ and $v(S,T) = (1-S)\pi f(T)$, where $F(T)$ and $f(T)$ fulfill the requirements of the S*-games. A unique solution S^* can be determined for these functions,[49] completely independent of the size of π. Hence an agreement on any quantity will lead to the *same* S^* as an agreement on the Pareto-optimal quantity of the contract curve. This means that no party can reach a more favorable temporary agreement *outside* the contract curve than the final agreement on the contract curve. This in turn implies that we can exclude the

[47] $S\pi=\pi(P-C)/(p-C)=q(p-C)(P-C)/(p-C)=q(P-C)=W$ and $(1-S)\pi=q(p-C)(1-(P-C)/(p-C))=q((p-C)-(P-C))=q(p-P)=w$.

[48] If $A=0$ or B is large in relation to A, sales will decrease.

[49] As long as $F^*(0)+f^*(0)\geqslant\mu$ (cf. p. 117).

possibility that a party can influence the final agreement in his favor by first reaching a preliminary agreement on the contract curve.

Furthermore, if the parties limit themselves to bids on the contract curve an agreement will be reached in one of the first two periods, while if bids outside the curve are delivered first, an agreement will be reached later on. Hence, there are no advantages – only disadvantages – to either party from bidding in any period outside the contract curve.

In these circumstances, the solution of the bilateral monopoly problem is that the parties agree on a P such that $(P-C)/(p-C) = S^*$, i.e. $P = (p-C)S^*+C$. Hence, the transfer price P is such that the seller H obtains his unit variable cost C *plus* his share S^* of the difference between the market price p and his unit variable cost C, where p is that price which maximizes $q(p-C)$.

8.4.2 Determining the Most Preferred Alternative for Each Party

I_8 consists of the following assumption: In the bargaining game the parties are bound to bid alternatives such that one given alternative is the most favorable one for H and another given alternative is the most favorable one for L.

L's and H's most favorable alternatives have often been designated alternative 1 and alternative n, respectively. The way in which these two alternatives can be determined will be discussed briefly. We concluded earlier (see p. 163) that in period $j-1$, L will not bid an alternative y such that $\overline{y}_j \leqslant \overline{0}_j$ and H will not bid an alternative y such that $\underline{y}_j \leqslant \underline{0}_j$. Hence we can rule out any alternative y such that $\overline{y}_j \leqslant \overline{0}_j$ or $\underline{y}_j \leqslant \underline{0}_j$ for *every* j.

The implications of this requirement will be studied here only in relation to pay-off function 1 without an investment component and with the break up pay-off leading to an annual flow of w^* or W^* up to time Z. The investigation of other cases is left to future research. For L, the requirement that $\underline{y}_j > \underline{0}_j$ for $j = T\Delta t$ can be written as $(s\pi - w^*) \int_T^Z e^{-rt} dt > 0$. This means we require that $s\pi - w^* > 0$, i.e. $s > w^*/\pi$. Hence the minimum value of s which is acceptable to L, is $w^*/\pi + \epsilon$. In the particular case when $w^*=0$, i.e. when the break up leads to zero profits, the lowest value of s is ϵ, where ϵ is the smallest difference between those values of s which can be covered by the contract. Thus when a certain sum is divided into whole percentages, L's least favorable alternative is 1 per cent to L, 99 per cent to H.

8.4.3 Determining the Number of Alternatives

8.4.3.1 Introduction

We now turn to an investigation of assumption I_9: The alternatives in the subset determined by I_7 and I_8 which can possibly be bid in this game are determined prior to the bargaining game. This assumption implies that the number of alternatives, n, is determined *prior* to the actual bargaining game. Hence $N=n+1$ is given. This assumption will be discussed here with reference to S*-games.[50]

Our earlier discussion on pp. 151–152 can be used to infer that this assumption is of little importance to some S*-games and of great importance to others. As shown for pay-off functions 1 and 3, the solution is completely independent of the size of $\mu=N\Delta t$ and hence also of N, when $r=R$. Furthermore, if we know that N is of a moderate size, while Δt is very small, it can be determined that the solution is independent of the exact size of μ for several cases where $r \neq R$ (cf. p. 152). But when $r \neq R$ and μ is *not* very small, we find that the solution, if existent, will probably be dependent on the size of μ. As shown e.g. for pay-off function 3 (see p. 127), the share to be obtained by H when $r > R$ will be higher, the *larger* μ is. Then H will be interested in making μ as large as possible, while L wants to make it as small as possible. Correspondingly, $R > r$ implies that L wants to make μ as large as possible, while H wants to keep it small.

It is reasonable to assume that the solution is dependent on μ not only in the case mentioned above, but also in many other types of games. However, the exact scope of this problem must be left for future research. It suffices for our purposes here to define the following pay-off assumption, $\hat{P}$:

The pay-offs of the S*-game are such that the solution y^*/N varies monotonously with μ.

When the total number of alternatives is given by outside forces such as trade usage, $\hat{P}$ will *not* cause any problem. But if the parties are free to determine the size of μ among themselves, $\hat{P}$ raises the question of how the size of μ should be determined. Since the parties have opposing interests the size of μ must be part of the actual bargaining and hence μ cannot be determined in the pre-bargaining phase of the negotiation.

For the sake of simplicity we assume first – in 8.4.3.2 – that Δt is given (e.g. in the way described on p. 150) and that the total variation in $\mu=N\Delta t$ is due to variations in N. Then the case where Δt can also vary is discussed in 8.4.3.3.

[50] By studying S*-games we limit our study to games where an equal distance along some scale is assumed between each alternative. Then the only variance to be disputed is the total number of alternatives.

8.4.3.2 Δt is Given

The case studied here involves S*-games in which the exact value of N cannot be determined in the pre-bargaining phase because the parties want different values of N. The question then is which alternatives can be bid in the bargaining game. The following assumption appears to be the most reasonable:

I'_9: If *no* agreement is reached in the pre-bargaining phase of the negotiation as regards which alternatives can be bid in the bargaining game, each party is free to bid any alternative he wants to, within the limits given by assumptions I_7, I_8 and I_{13} and provided the alternative implies meaningful agreements.[51]

Assumption I'_9 can be explained as follows: Let us assume that the bargaining concerns the share of a given sum to be obtained by each party during a certain number of periods. The alternatives can be defined unambiguously by a real number indicating the annual amount $S\pi$ going to H. Under these circumstances I'_9 implies that the parties are free to name any amount $S\pi$ on which an agreement can be reached. Let us study a case where $R > r$ and where L wants N to be as large as possible. Then, if an annual sum of \$ 100,000 is to be divided, L can in principle suggest an agreement on any amount as low as one cent or, correspondingly, as low as the fraction $1/100{,}000 \cdot 100 = 0.000\,000\,1$. In this case N could be as high as 10 millions. Denoting the highest possible value of N as N^{max}, we obtain that $N^{max} = 10{,}000{,}000$. H cannot stop L from setting $N = N^{max}$. H can in turn propose whatever type of agreement he likes, e.g. involving only whole \$ 1,000, implying $N = 100$.

The next question is what effect this difference in the desired size of N has on the outcome of the bargaining game. We begin by looking at the simple 3-alternative game (5,7) presented on p. 81. Let us assume that L wants this to be a *three*-alternative game, while H wants it to be a *two*-alternative game, with 5 and 7 as the only biddable alternatives.

In period 1, H will only bid 5 or 7. But H cannot prevent L from bidding 6 in period 2. If L bids 6, H has the choice in period 3 of accepting 6 and getting $\overline{6}_3$ or insisting on 7 and at best getting $\overline{7}_4$. Since $\overline{6}_3 > \overline{7}_4$ H will accept 6. Hence L can obtain $\underline{6}_3$ by bidding 6 and since $\underline{6}_3 > \underline{7}_2$, L will *not* accept 7 in period 2. Since H's bid 7 in period 1 thus leads to $\overline{6}_3$ and $\overline{5}_1 > \overline{7}_3$ H will accept 5 in period 1. Hence the outcome is the same as in the original case, even if H is *not* willing to be the *first* party to bid alternative 6. Once alternative 6 has been bid H must regard it as an alternative and the original three-alternative game is obtained. Thus the only difference as compared to the original case is that H will *not* bid 6 in period 1, although this will *not* affect the conclusion that H accepts 5.

[51] For example, it is *not* meaningful to suggest an alternative involving fractions of a cent.

In a similar manner it can be deduced, for every S*-game characterized by $\hat{\mathrm{P}}$, that $(x,x+k)_{j-1} = x_j$ (for every $j < i$) even if H refuses to be the first to bid a great many alternatives between x and $x+k$. A proof of this conclusion is given in the appendix on p. 284. This conclusion implies that if H has bid $x+k$ in some period, then by bidding x, L can *enforce* an agreement on x regardless of how much H wants to limit the number of alternatives.

Likewise, if H is the party desiring a high value of N, and L has bid $x-m$, then H can enforce an agreement on x. Hence with respect to S*-games characterized by $\hat{\mathrm{P}}$, we conclude that the party desiring a large N *can* enforce an agreement on the most favorable x he can obtain by making n as large as possible, with the restriction that P and P' hold for a finite $i \geqslant 3$.

For S*-games with a great many periods, this restriction implies that n must be such that $F^*(0)+f^*(0) \geqslant \mu$ (cf. p. 117). For the sake of simplicity we can let μ^* denote $F^*(0)+f^*(0)$ and write this requirement as $\mu^* \geqslant \mu$. Since $\hat{\mathrm{P}}$ implies that y^*/N is a monotonous function of μ, this restriction determines the value of μ, i.e. the party desiring a large value of N would make μ equal to μ^*.[52] Therefore the highest value of y, called $\hat{y}$ is that established by $F^*(T^*)N/\mu^*$.[53] When $\mu=\mu^*$, $T^*=0$[54] and hence $\hat{y} = F^*(0)N/\mu^* = F^*(0)N/(F^*(0)+f^*(0))$.[55]

For pay-off function 1 $\hat{y}/N$ is $1/(1+\int_0^Z e^{-rt}dt/\int_0^Z e^{-RT}dt)$.[56] When $Z \to \infty$, $\int_0^Z e^{-rt}\,dt \to 1/r$ and then $\hat{y}/N = 1/(1+R/r) = r/(r+\mathrm{R})$.

For pay-off function 3 the largest enforceable value of S, called $\hat{S}$, is $\hat{y}/N = (r-\beta_0)/(r+R-2\beta_0)$.

8.4.3.3 *Variations in Δt*

In the preceding section we established a value which one party can enforce when he makes μ as large as possible. We assumed that μ varied only with the number of

[52] If this restriction did not apply, the party desiring a large N would let μ approach $N^{max}\Delta t$.

[53] $y=F^*(T)/\Delta t$ (cf. p. 116) $\Rightarrow y/N=F^*(T)/\mu$. We also require that $\mu^* < N^{max}\Delta t$ in order for $\hat{y}$ to be the highest enforceable value of y.

[54] T^* is that value of T for which $F^*(T)+f^*(T) = \mu$. Since S_4^* implies that $d(F^*(T)+df^*(T))/dT < 0$, T^* must be 0, if $\mu = \mu^*$.

[55] Foldes' model (presented on pp. 238 ff.) – though derived in a completely different way – would lead to the same solution if applied to S*-games. See furthermore footnote 74 on p. 240. Foldes, however, seems to rule out the application of his model to S-games (see p. 243).

[56] Cf. p. 271. When W, w, K_T, k_T, C *and* $c = 0$, $\hat{y}/N = \hat{S}$, i.e. the largest enforceable value of S.

alternatives and that the length of each period Δt is given. The question then is whether the conclusions arrived at above can be retained if Δt is also allowed to vary.

We notice that if $r > R$, then for e.g. pay-off function 3, H wants μ to be as large as possible, while L wants μ to be as small as possible. Since H can make N as large as he likes up to N^{max}, L will try to make Δt as small as possible. A value Δt^{min} can be established for a specific game through a process similar to the one used to establish the value N^{max} on the basis of institutional assumptions.

The question then is whether $N^{max}\Delta t^{min} \geqslant \mu^*$. If this is true H can set $\mu = \mu^*$, even if L would set $\Delta t = \Delta t^{min}$. On the other hand, if $N^{max}\Delta t^{min} < \mu^*$, the size of Δt might become significant. We hypothesize, however, that for a great many games Δt^{min} is at least a few hours, due to reasons similar to those discussed on pp. 150 and 154. This implies that $N^{max}\Delta t^{min}$ is large in relation to probable values of μ^*.[57] Since it therefore appears reasonable to assume that the situation where $N^{max}\Delta t^{min} < \mu^*$ is *not* of overwhelming importance we leave this case for future research and – for the time being – regard it as not having a solution.

8.4.3.4 Does $\hat{y}$ Constitute a Solution?

With reference to certain cases characterized by $\hat{\mathrm{P}}$ (defined on p. 176), we have been able to establish a highest value of y, called $\hat{y}$, which one party e.g. H – according to our model – can enforce.[58] But without further deductions we can *not* conclude that an agreement will really be reached on $\hat{y}$. The possibility still remains that H would try to obtain an even more favorable agreement than $\hat{y}$ by making $\mu > \mu^*$. As noted above, L cannot stop H from doing this, if $\mu \leqslant N^{max}\Delta t^{min}$.

This brings us to the question of whether H, under pay-off assumption $\hat{\mathrm{P}}$, will attempt to reach a more favorable agreement than $\hat{y}$. No conclusions can be drawn by relying only on our *special* model since $\mu > \mu^*$ violates the assumptions on which this model is based.[59] This means we have to return to the general model discussed in Chapter 4. The computer program version of the general model was run for some simple games based on pay-off functions 1 and 3. On the basis of these runs it appears reasonable to hypothesize the following with respect to at least some S*-games: L can enforce an agreement on an alternative slightly lower than $\hat{y}$, even if H sets μ considerably higher than μ^* (e.g. $\mu = 2\mu^*$). The extent to which this might lead to more general conclusions has to be left for future research.

[57] For pay-off function 3, $\mu^* = 1/(r-\beta_0)+1/(R-\beta_0)$ and hence for $r-\beta_0$ and $R-\beta_0 \geqslant 0.04$, $\mu^* \leqslant 50$. Then for $\Delta t^{min} > 0.0005$ and $N^{max} \geqslant 100{,}000$, $\mu^* < N^{max}\Delta t^{min}$.

[58] On p. 177 L was the party who could enforce $\hat{y}$.

[59] The existence of a solution requires that $\mu \leqslant \mu^*$.

If $\mu > \mu^*$ implies that L can enforce an agreement on an alternative only slightly better than $\hat{y}$ for H, while H possibly runs the risk of no agreement or a much poorer result than $\hat{y}$[60], then H would quite likely be content to obtain $\hat{y}$ by setting $\mu=\mu^*$. Against this background we hypothesize – open for future research to reject or confirm – that for a large number of different S*-games characterized by $\hat{P}$, the solution will be an agreement on $\hat{y}$.

8.5 Complete Information

The most severe restriction as regards the institutional assumptions is probably assumption I_5 that each party has complete information about the other party's pay-off. This assumption is rarely fulfilled in reality. But this does not render an analysis of the complete information case uninteresting, due to the following reasons:

1. As in many other areas of economic theory it appears necessary to *base* any theory, which assumes rational expectations but less than complete information, on a theory of complete information. An important area for further research should be the gradual removal of the complete information assumption.[61]

2. It should again be stressed that our assumption of complete information is not as strong as in some other bargaining theories such those of Nash and Zeuthen. Since our model relies solely on an *ordinal* utility, it is sufficient to know how the parties rank their outcomes. Hence, if utility would be a monotonous function of the monetary value of the outcomes, it would be sufficient in our theory to know the monetary value – e.g. profit – of each outcome. In the other theories mentioned above, each party would also have to be assumed to know not only his own, but also the other party's cardinal utility function for money.[62] This appears to be a very strong assumption in many cases.

3. It should also be emphasized that the special model which solves S-games relies on a weaker assumption than that of complete information. In this case it is sufficient to assume that both parties know

   a) that the pay-off decreases over time and
   b) whether each period is critical, semicritical or uncritical for either party as regards each possible two-alternative subgame.

[60] If H makes $\mu > \mu^*$, the special model can *not* be employed to conclude that H can enforce $\hat{y}$..

[61] The author has made some preliminary attempts at such a theory for a very simple type of bilateral monopoly situation. As in 8.4.1, it is assumed that $w=q(p(q)-P)$ and $W=q(P-C)$, that the pay-off functions are the simplest possible version of pay-off function 1, i.e. $V=S\pi(Z-T)$ and $v=(1-S)\pi(Z-T)$. Furthermore, the lack of information is limited to L not knowing the size of H's unit cost C, but only that it lies within a certain interval.

[62] See also o. 218.

When both parties are aware that the game is an S*-game, it is generally sufficient for a solution to assume that both parties agree on that point (y^*, T^*) for which $y = F^*(T)/\Delta t$ and $(N-y) = f^*(T)/\Delta t$. This reduces the amount of information required. As regards the particular pay-off functions 1–3 in 6.4, the information required for establishing the solution y^* is not very large, provided the general form of the pay-off function is known. We notice that it is *not* necessary to know the size of Z for pay-off function 1 or β' and γ for pay-off function 3.

4. Even if the parties do not initially have complete information prior to the negotiation, they can obtain it in the pre-bargaining phase. In order to ensure a rapid agreement the parties might agree on an exchange of information. In other cases a mediator, with access to both parties' information, might supply each party with the necessary information. In some instances, this might be the most important phase of the mediation. In situations such as the determination of intra-company prices by bargaining, this type of information sharing seems likely. Central management usually has an interest in obtaining, and the ability to ensure, complete information.

5. Finally it should be stressed that even if the information requirements are not and cannot be fulfilled, our model is of direct interest to the extent that it can determine the *limits* within which the solution will lie. Let us illustrate this using the simple example on p. 122. If H does not know that w = $ 10,000, but that w lies between $ 6,000 and $ 14,000[63], H can determine that he will obtain between $ 60,000 and $ 64,000.

[63] We also assume that L knows that H's information about w is limited to this interval.

Chapter 9
Three Examples of Applications

9.1 Introduction

In this chapter we return to the three examples introduced at the beginning of Chapter 1. These three examples were presented with the following purposes in mind:

1. The examples should indicate whether our model's institutional assumptions are of interest in the sense discussed in Chapter 2 (p. 22).

2. The examples should illustrate the implications of the assumptions behind the S*-games.

3. The examples should describe some possible areas of application and some of the main problems encountered when the model is applied to these areas.

4. The examples should provide a preliminary basis for discussing the face validity of our model (cf. p. 25).

It should be stressed, however, that these examples of applications do not imply that the model is intended for direct use in real-life applications. As discussed in Chapter 2, our model has a more fundamental research purpose. The model should be thoroughly tested, e.g. in laboratory experiments, before it can be used in real-life situations, e.g. for mediation. Undoubtedly real-life applications also require the model to be extended in several of the ways discussed in Chapter 8.

9.2 Mergers

9.2.1 General Analysis

In the first example on p. 1, a corporation is in the process of purchasing another corporation. The question to be resolved by bargaining is how many shares of the purchasing corporation should be given for each share of the purchased corporation.

A more general merger situation will be dealt with in this section. We then return to the specific example in 9.2.2. A general merger situation involves two parties who pool certain resources (e.g. merge two corporations) to form a new corporation. The issue to be resolved by bargaining is each party's share of the new corporation. The example on p. 1 can be regarded as a special case, where each party puts an old corporation into the new corporation, which is given the name of one of the old corporations.

In order to facilitate comparisons with earlier chapters, we call the two parties L and H and the parties' shares of the new corporation s and S, such that $s+S=1$.

We start the analysis by looking at L's pay-off $v(S,T)$ from an agreement on S at time T. Part of this pay-off consists of a *pre-agreement* profit component. Up to time T, when the new corporation is formed, L uses his resources independently. We then assume that in each period, L obtains a profit which, calculated on an annual basis, is w at the start of the bargaining and increases by $100\,\beta$ per cent annually so as to become $we^{\beta t}$ at time t. It is also assumed that these profits, before being distributed to L are subject to a corporate income tax of $100\,\tau$ per cent – possibly 0 – i.e. L obtains $(1-\tau)we^{\beta t}$ at time t.[1] If L's discount rate is r, L's present value of his receipts from these resources prior to the agreement are $(1-\tau)w\int_0^T e^{\beta t}e^{-rt}dt.$

The pay-off $v(s,T)$ also consists of an *after-agreement* profit component, i.e. the profits or dividends obtained after the merger at time T. It is assumed here that the profits of the new corporation at time t consist not only of the profits $(W+w)e^{\beta t}$ that H and L could make on their own from the resources they put into the new corporation, but that, just as in the example on p. 1, there is also a "synergistic" merger profit which amounts to $\pi e^{\beta t}$ (computed at an annual rate) at time t. But this merger profit is not assumed to be obtainable indefinitely. Due to various factors such as the expiration of a patent, the saturation of a market, etc., it will only be generated up to time Z. Thus at a time t, such that $T \leqslant t \leqslant Z$, the annual profit of the new corporation is $(W+w+\pi)e^{\beta t}$ and L's dividend after tax is $(1-\tau)s(W+w+\pi)e^{\beta t}$. Hence, L's present value of all receipts from time T to time Z is $(1-\tau)s(W+w+\pi)\int_T^Z e^{\beta t}e^{-rt}dt.$

Finally, there are the profits obtained after time Z. At this stage the profits of the new corporation consist only of the profits of the old corporations $(W+w)e^{\beta t}$. Thus L's present value of all dividends after time Z is $(1-\tau)s(W+w)\int_Z^\infty e^{\beta t}e^{-rt}dt$. It is

[1] If H's resources constitute a corporation it is assumed for the sake of simplicity that the dividends are equal to profits after tax. This is *not* a necessary condition, but it simplifies the analysis.

assumed here that these original profits are obtainable indefinitely. With $r-\beta$ positive this assumption is a good approximation if the profits are obtained for a fairly large number of years.

Summing up the three terms, L's pay-off $v(s,T)$ can be written as

$$(1-\tau)w\int_0^T e^{\beta t}e^{-rt}dt+(1-\tau)s(W+w+\pi)\int_T^Z e^{\beta t}e^{-rt}dt+(1-\tau)s(W+w)\int_Z^\infty e^{\beta t}e^{-rt}dt.$$

Replacing s by S, w by W and r by R and letting τ denote H's tax rate, we obtain H's pay-off function $V(S,T)$.

As regards the general solution, we note first of all that $(1-\tau)$ is common to all terms, i.e. $v(S,T)$ can be written as a function of s and T times the constant $(1-\tau)$. Since the solution of a bargaining game according to our model is independent of the size of the constant by which the whole pay-off is multiplied,[2] the solution is independent of the tax rates which H's and L's dividends are subject to.

We can also prove that the pay-off functions presented above are such that they fulfill requirements $S_1^*-S_4^*$[3] implying that $V(S,T)$ can be written as $AyF(T)+B$, where $y=N(S(\pi+W+w)-W)/\pi$, i.e. $S=y\pi/N(\pi+W+w)+W/(\pi+W+w)$ and

$$F(T)=(e^{-(R-\beta)T}-\frac{\pi e^{-(R-\beta)Z}}{W+w+\pi})/(R-\beta).$$

Since the solution y^* is dependent only on $F(T)$ and $f(T)$ for a given value of μ (cf. p. 116), we see that the solution y^* depends on $R-\beta$, $r-\beta$, W, w, π and Z.

We refrain from a general analysis of how the solution varies with these parameters and proceed instead to the specific example described in Chapter 1.

9.2.2 Analysis of a Merger Example

The model presented in the preceding section will now be applied to the example of a merger negotiation on p. 1.

First, we make the simplifying assumption that the original shareholders of each corporation act as one single person. This is likely to be a good approximation if the number of stockholders is limited to a small homogeneous group with roughly the same borrowing and investment possibilities and hence roughly the same rate of time discount. This assumption means that the bargaining can be regarded as taking

[2] Since the model relies on an ordinal utility we can in principle let any monotonous transformation of the pay-off represent the original pay-off.

[3] The proof is given in the mathematical appendix, pp. 285–286.

place between two parties, H and L. We again use the principle of calling the seller (=the owner of S) who wants a high price, H, and the buyer (=the owner of B) who wants a low price, L.

Next we determine the parameters of the model, namely W, w, π, r, R,β and Z. L's annual profit before tax at the start of the bargaining w is \$ 6 mill., H's corresponding profit W is \$ 2 mill. Furthermore, the initial merger profit (before tax), π, is as large as W, i.e. \$ 2 mill. Hence we conclude that $\pi/(W+w+\pi)= 0.2$ and $y/N=S(\pi+W+w)/\pi-W/\pi=5S-1$, i.e. $S=0.2+0.2y/N$. L's discount rate r is equal to his rate of return on alternative investments = 0.25 and H's discount rate R=0.15. The growth rate of the profits β=0.05. Finally, the merger profit will be generated for 10 years, i.e. Z=10.

Before applying the model in the preceding section to this situation, we have to answer the question of whether we are justified in doing so. The most problematic aspect of applying the pay-off functions in the preceding section appears to be the assumption that the whole profit is distributed as a dividend (cf. footnote 1 on p. 183). But it does seem natural that the owners of B, having alternative investment possibilities with a 25 per cent rate of return, are less interested in plowing back profits into B than investing them elsewhere. As for the owners of S, who under all circumstances will obtain a smaller part of the share capital of the enlarged corporation, it appears advisable to distribute the whole profit w to themselves prior to the merger.

Next we ask whether I_1-I_{13} are reasonable assumptions. There does not seem to be anything in the situation above which could rule out any *particular* one of these assumptions although the limitations imposed by them (discussed in Chapter 8) have to be kept in mind. The most problematic assumption is probably I_5, i.e. both parties have complete information. However, in the example on p. 1, this assumption poses a much less serious problem since the assistance of a certified public accountant with access to the accounting and budgeting data of both corporations is assumed.

We now proceed to deduce that $y^*/N=(12-2\sqrt{16-\mu})/\mu$ [4].

The solution cannot be established until $\mu=N\Delta t$ has been determined. There does not appear to be any natural value of either N or Δt. With \$ 6 mill. shares of B, 2 mill. shares of S and limited ownership on both sides, an extremely large number of ratios of exchange are possible.[5] Furthermore, even if the parties would want to limit the exchange ratios so that there would be a low number in both the numerator and denominator, the merger conditions could also involve some cash

[4] The details of this deduction are given on p. 286.
[5] Such as 500 shares of S for 600 shares of B, 501 S for 600 B, 501 S for 601 B, etc.

settlements. Thus it appears reasonable to assume that N can be made very large and that y/N and hence also S can be regarded as virtually continuous variables.

We furthermore assume that Δt is of moderate size, e.g. $\geqslant 0.001$. This would be the case, for example, when each party could deliver at most one bid every day. This appears to be a resonable assumption in reference to this type of negotiation. Several board members would probably have to be consulted before a new proposal for settlement could be made. It also appears reasonable to assume (cf. p. 154) that the new corporation would have to start operating on a specific day, e.g. the first of the month or at least at a specific hour of the day regardless of when an agreement is reached.

With Δt of moderate size and N possibly very large we can rely on assumption I_9' to conclude that H (with the lower rate of interest) can determine the size of μ and that he can enforce a solution given by $\mu=\mu^*$ (cf. p. 178). In this case, with $R = 0.15 < r = 0.25$, H can enforce a solution by setting $\mu = \mu^* = 14.129$.[6]

By inserting this value of μ into the formula on p. 185, we obtain $y^*/N = (12-2\sqrt{16-14.129})/14.129 = 0.66$. Hence $S^* = 0.2(1+y^*/N) = 0.33$.[7]

Likewise L will obtain a share $1-S^*=0.67$.

This implies that H, the owner of S, can enforce the following solution: L will receive 2/3 and H 1/3 of the capital of the enlarged corporation B. Since L's 6 million shares remain unchanged H will obtain 3 million new shares of B in exchange for his 2 million shares of S. Hence, the exchange ratio will be three shares of B for two shares of S, i.e. the exchange ratio originally desired by H, the owner of corporation S.

It should be stressed that the analysis above is based on the assumption that the owners of S, party H, cannot sell S to someone else for a higher price. H's pay-off from the agreement on $S = 0.33$ is $0.33 \cdot 10 \int_0^{10} e^{-0.1t}dt + 0.33 \cdot 8 \int_{10}^{\infty} e^{-0.1t}dt = 30.6$, i.e. it is worth \$ 30.6 millions. Thus H will arrive at this agreement unless some other buyer offers more than \$ 30.6 million for corporation S. Any offers to H below this sum will not affect the settlement between H and L. On his own, H can only ensure $2 \int_0^{\infty} e^{-0.1t}dt = 20$, i.e. \$ 20 mill.

Likewise the value of the merger agreement to L is estimated at \$ mill. 32.6 $(0.67 \cdot 10 \int_0^{10} e^{-0.2t}dt + 0.67 \cdot 8 \int_{10}^{\infty} e^{-0.2t}dt = 32.6)$. Without purchasing S, L can

[6] Since $\mu^* = (1-0.2e^{-2})/0.2 + (1-0.2e^{-1})/0.1$

[7] Cf. p. 185.

obtain $6 \int_0^\infty e^{-0.2t}dt = 30$ from B, i.e. L's "surplus" from obtaining corporation S is \$ 2.6 mill. L would be unwilling to reach the agreement with H presented above only if he could obtain a higher "surplus" by purchasing resources similar to those of S elsewhere.

9.3 Duopoly Theory

9.3.1 Introduction

Example 2 in Chapter 1 (p. 1) illustrates another important area for the application of bargaining theory, namely duopoly theory. This concerns a market with two competing corporations. Some of the main problems of duopoly are discussed in general in this section. Then a simple kind of duopoly game, i.e. a Prisoner's Dilemma game, is presented in 9.3.2, after which we turn (in 9.3.3) to the type of duopoly game exemplified in Chapter 1.

The literature on duopoly is characterized by two different approaches with respect to methods for analyzing duopoly problems. Some authors have stressed the *cooperative* aspect of duopoly and assumed (or deduced) that the two players would agree to play a pair of strategies such that their joint profits would be maximized and that the parties would then divide these maximal profits between them. Other authors have stressed the *non-cooperative* aspect of the game and have presented some kind of equilibrium point as the solution.

The main problems associated with the cooperative approach are as follows:

1. It has rarely been possible to determine how the maximal joint profits should be divided.

2. Formal cooperation is often outright prohibited.

3. It might be difficult to ensure compliance to the agreement.

The main problem encountered in the non-cooperative solution to duopoly is that the solution is generally not Pareto-optimal. Both parties can improve their profits by turning to some other combination of strategies. This can often take the form of "implicit" cooperation, i.e. cooperation without a formal agreement.

Some attempts have been made to unite these two approaches; e.g. by Shubik and Krelle. The Shubik model uses the Nash model for division of the joint profits. Krelle (1968) presents a duopoly game which can be divided into rounds, where each round consists of two subgames – one bargaining subgame and one subgame in which the original game is played non-cooperatively. The second subgame is played only if an agreement as to how the joint profit should be divided is not reached in the bargaining game.

In this chapter it is shown how certain duopoly games can be given a unique solution by a method which is similar to Krelle's in the sense that it also assumes "rounds" or periods with one bargaining phase and one non-cooperative game phase. There are, however, several differences.[8] In particular, our solution is deduced on the basis of simpler behavioristic assumptions, namely those of B_4.[9]

9.3.2 An Iterated Prisoner's Dilemma Game

9.3.2.1 Presentation of the Situation

In order to illustrate the main ideas behind the application of our model to duopoly problems, we study the following very simple situation: Two corporations, H and L, have obtained licences for producing and selling a similar novelty item for a particular market. No competitors are foreseen. Both parties have a variable unit cost of 2.5 ¢. The pricing strategies are limited to either a nickel (5 ¢) or a dime (10 ¢).

After some months of competition during which the prices of the two parties have varied up and down, the presidents of the two corporations meet in order to discuss how to avoid what they regard as "excessive competition". They share each other's experiences and find that the market has followed roughly the following pattern: If one of the products was priced at 5 ¢ and the other at 10 ¢, the one selling for 5 ¢ sold 1.6 million units per month and the one selling for 10 ¢ only 0.1 million. If both quoted a price of 10 ¢, each sold 0.4 million units per month and if both quoted the 5 ¢ price, each sold 1 million units. Both parties regard this market pattern as fairly stable during the eight years (100 months) that the novelty item is expected to be of interest.

[8] Krelle (1968) deals with how two parties should mix their strategies when there are two strategy combinations which both lead to a joint maximum but no side-payments are allowed. Bargaining is assumed to be characterized by simultaneous bidding. In a recent paper Krelle has extended his model to the alternating bidding case. See Krelle (1971). Our model presented below is similar in many respects although there are several differences as regards the solution of the bargaining game. Krelle's behavioristic assumptions go beyond B_4 (cf. footnote 14, p. 65). Furthermore, Krelle assumes here that all duopoly subgames are also played alternatingly (i.e. with *perfect* information). In many cases this will lead to problems as to who should start playing in the duopoly subgame.

[9] Cf., however, footnote 20 on p. 195.

Each party calculates with a monthly fixed sales and administration cost of \$ 23,000. The machinery, which will last at least eight years, has negligible salvage value.

Each party is willing to enter into an agreement on prices if this turns out to be profitable and can be accomplished in some legal form such as by the formation of a jointly owned corporation which would handle sales. The question centers on whether the parties should split the joint maximal profits into two halves or whether one of the parties should pay the other a side-payment in order to induce him to enter into the collusion agreement.

We start by analyzing the monthly pay-off situation. As the cost of machinery is a sunk one, it can be disregarded. The following table (Table 3) is then obtained for determining the pay-offs.

1 H's price	2 L's price	3 H's sales (in mill. units)	4 H's unit contribution (Col.1–2.5 ¢)	5 H's total contribution (Col.3 · Col.4)	6 H's profits (Col. 5– \$ 23,000)
5 ¢	5 ¢	1	2.5 ¢	\$ 25,000	\$ 2,000
5 ¢	10 ¢	1.6	2.5 ¢	\$ 40,000	\$ 17,000
10 ¢	5 ¢	0.1	7.5 ¢	\$ 7,500	–\$ 15,500
10 ¢	10 ¢	0.4	7.5 ¢	\$ 30,000	\$ 7,000

Table 3 Calculation of H's pay-off

L's pay-offs can be obtained from Table 3 by letting H and L change names. Table 3 can then be used to construct the following matrix for monthly profits measured in \$ 1,000 (Table 4).

H \ L	5 ¢	10 ¢
5 ¢	2 / 2	–15.5 / 17
10 ¢	17 / –15.5	7 / 7

Table 4 Pay-off matrix of PD-game (Pay-offs in \$ 1,000)

This is a typical Prisoner's Dilemma Game.[10] If it is played only *once* without possibility of cooperation, the solution – due to $\boldsymbol{B}_1$ – is that both play 5 ¢, since this price strictly dominates 10 ¢.[11] The cooperative solution on the other hand is that the parties reach an agreement which maximizes profits by both quoting 10 ¢. If no side-payments are possible, the agreement would *only* involve both playing 10 ¢, whereby each party would get $ 7,000. If side-payments *are* allowed, the cooperative solution would still be an agreement on both quoting 10 ¢, but it would now *not* be certain whether each party would obtain exactly $ 7,000 each month. As we assume that side-payments *are* possible, one party might demand such a payment from his opponent before complying to the cooperative solution.

9.3.2.2 Solution of the Cooperative Game

In order to determine how the joint profit is to be divided if side-payments are possible, we proceed as follows:

The parties play a bargaining game of the alternating bidding type over a certain number of periods, e.g. months. One party delivers a bid in each period; if this bid does *not* lead to an agreement, then the parties play a *single* PD-game, i.e. set the prices for the coming month, without any possibility of reaching an agreement on how this single PD-game should be played. Hence each period j can be divided into:

1. A bargaining phase B(j), in which *one* party delivers a bid and
2. A duopoly game phase, D(j), in which each of the two parties chooses a price strategy.

While a possible agreement in B(j) will concern not only period j but also every subsequent period, the price decision in D(j) will only concern period j. It should be stressed that D(j) will be played only if no agreement has been reached in B(j) or earlier.

A possible sequence of B(j) D(j) for 1 . . . n can be termed a *supergame*. In order to analyze this supergame we deal first with the following simple two-alternative bargaining game: If the parties are to reach an agreement on playing (10 ¢, 10 ¢), implying that they both obtain the maximum joint profit of $ 14,000, H wants a distribution $ 9,000 to H, $ 5,000 to L, while L wants the somewhat more equitable distribution $ 8,000 to H, $ 6,000 to L. An agreement has to be reached on a whole number of thousand dollars, i.e. there are only *two* alternatives.

[10] We define a PD-game as a two-person game in which each party has only two strategies, the pay-off matrix is symmetric (i.e. the parties have identical pay-off functions) and there is a solution due to strict dominance, which gives a lower combined pay-off than the joint maximum.

[11] A strategy σ strictly dominates another strategy σ' if σ is a better reply than σ', regardless of the strategy selected by the other party. (For a definition of "better reply" see p. 254.) Thus we object to N. Howard's viewpoint proposed e.g. in Howard (1966) that both parties would quote 10 ¢ even if the game is played only *once*. Howard implicitly refutes B_1. For a further discussion, see Shubik (1970).

Assuming e.g. that this supergame can be played over a maximum of 100 periods = months, we begin the analysis by looking at the very last period, period 100. Since we start at the end of the game we first study D(100), i.e. the case where no agreement has been reached in any month so that even the last month – period 100 – has to be played non-cooperatively. At the time the decisions are made, D(100) can be regarded as a *single independent game*. The price strategies 5 ¢, 5 ¢ will be chosen and each party will obtain $ 2,000. Let us now assume that L bids in B(100). L will accept the distribution $ 9,000 to H, $ 5,000 to L and get $ 5,000 rather than insist on his own terms so as to carry the game into D(*j*) which would give him only $ 2,000. Next we go backwards one period and study D(99). Since all choices in period 100 are determined, the parties will regard the game at time D(99) as a single independent game and hence play 5 ¢, 5 ¢. Then in B(99), H has to choose between accepting L's terms and getting $ 8,000 *twice*, i.e. $ 16,000 or insisting on his own terms. In the latter case he will get $ 2,000 in D(99) and $ 9,000 in B(100), i.e. a total of $ 11,000. Hence H will accept L's terms.

We can continue backwards in this manner, one period at a time, and determine the following for each period j:[12]

1. If no agreement is reached in B(j), both parties can regard D(j) as a single independent game, since the choices in B(j') and D(j') are determined for every $j'>j$. This means that each party will play 5 ¢, 5 ¢, and obtain $ 2,000.

2. A choice can be determined for B(j). This can be either an acceptance bid implying agreement or an insistance bid, implying that D(j) will be played.

Regardless of whether or not an agreement is reached in B(j), each party can assure himself of $ 2,000 in each period. Hence out of the maximum joint profit of $ 14,000, only $ 10,000 are subject to real bargaining, since no party will agree to getting less than $ 2,000. The $ 10,000 can be regarded as constituting a kind of merger profit. This amounts to regarding the bargaining as a matter of $ 10,000. Then, in this two-alternative example, the two alternatives will be a distribution $ 6,000 to H, $ 4,000 to L and $ 7,000 to H, $ 3,000 to L. If an agreement is not reached in a certain period there will not be any merger profit and both obtain 0. We then obtain the same type of situation as in examples 3 and 4 in Chapter 3. It can be established that H will accept 6 in each of his bids, while L will insist on 6 in each of his bids except B(98) and B(100).[13] This implies that an agreement will be reached in period 1 on a division for every period of the merger profit $ 10,000, so

[12] The proof of this is given below.

[13] For the case of six periods it was shown that 2=u(6); 3=C(6); 4=c(6); see p. 38. With period 6 in example 3 corresponding to period 100 in the example above, we deduce 96=u(6), 97=C(6) and 98=c(6). This implies, due to T_3, that the choice is determined in each period prior to 98, whereby H accepts 6 and L insists on 6, and that L accepts 7 in B(98).

that H gets \$ 6,000 and L gets \$ 4,000. This in turn implies that the original \$ 14,000 will be divided so that H gets \$ 6,000 + \$ 2,000 = \$ 8,000 and L gets \$ 4,000 + \$ 2,000 = \$ 6,000.

This can now be generalized to apply to games with n alternatives. Let us assume that for a specific set of values of r, R and μ, every subgame $(y, y')_j$[14] with respect to dividing the \$ 10,000 leads to a unique choice. It can then be proved that the supergame above, concerning the division of the \$ 14,000 in 100 periods will lead to a unique solution. D(100) is solved first, giving each party \$ 2,000. B(100) can be regarded as concerning the division of \$ 10,000 and, due to our initial assumption, can be solved regardless of what was bid prior to period 100. With the choice in B(100) determined, D(99) can be regarded as played in isolation and solved, implying that each party gets \$ 2,000. B(99) concerns the division of \$ 10,000 once again and can also be solved. We proceed backwards in this way one period at a time, deducing that every subgame D(j) will be played cooperatively.[15] This procedure can be used to solve the entire supergame.

This analysis relies on the assumption that a unique choice can be established in every subgame $(y,y')_j$ with respect to the division of the \$ 10,000. This assumption requires that, in any subgame $(y,y')_j$, the highest pay-off for the bidding party is obtained by only *one* alternative, i.e. we do not allow for indifference due to the fact that *several* alternatives lead to the highest pay-off. Whether or not this requirement is fulfilled can be tested using the computer program based on the general model in Ståhl (1972). After some test runs with different values of r, R and μ it appears reasonable to hypothesize that the problem of indifference is not especially great for games in which r and $R > 0$ and the number of alternatives is moderate.

Furthermore, even if no unique choice can be determined in period j for some situation $(y,y')_{j-1}$ due to indifference, a choice might be determined in e.g. period $j-1$. This is true in example 6, where $r=R$ (cf. p. 54). Then D($j-1$) combined with D(j) can be regarded as played in isolation and it appears reasonable to assign the solution 5 ¢, 5 ¢ to these two duopoly subgames and thus assume that the analysis above would still apply.

[14] j represents both $\bar{j}$ and $\underline{j}$.

[15] For the case where $r<R$, as on p. 193, we note the following: Although H would be interested in having each subgame D(j) played cooperatively, L would prefer to have them played non-cooperatively. If all games D(j) were played cooperatively, the joint maximum profit would be shared equally. But if all D(j)-subgames were played non-cooperatively, the joint profit would be divided so that L, with the lower rate of interest, would obtain a larger share. Thus our supergame, involving both a bargaining phase B(j) and a duopoly phase D(j) in each period, will differ significantly from the traditional iterated PD-game, where *both* parties at the start would be *interested* in playing every subgame D(j) cooperatively.

It can thus be assumed more generally that there will a merger profit of \$ 10,000 to divide in each period. We know from our analysis on p. 122 that this merger profit will be evenly split if $r=R$, i.e. if the parties have identical rates of interest. This in turn implies that the maximum joint profit is also evenly split with each party obtaining \$ 5,000 + \$ 2,000 = \$ 7,000.

In the case of unequal interest rates, it is easily understood – since we are dealing with pay-off function 1 (with only an agreement pay-off component, cf. p. 119) – that the solution will be dependent on μ. With each period set at one month, i.e. $\Delta t= 1/12$[16] and if r=0.18 and R=0.25, we can establish that $\mu = 8.3$, either by assuming that N=100 or by setting $\mu=\mu^*$, in accordance with the kind of reasoning outlined on p. 178.[17] We then obtain $S = y/N = (1-e^{-ZR})/R\mu = 4(1-e^{-2.5})/8.3 = 0.44$, i.e. H obtains 4,400 + 2,000 =\$ 6,400 and L \$ 7,600 of the monthly maximum joint profit.

9.3.3 Analysis of Other Iterated PD-games

Other iterated PD-games, i.e. sequences of PD-games with an identical pay-off matrix, can be solved in a similar manner. More generally, let w denote L's pay-off from the solution, i.e. the combination of the two dominating strategies, and π the joint maximum pay-off. Then the "merger pay-off" $\pi-2w$ is to be divided and L's pay-off function can be written as $s(\pi-2w)\int_T^Z e^{-rt}dt$, where s is L's share of the joint profit. By substituting $\pi-2w$ for \$ 10,000 in the analysis above, it can be deduced that the game concerning the division of π can be solved, provided a unique choice can be assigned to (almost) every subgame $(y,y')_j$ of the bargaining game involving the division of $\pi-2w$. The solution is that H will obtain a share S given by

(1) $(1-e^{-R(Z-T^*)})/R\mu$, where T^* is that value of T for which

$(1-e^{-R(Z-T)})/R+(1-e^{-r(Z-T)})/r=\mu$[18].

9.3.4 Other Duopoly Games

An iterated Prisoner's Dilemma game was solved in the preceding section. Other duopoly games can be solved by applying a similar methodology. The basic

[16] We can assume that the prices are changed only *once* a month, e.g. on the first day of the month, and that bargaining bids have to be determined by a board of directors who meet once a month.

[17] $\mu^*=(1-e^{-1.8})/0.18+(1-e^{-2.5})/0.25 = 8.3$.

[18] Cf. pp. 270 and 272.

characteristics of the solution of a PD-game, applicable to other duopoly games as well, are as follows:

1. The duopoly subgame is played once in each of a given, known and finite number of periods, provided no agreement is reached earlier.

2. A bargaining game takes place concurrently, with one bid in each period. This game is characterized by alternating bidding and the other institutional assumptions ($I_1 - I_{13}$) of our model.

3. In each period j a bargaining phase B(j) precedes a duopoly subgame D(j).

4. If a subgame D(j) is played in isolation a solution can be determined giving H a pay-off W and L a pay-off w.

5. The joint maximum profit is π.

6. A choice can be determined in (virtually) every period of a game in which a periodic profit $\pi - w - W$ is to be divided.

7. Threats pertaining to any duopoly subgame D(j) are regarded as non-credible for reasons similar to those discussed in Section 8.3.4.

If these seven conditions hold then we can deduce, in a manner similar to that presented in the preceding sections, that H will obtain a periodic profit of $S(\pi - W - w) + W$ and L, $(1-S)(\pi - W - w) + w$, where S is determined by equation (1) on p. 193.

The only difference as compared to the iterated P D -game is the determination of the D(j)-game when played in isolation, i.e. the establishment of W and w. Since we are only interested in exemplifying applications of our bargaining model, we shall not attempt any general analysis of the determination of w and W. Instead, we study one type of games closely connected with our example on p. 1, namely a game of the so-called Bertrand type.[19]

9.3.5 The Bertrand Game

9.3.5.1 Presentation of the Game

The Bertrand Game, first proposed by J. Bertrand in 1883, involves a duopoly situation where the duopolists produce identical goods. This game is based on the

[19] As regards the solution of an independently played Cournot-game, where the production quantity is the choice variable, see e.g. Cyert & de Groot (1970).

assumption that each of the two duopolists quotes a price. If one party quotes a lower price than the other, he will sell the same quantity as if he were a monopolist. If he quotes the same price as the other party, he will sell half of the "monopolist's quantity" and if he quotes a higher price he will sell nothing.

It is also assumed that each party produces exactly the amount which he can sell. This requirement is fulfilled if the duopolists produce to order or if the production period is so short that output can be adjusted to demand more or less immediately.

Let us study the following simple example:

The two parties are called H and L. Each *month* H sells a quantity Q and L a quantity q – measured e.g. in millions of units – and the prices quoted are P and p.

We assume that the demand is such that

$q = 14-p$, and $Q = 0$, if $p < P$;
$Q = 14-p$, and $q = 0$, if $p > P$ and
$q = Q = 0.5\ (14-p)$, if $p = P$.

We assume that the parties cannot change their prices more often than once a month, e.g. since they send out price lists. It is also assumed that the parties have fixed variable unit costs C and c, respectively. In particular we assume that the parties in this example have the same variable costs, such that c=C=6.

Since the parties can always guarantee themselves zero contributions, (i.e. profits before the deduction of fixed costs) by selling nothing, it is meaningless e.g. for party L ever to accept a $p < c$. Hence, we require that $p \geqslant c$. Likewise we require that $P \geqslant C$. We can also determine that p=c is dominated, since it leads to v=0 for every value of P, while $p > c$ will lead to a positive value of v for $P > p$ and lead to v=0 only for $P \leqslant p$. This means we deduce that $p > c$=6 and $P > C$=6.[20]

Furthermore, an optimal value of p, called p^*, can be established if p is unconditionally lower than P. This p^* is obviously equal to the monopoly price

[20] It should be stressed that assumption set B_4 is used here in combination with the following revised, somewhat more demanding version of assumption B_{11}: If a party has no grounds for establishing the other party's choice between two strategies on the basis of the other behavioristic assumptions of B_4, he will not regard the choice of a specific strategy as certain. This implies in turn that a strategy which is weakly dominated (i.e. *not* strictly dominated; cf. p. 301) can be eliminated. This is because, for *each* of the other party's strategies, some – not infinitely small – probability would be assigned to the event of this strategy being chosen. Furthermore, in order to arrive at a solution, the order of elimination of the strategies may not affect the establishment of the solution. It can be shown that the solution of the game above will also be the same if p=c=6 is eliminated later in the process.

obtained from setting $dw/dp=0$, where w is party L's monthly profit prior to the deduction of fixed costs. With $q=14-p$ we obtain $p^*=10$.[21] Likewise, party H's optimal price, P^*, if $P<p$, is 10.

Next, we can establish that every price p *higher* than p^* is *dominated* by p^*. If $P<p^*$, both p^* and any higher p will give party L zero profits. If $P=p^*$, then p^* is the only price that will give party L a positive profit. Finally, if $P>p^*$, then p^* is his optimal price. Likewise we can deduce that $P=P^*$ dominates every higher price P. Hence we determine that $C<P\leqslant P^*$ and $c<p\leqslant p^*$. If it is assumed that only integer values of P and p are allowed, then only prices such that $7\leqslant P\leqslant 10$ and $7\leqslant p\leqslant 10$ remain. The following pay-off matrix, showing each party's monthly pay-off (profit before deducting fixed costs) is then obtained.

Party H chooses P \ Party L chooses p	7	8	9	10
7	3.5 / 3.5	7 / 0	7 / 0	7 / 0
8	0 / 7	6 / 6	12 / 0	12 / 0
9	0 / 7	0 / 12	7.5 / 7.5	15 / 0
10	0 / 7	0 / 12	0 / 15	8 / 8

Table 5: Original pay-off matrix of Bertrand game

9.3.5.2 Simultaneous Choice of Prices

Let us first analyze the case where the parties establish their prices for the month without knowing what price the other party has chosen. A solution can then be obtained by the *iterative* elimination of dominated strategies in three steps.[22]

Step 1: $P=9$ dominates $P=10$ and $p=9$ dominates $p=10$. After the dominated strategies are eliminated the following reduced matrix remains.

[21] $w = (p-6)(14-p) = 20p-p^2-84 \Rightarrow dw/dp = 20-2p \Rightarrow p^* = 20/2 = 10$.

[22] Cf. pp. 254–255.

		p = 7	p = 8	p = 9
P	7	3.5, 3.5	7, 0	7, 0
	8	0, 7	6, 6	12, 0
	9	0, 7	0, 12	7.5, 7.5

Table 6: Reduced pay-off matrix of Bertrand game after step 1

Step 2: P=8 dominates P=9 and p=8 dominates p=9. After the elimination of P=9 and p=9 the following matrix remains.

		p = 7	p = 8
P	7	3.5, 3.5	7, 0
	8	0, 7	6, 6

Table 7: Reduced pay-off matrix of Bertrand game after step 2

Step 3: P=7 dominates P=8 and p=7 dominates p=8. After the elimination of P=8 and p=8, a unique solution has been arrived at, namely that P=7 and p=7, implying that W=3.5 and w=3.5.

9.3.5.3 Alternating Choice of Price

We next look at the case where one party knows what price the other party has chosen when he sets his price.

Let us start with the case where H first chooses P and L selects p with knowledge about H's choice. This implies that for every value of P, L will choose that p which maximizes his profit. From Table 5 we see that for P = 8, 9, 10 this is P–1, i.e. the closest price below P, implying that L will cover the entire market. When P=7, however, L will choose to share the market with H by setting $p=7$, giving L 3.5, rather then sell everything at cost, giving 0 profits. H, forecasting L's reaction to his choice of P, thus realizes that $P=7$ is the only choice which will give him any profit. Hence H will choose $P=7$ and L will have to set $p=7$.

Since the pay-off matrix is symmetric, it can similarly be deduced that the solution is also $P=7$, $p=7$, when L starts to choose. We thus conclude that regardless of whether or not the prices are set simultaneously, the choice is the same – $P=7, p=7$ – and we can determine that $W=3.5$; $w = 3.5$.[23]

9.3.5.4 Generalization of the Conclusions concerning D(j)

In the mathematical appendix we study this solution method of iterative elimination of dominated strategies as applied to any Bertrand game such that

1. Average variable costs are constant and equal, i.e. $C=c$
2. ϵ is the smallest difference allowed between each price and
3. $f'(p) = f'(P) < 0$, where $f(p)$ is L's sales when $p<P$ and $f(P)$ is H's sales when $p>P$.[24]

For any such game we prove, for the case of simultaneous choice of prices (on p. 287), that $p=P=c+\epsilon=C+\epsilon$. The same conclusion is obtained for the case of alternating choice.[25] If 6 is substituted for C and 1 for ϵ then the proof of this is identical to that in 9.3.5.3. When any price is allowed and the smallest difference ϵ between each price is negligible, i.e. when price can be regarded as a continuous variable, the price will be set equal to the average variable cost, leading to zero contributions (i.e. zero profits before the deduction of fixed costs).[26]

Thus the profits w, W obtained in each period (=month), can be determined, if no agreement has been reached prior to the start of this period. As can be seen in the example above (Table 5), $w+W=7$ is far smaller than the joint profit obtained when

[23] $P=7, p=7$ constitutes a pair of equilibrium strategies. It is not unique, however, since $P=6$, $p=6$ is also a pair. Since both parties prefer $P=7, p=7$ to $P=6, p=6$ the game has a solution in the strict sense (see p. 256), but since the equilibrium pairs are not interchangeable the game lacks a non-cooperative solution in Nash's sense (see p. 256).

[24] It is also assumed that $q = Q = f(p)/2$, if $P = p$ and that $(p^*-c)f''(p^*)+2f'(P^*)< 0$. implying that $d^2v/dp^2 < 0$, when $p = p^*$. This last assumption is necessary for the establishment of p^*. Likewise it is assumed that $(P^*-C)f''(P^*)+2f'(P^*) < 0$.

[25] In other games the alternating and simultaneous cases might not lead to the same solution. There is also the problem that alternating choice often leads to a different solution depending on the order of choice. This is e.g. true for the Cournot duopoly game iterated a finite number of times; see furthermore von Stackelberg (1934) and Cyert & de Groot (1970).

[26] Thus we obtain the conclusion arrived at by Bertrand in 1883, but derived in a different manner from a more general set of behavioristic assumptions.

at least one party bids his optimum price and this price is not higher than the price quoted by the *other party*. Table 5 contains only the case where $p=P=p^*$, leading to a joint profit of 16. If e.g. $p=p^*$, but $P>p^*$, then L will sell as much as L and H combined when $p=P=p^*$. This implies – since $c=C$ – that L obtains the total joint maximum profit.

Generally, the maximum joint profit is $f(p^*)(p^*-c)$. The merger profit to be divided is hence $f(p^*)(p^*-c)-(f(c+\epsilon)\epsilon)$. When price is a continuous variable, implying that $\epsilon \to 0$, the merger profit is $f(p^*)(p^*-c)$, i.e. the maximum joint profit.

9.3.5.5 Solution of B(j)

We can now turn to the bargaining solution of the iterated Bertrand game. Assuming as in the PD-game that $r=0.18$ and $R=0.25$, H will obtain 44 per cent of the merger profit and L 56 per cent.[27] In the example on p. 196 where only prices of a whole dollar are quoted and the periodic merger profit is 16–7=9, H will obtain 3.5+0.44·9 =7.46 and L 3.5 +0.56·9=8.54. In the case where the price is a continuous variable. H will obtain 0.44·16 = 7.04 and L 0.56·16 = 8.96, all measured e.g. in millions of dollars. We see that the effect of the difference in interest rates is larger in the case where price is a continuous variable, since the merger profit will be larger.

As regards the specific example in Chapter 1, the conclusion when $R>r$ is that party H will obtain a *smaller* share of the total market.[28]

9.4 Labor-management Bargaining

9.4.1 Introduction

Throughout the history of bargaining theory, wage negotiations between the management of a corporation and a labor union have been regarded as one of the

[27] Cf. p. 193.

[28] In this context the following can be noted: When both parties have roughly the same production costs and given capacity, there is no incentive to merge the two corporations physically. On the contrary, transportation cost advantages and traditional bonds with customers might speak strongly against such a merger prior to time Z. Another question involves the possibility of a *pro forma* merger in order to make price collusion legal. Hence both parties can continue to produce for their share of the market during the remaining life of their existing machinery. Another question is what will happen after this. If new investments are made after this time, the plants can be better adjusted to demand. A physical merger might then become profitable. Unless new machinery will lead to considerably lower production costs there is no reason for such a merger to take place before time Z and hence the market division agreement for the time up to Z would not be affected. This means that future bargaining regarding a merger can be considered isolated from the present bargaining over market division and the agreement can be made independently of future merger bargaining.

most important areas of application.[29] It is therefore fitting that this area of application also be discussed in this study even though a formal bargaining theory, based on rationality assumptions, appears less interesting for this area of application than for others such as merger bargaining. This does not prevent us from briefly discussing the application of our bargaining theory in terms of the purpose for which it seems most suitable in this case, i.e. macroeconomic predictions. In this analysis we rely primarily on example 3 on p. 2. This example illustrates the relevancy of our model as regards parts of tax incidence theory, namely the incidence of a corporate income tax and a wage tax. However, as discussed further in Section 9.4.4, this analysis is severely limited in generality.

9.4.2 Description of a Specific Bargaining Situation

9.4.2.1 Assumptions

We deal with a situation involving bargaining between the management of a corporation and a union which represents the workers of this corporation. This situation is assumed to have the following characteristics:

1. The corporation's operations are such that they can be regarded as producing one standardized product.

2. The final products are sold on a market at a price p, which is given[30] and constant over time. This holds, for example, in an oligopolistic situation where a price leader – e.g. in another country – sets the price or where there is a more or less tacit price agreement. The corporation's methods of competition can then be limited to delivery service and product development.

3. The *total* market for this product is given as M units.

4. If no strike occurs, the corporation has a given market share $\phi(0)$ of the market M, i.e. the corporation will sell $\phi(0)M$ units annually.

5. If a strike does occur, the corporation will not only lose sales during the strike, but it will also have a lower market share once the strike is over. Some customers will switch *permanently* to other brands when unable to obtain the product of the corporation during the strike. If the delay in the agreement is denoted as T, the

[29] See e.g. Wicksell (1925), Zeuthen (1930), Hicks (1932), Pen (1959), Bishop (1964), Saraydar (1965), de Menil (1968) and Hieser (1970).

[30] This assumption can probably be replaced by more complicated assumptions which allow the corporation to change price and thereby influence its market share (cf. p. 204).

market share is written as a function of T, namely $\phi(T)$, where $\phi(0)$ is the market share prior to the strike. Thus it is assumed that $\phi'(T)<0$.

6. The following assumptions are made with respect to the way $\phi(T)$ decreases:

a) $\phi(T)$ decreases at a non-increasing rate, i.e. $\phi''(T) \geqslant 0$[31] implying that ϕ decreases most rapidly at the beginning of the strike. For instance, if 20 out of 100 original customers are permanently lost during the first month of a strike, it is assumed that the permanent loss of customers will not be larger than 20 in any subsequent month.

b) It is assumed that $d^2(\log\phi(T))/dT^2<0$. This implies that the permanent loss of customers during each month, in *per cent* of the customers remaining at the beginning of the month, will increase as the strike continues.[32] Hence if 20 out of each original 100 customers, i.e. 20 per cent, are permanently lost during the first month of a strike, one additional month of strike will imply a loss of more than 16 customers, i.e. more than 20 per cent of the 80 customers remaining at the start of the second month.[33]

If we temporarily refrain from giving $\phi(T)$ a definite form and substitute $\phi(T)$ for $f(T)$ in Figure 23 on p. 115, functions *a, b* and *c* illustrate some ways in which $\phi(T)$ is allowed to vary over T.

7. Plant capacity is given and larger than $\phi(0)M$. With a considerable remaining lifetime, no significant changes in the plant are contemplated. This in turn is assumed to imply that production per man hour is given and will not be changed by increased mechanization due to wage increases.

8. The cost of material, etc., and all other variable costs (except wages) per unit are denoted as c, which is given and assumed constant over time.

9. Bargaining concerns only the wage rate ω.

10. This wage rate ω is a piece rate, paid for one unit of production. With production per hour assumed given (cf. point 7), this implies that the hourly wage is directly proportional to ω.

[31] $\phi''(T) \geqslant 0$ and $\phi'(T)<0$ imply that $d|\phi'(T)|/dT \leqslant 0$, where $|\phi'(T)|$ is the absolute value of the rate of decrease.

[32] With $d(\log\phi(T))/dT = f'/f < 0$ the percentage loss of customers is negative. Hence $d^2(\log\phi(T))/dT^2 < 0$ implies that $d|d(\log\phi(T))/dT|/dT > 0$.

[33] Remaining in the sense that they would purchase the product of the corporation studied if the strike ended and deliveries began again.

11. The corporation has fixed annual costs – including e.g. depreciation – denoted as FC, which are incurred regardless of whether or not a strike takes place. For the basic one-contract model in this section we assume that FC=0. In Section 9.4.3 we allow for any value of FC.

12. If a strike occurs, the corporation will not produce anything and show a corresponding loss of sales. Since the corporation does not have an alternative use for its resources during the strike, it is not in a position to obtain a pre-agreement profit.

13. If a strike occurs the members of the union do not have any opportunities for alternative temporary employment. All compensations to the individual members during the strike come from the union itself.

14. The liquidity of the corporation and the strike fund of the union are large enough to allow for a very long strike.

15. The corporation is subject to a corporate income tax of $100\,\tau$ per cent.

16. The corporation pays a proportional wage tax of $100\,\tau'$ per cent on all wages.

17. The corporation attempts to maximize the present value of all its future profits after tax. Since the corporation desires low wages, we call it party L and its discount rate is denoted r.

18. The labor union attempts to maximize the present value of all future wages earned by its members, where the discount factor reflects the time discount of the various members.[34] Since the union desires high wages, we call it party H and its discount rate is denoted R.

19. The contract to be reached as a result of the negotiation is determined in advance to last Z years, e.g. due to trade usage, regardless of when an agreement is reached. Hence the contract will run from time T to time T+Z as in pay-off function 3 on p. 126.

20. The bargaining game is characterized by assumptions I_1–I_{13} and the bargaining parties fulfill the behavioristic assumptions of set B_4.

9.4.2.2 Solution

On the basis of these twenty assumptions, the corporation's pay-off $\nu(\omega,T)$ from an agreement on a wage ω is written as follows:

[34] This assumption is similar to the assumption that the union maximizes the total wage bill, made e.g. by Dunlop (1944).

$$v(\omega,T)=\int_T^{T+Z}(1-\tau)\,\phi(T)M(p-c-\omega(1+\tau'))\,e^{-rt}dt$$

where $p-c-\omega(1+\tau')$ is the contribution for each unit produced and sold and $\phi(T)M$ is the number of units sold each year.

Likewise the relevant pay-off of the union $V(\omega,T)$ from an agreement on ω at time T, i.e. the present value of the total wage bill from the contract, can be determined as $\int_T^{T+Z}\phi(T)M\omega e^{-Rt}dt$. It can now be proved that $v(\omega,T)$ and $V(\omega,T)$ as defined above fulfill requirements $S_1^*-S_4^*$. The proof, depending largely on the deductions for pay-off function 3, is given in the appendix (p. 288).

The solution value of ω, called ω^* is then $F^*(T^*)(p-c)/\mu(1+\tau')$[35], i.e. an immediate agreement is reached on ω^* thus defined.[36]

Since $F^*(T^*)$ depends solely on $F(T)$, $f(T)$ and μ and hence on R, r, $\phi(T)$ and μ, we conclude that ω^* is completely independent of the size of the corporate income tax τ. Instead ω^* is highly dependent on τ', the wage tax. In fact if the solution wage ω^*, when $\tau'=0$, is called ω^0, then $\omega^*=\omega^0/(1+\tau')$, since $F^*(T^*)$ is independent of τ'.

This means that if a tax τ' is introduced, management pays $\omega^*\tau'=\omega^0\tau'/(1+\tau')$ to the government, but $\omega^0/(1+\tau')$ instead of ω^0 to the union, i.e. $\omega^0(1-1/(1+\tau'))=\omega^0\tau'/(1+\tau')$ less to the workers. In other words, the *whole* wage tax is shifted backwards to the union.

9.4.2.3 Numerical Illustration

In order to determine the exact value of ω^* we must decide on a specific form for the function $\phi(T)$. If we assume that $\phi(T)=ke^{\beta_0 T-\beta' T^2}$, where $\beta_0<0$, r and $R>\sqrt{2\beta'}>0$ and k is a constant, then the requirements that $\phi'(T)<0$, $\phi''(T)\geqslant 0$ and $d^2(\log\phi(T))/dT^2<0$ are fulfilled (cf. p. 274). $f(T)$ can then be written as

[35] See p. 288 in the appendix for a detailed deduction.

[36] It should be noted that in our model, a strike is assumed to break out at the start of the bargaining game, i.e. at time 0. In reality bargaining usually starts *prior* to the expiration of the old contract and hence a strike will not start immediately. Up until the beginning of the strike, the delay in reaching an agreement will not affect either the total profits or the total wage bill, at least not to any significant extent. It is therefore quite conceivable that the bargaining agents will regard a certain delay in reaching an agreement as favorable. They have to justify their position as bargaining agents to the union members or the owners of the corporation. Hence, even if the bargaining agents perceive at the start of the bargaining game that an agreement will be reached on ω^*, they might use "sham bargaining" to prolong the bargaining session until the old contract expires.

$e^{-(r-\beta_0)T-\beta' T^2}$ and $F(T)$ as $e^{-(R-\beta_0)-\beta' T^2}$, i.e. we obtain pay-off function 3 and $F^*(T^*)=1/(1-\alpha+\sqrt{\alpha^2+1})$ where $\alpha=(R-r)\mu/2$ (cf. p. 126). Assuming that $R=0.1, r=2R=0.2$ and $\mu=2$, and furthermore that $p=20$ and $c=10$ we deduce that $\omega=5.25/(1+\tau')$. $\tau'=0$ gives $\omega^*=5.25$, while $\tau'=0.1$ gives $\omega^*=5.25/1.1=4.77$.

9.4.3 Modifications of the Assumptions

The model of a particular type of wage negotiations presented above no doubt suffers from great weaknesses with respect to the realism and hence the applicability of the assumptions.

First of all, however, it should be stressed that many of the assumptions made above can be replaced by other assumptions which would widen the scope of our model's applicability, without altering the conclusions. For at least some situations it appears that the assumptions of a given price and market share $\phi(0)$ can be replaced by assumptions which would allow p to be a variable affecting $\phi(0)$. Likewise we can probably introduce other assumptions such as that the labor union can maximize the wage bill after the deduction of a proportional personal income tax, that prices and costs can increase over time due to inflation, etc.

The conclusions regarding the incidence of the corporate income tax and the wage tax will hold as long as we have an S *-game with $f(T)$ and $F(T)$ independent of τ and τ' and with ν written as a linear function of $(1+\tau')\omega$. This implies that several different pay-off functions are possible. Hence assumptions 5–7, which appear restrictive, are not necessary.

The analysis of these and other modifications will be left for future research. Here we deal only with one particular modification, namely limitation of the analysis to one contract only and the subsequent assumption that FC=0. This assumption is particularly annoying, since it can be interpreted as implying that the bargaining parties are irrational. Furthermore, it is important from a more general point of view (cf. p. 134) to study the application of our model to a series of consecutive negotiations between two parties.

The problems related to applying the analysis to only one contract are due to the following with respect to management. Maximizing only the present value of the profits obtainable under the forthcoming contract period is not necessarily consistent with maximizing the present value of total long-term profits. Let us e.g. assume that management has to choose between an agreement on a wage rate ω at time T, and an agreement on another wage rate ω' at time T', where T' is later than T. This means management has to compare a contract on ω running between T and $T+Z$ and a contract on ω' running between T' and $T'+Z$. Before management can decide which of these outcomes is preferable, it has to contemplate what wage rate will prevail between $T+Z$ and $T'+Z$, if the first contract is chosen.

In order to determine this wage rate, we extend our discussion to the case of two contracts, where the first contract runs from period T_1 to T_1+Z and the second from T_1+Z+T_2 to T_1+T_2+2Z.[37] If the pay-off of each contract is written as a function of the negotiation variable and the agreement time, the total pay-off relevant to management, i.e. L, can be written as $\nu_1(\omega_1,T_1)+\nu_2(\omega_2,T_1+Z+T_2)$ where ν_i is the separate pay-off of contract i and ω_i is the wage in contract i. This problem can be solved by means of backwards deduction. Assume that the first agreement was reached at a certain value of T_1. The problem then is to determine the values of ω_2 and T_2 by applying our model to $\nu_2(\omega_2,T_1+Z+T_2)$, where T_1+Z is a parameter and the only variables are ω_2 and T_2.

This involves a closer look at $\nu_2(\omega_2,T_1+Z+T_2)$. The same values of p, c, production capacity, etc. as in Section 9.4.2, are assumed. Furthermore we assume that the market share at time T_1+Z+T_2 can be written as $\phi(T_1)\phi(T_2)/k$. When a new strike starts at time T_1+Z, $\phi(T_1)\phi(T_2)$ will decrease from $\phi(T_1)\phi(0)$ in the same manner as $\phi(T)$ was assumed to decrease from $\phi(0)$ (cf. p. 201). This assumption appears reasonable considering that deliveries have taken place during Z years and customer preferences have stabilized.

This assumption about the market share implies that $\phi(T_1)$ can be regarded as a constant for a given value of T_1. This in turn implies that the solution ω_2^* will be the same for every value of T_1 and hence the solution of the second contract is *not* influenced by the determination of the first contract. The first contract can be determined without taking the second contract into consideration. The same wage ω_1^* is obtained for the first contract, when determined with explicit consideration of the second contract, as when no such consideration was included in the previous section.

Any number of future contracts can be taken into consideration in this way, provided the market share at the *start* of the nth contract can be written as $\phi(T_1)\phi(T_2)\ldots\phi(T_n)/k^{n-1}$and $\phi(T_j)$ declines just as $\phi(T)$ on p. 201. We can then deduce that nth contract leads to ω^*, as determined on p. 203, regardless of the value of $T_1+T_2\ldots T_{n-1}$. Hence the nth contract will not affect contract $n-1$, which can be solved in isolation, leading to ω^*. Going backwards in this manner, contract after contract, we can deduce that each contract leads to ω^*, regardless of how many future contracts are taken into consideration[35].

[37] T_2 is the ***amount*** of time between the expiration of the first contract (at time T_1+Z counted from time 0) and the agreement on the second contract.

[38] With $f(T)$ unchanged, $f^*(T^*)$ will be the same for every contract. We can also allow for *different* values of p and c and obtain a determined value ω_i^*, possibly different for each contract i.

In this context we can also modify our earlier temporary assumption that the fixed cost, including depreciation, was 0. The total fixed annual cost FC, incurred from time 0 up to the expiration of the nth contract, can be subtracted from management's total pay-off of an agreement at time T_1.

Hence we subtract the present value of all fixed costs incurred at least up to nZ and at most up to T_1+T_2 . .$+T_n+nZ$. With nZ going towards ∞ and r positive and not infinitely small, the size of the fixed cost component to be subtracted will not be affected by values of $T_1 \ldots T_n$ and will not affect the determination of ω_1^*.[39]

9.4.4 Limitations of the Analysis

Even after possible modifications, the twenty assumptions in Section 9.4.2 will no doubt still appear restrictive. When discussing the restrictivity of these assumptions we must distinguish between institutional and behavioristic assumptions.

As regards an *institutional* assumption, this restrictivity implies that only a certain, perhaps small, percentage of all situations fulfill the assumption. This limitation is less serious since our purpose is not to establish a general analysis. In line with our discussion on p. 23, we are satisfied if our model can be applied to some situations that appear interesting.

As regards the conclusions for tax incidence theory, our analysis above cannot in any way be used to make general conclusions that the corporate income tax will never be shifted backwards or that the wage tax will always be shifted backwards. But our example can be used to cast doubt upon or refute some other authors' conclusions or theorems which were intended to be more general. Thus de Menil's conclusion, for example, that a proportional corporate income tax will at least partly be shifted backwards to the union if management has a quadratic utility function implying a decreasing marginal utility for money, does not hold generally.[40]

The restrictivity as regards the *behavioristic* assumptions is of a more fundamental nature. Behavioristic assumptions refer primarily to assumptions about the parties' goals. The assumption that management attempts to maximize the corporation's

[39] In principle FC $\int_0^\infty e^{-rt}dt$ also has to be subtracted from the pay-off of *each* contract in order to perform the backwards deduction presented above. Although this implies that too large a cost is subtracted with regard to the nth and immediately preceding contracts, when this error is discounted backwards to time 0 it will be negligible due to the size of nZ. Thus it does not affect the establishment of ω_1^*, the only value which we are really interested in.

[40] See de Menil (1968, p. 119). However, in a recent book (1971) de Menil reaches the conclusion that the corporate income tax will *not* be shifted backwards.

profits is common in most economic theory. Although this assumption is heavily criticized, one can safely say that there exist a great many corporations for which it is a good approximation of real goals.

There are more serious problems with regard to the labor union's goals. Although some authors who have attempted to construct predictive models for labor-management bargaining have used assumptions similar to ours[41], many others have suggested that unions in general exhibit completely different modes of behavior[42]. In particular, non-monetary factors as well as the personal goals of the union-leaders are introduced into the union goal function. Different goals are assumed for different situations. Since the introduction of these new factors can have contrary effects on different situations it appears reasonable to assume that these non-monetary and personal goals will "even out" to some extent with regard to the average behavior in a great number of negotiations.[43] The problems related to our assumptions about the union's goals thus appear to be greater when applied to one particular bargaining situation than to macroeconomic predictions.

[41] Cf. e.g. Zeuthen (1930), Harsanyi (1956) and de Menil (1968).

[42] Cf. e.g. Ross (1948) and Walton & McKersie (1965).

[43] Some unions might be assumed to be strike prone, e.g. since strikes can increase union unity. Other unions might be strike averting, since strikes might be contrary to the political aims of the union leaders.

Chapter 10

Conclusions and Future Research

10.1 Introduction

In Chapter 2 we stated that the main purpose of this study is to answer the following question: "Can a model of rational behavior be constructed which would lead to a solution for at least some bargaining situations of interest?"

We tried to answer this question in the chapters which followed. It was shown that with respect to a set of simple behavioristic assumptions implying rationality, a solution could be established for some set of institutional and pay-off assumptions. Our discussion in Chapters 8 and 9 indicated that these institutional assumptions could hardly be regarded as too extreme to make the model uninteresting. Furthermore our examples of applications in Chapters 6 and 9 indicated that the set of situations to which the pay-off assumptions of our model are relevant were fairly large, including several situations of genuine interest. Hence even if our model is far from being a general model of rational bargaining, it would constitute a special case of a general model, if such a general model were ever constructed (cf. p. 23).

However, the question of whether people in situations characterized by our institutional and pay-off assumptions really bargain or could be made to bargain in accordance with the model remains to be investigated. Although the behavioristic assumptions are simple, we cannot *a priori* accept their validity. One of the most important steps for future research in this area appears to consist of testing the model by experimental bargaining games (discussed in Section 10.2). A test of the predictive power of the model in certain real situations, at least as compared to other bargaining models, seems suitable as a complement (dealt with in Section 10.3). Finally the relevance of our model in terms of the possible outcome of these tests is briefly discussed in Section 10.4.

10.2 Experimental Testing of the Model

10.2.1 Introduction

Experimental tests of the model should be made in view of the various purposes for which a bargaining model might be used. In particular, we can distinguish between the following purposes:

1. Prediction
2. Normative for one player
3. Mediation

Regardless of the purpose of the test, the following procedure is envisaged: A large number of persons, grouped in pairs, play a number of simple bargaining games. The first set of bargaining games used in these experiments should be very simple, involving a small number of alternatives. If the model is *not* applicable to these very simple situations, then it is probably not applicable to more complicated situations either.

One factor which appears to contribute to rationality is the rewards connected with the outcomes (see p. 130). Subjects were believed to be more likely to adhere to B_1, if they bargained over significant sums of money rather than just for "fun and honor". Although no large sums can obviously be paid, monetary prizes should be used in at least a substantial part of the experiments.

10.2.2 Testing the Predictive Ability of the Model

It is natural that the main emphasis is on testing the ability of the model to predict the behavior of the parties in an experimental situation. In this context it is reasonable to hypothesize that the parties' behavior will become more rational when they gain more insight into the bargaining situation. Learning effects are likely. Such learning is facilitated by the fact that there is continuous feedback in the form of information about the other party's bids. In order to measure these effects each experimental subject should participate in a series of experimental games. The subject should play a new opponent in each of these games so that the series of games cannot be regarded as *one* supergame. Furthermore in order to reduce the effects of personal factors on the outcome of the experiments it appears desirable that the subjects in each pair remain unknown to each other in at least some of the experiments.

10.2.3 Testing the Model's Value for Normative Use

A model fulfills the normative purpose if a party, when presented with the model, will use it and, after feed-back about the results, continue to use it (cf. p. 142). According to this definition a test of the usefulness of the model for the normative purpose should be made by giving at least one of the two parties a short description of the model as applied to the particular situation along with the recommended mode of behavior according to the model.

It might also be of interest to test the normative value of our model as compared to the normative value of other bargaining models. This could be done by presenting some subjects with our model and some subjects with another model, in a situation where different modes of behavior are suggested by the two models. It would then be possible to test which mode of behavior is adhered to the most.

10.2.4 Testing the Mediation Purpose

The mediation purpose can be tested in two different ways:

The first way is focused on the question of whether a skilled mediator can get the parties to agree on the terms suggested by the model. The experiment leader could assume the rôle of mediator. The question then is whether he can persuade the parties to reach an agreement on the terms of the model. In order to measure the suitability of our model for the mediation purpose–and *not* the experiment leader's ability to persuade–it is particularly important in this case that other models, such as the Nash model, are also tested in the same way as our model. This will give some indication of whether our model is more suitable for mediation than other models.

The second way of testing the mediation ability of the model is to test whether an experimental subject, instructed to act as a mediator, will want to use the model when presented with it. In this instance it is also reasonable that our model should compete with other models to see which model will be most readily adopted by the parties acting as mediators.

10.3 Preliminary Testing of the Model's Predictive Power in Real World Situations

In its present stage of development, our model can to a limited extent be tested in real world situations. First of all, there is the possibility of conducting a rough face validity test of the model, e. g. with the aid of the examples in Chapters 6 and 9. Furthermore the games in Chapter 6 can also be used to derive certain hypotheses, which can be tested without detailed knowledge of all relevant factors. In general it will be difficult to test whether the real outcome is exactly the same as that of the proposed model, due to insufficient knowledge about the values of the parameters and about the outcome of the real negotiation. We have to limit ourselves to *qualitative* hypotheses about the effects of certain parameters. Among these hypotheses, it is probably easiest to test those concerning the effects of r and R (as regards pay-off functions 1 and 3) and of w and W (as concerns pay-off function 1). For cases when no investment is made at the time of the agreement, we deduced the following two hypotheses:

1. The party with the higher interest rate will, *ceteris paribus*, obtain the smaller share of the joint profit.

2. The party with the higher pre-agreement profit will, *ceteris paribus*, obtain the higher share of the joint profit.

These hypotheses can also be used to test whether our model is a better predictor than some of the more generally accepted bargaining solutions, i.e. the Nash solution and the "equal-split" solution. The Nash solution is not affected by differences in interest rates, while the equal-split solution remains unchanged by differences in either the interest rate or the pre-agreement profit (cf. p. 147).

10.4 Concluding Remarks

Finally, something should be said about the relevance of our model with respect to the possible outcome of the testing described in the previous sections.

If the experimental tests indicate that the model is of value as a tool for prediction, for normative use or for mediation, it appears reasonable to continue constructing the model in more complex versions incorporating different features such as considerations of incomplete information (cf. Chapter 8).

On the other hand, if the experiments imply that very few persons behave or are willing to behave according to our model, and if real world tests prove that the model does not have qualitative predictive ability, then the model probably has to be refuted. Such a refutal, however, would also be of great interest since our model can be regarded as a *special case* of any reasonable attempt at a general model of bargaining. The behavioristic assumptions of B_4 could then not be regarded as suitable even as an approximation of the behavior in a bargaining situation. Since it appears impossible to construct an interesting bargaining model on a smaller set of behavioristic assumptions implying rationality (cf. Chapter 3), any other set of rational behavioristic assumptions would most probably have to include B_4.

Furthermore it is difficult to replace any of assumptions $I_1 - I_{13}$ by other assumptions that do *not* include the assumption thus omitted as a special case, without seriously limiting the applicability of the model and/or increasing the required set of behavioristic assumptions. For example, if the assumption of complete information is replaced by one of *necessarily* incomplete information, we would no longer be able to analyze the case of complete information. The only assumption that has a substitute on roughly the same level of generality is I_4, i. e. the assumption of alternating bidding. This assumption can be replaced by an assumption of simultaneous bidding. As noted earlier, however, the establishment of a solution under simultaneous bidding requires additional behavioristic assumptions of a demanding nature. A convincing reason for ruling out alternating bidding in every period would also have to be established.

Thus it appears well-motivated to hypothesize that if our model of bargaining is refuted on the basis of empirical evidence, then every other attempt to construct a bargaining model consistent with rationality will be doomed in advance.

Literature Appendix

L.1 A General Taxonomy and Literature Survey

L.1.1 Introduction

This survey of the main works in the field of bargaining theory is aimed at facilitating a comparison between our model and other models in the literature. Another purpose of this appendix is to illustrate and clarify certain points discussed in the text.

First, a taxonomy for analyzing bargaining situations is introduced. This is followed by a brief review of how different models are related to this taxonomy.[1] Finally, and most important, the bargaining models which are of particular interest for us, namely those of Nash, Zeuthen, Bishop, Foldes, Cross, Coddington and Hicks are discussed in greater detail in Sections L.2 – L.6.[2]

L.1.2 A Taxonomy for Bargaining Models

There are no doubt several possible ways in which a taxonomy of bargaining can be constructed. Since our main purpose is to provide a basis for comparing our model with other models, the taxonomy presented here is closely related to the types of assumptions used in our model. Hence the main principle is that models should be classified according to the assumptions on which they are based. As in the main text, we distinguish between behavioristic assumptions, institutional assumptions and pay-off assumptions. We do not refer to each of the assumptions presented earlier, but only to those which appear to be significant when comparing different bargaining models. Some additional behavioristic assumptions will also be intro-

[1] An extensive survey of bargaining literature prior to 1943 is given in Denis (1943).

[2] The Nash and Zeuthen models, which lead to the Nash solution, are important because this solution is well-known and because we refer to it several times in the text. The Bishop-Foldes model and its predecessor, the Hicks model, are important since they constitute attempts at establishing a solution for a situation with rational bargainers, decreasing pay-offs over time and ordinal utility, i.e. the cornerstones of our theory. The Cross-Coddington model, which is also based on decreasing pay-offs over time and ordinal utility, is of interest since it illustrates the great problems encountered when B_{10} is violated (cf. p. 244).

duced. Thus, bargaining games can be classified according to the following dimensions:

A. Types of assumptions used

a)	Solution given by behavioristic, institutional and pay-off assumptions	b)	Establishment of a solution requires assumptions which describe desirable properties of the solution

B. Behavioristic assumptions

1.	a)	B_1 holds	b)	B_1 does *not* hold
2.	a)	B_2 holds	b)	B_2 does not hold
3.	a)	B_3 holds	b)	B_3 does not hold
4.	a)	Ordinal utility sufficient	b)	Cardinal utility also required
5.	a)	No *ad hoc* assumptions of subjective probabilities or concession rates	b)	Subjective probabilities or concession rates determine the solution
6.	a)	No interpersonal utility comparisons	b)	Interpersonal utility comparisons

C. Institutional assumptions

1.	a)	Alternating bidding (I_4 holds)	b)	Simultaneous bidding
2.	a)	Complete information (I_5 holds)	b)	Incomplete information
3.	a)	Bargaining game in *one* dimension (I_7 holds)	b)	Bargaining game in several dimensions
4.	a)	Threats *not* included (I_{10} holds)	b)	Explicit consideration of threats
5.	a)	No liquidity restrictions (I_{11} holds)	b)	Liquidity restrictions affect solution

D. Pay-off assumptions

1.	a)	Decreasing pay-offs over time (S_1 holds)	b)	Constant pay-offs over time
2.	a)	Play-*in*dependent pay-offs (G_1 holds)	b)	Possibly play-dependent pay-offs

This classification is such that in every instance our model belongs to the a-classification.

L.1.3 Discussion of the Taxonomy

Next we proceed to the implications of the different classification variables. Some models which are described to a considerable extent by the characteristic under discussion will also be referred to.

A. Types of assumptions used

The first characteristic concerns whether the three types of assumptions used in our model are sufficient for deducing a solution or whether other kinds of assumptions, which deal directly with desirable properties of a solution, are required for determining this solution. The latter applies to a group of models, generally called arbitration models, because their main purpose is for use in arbitration. The best-known model in this group is Nash's. Other fairly well-known arbitration models are those of Shapley, Raiffa and Braithwaite.[3] They are discussed very briefly at the end of Section L.2. This group contains models which require specifications of desirable properties of the solution. It also includes the "equal split" solution according to which the parties split the joint profits in two equal parts, due to a very simple principle of equity.

B. Behavioristic assumptions

B.1 *Individual Rationality*

1. The first characteristic concerns whether or not the parties are assumed to be rational in the most basic microeconomic sense, i.e. whether the parties are maximizers or whether they should be described in more psychologically oriented terms. All models discussed in L.2 – L.6 have characteristic B1a. There is extensive literature on characteristic B1b, mainly of the descriptive type. Some examples are Ross (1948), Douglas (1962), Stevens (1963), Walton & McKersie (1965), Aschenfelter & Johnson (1969) and Gustafsson (1970).[4] Most of this literature concerns labor management bargaining – often with an emphasis on institutional assumptions[5] – but there are also studies aimed at other areas of application such as political bargaining, e.g. Iklé & Leites (1964).

[3] One of the less well-known arbitration models is Rådström's (1959). A special case of arbitration is discussed by Contini (1967). The arbitration solution will be imposed only if no agreement is reached by the bargainers themselves.

[4] According to Aschenfelter & Johnson labor does not behave according to B_1, but management does.

[5] For a review, see Rothschild (1957), Pen (1959) and Walton & McKersie (1965).

Most of the literature based on B1b is not aimed further than establishing some general concepts for describing the bargaining process.[6] There are few attempts at operationalization and virtually no attempts at establishing any solution. Some attempts to establish a more formal theory include Siegel & Fouraker's aspiration level model (1960) and Grubbström's model (1972), based on the assumption that a party will continue to change his decision variable in a certain direction if earlier changes in this direction led to higher profits. In certain bilateral monopoly situations this model leads to an agreement on a quantity $q=0$. Bartos' model (1972), based on extensive experimentation, and Schenitzki's model (1962; described on p. 221), should also be included here.

B.2 *Rational expectations of an opponent's behavior*

All models implying that the solution is Pareto-optimal are based on B_2 (cf. Section 8.4). The models which do not lead to a Pareto-optimal solution either reject B_2 or are based on very strong and specific institutional assumptions. Among the best-known models which do not lead to Pareto-optimality are those of Wicksell (1925) and Bowley (1924 and 1928).[7] These models concern a bilateral monopoly situation of the type discussed on p. 171. A seller H (e.g. a manufacturer) sells a quantity q of some products at price P to a buyer L, who then resells them at price p on a market where the demand function $p=f(q)$, such that $f'(q)\neq 0$, holds. H quotes a price P and L a quantity q.

According to the Bowley model, L will violate B_2. When trying to maximize $v=q(f(q)-P)$ L will regard P as given and for each value of P, respond with that q for which v is maximized, implying that $P=f(q)+qf'(q)$.[8] Realizing this H maximizes his profit $V=qP-\text{TC}$ under the restriction that $P=f(q)+qf'(q)$, implying that q is given by $f(q)+3qf'(q)+q^2f''(q)=\text{MC}$.[9] With $f'(q)\neq 0$ this generally[10] leads to a q which is not the Pareto-optimal one established when the joint profit $v+V=q\,f(q)-\text{TC}$ is maximized, i.e. by $f(q)+qf'(q)=\text{MC}$.[11]

In the Wicksell model it is H who violates B_2 by regarding P as given, implying that he sets MC=P. This also results in a non-Pareto-optimal solution. The question of

[6] Several bargaining experiments can be regarded as closely related to this literature, which stresses the psychological aspects of bargaining. See e.g. Deutsch & Krauss (1962).

[7] The model presented by Hieser (1970) is one of several recent models which imply maximization behavior but do not lead to a Pareto-optimal outcome. As regards the solution of the distribution problem, this model resembles Hicks' model according to Shackle's interpretation (cf. p. 249).

[8] This "reaction curve" is illustrated by line dd' in Figure 25 (p. 172).

[9] This solution is given by the point ($q=1, P=6$) in Figure 25, where dd' is a tangent to H's curve implying that $V=2$. Hence 2 is the highest profit H can obtain.

[10] For an exception see Tintner (1939,(p. 172).

[11] In Figure 25, $q=2$ is Pareto-optimal.

why B_2 is discarded cannot be avoided unless very strong institutional assumptions, e.g. that each party only makes *one* single bid, are adopted.[12] This limitation of the bargaining process constitutes a violation of B_2 if the parties are free to determine the bargaining procedure among themselves.

B.3 *The insight assumption*

While most models based on B_1 also accept B_2, there are several models based on B_2 which violate B_3, i.e. violate assumption B_{10}. This means that in these models the parties entertain irrational *expectations* concerning the other party's expectations. Cross' and Coddington's models are examples where this violation is obvious (see Section L.5). There is also a violation of B_{10} in the n-alternative version of the models of Zeuthen (cf. p. 237) and Bishop (cf. p. 243).

Furthermore, B_{10} is violated by the marginal intersection model.[13] This model refers to a bilateral monopoly situation of the type discussed above. According to this model both H and L assume that P is given and they will both bid that q which for the contemplated or current value of P, maximizes their profits. Thus all of L's bids are such that $P = f(q)+qf'(q)$ and all of H's bids such that $P = \mathrm{MC}$. This implies that the solution is given by $f(q) + qf'(q) = \mathrm{MC}$, i.e. by the equation for the Pareto-optimal curve (see p. 216). Hence a specific Pareto-optimal solution is obtained but the model violates B_{10}, since *if* H knew that L assumes that P is given, he himself would *not* assume that P is given, if he would adhere to B_{10}.[14]

The question then arises as to whether there are institutional assumptions such that the marginal intersection solution would be obtained for parties who obey B_3. One possibility is bargaining in the form of a "market simulation" process carried out with the aid of a referee. The referee states a price and the parties then simultaneously state the quantities they desire at this price. If these quantities differ, the referee names a new price, chosen e.g. at random. Then, if each party bids the quantity which maximizes his profit at every price, an agreement will eventually be reached at the intersection of the marginal curves.[15]

[12] If there were more than two bids the parties could – after a preliminary agreement on e.g. the Bowley solution – move towards a Pareto-optimal solution. For experimental evidence against the Bowley solution see e.g. Siegel & Fouraker (1960), Fouraker & Siegel (1963) and Arvidsson (1972).

[13] This model is discussed e.g. by Schneider (1932), Henderson (1940, who calls it a pseudo-equilibrium), Boulding (1950), Fouraker (1957) and Grubbström (1972, who calls it the Economic Equilibrium Solution).

[14] H would then rather maximize V under the restriction that $P = f(q)+qf'(q)$, leading to the Bowley solution discussed above.

[15] In this case it is not necessarily inconsistent with B_3 to regard P as given, even if the other party regards it as given. Similar, but not identical, schemes are discussed in Marschak (1965).

An important problem, however, is that this will not lead to a truly Pareto-optimal solution. If H knows that the solution of the marginal intersection model will be reached, e.g. through a "market simulation procedure" of the type mentioned above, H will find it advantageous to increase his costs. Let us call the solution, obtained by this model when the original cost functions are retained, q^*. H can then obtain a solution on a quantity $q' < q^*$ in e.g. the following way: Immediately prior to the start of the bargaining, H announces a new wage rate which increases rapidly with production after total production has reached $q' - \epsilon$. The new wage rate is such that TC remains unchanged for $q < q' - \epsilon$ and MC=MR for $q = q'$. Then the solution according to the marginal intersection model is that an agreement will be reached on q'. Since ϵ is very small TC is virtually the same at this quantity as it was before the cost increase. The optimal value of q' for H is hence approximately the q at which H maximizes V – with the original cost functions – under the restriction that $P = f(q) + qf'(q)$. This is the Bowley solution, which, as noted above, is *not* Pareto-optimal.[16] Unless the condition that a party can increase his costs is explicitly ruled out, the marginal intersection model will *not* lead to a Pareto-optimal agreement. Then the parties cannot choose such a bargaining procedure without violating B_3.

B.4 *Type of utility assumed*

We next turn to the question of whether an ordinal or cardinal utility is used. While the models of Edgeworth (1881), Wicksell (1925), Bowley (1928), Hicks (1932), Bishop (1964), Foldes (1964), Cross (1965), Coddington (1968) as well as the marginal intersection model and our own model require only ordinal utility[17], those of Zeuthen (1930), Nash (1950 and 1953), Raiffa (1953), Braithwaite (1955), van der Ster (1957), Saraydar (1965), Krelle (1968), Contini (1970) and Krelle (1970 and 1971), – just to mention some examples – are based on *cardinal* utility of the von Neumann-Morgenstern type. The models of Pen (1952) and Shackle (1949) are based on more complex types of cardinal utility functions. The models of ordinal utility get by with a less restrictive implication of the complete information assumption (see p. 180). In the cardinal utility case the complete information assumption requires knowledge about not only the other party's monetary pay-off from various outcomes but also his cardinal utility for money.

[16] In the example in Figure 25, the marginal intersection solution for the case of the original cost functions implies that P=MC=C=4, i.e. that H obtains 0 profit. By increasing his costs so that MC=$4+2q$ for $q > 1-\epsilon$, H can obtain an agreement on q=1, with P=6, giving H a profit of 2.

[17] Defined over outcome factors, not lotteries as in Wagner (1957).

B.5 *Ad hoc assumptions regarding probabilities and concession rates*

In our model we showed that set B_3 (sometimes enlarged to the slightly more demanding set B_4) was sufficient for determining the solution for many situations. Models which rely only on B_4 – as our model – have the advantage that B_4 only contains assumptions also applicable to decision situations other than bargaining. Many other models also require some form of *ad hoc* assumptions, specific to behavior in bargaining situations. These can either be assumptions about subjective probabilities regarding the likelihood that the other party will concede in a certain way (as in the models of Zeuthen, Pen, van der Ster, Saraydar, Contini and the Bishop-Foldes model[18]) or assumptions about the rate at which one expects the other party to concede (as in the models of Cross, Coddington and possibly also Hicks, cf. p. 251).

B.6 *Interpersonal utility comparisons*

While most models assume that the parties do not compare each other's utility as such, there are some models which rely upon this kind of interpersonal utility comparison for their solution. This refers either to "behavioristic" models, i.e. those based on type Aa, e.g. Coen's model (1958) and possibly also Krelle's model (1961) or to arbitration models, e.g. the models of Raiffa and Braithwaite.[19].

C. *Institutional Assumptions*

C.1 *Mode of bidding*

The mode of bidding is an important variable. Our model and e.g. those of van der Ster (1967), Contini (1970) and Krelle (1970 and 1971) explicitly rely on alternating bidding, while those of e.g. Zeuthen (1930)[20] and Krelle (1968) explicitly rely on simultaneous bidding. In many models it is not explicitly stated whether alternating or simultaneous bidding is assumed. Some, such as the Bishop-Foldes model, have to be interpreted as relying on simultaneous bidding in

[18] While the Zeuthen and Bishop-Foldes models (as discussed later on) can be used for predictions, the models of Pen, van der Ster, Saraydar and Contini lack predictive capability. This is because the subjective probabilities in the former models are determined endogenously (in the Bishop-Foldes model taking the value 1 or 0) and given exogenously in the latter models.

[19] Coen's analysis is based on the behavioristic assumption that the party who expects to suffer the greater disutility from deadlock will be the one which feels compelled to make concessions. Krelle (1961) assumes that a certain concession by one party is regarded as *equal* to a certain concession from the other party.

[20] At least according to Harsanyi's interpretation of it (cf. p. 230).

order to be as consistent as possible with B_3 (cf. p. 240). The solution of some models, such as those of Cross and Coddington, are not affected significantly by which mode of bidding is assumed.

C.2 *State of information*

Most other models of bargaining are based – as is our model – on the assumption that each party has complete information. This pre-eminence of assumptions of complete information is probably due to the fact that most authors regard the development of a model based on complete information as a necessary first step towards a more complex theory of bargaining. It is then not surprising that most models which actually incorporate considerations of incomplete information are based on an existing bargaining model for complete information. Hence the models of Harsanyi (1967–68) and Harsanyi & Selten (1972) rely on the Nash model while Pen's model (1952) is partly based on Zeuthen's model. Krelle's model (1970 and 1971) can also be applied to the case of complete information, in which case it should be regarded primarily as a special case of Zermelo's theorem (cf. p. 257). An example of a model which has no correspondence in the case of complete information is Iklé-Leites'. Some models, such as Cross' and Schenitzki's, do not specify which kind of information about the pay-off is assumed, since the solution is not determined in conjunction with considerations of the other party's pay-off.

C.3 *Number of dimensions in the bargaining game*

In our model, and its extension in Chapter 8, we assumed that the negotiation was solved in two phases. The efficiency problem of the negotiation was solved in a pre-bargaining phase where the parties agreed to bargain only over Pareto-optimal agreements in the second phase – the bargaining game (cf. p. 59). In a similar manner, many bargaining models assume that the efficiency problem is solved prior to the bargaining game and hence assume that bargaining takes place in only one dimension. Some models, however, assume that bargaining concerns *several* dimensions simultaneously.[21] Noteworthy among these are some models aimed at solving both the efficiency and distribution problems. These include the Bowley, Wicksell and marginal intersection models, discussed above. Other models are limited to establishing the solution of the efficiency problem, such as e.g. the models of Edgeworth, von Neumann-Morgenstern and Schenitzki.

It is interesting to note that these last three models, although of different types, all lead to a Pareto-optimal outcome. The Schenitzki model only assumes that each

[21] Avoidance of the explicit assumption that all bids are Pareto-optimal is particularly common among bargaining models which focus primarily on labor-management bargaining.

party adheres to a concession pattern which is as follows: H starts by making a bid that implies a high V. Before making a bid leading to a lower V than before, H first makes all (or at least a great many different) bids, implying the same V. L behaves in a similar way. Figure 26 illustrates a case of bargaining regarding the division of a fixed amount. The numbers 1–4 refer to the periods in which the different bids might be made.[22]

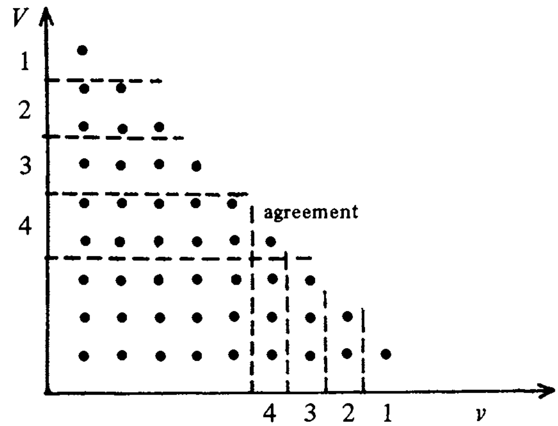

Figure 26 The Schenitzki model

The first time the parties' bids are identical is on the Pareto-optimal contract line.[23]

C.4 *Consideration of threats*

Threats of breaking up the game if the opponent does not accept a certain agreement were ruled out in our model. Threats are likewise ruled out either explicitly or implicitly in many other models, such as those of Hicks, Bishop, Foldes, Cross and Coddington. In other models, mainly those of Nash (1953) and Schelling (1960), explicit consideration of threats is fundamental for the establishment of a solution.

C.5 *Liquidity considerations*

The last of our institutional assumptions to be dealt with here has to do with liquidity reserves being so large that a delay in reaching an agreement will *not*

[22] The periods are of such length that several bids can be made in each period.

[23] It should be stressed that, definition-wise, no agreement can be reached on a (P,q)-combination *above* the contract-line, denoting the joint maximal profits. Any point above this line would symbolize one set of (P,q)-bids for L and a completely different set for H.

jeopardize the survival of the parties (e.g. two corporations). Our model seems to share this assumption with most other bargaining models. A notable exception is Krelle's model (1961) in which liquidity considerations can become significant.

D. Pay-off Assumptions

D.1 *Variations in the pay-offs over time*

A fundamental assumption for our model is S_1 – that the pay-offs of both parties decrease steadily over time. The same assumption is fundamental for several other models, particularly those of Hicks, Bishop, Foldes, Cross, Coddington and Contini. Other models seem to be concerned mainly with a static case where the pay-off is *not* dependent on when an agreement is reached. This is e.g. true for the arbitration models and the models of Edgeworth, Bowley, Wicksell, Zeuthen, Pen and Saraydar. These models can also be used for cases of variable pay-offs[24] with a varying degree of success. In many instances they will then be completely insensitive to differences in the parties' rate of time discount.

D.2 *Play-dependence of pay-offs*

The assumption of play-independent pay-offs is important as regards the possibility of finding the solution for any game with a large number of alternatives and periods (cf. p. 67). This assumption has no relevance to many models for which the extensive form cannot even be contemplated (e.g. the arbitration models). The assumption of play-independent pay-offs appears to be made, at least implicitly, in most other models (e.g. Bishop, Foldes, Cross, Coddington, etc.). Examples of exceptions are the models of Pen, Shackle and Krelle (1970).

L.2 The Nash Model

L.2.1 Presentation of the Model

L.2.1.1 The Fixed Threat Case

Nash's original model from 1950 deals with the fixed threat case, i.e. in the event of a conflict, the pay-offs of both parties are given. A solution is deduced with the aid of four axioms.

[24] For the Nash model (p. 225).

According to axiom 1, both parties – H and L – assign a cardinal utility[25], U and u, respectively, to each outcome. The conflict utilities are initially placed at origo. All possible outcomes are then represented by the lined area in Figure 27 below. The set of possible contracts is assumed to be convex[26] *and* compact[27].

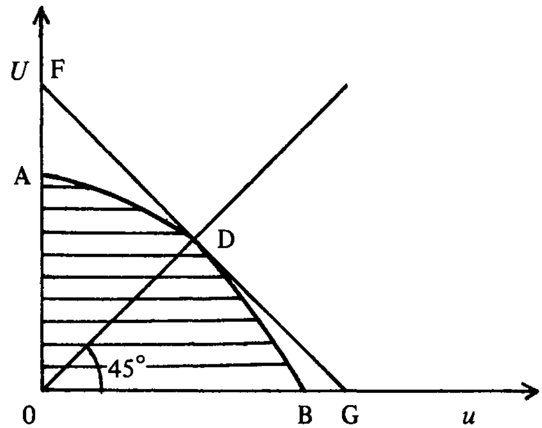

Figure 27. The set of outcomes in the Nash model

Axiom 2 – that the solution is Pareto-optimal – is used next for deducing that the solution must lie on the curve ADB in Figure 27. We call this the *contract curve*. Both parties can increase their utility by moving from every point that is *not* on the contract curve to some point on this curve.

We now introduce axiom 3 – the symmetry axiom: If, for every point $u=x$, $U=y$ belonging to a set, the point $u=y$, $U=x$ also belongs to the set, then the solution (u^*, U^*) is such that $u^*=U^*$[28]. This implies that if a compact contract set were

[25] In the von Neumann-Morgenstern sense.

[26] A set is convex if every straight line drawn between two points in the set belongs to the set.

[27] In a compact set every neighborhood of each point in the set also contains other points in the set, where a neighborhood of a point x is the set of all points y such that the distance between y and x is very small. (See Lancaster, 1968, pp. 220–221.) The assumption of compactness is critical for the interpretation of axiom 4. If the set is *not* compact, then virtually every point in the set can be eliminated without affecting the conflict point and hence the solution.

[28] One way of interpreting the symmetry axiom involves asking – after both parties have assigned a utility index to each outcome – whether there exists some transformation of one party's utility index such that the outcome set can be made symmetric. Contrary to e.g. Bishop (1963) the author claims that this does *not* involve any interpersonal utility comparison. The following example illustrates this: H and L, both having a linear utility for money, are to divide \$ 1 mill. H as a 50 per cent tax rate, while L pays 0 per cent. If each party originally assigns the utility index 1 to \$ 1 *after* tax, symmetry is obtained by doubling H's original utility. This in *no* way implies that the utility of \$ 2 after tax for L is regarded as equivalent to the utility of \$ 1 after tax for H.

completely symmetric around the 45°-line, then the solution must also lie on the 45°-line. In other words, if we can let H and L change places on the axes without affecting the set of possible outcomes, then the solution is on the 45°-line. An example of such a symmetric outcome set is the set constituted by the triangle OFGO. Hence axioms 2 and 3 combined assign the point D, i.e. the point of intersection between the 45°-line and the FG-line, as the solution to the set OFGO.

Since the distance OF is equal to the distance OG, FG has a slope of -1. Temporarily setting $U=u=1$ at point D, line FG obtains the form $U=2-u$. Since $uU=2u-u^2$, we find that uU obtains its maximum value when $u=1$[29], i.e. at point D.

Finally, we introduce axiom 4: "The axiom of independence of irrelevant alternatives". According to this axiom the solution shall not be affected by the subtraction of outcomes constituting neither the solution nor the conflict point. Hence the subtraction of all alternatives belonging to the set OFGO, but not to the set OABO, will leave point D as the solution for set OABO as well. Furthermore, for a given value of u, the value of U on or below the curve AB is not larger than the value of U on the FG-line. Hence the product uU cannot be larger at any point on or below the AB-curve than at that point on the FG-line at which uU is maximized, i.e. point D. Since D also belongs to OABO, we establish that the product uU is maximized also for this set at point D.

Axiom 1 can also be used to make a linear transformation. We set $\bar{U}=U^c+aU$ and $\bar{u}=u^c+bu$, where $ab>0$ and U^c and u^c are the utilities at the conflict point. We then note that $(U-U^c)(u-u^c)$ is maximized[30] at the point of solution D.

L.2.1.2 The Variable Threat Case

If we abandon the assumption that the conflict point is given, but retain the four axioms above, a unique pair (u^*,U^*), namely the one for which $(U-U^c)(u-u^c)$ is maximized, can be assigned to each possible conflict point (u^c,U^c) lying within the contract set.

It can now be shown that the larger u^c is and the smaller U^c is, the larger u^* will be.[31] The proof is simple for the case we are primarily interested in, i.e. when the

[29] $d(uU)/du=0 \Rightarrow 2-2u=0 \Rightarrow u=1$.

[30] The conclusion that Uu is maximized implies that $U^*u^* > Uu$ holds for every $U \neq U^*$ and $u \neq u^*$. In turn, $U^*u^* > Uu \Rightarrow abU^*u^* > abUu \Rightarrow (U^c + aU^*-U^c)(u^c + bu^*-u^c) > (U^c + aU-U^c)(u^c + bu-u^c) \Rightarrow (\bar{U}^*-U^c)(\bar{u}^*-u^c) > (\bar{U}-u^c)(\bar{u}-u^c)$.

[31] For a general proof, see Nash (1953).

contract curve is linear (see below).[33] Thus L wants to keep u^c high and U^c low. H's interests are exactly the opposite. Nash now assumes that a *threat game* for determining the conflict point will take place prior to the actual bargaining. In this game each party will announce a threat in the form of committing himself to the strategy he will use if *no* agreement is reached. H tries to maximize $U^c - u^c$ in this threat game, while L tries to maximize $u^c - U^c$. This leads to a zero-sum game for which a unique pair of equilibrium strategies can be determined.

L.2.2 Application of the Nash Model to a Simple S*-game

L.2.2.1 Introduction

The Nash model does not explicitly cover the possibility that the value of a contract may decline due to a delay in reaching an agreement. But a great many bargaining situations seem to be characterized by this condition. Thus an important question with regard to the Nash model is whether it is applicable to such a situation.

A simplified pay-off function 1 with only an agreement profit component (cf. p. 119) will be used to exemplify how the Nash model works. The same notations will be retained, where S is H's share of the annual profit. Setting $\pi=1$ we obtain H's pay-off $V(S,T) = S\int_T^Z e^{-Rt}dt$[33] and L's pay-off $v(S,T) = (1-S)\int_T^Z e^{-rt}dt$. Let us now assume that both parties have a linear utility for money. If an agreement is reached immediately, there will be an outcome on the contract curve. Writing $U=V(S,0)=S\int_0^Z e^{-Rt}dt=S(1-e^{-RZ})/R$ as SK and likewise $u=v(S,0)$ as $(1-S)k$, this curve is given by $U = K-uK/k$, i.e. the contract curve is linear.[34] If no agreement is reached prior to time Z, both parties obtain 0, i.e. the outcome is located at origo. Every agreement between time 0 and time Z will result in an outcome in the interior of the set.

[32] With $U = a-bu$, u^* is given by $(U-U^c)(u-u^c) = \max$, i.e. by $0 = d(a-bu-U^c)(u-u^c)/du = a - 2bu - U^c + bu^c$ implying that $u^* = (a-U^c+bu^c)/2b$.

[33] Since Z is *not* required to be finite in this case, $V(S,T)$ also covers the case where $V(S,T)=S\int_T^\infty e^{-Rt}dt=Se^{-RT}/R$. With R constant this can also represent the case where $V(S,T) = Se^{-RT}$.

[34] $u = (1-S)k \Rightarrow S = 1-u/k \Rightarrow U/K = 1-u/k \Rightarrow U = K-uK/k$.

L.2.2.2 *Solution of the Fixed Threat Case*

In the fixed threat case, it is assumed that each party can only threaten not to reach an agreement prior to Z. The threat pay-off of each party will be 0 and the threat point is located at origo. Then, according to Nash's model we obtain the solution $u^* = k/2$ and $U^*=K/2$[35] implying that $S=1/2$[36]. Hence, a situation with this simple type of pay-off function, a utility linear with money and the threat point (0,0) will – according to Nash's model – result in the parties getting 50 per cent each, ***regardless*** of differences in interest rates.

L.2.2.3 *Solution of the Variable Threat Case*

Above the threat point was assumed *ad hoc* to be the point (0,0). If we now allow for any possible threats, the question arises as to how the threat point should be determined. According to Nash (1953) each party chooses a certain threat strategy[37] if no immediate agreement is reached. The conflict point (u^c,U^c) is determined by the pair of outcomes (u,U) from the choice of these two threat strategies.

Let σ denote any of H's threat strategies with the following characteristics: H will make a bid in some period such that for at least one of L's threat strategies there will be a conflict point such that $u^c > U^c$. If H plays σ, L's best reply[38] is to play a threat strategy σ' leading to a conflict point where $u^c > U^c$. Since $u^c=U^c$ implies that $S=1/2$[39] and L obtains a higher pay-off the higher u^c is[40], L will get a share $(1-S) > 1/2$ by using σ'. But H has a better reply to σ' than σ, namely to refuse any agreement until period Z. Since this places the conflict point at origo, it leads as noted to $S=1/2$. Since σ' is the best reply to σ, but σ is ***not*** the best reply to σ', no strategy σ can be part of an equilibrium pair[41] of threat strategies. We can likewise prove that none of L's strategies possibly leading to $U^c > u^c$ can be part of an equilibrium pair of threat strategies.

This implies that no equilibrium pair of threat strategies will lead to a conflict point with $U^c \neq u^c$. We also conclude that there exists at least one pair of equilibrium

[35] $uU = uK-u^2K/k \Rightarrow duU/du = K-2uK/k = 0 \Rightarrow u^* = k/2$.
Then $U^* = K-u^*K/k = K-kK/2k = K/2$.

[36] $S = U/K = K/2K = 1/2$.

[37] For a definition of the strategy concept see p. 253.

[38] The best reply is a better reply as compared to any other choice. For a definition of the concept of *better reply* see p. 254.

[39] If both parties have a linear utility for money, $U+u$ is a constant on the contract line. Then $d(U-U^c)(u-u^c)/du = d(Uu-(U+u)U^c+(U^c)^2)/du = dUu/du$. In this case dUu/du leads to $S = 1/2$ (see L.2.2.2).

[40] See p. 224.

[41] For a definition of this concept see p. 256.

threat strategies leading to a conflict point with $U^c = u^c$, namely that both parties refuse an agreement prior to Z.[42] Since the conflict point is established by some set of equilibrium strategies, we conclude that $U^c = u^c$ at the conflict point and that the solution of the real bargaining game will be $S=1/2$, regardless of the size of the differences in the interest rate.

L.2.3 Comments

The conclusion that every negotiation of the type described above will lead to $S=1/2$ causes us to criticize the contention that the Nash model of bargaining can be used generally as a positive theory for bargaining situations. The suggestion that the outcome is completely unaffected by differences in time preference rates seems dubious. For the specific case where Z is finite and the bargaining takes the form of alternating bidding, the Nash model will – as proved by our model – run contrary to our simple behavioristic assumptions. There does not seem to be any reason to believe that this discrepancy is due only to our assumption of alternating bidding.

The question then arises as to the reason for this weakness in the Nash model as regards its application to the dynamic case. Our model very well allows both parties to have a linear utility for money so that the discrepancy between our results and those of Nash is not due to axiom 1. Since the contract curve is a straight line, axiom 4 of "independence of irrelevant alternatives" is not involved (cf. p. 224).[43] Since the solution according to our model is also Pareto-optimal in the widest sense of the word[44], the discrepancy is not due to axiom 2 either.

Instead the main criticism has to be directed towards axiom 3 – the symmetry axiom. To demonstrate that this axiom is unreasonable in certain cases, we first study a simple four-alternative game covering five periods. We assume that $V(S,T) = S \int_T^5 e^{-0t}dt = S(5-T)$, that $v(S,T) = (1-S) \int_T^5 e^{-0.2t}dt$ and that both parties' utility for money is linear, implying that $U = V$. L's profits are transformed so that the contract curve has a slope of -1.[45] Figure 28 is then obtained.

[42] If e.g. H threatens to delay an agreement up to time Z, L's best reply in the threat game is also to threaten a delay up to Z, since every lesser threat will lead to a conflict point with $U^c > u^c$.

[43] This axiom is criticized elsewhere, see e.g. Luce & Raiffa (1957).

[44] At least in the continuous case, where periods 1 and 2 are infinitely close to period 0, H and L's joint pay-off from the agreement will be maximal.

[45] This is done by making $U(1/2,0)$ equal to $u(1/2,0)$. v is therefore multiplied by a constant k such that $\int_0^Z e^{-Rt}dt = k \int_0^Z e^{-rt}dt$; in this case such that $5 = k \int_0^5 e^{-0.2}dt = k5 \cdot 0.632$, implying that $k = 1.582$.

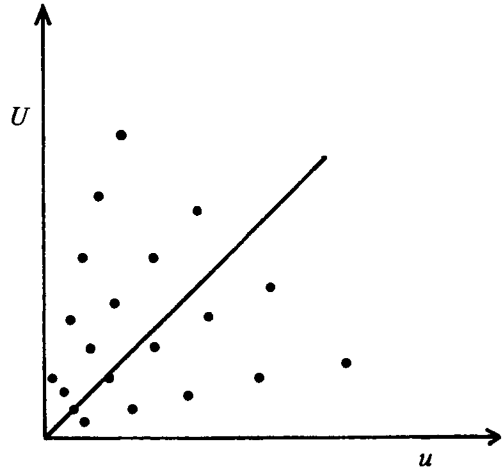

Figure 28. An asymmetric Nash game

Figure 28 clearly shows that the agreement points are *not* symmetrically distributed around the 45°-line. Most points *below* this line lie *closer* to the line than the corresponding points above the line.

Next the number of alternatives is increased by successively locating new alternatives between every pair of old alternatives. Likewise, the number of time-points is successively increased. The number of outcome points above the 45°-line will soon be larger than the number of points below the 45°-line.[46] This asymmetry will remain as long as the number of agreement points is finite. Hence, for any *finite* number of outcome points, the parties cannot change axes without affecting the outcome set. The set of outcomes above the 45°-line does not become equivalent to the set of outcomes below this line so that symmetry is obtained, until we come to the case of *infinitely* many alternatives and time-points[47], i.e. when the *compact* outcome space of the Nash theory is obtained.

If the symmetry axiom cannot be used for a negotiation having a great many – but *not* infinitely many – outcomes, it seems inappropriate to apply the symmetry

[46] The value of S on the 45°-line is given by $S \int_T^Z e^{-Rt}dt = k(1-S) \int_T^Z e^{-rt}dt$, where k is such that $\int_0^Z e^{-Rt}dt = k \int_0^Z e^{-rt}dt$ (see footnote 45).
If $r > R$, then for $T > 0$: $\int_T^Z e^{-Rt}dt > k \int_T^Z e^{-rt}dt$ and hence $S < 1-S$, i.e. $S < 0.5$. Let us assume that the number of alternatives, n, is not very small and that r is significantly larger than R. There will be more than $n/2$ alternatives on that part of the line, indicating an agreement at time T, which lies above the 45°–line.

[47] To be precise, we assume a non-denumerable set of alternatives and time points. The same effect is obtained if we allow for the random mixing of strategies with any probability in the interval [0,1].

axiom. This implies that we object to the use of the Nash theory – except possibly for compulsory arbitration – for all cases where $dU/dT \neq du/dT$ for some T.[48] Since we believe that most begotiations which can be carried out over a long period of time have this characteristic, the area of applicability of the Nash theory appears to be strongly reduced.

L.2.4 Other Arbitration Models

After having discussed the Nash model in detail it is easier to comment briefly on some other fairly well-known arbitration models. Shapley's model (when applied to two-person bargaining) is similar to Nash's model. It differs from that of Nash only with respect to establishment of the conflict point. The Shapley conflict point is obtained on the basis of the pay-off which each party can with certainty obtain on his own. In the example above, where no party can secure more than 0, this conflict point is the same as in the Nash model.

The arbitration models of Nash and Shapley both have the characteristic that establishment of the solution does *not* involve any comparisons of the two parties' utilities. Such interpersonal utility comparisons are, however, fundamental in the arbitration models of Raiffa[49] and Braithwaite. In a bargaining situation of the type presented on p. 225 (i.e. with u as a continuous function of U on the contract curve) the solution according to these models is obtained by setting U equal to u. In order for this to lead to a unique solution, some particular transformation of the utilities has to be established.

Raiffa suggests the following transformation: The utility of each party's worst pay-off is set as 0 and the utility of his best pay-off as 1. In the example above (p. 225), this implies that U=0, u=0 at origo and U=1, u=0 and U=0, u=1 at the two points where the contract curve and the axes intersect. This implies in turn that the joint profit is again divided into two equal parts.

In Braithwaite's model the utility of each party's maximin strategy is set as 0 and the party's utility of his minimax strategy is set as 1, when the other party plays his maximin strategy. In our example above we would then assign both the utility 1 and 0 to V=0. Hence the Braithwaite model can *not* be applied to this example.

[48] Since U and u are adjusted so that $U(1/2,0)=u(1/2,0)$, the assumption $dU/dT \neq du/dT$ for some T implies that there exists a T such that $U(1/2,T) \neq u(1/2,T)$. For the case of many – but not infinitely many – outcome points, the set of outcomes will be asymmetric.

[49] Raiffa (1953) suggested several arbitration models. The one presented above, however, appears to be the best known. See Luce & Raiffa (1957) and Bishop (1963).

L.3 The Zeuthen Model

L.3.1 Presentation of the Two-alternative Game

The main points of this model were presented by F. Zeuthen in 1930. It was further developed by J. Harsanyi, especially as regards the question of necessary behavioristic assumptions.[50] We shall first present the model for a two-alternative bargaining game and then proceed to the n-alternative case.

L.3.1.1 Basic Assumptions

We study a bargaining game with the following characteristics:

1) There are two parties, H and L.

2) Both parties can determine a von Neumann–Morgenstern cardinal utility for each outcome, with U as H's utility and u as L's utility.

3) There is complete information.

4) There are two alternatives, 1 and 2. L wants an agreement on alternative 1 and H an agreement on alternative 2.[51]

5) Bidding takes place simultaneously, i.e. when a party delivers his bid in a period, he does *not* know what the other party bids in this period.

6) There is a finite number of periods.

If both parties bid 1 in the last period, an agreement is reached on alternative 1. Likewise, if both bid 2, an agreement is reached on alternative 2. But if both parties insist on their own terms, i.e. L bids 1 and H bids 2, no agreement is reached and the outcome is a conflict.[52] Since the utility is invariant to a linear transformation, the utility scales can be transformed for the sake of simplicity so that 0 utility is assigned to both parties' pay-offs from this conflict.

If both parties suggest an agreement on the other party's terms, i.e. when L bids 2 and H bids 1, it would clearly be inefficient to let this situation lead to a conflict.

[50] See Harsanyi (1956), (1961), (1962), (1965) and (1966).

[51] This implies that $U_2 > U_1$ and $u_1 > u_2$, where U_2 is H's utility from alternative 2.

[52] The assumption that there is a conflict if both parties insist on their own terms proves that the last period is studied. Otherwise both parties would prefer that these bids carry the bargaining one period further without agreement.

The parties will therefore come to some prior agreement on what to do in this case. Zeuthen assumes that each party gets the same pay-off he would have received if he alone had made a concession, i.e. H gets the pay-off of an agreement on alternative 1 and L the pay-off of an agreement on alternative 2.

This leads to the following pay-off matrix:

		Party L	
		1	2
Party H	1	U_1 \ u_1	U_1 \ u_2
	2	0 \ 0	U_2 \ u_2

Table 8 Pay-off matrix of a two-alternative Zeuthen game

L.3.1.2 Deduction of a Solution

The following four steps illustrate how Harsanyi determines the choice in this situation.

Step 1: Maximization of expected utility

Since no party has any dominating strategy in Table 8 above, the solution can only be obtained by determining subjective probabilities for the different bids.[53] If L's subjective probability that H accepts 1 is denoted as p[54], then L's expected value of insisting on 1 is pu_1. L will accept 2 if $u_2 > pu_1$, i.e. if $u_2/u_1 > p$. For simplicity u_2/u_1 is called q and we thus deduce that L accepts 2, if $q > p$ and that L insists on 1, if $q < p$. Likewise if H's subjective probability that L accepts 2 is denoted as P and U_1/U_2 as Q, we deduce that H accepts 1, if $Q > P$ and insists on 2, if $Q < P$.

Step 2: Formation of a consistent set of probabilities

In step 1 each party's subjective probability that the other party will concede was assumed given. A fundamental question, however, is how these subjective

[53] See also p. 157.

[54] It should be stressed that our notations differ from those used by Zeuthen and Harsanyi. In particular, it should be noted that – for reasons of simplicity – we use p as the probability that H concedes, while Zeuthen and Harsanyi use it for the probability that a party insists on his terms. Zeuthen also assumed that parties maximized expected monetary value. We use Harsanyi's more general utility maximization interpretation.

probabilities are determined. A system for determining these probabilities is presented by Harsanyi[55]. He argues that this can be deduced from a set of – in his opinion – simple and plausible assumptions about rational behavior and rational expectations. These assumptions will be introduced as we proceed.

First Harsanyi investigates the conditions under which the probability estimates of the parties can be logically consistent. Harsanyi originally (1956) relied on the assumption that each party has complete information regarding his opponent's estimates. This assumption is not explicitly required in later articles (e.g. 1962). In order to keep our exposition simple, our analysis will be based on the assumption of complete probability information. Later on we discuss whether less demanding assumptions can be substituted for this assumption.

The complete probability assumption implies e.g. the following: If $q > p$, L will accept 2. Realizing this H sets P, the probability that L accepts 2, equal to 1. Since $U_2 > U_1$, i.e. $1 > U_1/U_2 = Q$, we obtain $P > Q$. Thus $q > p$, $Q > P$ are inconsistent expectations. Furthermore, P will either be 1 or 0, if $p \neq q$. Hence, since $0 < Q < 1$, $p \neq q$, $P = Q$ are inconsistent. Likewise, it can be proved that both of the pairs $p > q$, $P > Q$ and $P \neq Q$, $p = q$ are inconsistent. Only the following three pairs of expectations can possibly be consistent.

1) $p=0, P=1$, leading to an agreement on 2.[56]

2) $p=1, P=0$, leading to an agreement on 1.

3) $p=q$, $P=Q$, implying that each party uses a random mixed strategy, L conceding with probability Q and H with q.

Step 3. Establishing the relationship between q, Q, p and P.

Next, Harsanyi introduces a symmetry assumption implying that a party does *not* expect his opponent to concede, if he himself would not concede in the *same* situation. Furthermore, due to an assumption that the parties do not take "irrelevant variables" into consideration, both parties regard themselves as being in the *same* situation when $q=Q$. These two assumptions combined imply that when $q=Q$, one party cannot be expected to concede and the other to insist on his own terms. Hence, the pairs, $p=0$, $P=1$ and $p=1$, $P=0$ are ruled out when $q=Q$. The only consistent pair remaining in this case is $p=q$, $P=Q$.

[55] Harsanyi (1956 and 1962).

[56] This follows from the case where $q > p$ and $Q < P$, since $q > p$ leads to $P=1$ and $Q < P$ leads to $p = 0$.

Furthermore, Harsanyi introduces an assumption of "acceptance of higher profits" which implies the following: If H is willing to accept 1 when he gets some value of $U_1=U_1'$, then, with u_1, u_2 and U_2 remaining unchanged, H will be at least as willing to accept 1 in a situation where $U_1 > U_1'$, i.e. when he obtains even more from accepting 1. This implies that for a given value of q, L's probability of H accepting 1 will be a non-decreasing function of $U_1/U_2=Q$[57].

Since $p=q$ when $q=Q$, then $p \geqslant q$, i.e. p is either q or 1, when $Q > q$. Likewise p is either 0 or q when $Q < q$. Since similar conditions hold for P, the diagram shown in Figure 29 is obtained.[58]

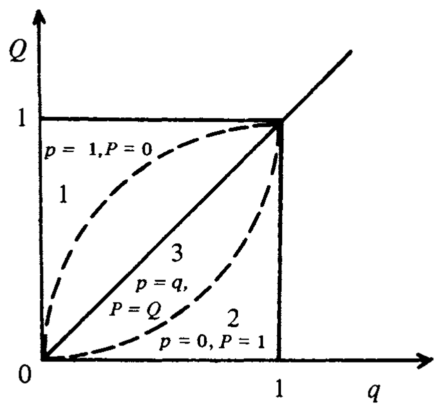

Figure 29 Relationship between q, Q, p and P.

In region 1, where $Q > q, p = 1$ and $P=0$, i.e. both parties bid 1; in region 2 $p=0$ and $P=1$, i.e. both bid 2 and in region 3 H bids 1 with the probability q and L bids 2 with the probability Q. The choice can now be determined, if a decision rule can be established for each party which indicates how he should behave for each possible (q,Q)–combination.

Step 4. Establishing the Zeuthen concession rule

In analyzing this figure further, Harsanyi notes that a conflict will occur in region 3 with the probability $(1-q)(1-Q)$. Next Harsanyi states that the "most efficient pair of decision rules" for the parties is that which minimizes region 3, by limiting it to the $Q=q$-line.[59] All points to the left of this line are assigned to region 1 and all

[57] If the function were differentiable we would obtain $dp/dQ \geqslant 0$.

[58] This is reproduced, after notational changes, from Harsanyi (1961). It should be stressed that the exact shapes of the three regions are not known in step 3.

[59] In region 3, H obtains $qU_1 + (1-q)QU_2 = qU_1 + (1-q)U_1U_2/U_2 = U_1$ and L obtains u_2.

points to the right to region 2. In order to get both parties to adopt this new set of rules, Harsanyi requires an axiom of efficiency implying that the parties will agree among themselves to select the most efficient pair of decision rules.[60]

Finally, Harsanyi also avoids conflict on the q=Q-line by introducing a fairly demanding axiom, implying that both parties accept the other's terms when q=Q. Since the probability that q=Q is small, Harsanyi regards this as unimportant to the analysis.

Summing up, Harsanyi has used his axioms to deduce that if $q \leqslant Q$, H accepts 1 and if $q \geqslant Q$, L accepts 2. This is the original Zeuthen concession rule, which Zeuthen introduced *ad hoc*.

L.3.2 Presentation of the *n*-alternative Model

Zeuthen continues his analysis by dropping the assumption that the two parties have only two alternatives. Let the starting point still be that H bids alternative 2 and L bids alternative 1. If $Q \geqslant q$, H should – according to the Zeuthen concession rule – accept alternative 1. In the *n*-alternative case, however, a great many alternatives will lie between the alternatives called 1 and 2. Instead of accepting 1, H will bid an alternative i with the following characteristics: Alternatives 1 and i form a two-alternative game in which L, on the basis of the Zeuthen concession rule, will accept i, i.e. $U_1/U_i < u_i/u_1$. However, L will not accept i but instead bid an alternative j lying between alternative 1 and alternative i such that $U_j/U_i > u_i/u_j$, i.e. such that $U_j u_j > U_i u_i$. It will then be H's turn to concede again. In this way the parties will alternate conceding and approach each other more and more.

We note that $U_n u_n > U_m u_m$ holds for any sequence of bids m and n, i.e. the product of the parties' utilities will constantly increase until it reaches its maximum.[61] This implies that the Nash product Uu is maximized, i.e. the result obtained according to the Nash model is also arrived at by using the Zeuthen model.

L.3.3 Comments on the Two-alternative Model

The Zeuthen model will now be analyzed with considerations of a fundamental principle of model construction. If a model is constructed step by step, a logical

[60] Harsanyi (1961 p. 183).
[61] This holds provided the utility functions are "well-behaved". See e.g. Coddington (1968, pp. 31–34).

requirement is that step k in the model is *not* influenced by a step $k+j$.[62] Therefore, each step in the analysis of a model must be judged solely on the basis of the steps and assumptions already presented.

L.3.3.1 Comments on Step 1

The main comment on step 1 concerns the assumption about the pay-off each party obtains if both parties concede. Zeuthen's assumption that party H obtains the pay-off of an agreement on alternative 1 and party L the pay-off of an agreement on alternative 2, will imply that the parties are contemplating the possibility of a non-Pareto-optimal agreement. Let us study our old case of splitting \$ 10 where H demands a 7,3 division and L a 6,4 division. The rule above would imply that if both concede, H gets \$ 6 and L gets \$ 3. The question then arises as to who gets the last dollar. Due to the requirement that step k must *not* rely on step $k+j$, this problem is an important one, even if it can later be proved that the parties will almost never concede simultaneously.

In private conversations with the author in 1970 Harsanyi has suggested an alternative rule that would avoid this problem: If both parties concede, some random event with equal probability will determine on which of the two alternatives the agreement is reached. Hence, only Pareto-optimal solutions will be contemplated. It can now be proved that the original Zeuthen rule – that H accepts 1, if $Q=U_1/U_2 \geqslant q=u_2/u_1$ – can also be deduced on the basis of this assumption.[63]

L.3.3.2 Comments on Step 2

The assumption of complete information regarding the other party's probability estimates seems very restrictive. This prompts us to ask whether other assumptions will lead to the establishment of the same set of consistent probability estimates. This question is not answered in the literature, but the following two assumptions appear to be sufficient.

1. Each party has a single-valued expectation as regards the other party's probability estimate. H makes an estimate of p called p^e, and L an estimate of P called P^e.

[62] Relying on step $k+j$ when motivating step k and relying on step k when motivating step $k+j$ implies circular reasoning.

[63] Party H's pay-off when both concede is $(U_1+U_2)/2$. Hence, H will accept 1, if $(1-P)U_1+P(U_1+U_2)/2 > PU_2$, i.e. $U_1-PU_1/2 > PU_2/2$, i.e. $2U_1/(U_1+U_2) > P$. Likewise, L accepts 2, if $2u_2/(u_1+u_2) > p$. If we now set $Q=2U_1/(U_1+U_2)$ and $q=2u_2/(u_1+u_2)$, the deductions in steps 2–4 can be used to deduce the rule that H accepts 1, if $Q \geqslant q$ i.e. if $2U_1/(U_1 + U_2) \geqslant 2u_2/(u_1+u_2)$, i.e. $u_1/u_2+1 \geqslant 1+U_2/U_1$, i.e. $U_1/U_2 \geqslant u_2/u_1$.

2. L assumes that H makes a correct estimate of p, i.e. L *believes* that $p^e = p$. Likewise H believes that $P^e = P$.

We can then deduce that only the three pairs of probability estimates on p.232 are consistent.[64]

Assumption 2, however, is not much weaker than the original assumption of complete probability information. The question then is whether assumption 2 can be replaced by some other, less demanding assumption. This is a complex question which we cannot answer definitely in this context. But it does appear difficult to carry out the analysis of rejecting certain pairs of probability estimates as logically inconsistent if e.g. H has reason to suspect that L might estimate P incorrectly.

L.3.3.3 Comments on Step 3

Step 3 rests on assumptions of "symmetry" and "exclusion of irrelevant variables". The symmetry assumption implies that each party will expect the other to act exactly as he would in a similar situation. The "exclusion" assumption defines this similarity only in terms of q and Q. These assumptions assign a very strong signification to the concept of "rationality"[65].

To illustrate this, let us take a situation in which H is known from earlier negotiations to have conceded much more often than he has insisted on his own terms, while L is known to have acted in an opposite manner. The two assumptions above would still require that L would expect H to concede in a situation where L himself would concede. Obviously the two assumptions explicitly rule out any establishment of probabilities on the basis of historical "events".[66]

L.3.3.4 Concluding Comments on the Two-alternative Game

We have no special comments on step 4. Summing up, our main conclusion about the deduction of the Zeuthen-concession rule is that the assumptions required for deducing this rule are very restrictive. They are in fact more restrictive than those in the literature on the Zeuthen-Harsanyi model, which are already much more restrictive than the assumptions on which our model is based.

[64] E.g. $q > p$ implies the following: L believes that H thinks that $q > p^e$ and hence that H sets P=1. In L's mind this rules out the pair $q > p$, $Q > P$.

[65] A considerable amount of criticism is directed against these assumptions in Saraydar (1968, pp 51–54).

[66] It should be stressed that this is not *a priori* ruled out by our model.

The deduction of the Zeuthen concession rule does, however, seem to follow logically from some set of rationality assumptions as regards games with a finite number of periods. If the assumptions behind this rule are accepted, then a determinate solution is given to the last period of the two-alternative bargaining game. If the study is limited to the completely static case where it does not matter when an agreement is reached – as Zeuthen and Harsanyi appear to do – then a solution can be given to the *whole* bargaining game. If an agreement x is to be reached in the last period, if bargaining is carried this far, then H will not be willing to accept any $y < x$ nor L any $y > x$ in any earlier period. This means that if we can assign the solution x to the *last* period, then the solution of the whole bargaining game is an agreement on x.

L.3.4 Comments on the *n*-alternative Model

Our criticism becomes more fundamental when we move to the n-alternative model. This step has been severely criticized by several authors such as Bishop (1964), Saraydar (1965 and 1968) and Coddington (1968). We agree with their criticism.

As noted above the Zeuthen concession rule is based on the assumption that each party compares his pay-off from getting an agreement on the alternative he himself proposes and the pay-off from an agreement on the alternative proposed by the other party. This comparison becomes irrelevant, however, in the n-alternative case, since the conceding party will not concede all the way to the alternative proposed by the other party. The parties behave as "ineducable" persons. They will assume that the choice in each period stands between complete intransigence and complete surrender, in spite of the fact that they alternated in making a partial concession in all earlier periods.

In order to show more clearly that the comparisons in the two-alternative case are irrelevant, let us investigate the following three-alternative game. L wants alternative 1 and H wants alternative 3. Alternative 2 lies in-between and this alternative maximizes the utility product. Let us call H's probability that L bids 2, P_2 and that L bids 3, P_3. Let us furthermore assume that H will get U_2 if H bids 2 and L 3.[67]

In the last period, H has the following expected pay-off from bidding

alternative 1: U_1;
alternative 2: $(P_2+P_3)U_2$;
alternative 3: P_3U_3.

[67] This simplifying assumption is not critical for the analysis. We could just as well have assumed that H then gets $(U_2+U_3)/2$.

In order to determine that H bids alternative 2, we require that $U_1 < (P_2+P_3)U_2$, i.e. $U_1/U_2 < P_2+P_3$ and that $(P_2+P_3)U_2 > P_3U_3$, i.e. $U_2/U_3 > P_3/(P_2+P_3)$. The conclusions based on the Zeuthen concession rule, that L would accept 2 in the game (1,2) if $U_1/U_2 \leqslant u_2/u_1$ and H would accept 2 in (2,3), if $U_2/U_3 \geqslant u_3/u_2$, provides no guidance for determining P_2 and P_3 as defined above.

On the other hand, the Nash solution can be obtained with the aid of the Zeuthen concession rule, if the *institutional* assumptions concerning the bargaining procedure are changed radically. The following can be assumed: Let each party propose *one* of the n alternatives, where L could propose y' and H y''. If $y' > y''$, they choose again. If $y' = y''$ an agreement is reached. If $y'' > y'$ a *two*-alternative game (y',y'') is obtained. Its solution is then determined using the Zeuthen concession rule.[68] It can now be proved that both parties will suggest x, and that an agreement will be reached on x, where x is that y for which Uu is maximized.[69] The question arises, however, as to *why* the parties would agree on such a bargaining procedure. It appears difficult to rule out the possibility that one party will believe he can obtain a better outcome by some other bargaining procedure.

L.4 The Bishop-Foldes Model

L.4.1 Presentation of the Two-alternative Game

In 1964, R. Bishop and L. Foldes published separate models of bargaining in which each party explicitly took the other party's ordinal utility of the outcomes into account. These models were also based on the assumption that both parties' pay-offs from a certain agreement decrease over time. The two models are virtually identical for the two-alternative case.[70] They differ somewhat as to the development of the n-alternative case, although the conclusions are nearly the same.

As in the Zeuthen model, Bishop and Foldes deduce a concession rule from the two-alternative case. This rule – the BF concession rule – can be deduced in three steps for a game (1,2), where L prefers alternative 1 and H prefers alternative 2.

Step 1: At the very beginning of the bargaining game each party can obtain an agreement on his least preferred alternative by accepting it in his first bid. H can obtain $\bar{1}_0$ and L $\underline{2}_0$.[71]

[68] This is similar to the procedure proposed by Foldes. See p. 244.

[69] Since $U_x u_x > U_y u_y$, i.e. $U_x/U_y > u_y/u_x$, L can enforce x, if H bids any $y > x$. Likewise, H can enforce x, if L bids any $y < x$. Hence both parties can ensure an agreement on x.

[70] Besides using a somewhat different kind of terminology and motivations, Foldes explicitly assumes an ordinal utility scale, which Bishop does not. Bishop limits himself to the case where H's present value of an agreement on y at time T can be written as either $(Z-T)y$, $\int_T^{\infty} ye^{-Rt}dt$ or ye^{-RT} times a constant.

[71] In this instance we let the index of the agreement denote the *time* of the agreement, instead of the period number. With the periods very short, i.e. with $\Delta t \rightarrow 0{:}\bar{1}_0 = \bar{1}_{\Delta t} = 1_{2\Delta t}$.

Step 2: Prior to the bargaining process, H determines a "maximum delay time" D. According to Bishop (p. 414), this is "the maximum duration of strike that he might be willing to endure for a complete victory" and, according to Foldes (p.120) "the longest delay . . . which [H] would consider worth while in order to obtain . . . [alternative 2] . . .with certainty, rather than accept . . .[alternative 1] . . .immediately". D is given by $\overline{1}_0=\overline{2}_D$, i.e. by H's pay-off from an immediate agreement on 1 being equal to his pay-off from an agreement on 2 at time D.[72] Likewise L determines a maximum delay time d, given by $\underline{2}_0=\underline{1}_d$. We note in this context that the model only requires an ordinal utility.

Step 3: The BF-concession rule is presented: If $d \neq D$, then the party with the shorter delay time will concede, i.e. if $d < D$, L accepts 2 and if $d > D$, H accepts 1.

Bishop justifies this rule by stating that a party concedes "when he is not willing to risk a longer strike than the other".

Foldes has two alternative justifications for the BF-concession rule:

1. "Psychological" restrictions are introduced with regard to the propensity of the parties to make or believe threats. Foldes regards it as sufficient to assume (p. 123):

   "(i) that a threat will be disbelieved if its execution (even if successful) would leave the threatening party worse off than he would be if he gave way immediately, and

   (ii) that both parties are aware of this fact."

2. Bargaining is subject to "suitable rules of procedure". Foldes suggests the following rules:

   "(i) The parties simultaneously announce threats . . .[of how long an agreement will be delayed]. If the threats are equal, the parties may choose again.

   (ii) The party whose threat is the greater[73] then has the option of maintaining his threat or giving way, i.e., changing to a zero threat.

   (iii) Threats are then carried out until the lesser (revised) period expires, when trade takes place on the terms preferred by the party whose threat was greater."

[72] If e.g. alternatives 1 and 2 give H an annual profit of \$ 5 and \$ 6, respectively, and H's pay-off is $ye^{-0.1T}$ then D is given by $5=6e^{-0.1D}$, implying that $D=(\ln 1.2)/0.1 = 1.82$.

[73] It appears that Foldes really means "lesser" instead of "greater".

L.4.2 Presentation of the n-alternative Model

L.4.2.1 Development according to Bishop

Bishop's model is very similar to Zeuthen's. If $D > d$, L will concede, but only so much that the inequality sign is reversed, i.e. so that d becomes larger than D. Next H concedes, but only so much that the inequality sign again changes. In this way the parties take turns making concessions until their demands ultimately meet.

L.4.2.2 Development according to Foldes

As opposed to Bishop's development, Foldes' is more static. If a *two*-alternative game (y, y') will lead to an agreement on y' according to the BF-concession rule, Foldes calls y' *enforceable* against y. According to Foldes an immediate agreement will be reached on an alternative x, which is enforceable against all other alternatives. Next Foldes defines a marginal delay time for party H as "the delay which will just off-set the advantage to him of a small increase in [y]." The definition for L is analogous. Finally Foldes proves that the marginal delay times at x will be equal.[74]

L.4.3 Comments on the Two-alternative Model

L.4.3.1 Comments on Step 1

In step 1 it is assumed that prior to the first period, both parties believe they can obtain an agreement on their less preferred alternative in the very first period. The question then is under what institutional assumptions this assumption can possibly be consistent with rational behavior, e.g. as defined by our set B_3.[75]

It appears that the bargaining game has to be characterized by simultaneous bidding. If it were characterized by alternating bidding, the following would be true: In his estimates prior to the start of the bargaining, the party bidding second

[74] As proved by Foldes this implies that the solution is found by the equation

$$\frac{\partial V(y,0)/\partial y}{\partial V(y,0)/\partial T} = -\frac{\partial v(y,0)/\partial y}{\partial v(y,0)/\partial T}$$

For S*games, i.e. with $V(y,T) = AyF(T)+B$, we obtain $AF(0)/AyF'(0) = af(0)/a(N-y)f'(0)$, i.e. $(N-y)F^*(0) = yf^*(0)$ implying that $y/N=F^*(0)/(F^*(0)+f^*(0))$. This solution is the same as for the special case of our model discussed on p. 178.

[75] It should be stressed that the assumptions presented below are not explicitly made by Bishop or Foldes. If the parties are *not* assumed to be rational, the institutional assumptions presented here are not necessary.

– e.g. H – would take into account the probability of L accepting 2 in his first bid. H's expected value of accepting 1 will be equal to $\overline{1}_0$, only if this probability is 0. This probability cannot generally be set at 0 since, due to the BF-criterion, L will accept 2 immediately if $D > d$. In other words, an assumption of 0 probability would be inconsistent with the behavior it induces.

According to the interpretation above, bargaining is characterized by simultaneous bidding. This means it also has to be assumed that a non-Pareto-optimal agreement will be reached if both parties concede. H then gets the pay-off of alternative 1 and L gets the pay-off of alternative 2. This is the same problem we encountered in connection with the Zeuthen model (see p. 235). The only difference is that in this case, we can *not* resort to a probabilistic interpretation without changing the ordinal utility character of the BF-model.

L.4.3.2 Comments on Step 2

According to step 2 the *maximum* delay time for H is the amount of time for which $\overline{1}_0 = \overline{2}_D$. The question is to what extent time D has any particular significance. This calls for an attempt to interprete the term "maximum delay". The most reasonable interpretation of this term, as applied to D, is as follows[76]: D is the ***highest*** value of T for which the strategy "insist on 2 in every bid up to time T, after which accept 1" is a better strategy for H than "accept 1 immediately."

But this interpretation can hardly be correct if H is rational. There are no rational grounds for ruling out the possibility that L might accept 2 prior to D. With some positive probability that L accepts 2 prior to D, H's expected value of adhering to the "insistance strategy" will also be dependent on various pay-offs $\overline{2}_T$, where $T < D$. Since the pay-offs decrease over time so that $\overline{2}_T > \overline{2}_D$, the expected value of insisting on 2 up to time D *might* be larger than $\overline{2}_D$. The expected value $\overline{2}_D$ might instead be obtained by a strategy of insisting on 2 up to period $T' > D$[77]. H might therefore very well insist on 2 *longer* than up until D, rather than accept 1 immediately. D is hence *not* the longest conceivable time that H will insist on 2. Thus the relevancy of a comparison between $\overline{2}_D$ and $\overline{1}_0$ in order to establish some maximum delay time is questionable. It appears that D is simply a time such that $\overline{1}_0 = \overline{2}_D$, without having any particular significance, except under Foldes' very special institutional assumptions (see p. 239).

[76] This appears to be the precise interpretation which is closest to the two fairly vague statements presented on p. 303. It should be stressed that each party himself has control over the delay time. H can always get an agreement by accepting L's terms. Hence the maximum delay time D must be established by comparing two of H's bargaining strategies.

[77] H would then have some chances of obtaining $\overline{2}_T > \overline{2}_D$ as well as of obtaining $\overline{2}_{T'} < \overline{2}_D$.

L.4.3.3 Comments on Step 3

The BF-rule seems fairly arbitrary. This is due, first of all, to the lack of significance of D and d discussed above.

Looking at Bishop's justification that H accepts 2 immediately "when he is not willing to risk a longer strike than" L, we again encounter the problem that H might very well be willing to insist on 2 longer than D. But it is impossible to say how much longer.

We now turn to Foldes' two assumptions implying a "psychological justification" of the BF-concession rule (see p. 239). Even if these strong assumptions hold, they are *insufficient* for deducing this rule. There is no rational reason, for example, why H should commit himself not only to striking up to a certain time T but also to conceding at this time T. Instead H could threaten L as follows: "I commit myself to striking up to time T, unless you accept 2. At time T I shall decide whether or not to continue striking." Whether or not L will concede when faced with this threat will depend not only on the credibility of H's threat of striking up to time T, but also on L's subjective probability that H will strike also after T.

The three assumptions in Foldes' "institutional" justification (see p. 239) no doubt lead to the BF-rule. However, Foldes has already admitted that these assumptions are artificial. Furthermore, if the parties are free to determine the rules for the bargaining procedure among themselves, the question arises – especially in games with only two alternatives – as to why the conceding party would agree to such a procedure.

A final problem related to the BF-concession rule when applied to certain pay-off functions, is that it is based solely on the size of D and d, computed at the very *start* of the bargaining game. Let us look at a case where $D > d$, computed at time 0, but where $D < d$ when the computation is made at some later time T and $\underline{2}_0 < \underline{1}_T$[78]. In this case L will prefer to delay an agreement until T and have the choice at T based on the BF-concession rule with D and d computed at time T.

This leads to a problem since H will prefer an immediate agreement with D and d computed at time 0. Foldes appears to avoid this problem by restricting his analysis

[78] The following example is based on pay-off function 3 (cf. p. 126) with β_0=0, β'=0.005, R=0.1 and r=0.14. When a periodic income of \$ 10 is to be divided, with H insisting on \$ 6 to H, \$ 4 to L and L insisting on \$ 5 to each, D is given by $6e^{-0.1D-0.005D^2}=5$, implying that D =1.68. d is given by $5e^{-0.14d}=4$, i.e. d = 1.59. Hence $D > d$. Next, if the comparison is made at time T = 1, d is unchanged, but D is given by $6e^{-0.1(1+D)-0.005(1+D)^2} = 5e^{-0.105}$, leading to D = 1.55, i.e. $D<d$. Since $5e^{-0.14}$ = 4.35 > 4, L will prefer to wait with the comparison until T = 1.

to pay-off functions such that $\bar{y}_j = \bar{y}'_{j'} \Rightarrow \bar{y}_{j+k} = \bar{y}'_{j'+k}$ (Foldes, 1964, p. 219). This implies that S_2 must *not* hold and that Foldes' model is *not* applicable to any S-games.

L.4.4 Comments on the *n*-alternative Model

L.4.4.1 Comments on Bishop's Version

All the comments directed against the Zeuthen model also apply to Bishop's development. The concession rule is based on the assumption of complete concession and not on the partial concession which actually takes place. Thus the parties prove to be completely ineducable (cf. p. 237).

L.4.4.2 Comments on Foldes' Version

The fundamental question with regard to Foldes' development of the *n*-alternative model is: If we accept the BF-concession rule, is it then sufficient to assume that an alternative is enforceable against every other alternative in order to prove that an immediate agreement will be reached on this alternative? Let the following three-alternative game (1,3) illustrate this question. We assume that according to the BF-rule the two-alternative game (2,3), and the *two*-alternative game (1,3) in which alternative 2 is *not* at all contemplated, lead to 3. According to Foldes, an immediate agreement would then be reached on alternative 3 in the *three*-alternative game (1,3).

However, this conclusion requires an assumption that L will rule out *any* possibility of H bidding alternative 2. If there would be some chance of H bidding 2, L might find it profitable to insist on 1 during a number of periods, hoping that H would bid 2. Likewise, if H's subjective probability of L insisting on alternative 1 during a number of periods is not 0, H might bid 2 instead of holding out until L accepts 3. But it is impossible to rule out the possibility of H bidding alternative 2 solely on the basis of the previous two-alternative analysis. It is insufficient to conclude that, if *only* alternatives 1 and 3 are available, L's subjective probability of H insisting on 3 in the next few periods is 1 according to the BF-rule. In the three-alternative game the "compromise" alternative 2 *does* exist and the BF-rule does not cover this game.

More generally, Foldes' *n*-alternative theory requires a further specific behavioristic assumption of the following type: If y' is enforceable due to the BF-concession rule in a bargaining game where L wants y and H wants y', then L's estimated probability of H bidding any alternative between y and y' is 0 for a great number of future periods.

Finally, it should be mentioned that the problem discussed above can be avoided by a bargaining procedure with the same rules as suggested on p. 238 for the Zeuthen model.[79] These rules are sufficient for deducing the solution x[80]. The question still remains however, as to why both parties would agree on this type of bargaining procedure.

L.5 The Cross-Coddington Model

L.5.1 Presentation of the Model

In 1965, J. Cross presented a model focused on the bargaining *process* and in particular on how the expectations concerning the other party's concessions change over time.[81] In our opinion, this model clearly demonstrates the great problems related to construction of a deductive model which is *not* based on assumptions of rational expectations. In 1968, A. Coddington presented a model that can be regarded as a modification of Cross' model, including corrections of some of its minor deficiencies.[82] We present the main points of Cross' original model and only occasionally refer to the modifications suggested by Coddington. In order to make the presentation easily understandable we use the simplest possible version of the model, assuming *inter alia* that there are no bargaining costs.[83] It should be stressed that our notations differ from those used by Cross.

Party H and party L bargain over how to divide a certain sum – for the sake of simplicity \$ 1.00. H demands a share S and L a share s. If $s+S \leqslant 1$, an agreement is reached. Otherwise, L assumes that H will concede by a constant annual rate C in the future, i.e. that H will lower the share he demands by C each year.[84] Hence, if L keeps insisting on s an agreement will be reached at a time $T=(s+S-1)/C$ from now.[85] With r as L's rate of interest, L's present value of such an agreement will be se^{-rT}.

79 This is dealt with by Foldes (1964) in a footnote on p. 124.

80 The proof of this statement is similar to that in footnote 69 on p. 238.

81 See Cross (1965). The model is explained in greater detail in Cross (1969).

82 See Coddington (1968). Coddington has also discussed various problems related to the model. See Coddington (1968), (1970) and (1972).

83 The exclusion of these costs, included in Cross' but not Coddington's analysis, simplifies the analysis considerably, but it does not affect the general principles of the model.

84 Coddington does not require the future expected concession to be the same for each period. He assumes more generally that at each time-point L makes a forecast of how much H will concede in each future period.

85 If e.g. s=0.7, S=0.6 and C=0.1, then T=(1.3−1)/0.1=3.

L also assumes that he will *not* concede during the rest of the bargaining. His optimal demand is then found by setting $dse^{-rT}/ds=0$, implying that $s=C/r$.[86] Likewise, under similar assumptions and with c as the rate at which H expects L to concede, the optimal bid for party H is $S=c/R$.[87]

Thus both parties will expect the other party to make all the concessions in each period. Since both parties cannot be right – in fact both turn out to be wrong – they will both successively modify their expectations as regards the other party's future concessions. The higher the discrepancy between the actual rate of concession and the expected rate, the more their original expectations will change. The following equation describes how H's expectations concerning L's concessions change over time: $\partial c/\partial T=A(-\partial s/\partial T-c)$ where $\partial c/\partial T$ is the change in H's expectation of L's concession rate, A is a parameter measuring H's learning, $\partial s/\partial T$ is L's actual concession rate and c is H's original expectation about L's concession rate.[88] Due to this $S=c/R$ will also change over time. An equation for how s will change over time is obtained in a similar manner. These equations provide a pair of differential equations. When these are solved, s and S are obtained as functions of T. Neither party will hold his demands constant between any two periods. As time passes, both s and S will eventually decrease over time.[89]

Then the demands of the parties gradually approach each other until a solution is reached. Cross computes a solution for the case where $r=R$ and $a=A$. L's share obtained at the actual time of agreement T^* is a function not only of r and A but also of c_1 and C_1, which are the *initial* values of c and C. With $r=R=0.25$; $a=A=0.2$; $c_1=0.125$ and $C_1=0.175$, L obtains 52 ¢ at $T^*= 1.67$.[90]

L.5.2 Comments on the Model

1. The foundation of the model is that in each period, L assumes that he will *not* make any further concession. Yet after having investigated how H's real concession differs from his expected concession, L will change his demand in every period. In the case of functions leading to convergence, this will eventually be in the form of a concession. Since this also applies to H, we conclude that the bargainers are assumed to be unable to learn.[91]

[86] $0 = d(se^{-rT})/ds = d(se^{-r(s+S-1)/C})/ds = e^{-rT} - sre^{-rT}/C = 0 \Rightarrow 1 = sr/C$.

[87] In a chapter on bluffing, Cross argues that it will not be profitable at any time for L to demand more than this expected outcome C/r. (See Cross, 1969, p. 171). Furthermore the whole maximization procedure above will be meaningless, if the demand and the expected outcome are not equal.

[88] See also the same equation for the discrete case on p. 246.

[89] Provided certain conditions hold, e.g. that a/r and $A/R < 1$.

[90] See Cross (1969, p. 85). We note in this context that equation (24) on p. 83 in Cross (1969) is wrong. This error is due to a fault in equation (21) on p. 80. Similar faults appear in Cross (1965). When $a = A$ and $r = R, c_1 < C_1 \Rightarrow s^* > S^*$.

[91] Coddington discusses this weakness in the model on pp. 62–65 (1968) and also on p. 1212 (1970).

2. Similar – but less serious – criticism of the inability to learn can also be directed against the assumption about each party's expectations of the *other* party's behavior. According to Cross L assumes that H will concede by a *constant* amount each period.[92] However, the model will generally lead to H conceding by different amounts in different periods. Though L's expectation of H conceding by a constant amount is proved erroneous in each period, L does not alter his expectation.

3. Cross works *only* with the continuous case, i.e. with infinitely short periods. This limitation has the following disadvantages, as compared to *also* using a discontinuous method. A discontinuous model would more clearly reveal the weakness of assuming that the parties do *not* contemplate how the other party forms his expectations. This is discussed under point 4. Secondly, the discontinuous method has several advantages from a computational point of view. A computer program for the discontinuous case is very easy to write.[93] Such a program can provide the solution for any values of a, A, r and R. Relying on analytical methods, Cross finds it difficult to compute the solution except for the case where a=A and r=R. Furthermore, the exact solution can also be found for cases when a small number of bids are delivered each year.[94]

The discontinuous method can be outlined briefly as follows, starting with the learning condition. The principle is that H's expectations at time T+2 regarding L's future concessions are adjusted according to how L's last known concession, i.e. his concession between T+1 and T, $-(s_{T+1}-s_T)$, compares with c_{T+1}, H's prior expectation regarding L's concessions. With A as the parameter measuring H's "learning" we obtain

$$(c_{T+2}-c_{T+1})/\Delta t = A(-(s_{T+1}-s_T)/\Delta t-c_{T+1})^{95},$$

i.e. $c_{T+2}-c_{T+1}=A(-(s_{T+1}-s_T)-c_{T+1}\Delta t)$.

A similar equation is obtained for C. By using these equations iteratively in a computer program and the initial values c_1 and C_1[96] as a basis, we can deduce

[92] In Coddington's model L is assumed to make a specific forecast of how much H will concede each period as a function over time. Then perhaps only the parameters – not the form – of the function might have to be changed.

[93] The author has written such a program in FORTRAN IV.

[94] Sometimes the continuous model will yield too rough an approximation of a real "discontinuous" solution. If e.g. c=0.02, C=0.2, r=R=0.21 and a=A=0.25, the continuous method gives more than 10 per cent error when there are less than 40 bidding rounds a year.

[95] This can be written as $(c(T+2\Delta t)-c(T+\Delta t))/\Delta t = A(-(s(T+\Delta t) - s(T))/\Delta t - c(T+\Delta t))$. When $\Delta t \rightarrow 0$, Cross' original equation on p. 245 is obtained.

[96] In the case of alternating bidding with H bidding first, we set c_1=c_2. This seems to be the only assumption consistent with Cross' learning assumptions and his conclusion that S^* depends only on c_1, C_1, a, A, r and R. Since H, when bidding in round 2, has *not* observed any concessions from L his expectation regarding L's future concessions must remain unchanged. For the case of simultaneous bidding we likewise assume that c_1=c_2 *and* C_1=C_2.

$s_T = C_T/r$ and $S_T = c_T/R$ for successively higher values of T until we obtain a $T = T^*$ such that $s_T + S_T \leqslant 1$. This gives us the solution. With a very large number of bidding rounds, this program generally seems to lead to the same result as Cross' continuous model.

4. Models of conflicts differ as to the level of insight assumed with respect to the other party's behavior and expectations. The simplest case – level 1 – is that neither party makes any forecast of how the other party will act, but each acts on some other grounds, determined "exogenously". The next level – level 2 – is that each party makes a forecast of how the other party will act – assuming that the other party's action is determined "exogenously", i.e. *without* regard to how his opponent acts. This is the level on which both parties operate in Cross' and Coddington's models.

A more sophisticated level of analysis – level 3 – would imply that L makes a forecast of how H will act on the basis of H's forecast of how L in turn will act, determined on exogenous grounds. According to Coddington such a procedure would be very complicated.[97] However, as concerns Cross' model with constant concession rates, it is easy to find a decision rule which, when used against a party acting according to the Cross model, leads to a much better result than the decision rule prescribed by this model.[98]

This can be exemplified by the numerical example presented on p. 245, where L obtained 52 ¢, 1.67 years from now. With a 25 per cent interest rate, this represents a present value of 34 ¢ at the start of the bargaining. If H is known to act in accordance with the Cross model, the following decision rule is better for L[99]: First L demands slightly more than the 50 ¢ initially offered him by H[100], thus avoiding an immediate agreement. In the second round of bidding L demands the whole dollar. H now notices that L has increased his demand by 50 ¢, i.e. by 0.5. Having earlier thought that L would concede at a rate c_1 of 0.125, H will – according to the equation presented on p. 246 – reduce this rate to $c_3 = 0.125 + 0.2(-0.5-0.125\Delta t)$ in round 3. For small values of Δt, $c_3 = 0.025$. Hence, in round 3, H demands $S = c_3/R = 0.025/0.25 = 0.1$, i.e. 10 ¢. By accepting this demand in round 3, L will get 90 ¢.

[97] Coddington (1968, pp. 62 and 64).

[98] It should be stressed that this is not a plea for an analysis on the third level of insight, but rather on the *n*th level of insight.

[99] We assume that bargaining is characterized by alternating bidding and that H bids first and L second in each round. This assumption is not critical in terms of the main conclusions of the discussion which follows, but the analysis is more complicated in cases where L bids first or there is simultaneous bidding.

[100] $S_1 = c_1/R = 0.125/0.25 = 0.5$.

This means that in the case of complete information, L can obviously do far better if he adheres to the decision rule proposed here rather than the Cross rule. Although L is not likely to know the true values of c_1 and A and perhaps not even the exact value of R, one might be able to assume that L can establish limits within which the parameters vary and that these limits are not very far away from the true values. This is probably sufficient for deducing that our decision rule is better than the Cross rule.[101]

5. Without knowledge about the parameters c_1, C_1, a and A which are specific for the bargaining situation, the Cross model lacks predictive capability. This also implies that rigorous empirical testing of the model in laboratory experiments will be very difficult.

Summing up, the main points of criticism – 1 and 4 – concerned the behavioristic assumptions behind the Cross model. The parties adhere to assumptions of rational *behavior*, since they *maximize* their expected value. They are irrational, however, with respect to the formation of their *expectations*. We believe this to be a fundamental discrepancy. If parties are maximizers, which implies that they have substantial computational capability, they are most likely not ineducable when it comes to forming their expectations; nor are they unaware of the other party's expectations about their own behavior.

L.6 The Hicks Model

L.6.1 Presentation of the Original Model

Hicks' contribution to bargaining theory is that he seems to be the first author who attempted to formulate a bargaining model with the following three components:

1. The search for a solution relies only on the use of *ordinal* utility functions,

2. The fact that the value of a certain agreement varies over *time* is explicitly taken into account and

3. A delay in reaching an agreement occurs only due to irrationality or incomplete information.

As noted above these three components are basic to both Foldes' model and ours.

[101] If L believes that $R \leqslant 0.3$, $A \geqslant 0.10$ and $c_1 \geqslant 0.075$ in the numerical example above (real values are 0.25, 0.2 and 0.125), then by running the computer model mentioned on page 246 it can be shown that L obtains more than 34 ¢ in period 3. Even if there would be some probability that the values of the parameters fall outside of these limits, L's expected monetary pay-off from adhering to the rule above would probably be larger than that obtained by following Cross' rule.

The details of Hicks' model have been widely and severely criticized.[102] The attacks have come from different angles, mainly because there is no consensus on how this model should be interpreted. The model is so vague that it lends itself to different interpretations. In trying to present it first without interpreting it, we rely mainly on quotations from Hicks' original version. In order to facilitate comparisons with our study, we call the union desiring *high* wages, H and management, desiring *low* wages, L.

"We . . . construct a schedule of wages and lengths of strike, setting opposite to each period of stoppage the highest wage [L] will be willing to pay rather than endure a stoppage of that period. At this wage, the expected cost of the stoppage and the expected cost of concession (accumulated at the current rate of interest) just balance. At any lower wage, . . . [L] . . . would prefer to give in; at any higher wage, he would prefer that a stoppage should take place. This we may call . . . [L's] . . . 'concession schedule' . . .We . . . [also] . . .draw up . . . a 'resistance schedule', giving the length of time . . . [H] . . . would be willing to stand out rather than allow . . . [his] . . .remuneration to fall below the corresponding wage . . . [L's] concession curve and . . . [H's] resistance curve will cut at a point . . . and the wage . . . corresponding to this point is the highest wage which skilful negotiation can extract from . . . [L]".[103]

In order to further facilitate comparisons we generalize Hicks' model by denoting the bargaining variable S, i.e. H's share. This does not imply any significant change. In labor-management bargaining the union's – i.e. H's – share S of the joint sum of profits and wages is, with given input of labor, a linear transformation of the wage rate. Hicks' model can then be illustrated by Figure 30 below where S^o denotes the value of S at the point where the two curves intersect and $S^\prime$ the lowest contemplated value of S.

Two of the many possible ways Hicks' model may be interpreted are presented below.

L.6.2 Interpretation 1

This interpretation is to a large extent equivalent to Shackle's.[104] According to this interpretation, Hicks relies on the same kind of reasoning used in the Bishop-Foldes model. Each party compares an immediate agreement on one particular S to an

[102] Shackle (1957, pp. 299–305), Pen (1952, p. 25 and 1959, pp. 115–117), Bishop (1964, p. 413) and Harsanyi (1956, p. 154).

[103] Hicks (1932, pp. 141–144). Fig. 30 is also based on Hicks with some notational changes.

[104] See Shackle (1957, p. 301).

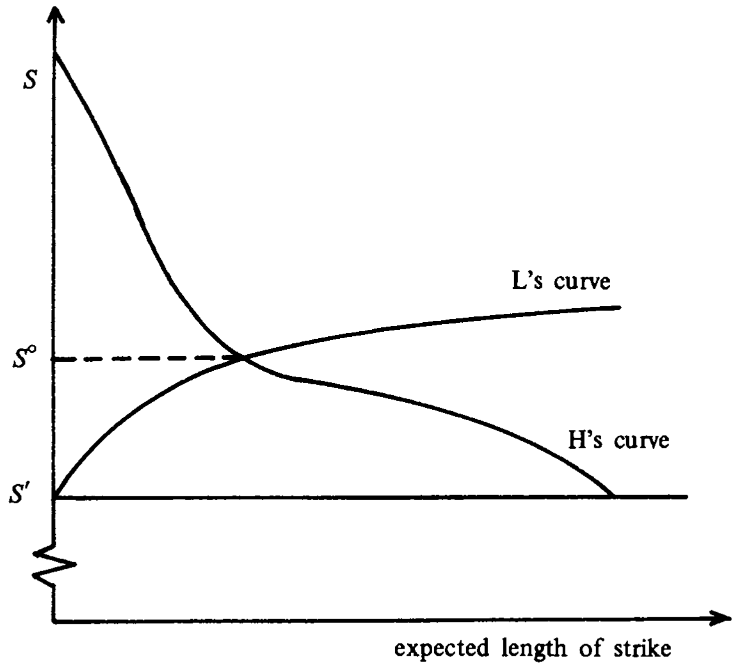

Figure 30 Hicks' original diagram

agreement on a more favorable S at time-point T. Since this comparison is illustrated in a two-dimensional diagram with T on one axis, only *one* S can be a variable.

It is assumed that only H can propose different values of S[105], while L consistently insists on a given value of S, S'. L's "resistance" curve will depict L's indifference between an immediate agreement on S and an agreement at time T on S'. In Bishop-Foldes' terminology, the T associated with each S would correspond to L's "maximum delay time". Analogously, H's curve would depict H's indifference relation between an immediate agreement on S' and an agreement on S at time T.

According to this interpretation and with decreasing pay-offs over time, H's curve becomes monotonically *increasing* with T[106] instead of decreasing with T as assumed by Hicks in Figure 30. When both curves increase it is impossible to decide whether there will be one, several or no intersection points without further specification of the functions. However, let us assume *one* unique intersection point, as illustrated by Figure 31.[107]

[105] This is in line with Bishop's interpretation. See Bishop (1964, p. 413).

[106] According to this interpretation, H's indifference function is given by $V(S',0)=V(S,T)$. Since $\partial V/\partial S > 0$ and $\partial V/\partial T < 0$ and since $V(S',0)$ is constant, S must increase when T increases, in order to keep also $V(S,T)$ constant. L's indifference function is given by $v(S,0)=v(S',T)$. If T increases, $v(S',T)$ decreases. In order for $v(S,0)$ to decrease, S must increase since $\partial v/\partial S < 0$.

[107] This is borrowed from Shackle (1957, p. 302) after some notational changes.

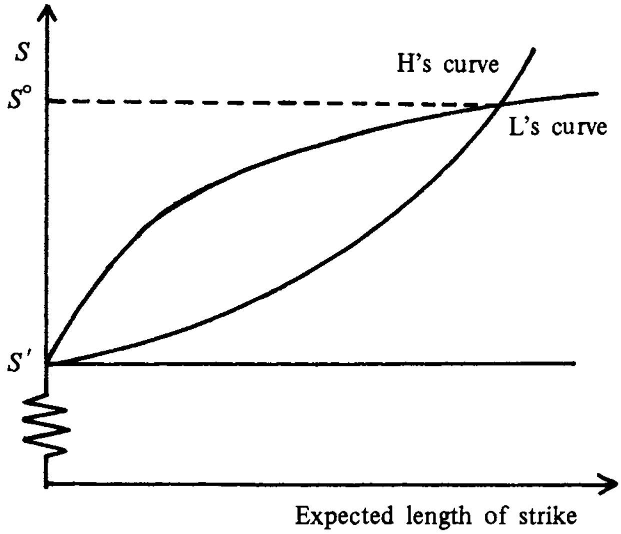

Figure 31 Shackle's modification of Hicks' model.

If each party adheres to the Bishop-Foldes concession rule, i.e. that the party with the shortest maximum delay time will accept the other's terms, then S^o is the *highest* S for which H has a longer delay time than L[108] and which L will thus accept. This means that H bids S^o, which becomes the solution.

According to this interpretation Hicks' model becomes subject to the criticism that we earlier directed against the Bishop-Foldes model. Furthermore, the question arises as to why L would limit himself to only two strategies, sticking to the original proposal S' or accepting H's proposal.

L.6.3 Interpretation 2

The second interpretation – which appears to be new – is that Hicks' reasoning is more similar to that of Cross. Each party expects the other to concede at a given rate, regardless of what the party himself will bid. It is outside the scope of this review to make a more general analysis, but the following example illustrates the main idea:

H expects L to concede at an annual rate c. If H does not accept L's proposal S at time T, H expects an agreement at time $T+D$ on $S+cD$. If H's pay-off is

[108] To be exact: $S^o - \epsilon$, where ϵ is very small.

$S(Z-T)$[109], we can deduce that H's optimal prolongment of the strike $D^*=(Z-T-S/c)/2$[110].

H will compare L's bid S with the agreement S^* that H would obtain if he prolonged the strike D^* more periods. The lowest value of S that H is willing to accept in period T is given by $V(S,T)=V(S^*,T+D^*)$[111] implying that $S=c(Z-T)$[112]. Hence S is obtained as a decreasing function of T. H's "resistance" curve will have a negative slope just as in Hicks' original diagram (Fig. 30).

Assuming that L's pay-off function is $(1-S)(Z-T)$ and that L assumes that H concedes at a rate C we can likewise deduce a "resistance" function of the form $1-S=C(Z-T)$, i.e. $S = 1-CZ+CT$. This is an *increasing* function of T as assumed in Figure 30. While H's curve denotes the lowest S that H will accept for each T, L's curve denotes the highest S that L is willing to accept. Hence, no agreement can be reached as long as H's curve lies above L's curve. An agreement is not possible until the two curves intersect.

The assumption that each party expects the other party to concede at a given rate is an *ad hoc* assumption of non-rational behavior. Much of the criticism against this type of assumption, discussed in conjunction with Cross' model, is applicable here. In particular we note the following: On the basis of the assumption that L will concede at a rate c, H determines that the lowest S he will accept decreases at a rate c. Likewise, on the basis of the assumption that H will concede at a rate C, L deduces that the highest wage he will accept increases at a rate C. These expectations will be consistent with the behavior they induce, only if $c=C$ and if each party will adhere to his most concessive strategy within the limits set by his "resistance" curve.

[109] This is equivalent to pay-off function 1 with only an agreement profit component, when $r = 0$ and $\pi = 1$.

[110] Setting $Z-T= Z'$, D^* is given by $d((Z'-D)(S+cD))/dD = d(Z'S+Z'cD-DS-cD^2)/dD$ $Z'c-S-2cD = 0$, i.e. by $D = (Z'-S/c)/2$.

[111] If $V(S,T) < V(S^*, T+D^*)$ H will prefer to strike until $T+D^*$ and get an agreement on S^*.

[112] With $Z' = Z-T$, the indifference relation implies that $Z'S = (Z'-D^*)(S+cD^*) = Z'S+Z'cD^*-D^*S-c(D^*)^2 \Rightarrow S = Z'c-cD^*$ With $D^* = (Z'-S/c)/2$, we obtain $S = Z'c-Z'c/2+S/2$ implying that $S = cZ'$.

Mathematical Appendix

M.1 Analysis of a Two-alternative Game in Normal Form

M.1.1 Introduction

The solutions deduced in the text for the two-alternative situation relied on an analysis of what is called the *extensive form*[1] of a bargaining game. Some aspects of the same type of two alternative bargaining games in the *normal form* will be discussed here.[2] The main reason for presenting a bargaining game in the normal form is that this analysis facilitates comparison of the solution concept used above and a solution concept generally used in game theory known as the equilibrium pair concept. Furthermore, presentation of basically the same method of solution in two different ways might serve to make our reasoning easier to understand. Example 4, analyzed in the extensive form on pp. 41 ff. will again be studied here.

M.1.2 Determination of the Outcome Matrix

L's strategy in this bargaining game (6,7) can be regarded as a plan[3] made prior to the start of the actual bargaining with regard to the period in which L will accept 7, provided H has not accepted 6 in an earlier period and thus brought the game to an end. This period, and hence also L's strategy, is denoted by a number j. Likewise H's strategy can be represented by a number J, which denotes the period in which H will, if ever, accept 6.

[1] In the *extensive form* each potential choice (each bid in the bargaining game) is described separately. The extensive form of a game can be represented by a game tree.

[2] In the *normal form*, the potential choices are incorporated into strategies, and "each party has exactly one move (a choice among his several strategies) and he makes his choice in absence of any certain knowledge about the choices of the other player". (Luce-Raiffa, 1957, p. 53.) The normal form of a game can be represented by a game matrix. For a more thorough discussion of these two forms of a game see e.g. Luce-Raiffa (1957).

[3] More strictly, a party's strategy in this case is a function that assigns either an acceptance bid or an insistence bid to each of the party's possible decision points in the game tree up to and including the first decision point that is assigned an acceptance bid. As the game is over as soon as one party makes an acceptance bid, it is meaningless to include in the strategy what the party will bid after the game has finished.

Next we establish a pay-off matrix. The pay-offs of each cell in the matrix are determined in the following way: If $j < J$, L will concede first and an agreement will be reached on 7 in period j, i.e. H obtains $\overline{7}_j$, L $\underline{7}_j$. If $J < j$, H concedes first and an agreement is reached on 6 in period J. H then obtains $\overline{6}_J$ and L $\underline{6}_J$. If no agreement is reached prior to period 9, the period when the game is broken up, both parties obtain 0. With $\overline{y}_j = y(9-j)$ and $\underline{y}_j = (10-y)(9-j)$ we determine the following pay-off matrix, where H's pay-off is in the lower left-hand corner of each cell and L's pay-off in the upper right-hand corner.[4]

		L				
		$j=2$	$j=4$	$j=6$	$j=8$	$j>8$
	$J=1$	32 48	32 48	32 48	32 48	32 48
	$J=3$	21 49	24 36	24 36	24 36	24 36
H	$J=5$	21 49	15 35	16 24	16 24	16 24
	$J=7$	21 49	15 35	9 21	8 12	8 12
	$J>7$	21 49	15 35	9 21	3 7	0 0

Table 9 Pay-off matrix of example 4

M.1.3 Determination of the Solution

On the basis of assumption set $\boldsymbol{B}_3$ a process involving iterative elimination of dominated strategies can be carried out. Before applying this process we have to define the concept of *better reply*: H's strategy J is a better reply to j than J' if $V(J) > V(J')$, when j is chosen. We also define the *domination* concept: H's strategy J dominates J', if J is a better reply to at least one of L's strategies than J', while J' is *not* a better reply to any of L's strategies than J.[5] We see e.g. that $J = 3$ dominates $J = 5$, since for $j = 2$, $J = 3$ and $J = 5$ give H the same pay-off, while for

[4] We have $\overline{6}_1 = 48$; $\underline{6}_1 = 32$; $\overline{6}_3 = 36$; $\underline{6}_3 = 24$; $\overline{6}_5 = 24$; $\underline{6}_5 = 16$; $\overline{6}_7 = 12$; $\underline{6}_7 = 8$; $\overline{7}_2 = 49$; $\underline{7}_2 = 21$; $\overline{7}_4 = 35$; $\underline{7}_4 = 15$; $\overline{7}_6 = 21$; $\underline{7}_6 = 9$; $\overline{7}_8 = 7$; $\underline{7}_8 = 3$.

[5] The domination concept can also be defined directly. H's strategy J dominates J', if for every one of L's strategies, J gives H at least as high a pay-off as J' and if J gives H a higher pay-off for at least one of L's strategies.

$j = 4$, $j = 6$, $j = 8$ and $j > 8$, $J = 3$ gives H higher pay-offs than $J = 5$. The process of iterative elimination of dominated strategies is carried out in three steps:

Step 1: $J = 3$ dominates not only $J = 5$, as shown above, but likewise also $J = 7$ and $J > 7$. Furthermore $j = 6$ dominates $j = 8$ and $j > 8$. After eliminating the dominated strategies, the following reduced pay-off matrix is obtained.

		L		
		$j = 2$	$j = 4$	$j = 6$
H	$J = 1$	48 / 32	48 / 32	48 / 32
	$J = 3$	49 / 21	36 / 24	36 / 24

Table 10 Reduced pay-off matrix

Step 2: Table 10 shows that $j = 4$ dominates $j = 2$. After eliminating $j = 2$, the following pay-off matrix remains.

		L	
		$j = 4$	$j = 6$
H	$J = 1$	48 / 32	48 / 32
	$J = 3$	36 / 24	36 / 24

Table 11: Pay-off matrix further reduced

Step 3: Table 11 shows that $J = 1$ dominates $J = 3$. This means that H has only one strategy remaining, namely $J = 1$, implying that H accepts 6 in period 1. The conclusion arrived at is the same as the one on p. 44.

M.1.4 Comparisons with an Equilibrium Pair Solution

M.1.4.1 Definition of the Equilibrium Pair Concept

The type of solution presented e.g. in Chapter 3 differs in various respects from the type of solution generally employed in two-person game theory, namely the equilibrium pair of strategies. An equilibrium pair of strategies is a pair of strategies such that there is *no* better reply to the other party's equilibrium strategy than one's own equilibrium strategy.[6]

M.1.4.2 Is our Solution an Equilibrium Pair?

We first note that the two pairs in our solution ($J = 1, j = 4$) and ($J = 1, j = 6$) are both equilibrium pairs. As shown in Table 9, when H chooses $J = 1$, L can do no better than choose $j = 4$ or $j = 6$, and when L chooses $j = 4$ or $j = 6$, H can do no better than $J = 1$. In other words, our solution in the example above *does* constitute an equilibrium pair of strategies. It can in fact be proved that every bargaining game which can be solved for a given order of bidding using our basic model has an equilibrium pair of strategies equivalent to the strategies chosen according to our solution, provided there is a last period within finite time.[7]

M.1.4.3 Are there Equilibrium Pairs which do not Belong to the Solution Set?

Our next question in whether there exist equilibrium pairs other than those which constitute the solution according to our model. Table 9 shows that e.g. ($J = 5, j = 2$) also constitutes an equilibrium pair but does not belong to our solution. We note that both parties' profits from this equilibrium pair ($V = 49; v = 21$) differ from the profits of the solution ($V = 48; v = 32$). Since ($J = 5, j = 2$) and ($J = 1, j = 4$ (or 6)) are not interchangeable,[8] the bargaining game presented above lacks a non-cooperative solution in Nash's sense.[9] Furthermore since neither of these equilibrium pairs is such that both parties prefer one pair to the other – H prefers ($J = 5, j = 2$) while L prefers ($J = 1, j = 4$ (or 6)) – the bargaining game does not possess what in game theory is called a solution in the strict sense.[10]

[6] The strategy pair (J^*, j^*) would constitute an equilibrium pair, provided that:
1. If L chooses j^*, H can *not* get a higher pay-off from using a $J \neq J^*$ than from using J^*.
2. If H chooses J^*, L can *not* get a higher pay-off from using a $j \neq j^*$ than from using j^*.

[7] The proof is given with the aid of Zermelo's theorem, implying that each game with perfect information and a finite number of periods (moves) has at least one equilibrium pair. This pair is determined using backwards deduction, starting with the last period. See furthermore Zermelo (1912), Mc Kinsey (1952, p. 130) and Luce-Raiffa (1957, p. 68).

[8] ($J = 1, j = 4$) and ($J = 5, j = 2$) would be interchangeable if ($J = 5, j = 4$) and ($J = 1, j = 2$) were also equilibrium pairs. This, however, is not the case.

[9] This solution concept should be clearly distinguished from Nash's solution of the cooperative game (see p. 230). A non-cooperative game is solvable in Nash's sense, if every pair of equilibrium pairs is interchangeable. See Nash (1951) and Luce-Raiffa (1957, p. 106).

[10] A solution in the strict sense requires all equilibrium pairs which are not regarded by both parties as less desirable than some other requilibrium pair, to be interchangeable and to lead to the same pay-off for both parties. See Luce-Raiffa (1957, p. 107).

The existence of an equilibrium pair in a bargaining game is *not* an indication of the existence of a solution in our sense. According to Zermelo's theorem[11] every game characterized by perfect information and a finite number of periods has an equilibrium pair. However, even though the example in figure 32 below fulfills these requirements, it lacks a solution in our sense.

Our main conclusion is that the ordinary equilibrium pair concept, so widely used in game theory, is of little value for our purposes.[12] This means that there is no reason for us to continue studying bargaining games using the normal form.

M.2 Examples of Two-alternative S-games, Unsolvable using B_4

The following example is the same as example 6 on p. 53 except that the value of $\overline{6}_2$ is increased somewhat.

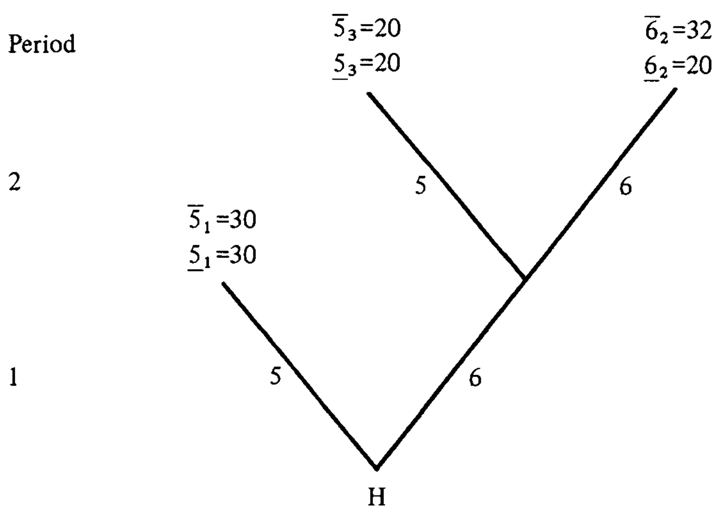

Figure 32 A game which cannot be solved using B_4

L will be indifferent between accepting 6 and insisting on 5 in period 2. According to the assumptions of B_4 this implies that H can assign a V lying above 20 but below 32 to his bid 6. Since this V can be lower or higher than $\overline{5}_1 = 30$, nothing can be determined with respect to what H will choose in period 1. H's behavior in this period cannot be determined until our behavioristic assumptions have been

[11] See footnote 7.
[12] A more useful concept for our purposes is the Selten "perfect equilibrium pair" (see Selten, 1965). However, this concept relies on the analysis of the game in extensive form.

extended even further. The following three assumptions appear to be the most natural and least restrictive ones required for solving the situation described above.

B_{13}. *Cardinal utility assumption*: Each party assigns a cardinal utility – in the von Neumann–Morgenstern sense – to each possible outcome.[13]

B_{14}. *Criteria for choice under uncertainty*: In the case of genuine uncertainty a party will behave according to either the Laplace criterion[14], the maximin criterion[15] or the minimax regret criterion[16].

B_{15}. *Establishment of uncertainty*: If a party makes use of all other behavioristic assumptions, but can *not* determine which of two events – subject to the other party's control – will occur, he will regard the outcome as subject to genuine uncertainty.

Assuming that H has a linear utility for money we can deduce that H accepts 5 in period 1. According to the Laplace criterion H has a 50 per cent chance of \$ 20 and a 50 per cent chance of \$ 32, which means that H's expected pay-off from insisting on 6 is (32+20)/2 = 26, i.e. lower than 30. According to the maximin criterion, alternative 5 is assigned a minimum value of \$ 30, while alternative 6 is assigned a minimum value of \$ 20. Hence alternative 5, with the higher minimum value, is chosen. According to the minimax regret criterion, alternative 5 leads to a possible maximum regret of 32–30 = 2, while alternative 6 leads to a possible maximum regret of 30–20 = 10. Hence H chooses 5 with the lowest maximum regret.

However, it should be stressed that every two-alternative S-game cannot be solved even after this extension of the behavioristic assumptions. For example, if we set $\overline{6}_2$ = 42 instead of 32 in Figure 32, no solution can be obtained. The Laplace criterion would then make H insist on 6, giving H an expected pay-off of (42+20)/2 = \$ 31, while the maximin criterion would still suggest that H accepts 5. Thus some further

[13] This assumption, in turn, depends on other more fundamental assumptions or axioms. Several different axiom systems lead to B_{13}. Some of these axioms are already included in our assumption set B_4, but several more are required. If the axiom system presented in Luce-Raiffa (1957) is used, assumption 2 (reduction of compound lotteries), assumption 3 (continuity) and assumption 4 (substitutability) on pp. 26–27 in their book would be required.

[14] The Laplace criterion implies that the party regards each of the n possible events as equally likely and hence the probability $1/n$ is assigned to each event. The party then chooses the alternative which has the highest expected utility, determined according to assumption B_{13}.

[15] The maximin criterion implies that the lowest attainable utility is established for each alternative. The act which has the ***highest*** of these minimum utilities is then chosen.

[16] The minimax regret criterion implies that a party establishes a regret pay-off for each possible outcome determined by the event and the alternative chosen. The regret pay-off is the ***difference*** between the utility of this outcome and the highest utility that could be obtained if this event became true. The party then establishes the highest possible regret for each alternative. He finally chooses the alternative with the lowest of these maximum regret values.

assumption is required which would rule out either the maximin criterion or the Laplace criterion.[17]

M.3 Proof that there are More than $(n-2)^j/j!$ Branches in Period j

$(k,m)_{\underline{0}}$ leads to $m-k$ situations $(k,m')_{\bar{1}}$, where $m' = k+1 \ldots m$.

$(k,m')_{\bar{1}}$ leads to $m'-k$ situations $(k',m')_{\underline{2}}$, where $k' = k \ldots m'-1$. Hence

$(k,m)_{\underline{0}}$ leads to $\sum_{m'=k+1}^{m} m'-k$ situations $(k',m')_{\underline{2}}$.

$(k',m')_{\underline{2}}$ leads to $m'-k'$ situations $(k',m'')_{\bar{3}}$, where $m'' = k'+1 \ldots m'$.

Hence $(k,m)_{\underline{0}}$ leads to $\sum_{m'=k+1}^{m} \sum_{k'=k}^{m'-1} m'-k'$ situations $(k',m'')_{\bar{3}}$.

$(k',m'')_{\bar{3}}$ leads to $m''-k'$ situations $(k'',m'')_{\underline{4}}$, where $k'' = k' \ldots m''-1$. Hence

$(k,m)_{\underline{0}}$ leads to $\sum_{m'=k+1}^{m} \sum_{k'=k}^{m'-1} \sum_{m''=k'+1}^{m'} m''-k'$ situations $(k'',m'')_{\underline{4}}$.

Since $\sum_{m''=k'+1}^{m'} m''-k' = \sum_{m''-k'=1}^{m'-k'} m''-k' = \sum_{k_0=1}^{m'-k'} k_0$,

$\sum_{k'=k}^{m'-1} f(m'-k') = \sum_{m'-k'=m'-k}^{1} f(m'-k') = \sum_{k_1=1}^{m'-k} f(k_1)$ and

$\sum_{m'=k+1}^{m} g(m'-k) = \sum_{m'-k=1}^{m-k} g(m'-k) = \sum_{k_2=1}^{m-k} g(k_2)$,

$(k,m)_{\underline{0}}$ leads to $\sum_{k_2=1}^{m-k} \sum_{k_1=1}^{k_2} \sum_{k_0=1}^{k_1} k_0$ situations $(k'',m'')_{\underline{4}}$.

Generalized: $(1,n)_0$ leads to $\sum_{k_{j-2}=1}^{n-1} \sum_{k_{j-3}=1}^{k_{j-2}} \ldots \sum_{k_0=1}^{k_1} k_0 >$

$\int_1^{n-1} \int_1^{k_{j-2}} \ldots (\int_1^{k_1} k\,dk)dk_1, dk_2$, etc. $>$

$\int_0^{n-2} \int_0^{k} \ldots (\int_0^{k} k dk)dk,\ dk$, etc. $= \frac{(n-2)^j}{j!}$ situations $(k,m)_j$.

[17] Choice of the Laplace criterion would be in line with an "equiprobability postulate" implying that, if a party expects two strategies to yield him the same pay-off, then he will be equally likely to use any of these strategies. However, this postulate has a somewhat *ad hoc* character, since it cannot be applied to e.g. zero-sum games. According to Harsanyi (1966, p. 620) this postulate is subordinated to a maximin postulate.

For $n = 99$, $j = 100$ there will be more than $\frac{97^{100}}{100!} = \frac{97^{100}}{9.33 \cdot 10^{158}} =$ $0.509 \cdot 10^{40} > 10^{39}$ branches in period j.

M.4 Special Problems of the General Model

M.4.1 Games without a Last Period

We also want to solve games in which a determined pay-off pair V or v can *not* be assigned to some specific period, e.g. z (see p. 74). This can be accomplished by using *upper limit* values denoted V' and v'. These are pay-off values assigned to a specific bid, e.g. y in period j such that $V \leqslant V'$ and $v \leqslant v'$, where V and v are the true – but possibly unknown – pay-offs resulting from the bid. For bargaining games without a last period within finite time, some period with a finite number can be chosen and, due e.g. to assumption S_1, a pair V', v' can be assigned to each bid that is *not* an acceptance bid.[18]

Next, we note that even if a unique outcome cannot be assigned to any choice for any situation in this period, we might still very well be able to derive uniquely determined outcomes for every situation in some earlier period. This is due to the fact that the determined values in the earlier periods, obtained e.g. from acceptance bids, can be larger than the highest undetermined values.[19] The following method can be used to determine a choice in a period in which some bids lead to undetermined values:

Let us study a specific situation $(y,y')_{\overline{j-1}}$, i.e. where $j \in L$.[20] We assume that either a determined value of L's pay-off, v, or an upper limit value on this pay-off, v' is assigned to each of L's bids *and* either a determined value V or an upper limit value V' to each of H's bids.

Next, we compare all the determined values v of L's pay-offs from these bids and establish the *highest* of these values, called v^d. We also compare all of L's upper limit values v', calling the highest of these values v^u. Now if there is only *one* bid leading to v^d and if $v^d > v^u$, L's choice in this period is determined.[21] v^d can then

[18] In e.g. example 7 on p. 71, a value $V' = 3 \cdot 6 = 18$ can be assigned to H's bid 6 and $V' = 3 \cdot 7 = 21$ to H's bid 7 in period 3, since H cannot possibly get a better agreement than L accepting H's proposal in period 4.

[19] Every two-alternative game without a last period z, but with a critical period i, is an example of this.

[20] The case where $j \epsilon H$ is treated in an equivalent manner.

[21] The case where several bids lead to v^d and the case where $v^u = v^d$ are discussed in M.4.2 below.

be assigned to $\underline{(y,y')}_{\overline{j-1}}$ as a determined value. If the alternative leading to v^d has a determined value V, this V is assigned to $\overline{(y,y')}_{\overline{j-1}}$ as a determined value. If the alternative which leads to v^d has an upper limit value V', then V' is assigned to $\overline{(y,y')}_{\overline{j-1}}$ as an upper limit value.

On the other hand, if $v^u > v^d$, we have to be content with assigning v^u as an upper limit value to $\underline{(y,y')}_{\overline{j-1}}$. Since L *might* in this case choose any alternative leading to an undetermined value $v' > v^d$, *or* the alternative leading to v^d [22], we assign the highest V or V' of those alternatives which lead to a $v' > v^d$ or to v^d as an upper limit value to $\overline{(y,y')}_{\overline{j-1}}$.

The choice of the last period to be studied might be difficult. Assume that no solution has been found in the case where backwards deduction is started from a certain period, and where upper limit values V' and v' are assigned to every bid which is not an acceptance bid in this period. In general, it is then impossible to conclude whether the game lacks a solution or whether one should try again, starting with a later period. However, a last interesting period, called z^*, beyond which it is unnecessary to go, can always be established for S-games (see p. 103).

M.4.2 Games Involving Cases of Indifference

The simple method of deduction presented in Section 4.4 in the text did not allow for the case where a party would be indifferent between two alternatives. This case can also be handled with the aid of the upper limit values V' and v', introduced in M.4.1 above. Let us again study a situation $(y,y')_{\overline{j-1}}$. When bidding in period j, L will be indifferent:

1. if several alternatives lead to the same value v^d and
2. if $v^u = v^d$.

1. The case where several alternatives lead to the same value v^d is exemplified by period 2 in example 6 on p. 54. Alternatives 5 and 6 both lead to $v^d = 20$. L's pay-off assigned to this situation $(5,6)_{\bar{1}}$ and hence to H's bid 6 in period 1, is obviously 20. But H's pay-off presents a problem. H can get either 20 or 30. Due to B_4, H runs some risk of getting 20. Hence the highest value of 20 and 30, *minus* ϵ, where ϵ is some very small number, is assigned to $\overline{(5,6)}_{\bar{1}}$ as an upper limit value. More generally, in a situation $(y,y')_{\overline{j-1}}$, where several alternatives lead to the *same* value v^d, we assign

[22] It can very well happen that the real v of an alternative with a $v' > v^d$ is smaller than v^d.

v^d as a determined value to $\underline{(y,y')_{j-1}}$ and
the *highest* of those values V and V' which are assigned to L's bids leading to v^d, *minus* ϵ, as an upper limit value to $\overline{(y,y')}_{\overline{j-1}}$[23].

2. If $v^u = v^d$, a determined value v^d is assigned to $\underline{(y,y')_{j-1}}$. Since L might in this case choose either the alternative(s) leading to v^d or the alternative(s) leading to v^u, we assign the highest of H's pay-offs from the alternatives leading to v^u and v^d as an upper limit value to $\overline{(y,y')}_{\overline{j-1}}$.

M.5 Proof that $P \Rightarrow (x,x+k)_{\underline{i-2}} = x_{i-1}$

Figure 33 exemplifies H's choice in period $i-1$.

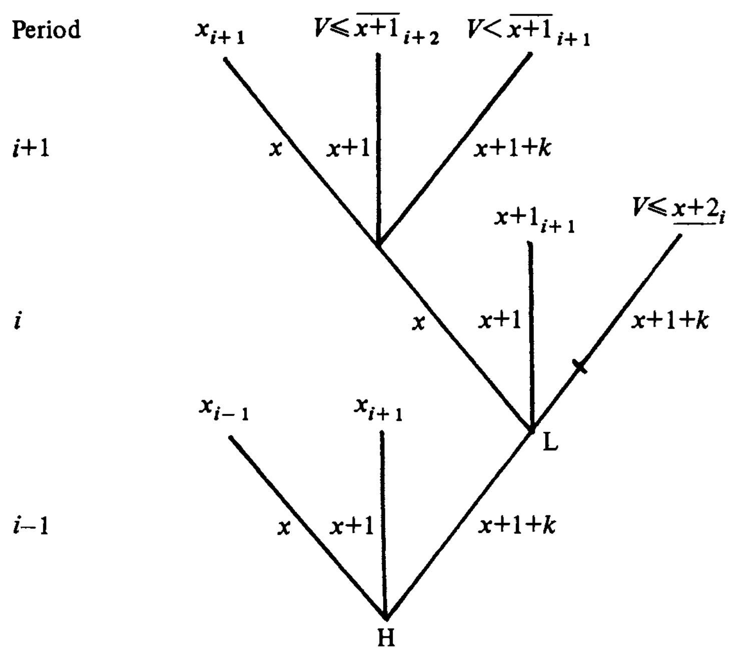

Figure 33 H's choice in period $i-1$.

First, we look at the pay-offs of H's bids in *period* $i+1$:

If H accepts x, an agreement is reached on x_{i+1}.

If H bids $x+1$, H can at best get $\overline{x+1}_{i+2}$.

[23] If all alternatives leading to v^d result in *the same value* V or V', this value is assigned to $\overline{(y,y')}_{\overline{j-1}}$.

If H bids $x+1+k$, we obtain a situation $(x,x+1+k)_{\overline{i+1}}$. It was proved (see p. 92) that P implies $\overline{(y-m',y+k)}_{\overline{i-x+y}} < \overline{y}_{i-x+y}$, where $m' = 0 \ldots y-1$. Setting $y = x+1$ and $m' = 1$, we obtain $\overline{(x,x+1+k)}_{\overline{i+1}} < \overline{x+1}_{i+1}$.

Next, we study L's bids in *period i*:

Setting $y = x+1$ in the conclusion on p. 92 that $(y,y+k)_{i-x+y-1} = y_{i-x+y}$, we obtain $(x+1,x+1+k)_{\underline{i}} = \overline{x+1}_{i+1}$. Hence L's bid $x+1$ leads to $\overline{x+1}_{i+1}$, i.e. L gets $\underline{x+1}_{i+1}$.

By bidding $x+1+k$, L can at best get an agreement on $x+2$ in this period, i.e. L gets a $v \leqslant \underline{x+2}_i$.

Since $\underline{x+1}_{i+1} > \underline{x+2}_i$[24], L prefers bidding $x+1$ rather than $x+1+k$ and L's bid $x+1+k$ is eliminated.

Finally, we turn to the pay-off of H's bids in *period* $i-1$.

H's acceptance of x in period $i-1$ leads to x_{i-1}.

$x+1$ leads to $(x,x+1)_{\overline{i-1}}$. Since $i = \mathrm{u}(x)$ and $i+1 = \mathrm{C}(x)$, we obtain – from T_3 – $(x,x+1)_{\overline{i-1}} = x_{i+1}$.

Since $\overline{x}_{i-1} > \bar{x}_{i+1} > \overline{x+1}_{i+1} > \overline{x+1}_{i+2}$, H's bid x gives him his maximum pay-off, i.e. H accepts x in i–1.

Conclusion: $P \Rightarrow (x,x+k)_{\underline{i-2}} = x_{i-1}$.

M.6 Proof of Theorem T_{12}

Theorem T_{11} can be written as follows:

P: $i+k' = \mathrm{Cu}(x+k')$, allowing for $i = \mathrm{S}(x)$, if $i+1 = \mathrm{su}(x)$, where $k' = 0 \ldots n-x-1$,

implies for every $j \leqslant i-1$ and for $k = 1 \ldots n-x$ that

$(x,x+k)_{\overline{j-2}} = x_j$ and $(x,x+k)_{\underline{j-1}} = x_j$.

[24] $i+1 = \mathrm{u}(x+1) \Rightarrow$, due to S_2, that $i = u(x+1)$, i.e. that $\underline{x+2}_i < \underline{x+1}_{i+1}$.

We can now establish theorem T_{12} as the mirror picture of theorem T_{11}. This is done by establishing a mirror picture of the assumptions and conclusions. The idea behind establishing a mirror picture of a certain theorem is as follows:

Since H and L are only temporary names given to the two parties, either of two notational systems can be applied:

System 1: Party 1 is called L and party 2 is called H.[25]

System 2: Party 1 is called H and party 2 is called L.

Either one of these two systems is allowed. Let us assume that we have previously used system 1. The first step in obtaining a mirror picture then involves renumbering the alternatives. System 1 implies that party 1's best alternative is called 1, while system 2 implies that party 1's best alternative is called n. We obtain the following relationship between the names of the alternatives according to the two systems of notation.

	Party 1's best alternative						Party 2's best alternative
System 1:	1	2	x	$x+1$	$x+k'$	$x+k'+1$	n
System 2:	n	$n-1$	$n+1-x$	$n-x$	$n+1-x-k'$	$n-x-k'$	1

Table 12 Relations between the numbers of the alternatives in the notational systems

Next we note the effect of the two systems on denoting a certain bargaining game. Let us study the case where according to system 1, we have a game $(x+k',x+k)$, where $k > k'$. According to system 2 in Table 12 above alternative $x+k'$ is called $n+1-x-k'$. Likewise alternative $x+k$ is called $n+1-x-k$. Retaining the principle that the lowest alternative is placed to the left, this game, according to system 2, is called $(n+1-x-k,\ n+1-x-k')$. By setting $(k' = 0,\ k = 1)$, $(k' = k',\ k = k'+1)$ and $(k' = 0,\ k = k)$, respectively, we obtain the following relation between games described by system 1 and system 2.

System 1:	$(x,x+1)$	$(x+k',\ x+k'+1)$	$(x,x+k)$
System 2:	$(n-x,\ n+1-x)$	$(n-x-k',\ n+1-x-k')$	$(n+1-x-k,\ n+1-x)$

Table 13 Relations between the games in the notational systems

[25] We assume that "party 1" and "party 2" represent the real names of the parties.

We can then determine the mirror picture of some pay-off assumptions. $i+k' = C(x+k')$ implies that period $i+k'$ is critical for H as regards the two-alternative game $(x+k', x+k'+1)$. Looking at Table 13 we see that according to system 2 this is equivalent to: $i+k'$ is critical for L as regards the game $(n-x-k', n+1-x-k')$. Denoting the critical characteristic according to the *lowest* numbered alternative, this is equivalent to $i+k' = c(n-x-k')$. Substituting uncritical and semicritical for critical we deduce, on the basis of Table 13, the following relations between notations for critical characteristics in the two notational systems:

System 1:	$i+k' = \mathrm{Cu}(x+k')$	$i = \mathrm{S}(x)$	$i+1 = \mathrm{su}(x)$
System 2:	$i+k' = \mathrm{cU}(n-x-k')$	$i = \mathrm{s}(n-x)$	$i+1 = \mathrm{SU}(n-x)$

Table 14 Relations between critical characteristics in the notational systems

We note that a capital letter in system 1 becomes a lower case letter in system 2 and vice versa, since the parties change names.

Next, on the basis of Table 14, we can deduce that P, described according to system 1 as: $i+k' = \mathrm{Cu}(x+k')$, allowing for $i = \mathrm{S}(x)$, if $i+1 = \mathrm{su}(x)$, described according to system 2 is: $i+k' = \mathrm{cU}(n-x-k')$, allowing for $i = \mathrm{s}(n-x)$, if $i+1 = \mathrm{SU}(n-x)$.

Furthermore, the period indici of the game situations have to be changed since they refer to the party who bids last. Hence $\underline{j-1}$ and $\overline{j-2}$ according to system 1 become $\overline{j-1}$ and $\underline{j-2}$ according to system 2. Finally, on the basis of Table 13, we can determine that the conclusion in system 1 that $(x,x+k)_{\underline{j-1}} = x_j$ in system 2 is that $(n+1-x-k, n+1-x)_{\overline{j-1}} = n+1-x_j$.

Conclusion: According to the notations of system 2 theorem T_{11} can be written as: $i+k' = \mathrm{cU}(n-x-k')$, allowing for $i = \mathrm{s}(n-x)$, if $i+1 = \mathrm{SU}(n-x)$, implies for $j \leqslant i-1$ that $(n+1-x-k, n+1-x)_{\overline{j-1}}$ and $(n+1-x+k, n+1-x)_{\underline{j-2}} = n+1-x_j$.

However, since we are interested in a situation where H insists on x we want to replace $n+1-x$ in the conclusion above by x. Since x can represent any alternative this is a legitimate substitution provided it is done consistently throughout the theorem. This exchange implies that $n-x$ is replaced by $x-1$. Since k' is such that $0 \leqslant k' \leqslant n-x-1$ this implies that $n-x-k'$ is replaced by $x-1-m'$, where m' in critical characteristics is such that $0 \leqslant m' \leqslant x-2$.

After this substitution the following set of pay-off assumptions is obtained: P': $i+m' = \mathrm{cU}(x-1-m')$ allowing for $i = \mathrm{s}(x-1)$, if $i+1 = \mathrm{SU}(x-1)$. We regard P' as the *mirror* picture of assumption P. The mirror picture of an assumption is hence obtained by rewriting the assumption in the other notational system and replacing $n+1-x$ by x. We notice in this context that P in turn can be regarded as the mirror picture of P'.[26]

The mirror picture of e.g. the conclusion that $(x,x+k)_{\underline{j-1}} = x_j$, obtained after replacing $n+1-x$ by x and k by m[27] in $(n+1-x-k,\ n+1-x)_{\overline{j-1}} = n+1-x_j$ is hence that $(x-m,\ x)_{\overline{j-1}} = x_j$.

Summing up, we obtain the following mirror picture of theorem T_{11}:

Theorem T_{12}: P' implies for $j \leqslant i-1$ that $(x-m,x)_{\overline{j-1}} = x_j$ and $(x-m,\ x)_{\underline{j-2}} = x_j$.

M.7 Proof that P and P' Imply that $\overline{(x-m,x+k)}_{\underline{i-1}} < \overline{x}_i$

We study H's choice in period i, when L has insisted on $x-m$ in $i-1$. This situation is illustrated in Figure 34.

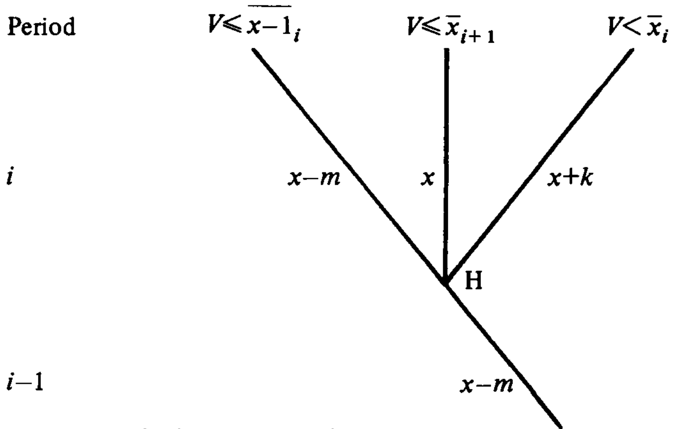

Figure 34 H's choice in period i

If H bids $x-m$, i.e. some alternative $x-1$, $x-2$, etc., he can at best obtain an agreement in period i on $x-1$, i.e. H obtains a $V \leqslant \overline{x-1}_i$.

If H bids x, he can at best obtain an agreement in the next period on x, i.e. H gets a $V \leqslant \overline{x}_{i+1}$

If H bids $x+k$ we rely on the conclusion that $\overline{(y-m',y+k)}_{\overline{i-x+y}} < \overline{y}_{i-x+y}$ holds

[26] Setting $n+1-x = x'$ is equivalent to setting $n+1-x' = x$.

[27] Since $1 \leqslant k \leqslant n-x$ in game notations, we obtain, when replacing $n-x$ by $x-1$, a k such that $1 \leqslant k \leqslant x-1$. Since $1 \leqslant m \leqslant x-1$ in game notations, k can be replaced by m.

for $y = x \ldots n-1$ (see p. 112). For $y = x$ and $m' = m$ we obtain $\overline{(x-m,x+k)}_{\bar{i}} < \bar{x}_i$ and hence H's bid $x+k$ will give him less than $\bar{x}_i$.

Since $\bar{x}_i > \bar{x}_{i+1}$ and $\bar{x}_i > \overline{x-1}_i$, L's bid $x-m$ in period $i-1$ will give H less than x_i, i.e. $\overline{(x-m,x+k)}_{\underline{i-1}} < \bar{x}_i$.

M.8 Proof that $j^* = \mathrm{s}(y^*-1)\mathrm{S}(y^*) \Rightarrow i'+1 = \mathrm{cU}(x'-1)\mathrm{Cu}(x'+1)$

According to the definition on p. 136: $x' \leqslant y^* < x'+1$ and $i' \leqslant j^* < i'+1$. We also assume that $y^* = x'$ and $j^* = i'$ do not both hold.

The proof consists of six steps:

1. Setting $j = j^*$, $j' = i'+1$:
   S'_{2A} implies (when $y = y^*$) that $j^* = \mathrm{S}(y^*) \Rightarrow i'+1 = \mathrm{C}(y^*)$ and
   S'_{2a} implies (when $y = y^*-1$) that $j^* = \mathrm{s}(y^*-1) \Rightarrow i'+1 = \mathrm{c}(y^*-1)$.

2. Setting $j = i'+1$:
   S'_{3A} implies (when $y = y^*$ and $y' = x'+1$) that $i'+1 = \mathrm{C}(y^*) \Rightarrow i'+1 = \mathrm{C}(x'+1)$ and
   S'_{3a} implies (when $y' = y^*-1$ and $y = x'-1$) that $i'+1 = \mathrm{c}(y^*-1) \Rightarrow i'+1 = \mathrm{c}(x'-1)$.

3. Setting $j' = j^*$ and $j = i'$:
   S'_{2B} implies (when $y = y^*$) that $j^* = \mathrm{S}(y^*) \Rightarrow i' = \mathrm{SU}(y^*)$[28] and
   S'_{2b} implies (when $y = y^*-1$) that $j^* = \mathrm{s}(y^*-1) \Rightarrow i' = \mathrm{su}(y^*-1)$.

4. Setting $j = i'$:
   S'_{3B} implies (when $y' = y^*$ and $y = x'$) that $i' = \mathrm{SU}(y^*) \Rightarrow i' = U(x')$[29] and
   S'_{3b} implies (when $y = y^*-1$ and $y' = x'$) that $i' = \mathrm{su}(y^*-1) \Rightarrow i' = \mathrm{u}(x')$.

5. The conclusions in step 4 can be further extended on the basis of the conclusion in footnote 8 on p. 132 that $j = \mathrm{U}(y+1) \Rightarrow j+1 = \mathrm{U}(y)$ and its mirror picture that $j = \mathrm{u}(y-2) \Rightarrow j+1 = \mathrm{u}(y-1)$. Setting $j = i'$, this implies (when $y = x'-1$) that $i' = \mathrm{U}(x') \Rightarrow i'+1 = \mathrm{U}(x'-1)$ and (when $y-2 = x'$) that $i' = \mathrm{u}(x') \Rightarrow i'+1 = \mathrm{u}(x'+1)$.

6. Combining the conclusions in steps 1–5 we obtain $j^* = \mathrm{s}(y^*-1)\mathrm{S}(y^*) \Rightarrow i'+1 = \mathrm{cU}(x'-1)\mathrm{Cu}(x'+1)$.

[28] $i' = \mathrm{S}(y^*)$, if $j^* = i'$.

[29] $i' \neq \mathrm{S}(x')$, since $i' = \mathrm{S}(y^*)$, only if $j^* = i'$ (see footnote 28) and $i' = \mathrm{S}(y^*) \Rightarrow i' = \mathrm{S}(x')$, only if $y^* = x'$ and since $y^* = x'$ and $j^* = i'$ do not *both* hold.

M.9 Proof that $S_1^*-S_4^*$ Fulfill $S_1' - S_5'$

In this proof we distinguish between Δt and dt, between Δy and dy, and between Δj and dj. While Δt, Δy and Δj are differences in $t = T$, y and j of *any* size, dt, dy and dj are *very small* differences as regards these variables. We assume as noted earlier, that Δt is the length of each period implying that $(T+\Delta t)/\Delta t = j+1$, $(T+2\Delta t)/\Delta t = j+2$, etc. It is also assumed that $dt/\Delta t = dj$.

As regards the correspondence between $\bar{y}_j$ and $AyF(T)+B$, we note that the alternative number y is exactly the same in both expressions and that the value of j is obtained by dividing T by Δt.

We give the proofs for H's functions, listing the deductions in order of complexity. The proofs for L's functions can be deduced from this, using the mirror picture technique on pp. 337–339.

1. $S_1^* \Rightarrow S_5'$:
   $y+2-(y+1) = y+1-y \Rightarrow A(y+2)F(T)+B - (A(y+1)F(T)+B) =$
   $= A(y+1)F(T)+B - (AyF(T)+B) \Rightarrow$ (due to S_1^*) $\overline{y+2}_j - \overline{y+1}_j = \overline{y+1}_j - \bar{y}_j$,
   i.e. that S_5' holds.

2. S_1^* and $S_2^* \Rightarrow S_1'$:
   S_2^*: $F'<0 \Rightarrow F(T+dt)-F(T) < 0 \Rightarrow F(T+\Delta t) < F(T)$[30] $\Rightarrow AyF(T+\Delta t)+B <$
   $AyF(T)+B \Rightarrow$ (due to S_1^*) $\bar{y}_{j+1} < \bar{y}_j$, i.e. that S_1' holds.

3. S_1^* and $S_3^* \Rightarrow S_4'$:
   $F''(T) \geqslant 0 \Rightarrow F(T+2dt) - F(T+dt) \geqslant F(T+dt) - F(T) \Rightarrow$
   $F(T+2\Delta t) - F(T+\Delta t) \geqslant F(T+\Delta t) - F(T)$[31] $\Rightarrow AyF(T+2\Delta t)+B - (AyF(T+\Delta t)+B) \geqslant$
   $AyF(T+\Delta t)+B - (AyF(T)+B) \Rightarrow \bar{y}_{j+2} - \bar{y}_{j+1} \geqslant \bar{y}_{j+1} - \bar{y}_j$, i.e. that S_4' holds.

4. S_1^* and $S_2^* \Rightarrow S_3'$:
   $S_2^*: F'<0 \Rightarrow F(T) > F(T+\Delta t) \Rightarrow F(T)dy > F(T+\Delta t)dy$
   $\Rightarrow (y+dy)F(T) - yF(T) > (y+1+dy)F(T+\Delta t) - (y+1)F(T+\Delta t)$

[30] Substituting $T+dt$ for T in (1) $F(T+dt) < F(T)$, we obtain (2) $F(T+2dt) < F(T+dt)$. Combining (1) and (2), we obtain $F(T+2dt) < F(T)$. By iteratively substituting $T+2dt, T+3dt$, etc. for T in (1) and combining with the earlier conclusions we obtain $F(T+kdt) =$ $= F(T+\Delta t) < F(T)$.

[31] Substituting $T+dt$ for T in (1) $F(T+2dt)-F(T+dt) \geqslant F(T+dt)-F(T)$, we obtain (2) $F(T+3dt)-F(T+2dt) \geqslant F(T+2dt)-F(T+dt)$. Merging (1) and (2) we obtain $F(T+3dt)-F(T+2dt) \geqslant F(T+dt)-F(T)$, implying (3) $F(T+3dt)-F(T+dt) \geqslant F(T+2dt)-F(T)$. Substituting $T+dt$ for T in (3) we obtain (4) $F(T+4dt)-F(T+2dt) \geqslant F(T+3dt)-F(T+dt)$. Merging (3) and (4) we obtain (5) $F(T+4dt)-F(T+2dt) \geqslant F(T+2dt)-F(T)$. By substituting $2dt$ for dt in the deduction above, we deduce that $F(T+8dt)-F(T+4dt) \geqslant F(T+4dt)-F(T)$. Continuing in this manner, we deduce that $F(T+2kdt)-F(T+kdt) \geqslant F(T+kdt)-F(T)$, i.e. $F(T+2\Delta t)-F(T+\Delta t) \geqslant F(T+\Delta t)-F(T)$.

$\Rightarrow (y+dy)F(T) - (y+1+dy)F(T+\Delta t) > yF(T) - (y+1)F(T+\Delta t) \Rightarrow$ (due to S_1^*) $\overline{y+dy}_j - \overline{y+1+dy}_{j+1} > \bar{y}_j - \overline{y+1}_{j+1}$. This implies in turn:

a. $\bar{y}_j - \overline{y+1}_{j+1} \geqslant 0 \Rightarrow \overline{y+dy}_j - \overline{y+1+dy}_{j+1} > 0$, i.e. $\bar{y}_j \geqslant \overline{y+1}_{j+1} \Rightarrow \overline{y+dy}_j > \overline{y+1+dy}_{j+1}$, i.e. $j = SC(y) \Rightarrow j = C(y+dy)$, i.e. $j = SC(y) \Rightarrow j = C(y')$,[32] i.e. S_{3A} holds.

b. $0 \geqslant \overline{y+dy}_j - \overline{y+1+dy}_{j+1} \Rightarrow 0 > \bar{y}_j - \overline{y+1}_{j+1}$, i.e. $\overline{y+1+dy}_{j+1} \geqslant \overline{y+dy}_j \Rightarrow \overline{y+1}_{j+1} > \bar{y}_j$, i.e. $j = SU(y+dy) \Rightarrow j = U(y)$, i.e. $j = SU(y') \Rightarrow j = U(y)$, i.e. S'_{3B} holds.

5. S_1^* and $S_4^* \Rightarrow S_2'$

$S_4^*: d(-F/F')/dT < 0 \Rightarrow -F'/F' + FF''/(F')^2 < 0 \Rightarrow (F')^2 > FF'' \Rightarrow (F(T+dt))^2 > F(T)F(T+2dt)$[33] $\Rightarrow F(T+dt)/F(T+2dt) > F(T)/F(T+dt) \Rightarrow F(T+dt)/F(T+\Delta t+dt) > F(T)/F(T+\Delta t)$[34] $\Rightarrow yF(T+dt)/(y+1)F(T+\Delta t+dt) > yF(T)/(y+1)F(T+\Delta t)$.

This in turn leads to:

a. $yF(T)/(y+1)F(T+\Delta t) \geqslant 1 \Rightarrow yF(T+dt)/(y+1)F(T+\Delta t+dt) > 1$, i.e. $yF(T) \geqslant (y+1)F(T+\Delta t) \Rightarrow yF(T+dt) > (y+1)F(T+\Delta t+dt)$ implying (due to S_1^* and since $T/\Delta t = j$ and $dt/\Delta t = dj$) $\bar{y}_j \geqslant \overline{y+1}_{j+1} \Rightarrow \bar{y}_{j+dj} > \overline{y+1}_{j+1+dj}$, i.e. $j = SC(y) \Rightarrow j+dj = C(y)$. Applying this conclusion iteratively [35] we deduce $j = SC(y) \Rightarrow j' = C(y)$ for any $j' > j$, i.e. S'_{2A} holds.

b. $1 \geqslant yF(T+dt)/(y+1)F(T+\Delta t+dt) \Rightarrow 1 > yF(T)/(y+1)F(T+\Delta t)$ i.e. $(y+1)F(T+\Delta t+dt) \geqslant yF(T+dt) \Rightarrow (y+1)F(T+\Delta t) > yF(T)$ $\Rightarrow \overline{y+1}_{j+1+dj} \geqslant \bar{y}_{j+dj} \Rightarrow \overline{y+1}_{j+1} > \bar{y}_j$, i.e. $j+dj = SU(y) \Rightarrow j = U(y)$. Applying this iteratively, we deduce for any $j' > j$ that $j' = SU(y) \Rightarrow j = U(y)$ i.e. S'_{2B} holds.

[32] Substituting $y+dy$ for y in (1) $j = SC(y) \Rightarrow j = C(y+dy)$ we obtain (2) $j = SC(y+dy) \Rightarrow j = C(y+2dy)$. Merging (1) and (2) we obtain $j = SC(y) \Rightarrow j = C(y+2dy)$. By iteratively substituting $y+2dy$, $y+3dy$ for y, etc. and merging, we obtain $j = SC(y) \Rightarrow j = C(y+kdy) = C(y')$.

[33] $(F')^2 > FF'' \Rightarrow F^2 + 2FF' + (F')^2 > F^2 + 2FF' + FF'' \Rightarrow (F+F')^2 > F(F+F'+(F'+F'')) \Rightarrow (F(T+dt))^2 > F(T)F(T+2dt)$.

[34] Substituting $T+2dt$ for $T+dt$ in $F(T+dt)/F(T+2dt) > F(T)/F(T+dt)$ we obtain $F(T+2dt)/F(T+3dt) > F(T+dt)/F(T+2dt)$. Merging these conclusions, we obtain $F(T+2dt)/F(T+3dt) > F(T)/F(T+dt)$. By iteratively replacing T with $T+2dt$, $T+3dt$, etc. and merging the conclusions, we deduce that $F(T+kdt)/F(T+kdt+dt) > F(T)/F(T+dt)$. Setting $kdt = \Delta t$, we obtain $F(T+\Delta t)/F(T+\Delta t+dt) > F(T)/F(T+dt)$, implying that $F(T+dt)/F(T+\Delta t+dt) > F(T)/F(T+\Delta t)$.

[35] Substituting $j+dj$ for j, we obtain $j+dj = SC(y) \Rightarrow j+2dj = C(y)$. Combining this with the original conclusion, we obtain $j = SC(y) \Rightarrow j+2dj = C(y)$. Likewise, by iteratively substituting $j+2dj$, $j+3dj$, etc. for j and merging the conclusions we deduce that $j = SC(y) \Rightarrow j+kdj = j' = C(y)$.

M.10 Further Analysis of S*-games

M.10.1 Establishing Requirement S_1^*

Each of the three pay-off functions in Section 6.4 can be written as a function of the share that each party obtains and the time of agreement T. Hence we denote H's and L's pay-off functions as $V(S,T)$ and $v(s,T)$, respectively.

First we prove that every pay-off function $v(s,T)$ and $V(S,T)$ fulfills S_1^*, if $V(S,T) = \pi(S+Q)F(T) + B$ and $v(s,T) = \pi(s+q)f(T) + b$, where S and s are the shares obtained by H and L, respectively, such that $S+s = 1$, while π, Q, q, B and b are constants.

That S_1^* holds is seen by setting $a = A = \pi(1+q+Q)/N$ and
$y = N(S+Q)/(1+q+Q) = \pi(S+Q)N/\pi(1+q+Q) = \pi(S+Q)/A$.

$a = A = \pi(1+q+Q)/N \Rightarrow aN = AN = \pi(1-S+q+S+Q) = \pi(s+q) + \pi(S+Q) \Rightarrow$
$\Rightarrow N = \pi(s+q)/a + \pi(S+Q)/A \Rightarrow \pi(s+q)/a = N - \pi(S+Q)/A = N-y$.

Hence $Ay = \pi(S+Q)$ and $a(N-y) = \pi(s+q)$ implying that
$\pi(S+Q)F(T) + B = AyF(T) + B = V(S,T)$ and
$\pi(s+q) f(T) + b = a(N-y)f(T)+b = v(s,T)$.

M.10.2 Determining the Solution

$y = N(S+Q)/(1+q+Q)$ and $y^* = F^*(T^*)/\Delta t$ (cf. p. 116) $\Rightarrow$
$N(S^*+Q)/(1+q+Q) = F^*(T^*)/\Delta t \Rightarrow (S^*+Q)/(1+Q+q) = F^*(T^*)/\mu \Rightarrow$
$S^* = F^*(T^*)(1+Q+q)/\mu - Q$.

M.11 Analysis of Pay-off Function 1

M.11.1 Analysis of Requirements S_1^*–S_4^*

Case a: $r > 0$

We develop L's pay-off $v(s,T)$ of an agreement on s at time T.

Setting $\bar{w} = w-c$, $v(s,T) = s\pi \int_T^Z e^{-rt}dt + \bar{w}\int_0^T e^{-rt}dt - k_T e^{-rT} + k_Z e^{-rZ} =$
$s\pi(e^{-rT}-e^{-rZ})/r + \bar{w}(1-e^{-rT})/r - k_T e^{-rT} + k_Z e^{-rZ} =$
$s\pi(e^{-rT}-e^{-rZ})/r + \bar{w}/r - (\bar{w}+rk_T)(e^{-rT}-e^{-rZ})/r - (\bar{w}+rk_T)e^{-rZ}/r + k_Z e^{-rZ} =$
$(s\pi - (\bar{w}+rk_T))(e^{-rT}-e^{-rZ})/r + \bar{w}/r - (\bar{w} + rk_T)e^{-rZ}/r + k_Z e^{-rZ} =$

$\pi(s+q)f(T)+b$, where $q = -(\bar{w}+rk_T)/\pi = -(w-c+rk_T)/\pi$,
$f(T) = (e^{-rT}-e^{-rZ})/r$ and $b = \bar{w}/r - (\bar{w}+rk_T)e^{-rZ}/r+k_Z e^{-rZ}$.

Likewise H's pay-off $V(S,T) = \pi(S+Q)F(T)+B$, where $Q = -(W-C+RK_T)/\pi$ and $F(T) = (e^{-RT}-e^{-RZ})/R$.

Hence S_1^* holds.

S_2^* holds, since $0 > -e^{-rT} = d((e^{-rT}-e^{-rZ})/r)/dT = f'$.

S_3^* holds, since $0 \leqslant re^{-rT} = d(-e^{-rT})/dT = df'/dT = f''$.

S_4^* holds, since $0 > (-e^{-r(Z-T)}) = d((1-e^{-r(Z-T)})/r)/dT = df^*/dT$,
since $f^* = -f/f' = (e^{-rT}-e^{-rZ})/re^{-rT} = (1-e^{-r(Z-T)})/r$.

It should be noted in this contoxt that if $Z\to\infty$, then $e^{-r(Z-T)}\to 0$ and S_4^* does *not* hold. Hence $f(T) = \int_T^\infty e^{-rt}dt = e^{-rT}/r$ is *not* allowed.

Case b: r = 0

$v(s,T) = s\pi \int_T^Z e^{-0t}dt + \bar{w}\int_0^T e^{-0t}dt - k_T e^{-0T} + k_Z e^{-0Z} = s\pi(Z-T)+\bar{w}T-k_T+k_Z$
$= (s\pi-\bar{w})(Z-T)+\bar{w}Z-k_T+k_Z = \pi(s+q)f(T)+b$,
where $q = -(w-c+rk_T)/\pi$, $f(T) = (Z-T)$ and $b = \bar{w}Z-k_T+k_Z$.

Likewise $V(S,T) = \pi(S+Q)F(T) + B$, where $Q = -(W-C+RK_T)/\pi$ and $F(T) = Z-T$.
Hence S_1^* holds.

S_2^* holds, since $0 > -1 = d(Z-T)/dT = f'$.

S_3^* holds, since $0 = d(-1)/dT = df'/dT = f''$.

S_4^* holds, since $0 > -1 = d(Z-T)/dT = d(f/-f')/dT = df^*/dT$.

M.11.2 Proof that $\partial S^*/\partial R < 0$

For $\zeta = Z-T^*$, $F^*(T^*)+f^*(T^*) = (1-e^{-R(Z-T^*)})/R + (1-e^{-r(Z-T^*)})/r = \mu \Rightarrow$
$1/R - e^{-R\zeta}/R - e^{-r\zeta}/r + 1/r - \mu = 0 \Rightarrow$
$(-1/R^2 + \zeta e^{-R\zeta}/R + e^{-R\zeta}/R^2)dR + (e^{-R\zeta} + e^{-r\zeta})d\zeta = 0 \Rightarrow$
$(1-(\zeta R+1)e^{-R\zeta})/(e^{-R\zeta}+ e^{-r\zeta})R^2 = d\zeta/dR = \zeta' \Rightarrow$
$\zeta' < (1-(\zeta R+1)e^{-R\zeta})/e^{-R\zeta}R^2 = (e^{R\zeta} - \zeta R-1)/R^2 \Rightarrow$
$1+R\zeta+R^2\zeta' < 1+R\zeta+e^{R\zeta}-R\zeta-1 = e^{R\zeta} \Rightarrow 1 > e^{-R\zeta}(1+R\zeta+R^2\zeta') \Rightarrow$

$0 > (-1+e^{-R\zeta}(1+R\zeta+R^2\zeta'))/R^2 = d(1/R - e^{-R\zeta}/R)/dR \Rightarrow$
$0 > ((1+Q+q)/\mu)\,(dF^*(T^*)/dR) = d((1+Q+q)F^*(T^*)/\mu - Q)/dR = \partial S^*/\partial R.$

M.11.3 Determining the Effect of W, C and RK_T on S^*

$f^*(T^*) > 0$ and $F^*(T^*)+f^*(T^*) = \mu \Rightarrow F^*(T^*)/\mu < 1 \Rightarrow 0 > F^*(T)/\mu - 1$
$= \partial(F^*(T^*)\,(1+Q+q)/\mu - Q)/\partial Q = \partial S^*/\partial Q$. Since $\partial Q/\partial W = (\partial(C-W-RK_T)/\partial W)/\pi = -1/\pi < 0$
we obtain $\partial S^*/\partial W = (\partial S^*/\partial Q)(\partial Q/\partial W) \geqslant 0.$

Likewise we deduce that $\partial S^*/\partial(RK_T) > 0$ and that $\partial S^*/\partial C < 0$.

M.11.4 Determining the Solution when $r = R$

$f(T^*) = F(T^*)$ and $f^*(T^*)+F^*(T^*) = \mu \Rightarrow F^*(T^*) = \mu/2 \Rightarrow S^* = F^*(T^*)(1+Q+q)/\mu - Q$
$= (1+Q+q)/2 - Q = (1-Q+q)/2$ and $S^*\pi = (\pi+W-C+RK_T-w+c-rk_T)/2.$

M.11.5 Determining the Solution when $r = 2R$

$f^*(T) + F^*(T) = \mu \Rightarrow (1-e^{-r(Z-T)})/r + (1-e^{-R(Z-T)})/R = \mu \Rightarrow (1-e^{-2R(Z-T)})/2R$
$+ (1-e^{-R(Z-T)})/R = \mu \Rightarrow 1-e^{-2R(Z-T)} + 2 - 2e^{-R(Z-T)} = 2R\mu$
$\Rightarrow (e^{-R(Z-T)})^2 + 2e^{-R(Z-T)} + (2R\mu-3) = 0$
$\Rightarrow e^{-R(Z-T)} = (-2+\sqrt{4-4(2R\mu-3)})/2$[36] $= -1+\sqrt{4-2R\mu} \Rightarrow$
$F^*(T^*) = (1-e^{-R(Z-T^*)})/R = (2-\sqrt{4-2R\mu})/R \Rightarrow$
$F^*(T^*)/\mu = (2-\sqrt{4-2R\mu})/R\mu = 2(1-\sqrt{1-R\mu/2})/R\mu = (1-\sqrt{1-R\mu/2})/(R\mu/2) =$
$(1-\sqrt{1-\alpha})/\alpha$, where $\alpha = R\mu/2$.

Hence $S^* = (1+Q+q)(1-\sqrt{1-\alpha})/\alpha - Q$ and
$S^*\pi = (\pi - W+C-RK_T-w+c-rk_T)(1-\sqrt{1-\alpha})/\alpha + W - C + RK_T.$

M.11.6 Existence of a Solution

As noted on p. 117, an S*-game will have a solution in the continuous case if the following conditions hold:

1. $F^*(0)+f^*(0) \geqslant \mu$. For pay-off function 1 this implies that

$$(1-e^{-RZ})/R+(1-e^{-rZ})/r = \int_0^Z e^{-Rt}dt + \int_0^Z e^{-rt}dt = \int_0^Z (e^{-Rt}+e^{-rt})dt \geqslant \mu.$$

[36] The root $e^{-R(Z-T)} = (-2-\sqrt{4-4(2R\mu-3)})/2$ is ruled out since $e^{-R(Z-T)}$ must be a positive, real number.

2. There exists a T such that $F(T) > 2F(T+\Delta t)$ and $f(T) > 2f(T+\Delta t)$. Since $F(Z) = 0$ for this function, we can set $T+\Delta t = Z$[37] and deduce that $F(T+\Delta t) = 0$. Due to S_2^*, $F(T) > 0$ and hence $F(T) > 2F(T+\Delta t)$. It can likewise be proved that $f(T) > 2f(T+\Delta t)$.

M.12 Analysis of Requirements $S_1^* - S_4^*$ of Pay-off Function 2

S_1^* holds, since $\nu(s,T) = \pi(s+q)f(T)+b$, where $q = 0$, $b = 0$ and $f(T) = e^{-\gamma' T} \int_T^{Z+\theta T} e^{-\bar{r}t}dt$, where $\gamma' = \gamma-\beta$ and $\bar{r} = r-\gamma > 0$.

As $\bar{r} \int_0^{Z+\theta T} e^{-\bar{r}t}dt = \bar{r}[-e^{-\bar{r}t}/\bar{r}]_T^{Z+\theta T} = e^{-\bar{r}T}-e^{-\bar{r}(Z+\theta T)} = e^{-\bar{r}T}-e^{-\bar{r}\theta T}E$,
where $E = e^{-\bar{r}Z}$,

$\bar{r}f = e^{-\gamma' T}(e^{-\bar{r}T}-e^{-\bar{r}\theta T}E) = e^{-(\gamma'+\bar{r})T}-e^{-(\gamma'+\bar{r}\theta)T}E = e^{-mT}-e^{-nT}E = \bar{m}-\bar{n}$,
where $\bar{m} = e^{-mT}$, $\bar{n} = e^{-nT}E$ and where $m = \gamma'+\bar{r}$ and $n = \gamma'+\bar{r}\theta$.
$\bar{r} = r-\gamma > 0 \Rightarrow m = \gamma'+\bar{r} > \gamma'+\bar{r}\theta = n$, since $\theta < 1$.

S_2^* holds: $\bar{r}f' = (e^{-mT}-e^{-nT}E)' = -me^{-mT}+ne^{-nT}E = -(m\bar{m}-n\bar{n})$
As $\bar{r} \geqslant \bar{r}(1-\theta) = \bar{r}-\bar{r}\theta = \gamma'+\bar{r}-(\gamma'+\bar{r}\theta) = m-n$.
$e^{-(m-n)T} \geqslant e^{-\bar{r}T} > e^{-\bar{r}Z} = E \Rightarrow e^{-mT} > e^{-nT}E$ i.e. $\bar{m} > \bar{n}$ and
as $m > n$, $m\bar{m} > n\bar{n}$, i.e. $0 > -(m\bar{m}-n\bar{n})$, i.e. $\bar{r}f' < 0$, i.e. as $\bar{r} > 0, f' < 0$

S_3^* holds: $\bar{r}f'' = (\bar{r}f')' = (-me^{-mT}+ne^{-nT}E)' = m^2e^{-mT}-n^2e^{-nT}E = m^2\bar{m}-n^2\bar{n}$.
As $\bar{m} > \bar{n}$ and $m > n$, $\bar{r}f'' > 0$, i.e. $f'' > 0$.

S_4^* holds: $(m-n)^2 > 0 \Rightarrow m^2-2mn+n^2 > 0 \Rightarrow m^2+n^2 > 2mn \Rightarrow$
$(m^2+n^2)\bar{m}\bar{n} > 2\bar{m}\bar{n}mn \Rightarrow m^2\bar{m}^2 - 2\bar{m}\bar{n}mn + n^2\bar{n}^2 > m^2\bar{m}^2 - \bar{m}\bar{n}(m^2+n^2)+n^2\bar{n}^2$
$\Rightarrow (-(m\bar{m}-n\bar{n}))^2 > (\bar{m}-\bar{n})(m^2\bar{m}-n^2\bar{n}) \Rightarrow (f')^2 > ff'' \Rightarrow df^*/dT < 0$ (cf. point 5, p. 344).

M.13 Analysis of Pay-off Function 3

M.13.1 Analysis of Requirements $S_1^* - S_4^*$

$\nu(s,t) = s\pi e^{-\bar{r}(T)T}k$, where
$\bar{r}(T) = r-\beta(T)$ and k is a constant, can be written as $\pi(s+q)f(T)+b$, where $q = 0$, $f(T) = e^{-\bar{r}(T)T}$ and $b = 0$, implying that S_1^* holds.

Setting $h = h(T) = T\bar{r}(T)$: $f(T) = e^{-h}$.

[37] By assuming that $\_ = (Z+\Delta t)/\Delta t$, we allow bids to be made at time Z and the pay-off $V(S,Z)$ is defined.

The following assumptions are made with respect to $\bar{r} = \bar{r}(T)$:

1. $\bar{r} > 0$.
2. $\bar{r}' > 0$, implying, since $\bar{r} > 0$, that $\bar{r}+\bar{r}'T = (\bar{r}T)' = h' > 0$.
3. $\bar{r}'' \geqslant 0$, implying that $2\bar{r}' + T\bar{r}'' = (\bar{r} + \bar{r}'T)' = (\bar{r}T)'' = h'' > 0$.
4. $(\bar{r} + \bar{r}'T)^2 \geqslant 2\bar{r}' + T\bar{r}''$, i.e. that $(h')^2 \geqslant h''$.

We note that it is sufficient, but *not* necessary, to assume that $\bar{r}^2 \geqslant 2\bar{r}'$ and $\bar{r}'' = 0$, in order for 4 to hold.
We now can determine that

S_2^* holds, since $h' > 0 \Rightarrow e^{-h}h' > 0 \Rightarrow 0 > -h'e^{-h} = (e^{-h})' = f'$.

S_3^* holds, since $(h')^2 \geqslant h'' \Rightarrow 0 \leqslant (h')^2 e^{-h} - h''e^{-h} = (-h'e^{-h})' = f''$.

S_4^* holds, since $h'' > 0 \Rightarrow 0 > -h''/(h')^2 = (1/h')' = (e^{-h}/h'e^{-h})' = (f/-f') = df^*/dT$.

We next note that if $\bar{r}(T) = a+bT/2$, where $a > 0$ and $b > 0$, then $\bar{r} > 0$, $\bar{r}' = b/2$ and $\bar{r}'' = 0$, i.e. assumptions 1, 2 and 3 above hold. Furthermore, if $a^2 \geqslant b$ then $(a+bT/2)^2 > b$, i.e. $\bar{r}^2 \geqslant 2\bar{r}'$ holds and hence, also assumption 4.

This implies in turn that if $a = r-\beta_0 > 0$, $b/2 = \beta'$, and $(r-\beta_0)^2 > 2\beta'$, then S_2^* – S_4^* hold.

M.13.2 Determining the Solution

T^* is given by $f^*(T)+F^*(T) = \mu$ (see p. 143), i.e. with $f^* = e^{-h}/h'e^{-h} = 1/h'$, by $1/h'(T) + 1/H'(T) = \mu$.

For $\bar{r}(T) = a+bT/2$, $h = \bar{r}(T)T = aT+bT^2/2$ and $h' = a+bT$, T^* is given by $1/(a+bT) + 1/(A+bT) = \mu$.

We develop this further into:

$$\mu(A+bT)(a+bT) - (A+bT) - (a+bT) = 0 \Rightarrow$$
$$\mu(Aa+AbT+abT+b^2T^2) - (a+A) - 2bT = 0 \Rightarrow$$
$$b^2T^2 + ((A+a)b - 2b/\mu)T + Aa - (a+A)/\mu = 0 \Rightarrow \text{(setting } \bar{a} = A+a \text{ and } 1/\mu = \bar{\mu})$$
$$T^* = (-(\bar{a}b-2b\bar{\mu}) \pm \sqrt{(\bar{a}b-2b\bar{\mu})^2 - 4b^2(Aa-\bar{a}\bar{\mu})})/2b^2 =$$
$$= (-\bar{a}+2\mu \pm \sqrt{(\bar{a}-2\bar{\mu})^2 - 4(Aa-\bar{a}\bar{\mu})})/2b = (-\bar{a}/2 + \bar{\mu} \pm \sqrt{(\bar{a}/2-\bar{\mu})^2 - Aa + \bar{a}\bar{\mu}})/b.$$

Since $(\bar{a}/2-\bar{\mu})^2 - Aa + \bar{a}\bar{\mu} = \bar{a}^2/4 - \bar{a}\bar{\mu} + \bar{\mu}^2 - Aa + \bar{a}\bar{\mu} = \bar{a}^2/4 + \bar{\mu}^2 - Aa =$
$= (a^2+2Aa+A^2)/4 + \bar{\mu}^2 - Aa = (a^2-2Aa+A^2)/4 + \bar{\mu}^2 = ((a-A)/2)^2 + \bar{\mu}^2 =$

$=\mu^2(((a-A)/2\bar{\mu})^2+1) = \bar{\mu}^2(\alpha^2+1)$, where $\alpha = (a-A)/2\bar{\mu} = (r-\beta_0-R+\beta_0)\mu/2 = (r-R)\mu/2$,

$T^* = (-\bar{a}/2+\bar{\mu}\pm\bar{\mu}\sqrt{\alpha^2+1})/b$ and
$A+bT^* = A-(a+A)/2+\bar{\mu}\pm\bar{\mu}\sqrt{\alpha^2+1} = \bar{\mu}-(a-A)/2\pm\bar{\mu}\sqrt{\alpha^2+1} = \bar{\mu}(1-\alpha\pm\sqrt{\alpha^2+1})$.

Hence $S^* = F^*(T^*)/\mu$[38] $= 1/\mu(A+bT^*) = 1/\mu\bar{\mu}(1-\alpha\pm\sqrt{\alpha^2+1}) = 1/(1-\alpha\pm\sqrt{\alpha^2+1}) =$
$= 1/(1-\alpha+\sqrt{\alpha^2+1})$.[39]

M.13.3 Proof that $\partial S^*/\partial\mu > 0$ and $\partial S^*/\partial R < 0$

$\alpha^2+1 > \alpha^2 \Rightarrow \sqrt{\alpha^2+1} \geqslant \alpha \Rightarrow 1 > \alpha/\sqrt{\alpha^2+1} \Rightarrow 0 < 1-\alpha/\sqrt{\alpha^2+1} =$
$= 1-(2\alpha)(1/2)(1/\sqrt{\alpha^2+1}) = d(-1+\alpha-\sqrt{\alpha^2+1})/d\alpha = -d(1-\alpha+\sqrt{\alpha^2+1})/d\alpha \Rightarrow$
$0 < (-d(1-\alpha+\sqrt{\alpha^2+1})/d\alpha)/(1-\alpha+\sqrt{\alpha^2+1})^2 = d(1/(1-\alpha+\sqrt{\alpha^2+1}))/d\alpha = dS^*/d\alpha$.

Since $d\alpha/d\mu = (r-R)/2 > 0$, $\partial S^*/\partial\mu = (dS^*/d\alpha)(d\alpha/d\mu) > 0$ and
since $d\alpha/dR = -\mu/2 < 0$, $\partial S^*/\partial R = (dS^*/d\alpha)(d\alpha/dR) < 0$.

M.13.4 Existence of a Solution

1. Condition 1 (on p. 117) that $F^*(0)+f^*(0) \geqslant \mu$ requires that $1/A+1/a \geqslant \mu$, i.e. that $1/(R-\beta_0)+1/(r-\beta_0) \geqslant \mu$.

2. Since b and Δt are not infinitely small (cf. pp. 127 and 151), $bT\Delta t$ will for some finite value of T, reach $\ln 2$ and $e^{bT\Delta t}$ hence 2. With $j = T/\Delta t$ (cf. p. 114) and setting $b' = (b/2)(\Delta t)^2$ (i.e. $b'j^2 = (b/2)(\Delta t)^2T^2/(\Delta t)^2 = (b/2)T^2$ and $b'j = (b/2)(\Delta t)^2 T/\Delta t = (b/2)T\Delta t$) and $A' = A\Delta t$ (i.e. $A'j = AT\Delta t/\Delta t = AT$), this implies that
$2 = e^{b'j} < e^{A'+2b'j+b'} = e^{-(A'+b'j)j}e^{(A'+b'j)j+b'j+A'+b'j+b'} =$
$= e^{-(A+b'j)j}/e^{-(A'+b'(j+1))(j+1)} = e^{-A'j-b'j^2}/e^{-A'(j+1)-b'(j+1)^2} =$
$= e^{-AT-(b/2)T^2}/e^{-A(T+\Delta t)-(b/2)(T+\Delta t)^2} = F(T)/F(T+\Delta t)$.

Hence $F(T) > 2F(T+\Delta t)$, i.e. condition 1 on p. 117 is fulfilled.

[38] When $Q = q = 0$, $S^* = F^*(T^*)/\mu$; cf. p. 270.
[39] $r > R \Rightarrow \alpha > 0 \Rightarrow 1-2\alpha+\alpha^2 < 1+\alpha^2 \Rightarrow (1-\alpha)^2 < \alpha^2+1 \Rightarrow 1-\alpha < \sqrt{\alpha^2+1} \Rightarrow$
$1-\alpha-\sqrt{\alpha^2+1} < 0 \Rightarrow 1/(1-\alpha-\sqrt{\alpha^2+1}) < 0$. Since $S^* \geqslant 0$, $1/(1-\alpha-\sqrt{\alpha^2+1})$ is ruled out.

M.14 Games with Many Bidding Periods in Each Pay-off Period

M.14.1 A Two-alternative Game

A game (6,7) will be studied here.

Step 1.1 First, we study the case where H bids in the last b.p. (bidding period) of p.p. (pay-off period) 3 and where 3 = C(6) and 2 = u(6), with the pay-offs referring to the pay-off periods.

If H accepts 6 in the last b.p. of p.p. 3, he gets $\overline{6}_3$ and if he insists on 7 the game will continue into p.p. 4 and H will at best get $\overline{7}_4$. Since 3 = C(6), i.e. $\overline{6}_3 > \overline{7}_4$, H will accept 6 in the last b.p. of p.p. 3.

Since L knows this, he will *not* accept 7 in any b.p. in p.p. 3, since by insisting on 6 he can get H to accept this. Hence, if L – when bidding in p.p. 2 – insists on 6 in every b.p. he can be assured of $\underline{6}_3$. Since 2 = u(6), i.e. $\underline{6}_3 > \underline{7}_2$ L would prefer insisting on 6 in every b.p. of p.p. 2 and get $\underline{6}_3$ to accepting 7 and get $\underline{7}_2$. Realizing this, H will accept 6 in p.p. 2 and get $\overline{6}_2$, rather than obtain $\overline{6}_3$ by insisting on 7 in every b.p. of p.p. 2.

This means that in each b.p. of p.p. 1, L in turn will insist on 6, thereby getting $\underline{6}_2$ rather than accepting $\underline{7}_1$, since 2 = u(6) ⇒ 1 = u(6), i.e. $\underline{7}_1 < \underline{6}_2$. Realizing this H will accept 6 in p.p. 1 rather than in p.p. 2.

Conclusion: If 3 = C(6) and 2 = u(6) and H bids in the last b.p. of p.p. 3, H will accept 6 in p.p. 1.

Step 1.2. By substituting i' for 3 and continuing backwards in the same manner, it can be deduced somewhat more generally that, if H bids in the last b.p. of p.p. i', then i' = C(6) and $i'-1$ = u(6) imply that H will accept 6 in the first p.p.

Step 1.3. Next we look at the other order of bidding, namely when L bids in the last b.p. of p.p. 3 and H hence bids in the last b.p. of p.p. 4. On the basis of the assumption that 4 = C(6) and 3 = u(6) and by substituting period 4 for i' in step 1.2, it can be deduced that H accepts 6 and L insists on 6 in p.p. 1.

Step 1.4. Due to steps 1.1 and 1.3, and on the basis of the assumption that 3 = Cu(6) – which implies that 2 = u(6) and 4 = C(6) – it can be deduced that H will accept 6 in the first p.p., regardless of whether H bids in the last b.p. of p.p. 3 or p.p. 4 or whether H bids in the last b.p. of both p.p. 3 *and* p.p. 4.

Step 1.5. When generalized to a game (x,x+1), the conclusion in step 1.4 is that

$i = \mathrm{Cu}(x)$ implies that H accepts x in p.p. 1, provided H bids in the last b.p. of p.p. i and/or p.p. $i+1$.

M.14.2 A Three-alternative Game

Just to show that the procedure presented above is *not* limited to two-alternative games, a three-alternative game (5,7) will also be presented. It suffices for our purpose here to analyze the case where H bids in the last b.p. of p.p. 1 and 3 and L in the last b.p. of p.p. 2. We assume that $1 = \mathrm{Cu}(5)$ and $2 = \mathrm{Cu}(6)$.

Step 2.1. On the basis of $2 = \mathrm{C}(6)$, implying that $3 = \mathrm{C}(6)$, and step 1.1 in the analysis of the two-alternative game (6,7) above the following is deduced: If L bids 6 in p.p. 2, H will accept this in p.p. 3 at the latest.

Step 2.2. Since L can ensure himself of $\underline{6}_3$ (see step 2.1), $2 = \mathrm{u}(6)$, i.e. $\underline{6}_3 > \underline{7}_2$ implies that L will insist on 6 throughout p.p. 2 and get $\underline{6}_3$ rather than accept 7 and get $\underline{7}_2$.

Step 2.3. We study H's choice in his last b.p. of p.p. 1, when L has insisted on 5 and H on 7 up until this bid.

H can obtain $\overline{5}_1$ by accepting 5.

H can at best obtain $\overline{6}_2$ by bidding 6.

H can at best obtain $\overline{7}_3$ or $\overline{6}_2$ by bidding 7 (see step 2.2).

Since $1 = \mathrm{C}(5)$ and $2 = \mathrm{C}(6)$ imply that $\overline{5}_1 > \overline{6}_2 > \overline{7}_3$, H will accept 5.

Step 2.4. Realizing that he can obtain $\underline{5}_1$ by insisting on 5 throughout p.p. 1, L will do so.

Conclusion: H will accept 5 in p.p. 1 just as in the basic model.

M.14.3 Case when the Selection of the Party to Bid in the Last Bidding Period is Made at Random

We take a brief look at the case where nothing particular is assumed about who bids in the last b.p. of each p.p. except that this is a random variable with equal probability for H and L.

In step 1.5 on p. 276 it was deduced that if $i = \mathrm{Cu}(x)$ and H bids in the last b.p. of p.p. i and/or p.p. $i+1$, then H accepts x in p.p. 1. Next it can be determined that the probability of L bidding in the last b.p. of *both* p.p. i and p.p. $i+1$ is $(1/2)^2 = 1/4$. Hence the probability of H bidding in the last b.p. of p.p. i and/or $i+1$ is 3/4.

Furthermore, as regards S′-games with $3 = \mathrm{Cu}(x)$, $S'_2 - S'_3$ imply that $3 = C(x+5)$, $4 = C(x+5) \ldots 9 = C(x+5)$ and $2 = u(x+5)$, $3 = u(x+5) \ldots 8 = u(x+5)$. Then on the basis of step 1.1 above it can be deduced that in the game $(x+5,x+6)$, H accepts $x+5$ in p.p. 1 provided H bids in the last b.p. of p.p. 3, 4 . . and/or 9. The probability of this is $1-(1/2)^7 > 0.99$. When both parties realize this, it appears very likely that H will accept $x+5$ as early as period 1. Finally we note that if n is large, $x+5$ will constitute a good approximation of x.

M.15 Proof that P and P' Imply that $(1,n)_0 = x_1$, if All Periods 1, . . . i' (where $i' \leqslant i-3$) are Played with Simultaneous Bidding

Let j be any period played with s.b. (simultaneous bidding) and $i' \leqslant i-3$ the last period played with s.b.

Step 1: (1) $j = u(x)$ and (2) $(x,x+k)_j = x_{j+1}$
imply that $(x,x+k)_{j-1} = x_j$.[40]

This can be proved with the aid of the following game tree:

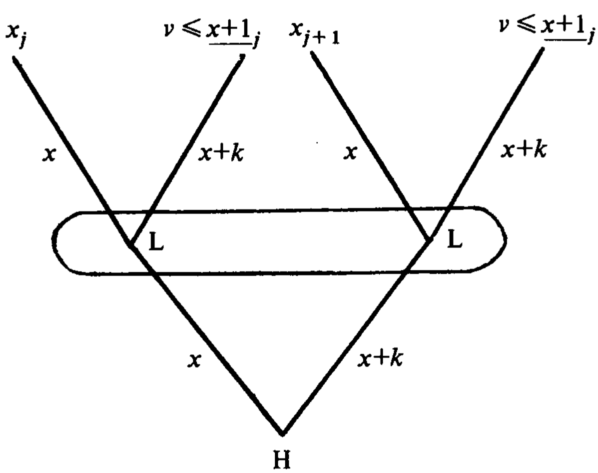

Figure 35 The choice in period j, characterized by simultaneous bidding

[40] In M.15 j without a bar is used as an index to a game to indicate that both parties bid in period j.

First, we look at the pay-offs at the end nodes.

If both bid x an agreement is reached on x_j.

If L bids $x+k$ he can at best get $\underline{x+1}_j$.

If H bids $x+k$ and L bids x an agreement is reached on x_{j+1} due to assumption (2) above.

Since $\underline{x}_j > \underline{x+1}_j$ and $j = \mathrm{u}(x) \Rightarrow \underline{x+1}_j < \underline{x}_{j+1}$, L will bid x regardless of what H bids. Hence by bidding x, H will obtain $\bar{x}_j$ and by bidding $x+k$ obtain $\bar{x}_{j+1}$. Since $\bar{x}_j > \bar{x}_{j+1}$ H also bids x, implying that $(x,x+k)_{j-1} = x_j$.

Step 2: $(x,x+k)_{i'} = x_{i'+1}$.

If no agreement is reached in i' a game with alternating bidding starts in period $i'+1$. The question then is who starts bidding. This is determined after the bids in i' have been delivered and, due to I_6, one specific party starts if both H and L want him to start. In the situation $(x,x+k)_{i'}$, H starts bidding, since due to T_{12} $\underline{(x,x+k)}_{i'} = x_{i'+1}$, while $\overline{(x,x+k)}_{i'} = x_{i'+2}$. Hence $(x,x+k)_{i'} = x_{i'+1}$.

Step 3: $\boldsymbol{P}$ and $\boldsymbol{P}'$ imply for every j that $(x,x+k)_j = x_{j+1}$.

3.1 $\boldsymbol{P} \Rightarrow i = \mathrm{u}(x) \Rightarrow$, due to S_2, $j = \mathrm{u}(x)$, i.e. assumption 1 in step 1 holds.

3.2 Due to step 2 $(x,x+k)_{i'} = x_{i'+1}$.

Hence assumption (2) in step 1 holds for $j = i'$.

> Due to 3.1 and 3.2 we conclude on the basis of step 1 that $(x,x+k)_j = x_{j+1}$, i.e. assumption 2 in step 1, holds for $j = i'-1$. Using step 1 again, this implies that the same conclusion in turn holds for $j = i'-2$. In this way we can go backwards step by step to $j = 1$, deducing that $(x,x+k)_j = x_{j+1}$ holds for every $j \leqslant i'$.

Step 4: As the mirror picture of the conclusion in step 3 $\boldsymbol{P}'$ implies for every j that $(x-m,x)_j = x_{j+1}$.

Step 5: $\boldsymbol{P}$ and $\boldsymbol{P}'$ imply that in the situation $(x-m,x+k)_{j-1}$, H will bid x or $x+k$, i.e. H will *not* bid $x-m$ in any period j. This is shown by the following matrix with H's pay-offs in the cells.

		L		
		$x-m$	x	$x+k$
H	$x-m$	$V \leqslant \overline{x-1}_j$	$V < \overline{x}_j$	$(1/2, \overline{x-m}_j; 1/2, \overline{x+k}_j)$
	x	$\overline{x}_{j+1}$	$\overline{x}_j$	$(1/2, \overline{x}_j; 1/2, \overline{x+k}_j)$

Table 15 H's pay-offs from x and $x-m$

The pay-offs in the different cells can be commented on as follows:

If H bids $x-m$:

L's bid $x-m$ gives H at best $\overline{x-1}_j$.
L's bid x *gives* H a lottery *with* $P = 1/2$ (i.e. a probability of 1/2) of $\overline{x-m}_j$; and $P = 1/2$ of $\overline{x}_j$ (cf. p. 157) implying, due to B_{12}, that $V < \overline{x}_j$.
L's bid $x+k$ gives H a lottery with $P = 1/2$ of $\overline{x-m}_j$ and $P = 1/2$ of $\overline{x+k}_j$. This is written as $(1/2, \overline{x-m}_j; 1/2, \overline{x+k}_j)$.

If H bids x:

L's bid $x-m$ leads to $(x-m,x)_j = x_{j+1}$ (step 4).
L's bid x implies an agreement on x_j.
L's bid $x+k$ gives H a lottery with $P = 1/2$ of $\overline{x}_j$, $P = 1/2$ of $\overline{x+k}_j$.

Since $j = U(x-1)$, i.e. $\overline{x}_{j+1} > \overline{x-1}_j$. H's strategy x dominates his strategy $x-m$.

Step 6: As the mirror picture of step 5 we deduce that P and P' imply that in the situation $(x-m,x+k)_{j-1}$, L will bid $x-m$ or x.

Step 7: $(\overline{x-m,x+k})_1 \leqslant \overline{x}_2$ and $(\underline{x-m,x+k})_1 \leqslant \underline{x}_2$.

Since H bids x or $x+k$ in $1 \ldots i'$ (step 5), L can at best get an agreement on x in $2 \ldots i'+1$ or $(\underline{x-m,x+k})_{i'} = \underline{x}_{i'+2}$ (due to theorem T_{13}). Hence $(\underline{x-m,x+k})_1 \leqslant \underline{x}_2$. Likewise we can prove that $(\overline{x-m,x+k})_1 \leqslant \overline{x}_2$.

Step 8: $(1,n)_0 = x_1$.

As noted in steps 5 and 6, H will bid x or $x+k$ and L $x-m$ or x in any period $j \leqslant i'$. Hence the following choice matrix is obtained for period 1:

		L: $x-m$	L: x
H	x	x_2	x_1
	$x+k$	$v \leqslant \underline{x}_2$ / $V \leqslant \bar{x}_2$	x_2

Table 16 Choice matrix of $(1,n)_0$

The outcome of cell $(x-m,x)$ is determined in step 4, of cell $(x,x+k)$ in step 3 and the pay-offs of cell $(x-m,x+k)$ in step 7.

Since $\bar{x}_1 > \bar{x}_2$ and $\underline{x}_1 > \underline{x}_2$, H's choice x dominates $x+k$ and L's choice x dominates $x-m$. Hence an agreement is reached on x in period 1.

M.16 Proof that $(1,n)_0 = x_2$ for S-games with P and P' if Every Period $j \leqslant i-3$ is Played with Bad-faith Bargaining

Let j be any period such that $j \leqslant i'$, where $i' \leqslant i-3$. Let us also assume that every such period j is played with b.b.(bad-faith bargaining). We note that P and P' imply $j = \mathrm{U}(x-1)\mathrm{u}(x)$.

Step 1.

1.1 $(x-m,x)_{\bar{j}} = x_{j+1}$
1.2 $(x, x+k)_{\bar{j}} = x_{j+2}$
1.3 $(x-m,x+k)_{\bar{j}} = x_{j+2}$
1.4 $\overline{(x-m,x-m'')}_{\bar{j}} < \bar{x}_{j+1}$, where $m'' \neq m$ and $m'' = 1 \ldots x-1$

imply

1.5 $(x-m,x+k)_{\underline{j-1}} = x_{j+1}$, $(x-m,x)_{\underline{j-1}} = x_{j+1}$ and $\underline{(x-m,x-m'')}_{\underline{j-1}} < \underline{x}_j$.
1.6 $(x,x+k)_{\underline{j-1}} = x_j$.

Proof of 1.5:

If L has bid $x-m$ in period $j-1$ H chooses – in period j – between:

$x-m$, giving H $\overline{x-m}_j \leqslant \overline{x-1}_j$. Since $j = \mathrm{U}(x-1)$, i.e. $\overline{x-1}_j < \bar{x}_{j+1}$, H's bid $x-m$ gives H less than $\bar{x}_{j+1}$

$x-m''$, giving H a $V < \bar{x}_{j+1}$ (1.4)

x leading to $(x-m,x)_{\bar{j}} = x_{j+1}$ (1.1)

$x+k$ leading to $(x-m,x+k)_{\bar{j}} = x_{j+2}$ (1.3)

Due to S_1, H chooses x, leading to x_{j+1}.

This implies in turn that $\underline{(x-m,x-m'')}_{\underline{j-1}} = \underline{x}_{j+1} < \underline{x}_j$

Proof of 1.6:

In the situation $(x,x+k)_{\underline{j-1}}$, H chooses – in period j – between:

$x-m$ giving H less than $\bar{x}_j$, due to I_{12}.

x leading to x_j

$x+k$ leading to $(x,x+k)_{\bar{j}} = x_{j+2}$ (1.2).

Due to S_1, H chooses x, leading to x_j.

Step 2. The following is obtained as the mirror picture of step 1:

2.1 $(x,x+k)_j = x_{j+1}$
2.2 $(x-m,x)_{\underline{j}} = x_{j+2}$
2.3 $(x-m,x+k)_{\underline{j}} = x_{j+2}$
2.4 $\underline{(x+k'',x+k)}_{\underline{j}} < \underline{x}_{j+1}$ where $k'' \neq k$ and $k = 1 \ldots n-x$

imply

2.5 $(x-m,x+k)_{\overline{j-1}} = x_{j+1}$, $(x,x+k)_{\overline{j-1}} = x_{j+1}$ and $(x+k'',x+k)_{\overline{j-1}} < \bar{x}_j$
2.6 $(x-m,x)_{\overline{j-1}} = x_j$.

Step 3. Theorems $T_{11} - T_{13}$ can now be used to deduce that, for $i' \leqslant j \leqslant i-3$: $(x-m,x+k)_{\bar{j}} = x_{j+2}$, $(x,x+k)_{\bar{j}} = x_{j+2}$ and $(x-m,x)_{\bar{j}} = x_{j+1}$. Furthermore $\overline{(x-m,x-m'')}_{\bar{j}} < \overline{x-1}_j < \bar{x}_{j+1}$. Hence 1.1 – 1.4 hold for $j = i'$.

Step 4. Assuming that $i' \in H$, steps 1 and 3 can be applied to deduce that 1.5 – 1.6 hold for $j = i'$, i.e. for $j-1 = i'-1$. Since 1.5 – 1.6 for $j-1 = i'-1$ imply that 2.1 – 2.4 hold for $j = i'-1$ we deduce that 2.5 – 2.6 hold for $j = i'-1$. This is equivalent to assuming that 1.1 – 1.4 hold for $j = i'-2$. Due to step 1, we deduce that 1.5 – 1.6 hold for $j = i'-2$ implying in turn that 2.1 – 2.4 hold for $j = i'-3$, etc.

Going backwards in this manner, it can be deduced that $(x-m,x+k)_j = x_{j+2}$ for every $j \leqslant i'$ and hence that $(1,n)_0 = x_2$, when $i' \epsilon H$. Using the mirror picture technique, the same conclusion can be deduced for the case where $i' \epsilon L$.

M.17 Proof that in Period $j-1$ of an S′-game, L will n̲o̲t Bid an Alternative which would Give H a Profit Smaller or Equal to $\bar{0}_j$

Let us study an S′-game $(y',y)_{\overline{j-1}}$ and assume that according to the basic model, there is a unique alternative y^* leading to the highest pay-off. We add an alternative y'', such that $y'' = y' - \epsilon$ and $\bar{y}''_j \leqslant \bar{0}_j$, to the alternatives of this game, giving us the game $(y'',y)_{\underline{j-1}}$. Next we distinguish between the following three cases[41]:

1. $\bar{y}''_j < \bar{0}_j$, implying that H will break up the game rather than accept y'' in period j. This means that L cannot obtain a better agreement by bidding y'' instead of y'. This only allows him to delay an agreement and he will therefore abstain from it.

2. $\bar{y}''_j = \bar{0}_j$ and – according to the basic model – $(y'',y^*)_{\bar{j}} = y^*_{j+1}$, where $\bar{y}^*_{j+1} > \bar{0}_j$. Then H will *not* be willing to accept $\bar{y}''_j$ and L cannot improve his pay-off by bidding y''.

3. $\bar{y}''_j = 0_j$ and $(y',y)_{\overline{j-1}} = y'_j$, according to the basic model. If L bids y'' in period $j-1$, H will be indifferent between bidding y'' and 0 in period j. L will therefore, due to B_{11}, assign a positive – not extremely small – probability to the event that H will break up the game rather than accept y''. If L bids y', L can be assured of an agreement on y'_j. Then in period $j-1$, L can choose between obtaining $\underline{y}'_j$ for certain by bidding y' or a lottery with some probability of $\underline{y}''_j$ and some (not very small) probability of $\underline{0}_j$ by bidding y''. With $\underline{y}''_j$ only marginally better than $\underline{y}'_j$ but $\underline{0}_j$ significantly inferior, L will – due to B_{12} – bid y' rather than y''.

Hence y'' will not be bid in any of these three cases.

M.18 Proof that T_{11} Holds also when H can Refrain from Bidding Intermediate Alternatives

Step 1

The conclusions of the basic model obviously hold in the two-alternative case since there are no intermediate alternatives. Hence T_5 holds.

[41] It appears reasonable to assume that this is an exhaustive list of possibilities. However, the proof that this assumption is true is left for future research.

Step 2

The *three*-alternative game is studied with the aid of Figure 14 (p. 82). H can abstain from bidding 6 in period 1. This will not affect the conclusion that H chooses 5 (cf. also p. 177). Hence theorem T_7 still holds.

Step 3

Theorem T_8 holds. Steps 1 and 2 of the deduction of theorem T_8 (p. 83) refer to two and three-alternative games and are thus unaffected. Step 3 is studied with the aid of Figure 15 which shows that the elimination of H's bids 6 or 7 will not affect the conclusion that $\overline{(5,8)}_{\bar{l}} < \bar{5}_1$. Step 4 relies solely on steps 1–3 and thus is not affected either.

Step 4

Theorem T_9 holds. In Figure 16 (p. 87) we see that H will get a $V < \bar{7}_3$ from L's bids 5 and 6 in period 2, even if H does not bid 6 or 7 in period 3 or if 7+*k* only represents some of the alternatives 8 . . . *n*.

Step 5

T_5', T_7', T_8' and T_9' (p. 88) are also unaffected by the exclusion of some alternatives between 5 and H's most preferred alternative.[42]

T_5' is unaffected, since there are no alternatives between 5 and 6.

T_7' is unaffected, since
$\overline{(5-m, 7)}_{\underline{2}} \leqslant \max\,[\bar{4}_3, \bar{7}_4]$, even if H does not bid 6 in period 3.

T_8' is unaffected, since
$\overline{(5-m, 8)}_{\underline{2}} \leqslant \max\,[\bar{4}_3, \bar{8}_4]$ even if H does not bid 6 or 7 in period 3.

T_9' is not affected, since by substituting $5-m'$ for 5 in Figure 16 (p. 87) it can be seen that the exclusion of H's bids 6 or 7 or the limitation of the number of alternatives, represented by the branch 7+*k*, will (just as in step 4 above) not affect the conclusion that $\overline{(5-m', 8+k)}_{\bar{1}} < \bar{5}_1$.

Step 6

Since theorems T_5, T_7–T_9 as well as T_5', T_7'–T_9' still hold, the conclusions on pp.

[42] Since H's bid $5-m$ in period 3 will lead to $V < \bar{5}_1$, T_5' and $T_7'-T_9'$ are also unaffected by the exclusion of some alternatives between L's most preferred alternative and 5.

89 – 92 still hold, implying in turn that theorem T_{10} holds, i.e. that $(x,x+k)_{\underline{i-1}} = x_i$.

Step 7

The conclusion on p. 94 – that $(x,x+k)_{\underline{j-1}} = x_j$ implies that $(x,x+k)_{\overline{j-2}} = x_j$ and $(x,x+k)_{\underline{j-3}} = x_{j-2}$ – is not affected, since it involves only the two alternatives x and $x+k$. This implies in turn that it can still be deduced from T_{10} that T_{11} holds.

M.19 Mergers

M.19.1 Proof that S_1^*–S_4^* Hold

Setting $r-\beta = \bar{r} > 0$, L's pay-off function $v(s,T)$ on p. 184 is, after elimination of $1-\tau$, written as

$$\int_0^T we^{-\bar{r}t}dt + s\int_T^Z (w+W+\pi)e^{-\bar{r}t}dt + s\int_Z^\infty (w+W)e^{-\bar{r}t}dt.$$

Setting $w+W+\pi = \pi^*$, $\int_Z^\infty (w+W)e^{-\bar{r}t}dt = (w+W)e^{-\bar{r}Z}/\bar{r} = k$, $\int_T^Z e^{-\bar{r}t}dt = g(T)$,

$\int_0^T we^{-\bar{r}t}dt = w/\bar{r} - we^{-\bar{r}T}/\bar{r} = w/\bar{r} - (we^{-\bar{r}T} - we^{-\bar{r}Z})/\bar{r} - we^{-\bar{r}Z}/\bar{r} = b' - w\int_T^Z e^{-\bar{r}t}dt =$
$b' - wg(T)$,

we obtain $v(s,T) = (s\pi^* - w)g(T) + sk + b' = \pi^*(s - w/\pi^*)g(T) + sk + b' =$
$= \pi^*(s+q)g(T) + sk + b'$, where $q = -w/\pi^*$.

Likewise we write H's pay-off function $V(S,T)$ as $\pi^*(S+Q)G(T)+SK+B'$,
where $G(T) = \int_T^Z e^{-\bar{R}t}dt$, $Q = -W/\pi^*$ and $K = (w+W)\, e^{-\bar{R}Z}/\bar{R}$.

Setting $y = \dfrac{N(S(\pi+W+w)-W)}{\pi} = \dfrac{N\left(\dfrac{S(\pi+W+w)-W}{\pi+W+w}\right)}{\dfrac{\pi+W+w-(W+w)}{\pi+W+w}} = \dfrac{N(S-W/\pi^*)}{1-W/\pi^*-w/\pi^*} =$

$= \dfrac{N(S+Q)}{1+Q+q}$, i.e. $N-y = N - \dfrac{N(S+Q)}{1+Q+q} = \dfrac{N(1+Q+q-S-Q)}{1+Q+q} = \dfrac{N(s+q)}{1+q+Q}$ and

$a = A = \pi^*(1+Q+q)/N$:

$$\pi^*(S+Q) = \frac{\pi^*(1+Q+q)N(S+Q)}{N(1+Q+q)} = Ay \text{ and } \pi^*(s+q) = \frac{\pi^*(1+Q+q)N(s+q)}{N(1+Q+q)} = a(N-y).$$

Furthermore $y = \dfrac{N(S+Q)}{1+Q+q} \Rightarrow NS = y(1+Q+q) - NQ \Rightarrow SK = y(1+Q+q)K/N - QK =$
$= yC'+B''$, where $C' = (1+Q+q)K/N$. Likewise, $N-y = \dfrac{N(s+q)}{1+Q+q} \Rightarrow sk = (N-y)c'+b''$,

where $c' = (1+Q+q)k/N$.

Hence $\pi^*(S+Q)G(T)+SK+B' = AyG(T)+yC'+B''+B' = Ay(G(T)+C'/A)+B =$
$= Ay(G(T)+C)+B = AyF(T)+B$, where $F(T) = G(T)+C$ and
$C = C'/A = N(1+Q+q)K/N\pi^*(1+Q+q) = K/\pi^* = (W+w)e^{-\bar{R}Z}/\bar{R}\pi^* = \pi^Q e^{-\bar{R}Z}/\bar{R}$,
where $\pi^Q = (W+w)/\pi^*$,

i.e. $F(T) = \int_T^Z e^{-\bar{R}t}dt + \pi^Q e^{-\bar{R}Z}/\bar{R} = (e^{-\bar{R}T} - (1-\pi^Q)e^{-\bar{R}Z})/\bar{R} = (e^{-\bar{R}T} - \lambda e^{-\bar{R}Z})/\bar{R}$,

where $\lambda = 1-\pi^Q = 1-(W+w)/\pi^* = 1-(W+w)/(W+w+\pi) = \pi/(W+w+\pi) > 0$.

Likewise $\pi^*(s+q)g(T)+sk+b' = a(N-y)g(T)+(N-y)c'+b'' = a(N-y)f(T)+b$,
where $f(T) = (e^{-\bar{r}T} - \lambda e^{-\bar{r}Z})/\bar{r}$.

We hence see that S_1^* holds for both parties' pay-off functions.

Next we deduce that:

S_2^* holds, since $F' = -e^{-\bar{R}T} < 0$.

S_3^* holds, since $F'' = \bar{R}e^{-\bar{R}T} \geqslant 0$.

S_4^* requires for H that $(F')^2 > FF''$, i.e. that $e^{-2\bar{R}T} > \bar{R}e^{-\bar{R}T}(e^{-\bar{R}T} - \lambda e^{-\bar{R}Z})/\bar{R}$ $= e^{-2\bar{R}T} - \lambda e^{-\bar{R}(T+Z)}$, i.e. that $\lambda e^{-\bar{R}(T+Z)} > 0$, which holds since $\lambda > 0$ and $Z > T$ is finite.

M.19.2 Determining the Solution

Since $f^*(T) = (e^{-\bar{r}T} - \lambda e^{-\bar{r}Z})/\bar{r}e^{-\bar{r}T} = (1-\lambda e^{-\bar{r}(Z-T)})/\bar{r}$, $F^*(T) + f^*(T) = \mu$ implies that $(1-\lambda e^{-\bar{R}(Z-T)})/\bar{R} + (1-\lambda e^{-\bar{r}(Z-T)})/\bar{r} = \mu$.

For $\bar{r} = 2\bar{R}$ and setting $Z-T = Z'$ we obtain: $(1-\lambda e^{-\bar{R}Z'})/\bar{R} + (1-\lambda e^{-2\bar{R}Z'})/2\bar{R} = \mu \Rightarrow$

$2-2\lambda e^{-\bar{R}Z'} + 1-\lambda e^{-2\bar{R}Z'} = 2\bar{R}\mu \Rightarrow 0 = (e^{-\bar{R}Z'})^2 + 2e^{-\bar{R}Z'} - (3-2\bar{R}\mu)/\lambda \Rightarrow$

$e^{-\bar{R}Z'} = -1 + \sqrt{1 + (3-2\bar{R}\mu)/\lambda}$, since $e^{-\bar{R}Z'} > 0$.

$y^*/N = F^*(T^*)/\mu = \dfrac{1-\lambda e^{-\bar{R}Z'}}{\bar{R}\mu} = (1+\lambda-\lambda\sqrt{1+(3-2\bar{R}\mu)/\lambda})/\bar{R}\mu$.

With $\lambda = 0.2$, $y^*/N = (1.2-0.2\sqrt{16-10\bar{R}\mu})/\bar{R}\mu$.

M.20 Solution of the Bertrand Game

We assume a finite number of prices with a difference ϵ between each price.

1. After the elimination of strategies discussed on p. 196 a reduced pay-off matrix – matrix 1 – remains, with $c+\epsilon \leqslant p \leqslant p^*$ and $C+\epsilon \leqslant P \leqslant P^*$, where $c = C$ and $p^* = P^*$.

2. Calling L's *highest* price in matrix i p_i and H's highest price P_i, each matrix i with $P_i = p_i \geqslant c+2\epsilon$ can be reduced to a matrix $i+1$ with $p_{i+1} = p_i - \epsilon$ and $P_{i+1} = P_i$, since $p_i - \epsilon$ dominates p_i.

2.1 For $P < p_i - \epsilon$: $\nu(p_i) = \nu(p_i - \epsilon) = 0$.

2.2 For $P = p_i - \epsilon$: $\nu(p_i - \epsilon) > 0 = \nu(p_i)$.

2.3 For $P = p_i$: $\nu(p_i - \epsilon) > \nu(p_i)$.
When $P = p$: $f'(p) < 0$ and $p - c \geqslant 2\epsilon \Rightarrow f(p-\epsilon) - f(p) > 0$ and $p - c - 2\epsilon \geqslant 0 \Rightarrow$
$0 < (f(p-\epsilon) - f(p))(p-c) + f(p-\epsilon)(p-c-2\epsilon) = 2f(p-\epsilon)\,(p-c-\epsilon) - f(p)(p-c)$.
Since for $P = p$: $\nu(p) = f(p)(p-c)/2$, i.e. $f(p)(p-c) = 2\nu(p)$,
and $f(p-\epsilon)(p-c-\epsilon) = \nu(p-\epsilon)$, this implies that $0 < 2\nu(p-\epsilon) - 2\nu(p)$,
i.e. that $\nu(p-\epsilon) > \nu(p)$.

3. Each matrix $i+1$ with $p_{i+1} = p_i - \epsilon$ and $P_{i+1} = P_i = p_i \geqslant C+2\epsilon$ can be reduced to a matrix $i+2$ with $p_{i+2} = p_i - \epsilon$ and $P_{i+2} = P_i - \epsilon$, since $P_i - \epsilon$ dominates P_i.

3.1 For $p < P_i - \epsilon$: $V(P_i) = V(P_i - \epsilon) = 0$.

3.2 For $p = P_i - \epsilon$: $V(P_i - \epsilon) > 0 = V(P_i)$.

4. By iteratively applying steps 2 and 3, matrix 1 is reduced to matrix 3, matrix 3 to matrix 5, etc. This process continues until we obtain a matrix i such that $p_i = c+\epsilon = P_i = C+\epsilon$.

M.21 Labor-management Bargaining

M.21.1 Proof that S_1^*–S_4^* Hold

1. $v(\omega,T) = \int_T^{Z+T} (1-\tau)\phi(T)M(p-c-\omega(1+\tau'))e^{-rt}dt$ and $V(\omega,T) = \int_T^{Z+T} \phi(T)M\omega e^{-Rt}dt$ where $\phi'(T) < 0$, $\phi''(T) \geqslant 0$ and $d^2(\log\phi(T))/dT^2 < 0$.

Setting $1+\tau' = \tau''$ and $1-\tau = \tau^*$:
$(1-\tau)M(p-c-\omega(1+\tau')) = \tau^*M(p-c)(1-\omega\tau''/(p-c))$.
Setting $y/N = \omega\tau''/(p-c)$, $a = \tau^*M(p-c)/N$ and $A = M(p-c)/\tau''N$, we obtain $\tau^*M(p-c-\omega\tau'') = a(N-y)$ and $M\omega = Ay$. Hence, setting $f(T) = \phi(T)e^{-rT}$ and $F(T) = \phi(T)e^{-RT}$, S_1^* is fulfilled.

2. $\phi' < 0 \Rightarrow 0 > e^{-rT}(\phi' - r\phi) = (\phi e^{-rT})' = f'$, i.e. S_2^* holds.

3. $\phi'' \geqslant 0$ and $\phi' < 0 \Rightarrow 0 \leqslant e^{-rT}(\phi'' - 2r\phi' + r^2\phi) = (e^{-rT}(\phi' - r\phi))' =$
$= (\phi e^{-rT})'' = f''$, i.e. S_3^* holds.

4. $d^2(\log\phi)/dT^2 < 0 \Rightarrow (\phi'/\phi)' < 0 \Rightarrow (\phi/\phi')' > 0 \Rightarrow$[43]
$(-\phi/\phi')' < 0 \Rightarrow (\phi')^2 > \phi\phi''$[44] $\Rightarrow (\phi')^2 - 2r\phi\phi' + r^2\phi^2 > \phi\phi'' - 2r\phi\phi' + r^2\phi^2 \Rightarrow$
$(\phi' - r\phi)^2 > \phi(\phi'' - 2r\phi' + r^2\phi) \Rightarrow (e^{-rT}(\phi' - r\phi))^2 > \phi e^{-rT}(\phi'' - 2r\phi' + r^2\phi)e^{-rT}$
$\Rightarrow (f')^2 > ff''$, i.e. S_4^* holds.

M.21.2 Determining the Solution

$y^*/N = F^*(T^*)/\mu$ (cf. p. 116) i.e. since $y/N = \omega\tau''/(p-c)$: $\omega^*\tau''/(p-c) =$
$= F^*(T^*)/\mu$ i.e. $\omega^* = (p-c)F^*(T^*)/\tau''\mu = ((p-c)/(1+\tau'))F^*(T^*)/\mu$

[43] Cf. p. 114 (footnote 28).
[44] Cf. p. 269.

General Appendix

G.1 Behavioristic Assumptions

Set $\boldsymbol{B}_1$ is comprised of B_1–B_7, set $\boldsymbol{B}_2$ of B_1–B_9, set $\boldsymbol{B}_3$ of B_1–B_{10} and set $\boldsymbol{B}_4$ of B_1–B_{12}.

B_1 *Preference relations*: On the basis of various factors affecting each outcome a party can define preference relations for pairs of outcomes (a, b). The party can say whether he prefers a to b, or b to a or whether he is indifferent between the two.

B_2 *Completeness*: When comparing any two outcomes a party can *always* say whether he prefers one to the other and if so, which one he prefers, or whether he is indifferent between them.

B_3 *Continuity* in preference relations: If outcome a is *almost* identical to outcome b and a party prefers outcome a to some outcome c, clearly different from a, then the party will not like b less than c.

B_4 *Transitivity*: If a party prefers a to b and b to c he will also prefer a to c. If he is indifferent between a and b and indifferent between b and c, he is also indifferent between a and c.

B_5 *Optimization*: A party will not select an alternative to which an outcome b can be assigned with certainty, if there exists a choice that with certainty leads to an outcome a and he prefers a to b.

B_6 *Information utilization*: Both players utilize all relevant information available to them.

B_7 *Deductive capacity*: Each party is able to carry out complicated logical deductions and use any available computational aid.

B_8 *Mutual knowledge of the other party's rationality*: H knows that L is rational (i.e. behaves according to assumption set $\boldsymbol{B}_1$) and L knows that H is rational.

B_9 *Information about preference relations*: Full information about the other party's preference relations, established according to assumption B_1, is available to each party.

B_{10} *Mutual knowledge about the other party's knowledge about oneself*: H knows what L knows about H, and L knows what H knows about L.

B_{11} *Uncertain choice under indifference*: If a party is indifferent between two alternatives, the other party will *not* regard the choice of a specific one of these as certain.

B_{12} *Probability dominance*: If a party prefers an outcome y to an outcome y', the party will prefer receiving y for certain to obtaining a lottery involving y and y' and with some (positive, not extremely small) probability that y' will occur.

G.2 Institutional Assumptions

Only the assumptions of the basic model are given here.

I_1 The potential bargaining process takes time, i.e. it is distributed over a number of periods. In each period *one* – and only *one* – party delivers a *bid*. This is a written or oral proposal for an agreement on a certain set of terms called an *alternative*.

I_2 An agreement on a certain alternative is reached in any one of the periods, if the party bidding in this period bids the same alternative as the other party has bid in some preceding period. As soon as an agreement has been reached, the game is over. If an agreement is not reached in a period which is not the last one, the game continues into the next period. If it is the last period, the game is discontinued.

I_3 In a certain period each party has information about
- (a) the value of the factors (e.g. profits) that can possibly affect his preferences with regard to each future agreement
- (b) what he himself has bid in all of his preceding periods and
- (c) what the other party has bid in all of his preceding period.

I_4 The parties alternate bidding.

I_5 Each party has complete information about the other party's pay-offs, implying that he can assign – to every possible agreement – a correct value of the different outcome factors relevant for the other party's preferences.

I_6 The parties themselves determine who starts bidding. This is done in a pre-bargaining game before the start of the actual bargaining game. Each party suggests who should start bidding. If both suggest the same one, this party will start.

I_7 The parties are bound to limit the set of alternatives in the bargaining game to some subset of alternatives with the following characteristics as regards every pair y, y' in the set: If H prefers an agreement on y in some period, then L prefers an agreement on y' in this period.

I_8 The parties are bound to bid alternatives in the bargaining game such that one given alternative is the most favorable one for L and another given alternative is the most favorable one for H.

I_9 The alternatives in the subset determined by I_7 and I_8 that can possibly be bid in this game, are determined prior to the bargaining game.

I_{10} Prior to the bargaining game the parties agree to limit their bargaining procedure to proposals for an agreement and to adhere to such a procedure until an agreement is reached or bargaining is broken up in a period, called period z, determined prior to the bargaining game.

I_{11} The liquidity of each party is so large in relation to the possible losses from no-agreement that neither party will face bankruptcy, even if no agreement is reached prior to period z.

I_{12} If one party proposes an alternative y which is *more* favorable for the *other* party than the alternative y' that the other party proposed in the preceding period, then an agreement is reached on y.

I_{13} If party H bids y' in period j', H must bid $y \leqslant y'$ in every period $j > j'$ and if L bids y' in period j' L must bid $y \geqslant y'$ in every period $j > j'$.

G.3 General Pay-off Assumptions

G_1 The pay-offs $\bar{y}_j$ and $\underline{y}_j$, assigned to a specific agreement y_j are independent of the play by which this agreement is reached.

G_2 The present value of one dollar, discounted from period j back to period 0, is equal to the present value of the same dollar, discounted first back from period j to j' and then from j' to 0.

G_3 The discount factor for discounting y dollars back from period j to period $j-1$ is independent of y.

G.4 Special Pay-off Assumptions

S-games are characterized by

$S_1: \overline{y}_j > \overline{y}_{j+1}$ and $\underline{y}_j > \underline{y}_{j+1}$

S_2: consisting of the following assumptions concerning any two periods j and j' such that $j < j'$:

$S_{2A}: j = SC(y) \Rightarrow j' = C(y)$ and $S_{2a}: j = sc(y) \Rightarrow j' = c(y)$
$S_{2B}: j' = SU(y) \Rightarrow j = U(y)$ and $S_{2b}: j' = su(y) \Rightarrow j = u(y)$

S′-games, constituting a proper subset of all S-games, are characterized by

S'_1: Equivalent to S_1
S'_2: Equivalent to S_2, if j and j' are allowed to take any real values
S'_3: consisting of the following assumptions about y and y', not necessarily integers, such that $y' > y$.

$S'_{3A}: j = SC(y) \Rightarrow j = C(y')$ and $S_{3a}: j = sc(y') \Rightarrow j = c(y)$
$S'_{3B}: j = SU(y') \Rightarrow j = U(y)$ and $S'_{3b}: j = su(y) \Rightarrow j = u(y')$

$S'_4: \overline{y}_{j+2} - \overline{y}_{j+1} \geqslant \overline{y}_{j+1} - \overline{y}_j$ and $\underline{y}_{j+2} - \underline{y}_{j+1} \geqslant \underline{y}_{j+1} - \underline{y}_j$
$S'_5: \overline{y+2}_j - \overline{y+1}_j \leqslant \overline{y+1} - \overline{y}_j$ and $\underline{y-2}_j - \underline{y-1}_j \leqslant \underline{y-1}_j - \underline{y}_j$

S-games*, constituting a proper subset of all S′-games, are characterized by

$S^*_1: \overline{y}_j = AyF(T) + B$ and $\underline{y}_j = a(N-y)f(T) + b$
$S^*_2: F'$ and $f' < 0$
$S^*_3: F''$ and $f'' \geqslant 0$
$S^*_4: dF^*/dT$ and $df^*/dT < 0$

G.5 Theorems

(For definition of P^0_y (P^0_5–P^0_7), P_x, P etc. see the list of notations on pp. 294 ff.)

		Assumptions		
		B	I	S[1]
T_1	If H has a decreasing pay-off over time after period i (fulfilled in S-games) and $i = C(x)$, then $(x,x+1)_{\underline{i-1}} = x_i$	B_1	I_1–I_4	
T_2	1. $i = C(x)$ and $i \in H$ 2. H has a decreasing pay-off over time and 3. L has only uncritical periods prior to period i, imply for $j \leqslant i$: $(x,x+1)_{\underline{j-1}} = x_j$ and $(x,x+1)_{\overline{j-2}} = x_j$	B_3	I_1–I_5	
T_3	If H bids in period i, then $i = C(x)$ and $i-1 = u(x)$ imply for $j \leqslant i$: $(x,x+1)_{\underline{j-1}} = x_j$ and $(x,x+1)_{\overline{j-2}} = x_j$.	B_3	I_1–I_5	S
T_4	$i = Cu(x)$ implies: $(x,x+1)_0 = x_1$ $i = cU(x)$ implies: $(x,x+1)_0 = x+1_1$	B_3	I_1–I_6	S
T_5 and T'_5	P^0_5 implies: $\overline{(5-m',6)}_{\bar{1}} < \bar{5}_1$ and $(5,6)_{\underline{0}} = 5_1$	B_4	I_1–I_6	S
T_6	No unique solution can be established in a game with the following characteristics: a. The game is broken up in period z, which is known. Each party obtains a smaller pay-off from this than from an agreement on any of the possible alternatives b. All periods are *un*critical for both parties for all two-alternative subgames c. S_1 holds.	B_4	I_1–I_{13}	
T_7 and T'_7	P^0_5–P^0_6 imply: $\overline{(5-m',7)}_{\bar{1}} < \bar{5}_1$ and $(5,7)_{\underline{0}} = 5_1$	B_4	I_1–I_{13}	S
	General: P_x and P_{x+1} imply: $\overline{(x-m',x)}_{\bar{i}} < \bar{x}_i$ and $(x,x+2)_{\underline{i-1}} = x_i$	B_4	I_1–I_{13}	S
T_8 and T'_8	P^0_5–P^0_7 imply: $\overline{(5-m',8)}_{\bar{1}} < \bar{5}_1$ and $(5,8)_{\underline{0}} = 5_1$	B_4	I_1–I_{13}	S
	General: P_x – P_{x+2} imply: $\overline{(x-m',x+3)}_{\bar{i}} < \bar{x}_i$ and $(x,x+3)_{\underline{i-1}} = x_i$	B_4	I_1–I_{13}	S

[1] The type of game investigated by the theorem. Blank implies that theorem applies generally.

		Assumptions		
		B	I	S
T_9 and T'_9	$P^0_5 - P^0_7$, $\overline{(7-m',7+k)_{\overline{3}}} < \overline{7}_3$ and $(7,7+k)_{\underline{2}} = 7_3$ imply: $\overline{(5-m',8+k)}_{\overline{1}} < \overline{5}_1$ and $(5,8+k)_{\underline{0}} = 5_1$	B_4	$I_1 - I_{13}$	S
T_{10}	P implies: $\overline{(x,x+k)_{\overline{i}}} < \overline{x}_i$ and $(x,x+k)_{\underline{i-1}} = x_i$	B_4	$I_1 - I_{13}$	S
T_{11}	P implies for $j \leqslant i-1$: $(x,x+k)_{\overline{j-2}} = x_j$ and $(x,x+k)_{\underline{j-1}} = x_j$	B_4	$I_1 - I_{13}$	S
T_{12}	P' implies for $j \leqslant i-1$: $(x-m,x)_{\underline{j-2}} = x_j$ and $(x-m,x)_{\overline{j-1}} = x_j$	B_4	$I_1 - I_{13}$	S
T_{13}	P and P' imply for $j \leqslant i-2$: $(x-m,x+k)_{j-1} = x_{j+1}$	B_4	$I_1 - I_{13}$	S
T_{14}	P_x and P'_x imply for $j \leqslant i-1$: $(x-m,x+k)_{j-2} = x_j$	B_4	$I_1 - I_{13}$	S'
T_{15}	The game $(1,n)_0$ has a unique solution x if: 1. there exists a y such that $3 = \mathrm{su}(y-1)\mathrm{SU}(y)$ 2. there exists within finite time a period j such that $j = \mathrm{C}(1)\mathrm{c}(n-1)$ 3. P_x and P'_x hold	B_4	$I_1 - I_{13}$	S'

G.6 Frequently Used Notations

The notations included in this list appear in more than one section in the text and in the literature appendix. Notations that are used in only one section[1] are explained in that section and hence not listed below. The most frequent meaning of the notation is given in this list. References to sections where a particular notation has a different meaning are shown in parentheses.

For notations involving numbers, e.g. $(6,7)_{\overline{3}}$, $\overline{6}_1$, etc., see y.

[1] With two digits, e.g. 5.1.

a	a constant
A	a constant
b	a constant
B	a constant
$\mathrm{B}_1, \ldots, \mathrm{B}_{12}$	behavioristic assumptions (see p. 289)
$\boldsymbol{B}_1, \ldots, \boldsymbol{B}_4$	sets of behavioristic assumptions (see p. 289)
c	L's cost of bargaining, computed on an annual basis, or (in Sections 8.4, 9.3 and 9.4) L's average variable cost (different meaning in Sections L.5 and L.6)
$\mathrm{c}(y)$	a critical characteristic. $j = \mathrm{c}(y)$ implies that $\underline{y+1}_j > \underline{y}_{j+1}$, i.e. that period j is critical for L as regards $(y, y+1)$
$\mathrm{cU}(y)$	$j = \mathrm{cU}(y)$ implies $j = \mathrm{c}(y)$ *and* $j = \mathrm{U}(y)$
C	H's cost of bargaining, computed on an annual basis, or (in Sections 8.4, 9.3 and 9.4) H's average variable cost (different meaning in Sections L.5 and L.6)
$\mathrm{C}(y)$	a critical characteristic. $j = \mathrm{C}(y)$ implies that $\bar{y}_j > \bar{y}_{j+1}$, i.e. that j is critical for H as regards $(y, y+1)$
$\mathrm{Cu}(y)$	$j = \mathrm{Cu}(y)$ implies that $j = \mathrm{C}(y)$ *and* $j = \mathrm{u}(y)$
d	used to denote the derivative (different meaning in L.4)
e	the base of the natural logarithmic system
f	a function of T, denoting how L's pay-off varies over time
f'	df/dT
f''	d^2f/dT^2
f^*	$f/-f'$
F	a function of T, denoting how H's pay-off varies over time

F'	dF/dT
F''	d^2F/dT^2
F^*	$F/-F'$
G_1, G_2, G_3	general pay-off assumptions (see p. 291)
H	party H(igh)
H	the set of periods in which H bids for a given bidding order
i	a period, generally one that is Cu(x) or Su(x)
$I_1, \ldots, I_{13}$	institutional assumptions of the basic model (see p. 290)
j	the number of a period in general
j'	the number of a period $\neq j$
k	an integer $\geqslant 1$, used in the connection $y+k$; $k \leqslant n-y$, when used in a game $(y, y+k)$; $k \leqslant n-y-1$, when used in critical characteristics, e.g. $j = C(y+k)$
L	party L(ow)
L	the set of periods in which L bids for a given bidding order
m	an integer $\geqslant 1$, used in the connection $y-m$; $m \leqslant y-1$, when used in a game $(y-m, y)$; $m \leqslant y-2$, when used in a critical characteristic, e.g. $j = c(y-m-1)$
m'	an integer $\geqslant 0$, used in the connection $y-m'$, e.g. in games $(y'-m,y)$ such that $y-m' \geqslant 1$
n	the number of the highest alternative of the bargaining game
N	$n+1$
p	the price quoted by L (different meaning in L.3.)
P	the price quoted by H (different meaning in L.3.)
P_x	$i = $ Cu(x), allowing for $i = $ S(x), if $i+1 = $ su(x)

P_{x+k}	$i = Cu(x+k)$
P'_x	$i = cU(x-1)$, allowing for $i = s(x-1)$, if $i+1 = SU(x-1)$
P'_{x+m}	$i+m = cU(x-m-1)$
P^0_y	$-4+y = Cu(y)$, allowing for $1 = S(5)$, if $2 = su(5)$
$\boldsymbol{P}$	the set of P_x and all assumptions P_{x+k}
$\boldsymbol{P}'$	the set of P'_x and all assumptions P'_{x+m}
q	L's sales quantity (different meaning in L.3)
Q	H's sales quantity (different meaning in L.3)
r	L's rate of interest, computed on an annual basis
R	H's rate of interest, computed on an annual basis
s	L's share of the joint periodic profit
$s(y)$	a critical characteristic. $j = s(y)$ implies that $\underline{y+1}_j = \underline{y}_{j+1}$, i.e. that j is semicritical for L as regards $(y,y+1)$
$sc(y)$	$j = sc(y)$ implies $j = s(y)$ *or* $c(y)$
$su(y)$	$j = su(y)$ implies $j = s(y)$ *or* $u(y)$
S	H's share of the joint periodic profit
S^*	the solution value of S in a continuous S*-game
$S(y)$	a critical characteristic. $j = S(y)$ implies that $\overline{y}_j = \overline{y+1}_{j+1}$, i.e. that j is semicritical for H as regards $(y,y+1)$
$SC(y)$	$j = SC(y)$ implies $j = S(y)$ *or* $C(y)$
$SU(y)$	$j = SU(y)$ implies $j = S(y)$ *or* $U(y)$
S_1, S_2	the assumptions of the S-games (see p. 292)
$S'_1, \ldots, S'_5$	the assumptions of the S′-games (see p. 292)

$S_1^*, \ldots, S_4^*$	the assumptions of the S*-games (see p. 292)
t	time in general
Δt	the length of each period
T	the time of agreement
T^*	the value of T such that $T/\Delta t = s(y-1)S(y)$, i.e. in continuous S*-games such that $F^*(T)+f^*(T) = \mu$
u	L's cardinal utility (in the von Neumann–Morgenstern sense)
$u(y)$	a critical characteristic. $j = u(y)$ implies that $\underline{y+1}_j < \underline{y}_{j+1}$, i.e. that j is uncritical for L as regards $(y, y+1)$
U	H's cardinal utility (in the von Neumann–Morgenstern sense)
$U(y)$	a critical characteristic. $j = U(y)$ implies that $\overline{y}_j < \overline{y+1}_{j+1}$, i.e. that j is uncritical for H as regards $(y, y+1)$
v	L's pay-off
V	H's pay-off
w	L's pre-agreement profit, computed on an annual basis
W	H's pre-agreement profit, computed on an annual basis
x	an alternative, generally the one on which an agreement is reached (see also various combinations of y)
y	an alternative in general
y_j	an agreement on alternative y in period j
$\overline{y}_j$	H's pay-off of an agreement on y_j
$\underline{y}_j$	L's pay-off of an agreement on y_j
y'	an alternative $\neq y$
$(y,y')_{\bar{j}}$	a situation in which L has bid y in period $j-1$ and H has bid y' in period j

$(y,y')_{\underline{j}}$	a situation in which H has bid y' in period $j-1$ and L has bid y in period j
$(y,y')_j$	a situation, where $j > 0$, representing both $(y,y')_{\bar{j}}$ and $(y,y')_{\underline{j}}$
$(y,y')_0$	a situation prior to the actual bargaining game, where y is L's and y' H's most desired alternative and it has not yet been decided who starts bidding
$\overline{(y,y')}_j$	H's pay-off from the situation $(y,y')_j$
$\underline{(y,y')}_j$	L's pay-off from the situation $(y,y')_j$
y^*	the value of y such that $j = s(y-1)S(y)$
z	the period in which the bargaining game is broken up in the basic model
z^*	the last interesting period (see p. 302)
Z	a fixed amount of time; for pay-off function 1 the time from the start of the bargaining game to the expiration of the contract
α	$(r-R)\mu/2$
β	the annual rate at which π grows *prior* to time T in pay-off functions 2+3; in particular it is assumed that $\beta = \beta_0 - \beta' T$ in pay-off function 3
γ	the rate at which π grows *after* time T in pay-off functions 2+3
ϵ	a very small number > 0. In connection with H and L it implies "belongs to"
μ	$N\Delta t = (n+1)\Delta t$
π	the joint profit, computed on an annual basis

G.7 Definitions of Some Frequently Used Concepts

acceptance bid: A bid in which one party accepts the alternative that the other party likes the most.

agreement: An act by which two or more parties, simultaneously and after exchanging information, irrevocably commit themselves to certain future actions.

agreement profit: The profit obtained *after* an agreement has been reached.

alternating bidding: The parties deliver their bids in every other period.

alternative: An ordered set of terms with the following two characteristics:

1. If a party in some period j prefers an agreement on some alternative y to an agreement on some other alternative y', then the party will also prefer an agreement on y to an agreement on y' in any other period.

2. The terms applying if an agreement is reached are determined for each period.

arbitration: A process by which a third party imposes a settlement on the two bargaining parties.

bargaining game: Part of the process towards an agreement that consists of the bargaining process *and* the preceding determination of the bidding order, if this is determined by the parties.

bargaining process: A process by which the parties in a bargaining situation exchange a number of bids for the purpose of reaching an agreement.

bargaining situation:

(in general) A situation in which there is a potential pay-off that a party can obtain only by reaching an agreement with some other party and where such an agreement is possible.

(as used in Chapters 3–10) A pure two-person bargaining situation, i.e. a situation in which two parties, in terms of certain resources at their disposal, can obtain an agreement pay-off only by making an agreement with each other.

basic model: The bargaining model in Chapters 4–6, based on I_1–I_{13}.

behavioristic assumptions: Assumptions concerning the properties of the parties – their thought processes and patterns of behavior. These assumptions, which could also be called behavioral assumptions, concern factors that can *not* be controlled in experimental replication of the bargaining situation.

better reply: H's strategy σ is a better reply to a specific strategy of L's than σ' if σ gives H a higher pay-off than σ' does.

bid:
(as a noun) A proposal for agreement on a specific set of terms.
(as a verb) To deliver a proposal (orally or in writting) for an agreement on certain terms.

bidding order: A specific order in which the bids are delivered, specifying which party bids in each period.

bilateral monopoly: A situation with the following characteristics: One seller is the only producer or owner of a particular commodity, i.e. a product or a service, and the buyer is the only one interested in aquiring it.

complete information: (in the sense used in this study) For every possible agreement, each party can assign a correct value to the different outcome factors that are relevant for both his own and the other party's preferences over the outcomes.

critical characteristic: A notation characterizing a period in terms of whether it is critical, semicritical or uncritical for a certain party.

critical period: A period is critical for a party in a two-alternative game, characterized by I_1-I_4, if his pay-off from accepting the other party's terms in this period is *larger* than his pay-off from having the other party accept his terms in the *next* period.

distribution problem: The problem remaining in a bargaining situation after the parties have limited themselves to alternatives for which they have completely opposing preferences.

dominate: H's strategy σ dominates σ', if σ is a better reply to at least one of L's strategies than σ', while σ' is *not* a better reply to any of L's strategies than σ. If σ is a better reply to each of L's strategies than σ', then σ *strictly* dominates σ'.

duopoly: A situation with only two sellers on a market.

efficiency problem: The problem of establishing a set of alternatives for which the parties have completely opposing preferences.

equilibrium pair: A pair of strategies such that there is *no* better reply to the other party's equilibrium strategy than one's own equilibrium strategy.

game: (as used in Chapters 3–6) Bargaining game.

general model: A general version of our basic model, presented in Chapter 4, allowing for an analysis of any game based on I_1–I_{13} and G_1–G_3.

good-faith bargaining: A rule for the bargaining game implying that if a party has bid a specific alternative in a certain period, he may *not* bid an alternative that is less favorable to the other party in a later period.

insight assumption: Assumption B_{10}.

insistance bid: A bid in which a party insists on the alternative most favorable to him.

institutional assumptions: Assumptions that concern the properties of the bargaining situation, e.g. how and when bids are exchanged. In an experimental replication of the bargaining situation, the experimenter has control over the factors covered by these assumptions.

last interesting period: An abbreviation of "the last period that can possibly be interesting". This is a period, denoted as z^*, with the following characteristics: The exclusion of all periods after z^* will not affect the analysis, provided all periods are characterized by S_1 and all periods $1, \ldots, z^*$ are characterized by S_2.

last period: The last period of bidding in a bargaining game. In the next period, period z, the game is broken up.

mediation: Process by which a third party attempts to get the bargaining parties to agree by proposing and discussing possible solutions.

negotiation: A process involving a pre-bargaining phase and a bargaining game.

normative model: A model aimed at fulfilling the normative purpose for *one* party in a decision situation, characterized by the institutional assumptions of the model. This purpose is fulfilled if the party – after being thoroughly informed about the characteristics of the model and the mode of behavior recommended by the model – will want to follow this recommended mode of behavior, at least to a significant extent.

outcome factors: Factors which affect a party's preference for a certain agreement.

partial solution: The determination of how some party will choose in at least *one* period when the bidding order is given.

pay-off: An *index* for the combined effect of the outcome factors such that each party will always prefer an alternative with a higher pay-off to one with a lower pay-off.

period: (as used in Chapters 3–6) The amount of time, which may vary in length, during which one party delivers one bid. (In 8.2 this is called a bidding period.)

play: A sequence of choices in a game, one following the other, until the game is terminated.

pre-agreement profit: A profit made in each period prior to the agreement, but in no period after the agreement.

pre-bargaining phase: The part of a negotiation which takes place prior to the bargaining game and in which the efficiency problem is solved and the rules of the bargaining game established.

rational behavior:
(more generally) Behavior governed by extensive and explicit thought processes of an intelligent and purposive individual.
(more specifically) Behavior according to assumption set B_1, B_2, B_3 or B_4 or some larger set of assumptions of which B_4 is a subset. In this study the concept refers mainly to behavior according to set B_1, but in some instances it also includes rational expectations, covered by assumptions B_8–B_{10}.

semicritical period: A period is semicritical for a party in a two-alternative game, characterized by I_1–I_4, if the party's pay-off from accepting the other party's terms in this period is *equal* to his profit from having the other party accept his terms in the next period.

simultaneous bidding: The parties deliver their bids at (roughly) the same time so that one party does *not* know the other party's most recent bid.

solution: Determination of the *terms* of the agreement, the *period* of agreement and what each party has bid in each period prior to the agreement.

special model: A special version of the basic model, presented in Chapters 5 and 6, dealing exclusively with S-games.

stop rule: A rule implying that the game is broken up at the very latest in some specific period with a finite number.

subgame: (as used in Chapters 4–8) A part of a larger game with the characteristics that at least one party's best alternative in the subgame is different from his best alternative in the larger game.

terms: A set of values of certain factors determined by the agreement, e.g. price, quantity, etc.

uncritical period: A period is *uncritical* for a party in a two-alternative game, characterized by I_1–I_4, if the party's pay-off from accepting the other party's terms in this period is *smaller* than his profit from having the other party accept his terms in the next period.

unique solution: A solution which does not rely on the assumption that the party who starts bidding is given.

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